Math 534 Homework 2 (Part 1) - 20 points

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due 2024/02/07

Exercise GH-2.2: Do all parts (a) through (d)

Consider the density $f(x) = [1 - cos(x - \theta)]/2\pi$ on $0 \leq x \leq 2\pi$, where θ is a parameter between $-\pi$ and π. The following i.i.d. data arise from this density: 3.91, 4.85, 2.28, 4.06, 3.70, 4.04, 5.46, 3.53, 2.28, 1.96, 2.53, 3.88, 2.22, 3.47, 4.82, 2.46, 2.99, 2.54, 0.52, 2.50. We wish to estimate θ.

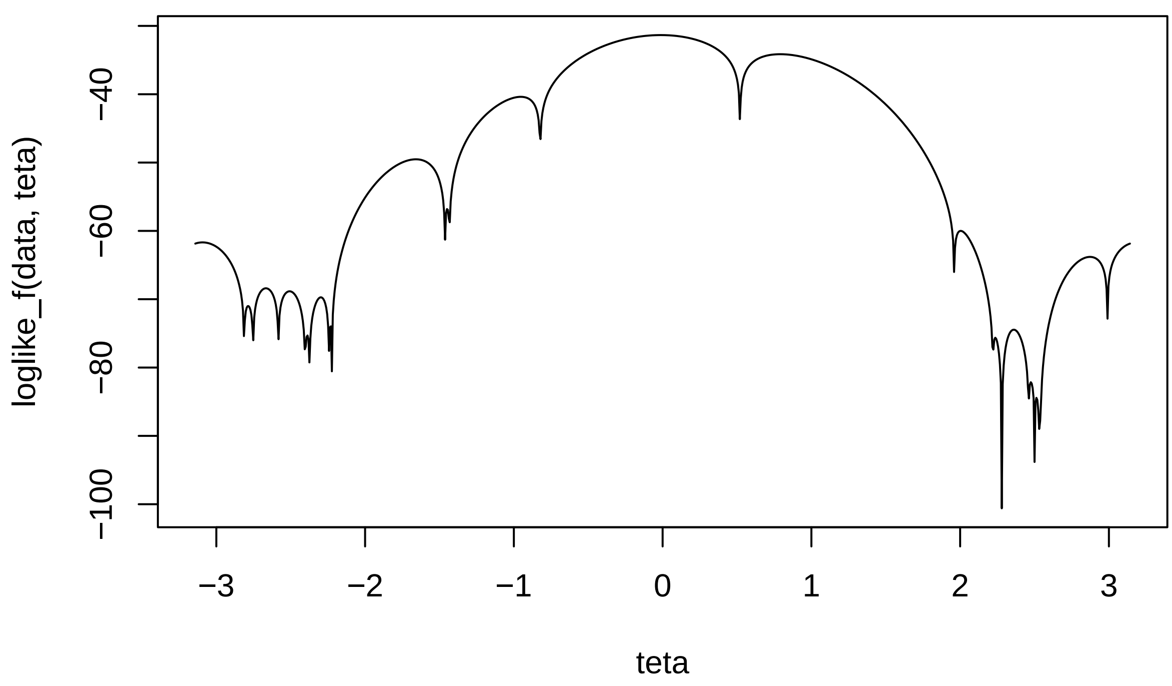
a. Graph the log likelihood function between $-\pi$ and π.

```
data = c(3.91,4.85,2.28,4.06,3.70,
         4.04,5.46,3.53,2.28,1.96,
         2.53,3.88,2.22,3.47,4.82,
         2.46,2.99,2.54,0.52,2.50)

teta <- seq(from=-pi,to=pi, length = 1000)

loglike_f <- function(data,teta){
  n = length(data)
  -n*log(2*pi) + rowSums(sapply(data,function(x) log(1-cos(x-teta))))
}

plot(teta, loglike_f(data,teta), type = "l")
```



b. Find the method-of-moments estimator of θ.

First, note $\frac{\sum_{i=1}^{n} x_i}{n}$ which is equal to the mean of the data.

```r
mean(data)
```

```
## [1] 3.2
```

Second, note that

$$E[X] = \int_0^{2\pi} x \frac{[1 - cos(x - \theta)]}{2\pi} dx = sin(\theta) + \pi.$$

Now, method of moments uses

$$\frac{\sum_{i=1}^n x_i}{n} = E[X].$$

So,

$$3.2 = sin(\theta) + \pi.$$

Therefore the method of moments estimate (MME) of θ is

$$\hat{\theta} = sin^{-1}(3.2 - \pi).$$

Or, using R for calculation gives us

```r
teta_hat <- asin(mean(data)-pi) #method of moments estimate
asin(3.2-pi)
```

```
## [1] 0.05844061
```

Finally, we have

$$\hat{\theta} = sin^{-1}(3.2 - \pi) \approx 0.05844061.$$

c. Find the MLE for θ using the Newton-Raphson method, using the result from (b) as the starting value. What solutions do you find when you start at -2.7 and 2.7?

```r
teta_start <- c(teta_hat,-2.7,2.7) # starting points

newton <- function(data, teta_start, print_each_it = TRUE, return_last_it = FALSE){

  teta_n <- teta_start # starting point
  maxit = 1000
  tolgrad = 1e-9
  tolerr = 1e-6
  it = 1
  stop = 0

  d1_loglike_f <- function(data,teta){
    sum(sapply(data,function(x) -sin(x-teta)/(1-cos(x-teta)) ))}

  d2_loglike_f <- function(data,teta){
    sum(sapply(data,function(x) 1/(cos(x-teta)-1) ))}

  #print
```

```
  if (print_each_it == TRUE) print(paste0("iteration", "    teta","   modified relative error", "   grad

  #core calculation
  while(it <= maxit & stop == 0){

    teta_n_new <- teta_n - d1_loglike_f(data,teta_n)/d2_loglike_f(data,teta_n)

    #stop calculation
    mod_rel_err = abs(teta_n_new-teta_n)/max(1,abs(teta_n_new))
    grad = d1_loglike_f(data,teta_n)
    if (mod_rel_err<tolerr & abs(grad) < tolgrad) stop=1

    #print
    if (print_each_it == TRUE) print(sprintf('%3.0f    %12.12f   %4.1e   %4.1e',it,teta_n_new,mod_rel_er

    #next iteration
    teta_n <- teta_n_new
    it = it + 1}

  if (return_last_it == TRUE) return(c(it-1,teta_start,teta_n_new,mod_rel_err,grad))
}

teta_hat <- asin(mean(data)-pi) #again #method of moments estimate
teta_start <- c(teta_hat,-2.7,2.7) #again #starting points
newton(data,teta_start[1], print_each_it = TRUE, return_last_it = FALSE) #  MME
```

```
## [1] "iteration    teta   modified relative error   gradient"
## [1] "  1    -0.009098573745   6.8e-02   -1.6e+00"
## [1] "  2    -0.011968737913   2.9e-03   -6.3e-02"
## [1] "  3    -0.011972002283   3.3e-06   -7.2e-05"
## [1] "  4    -0.011972002287   4.1e-12   -9.1e-11"
```

```
newton(data,teta_start[2], print_each_it = TRUE, return_last_it = FALSE) # -2.7
```

```
## [1] "iteration    teta   modified relative error   gradient"
## [1] "  1    -2.674113655831   9.7e-03   2.8e+01"
## [1] "  2    -2.666793927068   2.7e-03   5.5e+00"
## [1] "  3    -2.666699927130   3.5e-05   7.0e-02"
## [1] "  4    -2.666699926101   3.9e-10   7.6e-07"
## [1] "  5    -2.666699926101   1.7e-16   -4.0e-13"
```

```
newton(data,teta_start[3], print_each_it = TRUE, return_last_it = FALSE) #  2.7
```

```
## [1] "iteration    teta   modified relative error   gradient"
## [1] "  1    2.825724484570   4.4e-02   3.9e+01"
## [1] "  2    2.877549108301   1.8e-02   1.0e+01"
## [1] "  3    2.873184456115   1.5e-03   -1.1e+00"
## [1] "  4    2.873094549040   3.1e-05   -2.3e-02"
## [1] "  5    2.873094514245   1.2e-08   -8.7e-06"
## [1] "  6    2.873094514245   1.7e-15   -1.3e-12"
```

d. Repeat part (c) using 200 equally spaced starting values between $-\pi$ and π. Partition the interval between $-\pi$ and π into sets of attraction. In other words, divide the set of starting values into separate groups, with each group corresponding to a separate unique outcome of the optimization (a local mode). Discuss your results.

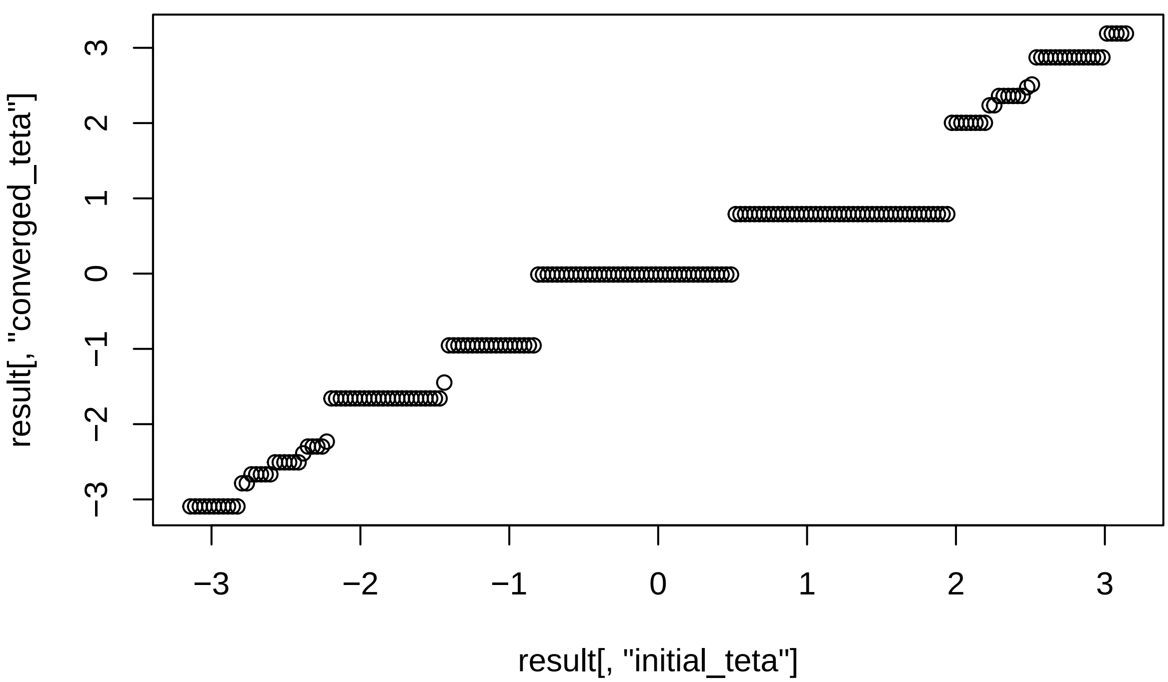
```r
#part d
teta_start <- seq(from=-pi,to=pi, length = 200)
result = matrix(0,nrow = length(teta_start) , ncol = 5)
colnames(result) <- c("iterations","initial_teta","converged_teta","mod_rel_error","gradient")

for(i in 1:length(teta_start)){

  result[i,] <- newton(data,teta_start[i], print_each_it = FALSE, return_last_it = TRUE)

}

plot(result[,"initial_teta"],result[,"converged_teta"])
```



Extra

```r
teta_start <- seq(from=-pi,to=pi, length = 100)
result_extra <- matrix(0,nrow = length(teta_start) , ncol = 5)
colnames(result_extra) <- c("iteration","initial_teta","converged_teta","modified_relative_error","grad

for(i in 1:length(teta_start)){

  result_extra[i,] <- newton(data,teta_start[i], print_each_it = FALSE, return_last_it = TRUE)

}
```

```r
length(unique(round(result[,"initial_teta"], digits = 3)))
```

```
## [1] 200
```

```r
length(unique(round(result[,"converged_teta"], digits = 3)))
```

```
## [1] 19
```

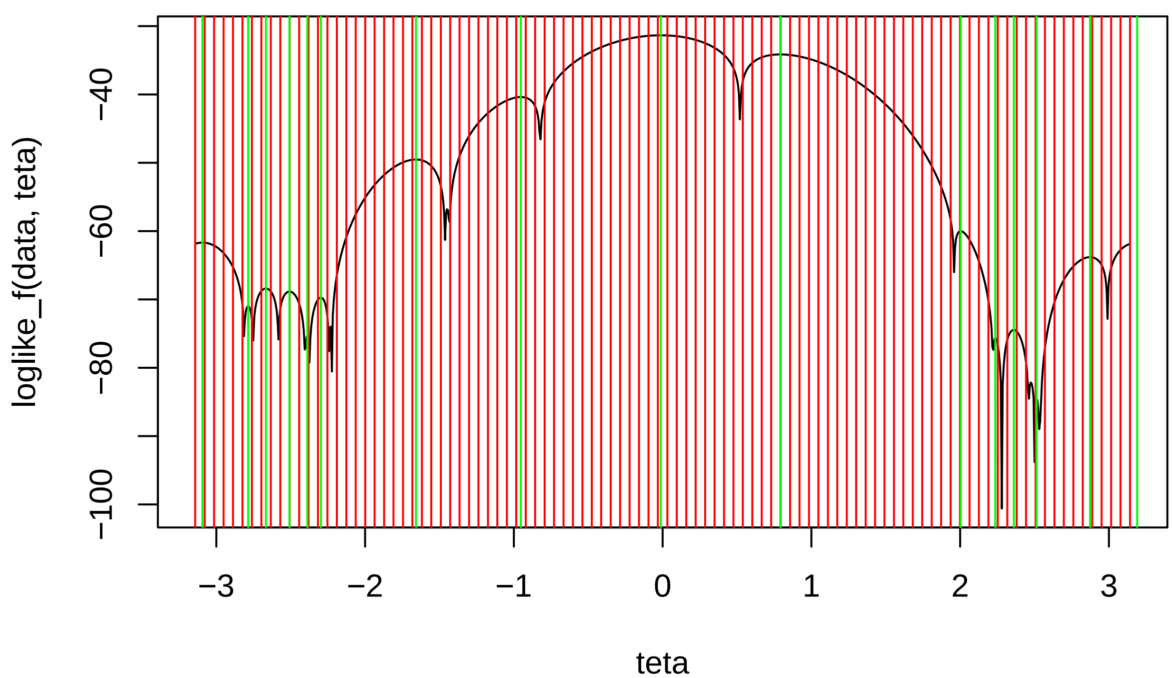
```r
length(unique(round(result_extra[,"initial_teta"], digits = 3)))
```

```
## [1] 100
```

```r
length(unique(round(result_extra[,"converged_teta"], digits = 3)))
```

```
## [1] 16
```

```r
plot(teta, loglike_f(data,teta), type = "l")
abline(v=unique(round(result_extra[,"initial_teta"], digits = 3)), col = 'red')
abline(v=unique(round(result_extra[,"converged_teta"], digits = 3)), col = 'green')
```




```r
result
```

```
##       iterations initial_teta converged_teta mod_rel_error      gradient
##   [1,]          5  -3.14159265     -3.0930917  0.000000e+00 -6.189493e-15
##   [2,]          4  -3.11001886     -3.0930917  1.579320e-15  6.884215e-13
##   [3,]          4  -3.07844506     -3.0930917  0.000000e+00 -6.189493e-15
##   [4,]          4  -3.04687127     -3.0930917  2.110546e-14  9.466788e-12
##   [5,]          5  -3.01529747     -3.0930917  0.000000e+00 -6.189493e-15
##   [6,]          5  -2.98372368     -3.0930917  0.000000e+00 -6.189493e-15
```

```
##   [7,]           5  -2.95214988     -3.0930917  7.178727e-16  3.506084e-13
##   [8,]           6  -2.92057608     -3.0930917  1.435745e-16  6.156187e-14
##   [9,]           6  -2.88900229     -3.0930917  1.435745e-16  6.156187e-14
##  [10,]           7  -2.85742849     -3.0930917  1.435745e-16  6.156187e-14
##  [11,]           8  -2.82585470     -3.0930917  4.881534e-15  2.192996e-12
##  [12,]           5  -2.79428090     -2.7861668  4.781723e-16  7.244344e-12
##  [13,]           6  -2.76270711     -2.7861668  9.563445e-16 -1.228216e-11
##  [14,]           6  -2.73113331     -2.6666999  1.831845e-15  3.695835e-12
##  [15,]           5  -2.69955952     -2.6666999  1.665314e-16  4.465490e-13
##  [16,]           3  -2.66798572     -2.6666999  1.432170e-14  2.846226e-11
##  [17,]           5  -2.63641193     -2.6666999  1.665314e-16  4.465490e-13
##  [18,]           6  -2.60483813     -2.6666999  3.330628e-16  7.660712e-13
##  [19,]           8  -2.57326433     -2.5076132  3.541927e-16  7.486095e-13
##  [20,]           6  -2.54169054     -2.5076132  1.770964e-16 -2.575856e-13
##  [21,]           4  -2.51011674     -2.5076132  1.770964e-16 -2.575856e-13
##  [22,]           5  -2.47854295     -2.5076132  1.770964e-16 -2.575856e-13
##  [23,]           6  -2.44696915     -2.5076132  1.770964e-16 -2.575856e-13
##  [24,]           7  -2.41539536     -2.5076132  1.239675e-15  2.299383e-12
##  [25,]           5  -2.38382156     -2.3882005  0.000000e+00  1.602690e-12
##  [26,]           6  -2.35224777     -2.2972562  1.279731e-13 -4.801218e-10
##  [27,]           5  -2.32067397     -2.2972562  0.000000e+00 -3.571587e-13
##  [28,]           5  -2.28910017     -2.2972562  5.799386e-16  2.197575e-12
##  [29,]           7  -2.25752638     -2.2972562  0.000000e+00 -3.571587e-13
##  [30,]           7  -2.22595258     -2.2321673  0.000000e+00  4.048317e-12
##  [31,]           9  -2.19437879     -1.6582832  1.339003e-16 -1.570966e-14
##  [32,]           7  -2.16280499     -1.6582832  7.930914e-13 -1.686520e-10
##  [33,]           8  -2.13123120     -1.6582832  0.000000e+00 -1.310063e-14
##  [34,]           7  -2.09965740     -1.6582832  1.339003e-16 -1.892930e-14
##  [35,]           7  -2.06808361     -1.6582832  5.356012e-16  1.088019e-13
##  [36,]           7  -2.03650981     -1.6582832  1.339003e-16 -1.892930e-14
##  [37,]           7  -2.00493602     -1.6582832  0.000000e+00 -1.310063e-14
##  [38,]           6  -1.97336222     -1.6582832  1.375156e-13 -2.925168e-11
##  [39,]           6  -1.94178842     -1.6582832  1.339003e-16 -1.570966e-14
##  [40,]           5  -1.91021463     -1.6582832  8.435718e-15 -1.807221e-12
##  [41,]           5  -1.87864083     -1.6582832  3.202895e-13 -6.810158e-11
##  [42,]           6  -1.84706704     -1.6582832  0.000000e+00 -1.310063e-14
##  [43,]           6  -1.81549324     -1.6582832  0.000000e+00 -1.310063e-14
##  [44,]           6  -1.78391945     -1.6582832  4.017009e-16 -8.237855e-14
##  [45,]           6  -1.75234565     -1.6582832  1.339003e-16 -1.892930e-14
##  [46,]           5  -1.72077186     -1.6582832  9.587261e-14 -2.037809e-11
##  [47,]           5  -1.68919806     -1.6582832  0.000000e+00 -1.310063e-14
##  [48,]           3  -1.65762426     -1.6582832  3.737425e-12 -7.947344e-10
##  [49,]           5  -1.62605047     -1.6582832  1.339003e-16 -1.892930e-14
##  [50,]           6  -1.59447667     -1.6582832  1.339003e-16 -1.570966e-14
##  [51,]           6  -1.56290288     -1.6582832  3.615308e-15 -7.777667e-13
##  [52,]           7  -1.53132908     -1.6582832  0.000000e+00 -1.310063e-14
##  [53,]           8  -1.49975529     -1.6582832  1.339003e-16 -1.570966e-14
##  [54,]          10  -1.46818149     -1.6582832  3.648783e-13 -7.759215e-11
##  [55,]           7  -1.43660770     -1.4474788  3.835024e-15 -1.003243e-10
##  [56,]          10  -1.40503390     -0.9533363  5.551115e-16  8.958459e-14
##  [57,]           8  -1.37346010     -0.9533363  3.060219e-12 -4.540373e-10
##  [58,]           8  -1.34188631     -0.9533363  2.220446e-16 -2.894560e-14
##  [59,]           7  -1.31031251     -0.9533363  2.220446e-16 -3.284525e-14
##  [60,]           8  -1.27873872     -0.9533363  5.551115e-16  8.554615e-14
```

```
##  [61,]          7 -1.24716492     -0.9533363  1.165734e-14 -1.724006e-12
##  [62,]          6 -1.21559113     -0.9533363  1.786460e-12 -2.650473e-10
##  [63,]          5 -1.18401733     -0.9533363  4.107825e-15 -6.070248e-13
##  [64,]          6 -1.15244354     -0.9533363  2.220446e-16 -3.172115e-14
##  [65,]          6 -1.12086974     -0.9533363  3.376965e-12 -5.010304e-10
##  [66,]          7 -1.08929595     -0.9533363  1.332268e-15  1.923219e-13
##  [67,]          6 -1.05772215     -0.9533363  7.218670e-13 -1.071067e-10
##  [68,]          6 -1.02614835     -0.9533363  6.661338e-16  9.206871e-14
##  [69,]          5 -0.99457456     -0.9533363  4.618750e-12 -6.852814e-10
##  [70,]          5 -0.96300076     -0.9533363  1.443290e-15  2.136936e-13
##  [71,]          5 -0.93142697     -0.9533363  2.442491e-15 -3.615129e-13
##  [72,]          6 -0.89985317     -0.9533363  5.551115e-16  8.554615e-14
##  [73,]          7 -0.86827938     -0.9533363  2.220446e-16 -2.894560e-14
##  [74,]          9 -0.83670558     -0.9533363  5.551115e-16  8.958459e-14
##  [75,]          9 -0.80513179     -0.0119720  1.703672e-14 -3.747835e-13
##  [76,]          8 -0.77355799     -0.0119720  8.153200e-17 -1.776357e-15
##  [77,]          7 -0.74198419     -0.0119720  2.959265e-14 -6.509931e-13
##  [78,]          7 -0.71041040     -0.0119720  1.942890e-16 -4.260481e-15
##  [79,]          6 -0.67883660     -0.0119720  1.156939e-13 -2.545020e-12
##  [80,]          6 -0.64726281     -0.0119720  1.104655e-13 -2.429987e-12
##  [81,]          6 -0.61568901     -0.0119720  2.130240e-15 -4.685141e-14
##  [82,]          6 -0.58411522     -0.0119720  1.942890e-16 -4.260481e-15
##  [83,]          6 -0.55254142     -0.0119720  3.989864e-17 -8.881784e-16
##  [84,]          5 -0.52096763     -0.0119720  1.379383e-13 -3.034364e-12
##  [85,]          5 -0.48939383     -0.0119720  4.592507e-14 -1.010261e-12
##  [86,]          5 -0.45782003     -0.0119720  2.960045e-13 -6.511458e-12
##  [87,]          5 -0.42624624     -0.0119720  1.895151e-13 -4.168929e-12
##  [88,]          5 -0.39467244     -0.0119720  4.997565e-14 -1.099371e-12
##  [89,]          5 -0.36309865     -0.0119720  7.108897e-15 -1.563749e-13
##  [90,]          5 -0.33152485     -0.0119720  4.874573e-16 -1.072753e-14
##  [91,]          5 -0.29995106     -0.0119720  2.949030e-16 -6.480927e-15
##  [92,]          5 -0.26837726     -0.0119720  2.949030e-17  6.661338e-16
##  [93,]          5 -0.23680347     -0.0119720  3.122502e-16  6.883383e-15
##  [94,]          4 -0.20522967     -0.0119720  1.102425e-12 -2.425099e-11
##  [95,]          4 -0.17365588     -0.0119720  3.694961e-16 -8.146261e-15
##  [96,]          4 -0.14208208     -0.0119720  1.918448e-13 -4.220194e-12
##  [97,]          4 -0.11050828     -0.0119720  2.660962e-13 -5.853540e-12
##  [98,]          4 -0.07893449     -0.0119720  5.897366e-14 -1.297296e-12
##  [99,]          4 -0.04736069     -0.0119720  1.118897e-15 -2.461920e-14
## [100,]          3 -0.01578690     -0.0119720  1.174756e-11 -2.584211e-10
## [101,]          4  0.01578690     -0.0119720  9.020562e-16 -1.983136e-14
## [102,]          4  0.04736069     -0.0119720  8.297911e-13 -1.825363e-11
## [103,]          5  0.07893449     -0.0119720  2.949030e-17  6.661338e-16
## [104,]          5  0.11050828     -0.0119720  2.949030e-16 -6.480927e-15
## [105,]          5  0.14208208     -0.0119720  2.949030e-16 -6.480927e-15
## [106,]          5  0.17365588     -0.0119720  3.526693e-15 -7.757683e-14
## [107,]          5  0.20522967     -0.0119720  1.566663e-13 -3.446327e-12
## [108,]          5  0.23680347     -0.0119720  4.994545e-12 -1.098693e-10
## [109,]          6  0.26837726     -0.0119720  2.949030e-17  6.661338e-16
## [110,]          6  0.29995106     -0.0119720  3.989864e-17 -8.881784e-16
## [111,]          6  0.33152485     -0.0119720  1.219511e-15 -2.683964e-14
## [112,]          6  0.36309865     -0.0119720  3.824996e-13 -8.414158e-12
## [113,]          7  0.39467244     -0.0119720  1.942890e-16 -4.260481e-15
## [114,]          7  0.42624624     -0.0119720  6.088879e-16 -1.339207e-14
```

```
## [115,]          8  0.45782003    -0.0119720  2.949030e-17  6.661338e-16
## [116,]          9  0.48939383    -0.0119720  7.112366e-17 -1.554312e-15
## [117,]         14  0.52096763     0.7906013  1.110223e-16  4.432218e-15
## [118,]          8  0.55254142     0.7906013  6.856404e-12  2.902990e-10
## [119,]          7  0.58411522     0.7906013  3.805734e-12  1.611363e-10
## [120,]          7  0.61568901     0.7906013  1.110223e-16  6.366435e-15
## [121,]          6  0.64726281     0.7906013  2.742251e-13  1.160983e-11
## [122,]          6  0.67883660     0.7906013  3.330669e-16 -1.328451e-14
## [123,]          5  0.71041040     0.7906013  5.712986e-12  2.418877e-10
## [124,]          5  0.74198419     0.7906013  4.440892e-16  2.008636e-14
## [125,]          4  0.77355799     0.7906013  2.895795e-12  1.226097e-10
## [126,]          4  0.80513179     0.7906013  5.384582e-13  2.279679e-11
## [127,]          5  0.83670558     0.7906013  1.110223e-16  4.756612e-15
## [128,]          5  0.86827938     0.7906013  5.495604e-14  2.326507e-12
## [129,]          5  0.89985317     0.7906013  5.351275e-12  2.265702e-10
## [130,]          6  0.93142697     0.7906013  3.330669e-16 -1.328451e-14
## [131,]          6  0.96300076     0.7906013  2.220446e-16  8.142792e-15
## [132,]          6  0.99457456     0.7906013  3.330669e-16 -1.328451e-14
## [133,]          6  1.02614835     0.7906013  4.440892e-16  2.008636e-14
## [134,]          6  1.05772215     0.7906013  2.553513e-15  1.068191e-13
## [135,]          6  1.08929595     0.7906013  5.884182e-15  2.497811e-13
## [136,]          6  1.12086974     0.7906013  5.107026e-15  2.148507e-13
## [137,]          6  1.15244354     0.7906013  2.220446e-15  9.571684e-14
## [138,]          6  1.18401733     0.7906013  6.661338e-16  2.622035e-14
## [139,]          6  1.21559113     0.7906013  1.110223e-16  4.432218e-15
## [140,]          6  1.24716492     0.7906013  1.110223e-16  4.756612e-15
## [141,]          5  1.27873872     0.7906013  4.383161e-13  1.855657e-11
## [142,]          5  1.31031251     0.7906013  2.220446e-16  8.142792e-15
## [143,]          5  1.34188631     0.7906013  1.699618e-11  7.196185e-10
## [144,]          6  1.37346010     0.7906013  3.330669e-16 -1.328451e-14
## [145,]          6  1.40503390     0.7906013  1.543210e-14  6.551096e-13
## [146,]          6  1.43660770     0.7906013  1.628142e-12  6.893760e-11
## [147,]          7  1.46818149     0.7906013  1.110223e-16  6.366435e-15
## [148,]          7  1.49975529     0.7906013  1.110223e-16  6.366435e-15
## [149,]          7  1.53132908     0.7906013  3.330669e-16 -1.328451e-14
## [150,]          7  1.56290288     0.7906013  1.887379e-15  7.773816e-14
## [151,]          7  1.59447667     0.7906013  5.551115e-15  2.328780e-13
## [152,]          7  1.62605047     0.7906013  4.218847e-15  1.807079e-13
## [153,]          7  1.65762426     0.7906013  1.110223e-16  6.366435e-15
## [154,]          7  1.68919806     0.7906013  1.110223e-16  6.366435e-15
## [155,]          6  1.72077186     0.7906013  2.220446e-16  8.142792e-15
## [156,]          7  1.75234565     0.7906013  1.110223e-16  4.756612e-15
## [157,]          7  1.78391945     0.7906013  1.333478e-11  5.645926e-10
## [158,]          8  1.81549324     0.7906013  6.661338e-16  2.622035e-14
## [159,]          8  1.84706704     0.7906013  4.218847e-15  1.792923e-13
## [160,]          7  1.87864083     0.7906013  3.108624e-15  1.298284e-13
## [161,]          9  1.91021463     0.7906013  1.443290e-15  5.975255e-14
## [162,]         10  1.94178842     0.7906013  3.441691e-15  1.467038e-13
## [163,]          7  1.97336222     2.0036449  1.152532e-13  2.748095e-10
## [164,]          4  2.00493602     2.0036449  5.097736e-15  1.217801e-11
## [165,]          7  2.03650981     2.0036449  2.216407e-16 -2.963740e-13
## [166,]          9  2.06808361     2.0036449  1.329844e-15  2.988859e-12
## [167,]          8  2.09965740     2.0036449  0.000000e+00  9.897638e-14
## [168,]          6  2.13123120     2.0036449  3.184977e-13  7.588564e-10
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## [169,]           10   2.16280499       2.0036449  2.216407e-16 -6.973588e-13
## [170,]           10   2.19437879       2.0036449  6.649220e-16 -1.501799e-12
## [171,]            7   2.22595258       2.2362194  9.929464e-16 -2.104972e-11
## [172,]            6   2.25752638       2.2362194  1.191536e-15 -2.737816e-11
## [173,]            8   2.28910017       2.3607182  4.702904e-15  1.287390e-11
## [174,]            6   2.32067397       2.3607182  0.000000e+00 -6.808790e-14
## [175,]            5   2.35224777       2.3607182  5.643484e-16  1.474515e-12
## [176,]            4   2.38382156       2.3607182  3.555395e-14  9.913058e-11
## [177,]            5   2.41539536       2.3607182  8.220676e-14  2.296043e-10
## [178,]            7   2.44696915       2.3607182  5.643484e-16  1.474515e-12
## [179,]            5   2.47854295       2.4753736  2.152835e-15 -7.173562e-11
## [180,]            5   2.51011674       2.5135932  0.000000e+00  1.234179e-12
## [181,]           11   2.54169054       2.8730945  1.545683e-16 -8.498757e-14
## [182,]            8   2.57326433       2.8730945  4.637048e-16 -3.262945e-13
## [183,]            7   2.60483813       2.8730945  7.728413e-15 -5.576317e-12
## [184,]            7   2.63641193       2.8730945  1.545683e-16 -7.421841e-14
## [185,]            6   2.66798572       2.8730945  4.637048e-16 -3.094747e-13
## [186,]            6   2.69955952       2.8730945  2.163956e-15 -1.517120e-12
## [187,]            6   2.73113331       2.8730945  1.545683e-16 -8.498757e-14
## [188,]            5   2.76270711       2.8730945  1.545683e-16 -7.421841e-14
## [189,]            5   2.79428090       2.8730945  2.473092e-15 -1.736278e-12
## [190,]            5   2.82585470       2.8730945  1.700251e-15 -1.256495e-12
## [191,]            5   2.85742849       2.8730945  1.545683e-16 -8.498757e-14
## [192,]            5   2.88900229       2.8730945  1.545683e-16 -7.421841e-14
## [193,]            6   2.92057608       2.8730945  1.545683e-16 -7.421841e-14
## [194,]            7   2.95214988       2.8730945  1.545683e-16 -8.498757e-14
## [195,]           10   2.98372368       2.8730945  1.545683e-16 -7.421841e-14
## [196,]            7   3.01529747       3.1900936  1.187173e-12  5.501190e-10
## [197,]            6   3.04687127       3.1900936  7.043967e-14  3.265663e-11
## [198,]            6   3.07844506       3.1900936  1.392088e-16 -3.347322e-14
## [199,]            5   3.11001886       3.1900936  2.923386e-15  1.347839e-12
## [200,]            5   3.14159265       3.1900936  1.392088e-16  3.225198e-14
```