Math 530 Quiz 4

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Note: Show your work on all problems. Each part of each problem is worth 5 points. A total of 25 points is possible.

1. Let X be a random variable with pdf

$$f(x) = \frac{1}{2\beta}e^{-|x-\alpha|/\beta}, \quad -\infty < x < \infty, \quad -\infty < \alpha < \infty, \quad \beta > 0.$$

Derive the mgf of the random X. State the domain where the mgf is defined.

Domain: $t < \frac{1}{\beta}$, $-\infty < \alpha < \infty$, $\beta > 0$

$$M_X(t) = E[e^{tx}] = \frac{1}{2\beta}\int_{-\infty}^{\infty} e^{tx - \frac{|x-\alpha|}{\beta}}\,dx$$

$$= \frac{1}{2\beta}\left(\int_{-\infty}^{0} e^{tx+\frac{x}{\beta}-\frac{\alpha}{\beta}}\,dx + \int_{0}^{\infty} e^{tx-\frac{x}{\beta}+\frac{\alpha}{\beta}}\,dx\right)$$

$$= \frac{1}{2\beta}\left(e^{-\alpha/\beta}\int_{-\infty}^{0} e^{\frac{t\beta+1}{\beta}x}\,dx + e^{\alpha/\beta}\int_{-\infty}^{0} e^{\frac{t\beta-1}{\beta}x}\,dx\right)$$

$$= \frac{1}{2\beta}\left(e^{-\alpha/\beta}\left[\frac{\beta}{t\beta+1}e^{\frac{t\beta+1}{\beta}x}\right]_{-\infty}^{0} + e^{\alpha/\beta}\left[\frac{\beta}{t\beta-1}e^{\frac{t\beta-1}{\beta}x}\right]_{0}^{\infty}\right)$$

$$= \frac{1}{2}\left(\frac{e^{-\alpha/\beta}}{t\beta+1} + \frac{e^{\alpha/\beta}}{t\beta-1}\right)$$

2. Let Y be a geometric random variable with parameter p, where p is the success probability. Show that as p approaches zero, the random variable $W = pY$ converges to the exponential distribution with parameter $\beta = 1$.

$$f_Y(y) = p(1-p)^{y-1}$$

we know that $g^{-1}(w) = w/p$ and $|\frac{d}{dw}(w/p)| = 1/p$, thus

$$f_W(w) = f_Y(g^{-1}(w))|\frac{d}{dw}(w/p)| = p(1-p)^{w/p-1} \cdot \frac{1}{p} = (1-p)^{w/p-1}$$

note that if $\lim_{x\to\infty}(1+\frac{\alpha}{x})^x = e^{\alpha}$ then $\lim_{p\to 0}(1+\alpha p)^{1/p} = e^{\alpha}$

$$\lim_{p\to 0}\Big[(1-p)^{w/p}(1-p)^{-1}\Big] = \lim_{p\to 0}\Big[\big((1+(-1)p)^{1/p}\big)^w(1-p)^{-1}\Big]$$

$$= \left(e^{-1}\right)^w(1-0)^{-1}$$

$$= (e^{-1})^w 1^{-1}$$

$$= e^{-w}$$

but notice this is the exponential distribution of random variable W where $\beta = 1$

$$f(W|\beta = 1) = \frac{1}{1}e^{\frac{-w}{1}} = e^{-w}$$

3. Theaters A and B compete for the business of 1000 customers. Assume that Theater A shows a more popular movie, and thus the probability that a randomly selected customer chooses Theater A is 3/4. Let n be the number of seats in Theater A. Write an equation that you would solve for n such that the probability of turning away a customer by Theater A, because of a full house, is less than 5%. Do not solve for n.

$P('notfull') = P(X \le n)$ where n is the number of seats, thus we must express where $1 - P(X \le n) < 0.05$ or $P(X > n) < 0.05$

$P(A) = 3/4 = p,\ P(B) = 1/4 = q$

so then $P(X > n) = 1 - P(X \le n) = 1 - \left(\sum_{i=0}^{n}\binom{n}{i}(\frac{3}{4})^i(\frac{1}{4})^{n-i}\right) < 0.05$

4. Let X have the standard normal distribution (i.e. $X \sim N(0,1)$). Use the moment generating function of X to obtain $E(X^4)$.

$$M_X(t) = e^{(0)t+\frac{1^2 t}{2}} = \sqrt{e}\cdot e^{\frac{t^2}{2}}$$

$$[M_X(t)]_{tttt} = \frac{d^4}{dt^4}\left(\sqrt{e}\cdot e^{\frac{t^2}{2}}\right) = \sqrt{e}\left(e^{t^2/2} + t^2e^{t^2/2} + 2e^{t^2/2} + 2t^2e^{t^2/2} + 3t^2e^{t^2/2} + t^4e^{t^2/2}\right)$$

$$E[X^4] = [M_X(0)]_{tttt} = \sqrt{e}(1+0+2+0+0) = 3\sqrt{e}$$

5. Let Y be a random variable with pmf

$$P(Y=\sqrt{3}) = P(Y=-\sqrt{3}) = 1/6, \quad P(Y=0) = 2/3.$$

Obtain $E(Y^4)$.

$$E[Y^4] = (\sqrt{3})^4\cdot\frac{1}{6} + (-\sqrt{3})^4\cdot\frac{1}{6} + (0)^4\cdot\frac{4}{6} = 3$$