Math 530 Quiz 2

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Note: Show your work on all problems. Each problem is worth 5 points. A total of 25 points is possible.

1. Prove or give a counter example: If A and B are two events, then $P(A|B) + P(A|B^c) = 1$.

Proof. Assume that A and B are two events. We want to show that

$$P(A|B) + P(A|B^c) = 1$$

Given the identity $P(A|B) = \frac{P(A \cap B)}{P(A)}$ we can rewrite the left side as

$$\frac{P(A \cap B)}{P(A)} + \frac{P(A \cap B^c)}{P(A)}$$
$$= \frac{P(A \cap B) + P(A \cap B^c)}{P(A)}$$

Because $A = (A \cap B) \cup (A \cap B^c)$ by finite additivity $P(A) = P(A \cap B) + P(A \cap B^c)$ thus

$$\frac{P(A \cap B) + P(A \cap B^c)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

Thus we have shown what we intended. □

2. We have two coins, each having $P(heads) = \alpha$, where $0 \leq \alpha \leq 1$. We flip these two coins continually and simultaneously until either two heads appear or two tails appear. What is the probability that two heads appear first; that is two heads appear before two tails appear. Compute the probability in terms of α.

denote HT as $\alpha(1-\alpha)$, HH as (α^2), TT as $(1-\alpha)^2$

Let E_1 be the event that two coins flip heads for the first time.
Let E_2 be the event that two coins flip tails for the first time.

Find $P(E_2^c|E_1)$ or $1 - P(E_2|E_1)$
$1 - P(E_2|E_1) = 1 - \alpha^2(1-\alpha)^2$ because flipping coins are independent events.
$1 - \alpha^2(1-\alpha)^2 = 1 - \alpha^2 + 2\alpha^3 - \alpha^4$

Probability is

$$1 - \alpha^2 + 2\alpha^3 - \alpha^4$$

3. Use mathematical induction to show that

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}).$$

You can assume that you know $P(A \cap B) = P(A|B)P(B)$.

Proof. (induction):

Base case (true for n=2):
Because we are given $P(A \cap B) = P(A|B)P(B)$ is true then we know that $P(A_1 \cap A_2) = P(A_2|A_1)P(A_1)$ also holds proving the base case.

Inductive Case ($P_k \to P_{k+1}$):
Assume that $P(A_1 \cap ... \cap A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)...P(A_k|A_1 \cap ... \cap A_{k-1})$ is true.
We want to show that $P(A_1 \cap ... \cap A_{k_1}) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)...P(A_{k+1}|A_1 \cap ... \cap A_k)$

For sake of simplicity, let $A_1 \cap ... \cap A_k = E$.
Thus we can write

$$P(A_1 \cap ... \cap A_k \cap A_{k_1}) = P(E \cap A_{k+1}) = P(A_{k+1} \cap E)$$

Because of Multiplication Rule

$$P(A_{k+1} \cap E) = P(E) \cdot P(A_{k_1}|E)$$

replacing E with $A_1 \cap ... \cap A_k$ returns

$$P(A_1 \cap ... \cap A_k \cap A_{k_1}) = P(A_1 \cap ... \cap A_k)P(A_{k+1}|A_1 \cap ... \cap A_k)$$

Because $P(A_1 \cap ... \cap A_k) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)...P(A_k|A_1 \cap ... \cap A_{k-1})$ we now have

$$P(A_1 \cap ... \cap A_k \cap A_{k_1}) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)...P(A_k|A_1 \cap ... \cap A_{k-1})P(A_{k+1}|A_1 \cap ... \cap A_k)$$

Because the Base case was shown and $P_k \to P_{k+1}$ is shown to be true, it can be concluded that

$$P(A_1 \cap ... \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)...P(A_n|A_1 \cap ... \cap A_{n-1})$$

is true $\forall n \in N$ $\square$

4. Let X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$ both of which have support in $(-\infty, \infty)$. Consider the fixed values a and b with $a < b$. Show that the following function is a pdf with support $[a, b]$:

$$g(x) = f(x)/[F(b) - F(a))]$$

notice that

$$\int_a^b g(x)dx = \int_a^b \frac{f(x)dx}{F(b) - F(a)} = \frac{1}{F(b) - F(a)} \cdot \int_a^b f(x)dx$$

but because X is continuous, by FTC

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

thus

$$\int_a^b g(x)dx = \frac{F(b) - F(a)}{F(b) - F(a)} = 1$$

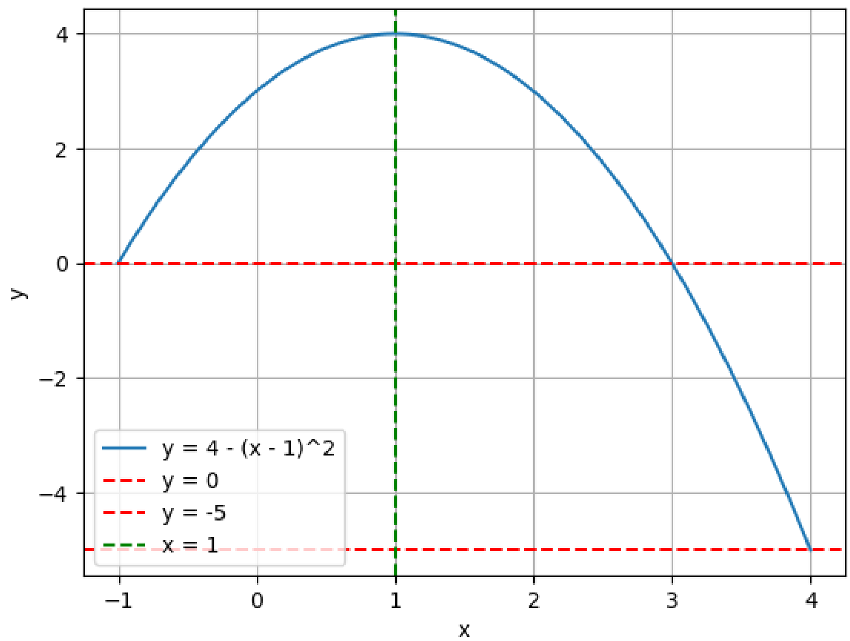
because $f(x)$ is already defined to be a pdf, we know $f(x) \geq 0, \forall x$ and thus so is $\frac{1}{F(b)-F(a)} \cdot \int_a^b f(x)dx$. Thus $g(x)$ is a pdf with support $[a, b]$.

5. Suppose that X is a continuous random variable with cumulative distribution function (cdf)

$$F(x) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{x+1}{5} & \text{if } -1 \le x \le 4 \\ 1 & \text{if } x > 4. \end{cases}$$

Obtain the cumulative distribution function of $Y = 4 - (x-1)^2$.

Graph $y = 4 - (x-1)^2$



notice that the $P(Y \ge 4) = 1$ and that $P(Y \le -5) = 0$
solving for that X in Y we get $X = 1 \pm \sqrt{4-y}$

In the case where $y \in [0,4]$:

$$P(Y \le y) = P(X_1 \le x \le X_2)$$

where $X_1 = 1 - \sqrt{4-y}, X_2 = 1 + \sqrt{4-y}$
thus

$$P(X_1 \le x \le X_2) = F(1+\sqrt{4-y}) - F(1-\sqrt{4-y}) = \frac{1}{5}(1+\sqrt{4-y}+1) - \frac{1}{5}(1-\sqrt{4-y}+1)$$

$$= \frac{1}{5}[2+\sqrt{4-y} - 2 + \sqrt{4-y}] = \frac{2}{5}\sqrt{4-y}$$

In the case where $y \in [-5,0]$:
$P(Y \le y) = P(X_2 \le x) = F(1+\sqrt{4-y}) = \frac{1}{5}(1+\sqrt{4-y}+1) = \frac{1}{5}(2+\sqrt{4-y})$
thus

$$F(y) = \begin{cases} 1 & \text{if } y > 4 \\ \frac{2}{5}\sqrt{4-y} & \text{if } 0 \le y \le 4 \\ \frac{1}{5}(2+\sqrt{4-y}) & \text{if } -5 \le y \le 0 \\ 0 & \text{if } y < -5 \end{cases}$$