Quiz 2 Takehome

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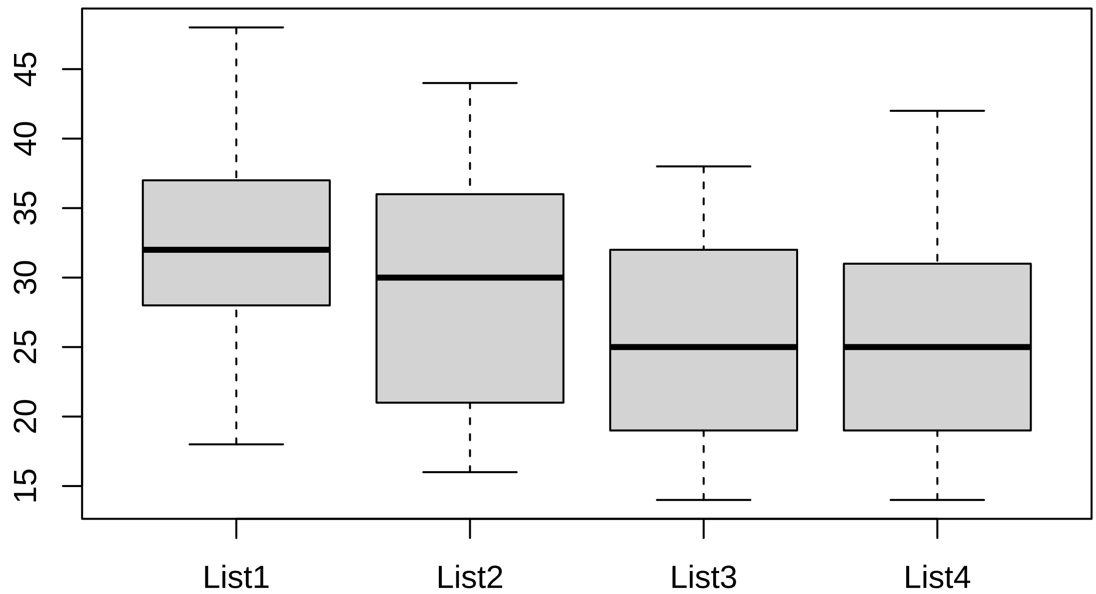
Question 1

```r
boxplot(hear)
```



(a)

```r
summary(hear)
```

```
##      List1           List2           List3           List4      
##  Min.   :18.00   Min.   :16.00   Min.   :14.00   Min.   :14.00  
##  1st Qu.:28.00   1st Qu.:21.50   1st Qu.:19.50   1st Qu.:19.50  
##  Median :32.00   Median :30.00   Median :25.00   Median :25.00  
##  Mean   :32.75   Mean   :29.67   Mean   :25.25   Mean   :25.58  
##  3rd Qu.:36.50   3rd Qu.:36.00   3rd Qu.:32.00   3rd Qu.:30.50  
##  Max.   :48.00   Max.   :44.00   Max.   :38.00   Max.   :42.00  
```

(b)

$$P(\theta, \mu, \sigma^2|\mathbf{y}) = P(\mu)P(\sigma^2)\prod_{j=1}^{J} P(\theta_j|\mu, \sigma^2) \prod_{j=1}^{J}\prod_{i=1}^{n_j} P(y_{ij}|\theta_j, \sigma^2)$$

(c) we care about θ_j so we extract $P(\theta_j|\mu, \sigma^2)\prod_{i=1}^{n_j} P(y_{ij}|\theta_j, \sigma^2)$

$$
\begin{aligned}
&P(\theta_i|\mu,\sigma^2)\prod_i^{n_j} P(y_{ij}|\theta_j,\sigma^2)\\
&\propto exp\Big[\frac{-1}{2\sigma^2}((\theta_j-\mu)^2+\sum_i^{n_j}(y_{ij}-\theta_j)^2)\Big]\\
&= exp\Big[\frac{-1}{2\sigma^2}(\theta_j^2-2\mu\theta_j+\mu^2+\sum_i^{n_j}(y_{ij}^2-2\theta_j y_{ij}+\theta_j^2)\Big]\\
&\propto exp\Big[\frac{-1}{2\sigma^2}(\theta_j^2-2\mu\theta_j+n_j\theta_j^2-2\theta_j\sum_i^{n_j} y_{ij})\Big]\\
&= exp\Big[\frac{-(n_j+1)}{2\sigma^2}(\theta_j^2-2\theta_j\frac{\mu+\sum_i^{n_j} y_{ij}}{n_j+1}+(\frac{\mu+\sum_i^{n_j} y_{ij}}{n_j+1})^2-(\frac{\mu+\sum_i^{n_j} y_{ij}}{n_j+1})^2)\Big]\\
&\propto exp\Big[\frac{-(n_j+1)}{2\sigma^2}(\theta_j^2-2\theta_j\frac{\mu+\sum_i^{n_j} y_{ij}}{n_j+1}+(\frac{\mu+\sum_i^{n_j} y_{ij}}{n_j+1})^2)\Big]\\
&= exp\Big[\frac{-(n_j+1)}{2\sigma^2}(\theta_j-\frac{\mu+\sum_i^{n_j} y_{ij}}{n_j+1})^2\Big]
\end{aligned}
$$

This gives us $N(\frac{\mu+\sum_i^{n_j} y_{ij}}{n_j+1}, \frac{\sigma^2}{1+n_j})$

Thus we have $P(\theta_j|\theta_{-j},\mu,\sigma^2,Y_{ij}) \sim N(\frac{\mu+\sum_i^{n_j} y_{ij}}{25}, \frac{\sigma^2}{25})$

(d) finding the $\mu|\theta_j,\mu,\sigma^2,\mathbf{Y}$

$$
\begin{aligned}
&P(\mu)\prod_j^4 P(\theta_i|\mu,\sigma^2)\\
&\propto exp\Big[-(\mu-30)^2-\frac{1}{2\sigma^2}\sum_i^4(\theta_j-\mu)^2\Big]\\
&= exp\Big[-\Big[\mu^2-60\mu+900+\frac{1}{2\sigma^2}\sum_j^4\theta_j^2-\frac{2\mu}{2\sigma^2}\sum_j^4\theta_j+\frac{24}{2\sigma^2}\mu^2\Big]\Big]\\
&\propto exp\Big[-\Big[\mu^2-60\mu-\frac{2\mu}{2\sigma^2}\sum_j^4\theta_j+\frac{24}{2\sigma^2}\mu^2\Big]\Big]\\
&= exp\Big[-\Big[\mu^2(1+\frac{12}{\sigma^2})-\mu(60+\frac{1}{\sigma^2}\sum_j^4\theta_j)\Big]\Big]\\
&= exp\Big[-\frac{1}{1+\frac{12}{\sigma^2}}\Big[\mu^2-\mu(\frac{60+\frac{1}{\sigma^2}\sum_j^4\theta_j}{1+\frac{12}{\sigma^2}})\Big]\Big]
\end{aligned}
$$

note:

$$
\begin{aligned}
&-\frac{1}{1+\frac{12}{\sigma^2}}=\frac{-\sigma^2}{2(\frac{\sigma^2}{2}+6)}\\
&\frac{60+\frac{1}{\sigma^2}\sum_j^4\theta_j}{1+\frac{12}{\sigma^2}}=\frac{60\sigma^2+\sum_j^4\theta_j}{\sigma^2+12}\\
&= exp\Big[\frac{-\sigma^2}{2(\frac{\sigma^2}{2}+6)}\Big[\mu^2-\mu(\frac{60\sigma^2+\sum_j^4\theta_j}{\sigma^2+12})+[\frac{1}{2}(\frac{60\sigma^2+\sum_j^4\theta_j}{\sigma^2+12})]^2-[\frac{1}{2}(\frac{60\sigma^2+\sum_j^4\theta_j}{\sigma^2+12})]^2\Big]\Big]\\
&\propto exp\Big[\frac{\sigma^2}{2(\frac{\sigma^2}{2}+6)}\Big[\mu-\frac{1}{2}(\frac{60\sigma^2+\sum_j^4\theta_j}{\sigma^2+12})\Big]^2\Big]\\
&\Rightarrow \mu|\theta,\sigma^2,Y\sim Norm\left(\frac{1}{2}(\frac{60\sigma^2+\sum_j^4\theta_j}{\sigma^2+12}),\frac{\sigma^2}{\frac{\sigma^2}{2}+6}\right)
\end{aligned}
$$

finding the distribution of $\sigma^2|\mu,\theta,\mathbf{Y}$

$$
\begin{aligned}
&P(\sigma^2)\prod_j^4 P(\theta_i|\mu,\sigma^2)\prod_j^4\prod_i^{24}P(y_{ij}|\theta_j,\sigma^2)\\
&=100(\frac{1}{\sigma^2})^3e^{\frac{-10}{\sigma^2}}\Big[\prod_j^4\prod_i^{24}\frac{1}{\sqrt{2\pi}\sigma^2}e^{\frac{-1}{2\sigma^2}(y_{ij}-\theta_j)^2}\Big]\Big[\prod_i^{24}\frac{1}{\sqrt{2\pi}\sigma^2}e^{\frac{-1}{2\sigma^2}(\theta_j-\mu)^2}\Big]\\
&\propto\left(\frac{1}{\sigma^2}\right)^{24(5)+3}exp\Big[\frac{-1}{\sigma^2}\Big(10+\frac{1}{2}\sum_j^4\sum_i^{24}(y_{ij}-\theta_j)^2+\frac{1}{2}\sum_i^{24}(\theta_j-\mu)^2\Big)\Big]\\
&\propto InvGamma\left(122,10+\frac{1}{2}\sum_j^4\sum_i^{24}(y_{ij}-\theta_j)^2+\frac{1}{2}\sum_i^{24}(\theta_j-\mu)^2\right)
\end{aligned}
$$