Math 530 Quiz 3

Name: Michael Pena

Note: Show your work on all problems. Each part of each problem is worth 5 points. A total of 25 points is possible.

1. Let $X \sim \text{gamma}(\alpha, \beta)$. Show that $EX^n = \beta^n \Gamma(n+\alpha)/\Gamma(\alpha)$, where n is a positive integer.

$$E(X^n) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \int_0^\infty x^n \cdot x^{\alpha-1} \cdot e^{-x/\beta} dx$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \int_0^\infty x^{\alpha+n-1} e^{-x/\beta} dx$$

notice the gamma kernel $(\alpha+n, \beta)$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \Gamma(\alpha+n)\beta^{\alpha+n}$$

$$= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \cdot \beta^{\alpha+n-\alpha}$$

$$= \frac{\Gamma(\alpha+n)\beta^n}{\Gamma(\alpha)}$$

2. Let X be a continuous random variable with pdf f_X and cdf F_X. Moreover, assume that f_X is symmetric about a point a.

(a) Show that the random variables $U = X - a$ and $W = a - X$ have the same distribution.

Want to show that $F_U(u) = F_W(w)$
$g_1^{-1}(u) = u + a$ and $g_2^{-1}(w) = a - w$
thus $\frac{d}{du}g_1^{-1}(u) = 1$ and $\frac{d}{dw}g_2^{-1}(w) = -1$
thus $f_U(u) = f_X(u) \cdot |1| = f_X(x-a)$ and $f_W(w) = f_X(w) \cdot |-1| = f_X(a-x)$

$$F_U(u) = \int_{-\infty}^{\infty} f_U(u)du$$
$$= \int_{-\infty}^{\infty} f_U(u)du$$
$$= \int_{-\infty}^{\infty} f_X(x-a)du$$

note: $du = dx$ while $dw = -dx$

$$= \int_{-\infty}^{\infty} f_X(x-a)dx$$

f_X is symmetric about a

$$= \int_{-\infty}^{\infty} f_X(a-x)dx$$
$$= \int_{-\infty}^{\infty} -f_X(a-x) \cdot -dx$$
$$= \int_{w(-\infty)=\infty}^{w(\infty)=-\infty} -f_W(w)dw$$
$$= \int_{-\infty}^{\infty} f_W(w)dw$$
$$= F_W(w)$$

Thus $F_U(u) = F_W(w)$ and thus U and W have same distribution

(b) Assuming that the k-th central moment of X exists, show that for an odd positive integer $E[X-a]^k = 0$.

$$E[x-a]^k = \int_{\infty}^{\infty} [x-a]^k f_X(x)dx$$

$u = x - a$ thus $x = a + u$ thus $du = dx$

$$= \int_{\infty}^{\infty} u^k f_X(a+u)du$$

let $g(u) = u^k f_X(a+u)$, notice

$$g(-u) = (-u)^k f_X(a-u)$$
$$= -u^k f_X(a-u)$$

f_X is symmetric at a so $f_X(a-u) = f_X(a+u)$

$$= -u^k f_X(a+u)$$

This shows that $g(u) = g(-u)$; g is odd.Thus $\int_{\infty}^{\infty} g(u)du = 0$
Which implies

$$E[x-a]^k = 0$$

3. Let X be a random variable with pmf $f_X(x) > 0$ for $x = 1, 2, 3, \cdots$ (positive integers), and $f_X(x) = 0$ for all other values of x. Then, the pmf of X_T, the random variable X truncated at $X = 1$, is given by

$$f_{X_T}(x) = \frac{f_X(x)}{P(X > 1)}, \text{ for } \quad x = 2, 3, \cdots.$$

(a) Verify that $f_{X_T}(x)$ is a pmf.

We know that $\sum_{x=1}^{\infty} f_X(x) = 1$ because it is defined as a pmf.
We want to show that $\sum_{x=2}^{\infty} \frac{f_X(x)}{P(X>1)} = 1$
notice

$$\sum_{x=2}^{\infty} \frac{f_X(x)}{P(X > 1)}$$

$$= \frac{1}{P(X > 1)} \cdot \sum_{x=2}^{\infty} f_X(x)$$

$$= \frac{1}{P(X > 1)} \cdot [1 - f_X(1)]$$

$$= \frac{1}{P(X > 1)} \cdot [1 - P(X \leq 1)]$$

$$= \frac{1}{1 - P(X \leq 1)} \cdot [1 - P(X \leq 1)] = 1$$

Notice that $\forall x \in N$, $f_X(x) > 0$. Thus $\forall x \in N - \{1\}$, $\frac{f_X(x)}{P(X>1)} > 0$ implying $\forall x \in N - \{1\}$, $f_{X_T} > 0$
This shows that $f_{X_T}(x)$ is a pmf.

(b) Assume that $f_X(1) = 1/4$, $E(X) = \mu$. Obtain $E(X_T)$ as a function of μ.

$E[X_T] = \sum_{x=2}^{\infty} x \cdot f_{X_T}(x) = \sum_{x=2}^{\infty} x \frac{f_X(x)}{P(X>1)} = \frac{1}{P(X>1)} \sum_{x=2}^{\infty} x f_X(x)$

Note that $E(X) = \mu = \sum_{x=1}^{\infty} x f_X(x) = f_X(1) + \sum_{x=2}^{\infty} x f_X(x) = (1/4) + \sum_{x=2}^{\infty} x f_X(x)$
Thus $\sum_{x=2}^{\infty} x f_X(x) = \mu - 1/4$
Thus

$$\frac{1}{P(X > 1)} \sum_{x=2}^{\infty} x f_X(x)$$

$$= \frac{1}{P(X > 1)} (\mu - 1/4)$$

$$= \frac{1}{1 - P(X \leq 1)} (\mu - 1/4)$$

$$= \frac{1}{1 - f_X(1)} (\mu - 1/4)$$

$$= \frac{\mu - 1/4}{3/4} = \frac{4\mu - 1}{3}$$

thus $E[X_T] = [4\mu - 1]/3$