Lock-chart solving

Dissertation defense

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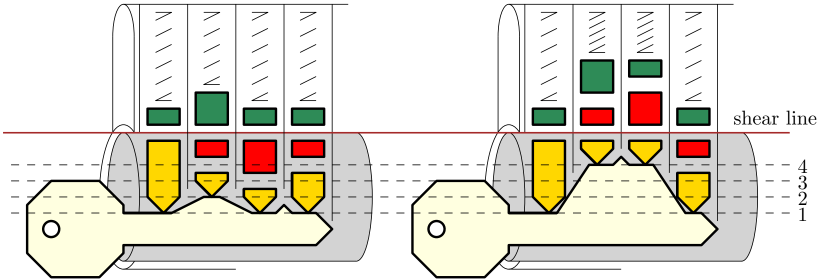
27/6/2018

1. Introduction

Tumbler lock



In this picture:

- There are 4 *positions* (vertical bars).
- There are 4 *cutting depths* (horizontal lines).

$$p = 4, d = 4$$

Lock-chart problem

Motivation

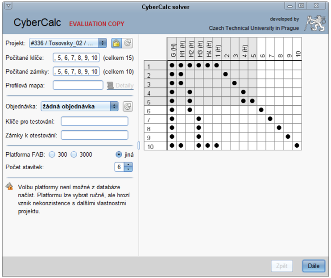
Reviewer's comment

Based on the presented research, has there been a progamme written that could be practically used to design master keyed systems?

Otázka oponenta

Byl na základě představeného výzkumu napsán program, který se v praxi použil na návrh zámků a klíčů, případně na nějaké dílčí úlohy?

Motivation

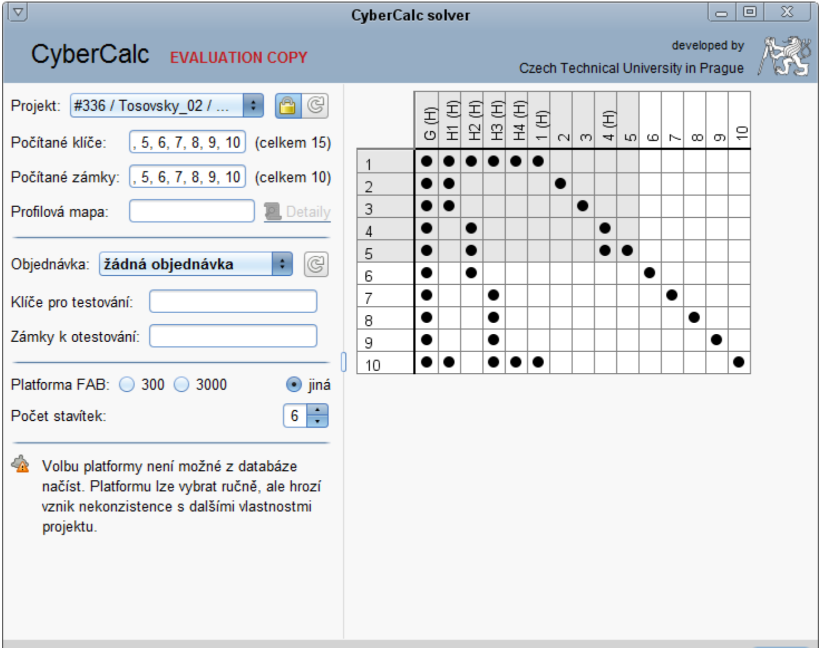


CyberCalc: Depth-first-search with constraint-satisfaction pruning and many heuristics.

Issues

- **Lack of insight** into heuristics' effects.
- Does not scale for **modern platforms** with $\geq 10^6$ key cuttings.
- Can some subproblems be **polynomial**? Under which assumptions?
- The **extensibility** problem.

Motivation


CyberCalc solver
CyberCalc
EVALUATION COPY
developed by
Czech Technical University in Prague
Projekt: #336 / Tosovsky_02 / ...
Počítané klíče: , 5, 6, 7, 8, 9, 10 (celkem 15)
Počítané zámky: , 5, 6, 7, 8, 9, 10 (celkem 10)
Profilová mapa:
Detaily
Objednávka: žádná objednávka
Klíče pro testování:
Zámky k otestování:
Platforma FAB: 300 3000 jiná
Počet stavítek: 6
Volbu platformy není možné z databáze načíst. Platformu lze vybrat ručně, ale hrozí vznik nekonzistence s dalšími vlastnostmi projektu.
G (H) H1 (H) H2 (H) H3 (H) H4 (H) 1 (H) 2 3 4 (H) 5 6 7 8 9 10
1 2 3 4 5 6 7 8 9 10

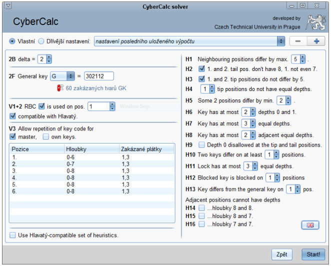
2. Complexity classes

Frameworks

4 constraint *frameworks* of increasing expressivity:

1. **Vanilla**: Number of positions p and cutting depths d.
2. (**Asymmetric**: Number of cutting depths varies between positions.)
3. **General**: Plus a set of *general constraints* (gecons).
4. (**Explicit**: List of valid key cuttings is algorithm's input.)

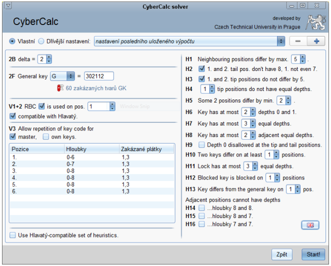
General framework



Why gecons?

- Most constraints “overlap”, **counting key cuttings** that satisfy all constraints is difficult.
- **Gecon** = a formally defined group of forbidden key cuttings.
- 15 rules can be each “compiled” to a set of gecons.

General framework



Counting key cuttings

- Derive number of key cuttings **invalidated** by 1 gecon.
- Define an **intersection** of 2 gecons (which is also a gecon).
- Design a **inclusion-exclusion** counting procedure.
- It can count $\approx 840 \cdot 10^6$ keys within 60 s.

General framework

$\mathcal{NP}$-completeness

- **Gecon** corresponds to 1 clause in the SAT problem.
- SAT is equivalent to solving a "1 key $\times$ 0 locks" lock-chart.
- Lock-chart solving in the general framework is $\mathcal{NP}$-complete.

Counting key cuttings

- Both reductions **preserve** the number of solutions.
- #SAT counts the number of valid key cuttings.
- Counting key cuttings is in $\#\mathcal{P}$.

$\mathcal{NP}$-complete classes

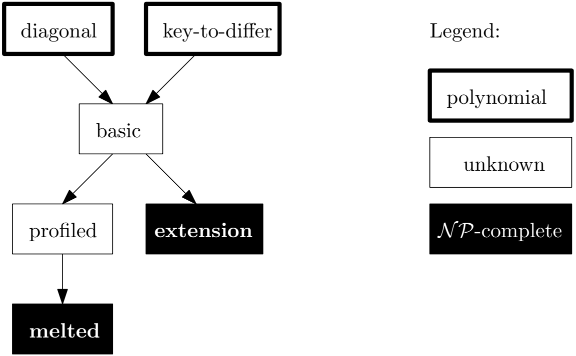
• = lock-chart problem instance grows in this parameter

	framework	lock-chart	positions	depths	gecons	keys	locks	
1.	general	all	•		•			
2.	vanilla	extension	•				•	Lawer (2004)
3.	vanilla	melted-profiles		•		•	•	

- Results 1. and 2. proved by a reduction from SAT.
- Result 3. by a reduction from k-coloring.

Vanilla framework

Complexity classes of lock-chart problem variants:



Vanilla framework

Diagonal lock-chart:

	k_G	k_1	k_2	k_3	k_4
l_1	■	■			
l_2	■		■		
l_3	■			■	
l_4	■				■

Contains exactly 1 *general* key which opens all locks and several *individual* keys which open exactly 1 lock.

Properties

- Finding cuttings for individual keys translates to the *maximum independent set* (MIS) problem.
- This is true for all frameworks.

Vanilla framework

Structure of the MIS graph

- The MIS graph can be partitioned into $S_0, \ldots, S_p$ **classes**, based on the distance to the cutting of the general key.
- Size of the q-th class was derived to be

$$|S_q| = \binom{p}{p-q} \cdot (d-1)^{p-q} \ ,$$

  maximized if $q = \hat{q}$, where

$$\hat{q} = \left\lfloor \frac{p+1}{d} \right\rfloor \ .$$

- Each S_q class is an independent set (true in every fw.).
- $S_{\hat{q}}$ is the maximum independent set (in vanilla fw.).
- The *rotating constant method* (RCM) was analysed.

Vanilla framework

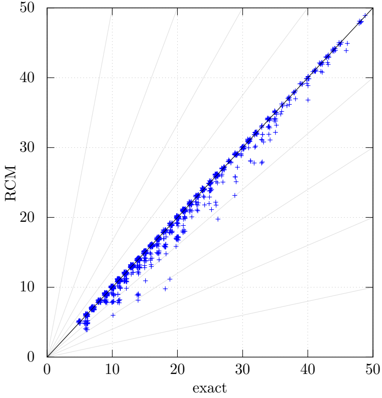
- Solution exists iff the number of keys is at most $|S_{\hat{q}}|$.
- Diagonal lock-charts in the vanilla framework are polynomial.

Other frameworks

- MIS is $\mathcal{NP}$-complete, but it can be approximated.
- Since each S_q is an independent set, RCM gives at least a **lower bound** on the MIS size.
- How tight is this bound?

Other frameworks

- Real-world constraints were randomly sampled.
- The MIS graph was constructed in memory.
- RCM was compared againts an exact algorithm (backtracking, exponential runtime).



Other frameworks

RCM gives a lower bound on the size of diagonal lock-charts:

1. **Vanilla**: Given by a closed formula.
2. **Asymmetric**: Calculated using dynamic programming (not discussed here, still in $\mathcal{P}$).
3. **General**: A modified inclusion-exclusion algorithm.

3. Practical algorithms

Use of cutting-counting

A typical real-world lc.:

```
**..*.....*........
*.*..*.....*....... <- Group 1
*...**.*....*......
***.....*....*.....
****...*......*....
**..*..........*...
*.*..*..........*.. <- Group 1
*...***.*........*.
**********........*
\____ ___/\___ ___/
     ˇ         ˇ
  MASTER  INDIVIDUAL
```

- Each *group* defined by a combination of master keys.
- RCM was generalized to count individual keys in each group.

All-different pruning

- *All-different* algorithm is inspired by *constraint satifaction problems*.
- If there are less cuttings for individual keys than the size of the group, **no solution can exist**.

All-different pruning

- The algorithm by Lawer (2004) (which optimizes a criterial function) was taken as a base-line.
- The all-different pruning reduces runtime
  - by 9% on a real-world dataset,
  - by 80% on a synthetic dataset (larger lock-charts).
- Keeps completeness and optimality.

SAT approach

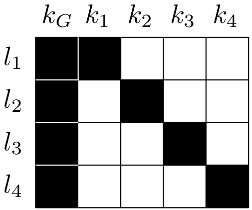
- Lock-chart problem was translated to SAT.
- The criterial function was approximated (no optimality guaranteed).
- The SAT instances were solved by MiniSAT Eén and Sörensson (2003).
- Only algorithm that solves **all lock-charts** in all datasets.
- Runtime is **2 orders of magnitude faster**.
- Optimal result achieved in **all but one** lock-chart.
- Suitable for small lock-charts (up to $\sim 100 \times 100$) due to many auxiliary variables.

4. Extensibility

General approach

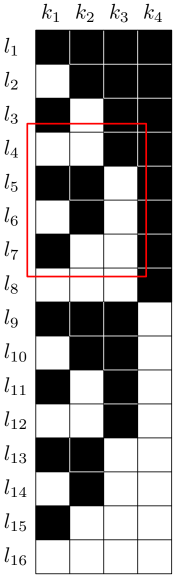
- **Extensibility:** dealing with unknown constraints, added after a solution is found.
- **Example:** Adding 1 floor to the building.
- Extensibility was vaguely defined and hard to quantify.
- Main idea: Given some assumptions, **extrapolate** the from-scratch lock-chart to the **largest solvable** lock-chart.
- Such lock-chart is called *extremal*.

Individual keys



- **Assumptions:**
  1. The lock-chart is diagonal.
  2. Only individual keys will be added in future.
- **Consequences:**
  1. Pick cuttings from $S_{\hat{q}}$.
  2. The extremal lock-chart can have at most $|S_{\hat{q}}|$ individual keys / locks.

Independent master keys



- **Assumption:** Lock-chart has at most $p \cdot (d-1) + 1$ keys.
- **Consequences:**
  1. Any lock can be added in future.
  2. There are $2^{p \cdot (d-1)+1}$ such locks.

Conclusions

- The computational complexity proof show that the $\mathcal{P} \times \mathcal{NP}$ boundary disects the lock-chart problem.
- Practical cutting-counting algorithm allowed by the "compilation to gecons".
- All-different pruning improves a state-of-the-art algorithm.
- SAT-based algorithm is the new baseline.
- Extensibility has been formalized.

Thank you for your attention.

Question #1

How many cylinders are there?

- Cutting is a p-dimensional vector with discrete values $\{1, \ldots, d\}$. There may be d^p of them.
- On every position, a cylinder may or may not have a cutting depth d. There are $p \cdot d$ such binary choices. Together $2^{p \cdot d}$ cylinders.

Question #2

Citation

15 “real” constraint can be polynomially translated to gecons.

Reviewer’s comment

For how many constraints is this true? Is there a common pattern among the remaining ones?

Otázka oponenta

Pro kolik podmínek to je pravda a co měly společné podmínky, pro které to neplatilo?

Question #3

All constraints that are translated to gecons, are translated to a polynomial number of gecons.

Example

"The first n positions may not be equal."
Generates $\mathcal{O}(d \cdot n^2)$ gecons:

$$\begin{array}{llll}
(1, 1, *, *, \ldots, *), & (2, 2, *, *, \ldots, *), & \ldots, & (d, d, *, *, \ldots, *) \\
(1, *, 1, *, \ldots, *), & (2, *, 2, *, \ldots, *), & \ldots, & (d, *, d, *, \ldots, *) \\
(*, 1, 1, *, \ldots, *), & (*, 2, 2, *, \ldots, *), & \ldots, & (*, d, d, *, \ldots, *) \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

Question #3

Some constraints are not translatable to gecons.

Example

"Some two cutting depths differ by at least n."

Solution in the solver

We use *existential constraints* (excons) Hořeňovský (2018). Used for constraints speaking about "existence" of a cutting depth.

Question #4

Reviewer's criticism

Theoretical results are straightforward, except for Theorem 52, whose proof is incomplete.

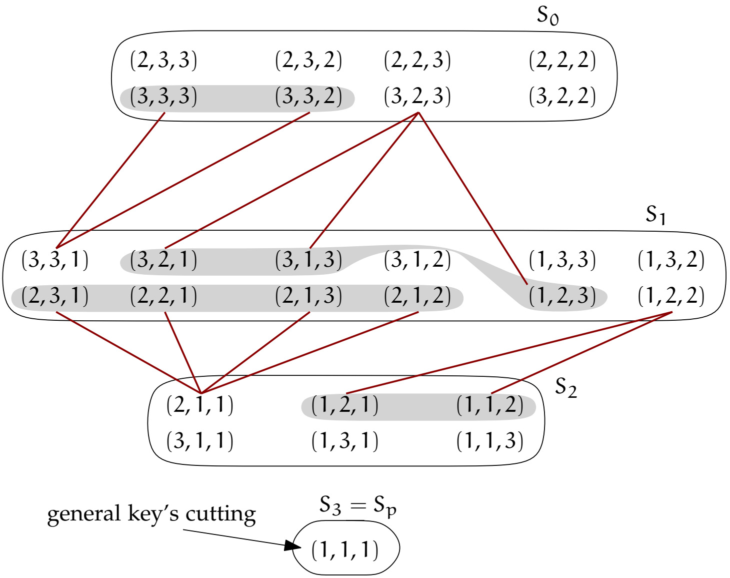
Výhrada oponenta

K teoretickým výsledkům mám výhrady: jsou povětšinou přímočaré s výjímkou věty 52, která má neúplný důkaz.

Question #4

- Proofs in the theoretical results are straightforward indeed.
- Most of them fit within a single A5 page (see page 37).
- The results are still novel.

Question #4



S_0
(2,3,3) (2,3,2) (2,2,3) (2,2,2)
(3,3,3) (3,3,2) (3,2,3) (3,2,2)
S_1
(3,3,1) (3,2,1) (3,1,3) (3,1,2) (1,3,3) (1,3,2)
(2,3,1) (2,2,1) (2,1,3) (2,1,2) (1,2,3) (1,2,2)
S_2
(2,1,1) (1,2,1) (1,1,2)
(3,1,1) (1,3,1) (1,1,3)
$S_3 = S_p$
general key's cutting
(1,1,1)

Question #4

- Let $0 \leq q < r \leq p$, $S'_q \subseteq S_q$, $S'_r \subseteq S_r$ and $S'_q \cup S'_r$ be an independent set.
- Proved that

$$|S'_q| + |S'_r| \leq \max\{|S_q|, |S_r|\} \ .$$

- Can $|S'_q| + |S'_r| + |S'_s| + \cdots \leq \max\{|S_q|, |S_r|, |S_s|, \ldots\}$?
- Proof is incomplete.
- There is no known counter-example.
- I am convinced that the theorem still holds.

Question #5

Reviewer's comment

The algorithms were tested in the “vanilla framework”, but the gecons alone cause NP-completeness. Are there resons to believe that the results would not change [if gecons were included]?

Otázka oponenta

Testování algoritmů proběhlo ve „vanilla framework“. Samotná omezení ve formě geconů zapřičiňují NP-úplnost. Existují důvody pro to, že by se výsledky podstatně nezměnily při porovnávání vstupních dat z praxe?

Question #5

- Let's assume that the Lawers' algorithm can (somehow) accomodate gecons.
- **Will SAT be still better?**

1. **Runtime**:
   - Finding a valid key cutting given real-world constraints is easy ($\leq 1\,\mathrm{ms}$).
   - Gecons are applied to each key separately (exploited by DPLL).
   - Finding valid key cuttings for ~ 100 keys is $\leq 1\,\mathrm{s}$.
2. **Criterial function:**
   - Some modifications to the strategy are needed.
   - It is tricky to pick a cutting depths to forbid.
   - This is done in the commercial product.
   - Lock-charts of size ~ 25 keys can still be optimized within $3\,\mathrm{s}$.
3. Also, the 2 orders-of-magnitude margin is very big.

Question #0

Reviewer's comment

Based on the presented research, has there been a progamme written that could be practically used to design master keyed systems?

Otázka oponenta

Byl na základě představeného výzkumu napsán program, který se v praxi použil na návrh zámků a klíčů, případně na nějaké dílčí úlohy?

Question #0

- Yes.
- The translation to SAT is already in production.
- A fully fledged all-different pruning is in development.

Eén, N. and Sörensson, N. 2003. An extensible sat-solver. In *Theory and applications of satisfiability testing*, pages 502–518. Springer.

Hořeňovský, M. 2018. Performance analysis of a master-key system solver. Master's thesis, Czech Technical University in Prague.

Lawer, A. 2004. Calculation of lock systems. Master's thesis, Royal Institute of Technology.