Extension of Dasgupta's Technique for Higher Degree Approximation

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Abstract

In the present paper, rational wedge functions for degree two approximation have been computed over a pentagonal discretization of the domain, by using an analytic approach which is an extension of Dasgupta's approach for linear approximation. This technique allows to avoid the computation of the exterior intersection points of the elements, which was a key component of the technique initiated by Wachspress. The necessary condition for the existence of the denominator function was established by Wachspress whereas our assertion, induced by the technique of Dasgupta, assures the sufficiency of the existence. Considering the adjoint (denominator) functions for linear approximation obtained by Dasgupta, invariance of the adjoint for degree two approximation is established. In other words, the method proposed by Dasgupta for the construction of Wachspress coordinates for linear approximation is extended to obtain the coordinates for quadratic approximation. The assertions have been supported by considering some illustrative examples.
Keywords: Adjoint; invariance; pentagonal discretization; wedge functions.

1 Introduction

Augustus Ferdinand Möbius [1] is attributed for initializing the concept of Barycentric coordinates. Initially, it was restricted to a triangle in two dimensional space. Generalization of these coordinates was first introduced by Wachspress [2] where the concept was extended from triangles to convex polygons, polycons [3] and further to 3-D elements [4–6]. Meanwhile, there have been efforts to present simpler and general form of barycentric coordinates [7–12] which may be computed and applied easily.

It has become a trend as well as a need to machine the calculation of geometric shapes and bodies using computers. Expansion of computer capacity has increased computational sophistication leading to methods with more precision and less time consumption and having newer fields of applications such as generalized barycentric coordinates on irregular polygons [13], interpolants within convex polygons [14], integration within polygonal finite elements [15], interpolations for temperature distributions [16], in the field of computer graphics, computational mechanics [17], etc. To obtain the desired shape using computers, the usual process is to interpolate the provided data using a certain class of functions. The Finite Element Method (FEM) algorithm in interpolation adds more perfection, as it allows to design by fragmenting the given function or data which is to be approximated with respect to the elements of the domain and then considering each fragment independently.

Wachspress' coordinates (cf. [5]) in addition, provides inter-element continuity, thus emerges as boon to shape formation. In this method (see [6]), the domain is discretized using convex polygons called elements, corresponding to each element, a basis of rational wedge functions is defined for linear approximation.

In the Wachspress' method, the computation of rational wedge functions relied on the geometry of the element under consideration, especially the denominator of the wedge functions (adjoint) was the curve passing through the exterior intersection points (EIPs) of the polygonal element. Thus, for each element of the domain, EIPs are to be calculated to obtain the adjoint, increasing the number of steps of the computation.

Dasgupta [14] simplified the task by proposing an analytic method for the calculation of the adjoint to the wedge functions. The rational wedge functions introduced by Wachspress [6] for the discretization with convex polygons of order m in a degree k approximation are of the form

$$\frac{P^{m+k-3}(x,y)}{P^{m-3}(x,y)} \tag{1}$$

where $P^n(x,y)$ denotes a bivariate polynomial of degree n. It has been identified in [14] that the adjoint is nothing but the sum of numerators of the wedge functions, having coefficients of terms of higher degree (higher than $m-3$, cf. Equation 1) equated to zero.

Dasgupta [14], considered the rational wedge functions for degree one approximation over a pentagonal discretization of the domain. In this paper, a formulation to compute wedge functions for degree two approximation has been proposed. It has been concluded that the adjoint function so computed is invariant, i.e., adjoints of quadratic wedge functions are the same as those of linear wedge functions. Also, they have been compared with the Wachspress' wedge functions and observed to be the same.

It may be noted that wedge functions for degree two approximation provide better approximant than the wedge functions for degree one approximation as the number of nodal points is increased and so is the precision.

In this paper, the work of Dasgupta [14] has been extended as follows:

- Dasgupta has imposed a constraint on the element that no side of the element should pass through the origin, whereas this paper covers the general case.
- The existence and uniqueness problem of the adjoint function has been studied in this paper which yields certain geometric conditions on the element for the existence of a unique adjoint function.

An algorithm based on a Mathematica program has been included in this paper, which identifies the geometric constraints of a particular element and also computes the approximation to the provided data. It is quite significant to note that the denominator involved in the wedge construction due to Wachspress was the curve passing through the exterior intersection points (EIP) of the convex polygon of the mesh, whereas Dasgupta's approach computes the denominator (adjoint) analytically without using the geometry of the element and later assures that the adjoint essentially passes through the EIP, thus establishing sufficiency of the condition for the existence of the denominator function, making it a well-defined one.

2 Prerequisites

In this section, some preliminaries which are needed for construction and analysis are recalled.

Consider a closed and convex polygon $P_m = (1, 2, \ldots, m)$, $m \geq 4$, with m vertices as nodes in $\mathbb{R}^2$. Let $i-1$ and i be the consecutive nodes of P_m and $i \in \mathbb{Z}_m$, where $\mathbb{Z}_m$ is the set of integers modulo m.

Definition 2.1. *Discretization* [18] of the domain $\Omega \subseteq \mathbb{R}^2$ using convex polygons is the process of subdividing Ω into non overlapping polygons P_m in such a way that:

- The union of all polygons in the discretization is equal to the domain Ω.
- The intersection of interiors of any two elements is an empty set.
- The boundaries of any two elements intersect only at a common edge or at a common node.
- The domain is simply connected.

Definition 2.2 ([6], see also [5])**.** Sides of the polygon containing the node i, are called *adjacent* to i and remaining sides are called *opposite* to the node i.

Definition 2.3 ([6], see also [5])**.** Let s_i be the straight line passing through the nodes $i-1$ and i where Cartesian coordinates of $i-1$ and i are (x_{i-1}, y_{i-1}) and (x_i, y_i) respectively. Then, the *linear form* of the line s_i is denoted by

$$l_i \cong l_i(x,y) = (i-1,i) \cong (x-x_i)(y_{i-1}-y_i) - (y-y_i)(x_{i-1}-x_i). \tag{2}$$

Definition 2.4 ([6], see also [5])**.** The *wedge function* corresponding to the i^{th} node of P_m is a regular function $N_i : P_m \to \mathbb{R}$ of the form

$$N_i(x,y) = K_i \frac{P^{m+k-3}(x,y)}{P^{m-3}(x,y)}, \quad (i \in \mathbb{Z}_m) \tag{3}$$

where $P^n(x,y)$ is a bivariate polynomial of degree n.

We now enumerate the properties of wedge functions described in [6].

Properties of wedge functions for degree one approximation [6] In order to obtain a linear approximation corresponding to an element, the class of wedge functions described in [6] satisfies the following properties:

1. There is a node at each vertex of the polygon. For each node there is an associated wedge within each polygon containing the node.
2. Wedge $N_i(x,y)$ associated with node i is normalized to unity at i $(i \in \mathbb{Z}_m)$.
3. Wedge $N_i(x,y)$ is linear on sides adjacent to i $(i \in \mathbb{Z}_m)$.
4. Wedge $N_i(x,y)$ vanishes on sides opposite to node i.
5. The wedges associated with P_m form a basis for degree one approximation over it. For the polygon P_m, there must be at least m nodes. For these to suffice, we must have (cf. G.15 of [19]):

$$\sum_{i=1}^{m} N_i(x,y) = 1, \tag{4}$$

$$\sum_{i=1}^{m} x_i N_i(x,y) = x, \tag{5}$$

$$\sum_{i=1}^{m} y_i N_i(x,y) = y. \tag{6}$$

6. Each wedge function and all its derivatives are continuous within the polygon for which the wedge is a basis function.

Definition 2.5 ([6])**.** If the domain under consideration is discretized using pentagons then each pentagon of the domain is termed as a *pentagonal element* (or simply an element).

Definition 2.6 ([6])**.** The node on a side of the polygon, such that it does not coincide with the vertices is said to be a *side node*.

3 Construction of wedge functions

Referring to Equation 3, the wedge functions for degree one and two approximation over a pentagonal element are computed in this section.

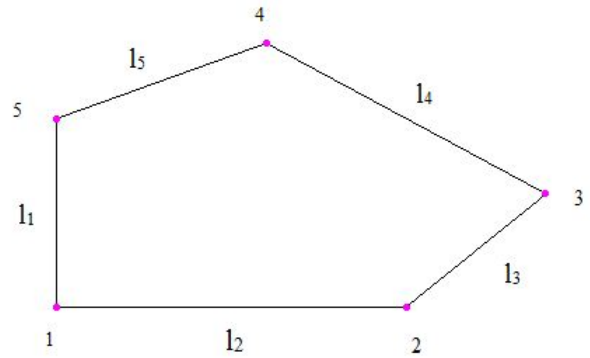

Figure 1: Pentagon

3.1 Degree one approximation

3.1.1 Construction

Applying the technique of Dasgupta (cf. [14]) and referring the generalized form of wedge functions (cf. Equation 3), we now determine the adjoint to the linear approximation over P_5.

The wedge functions N_i^1's corresponding to the nodes of P_5 (cf. Figure 1) have been defined as follows:

$$\begin{aligned} &N_1^1 = K_1 \frac{l_3 l_4 l_5}{D(x,y)}, \quad N_2^1 = K_2 \frac{l_1 l_4 l_5}{D(x,y)}, \quad N_3^1 = K_3 \frac{l_1 l_2 l_5}{D(x,y)}, \\ &N_4^1 = K_4 \frac{l_3 l_2 l_1}{D(x,y)}, \quad N_5^1 = K_5 \frac{l_3 l_4 l_2}{D(x,y)}. \end{aligned} \tag{7}$$

Where K_i's are appropriate normalizing constants.

In view of [14] and value of $\{N_i^1\}_{i=1}^5$ the denominator $D(x,y)$ can be obtained by following steps 1, 2, and 3:

Step 1 Normalize K_1 to 1 (cf. Equation 7).

Step 2 Sum the numerators of all the N_i^1's.

Step 3 In the above sum, equate the coefficients of terms of higher degree (higher than two) to zero and obtain a system of linear equations

$$\mathbf{AK} = \mathbf{M}, \tag{8}$$

where $\mathbf{A}$ is a 4×4 coefficient matrix, $\mathbf{K} = \begin{bmatrix} K_2 & K_3 & K_4 & K_5 \end{bmatrix}^{\mathbf{T}}$ and $\mathbf{M}$ is a 4×1 matrix of some real numbers. On solving this system of linear equations the denominator has been obtained (cf. [14]).

3.1.2 Existence

Our aim in this section is to obtain a solution of the system of linear equations (Equation 8). A unique solution to the system will exist if the determinant of $\mathbf{A}$ is non-zero. Consider Ω to be the pentagonal discretization of the domain $\mathbb{R}^2$ (cf. Figure 2).

The following theorem establishes the conditions under which a unique solution to the system (Equation 8) exists.

Theorem 3.1. Let $\Omega \subseteq \mathbb{R}^2$ be the pentagonal discretization of the domain and P_5^1 is an arbitrary element of Ω. Then the adjoint of the wedge functions for degree one approximation corresponding to the element P_5^1 exists and is unique if the following conditions hold:

(A) No two sides of P_5^1 are parallel.

(B) No three vertices of P_5^1 are co-linear.

Proof. Consider the pentagon $P_5^1 = (a, b, c, d, e)$ having Cartesian coordinates of the vertices as (a_1, a_2), (b_1, b_2), (c_1, c_2), (d_1, d_2), and (e_1, e_2) respectively. Without any loss of generality, the pentagon P_5^1 can be transformed into the pentagon $P_5 = (1, 2, 3, 4, 5)$, having Cartesian coordinates $(0, 0)$, $(x_2, 0)$, (x_3, y_3), (x_4, y_4), and (x_5, y_5) (cf. Figure 3).

Using the notation of linear forms described in Section 2, we now consider the following:

$$\begin{aligned}
l_1 &= (5,1) \cong x_5 y - x y_5, \\
l_2 &= (1,2) \cong y, \\
l_3 &= (2,3) \cong -(-x_2 + x_3)y + x y_3 - x_2 y_3 \\
l_4 &= (3,4) \cong -(x_3 + x_4)y + x_4 y_3 - x(y_3 - y_4) - x_3 y_4, \\
l_5 &= (4,5) \cong -(-x_4 + x_5)y + x_5 y_4 - x(y_4 - y_5) - x_4 y_5.
\end{aligned} \tag{9}$$

In view of wedge properties, the numerator of the wedge function corresponding to the i^{th} node Num_i, for (say) $i = 1, \ldots, 5$, will be:

$$\begin{aligned}
&\text{Num}_1 = K_1 l_3 l_4 l_5, \quad \text{Num}_2 = K_2 l_4 l_1 l_5, \quad \text{Num}_3 = K_3 l_1 l_2 l_5, \\
&\text{Num}_4 = K_4 l_1 l_2 l_3, \quad \text{Num}_5 = K_5 l_2 l_3 l_4.
\end{aligned} \tag{10}$$

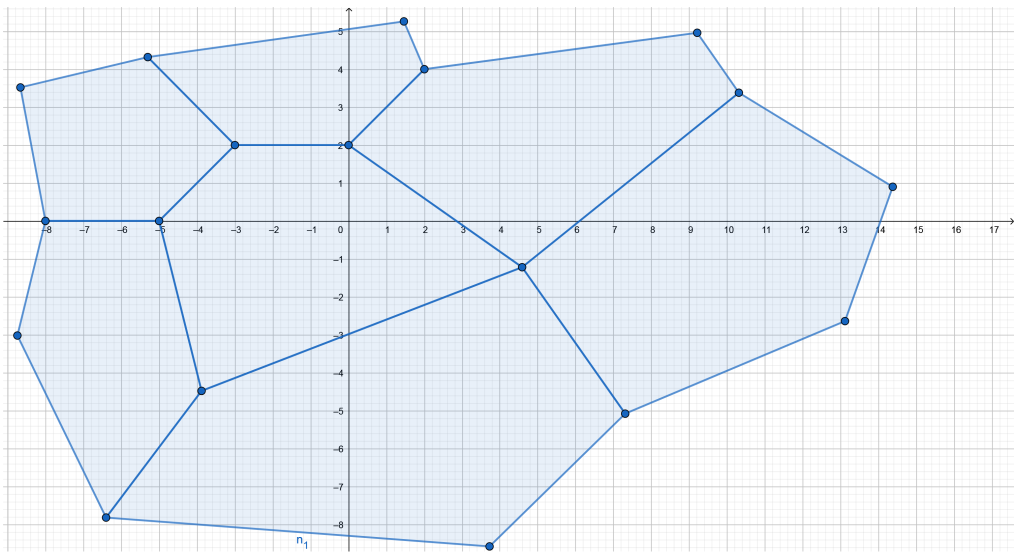

Figure 2: Pentagonal discretization

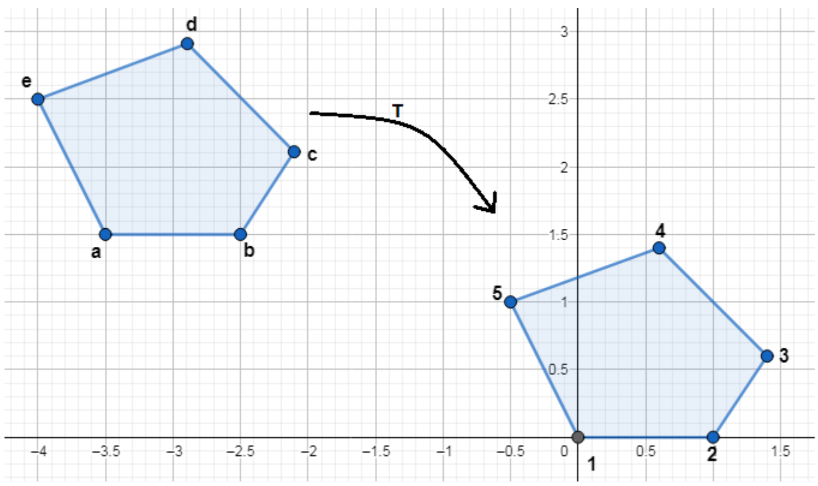

Figure 3: Transformation of the element

Using relation Equation 3, and applying the technique prescribed in [14] we equate the coefficients of terms $x^i y^j$ $(i+j=3,\ i,j=0,1,2,3)$ to zero. Thus, normalizing the constant K_1 to 1, a system having four equations in four unknowns viz $\{K_i\}_{i=2}^{5}$, of the form $\mathbf{AK}=\mathbf{M}$ (cf. Equation 8) is obtained. It may be verified easily that a unique solution to this system will exist only if the following conditions hold:

- $x_3 \neq x_2$,
- $y_3 \neq y_4$,
- $x_4 \neq \dfrac{-x_2y_4 + x_3y_4 + x_2y_5 - x_3y_5}{y_3}$,
- $x_3 \neq x_4$.

The above conditions in turn imply assertions (A) and (B). □

3.2 Degree two approximation

In order to define the wedge functions for degree two approximation the following construction will be required:

3.2.1 Construction

Define side nodes $i+5$ $(i=1,2,\ldots,5)$ on the line joining nodes $(i,i+1)$, $i \in \mathbb{Z}_5$. The linear form of the straight line joining nodes i and $i+1$ (by convention node11 = node6) is denoted by l_{i+1}, $i=6,7,8,9$, and 10 (cf. Figure 4b) which satisfies the equation $l_{i+1}=0$ on this line.

Inheriting the process of computation of wedge functions for linear approximation (see [19]), the following wedge properties for degree two approximation over P_m have been considered.

Properties of wedge functions for degree two approximation. In order to obtain a quadratic approximation corresponding to an element P_m, the class of wedge functions satisfy the following properties:

1. There is a node at each vertex of the polygon and the side nodes on the sides of the polygon. For each node (vertex node and side node) there is an associated wedge within each polygon containing the node.
2. Wedge $N_i(x,y)$ associated with node i is normalized to unity at node i.
3. Wedge $N_i(x,y)$ is quadratic on sides adjacent to i.
4. Wedge $N_i(x,y)$ vanishes on sides opposite to node i and at all nodes j for which $j \neq i$.

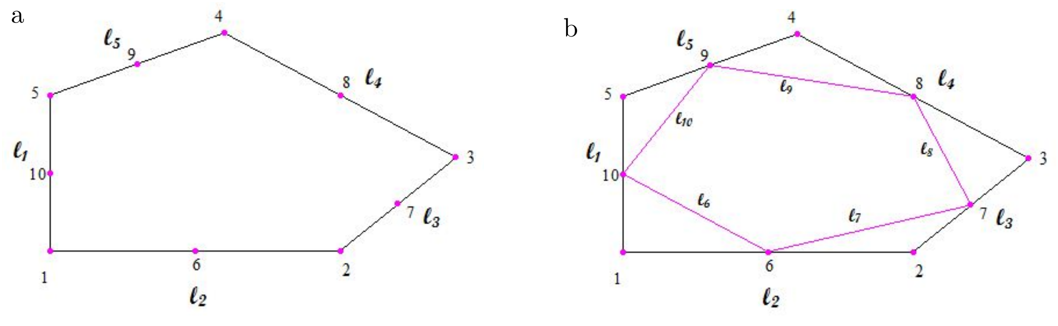

Figure 4: a Side Nodes. b Linear forms of straight lines through side nodes

5. The wedges associated with P_m form a basis for approximation of degree two over it. There must be at least m nodes and m side nodes in the polygon P_m. For these to suffice, we must have:

$$\begin{aligned} &\sum_{i=1}^{2m} N_i(x,y) = 1, && \sum_{i=1}^{2m} x_i N_i(x,y) = x, \\ &\sum_{i=1}^{2m} y_i N_i(x,y) = y, && \sum_{i=1}^{2m} x_i^2 N_i(x,y) = x^2, \\ &\sum_{i=1}^{2m} y_i^2 N_i(x,y) = y^2, && \sum_{i=1}^{2m} x_i y_i N_i(x,y) = xy, \end{aligned} \tag{11}$$

where (x_i, y_i) is a Cartesian coordinate of the vertex i of P_m.

6. Each wedge function and all its derivatives are continuous within the polygon for which the wedge is a basis function.

In view of the aforesaid properties the wedge function corresponding to the i^{th} node ($i \in \mathbb{Z}$) of P_5 will be,

$$N_i^2(x,y) = \begin{cases} \dfrac{K_i L_i l_{5+i}}{Q(x,y)}, & \text{for } i \leq 5, \\ \\ \dfrac{K_i L_{i-5} l_{i-5}}{Q(x,y)}, & \text{for } 5 < i \leq 10, \end{cases} \tag{12}$$

where

$$L_i = \prod_{j=1, j\neq i, j\neq i+1}^{5} l_j \qquad (\text{by convention } l_6 = l_1). \tag{13}$$

3.2.2 Computation of adjoint

Following the technique of Dasgupta [14], the unknowns $\{K_i\}_{i=2}^{10}$ are to be computed, to determine the adjoint function $Q(x,y)$ for degree two approximation.

In view of Equation 12 and Equation 3, it is evident that to achieve degree two approximation, $Q(x,y)$ must be a bivariate polynomial of degree two.

Hence in view of the above assertion, the technique of Dasgupta (cf. [14]) has been applied in the following steps:

Step 1 By property 5,

$$\sum_{i=1}^{10} N_i^2(x,y) = 1, \tag{14}$$

where $N_i^2(x,y)$ is defined in Equation 12.

Step 2 With reference to Equation 14, the sum of numerators of the rational forms $N_i^2(x,y)$ is held on the left and is equated with $Q(x,y)$.

Step 3 In relation Equation 3, it is quite clear that the coefficients of terms of degree higher than 2 must be equated to zero and, without loss of generality, the constant $K_1 (\neq 0)$ may be normalized to 1.

Thus, nine linear equations (with nine unknowns $\{K_i\}_{i=2}^{10}$) are left, which can be expressed in matrix form as

$$\mathbf{AK} = \mathbf{M}, \tag{15}$$

where $\mathbf{A}$ is a 9×9 square matrix, $\mathbf{K} = \begin{bmatrix} K_2 & K_3 & K_4 & K_5 & K_6 & K_7 & K_8 & K_9 & K_{10} \end{bmatrix}^{\mathbf{T}}$ and $\mathbf{M}$ is a 9×1 matrix of some real numbers.

The aim is to compute the adjoint $Q(x, y)$, for which using Equation 15 the values of K_i's are needed to be computed.

On solving these equations, if the solution exists, the values of K_i's are obtained and thus the exact value of $Q(x, y)$.

4 Invariance of Adjoint

In this section, the adjoints obtained by applying the technique of Dasgupta (cf. [14]) in Section 3, are compared with the denominator function introduced by Wachspress (see [6]).

The concept of computing adjoints by using the 'exterior intersection point' (EIP) was initiated by Wachspress [5], where an EIP is the point of intersection of the extended opposite sides of the pentagon which do not intersect within it (cf. Figure 5). In Figure 5, E_1, E_2, E_3, E_4, and E_5 are the EIPs.

According to Wachspress [5] the denominator function for the wedge construction of degree one over a pentagon, is a unique curve of degree two, passing through the five EIPs.

It was observed that the adjoint computed by the method of Dasgupta for Linear approximation and by Wachspress' EIP method are identical. In fact, even on increasing the degree of approximation the adjoint remains unchanged.

Theorem 4.1. For a pentagonal discretization Ω of the domain, the adjoint function computed by Wachspress' method is the same as that obtained by Dasgupta for degree one approximation. Moreover, the adjoint remains unchanged for degree two approximation, computed by inheriting the technique of Dasgupta.

Proof. According to Wachspress, the adjoint for linear approximation over a pentagon is a unique curve represented by a degree two polynomial which passes through the EIPs.

Let $D^*(x, y)$ be the denominator function of the wedge obtained by Wachspress method, $D(x, y)$ be the adjoint function obtained by Dasgupta's technique for degree one approximation, and $Q(x, y)$ the adjoint function obtained for degree two approximation by the technique of Dasgupta.

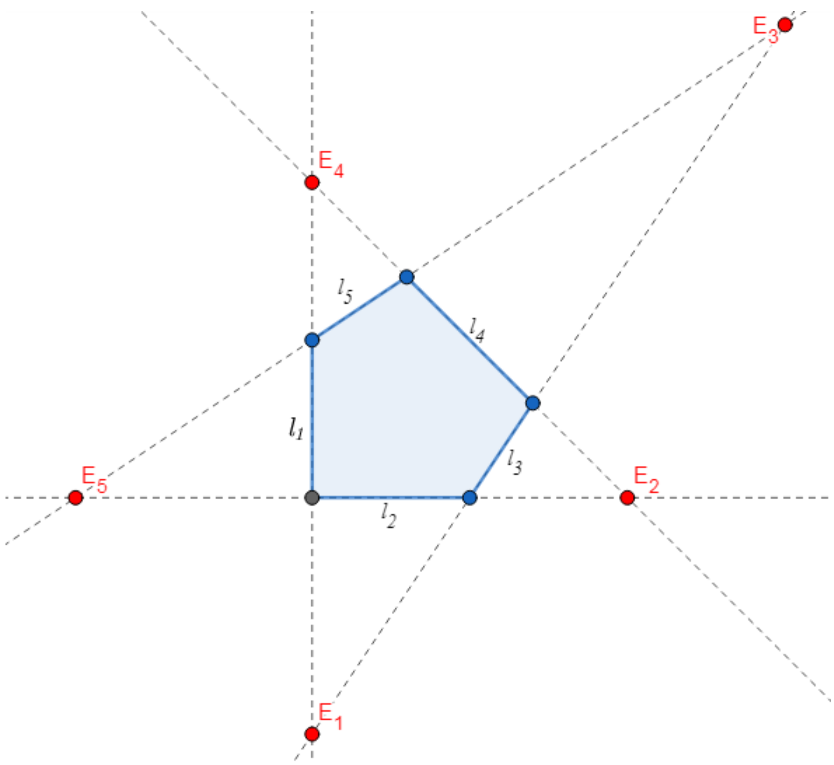

Figure 5: Pentagon representing exterior intersection points

It can be seen easily that the EIP E_i, is the intersection of the lines $l_i = 0$ and $l_{i+2} = 0$, $i = 1, \ldots, 5$ (here $l_6 = l_1$ and $l_7 = l_2$). Consider,

$$D(x,y) = \sum_{i=1}^{5} \mathrm{Num}_i^1(x,y) = \sum_{i=1}^{5} K_i l_{i+2} l_{i+3} l_{i+4} \text{ (cf. Section 3)} \tag{16}$$

and,

$$Q(x,y) = \sum_{i=1}^{10} \mathrm{Num}_i^2(x,y) = \sum_{i=1}^{5} K_i L_i l_{n+i} + \sum_{i=6}^{10} K_i L_{i-5} l_{i-5} \quad \text{(cf. Section 4)} \tag{17}$$

and either l_i or l_{i+2} is present in each component of the sum in the right side of Equation 16 and Equation 17.

Hence,

$$D(x,y)|_{E_i} = Q(x,y)|_{E_i} = 0 \quad (i = 1, \ldots, 5). \tag{18}$$

It implies that

$$D(x,y) \cong Q(x,y) \cong D^*(x,y). \tag{19}$$

□

5 Numerical Examples

In order to support the assertions, made in this paper an illustrative example is discussed here. The example is organized as follows:

1. Linear approximation over the given pentagon is computed for the function $\sin(xy)$.
2. Considering the same function $\sin(xy)$, degree two approximation has been computed on the pentagon as described in item 1.

Example 5.1. Consider the pentagon $P_5 = (1, 2, 3, 4, 5)$ with Cartesian coordinates of the vertices as (cf. Figure 6) $1 = (0, 0)$, $2 = (1, 0)$, $3 = \left(\frac{7}{5}, \frac{3}{5}\right)$, $4 = \left(\frac{3}{5}, \frac{7}{5}\right)$ and $5 = (0, 1)$.

Degree one approximation In reference to Equation 8, it is noticed that $|A| \neq 0$, hence the adjoint $D(x,y)$ exists uniquely, and

$$D(x,y) = \frac{1}{125}\left(24x^2 - 32xy + 24y^2 - 12x - 12y - 72\right). \tag{20}$$

For a non linear function, say $f(x,y) = \sin(xy)$, the approximant is

$$\phi(x,y) = \frac{5xy(-6 + x + y)\sin(21/25)}{6x^2 - 8xy + 6y^2 - 3x - 3y - 18}. \tag{21}$$

The function $f(x,y)$ over the element P_5 and its approximation is displayed in Figure 7.

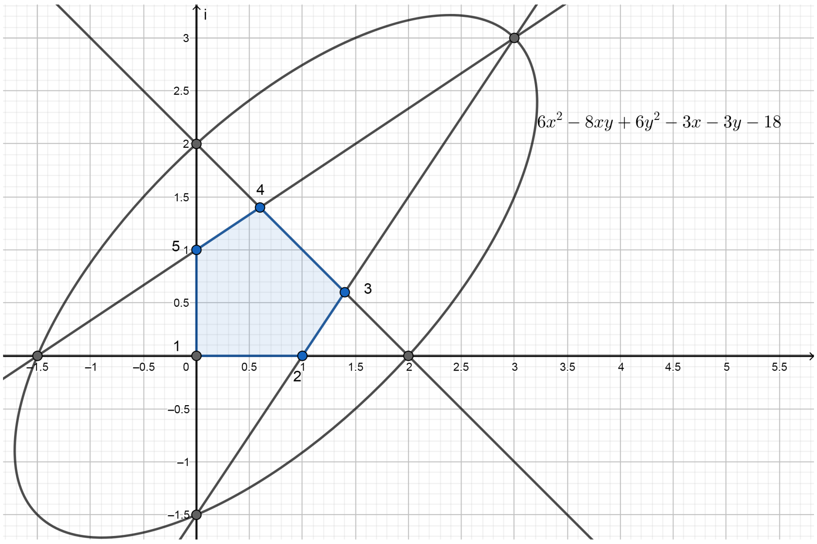

Figure 6: Pentagon with EIPs and curve

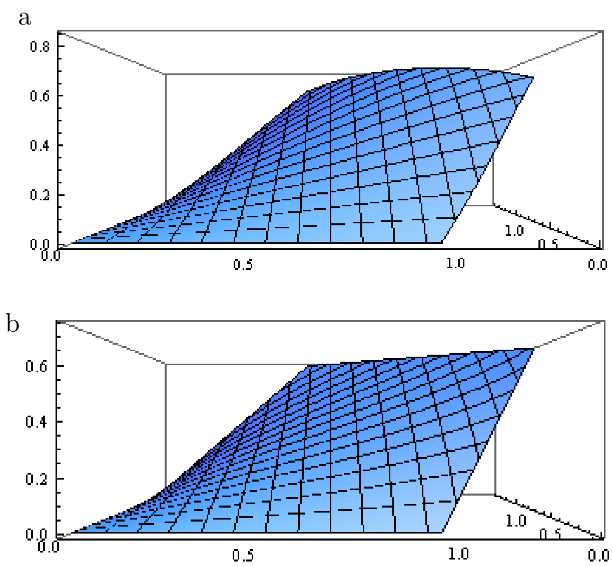

Figure 7: Comparison of the original curve and its linear approximation. a Function $\sin(xy)$. b Degree one approximation of $\sin(xy)$

Degree two approximation In order to define wedge functions for degree two approximations, the intermediate points with the following Cartesian coordinates on sides of the pentagon P_5 are introduced: $6 = \left(\frac{1}{2}, 0\right)$, $7 = \left(\frac{12}{10}, \frac{3}{10}\right)$, $8 = (1, 1)$, $9 = \left(\frac{3}{10}, \frac{12}{10}\right)$ and $10 = \left(0, \frac{1}{2}\right)$ (cf. Figure 8).

In view of Equation 15, the adjoint exists uniquely, and is given by

$$Q(x, y) = \frac{1}{125}(6x^2 - 8xy + 6y^2 - 3x - 3y - 9). \tag{22}$$

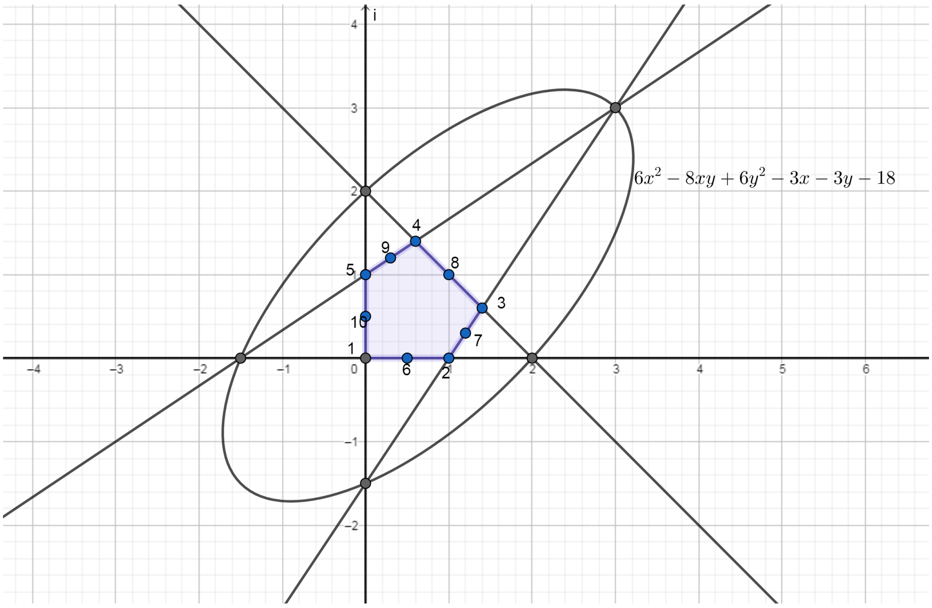

Figure 8: Pentagon with intermediate points, EIPs and curve

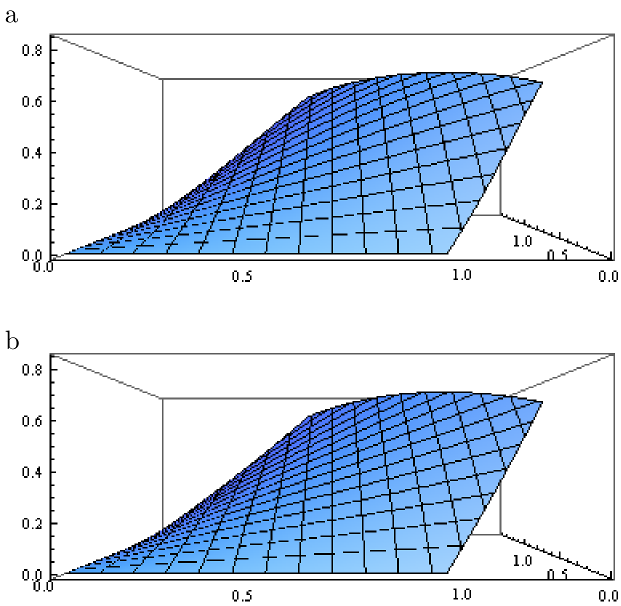

Figure 9: Comparison of the original curve its degree two approximation. a Function $\sin(xy)$. b Degree two approximation of $\sin(xy)$

For the same function $f(x, y) = \sin(xy)$ the approximant, say $\psi(xy)$, is obtained as

$$\psi(x, y) = \frac{5xy}{6x^2 - 8xy + 6y^2 - 3x - 3y - 18}\Bigg[-4(-6 + x + y)(-2 + x + y)\sin\left(\frac{9}{25}\right) \\ - (-27 + 4x^2 + x(18 - 17y) + 2y(9 + 2y))\sin\left(\frac{21}{25}\right) \\ + (3 + 2x - 3y)(-3 + 3x - 2y)\sin(1)\Bigg]. \quad (23)$$

The function $f(x, y)$ over the element P_5 and its approximation is displayed in Figure 9. From Equation 20 and Equation 22 it can be seen that the adjoint is invariant.

5.1 Comparative Study

In order to compare the linear and the quadratic approximations over the considered element we have computed the error, which is defined as follows:

$$\|e\| = \int_{\omega} |f(x,y) - A(x,y)|\, dxdy, \quad \text{where } A \text{ is the approximate value of } f. \tag{24}$$

- Error for linear approximation $= 0.036\,629\,9$.
- Error for quadratic approximation $= 0.012\,876$.

It is clear from the above computation that the quadratic approximation is closer to the function in comparison to the linear approximation.

Remark 5.2. Comparing Figure 7 and Figure 9, it may be observed that the degree two approximation is quite close to the given function $f(x,y) = \sin(xy)$.

6 Algorithm to compute the degree two approximation

An algorithm based on a program written in Mathematica is placed in this part of the paper, which illustrates the method for computing the approximant.

6.1 Description of the algorithm

Step 2 In this step we declare the variables, to assign the coordinates of the polygon as well as the edge nodes.

Step 3 Computes the linear forms, of all the edges.

Step 4 Multiply the linear forms and get a polynomial of degree four in the numerator, namely, $\text{num}[(xy),[i]]$, $i = 1, 2, \ldots, 10$, corresponding to each node.

Step 5 Define denominator as the sum of all the numerators $\text{num}[(xy),[i]]$.

Step 6 In the denominator polynomial, equate the coefficients of the terms of degree higher than two to zero.

Step 7 Solve the system of 9×9 equations obtained in Step 6, to get k_i's.

Step 8 Finally get the denominator polynomial and the wedge functions.

Step 9 Compute the error in approximation, i.e., integrate the modulus of difference in approximate and approximant over the considered element.

7 Conclusion

The method developed by Dasgupta gives a new approach to solving problems related to rational FEM. Being a technique different from the well established Wachspress' method, it gives a new perspective to the rational finite element methods with a scope of rebuilding the theory with the emergence of some new concepts, applications, and mainly easy computation of the denominator function. The theorem stated in this paper identifies the constraints in the geometry of the element required to be taken care of, to assure the existence of the wedge functions.

In addition to the conditions for the existence of the wedge functions, a method to compute wedge functions for degree two approximation has been proposed, a theorem has been stated claiming invariance of the adjoint functions in moving from degree one to degree two approximation; also the adjoint function is compared with the adjoint of Wachspress wedge functions and found to be the same. Wedge functions for degree two approximation increase the precision in approximation.

8 Conflict of Interest

The authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent arrangements), or non (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript. The authors declare that there are no conflicts of interest.

References

[1] Möbius AF. Der barycentrische Calcul0: ein neues Hülfsmittel zur analytischen Behandlung der Geometrie. Barth. 1827.

[2] Wachspress EL. A rational basis for function approximation. *Conference on Applications of Numerical Analysis*. Springer: 223–252, 1971.

doi: https://doi.org/10.1093/imamat/8.1.57.

[3] Wachspress EL. A rational basis for function approximation: II. Curved sides. *IMA Journal of Applied Mathematics*. 11.1: 83–104, 1973.

doi: https://doi.org/10.1093/imamat/11.1.83.

[4] Warren J. Barycentric coordinates for convex polytopes. *Advances in Computational Mathematics*. 6.1: 97–108, 1996.

doi: https://doi.org/10.1007/bf02127699.

[5] Wachspress EL and EL W. A rational finite element basis. 1975.

doi: https://doi.org/10.1016/s0076-5392(09)x6007-8.

[6] Wachspress E. Rational bases and generalized barycentrics. Springer. 2016.

doi: http://dx.doi.org/10.1007/978-3-319-21614-0.

[7] Gordon WJ and Wixom JA. Pseudo-harmonic interpolation on convex domains. *SIAM Journal on Numerical Analysis*. 11.5: 909–933, 1974.

doi: https://doi.org/10.1137/0711072.

[8] Meyer M, Barr A, Lee H, and Desbrun M. Generalized barycentric coordinates on irregular polygons. *Journal of graphics tools*. 7.1: 13–22, 2002.

doi: https://doi.org/10.1201/b10628-10.

[9] Floater MS. Mean value coordinates. *Computer aided geometric design*. 20.1: 19–27, 2003.

doi: https://doi.org/10.1016/s0167-8396(03)00002-5.

[10] Malsch EA and Dasgupta G. Shape functions for polygonal domains with interior nodes. *International Journal for Numerical Methods in Engineering*. 61.8: 1153–1172, 2004.

doi: https://doi.org/10.1002/nme.1099.

[11] Sukumar N. Construction of polygonal interpolants: a maximum entropy approach. *International journal for numerical methods in engineering*. 61.12: 2159–2181, 2004.

doi: https://doi.org/10.1002/nme.1193.

[12] Floater MS, Kós G, and Reimers M. Mean value coordinates in 3D. *Computer Aided Geometric Design*. 22.7: 623–631, 2005.

doi: https://doi.org/10.1016/j.cagd.2005.06.004.

[13] Meyer M, Barr A, Lee H, and Desbrun M. Generalized barycentric coordinates on irregular polygons. *Journal of graphics tools*. 7.1: 13–22, 2002.

doi: https://doi.org/10.1201/b10628-10.

[14] Dasgupta G. Interpolants within convex polygons: Wachspress' shape functions. *Journal of Aerospace Engineering*. 16.1: 1–8, 2003.

doi: https://doi.org/10.1061/(ASCE)0893-1321(2003)16:1(1).

[15] Dasgupta G. Integration within polygonal finite elements. *Journal of Aerospace Engineering*. 16.1: 9–18, 2003.

doi: https://doi.org/10.1061/(asce)0893-1321(2003)16:1(9).

[16] Malsch EA and Dasgupta G. Interpolations for temperature distributions: A method for all non-concave polygons. *International Journal of Solids and Structures*. 41.8: 2165–2188, 2004.

doi: https://doi.org/10.1016/j.ijsolstr.2003.11.037.

[17] Hormann K and Sukumar N. Generalized barycentric coordinates in computer graphics and computational mechanics. CRC press. 2017.

doi: https://doi.org/10.1201/9781315153452.

[18] Hjelle Ø and Dæhlen M. Triangulations and applications. Springer Science & Business Media. 2006.

doi: http://dx.doi.org/10.1007/3-540-33261-8.

[19] Dasgupta G. Finite Element Concepts. Springer. 2018.

doi: https://doi.org/10.1007/978-1-4939-7423-8.