homework 3 (part 2)

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2024-02-20

part (a).

```
#function to square root a matrix "A"
sqrtm <- function(A){
  a <- eigen(A)
  sqm <- a$vectors %*% diag(sqrt(a$values)) %*% t(a$vectors)
  sqm <- (sqm+t(sqm))/2
}

#function for generating data
gen <- function(n,p,mu,sigma,seed){
  #generate data from a p-variate normal with mean mu and covaraince sigma
  #set seed to 2024
  set.seed(seed)
  #generate data from normal
  z <- matrix(rnorm(n*p),n,p)
  datan <- z %*% sqrtm(sigma) + matrix(mu,n,p,byrow = TRUE)
  return(datan)
}
```

```
# putting in the data
sig <- matrix(c(1,0.7,0.7,0.7,1,0.7,0.7,0.7,1), nrow = 3, ncol = 3)
mu <- matrix(c(-1,1,2), nrow =3)

gen(200,3,mu,sig,2024)[1:3,]
```

```
##            [,1]      [,2]     [,3]
## [1,]  0.5341745 1.9975269 4.092011
## [2,] -0.1649303 1.8387117 3.010171
## [3,] -1.2914162 0.3417351 1.871737
```

part (b).

```
# compile Sigma and Mu into a single theta vector
to.theta <- function(mu,sig){
  p <- length(mu)
  theta <- matrix(0,nrow = (p + p*(1+p)/2),ncol = 1)
  theta[1:p] <- mu

  k = p + 1
  for(i in 1:p){
    for(j in 1:i){
      theta[k] <- sig[i,j]
      k = k + 1
    }
  }
  return(theta)
}
```

```
# turning theta back into mu and Sigma
from.theta <- function(p,theta){
  mu <- theta[1:p]
  sig <- matrix(0, nrow = p, ncol = p)

  k = p + 1

  for (i in 1:p){
    for (j in 1:i){
      sig[i,j] <- theta[k]
      sig[j,i] <- sig[i,j]
      k = k + 1
    }
  }
 list(mu = mu, sig = sig)
}
```

```r
# make gradient
gradient <- function(x,mu,sig){
  n <- nrow(x)
  p <- ncol(x)

  inv.sig <- solve(sig) # inverse sigma

  # make initials
  xi.sum <- matrix(0, nrow = p, ncol = 1)
  grad.mu <- xi.sum
  C.mu <- matrix(0, nrow = p, ncol = p)

  # take care of C.mu
  for (i in 1:n){
    xi <- as.numeric(x[i,] - mu)
    xi.sum <- xi.sum + xi
    C.mu <- C.mu + (xi %*% t(xi))
  }
  grad.mu <- inv.sig %*% xi.sum
  A <- (n* inv.sig) - inv.sig %*% C.mu  %*% inv.sig
  grad.sig <- matrix(0, nrow = nrow(A), ncol = ncol(A))

  for (i in 1:nrow(sig) - 1){
    grad.sig[i,i] <- -0.5*A[i,i]
  }
  for (i in 1:nrow(sig) - 1){
    for (j in (i+1):ncol(sig)){
      grad.sig[i,j] <- -1*A[i,j]
      grad.sig[j,i] <- -1*A[i,j]
    }
  }
  grad.norm <- norm(to.theta(grad.mu,grad.sig), type = '2')
  list(grad.mu = grad.mu, grad.sig=grad.sig, grad.norm = grad.norm)
}
```

```r
likemvn <- function (x,mu,sig) {
  # computes the likelihood and the gradient for multivariate normal
  # if gcomp=FALSE, then the gradient is not computed
  # x is the n by p data matrix
  # mu is the mean
  # sig is the covariance
  # gcomp if TRUE, the gradient with respect to mu will be output
  n = nrow(x)
  p = ncol(x)

  C.mu = matrix(0,p,p) # initializing sum of (xi-mu)(xi-mu)^T
  xi.sum = matrix(0,p,1) # initializing sum of xi-mu
  grad.mu = xi.sum; # initializing this sum is used for the gradient w.r.t. mu
  for (i in 1:n){
    xi = as.numeric(x[i,] - mu)
    xi = xi + 1
    C.mu = C.mu + xi %*% t(xi)
  }

  ell = -(n*p*log(2*pi)+n*log(det(sig)) + sum(solve(sig) %*% C.mu ))/2
  return(ell)
}
```

```
# Steepest ascent

optmvn <- function (x,mu,sig,maxit,tolerr,tolgrad) {
  header = paste0("Iteration",
                  "      Halving",
                  "       log-likelihood",
                  "       ||gradient||")
  print(header)

  for(it in 1:maxit){
    theta0 <- to.theta(mu,sig)
    L <- likemvn(x,mu,sig)
    grad.mu0 <- gradient(x,mu,sig)$grad.mu
    grad.sig0 <- gradient(x,mu,sig)$grad.sig
    grad.norm0 <- gradient(x,mu,sig)$grad.norm

    if (it == 1 | it ==2 | it == 499 | it == 500){
      print(sprintf('%2.0f                 --          %3.4f               %.1e',it,L,gr
ad.norm0))
    }

    direc <- to.theta(grad.mu0,grad.sig0) # get direction
    # get new components
    theta1 <-  theta0 + direc
    mu1 <- from.theta(length(mu), theta1)$mu
    sig1 <- from.theta(length(mu), theta1)$sig
    grad.norm1 <- gradient(x,mu1,sig1)$grad.norm

    if(all(eigen(sig1)$values > 0)){atmp = likemvn(x,mu1,sig1)}
    else{atmp = -Inf}

    halve = 0

    if(it == 1 | it ==2 | it == 499 | it == 500){
    print(sprintf('%2.0f                 %2.0f          %3.4f               %.1e',
                  it,  halve, atmp, grad.norm1))}

    while((all(eigen(sig1)$values <= 0) && halve < 20) || atmp < L){
      halve = halve + 1
      # mathematics
      theta1 <- theta0 + direc/(2^halve)
      mu1 = from.theta(length(mu), theta1)$mu
      sig1 = from.theta(length(mu), theta1)$sig

      if(all(eigen(sig1)$values > 0)){atmp = likemvn(x,mu1,sig1)}
      else{atmp = -Inf}

      grad.norm1 <-  gradient(x,mu1,sig1)$grad.norm

      if(it == 1 | it ==2 | it == 499 | it == 500){
          print(sprintf('%2.0f                 %2.0f          %3.4f               %.1e',
it,  halve,atmp, grad.norm1))
```

```
      }
    }
    if(it == 1 | it == 2 | it == 499){
      print("------------------------------------------------------------------")
      print(header)
    }
    r.e <- max(abs(theta0 - theta1)/abs(pmax(1,abs(theta0))))
    theta0 <- theta1

    if (r.e < tolerr & grad.norm1 < tolgrad){break}

    mu <- mu1
    sig <- sig1
  }
  return(list("mu.estimator" = mu,
              "sigma.estimator" = sig,
              "iteration" = it))
}
```

```
# putting in parameters
x <- gen(200,3,mu,sig,2024)
m <- c(0,0,0)
s <- diag(3)

optmvn(x,m,s,500,1e-6,1e-5)
```

```
## [1] "Iteration      Halving       log-likelihood       ||gradient||"
## [1] " 1                 --          -3846.7751              8.1e+02"
## [1] " 1                  0          -Inf                 5.2e+02"
## [1] " 1                  1          -Inf                 5.0e+02"
## [1] " 1                  2          -Inf                 4.5e+02"
## [1] " 1                  3          -Inf                 3.8e+02"
## [1] " 1                  4          -Inf                 3.0e+02"
## [1] " 1                  5          -Inf                 2.1e+02"
## [1] " 1                  6          -Inf                 1.3e+02"
## [1] " 1                  7          -Inf                 7.6e+01"
## [1] " 1                  8          -Inf                 1.6e+02"
## [1] " 1                  9          -Inf                 1.2e+04"
## [1] " 1                 10          -2965.7040              3.5e+02"
## [1] "-----------------------------------------------------------------"
## [1] "Iteration      Halving       log-likelihood       ||gradient||"
## [1] " 2                 --          -2965.7040              3.5e+02"
## [1] " 2                  0          -Inf                 4.7e+04"
## [1] " 2                  1          -Inf                 6.0e+04"
## [1] " 2                  2          -Inf                 1.1e+05"
## [1] " 2                  3          -Inf                 2.5e+06"
## [1] " 2                  4          -Inf                 4.7e+04"
## [1] " 2                  5          -Inf                 3.5e+03"
## [1] " 2                  6          -Inf                 6.9e+02"
## [1] " 2                  7          -Inf                 9.1e+02"
## [1] " 2                  8          -Inf                 2.5e+04"
## [1] " 2                  9          -2156.4052              2.3e+02"
## [1] "-----------------------------------------------------------------"
## [1] "Iteration      Halving       log-likelihood       ||gradient||"
## [1] "499                --             -501.0274              8.9e+05"
## [1] "499                 0             -Inf                  2.9e+00"
## [1] "499                 1             -Inf                  2.9e+00"
## [1] "499                 2             -Inf                  2.9e+00"
## [1] "499                 3             -Inf                  2.9e+00"
## [1] "499                 4             -Inf                  2.9e+00"
## [1] "499                 5             -Inf                  2.9e+00"
## [1] "499                 6             -Inf                  2.9e+00"
## [1] "499                 7             -Inf                  2.8e+00"
## [1] "499                 8             -Inf                  2.8e+00"
## [1] "499                 9             -Inf                  2.7e+00"
## [1] "499                10             -Inf                  2.6e+00"
## [1] "499                11             -Inf                  3.2e+00"
## [1] "499                12             -Inf                  5.8e+00"
## [1] "499                13             -Inf                  1.3e+01"
## [1] "499                14             -Inf                  3.7e+01"
## [1] "499                15             -Inf                  1.7e+02"
## [1] "499                16             -Inf                  5.0e+03"
## [1] "499                17             -Inf                  1.4e+02"
## [1] "499                18             -Inf                  9.4e+02"
## [1] "499                19             -1707.3445               1.9e+02"
## [1] "499                20             -1446.5475               1.6e+02"
## [1] "499                21             -1333.9782               2.1e+02"
## [1] "499                22             -1244.8847               5.4e+02"
```

```
## [1] "499                23          -1160.4694              2.4e+03"
## [1] "499                24          -1072.9957              9.7e+03"
## [1] "499                25          -977.5635              3.4e+04"
## [1] "499                26          -873.1483              9.6e+04"
## [1] "499                27          -766.6403              2.2e+05"
## [1] "499                28          -672.2395              3.9e+05"
## [1] "499                29          -601.6567              5.7e+05"
## [1] "499                30          -556.2846              7.0e+05"
## [1] "499                31          -530.0992              7.9e+05"
## [1] "499                32          -515.9554              8.4e+05"
## [1] "499                33          -508.5937              8.6e+05"
## [1] "499                34          -504.8367              8.8e+05"
## [1] "499                35          -502.9386              8.8e+05"
## [1] "499                36          -501.9847              8.9e+05"
## [1] "499                37          -501.5064              8.9e+05"
## [1] "499                38          -501.2670              8.9e+05"
## [1] "499                39          -501.1472              8.9e+05"
## [1] "499                40          -501.0873              8.9e+05"
## [1] "499                41          -501.0573              8.9e+05"
## [1] "499                42          -501.0423              8.9e+05"
## [1] "499                43          -501.0348              8.9e+05"
## [1] "499                44          -501.0311              8.9e+05"
## [1] "499                45          -501.0292              8.9e+05"
## [1] "499                46          -501.0283              8.9e+05"
## [1] "499                47          -501.0278              8.9e+05"
## [1] "499                48          -501.0276              8.9e+05"
## [1] "499                49          -501.0275              8.9e+05"
## [1] "499                50          -501.0274              8.9e+05"
## [1] "499                51          -501.0274              8.9e+05"
## [1] "499                52          -501.0274              8.9e+05"
## [1] "499                53          -501.0274              8.9e+05"
## [1] "499                54          -501.0274              8.9e+05"
## [1] "499                55          -501.0274              8.9e+05"
## [1] "499                56          -501.0274              8.9e+05"
## [1] "499                57          -501.0274              8.9e+05"
## [1] "499                58          -501.0274              8.9e+05"
## [1] "499                59          -501.0274              8.9e+05"
## [1] "499                60          -501.0274              8.9e+05"
## [1] "499                61          -501.0274              8.9e+05"
## [1] "499                62          -501.0274              8.9e+05"
## [1] "499                63          -501.0274              8.9e+05"
## [1] "499                64          -501.0274              8.9e+05"
## [1] "499                65          -501.0274              8.9e+05"
## [1] "499                66          -501.0274              8.9e+05"
## [1] "499                67          -501.0274              8.9e+05"
## [1] "499                68          -501.0274              8.9e+05"
## [1] "499                69          -501.0274              8.9e+05"
## [1] "499                70          -501.0274              8.9e+05"
## [1] "499                71          -501.0274              8.9e+05"
## [1] "499                72          -501.0274              8.9e+05"
## [1] "499                73          -501.0274              8.9e+05"
## [1] "499                74          -501.0274              8.9e+05"
```

```
## [1] "499                 75          -501.0274               8.9e+05"
## [1] "------------------------------------------------------------------"
## [1] "Iteration      Halving       log-likelihood       ||gradient||"
## [1] "500                 --          -501.0274               8.9e+05"
## [1] "500                  0          -Inf                2.9e+00"
## [1] "500                  1          -Inf                2.9e+00"
## [1] "500                  2          -Inf                2.9e+00"
## [1] "500                  3          -Inf                2.9e+00"
## [1] "500                  4          -Inf                2.9e+00"
## [1] "500                  5          -Inf                2.9e+00"
## [1] "500                  6          -Inf                2.9e+00"
## [1] "500                  7          -Inf                2.8e+00"
## [1] "500                  8          -Inf                2.8e+00"
## [1] "500                  9          -Inf                2.7e+00"
## [1] "500                 10          -Inf                2.6e+00"
## [1] "500                 11          -Inf                3.2e+00"
## [1] "500                 12          -Inf                5.8e+00"
## [1] "500                 13          -Inf                1.3e+01"
## [1] "500                 14          -Inf                3.7e+01"
## [1] "500                 15          -Inf                1.7e+02"
## [1] "500                 16          -Inf                5.0e+03"
## [1] "500                 17          -Inf                1.4e+02"
## [1] "500                 18          -Inf                9.4e+02"
## [1] "500                 19          -1707.3445                1.9e+02"
## [1] "500                 20          -1446.5475                1.6e+02"
## [1] "500                 21          -1333.9782                2.1e+02"
## [1] "500                 22          -1244.8847                5.4e+02"
## [1] "500                 23          -1160.4694                2.4e+03"
## [1] "500                 24          -1072.9957                9.7e+03"
## [1] "500                 25          -977.5635                3.4e+04"
## [1] "500                 26          -873.1483                9.6e+04"
## [1] "500                 27          -766.6403                2.2e+05"
## [1] "500                 28          -672.2395                3.9e+05"
## [1] "500                 29          -601.6567                5.7e+05"
## [1] "500                 30          -556.2846                7.0e+05"
## [1] "500                 31          -530.0992                7.9e+05"
## [1] "500                 32          -515.9554                8.4e+05"
## [1] "500                 33          -508.5937                8.6e+05"
## [1] "500                 34          -504.8367                8.8e+05"
## [1] "500                 35          -502.9386                8.8e+05"
## [1] "500                 36          -501.9847                8.9e+05"
## [1] "500                 37          -501.5064                8.9e+05"
## [1] "500                 38          -501.2670                8.9e+05"
## [1] "500                 39          -501.1472                8.9e+05"
## [1] "500                 40          -501.0873                8.9e+05"
## [1] "500                 41          -501.0573                8.9e+05"
## [1] "500                 42          -501.0423                8.9e+05"
## [1] "500                 43          -501.0348                8.9e+05"
## [1] "500                 44          -501.0311                8.9e+05"
## [1] "500                 45          -501.0292                8.9e+05"
## [1] "500                 46          -501.0283                8.9e+05"
## [1] "500                 47          -501.0278                8.9e+05"
```

```
## [1] "500                48          -501.0276              8.9e+05"
## [1] "500                49          -501.0275              8.9e+05"
## [1] "500                50          -501.0274              8.9e+05"
## [1] "500                51          -501.0274              8.9e+05"
## [1] "500                52          -501.0274              8.9e+05"
## [1] "500                53          -501.0274              8.9e+05"
## [1] "500                54          -501.0274              8.9e+05"
## [1] "500                55          -501.0274              8.9e+05"
## [1] "500                56          -501.0274              8.9e+05"
## [1] "500                57          -501.0274              8.9e+05"
## [1] "500                58          -501.0274              8.9e+05"
## [1] "500                59          -501.0274              8.9e+05"
## [1] "500                60          -501.0274              8.9e+05"
## [1] "500                61          -501.0274              8.9e+05"
## [1] "500                62          -501.0274              8.9e+05"
## [1] "500                63          -501.0274              8.9e+05"
## [1] "500                64          -501.0274              8.9e+05"
## [1] "500                65          -501.0274              8.9e+05"
## [1] "500                66          -501.0274              8.9e+05"
## [1] "500                67          -501.0274              8.9e+05"
## [1] "500                68          -501.0274              8.9e+05"
## [1] "500                69          -501.0274              8.9e+05"
## [1] "500                70          -501.0274              8.9e+05"
## [1] "500                71          -501.0274              8.9e+05"
## [1] "500                72          -501.0274              8.9e+05"
## [1] "500                73          -501.0274              8.9e+05"
## [1] "500                74          -501.0274              8.9e+05"
## [1] "500                75          -501.0274              8.9e+05"
```

```
## $mu.estimator
## [1] -1.2045485  0.6976222  2.3888200
##
## $sigma.estimator
##           [,1]      [,2]      [,3]
## [1,] 1.3859638 0.6606713 0.1281701
## [2,] 0.6606713 1.1462171 0.9588739
## [3,] 0.1281701 0.9588739 1.0000000
##
## $iteration
## [1] 500
```