Homework 1 pt. 2

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Problem 1

(a).

recall the observed information is: $-\nabla^2 \ell(\mu, \sigma)$

$\ell_\mu = \frac{\sum_{i=1}^n (x_i - \mu)}{\sigma^2}$

$\ell_\sigma = \frac{-1}{n} + \frac{\sum_{i=1}^n (x_i - \mu)^2}{\sigma^3}$

$\ell_{\mu\mu} = \frac{-n}{\sigma^2}$

$\ell_{\mu\sigma} = \ell_{\sigma\mu} = \frac{-2\sum_{i=1}^n (x_i - \mu)}{\sigma^3}$

$\ell_{\sigma\sigma} = \frac{n}{\sigma^2} - \frac{3\sum_{i=1}^n (x_i - \mu)^2}{\sigma^4}$

thus…

$$-\nabla^2 \ell(\mu, \sigma) = \begin{bmatrix} \frac{n}{\sigma^2} & \frac{2\sum_{i=1}^n (x_i - \mu)}{\sigma^3} \\ \frac{2\sum_{i=1}^n (x_i - \mu)}{\sigma^3} & \frac{-n}{\sigma^2} + \frac{3\sum_{i=1}^n (x_i - \mu)^2}{\sigma^4} \end{bmatrix}$$

(b).

let's express the Fisher Info. (Fisher Info: $E(-\nabla^2 \ell(\mu, \sigma))$)

$E(\frac{n}{\sigma^2}) = \frac{n}{\sigma^2}$

$E[\frac{2\sum_{i=1}^n (x_i - \mu)}{\sigma^3}] = \frac{-2}{\sigma^3} \sum_{i=1}^n E[x_i] - \mu = \frac{-2}{\sigma^3} \sum_{i=1}^n \mu - \mu = 0$

$E[\frac{-n}{\sigma^2} + \frac{3\sum_{i=1}^n (x_i - \mu)^2}{\sigma^4}] = \frac{-n}{\sigma^2} + \frac{3}{\sigma^4} \sum_{i=1}^n E[x_i^2] - 2\mu E[x_i] + \mu^2 = \frac{-n}{\sigma^2} + \frac{3}{\sigma^4} \sum_{i=1}^n \sigma^2 + \mu^2 - 2\mu^2 + \mu^2 = \frac{-n}{\sigma^2} + \frac{3n\sigma^2}{\sigma^4} = \frac{2n}{\sigma^2}$

$$E[-\nabla^2 \ell(\mu, \sigma)] = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{bmatrix}$$

(c).

let $\vec{\theta} = (\theta_1, \theta_2)$

let

$$g(\vec{\theta}) = \begin{bmatrix} \theta_1 \\ \theta_2^2 \end{bmatrix}$$

thus

$$J(\theta) = \begin{bmatrix} (\theta_1)_{\theta_1} & (\theta_1)_{\theta_2} \\ (\theta_2^2)_{\theta_1} & (\theta_2^2)_{\theta_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2\theta_2 \end{bmatrix}$$

in our case

$$J(\mu, \sigma) = \begin{bmatrix} 1 & 0 \\ 0 & 2\sigma \end{bmatrix}$$

$$I^{-1}(\vec{\theta}) = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{2n}{\sigma^2} \end{bmatrix}^{-1} = \frac{\sigma^4}{2n^2} \begin{bmatrix} \frac{2n}{\sigma^2} & 0 \\ 0 & \frac{n}{\sigma^2} \end{bmatrix} = \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^2}{2n} \end{bmatrix}$$

thus the Fisher information for $\ell(\mu, \sigma^2)$ is. . . .

$$[J(\vec{\theta})I^{-1}(\vec{\theta})J^T(\vec{\theta})]^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 2\sigma \end{bmatrix} \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^2}{2n} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2\sigma \end{bmatrix} \right)^{-1} = \left(\begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^3}{n} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2\sigma \end{bmatrix} \right)^{-1} = \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{n}{\sigma^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{bmatrix}$$

(d).

$$I^{-1}(\mu, \sigma) = \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^2}{2n} \end{bmatrix}$$

above shows us that

$$SE(\hat{\theta}_1 = \mu) = \sigma/\sqrt{n}$$

and that

$$SE(\hat{\theta}_2 = \sigma) = \frac{\sigma}{\sqrt{2n}}$$

$$I(\mu, \sigma^2) = \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{2\sigma^4}{n} \end{bmatrix}$$

above shows us that $SE(\theta_2^* = \sigma^2) = \sigma^2\sqrt{\frac{2}{n}}$

Problem 2

(a).