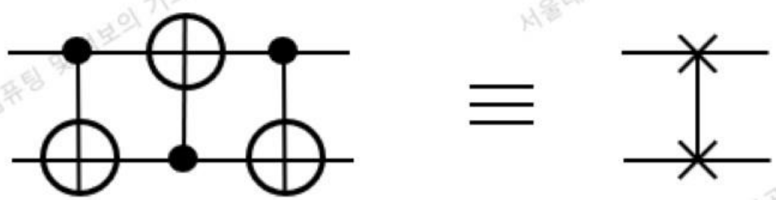
Summary of previous lecture

- Multiple qubits
  - Each qubit has its own Hilbert space
  - Combine multiple Hilbert spaces to create a new Hilbert space so that a single vector can represent the quantum state of multiple qubits
- Tensor product
  - Mathematical tool to represent multiple-qubit state
  - Combined new space: $\mathbb{V} \otimes \mathbb{W}$
  - Vector in the new space: $|v\rangle \otimes |w\rangle$ and their linear combinations
  - Linear operator: $A \otimes B$ and their linear combinations
    - $A \otimes B(|v\rangle \otimes |w\rangle) \equiv A|v\rangle \otimes B|w\rangle$
  - Matrix representation
  - Inner product: $\left(\sum_i a_i^* \langle v_i| \otimes \langle w_i|\right)\left(\sum_j b_j |v'_j\rangle \otimes |w'_j\rangle\right) \equiv \sum_{i,j} a_i^* b_j \langle v_i|v'_j\rangle\langle w_i|w'_j\rangle$

Quantum circuits

- Section 1.3.4
- Wire is not necessarily a physical wire
  - Can be a passage of time or path for the photon
- Example circuit
  - SWAP in C programming

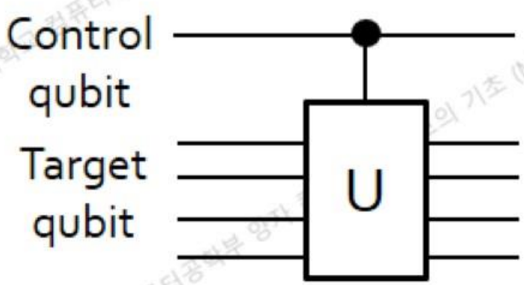
```
int a, b;
...
b = a + b;
a = b − a;
b = b − a;
```



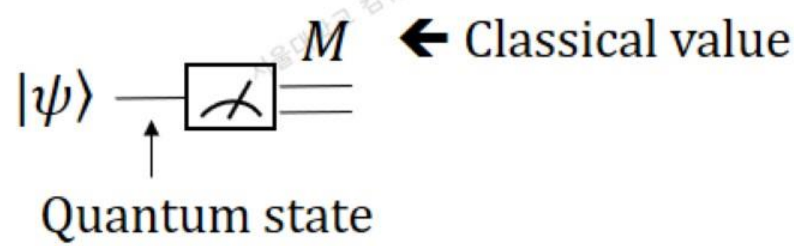
- $|a, b\rangle \rightarrow |a, a \oplus b\rangle$
  $\rightarrow |a \oplus (a \oplus b), a \oplus b\rangle = |b, a \oplus b\rangle$
  $\rightarrow |b, b \oplus (a \oplus b)\rangle = |b, a\rangle$

Quantum circuits

- Comparison with classical circuits
  - Feedback (or loop) is not allowed in quantum circuit ➔the circuit is acyclic
  - No FANOUT is allowed ➔ No copy of the information is allowed
  - No FANIN is allowed ➔ Not reversible
- Controlled-U gate



- Measurement symbol

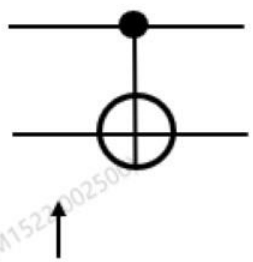


Copying qubit?

- We want to copy $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. Can CNOT make a copy?

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$|0\rangle$



$\rightarrow \alpha|0\rangle \otimes |0\rangle + \beta|1\rangle \otimes |1\rangle$

$$|\psi\rangle \otimes |0\rangle = \alpha|0\rangle \otimes |0\rangle + \beta|1\rangle \otimes |0\rangle$$

- When we say copy, we want

$$|\psi\rangle \otimes |\psi\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle)$$

- Can CNOT generate this result?

No cloning theorem

- Proof
  - Assume there exists a cloning machine.
  - This machine should follow the quantum mechanics rule and be able to copy any arbitrary input state $|\psi\rangle$.
  - $|\psi\rangle \otimes |Init\rangle \xrightarrow{U} |\psi\rangle \otimes |\psi\rangle \Leftrightarrow U(|\psi\rangle \otimes |Init\rangle) = |\psi\rangle \otimes |\psi\rangle$
    where $|Init\rangle$ is some initial state
  - $U(|0\rangle \otimes |Init\rangle) = |0\rangle \otimes |0\rangle$
  - $U(|1\rangle \otimes |Init\rangle) = |1\rangle \otimes |1\rangle$
  - Then $U\big((\alpha|0\rangle + \beta|1\rangle) \otimes |Init\rangle\big) = U(\alpha|0\rangle \otimes |Init\rangle) + U(\beta|1\rangle \otimes |Init\rangle)$
    $= \alpha(|0\rangle \otimes |0\rangle) + \beta(|1\rangle \otimes |1\rangle)$
  - However, we expect $(\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle)$.
  - Therefore, there cannot exist such kind of machine. ➔ Proof by contradiction!

Measurement in other bases

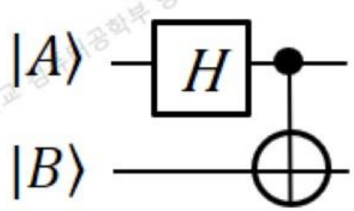
- Section 1.3.3 Measurement in bases other than the computational basis
- For example, how can we measure in $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ basis?
  - An arbitrary state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ can be re-expressed as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha\frac{|+\rangle+|-\rangle}{\sqrt{2}} + \beta\frac{|+\rangle-|-\rangle}{\sqrt{2}} = \frac{\alpha+\beta}{\sqrt{2}}|+\rangle + \frac{\alpha-\beta}{\sqrt{2}}|-\rangle$
  - Polarization

  ➔ rotate the polarizing beam splitter
  - Two-level atom

  ➔ apply the Hadamard gate before the measurement
- Why do we need to measure in other bases?
  - To find out the relative phase
  - Quantum teleportation
  - Etc…

Bell basis

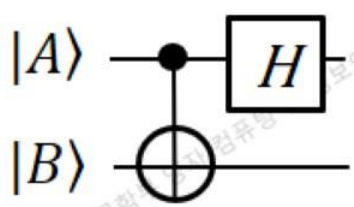
- Section 1.3.6
- Assume that there are two qubits A and B.
- $$\begin{cases} |\psi^+\rangle_{AB} = [|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B]/\sqrt{2} \\ |\psi^-\rangle_{AB} = [|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B]/\sqrt{2} \\ |\phi^+\rangle_{AB} = [|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B]/\sqrt{2} \\ |\phi^-\rangle_{AB} = [|0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B]/\sqrt{2} \end{cases}$$
- Are they orthonormal to each other?
- Are they complete basis?
- Called Bell basis, Bell state, EPR state, EPR pair, etc.
- How to create Bell state?
  - Entangling circuit



Input		Output
$\lvert A\rangle$	$\lvert B\rangle$	
$\lvert 0\rangle$	$\lvert 0\rangle$	$\lvert\phi^+\rangle = (\lvert 00\rangle + \lvert 11\rangle)/\sqrt{2}$
$\lvert 0\rangle$	$\lvert 1\rangle$	$\lvert\psi^+\rangle = (\lvert 01\rangle + \lvert 10\rangle)/\sqrt{2}$
$\lvert 1\rangle$	$\lvert 0\rangle$	$\lvert\phi^-\rangle = (\lvert 00\rangle - \lvert 11\rangle)/\sqrt{2}$
$\lvert 1\rangle$	$\lvert 1\rangle$	$\lvert\psi^-\rangle = (\lvert 01\rangle - \lvert 10\rangle)/\sqrt{2}$

Bell basis

- How to measure in Bell basis?
- Recall how we can measure in $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ for two-level atom.
- Use un-entangling circuit



Input	Output	
	$\|A\rangle$	$\|B\rangle$
$\|\phi^+\rangle = (\|00\rangle + \|11\rangle)/\sqrt{2}$	$\|0\rangle$	$\|0\rangle$
$\|\psi^+\rangle = (\|01\rangle + \|10\rangle)/\sqrt{2}$	$\|0\rangle$	$\|1\rangle$
$\|\phi^-\rangle = (\|00\rangle - \|11\rangle)/\sqrt{2}$	$\|1\rangle$	$\|0\rangle$
$\|\psi^-\rangle = (\|01\rangle - \|10\rangle)/\sqrt{2}$	$\|1\rangle$	$\|1\rangle$