Name (please print)________________________________

Important Notes:

- Show your work on all of the problems. A final solution without showing work will not receive any credit, even if it is correct.
- **Interview:** You maybe be interviewed after you take the exam. If during the interview, you cannot give a reasonable explanation of the solution that you have provided, this would constitute grounds for cheating. Please do not copy solutions!
- **Getting help:** You can request pre-grading of some of the questions before you submit your exam. If you choose to use this option, the following rules apply:
  - You can request pre-grading no later than Saturday, 12/9, at 11:59 p.m.
  - There will be a minimum reduction of one point, even if your solution is completely correct.
  - I will do my best to respond as quickly as I can, but I cannot guarantee a quick response.
- Common random variables referred to in this exam are as defined in the table of distributions in the Casella and Berger text; make sure to use the parameterization used in that table.
- Posting this exam's questions on the Internet or sharing them with others at any time (during or after the exam) is strictly prohibited.
- You must not talk, email, chat, or have any form of communication with any other individual about the exam questions before the exam's due date.
- You are allowed to use our textbook or your class notes.
- Copying solutions from the Internet is considered cheating.
- Unless stated otherwise, I'm not interested in algebraic details or details of integrations or differentiations. Provide simplified versions of solutions. In case you're using the kernel of a pdf or pmf for integration or summation, simply point it out.
- Print this exam and write your solutions in the space provided for each question. Then scan your solutions and upload them to Canvas. Ensure that your scanned document is clear. If you are unable to print the exam, write your solutions on blank pages, clearly indicating problem numbers and following the same sequence as the exam questions. If you are not answering a problem, write the problem number and leave some blank spaces. Answering your questions on an iPad or other electronic forms is acceptable, as long as I receive a single PDF file with your solutions written in the order that the problems appear on the exam.
- Upload your solutions as a single pdf file to Canvas.
- The exam is due on Monday 12/11/2023 at 8:00 a.m.
- A total of 75 points is possible.

GOOD LUCK!

1. Let $X_1, X_2, \cdots, X_n$ be a sample from a distribution with pdf

$$f(x|\theta) = \frac{x}{\theta^2} e^{-x^2/(2\theta^2)}, \quad \text{for } x > 0; \quad \theta > 0,$$

and zero otherwise. This sample and distribution are used for all parts of this problem (Problem 1).

   (a) Obtain a method of moment estimator for θ. Show details of integration for the required integral. Don't use software for integration in this problem. [5 pts]

(b) Obtain the maximum likelihood estimate for θ. [5 pts]

(c) Obtain the maximum likelihood estimate for $\eta = \theta^2$. Is your estimate an unbiased estimate of η? Justify your answer. Show details of integration for the required integral. Don't use software for integration in this problem. [5 pts]

(d) Obtain the Cramer-Rao lower bound for an unbiased estimator of θ. [5 pts]

(e) The estimator $\tilde{\theta} = \sqrt{2/\pi}\bar{X}$ is an unbiased estimator of θ, where $\bar{X}$ is the sample mean. Does variance of $\tilde{\theta}$ achieve the Cramer-Rao lower-bound? Justify your answer. [5 pts]

(f) Let $Y_n = \sum_{i=1}^{n} X_i^2/(2n)$. It can be shown that $\sqrt{n}(Y_n - \theta^2)$ converges in distribution to the normal distribution with mean zero and variance θ^4. Using this result determine the limiting distribution of $\sqrt{Y_n}$, as the sample size n approaches infinity. [5 pts]

(g) Show that $\sum_{i=1}^{n} X_i^2/\theta^2$ is a pivotal quantity. [5 pts]

(h) Explain, in detail, how you would use the pivot $\sum_{i=1}^{n} X_i^2/\theta^2$ to construct a $100(1-\alpha)\%$ confidence interval for θ. [5 pts]

(i) Consider the likelihood ratio test to test $H_0 : \theta = 1$ versus $H_a : \theta > 1$. Show that if $\sum_{i=1}^n X_i^2/(2n) \leq 1$ you will not reject H_0 [2 points], and if $\sum_{i=1}^n X_i^2/(2n) > 1$ the rejection region is of the form

$$\left\{(X_1, \cdots, X_n) : \sum_{i=1}^n X_i^2 < c_1 \text{ or } \sum_{i=1}^n X_i^2 > c_2\right\},$$

where c_1 and c_2 are constants. [6 points] [8 pts]

(j) Suppose that we take a sample of size 20. Determine c_1 and c_2 in part (i) so that the test is of size 0.05. You can use R or any other probability calculator to compute c_1 and c_2. To get complete credit, give details of how you obtain c_1 and c_2. [5 pts]

(k) For the test in part (h) assume that we have a sample of size 20, and use the rejection region

$$\left\{(X_1, \cdots, X_n) : \sum_{i=1}^{n} X_i^2 < 5 \text{ or } \sum_{i=1}^{n} X_i^2 > 15\right\}.$$

Write down the power function $\beta(\theta)$ in terms of the cdf of an appropriate random variable, and draw the power function for values of $0.05 < \theta < 1.5$ when $n = 20$. You can sketch the graph by hand or use R or any other software. If you sketch the graph make sure to mark x and y axes with appropriate values. [4 pts]

2. Let U_1, U_2, and U_3 be unbiased estimators of a parameter θ with $Var(U_1) = 1$, $Var(U_2) = 1/2$ and $Var(U_3) = 1/3$. Moreover, assume that $Cov(U_i, U_j) = 0$ for $i \neq j$. Consider the estimator $T = a_1U_1 + a_2U_2 + a_3U_3$, where a_i are constants. Determine a_1, a_2, and a_3 such that the estimator T is unbiased, and it has minimum variance. [5 pts]

3. Suppose that $X_1, \cdots, X_n$ are i.i.d from the Poisson distribution with mean θ. Moreover, assume a prior for θ that is proportional to $\theta^{-1/2}$.

   (a) Derive the posterior distribution of θ, given the data $X_1, \cdots, X_n$. [5 pts]

(b) Using squared error loss, what is the Bayes estimate of θ? [3 pts]

(c) Suppose that for a sample of size $n = 10$, $\sum_{i=1}^{10} X_i = 20$. Let $[a, b]$ be the $1 - \alpha$ highest posterior density (HPD) credible interval for θ. What equations do a and b satisfy? [Hint: the equations should be in terms of the cdf and pdf of an appropriate random variable.] [5 pts]