Homework 1 - Part 2 - Math 534

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Problem 1

Suppose that $X_1, X_2, \cdots, X_n$ are iid $N(\mu, \sigma^2)$. Then the corresponding log-likelihood for estimating μ and σ is given by

$$\ell(\mu,\sigma) = -\frac{n}{2}log(2\pi) - nlog(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Part i) Derive the observed information matrix for μ and σ.

Here, we need to find $-\nabla^2\ell(\mu,\sigma)$. Thus,

$$-\nabla\ell(\mu,\sigma) = -\begin{bmatrix} \frac{\partial\ell(\mu,\sigma)}{\partial\mu} \\ \frac{\partial\ell(\mu,\sigma)}{\partial\sigma} \end{bmatrix} = -\begin{bmatrix} \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) \\ \frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2 \end{bmatrix} = -\begin{bmatrix} \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i) - n\mu \\ \frac{-n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2 \end{bmatrix}$$

$$-\nabla^2\ell(\mu,\sigma) = -\begin{bmatrix} \frac{\partial^2\ell(\mu,\sigma)}{\partial\mu\partial\mu} & \frac{\partial^2\ell(\mu,\sigma)}{\partial\mu\partial\sigma} \\ \frac{\partial^2\ell(\mu,\sigma)}{\partial\sigma\partial\mu} & \frac{\partial^2\ell(\mu,\sigma)}{\partial\sigma\partial\sigma} \end{bmatrix} = -\begin{bmatrix} \frac{-n}{\sigma^2} & \frac{-2}{\sigma^3}(\sum_{i=1}^{n}x_i - n\mu) \\ \frac{-2}{\sigma^3}(\sum_{i=1}^{n}x_i - n\mu) & \frac{n}{\sigma^2} - \frac{3}{\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{n}{\sigma^2} & \frac{2}{\sigma^3}(\sum_{i=1}^{n}x_i - n\mu) \\ \frac{2}{\sigma^3}(\sum_{i=1}^{n}x_i - n\mu) & \frac{-n}{\sigma^2} + \frac{3}{\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2 \end{bmatrix}$$

Part ii) Obtain the Fisher-information matrix for μ and σ.

$$I(\mu,\sigma) = -E[\nabla^2\ell(\mu,\sigma)] = -E\begin{bmatrix} \frac{n}{\sigma^2} & \frac{2}{\sigma^3}(\sum_{i=1}^{n}x_i - n\mu) \\ \frac{2}{\sigma^3}(\sum_{i=1}^{n}x_i - n\mu) & \frac{-n}{\sigma^2} + \frac{3}{\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2 \end{bmatrix} =$$

$$= \begin{bmatrix} -E\left[\frac{-n}{\sigma^2}\right] & -E\left[\frac{2}{\sigma^3}(\sum_{i=1}^{n} x_i - n\mu)\right] \\ -E\left[\frac{2}{\sigma^3}(\sum_{i=1}^{n} x_i - n\mu)\right] & -E\left[\frac{-n}{\sigma^2} + \frac{3}{\sigma^4}\sum_{i=1}^{n}(x_i - \mu)^2\right] \end{bmatrix} = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{bmatrix}$$

Part iii) Using the transformation theorem, provided in the lecture, obtain the Fisher-information matrix for μ and σ^2.

The Fisher-information matrix of $g(\underline{\theta})$ is $\begin{bmatrix} g(\mu) = \mu \\ g(\sigma) = \sigma^2 \end{bmatrix}$

Now we need to calculate the following:

$$\left[J(\underline{\theta})I^{-1}(\underline{\theta})J^T(\underline{\theta})\right]^{-1}$$

Thus,

$$J(\underline{\theta}) = \begin{bmatrix} \frac{\partial g(\mu)}{\partial \mu} & \frac{\partial g(\mu)}{\partial \sigma} \\ \frac{\partial g(\sigma)}{\partial \mu} & \frac{\partial g(\sigma)}{\partial \sigma} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2\sigma \end{bmatrix}$$

Since our Jacobian is a diagonal matrix then $J(\underline{\theta}) = J(\underline{\theta})^T$

Also, we need to compute our inverse of the Fisher form from part b. Thus,

$$I^{-1}(\underline{\theta}) = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^2}{2n} \end{bmatrix}$$

Now, we can compute the Fisher-information matrix for μ and σ^2.

$$\left[J(\underline{\theta})I^{-1}(\underline{\theta})J^T(\underline{\theta})\right]^{-1} = \left[\begin{bmatrix} 1 & 0 \\ 0 & 2\sigma \end{bmatrix} \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^2}{2n} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2\sigma \end{bmatrix}\right]^{-1} = \left[\begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^3}{n} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2\sigma \end{bmatrix}\right]^{-1} =$$

$$= \left[\begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^4}{2n} \end{bmatrix}\right]^{-1} = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^4} \end{bmatrix}$$