MATH 538: Bayesian Data Analysis (2024)
Assignment #3

Due date: Wednesday, October 27, 2024

Book Problems Chapter 3: 6, 11 (use Metropolis MC instead of grid sampling), 15

Additional Problems

1. Consider the Normal model with reference prior:

$$\begin{aligned} y_i &\overset{iid}{\sim} N(\mu, \sigma^2) \quad i = 1, ..., n \\ p(\mu, \sigma^2) &\propto (\sigma^2)^{-1} \end{aligned}$$

   (a) Write out (on paper) the algorithm for obtaining the Monte Carlo samples from the joint posterior $p(\mu, \sigma^2|data)$

   (b) Simulate 50 values from a normal population of your choice.

   (c) Implement your algorithm in R to infer on μ and σ given your simulated observations. Provide plots of your Monte Carlo samples for the joint and marginal posterior distributions. Overlay the true marginal posterior densities for μ and σ^2 on their respective Monte Carlo approximations. Discuss your findings; were your MAP estimates close to the true population parameter values?

   (d) Use your posterior samples of μ and σ^2 to obtain a Monte Carlo approximation of the posterior predictive distribution. Overlay the Monte Carlo approximation density with the true posterior predictive density $p(\tilde{y}|data)$.

   (e) Can you apply Gibbs Sampling to this model? If so, derive the posterior conditionals, state your algorithm, and implement it to your simulated data, and provide posterior plots of your model parameters and provide a short discussion.

2. Consider the Gamma model for the artichoke plants from Homework #1:
   Suppose we observe 7 artichoke plants that have the following heights (in meters):

   1.8 1.6 1.4 1.6 1.4 1.5 1.2

   Let y_i be the number of artichokes on plant i, and suppose we want to model the y_i as independent realizations from a Gamma distribution with parameters α and β (i.e., with mean α/β and pdf $f(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\beta y}$). Recall that the Gamma distribution is a conjugate prior β, however, there is no conjugate prior for α. Consider a Gamma prior for β with $\alpha_\beta = 1$ and $\beta_\beta = 0.8$, and also consider a Gamma prior for α with $\alpha_\alpha = 2$ and $\beta_\alpha = 0.5$.

(a) Perform MCMC using a Gibbs step for β and a Metropolis- Hastings step for α. What proposal distribution did you use?

(b) Provide trace plots and autocorrelation plots of the posterior samples of α and β. Discuss your results. Report your acceptance ratio, burnin, and thinning/effective sample size. Tune your MCMC to maximize your acceptance while achieving good mixing (i.e. low autocorrelation).

(c) Obtain posterior plots of 2000 α and β. Also report the MAP and 95% credible estimates for these model parameters.

(d) Use your posterior samples to simulate a new observation from the posterior predictive distribution. Provide the density plot of the predicted value and report the MAP and 95% predictive credible interval.