Homework3part1

Henry Surjono

Part A

$\ell(\mu,\Sigma|x_1,x_2,...x_n) = -\frac{1}{2}nplog(2\pi) + nlog|\Sigma| + trace(\Sigma^{-1}C(\mu))$, $C(\mu) = \sum_{i=1}^{n}[(x_i-\mu)(x_i-\mu)^T]$.

for $\partial\ell(\partial\mu)$

$\partial\ell(\partial\mu) = trace\sum_{i=1}^{n}\Sigma^{-1}(x_i-\mu)(\partial\mu^T)$.

for $\partial\ell(\partial\sigma)$

$$\begin{aligned}
\partial\ell(\partial\Sigma) = &-\frac{n}{2}trace(\Sigma^{-1}\partial\Sigma) + \frac{1}{2}trace(\Sigma^{-1}(\partial\Sigma)\Sigma^{-1}\sum_{i=1}^{n}(xi-\mu)(xi-\mu)^T)\\
&= -\frac{n}{2}trace(\Sigma^{-1}\partial\Sigma) - \frac{1}{n}\Sigma^{-1}(\partial\Sigma)\Sigma^{-1}\sum_{i=1}^{n}(xi-\mu)(xi-\mu)^T)\\
&= -\frac{n}{2}trace(\Sigma^{-1}\partial\Sigma) - \frac{1}{n}\Sigma^{-1}\sum_{i=1}^{n}(xi-\mu)(xi-\mu)^T\Sigma^{-1}(\partial\Sigma))\\
&= -\frac{n}{2}trace(\Sigma^{-1}(\Sigma - \frac{1}{n}\sum_{i=1}^{n}(xi-\mu)(xi-\mu)^T)\Sigma^{-1}\partial\Sigma)
\end{aligned}$$

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Let A= $\Sigma^{-1}(\Sigma - \frac{1}{n}\sum_{i=1}^{n}(xi-\mu)(xi-\mu)^T\Sigma^{-1})$. $\partial(\partial\Sigma) = -\frac{n}{2}trace(A)\partial\Sigma$.

for $dl(\partial\mu,\partial\mu)$

$\partial\partial\ell(\partial\mu\partial\mu) = -ntrace(\Sigma^{-1}\partial\mu\partial\mu^T)$

for $dl(\partial\mu,\partial\sigma)$

$\partial\partial\ell(\partial\mu,\partial\Sigma) = trace(-\Sigma^{-1}(\partial\Sigma)\Sigma^{-1}\sum_{i=1}^{n}(x_i-\mu)\partial\mu^T$

for $\partial(\partial\sigma,\partial\sigma)$

$$\begin{aligned}
\partial\ell(\partial\Sigma\partial\Sigma) = &-\frac{n}{2}trace(-\Sigma^{-1}(\partial\Sigma)\Sigma^{-1}\partial\Sigma - \frac{1}{n}[-\Sigma^{-1}(\partial\Sigma)\Sigma^{-1}\partial\Sigma^{-1}(C(\mu) + \Sigma^{-1}(\partial\Sigma)(-\Sigma^{-1})(\partial\Sigma)\Sigma^{-1})])\\
&= ntrace[\Sigma^{-1}(\Sigma - \frac{C(\mu)}{n}) - \frac{1}{2}I)]\Sigma^{-1}(\partial\Sigma)\Sigma^{-1}(\partial\Sigma)
\end{aligned}$$

for dl/dmu

$\frac{\partial \ell}{\partial \mu_i} = \Bigg[\Sigma^{-1} \sum_{i=1}^{n} (x_i - \mu)\Bigg]_i$

for $dl/d\sigma_{i=j}$ and i does not equal j

Let A= $\Sigma^{-1}(\Sigma - \frac{1}{n}\sum_{i=1}^{n}(xi-\mu)(xi-\mu)^T\Sigma^{-1})$. $\frac{\partial \ell}{\partial \sigma_{ii}} = -\frac{n}{2}A_{ii}$ $\frac{\partial \ell}{\partial \sigma_{ij}} = -\frac{n}{2}\Bigg[A_{ij} + A_{ji}\Bigg]$

for $dl/d\mu_{ij}d\mu_{i=j}$ and i does not equal j

$\frac{\partial \partial \ell}{\partial \mu_i \partial \mu_j} = -n\Bigg[\Sigma^{-1}\Bigg]_{ij}$

for $dl/(d\mu, d\sigma)_{i=j}$ and i does not equal j

when i = j $\frac{\partial \partial \ell}{\partial \mu_k \partial \sigma_{ii}} = -\sum_{w=1}^{p}\Bigg[\Bigg[\Sigma^{-1}\Bigg]_{iw}\Bigg[\Sigma^{-1}\Bigg]_{ki}\Bigg[\sum_{z=1}^{n}(x_z-\mu)\Bigg]_w\Bigg]$

for $i \neq j$ $\frac{\partial \partial \ell}{\partial \mu_k \partial \sigma_{ij}} = -\sum_{w=1}^{p}\Bigg[\Bigg[\Bigg[\Sigma^{-1}\Bigg]_{iw}\Bigg[\Sigma^{-1}\Bigg]_{kj} + \Bigg[\Sigma^{-1}\Bigg]_{jw}\Bigg[\Sigma^{-1}\Bigg]_{ki}\Bigg]\sum_{z=1}^{n}(x_z-\mu)\Bigg]_w\Bigg]$

for $dl/d\sigma_{ij}, d\sigma_{kl}$

Let A= $\Bigg[\Sigma^{-1}(\Sigma - \sum_{i=1}^{n}(xi-\mu)(xi-\mu)^T - \frac{1}{2}I)\Sigma^{-1}\Bigg]$. Let i =1, j=2, k=3, l=4.

When i=j,k=l) $\frac{\partial \partial \ell}{\partial \sigma_{ij} \partial \sigma kl} = nA_{ki}[\Sigma^{-1}]_{ik}$

When$i \neq j$ and $k \neq l$

$\frac{\partial \partial \ell}{\partial \sigma_{ij} \partial \sigma kl} = n\Bigg[A_{kj}[\Sigma^{-1}]_{il} + A_{ki}[\Sigma^{-1}]_{jl} + A_{lj}[\Sigma^{-1}]_{ik} + A_{li}[\Sigma^{-1}]_{jk}\Bigg]$.

When$i \neq j$ and $k = l$ $\frac{\partial \partial \ell}{\partial \sigma_{ij} \partial \sigma kl} = n\Bigg[A_{kj}[\Sigma^{-1}]_{il} + A_{ki}[\Sigma^{-1}]_{jl}\Bigg]$.

When$i = j$ and $k \neq l$ $\frac{\partial \partial \ell}{\partial \sigma_{ij} \partial \sigma kl} = n\Bigg[A_{ki}[\Sigma^{-1}]_{il} + A_{li}[\Sigma^{-1}]_{ik}\Bigg]$.

Part B

for $E[dl/d\mu_{ij}d\mu_{i=j}]$ and i does not equal j

Since the expected value of Σ^{-1} is itself.

$E\Bigg[\frac{\partial \partial \ell}{\partial \mu_i \partial \mu_j}\Bigg] = n\Bigg[\Sigma^{-1}\Bigg]_{ii}$

$E\Bigg[\frac{\partial \partial \ell}{\partial \mu_i \partial \mu_j}\Bigg] = n\Bigg[\Sigma^{-1}\Bigg]_{ij}$ Since the expected value of Σ^{-1} is itself.

for $E[dl/(d\mu, d\sigma)_{i=j}]$ and i does not equal j

Since $E((x_i - \mu) = 0)$, then when i = j $\frac{\partial\partial\ell}{\partial\mu_k \partial\sigma_{ii}} = -\sum_{w=1}^{p} \left[\left[\Sigma^{-1} \right]_{iw} \left[\Sigma^{-1} \right]_{ki} \left[\sum_{z=1}^{n} (x_z - \mu) \right]_{w} \right]$

Since $E((x_i - \mu) = 0)$, then $E[dl/(d\mu, d\sigma)_{i=j}] = 0$ for i=j and $i \neq j$

for $E[dl/d\sigma_{ij}, d\sigma_{kl}]$

Let A= $\left[\Sigma^{-1}(\Sigma - \sum_{i=1}^{n}(xi - \mu)(xi - \mu)^T - \frac{1}{2}I)\Sigma^{-1} \right]$. Let E[A] = $-\frac{1}{2}\Sigma^{-1}$

When i=j,k=l) $E[\frac{\partial\partial\ell}{\partial\sigma_{ij}\partial\sigma kl}] = n\frac{1}{2}\Sigma^{-1}_{ki}[\Sigma^{-1}]_{ik}$

When$i \neq j$ and $k \neq l$

$E[\frac{\partial\partial\ell}{\partial\sigma_{ij}\partial\sigma kl}] = n\left[\frac{1}{2}\Sigma^{-1}_{kj}[\Sigma^{-1}]_{il} + \frac{1}{2}\Sigma^{-1}_{ki}[\Sigma^{-1}]_{jl} + \frac{1}{2}\Sigma^{-1}_{lj}[\Sigma^{-1}]_{ik} + \frac{1}{2}\Sigma^{-1}_{li}[\Sigma^{-1}]_{jk} \right]$.

When$i \neq j$ and $k = l$ $E[\frac{\partial\partial\ell}{\partial\sigma_{ij}\partial\sigma kl}] = n\left[\frac{1}{2}\Sigma^{-1}_{kj}[\Sigma^{-1}]_{il} + \frac{1}{2}\Sigma^{-1}_{ki}[\Sigma^{-1}]_{jl} \right]$.

When$i = j$ and $k \neq l$ $E[\frac{\partial\partial\ell}{\partial\sigma_{ij}\partial\sigma kl}] = n\left[\frac{1}{2}\Sigma^{-1}_{ki}[\Sigma^{-1}]_{il} + \frac{1}{2}\Sigma^{-1}_{li}[\Sigma^{-1}]_{ik} \right]$.