Common Vectors and Matrices

Name	Definition	Notation	Example
Scalar	$p = n = 1$	a,b	1
Column Vector	$p = 1$	$\boldsymbol{a}$, $\boldsymbol{b}$	$\begin{pmatrix}1\\2\end{pmatrix}$
Row Vector	$n = 1$	$\boldsymbol{a}^T$, $\boldsymbol{b}^T$	$\begin{pmatrix}1, & 2\end{pmatrix}$
Unit Vector	$\begin{pmatrix}1, & \cdots, & 1\end{pmatrix}^T$	$\mathbf{1}_n$	$\begin{pmatrix}1\\1\end{pmatrix}$
Zero Vector	$\begin{pmatrix}0, & \cdots, & 0\end{pmatrix}^T$	$\mathbf{0}_n$	$\begin{pmatrix}0\\0\end{pmatrix}$
Square Matrix	$n = p$	$\boldsymbol{A}_{pxp}$	$\begin{pmatrix}1 & 3\\4 & 5\end{pmatrix}$
Diagonal Matrix	$a_{ij} = 0$ for $i \neq j$; $n = p$	$diag(a_{ii})$	$\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}$
Identity Matrix	$diag(1, \cdots, 1)$	$\boldsymbol{I}_p$ or $\boldsymbol{I}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$
Unit Matrix	$a_{ij} = 1$; $n = p$	$\mathbf{1}_n\ \mathbf{1}_n^T$	$\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$
Symmetric Matrix	$a_{ij} = a_{ji}$	–	$\begin{pmatrix}3 & 2\\2 & 5\end{pmatrix}$
Null Matrix	$a_{ij} = 0$	$\mathbf{0}_p$ or $\mathbf{0}$	$\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$
Upper Triangular Matrix	$a_{ij} = 0$ for $j < i$	–	$\begin{pmatrix}3 & 2 & 1\\0 & 2 & 1\\0 & 0 & 0\end{pmatrix}$
Idempotent Matrix	$\boldsymbol{A}\boldsymbol{A} = \boldsymbol{A}$	–	$\begin{pmatrix}1 & 0 & 0\\0 & 1/2 & 1/2\\0 & 1/2 & 1/2\end{pmatrix}$
Orthogonal Matrix	$\boldsymbol{A}\boldsymbol{A}^T = c\boldsymbol{I}$	–	$\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$

Matrix Operations

Using the notation described earlier:

1. $\boldsymbol{A}^T = \left(a_{ji}\right)$
2. $\boldsymbol{A} + \boldsymbol{B} = \left(a_{ij} + b_{ij}\right)$
3. $\boldsymbol{A} - \boldsymbol{B} = \left(a_{ij} - b_{ij}\right)$
4. $c\boldsymbol{A} = \left(ca_{ij}\right)$
5. $\boldsymbol{AB}| = \boldsymbol{A}|_{n\times p}\boldsymbol{B}_{p\times m} = \boldsymbol{C}_{n\times m} = \left(\sum_{j=1}^{p} a_{ij}b_{jk}\right)$ Note: $\boldsymbol{AB} \neq \boldsymbol{BA}$
6. $(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$
7. $(\boldsymbol{A} + \boldsymbol{B}) = (\boldsymbol{B} + \boldsymbol{A})$
8. $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$
9. $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$
10. $(\boldsymbol{A}^T)^T = \boldsymbol{A}$
11. $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$
12. $(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC}$
13. $(\boldsymbol{A} - \boldsymbol{B})(\boldsymbol{C} - \boldsymbol{D}) = \boldsymbol{AC} - \boldsymbol{BC} - \boldsymbol{AD} + \boldsymbol{BD}$
14. $\boldsymbol{ABC} + \boldsymbol{ADC} = \boldsymbol{A}(\boldsymbol{B} + \boldsymbol{D})\boldsymbol{C}$
15. $\boldsymbol{x}^T\boldsymbol{x} - \boldsymbol{x}^T\boldsymbol{Ax} = \boldsymbol{x}^T(\boldsymbol{x} - \boldsymbol{Ax}) = \boldsymbol{x}^T(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x}$
16. $\boldsymbol{a}^T\boldsymbol{a} = \sum_{j=1}^{n} a_j^2; \quad \boldsymbol{aa}^T \begin{pmatrix} a_1^2 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2^2 & \cdots & a_2a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & \cdots & a_n^2 \end{pmatrix}$
17. $(\boldsymbol{x} - \boldsymbol{y})^T(\boldsymbol{x} - \boldsymbol{y}) = \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{y} + \boldsymbol{y}^T\boldsymbol{y}$
18. $(\boldsymbol{A} - \boldsymbol{B})^T(\boldsymbol{A} - \boldsymbol{B}) = \boldsymbol{A}^T\boldsymbol{A} - \boldsymbol{A}^T\boldsymbol{B} - \boldsymbol{B}^T\boldsymbol{A} + \boldsymbol{B}^T\boldsymbol{B}$
19. $(\boldsymbol{A} - \boldsymbol{B})^2 = \boldsymbol{A}^2 - \boldsymbol{AB} - \boldsymbol{BA} + \boldsymbol{B}^2$
20. $\boldsymbol{IA} = \boldsymbol{AI} = \boldsymbol{A}$
21. Quadratic Form: $\boldsymbol{y}^T\boldsymbol{Ay} = \sum_i a_{ii}y_i^2 + \sum_{i\neq j} a_{ij}y_iy_j$
22. $\boldsymbol{x}^T\boldsymbol{Ay} = \sum_{ij} a_{ij}x_iy_j$

Properties of Trace

1. $tr(c\boldsymbol{A}) = ctr(\boldsymbol{A})$ for $\boldsymbol{A}_{n\times n}$
2. $tr(\boldsymbol{A} \pm \boldsymbol{B}) = tr(\boldsymbol{A}) \pm tr(\boldsymbol{B})$ for $\boldsymbol{A}_{n\times n}$ and $\boldsymbol{B}_{n\times n}$
3. $tr(\boldsymbol{AB}) = tr(\boldsymbol{BA})$ for $\boldsymbol{A}_{n\times p}$ and $\boldsymbol{B}_{p\times n}$
4. $tr(\boldsymbol{B}^{-1}\boldsymbol{AB}) = tr(\boldsymbol{A})$ for $\boldsymbol{A}_{n\times n}$ and $\boldsymbol{B}_{n\times n}$
5. $tr(\boldsymbol{AA}^T) = \sum_{i=1}^{n} a_{ii}^2$ for $\boldsymbol{A}_{n\times n}$
6. $tr(\boldsymbol{x}^T\boldsymbol{Ay}) = \boldsymbol{x}^T\boldsymbol{Ay}$
7. $tr(\boldsymbol{ABC}) = tr(\boldsymbol{BCA}) = tr(\boldsymbol{CBA})$

Properties of Rank

1. $0 \leq rank(\boldsymbol{A}) \leq min(n, p)$ for $\boldsymbol{A}_{n\times p}$
2. $rank(\boldsymbol{A}_{n\times n}) = n$ only if $\boldsymbol{A}$ is non-singular.
3. $rank(\boldsymbol{A}) = rank(\boldsymbol{A}^T)$
4. $rank(\boldsymbol{A}^T\boldsymbol{A}) = rank(\boldsymbol{A})$
5. $rank(\boldsymbol{A} + \boldsymbol{B}) \leq rank(\boldsymbol{A}) + rank(\boldsymbol{B})$
6. $rank(\boldsymbol{AB}) \leq min\,(rank(\boldsymbol{A}), rank(\boldsymbol{B}))$
7. $rank(\boldsymbol{ABC}) = rank(\boldsymbol{B})$ for nonsingular $\boldsymbol{A}, \boldsymbol{C}$

Properties of Determinant

For $\boldsymbol{A}_{n\times n}$ and $\boldsymbol{B}_{n\times n}$

1. $|c\boldsymbol{A}| = c^n|\boldsymbol{A}|$
2. $|\boldsymbol{A}^{-1}| = |\boldsymbol{A}|^{-1}$
3. $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$
4. $|\boldsymbol{A}^{-1}| = |\boldsymbol{A}|^{-1}$
5. If we can partition $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_{22} \end{pmatrix}$
   $\Rightarrow |\boldsymbol{A}| = |\boldsymbol{A}_{11}||\boldsymbol{A}_{22}|$