Name (please print)________________________________

Note: Show your work on all problems. Please do not include any obvious algebraic details. A total of 20 points is possible.

1. `[4 Points]` Using **only** the Axioms of probability and the finite additivity property show that for any two events A and B if $A \subset B$, then $P(A) \leq P(B)$. State the properties that you use at each step. You don't need to prove set-theoretic properties. Note: You will not get any credit if you use other theorems on properties of $P(\cdot)$.

2. `[5 Points]` Suppose that $\mathcal{B}$ is a Borel field. Let $A_1 \in \mathcal{B}$ and $A_2 \in \mathcal{B}$. Use only the three properties of a Borel field, given in Definition 1.2.1 of your text (page 6), to show that $A_1 \cap A_2 \in \mathcal{B}$. Note you are only allowed to use set theory and the three properties in your proof. Cite the properties that you use at each step of your proof.

3. `[5 Points]` There are four children in a family. Their mom purchases six different gifts and decides to divide the gifts randomly between the children. Assuming that a child can receive no gift, or multiple gifts (up to all six gifts), What is the probability that each child receives at least one gift?

4. `[3 Points]` In the previous problem, thinking only about the number of gifts received by each child, how many different possibilities are there? [for example, (3, 1, 1, 1) is one possibility where child 1 receives 3 gifts and each of the other children receive one; or (1, 3, 1, 1) is another possibility where the second child receives three gifts and the remaining children receive one; yet (0, 0, 6, 0) is another choice where the third child receives all the gifts. I have given examples of three possibilities. The problem is asking for the total number of such possibilities.]

5. `[3 Points]` Suppose that we have collection of six numbers., $\{1, 2, 7, 8, 14, 20\}$. If we draw six numbers with replacement from this set, what is the probability that the mean of the six numbers drawn is 11?

Name (please print)____

Note: Show your work on all problems. Each problem is worth 5 points. A total of 25 points is possible.

1. Prove or give a counter example: If A and B are two events, then $P(A|B) + P(A|B^c) = 1$.

2. We have two coins, each having $P(heads) = \alpha$, where $0 \leq \alpha \leq 1$. We flip these two coins continually and simultaneously until either two heads appear or two tails appear. What is the probability that two heads appear first; that is two heads appear before two tails appear.Compute the probability in terms of α.

3. Use mathematical induction to show that

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

You can assume that you know $P(A \cap B) = P(A|B)P(B)$.

4. Let X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$ both of which have support in $(-\infty, \infty)$. Consider the fixed values a and b with $a < b$. Show that the following function is a pdf with support $[a, b]$:

$$g(x) = f(x)/[F(b) - F(a))]$$

5. Suppose that X is a continuous random variable with cumulative distribution function (cdf)

$$F(x) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{x+1}{5} & \text{if } -1 \le x \le 4 \\ 1 & \text{if } x > 4. \end{cases}$$

Obtain the cumulative distribution function of $Y = 4 - (x-1)^2$.

Name (please print)____

Note: Show your work on all problems. Each part of each problem is worth 5 points. A total of 25 points is possible.

1. Let $X \sim \text{gamma}(\alpha, \beta)$. Show that $EX^n = \beta^n \Gamma(n + \alpha)/\Gamma(\alpha)$, where n is a positive integer.

2. Let X be a continuous random variable with pdf f_X and cdf F_X. Moreover, assume that f_X is symmetric about a point a.

   (a) Show that the random variables $U = X - a$ and $W = a - X$ have the same distribution.

   (b) Assuming that the k-th central moment of X exists, show that for an odd positive integer $E[X-a]^k = 0$.

3. Let X be a random variable with pmf $f_X(x) > 0$ for $x = 1, 2, 3, \cdots$ (positive integers), and $f_X(x) = 0$ for all other values of x. Then, the pmf of X_T, the random variable X truncated at $X = 1$, is given by

$$f_{X_T}(x) = \frac{f_X(x)}{P(X > 1)}, \quad \text{for} \quad x = 2, 3, \cdots .$$

(a) Verify that $f_{X_T}(x)$ is a pmf.

(b) Assume that $f_X(1) = 1/4$, $E(X) = \mu$. Obtain $E(X_T)$ as a function of μ.

Name (please print)____

Note: Show your work on all problems. Each part of each problem is worth 5 points. A total of 25 points is possible.

1. Let X be a random variable with pdf

$$f(x) = \frac{1}{2\beta} e^{-|x-\alpha|/\beta}, \quad -\infty < x < \infty, \quad -\infty < \alpha < \infty, \quad \beta > 0.$$

Derive the mgf of the random X. State the domain where the mgf is defined.

2. Let Y be a geometric random variable with parameter p, where p is the success probability. Show that as p approaches zero, the random variable $W = pY$ converges to the exponential distribution with parameter $\beta = 1$.

3. Theaters A and B compete for the business of 1000 customers. Assume that Theater A shows a more popular movie, and thus the probability that a randomly selected customer chooses Theater A is $3/4$. Let n be the number of seats in Theater A. Write an equation that you would solve for n such that the probability of turning away a customer by Theater A, because of a full house, is less than 5%. Do not solve for n.

4. Let X have the standard normal distribution (i.e. $X \sim N(0,1)$). Use the moment generating function of X to obtain $E(X^4)$.

5. Let Y be a random variable with pmf

$$P(Y = \sqrt{3}) = P(Y = -\sqrt{3}) = 1/6, \quad P(Y = 0) = 2/3.$$

Obtain $E(Y^4)$.