Homework 3 (Part 1)

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Part (a)

$$\partial\ell(\partial\mu) = Tr\Big(\mathbf{\Sigma^{-1}}\sum_{i=1}^{n}\partial\mu(\mathbf{x}_i-\mu)^T\Big)$$

the ith element will be $\Big[\mathbf{\Sigma^{-1}}\sum_{i=1}^{n}(\mathbf{x}_i-\mu)\Big]_i$

$$\partial\ell(\partial\mathbf{\Sigma}) = \frac{-n}{2}Tr\Big(\mathbf{\Sigma}^{-1}(\mathbf{\Sigma}-\frac{C(\mu)}{2})\mathbf{\Sigma}^{-1}\partial\mathbf{\Sigma}\Big)$$

here let $A = \mathbf{\Sigma}^{-1}(\mathbf{\Sigma}-\frac{C(\mu)}{n})\mathbf{\Sigma}^{-1}$

when $i=j$

$$\frac{\partial\ell}{\partial\sigma_{ij}} = \frac{-n}{2}A_{ii}$$

when $i\neq j$

$$\frac{\partial\ell}{\partial\sigma_{ij}} = \frac{-n}{2}(A_{ij}+A_{ji})$$

$$\partial^2\ell(\partial\mu\partial\mu) = -Tr\Big(-\mathbf{\Sigma}^{-1}n\partial\mu\partial\mu^T\Big)$$

nonzero when $i\neq j$

$$\frac{\partial^2\ell}{\partial\mu_i\partial\mu_j} = -n\Sigma_{ij}^{-1}$$

$$\partial^2\ell(\partial\mu\partial\mathbf{\Sigma}) = Tr\Big((-\mathbf{\Sigma}^{-1}(\partial\mathbf{\Sigma})\mathbf{\Sigma}^{-1})\sum_{i=1}^{n}\partial\mu(\mathbf{x}_i-\mu)^T\Big)$$

when $i=j$

$$\frac{\partial^2\ell}{\partial\mu_k\partial\sigma_{ij}} = -\sum_{n=1}^{p}\Big[\Sigma_{in}^{-1}\Sigma_{ki}^{-1}\Big(\sum_{z=1}^{m}(x_z-\mu)^T\Big)\Big]$$

when $i\neq j$

$$\frac{\partial^2\ell}{\partial\mu_k\partial\sigma_{ij}} = -\sum_{n=1}^{p}\Big[\Big(\Sigma_{in}^{-1}\Sigma_{kj}^{-1}+\Sigma_{jn}^{-1}\Sigma_{ki}^{-1}\Big)\Big(\sum_{z=1}^{m}(x_z-\mu)^T\Big)\Big]$$

$$\partial^2\ell(\partial\mathbf{\Sigma}\partial\mathbf{\Sigma}) = nTr\Big(\mathbf{\Sigma}^{-1}(\mathbf{\Sigma}-\frac{C(\mu)}{n}+\frac{1}{2}I)\mathbf{\Sigma}^{-1}\partial\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\partial\mathbf{\Sigma}\Big)$$

let $A = \mathbf{\Sigma}^{-1}(\mathbf{\Sigma} - \frac{C(\mu)}{n} + \frac{1}{2}I)\mathbf{\Sigma}^{-1}$

when $i = j, k = l$

$$\frac{\partial^2 \ell}{\partial \sigma_{ij} \partial \sigma_{kl}} = \frac{n}{2} A_{ki} \Sigma^{-1}_{ik}$$

when $i \neq j, k = l$

$$\frac{\partial^2 \ell}{\partial \sigma_{ij} \partial \sigma_{kl}} = \frac{n}{2} \Big[A_{kj} [\Sigma^{-1}]_{il} + A_{ki} [\Sigma^{-1}]_{jl} \Big]$$

when $i = j, k \neq l$

$$\frac{\partial^2 \ell}{\partial \sigma_{ij} \partial \sigma_{kl}} = \frac{n}{2} \Big[A_{kj} [\Sigma^{-1}]_{il} + A_{li} [\Sigma^{-1}]_{ik} \Big]$$

when $i \neq j, k \neq l$

$$\frac{\partial^2 \ell}{\partial \sigma_{ij} \partial \sigma_{kl}} = \frac{n}{2} \Big[A_{kj} [\Sigma^{-1}]_{il} + A_{ki} [\Sigma^{-1}]_{jl} + A_{lj} [\Sigma^{-1}]_{ik} + A_{li} [\Sigma^{-1}]_{jk} \Big]$$

Part (b)

we know that $E[\Sigma^{-1}] = \Sigma^{-1}$ and $E[x_i - \mu] = 0$ when considering $E[\frac{\partial^2 \ell}{\partial \sigma_{ij} \partial \mu_k}]$ both cases have $\sum_{i=1}^{n} (x_i - \mu)$ being multiplied thus both cases are zero.

when considering $E[\frac{\partial^2 \ell}{\partial \mu_{ij} \partial \mu_{ij}}]$

$i \neq j$ case

$$E[\frac{\partial^2 \ell}{\partial \mu_i \partial \mu_i}] = n[\Sigma^{-1}]_{ij}$$

when considering $E[\frac{\partial^2 \ell}{\partial \sigma_{ij} \partial \sigma_{kl}}]$ we know that

when $i = j, k = l$

$$E[\frac{\partial^2 \ell}{\partial \sigma_{ij} \partial \sigma_{kl}}] = \frac{n}{2} \Sigma^{-1}_{ki} \Sigma^{-1}_{ik}$$

when $i \neq j, k = l$

$$E[\frac{\partial^2 \ell}{\partial \sigma_{ij} \partial \sigma_{kl}}] = \frac{n}{2} \Big[\Sigma^{-1}_{kj} [\Sigma^{-1}]_{il} + \Sigma^{-1}_{ki} [\Sigma^{-1}]_{jl} \Big]$$

when $i = j, k \neq l$

$$E[\frac{\partial^2 \ell}{\partial \sigma_{ij} \partial \sigma_{kl}}] = \frac{n}{2} \Big[\Sigma^{-1}_{kj} [\Sigma^{-1}]_{il} + \Sigma^{-1}_{li} [\Sigma^{-1}]_{ik} \Big]$$

when $i \neq j, k \neq l$

$$E[\frac{\partial^2 \ell}{\partial \sigma_{ij} \partial \sigma_{kl}}] = \frac{n}{2} \Big[\Sigma^{-1}_{kj} [\Sigma^{-1}]_{il} + \Sigma^{-1}_{ki} [\Sigma^{-1}]_{jl} + \Sigma^{-1}_{lj} [\Sigma^{-1}]_{ik} + \Sigma^{-1}_{li} [\Sigma^{-1}]_{jk} \Big]$$