Homework 2 – Part II (20 points)

Exercise J-2.1: A classic example of maximum likelihood estimation is due to Fisher (1925, *Statistical Methods for Research Workers*. Oliver and Boyd: Edinburgh.) and arises in a genetic problem. Consider a multinomial observation $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ with class probabilities given by

$$p_1 = (2+\theta)/4, \qquad p_2 = (1-\theta)/4, \qquad p_3 = (1-\theta)/4, \qquad p_4 = \theta/4,$$

where $0 < \theta < 1$. The parameter θ is to be estimated using maximum likelihood estimation based on the observed frequencies $x_1 = 1997,\ x_2 = 907,\ x_3 = 904, x_4 = 32$.

(a) **[10 pts]** Write an R function that applies the secant method for solving this problem. Here are some specific instructions: Write the required gradient as a separate R function. Stop your algorithm when **either** of the following criteria is satisfied:

  i. The number of iterations reaches 20
  ii. The absolute value of the gradient is less than *tolgrad*, **and** the modified relative error (as defined below) is less than *tolerr.* These tolerance values should be input to your program.

$$\text{Modified Relative Error (MRE)} = \frac{|\theta^{(n+1)} - \theta^{(n)}|}{\max(1, |\theta^{(n+1)}|)}$$

Run your program using the starting values $\theta^{(0)} = 0.02$ and $\theta^{(1)} = .01$, and use *tolerr = 1e-6* and *tolgrad=1e-9*.

You should nicely output the following quantities at each iteration:

- Iteration number n (2 digits, no decimals)
- Value of $\theta^{(n)}$ (12 decimal places)
- Value of the modified relative error (use 1 decimal with exponent notation, e.g., 2.0e-07)
- Value of the gradient at $\theta^{(n)}$(use 1 decimal with exponent notation, e.g., 2.0e-07)

(b) **[6 pts]** We know that the exact solution to this problem is $\theta^* = \left(-1657 + \sqrt{3728689}\right)/7680$. Using the same starting values and convergence criteria, modify your program in part (a) to output the following values:

- Iteration number n (2 digits, no decimals)
- Value of $\theta^{(n)}$ (12 decimal places)
- Convergence ratio $\left|\theta^{(n)} - \theta^*\right| / \left|\theta^{(n-1)} - \theta^*\right|^{(1+\sqrt{5})/2}$ (use 3 decimals with exponent notation)
- Number of significant digits of accuracy at iteration n, as defined by $-\log_{10}\left|\theta^{(n)} - \theta^*\right| / |\theta^*|$.

(c) **[2 pts]** Using Part (b), can you conclude that the secant method locally converges at least super-linearly? How can you check that the algorithm does not converge quadratically? Explain.

(d) **[2 pts]** Finally, obtain two initial values $\theta^{(0)}$ and $\theta^{(1)}$f such that your algorithm diverges. Set maxit = 10.