Chapter 6 Homework – Monte Carlo Integration Part II
25 Points

1. Let X be distributed as a truncated exponential distribution with density $f(x) = \frac{e^{-x}}{1-e^{-1}}$ for $0 \le x \le 1$. In this problem we use the importance functions shown below to obtain an approximation for the integral

$$E\big(h(X)\big) = \frac{1}{1-e^{-1}} \int_0^1 \frac{1}{1+x^2} e^{-x} dx.$$

using 20,000 random variates. Obtain an approximation for the standard error in each case.

(a) [3 points] $g(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$ [Cauchy]

(b) [3 points] $g(x) = \frac{4}{\pi(1+x^2)}, \quad 0 \le x \le 1.$ [Truncated Cauchy]

(c) [3 points] $g(x) = e^{-x}, x > 0,$ [The exponential distribution]

Hint: Use rcauchy() in R to generate random for Cauchy and Truncated Cauchy, and rexp() to generate values from exponetial.

2. [6 points] The tri-variate t distribution with parameters $\mu = 0$, covariance Σ, and 5 degrees of freedom has a density that is proportional to

$$f(\boldsymbol{x}; \Sigma) \propto \left(5 + \boldsymbol{x}^{\boldsymbol{T}} \Sigma^{-1} \boldsymbol{x}\right)^{-4}.$$

By generating 20,000 random variates from an appropriate tri-variate normal and using the importance sampling method (using standardized sampling weights) approximate the probability

$$P(-\infty \le X_1 \le 1, -\infty \le X_2 \le 4, -\infty \le X_3 \le 2),$$

where (X_1, X_2, X_3) come from the above tri-variate t with 5 degrees of freedom and covariance matrix

$$\Sigma = \begin{pmatrix} 1 & \frac{3}{5} & \frac{1}{3} \\ \frac{3}{5} & 1 & \frac{11}{15} \\ \frac{1}{3} & \frac{11}{15} & 1 \end{pmatrix}.$$

The true value of the probability to a good approximation is 0.79145379.

3. For generating $X \sim N(0,1)$ using an Accept/Reject algorithm, we could generate $U \sim \mathrm{uniform}(0,1)$ and $Y \sim \mathrm{double-exponential}\ (\lambda = 1)$. To generate values from Y, we generate an $\mathrm{exponential}\ (\lambda = 1)$, and attach a positive or negative sign to Y with equal probability. The density of Y is $g(y) = e^{-|y|}/2$, for -∞ $< x < \infty$.

(a) [2 points] Show that the density of the double exponential density with parameter $\lambda = 1$ intersects the density of the standard normal density at a value of $y > 0$ and at a value of $y < 0$. While a graph may help you understand this, you should answer this question analytically.

(b) [3 points] In light of part (a), to construct an envelope for the standard normal, using the double exponential with parameter $\lambda = 1$, we need to set $e(x) = \frac{g(x)}{\alpha}, for\ some\ 0 < \alpha < 1$. Find an optimal α that minimizes the number of rejections if we are to use the accept-reject algorithm to generate values from the standard normal, by using random values generated from the double-exponential.

(c) [5 points] Apply the Accept/Reject algorithm to generate values from the standard normal distribution, by generating 10,000 $U \sim \mathrm{uniform}(0,1)$ and 10,000 double exponential random variable. Write your algorithm, include your R code, and a histogram of the generated values.