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A VERSION OF THE PROJECTIVE GRID METHOD

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A stochastic discrepancy method is proposed for the construction of a solving system of equations of the projective grid method in Bubnov–Galerkin form. The interpolation polynomial on a finite element is viewed as the result of weighted averaging of the nodal parameters of the element. A simple posterior bound on discretization error is considered.

Projective grid methods [1] provide an efficient technique for numerical solution of many problems in mathematical physics. Various modifications of projective grid algorithms are available. In this paper, we use a probability interpretation of the barycentric coordinates of simplex elements in order to formulate a stochastic conception of the finite-element method (FEM).

The fruitful ideas of probabilistic simulation are not very widely used in computational mathematics (with the exception of the popular Monte-Carlo methods). Yet in many cases the numerical method has a definite probabilistic-statistical meaning, which helps to better understand the algorithm and grasp the underlying ideas of the method.

A numerical solution generally consists of a set of numbers, which can be used to construct a distribution of the dependent variable u. In this sense, a numerical method is similar to a laboratory experiment, which makes it possible to find a distribution of the observed quantity from the set of instrument readings. Researchers using numerical analysis, like experimental scientists, are usually satisfied with a result which produces a finite set of values, although in principle the number of values can be made sufficiently large for practical purposes. In numerical methods, the primary unknowns are usually represented by the values of u at some points of a given region, called nodes. In FEM, for instance, the information continuum is replaced with discrete information (a sample) obtained by discretizing the given region into elements. This is also typical of statistical research based on sampling. In FEM, even if the total number of nodes is constant, we can construct many different combinations of nodal values by altering the number of nodes and their arrangement on each element or the shape, size, and orientation of the elements. These factors are sufficient to ensure that a discrete collection of nodal parameters u_i may be viewed as a realization of the random variable U.

In ordinary statistical problems, the sample values of the sought variable are known. The sample is used to estimate the statistical parameters, to construct an empirical distribution function, and so on. In FEM, conversely, we solve the problem of constructing a discrete series of nodal values of the sought quantity given the distribution function on the element. The success of the method largely depends on a good choice of the distribution function on the element. In FEM, this distribution function – we call it the interpolation function – is actually constructed by weighted averaging of the nodal parameters of the elements.

Basis Functions and Geometrical Probability

Simplex elements are the most common in FEM and its numerous applications [1-3]. A simplex in k-dimensional space is a convex set Ω defined by k + 1 vertices (nodes) N_1, N_2, ..., N_{k+1} that do not lie in one (k – 1)-dimensional hyperplane. Examples of simplexes are a tetrahedron (four nodes) in three-dimensional space, a triangle (three nodes) in two-dimensional space, and a line segment (two nodes) in one-dimensional space. Local interpolation functions on simplex elements are conveniently constructed using special coordinates ξ_i, which are called natural or barycentric [2, 3]. These

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coordinates are represented as a ratio of measures. Depending on the dimension of the embedding space, they describe a linear change of relative length (for k = 1), area (k = 2), or volume (k = 3).

Let M be the current point on a simplex. Joining this point with the vertices N_i (i = 1, ..., k + 1), we partition the element into k + 1 subelements. The barycentric coordinates of the point M are defined as ξ_i = mes Ω_i/mes Ω, where mes Ω is the measure of the element; mes Ω_i is the measure of the subelement Ω_i opposite the node i. We see that these coordinates have the properties

$$0 \leqslant \xi_i \leqslant 1, \quad \sum_{i=1}^{k+1} \xi_i = 1. \tag{1}$$

The barycentric coordinates have an obvious probabilistic interpretation: ξ_i is the probability that a point tossed at random into the simplex Ω hits the subdomain Ω_i (geometrical probability). This means that a uniform distribution with density

$$p(M) = 1/\text{mes}\,\Omega \tag{2}$$

is defined within each element. The barycentric coordinates of a simplex supply the probabilities that should be assigned to the nodal values of the sought function so that its value at the current point M of the simplex is equal to the mean of the nodal values u_i. Local interpolation on an element now can be written as the mean

$$u = \sum_{i=1}^{k+1} u_i \xi_i, \tag{3}$$

and the properties of the basis functions ξ_i acquire a simple probabilistic interpretation [3].

When constructing higher order interpolations, the element is appropriately partitioned into simple subelements linked at the selected node. Then the sought basis function is formed by multiplying the geometrical probabilities calculated on each subelement. This technique has been successfully tried for one-, two-, and three-dimensional elements of various configurations [3-6].

Let us now consider the construction of a solving system of FEM equations.

Stochastic Discrepancy Method

In line with the standard FEM assumptions [1, 2], consider the finite-element solution of the differential equation

$$L(u_*) = f, \; M \in \sigma \tag{4}$$

with the boundary condition

$$F(u_*) = g, \quad M \in \Gamma. \tag{5}$$

We assume that the point M is contained in the planar region σ. By u_* we denote the exact solution. Partition the region σ into elements linked at the nodes. On each element, take an approximate solution of Eq. (4) in the form (3). Then the discrepancy ε = L(u) − f ≠ 0 is a function of the random arguments x and y. The nodal values of the approximation (3) should be "adjusted" so as to ensure the "best" approximation to the true distribution of the variable u. The usual criterion is to make the discrepancy equal zero on average. In this case, we naturally equate to zero the mean discrepancy M{ε} = 0. Expressing the mean discrepancy in terms of the density (2) and using the property (1), we obtain

$$M\{\varepsilon(x, y)\} = \iint_{\Omega} \varepsilon(x, y)\, p(x, y)\, dxdy = \frac{1}{\text{mes}\,\Omega} \sum_{i=1}^{k+1} \iint_{\Omega} \varepsilon(x, y)\, \xi_i(x, y)\, dxdy = 0,$$

whence follows the standard system of FEM equations in Bubnov−Galerkin form

$$\iint_{\Omega} \varepsilon(x, y)\, \xi_i(x, y)\, dxdy = 0, \quad i = \overline{1, k+1}.$$

Thus, zero mean discrepancy is equivalent to orthogonalizing the discrepancy with respect to the basis functions. The collection of all elements produces a global system of equations, which is transformed using the boundary conditions (5). The

procedure of generation of the solving system of FEM equations in this case may be interpreted as a stochastic discrepancy method.

Error Bounds

The standard technique is to express the order of discretization errors in terms of the characteristic grid spacing. This approach in itself does not provide an explicit way to determine the value of the discretization error. The local error on an element is the difference between the exact and the approximate solutions. An exact bound on this difference cannot be determined, because the exact solution is unknown.

Ideally, we should be able to assert for each discrete solution that the error does not exceed some prespecified value, which is a reasonable estimate of the true error. The probabilistic approach using the variance (standard deviation) of a random variable suggests a simple posterior bound in the form

$$\| u - u_* \| < C \max_{\Omega} | u_i - u_j |, \quad i \neq j,$$

where C depends on the number of nodes on an element.

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