Gibs and MH algorithm [85 points]

Problem 1: Gibbs Sampling (Bivariate Normal): Consider using Gibbs Sampling to generating random values from the random vector (X, Y) having a bivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix Σ.

a) [8 points] Write a general R program with the following input:

- n = the number of bivariate samples to be generated (including burn-in).
- mu = a 2 by 1 vector, indicating the mean of the random vector
- sigma = a 2 by 2 matrix, indicating the covariance of the random vector
- x0 = initial starting value to start the chain
- nburn = The number of generated values to burn.
- Seed = random number generation seed

Apply your code to generate from a bivariate normal with the follow mean and covariance:

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & .5 \\ .5 & .4 \end{pmatrix}$$

Specifically start with x0 = 7, set $n = 5{,}000$, seed = 534. Show the first five rows of your generated data.

b) [2 points] Graph the generated values of X and generated values of Y in separate graphs, where in each graph the horizontal axis shows the iteration number, and the vertical axis shows the corresponding generated value.

c) [3 points] Burn 1000 elements of the chain that you obtained and produce a bivariate plot of x values versus y values. Explain, why you expect to see the shape of the plot that you obtain.

Problem 2. (Gibbs Sampling) Do Problem 7.5 on page 231 of your text.

(a) [3 points], (b) [4 points], (c) [10 points] (d) [4 points] (e) [3 points], (f) [3 points], (g) [5 points]

Note: The dataset has a censored variable (let's use notation c_i) for that variable. $c_i = 0$ means that you have observed a recurrence time and $c_i = 1$ means that the value is a censored time. However, the likelihood $L(\theta, \tau|\boldsymbol{y})$ given in the book uses δ_i, where $\delta_i = 1 - c_i$.

Problem 3. [10 points] (MH-sampling) Consider generating random values from a double exponential distribution with density $f(x) = \frac{1}{2}e^{-|x|}$ for $x \in R$. Use M-H algorithm with the proposal density being the normal distribution with mean 0 and variance σ^2. Make a comparison of your chains for $\sigma^2 = .05, 0.5, 1, 3, 100$. Plot the chain, and explain the mixing of the chains for various values of σ^2.

Problem 4. [10 points] (MH-sampling) Consider generating random values from the exponential random variable with mean 1. (a) Use M-H algorithm with the proposal density being the normal distribution with mean 0 and variance σ^2. Use an appropriate σ^2. (b) Determine your acceptance rate and plot the chain. (c) Plot the running means and determine a burn-in value based on the plot. (d) Make a Q-Q plot of your generated values (after burn-in) against a sample of the same size generated using the `rexp` function in R.

Problem 5 [20 points] **(Simulating a bivariate normal).** In section 7.1 of the the paper "Understanding the Metropolis Hasting Algorithm" by Chib and Greenberg (*The American Statistician, Vol 49, 1995, click here*), examples of several candidate densities are given to generate from a bivariate normal with mean $(1,2)^T$, variances 1, and correlation 0.9. Read these examples, and implement the M-H using the four candidates given on page 333. Generate 10,000 values from each chain, and burn the first 2000. Then for the remaining values perform the following tasks:

1) Generate 8000 bivariate values from the target density, and plot the pairs in the x-y plane (as in top panel of Figure 3 in Chip and Greenberg). Then, draw a similar plot for the values you obtain for your four chains. Briefly comment on how these plots look.
2) Plot the chain x axis is t, and y-axis is the value generated. Connect each value to the next by a line. Judge how well each chain mixes?
3) Compute the acceptance rates for each of the chains. Are the acceptance rates all reasonable? Comment.