Chapter 6 Homework – Monte Carlo Integration Part I – 60 Points

1. Consider the integral

$$\int_0^1 \frac{1}{1+x^2} e^{-x} dx.$$

In each of the following cases, approximate the value of the integral, by generating 20,000 random values from the uniform distribution.

(a) [5 points] Use the basic Monte Carlo integration by generating Uniform(0,1) values to approximate the integral. Also obtain an approximation to the standard error of your estimate.

(b) [5 points] Use the antithetic method to approximate the integral. Obtain an approximation to the standard error of your estimate. Note here all you need is 10,000 uniform values.

(c) [2 points] Compute the percent reduction in variance when using the antithetic method in part (b) as compared to the Monte Carlo method in part (a).

(d) [3 points] By computing an appropriate correlation from your simulation in part (b), explain why you expect a smaller variance for the antithetic method as compared to the basic Monte Carlo method of part (a).

2. Now consider the truncated exponential random variable X with density

$$f(x) = e^{-x}/(1 - e^{-1}), \quad 0 < x < 1$$

(a) [4 points] Write an R function that uses the inverse transformation method to generate random values from X. {Hint, if F_X is the cdf of X and $U \sim Unif(0,1)$, then $F^{-1}(U) \sim X$.]

(b) [5 points] Approximate the integral in Problem 1, by generating 20,000 random values from X, using your function in part (a), and using the basic Monte Carlo method. Obtain an approximation to the standard error of your estimate.

(c) [5 points] Approximate the integral in Problem 1, by generating random values from X, using your function in part (a), and using the antithetic method. Obtain and approximation to the standard error of your estimate. Here again you will need to generate 10,000 uniform values.

3. Let (X_1, X_2, X_3)be random variables from a trivariate normal distribution with mean zero, and covariance matrix

$$\Sigma = \begin{pmatrix} 1 & 3/5 & 1/3 \\ 3/5 & 1 & 11/15 \\ 1/3 & 11/15 & 1 \end{pmatrix}.$$

Thus, for $\boldsymbol{x} = (x_1, x_2, x_3)^T$ the density of (X_1, X_2, X_3) is given by

$$f(\boldsymbol{x}; \Sigma) = (2\pi)^{-3/2} |\Sigma|^{-1/2} e^{-\frac{1}{2}\boldsymbol{x}^T \Sigma^{-1} \boldsymbol{x}}.$$

(a) [5 points] Use the hit-miss algorithm to approximate

$$P(-\infty \le X_1 \le 1, -\infty \le X_2 \le 4, -\infty \le X_3 \le 2) = E\left[I_{(-\infty \le X_1 \le 1, -\infty \le X_2 \le 4, -\infty \le X_3 \le 2)}\right],$$

by generating 20,000 random values from an appropriate trivariate normal distribution.

(b) [3 points] Estimate the standard error of your estimate.

(c) [3 points] Give a 95% confidence interval for the value of the integral. The true value of the probability to a good approximation is 0.8279849897…. Does your confidence interval contain the true value?

4. Consider the integral

$$\int_0^1 \frac{1}{1+x^2} e^{-x} dx.$$

(a) [5 points] Estimate this integral, using 20,000 uniform random numbers in the method of control variates with the control function

$$h(x) = \frac{e^{-0.5}}{1+x^2}.$$

(b) [4 points] Let $\hat{\theta}_{CV}$ be the estimate obtained in part (a), and $\hat{\theta}_{MC}$ be the estimate obtained using the basic Monte Carlo integration (as you obtained in Chapter 6 - Part I homework). Use the simulated values in part (a) to obtain an estimate for $Cor\left(\frac{e^{-U}}{1+U^2}, \frac{e^{-0.5}}{1+U^2}\right)$. Then show how you can use this correlation value and the standard error of $\hat{\theta}_{MC}$ to obtain an estimate of the standard errors of $\hat{\theta}_{CV}$.

(c) [3 points] Estimate standard error of $\hat{\theta}_{CV}$, using the 20,000 values that you simulated, and compare this estimate to your estimate in Part (b).

(d) [5 points] Now, consider a random variable X with density $f_X(x) = 2(1-x), 0 \le x \le 1$. Use 20,000 random values from X in the method of control variates with the control function as in part (a) to estimate the integral given in Problem 1.

(e) [3 points] Estimate the standard error of your estimate in part (d), using the 20,000 random values generated, and compare it to the standard error of $\hat{\theta}_{CV}$ obtained in part (c).