

Summary of previous lecture

- Properties of density matrix

- $\text{tr}(\rho^2) = \begin{cases} 1 & \Rightarrow \text{Pure state} \\ < 1 & \Rightarrow \text{Mixed state} \end{cases}$

- (1) (Trace condition) $\text{tr}(\rho) = 1$

- (2) (Positivity condition) ρ is a positive operator

- Decomposition of density matrix is not unique

- Possible decompositions are connected by unitary relation: $|\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\phi}_j\rangle$

- Reduced density matrix

- Partial trace: for $\rho^{AB} \equiv \sum_i p_i^{AB} |\psi_i^{AB}\rangle \langle \psi_i^{AB}|$ representing entire system, take trace only for part of the system $\rho^A \equiv \text{tr}_B(\rho^{AB})$

- Pure entangled state becomes mixed state when only part of the system is considered

- During quantum teleportation, without the measurement result in Bell basis, the information stored in the destination qubit is equivalent to completely random quantum state.

Quantification of Entanglement

- Are the following states entangled or separable?

- $|\psi_1\rangle = |1_A\rangle|2_B\rangle$
 - $|\psi_2\rangle = (|1_A\rangle|2_B\rangle - |2_A\rangle|1_B\rangle)/\sqrt{2}$
 - $|\psi_3\rangle = (|1_A\rangle|1_B\rangle - |1_A\rangle|2_B\rangle + |2_A\rangle|1_B\rangle - |2_A\rangle|2_B\rangle)/2$
 - $|\psi_4\rangle = (2|1_A\rangle|1_B\rangle - |1_A\rangle|2_B\rangle - |2_A\rangle|1_B\rangle + 2|2_A\rangle|2_B\rangle)/\sqrt{10}$

- Can we turn $|\psi_4\rangle$ into the form of $|\psi_2\rangle$?

- $2|1_A\rangle|1_B\rangle - |1_A\rangle|2_B\rangle - |2_A\rangle|1_B\rangle + 2|2_A\rangle|2_B\rangle$
 $\Rightarrow |\alpha_A\rangle|\alpha_B\rangle - |\beta_A\rangle|\beta_B\rangle$ where $|\alpha_A\rangle$ & $|\beta_A\rangle$ are orthogonal and $|\alpha_B\rangle$ & $|\beta_B\rangle$ are also orthogonal, respectively
 - If we define c_{ij} as $c_{11}|1_A\rangle|1_B\rangle + c_{12}|1_A\rangle|2_B\rangle + c_{21}|2_A\rangle|1_B\rangle + c_{22}|2_A\rangle|2_B\rangle$,
 $c_{11} = \frac{2}{\sqrt{10}}, c_{12} = -\frac{1}{\sqrt{10}}, c_{21} = -\frac{1}{\sqrt{10}}, c_{22} = \frac{2}{\sqrt{10}} \Rightarrow C = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$
 - Can we find unitary transform for A and B such that $C' = \begin{bmatrix} \blacksquare & 0 \\ 0 & \blacksquare \end{bmatrix}$?

Basis Transformation of a Matrix Representation

- Assume that \mathbb{O} is a matrix representation of an operator Ω in some orthonormal basis $|1\rangle, |2\rangle, \dots, |n\rangle$, where each component of \mathbb{O} is obtained by $\mathbb{O}_{ij} = \langle i|\Omega|j\rangle$.
- If we switch the basis from $|1\rangle, |2\rangle, \dots, |n\rangle$ to a new orthonormal basis $|I\rangle, |II\rangle, \dots, |N\rangle$, the new matrix representation \mathbb{O}' of the same operator Ω in the new orthonormal basis can be obtained by $\mathbb{O}' = \mathbb{U}^\dagger \mathbb{O} \mathbb{U}$ or $\mathbb{O} = \mathbb{U} \mathbb{O}' \mathbb{U}^\dagger$.
- \mathbb{U} is a matrix representation of $U = \sum_{m=1}^n |M\rangle \langle m|$ in the original basis $|1\rangle, |2\rangle, \dots, |n\rangle$.

Basis Transformation of a Vector Representation

- Assume that \mathbf{v} is a column vector representation of a vector $|v\rangle$ in some orthonormal basis $|1\rangle, |2\rangle, \dots, |n\rangle$, where each component of \mathbf{v} is obtained by $v_k = \langle k|v\rangle$.
- Consider a new orthonormal basis vectors $|I\rangle, |II\rangle, \dots, |N\rangle$ that have column vector representation of

$$\begin{bmatrix} I_1 \\ \vdots \\ I_n \end{bmatrix}, \begin{bmatrix} II_1 \\ \vdots \\ II_n \end{bmatrix}, \dots, \begin{bmatrix} N_1 \\ \vdots \\ N_n \end{bmatrix} \text{ with each component } J_i = \langle i|J\rangle \text{ in the original basis.}$$

- If we switch the basis from $|1\rangle, |2\rangle, \dots, |n\rangle$ to a new orthonormal basis $|I\rangle, |II\rangle, \dots, |N\rangle$, the new column vector representation \mathbf{v}' of the same vector $|v\rangle$ in the new orthonormal basis can be obtained by

$$v'_k = \langle K|v\rangle = \langle K|\left\{\sum_{i=1}^n |i\rangle\langle i|\right\}|v\rangle = \sum_{i=1}^n \langle K|i\rangle\langle i|v\rangle = \sum_{i=1}^n \langle K|i\rangle v_i = \sum_{i=1}^n W_{ki} v_i$$

- If we define $W = \sum_{m=1}^n |m\rangle\langle M|$ and W as a matrix representation of W in the original basis $|1\rangle, |2\rangle, \dots, |n\rangle$, then it automatically satisfies $W_{ki} = \langle K|i\rangle$ because

$$W_{ki} = \langle k|W|i\rangle = \langle k|\left\{\sum_{m=1}^n |m\rangle\langle M|\right\}|i\rangle = \sum_{m=1}^n \delta_{km}\langle M|i\rangle = \langle K|i\rangle$$

- Conventionally, it is more common to define $U \equiv W^\dagger = \sum_{m=1}^n |M\rangle\langle m|$
- Then U is a matrix representation of $U = \sum_{m=1}^n |M\rangle\langle m|$ in the original basis $|1\rangle, |2\rangle, \dots, |n\rangle$, and

$$U_{ij} = \langle i|U|j\rangle = \langle i|\left\{\sum_{m=1}^n |M\rangle\langle m|\right\}|j\rangle = \langle i|M\rangle\langle m|j\rangle = \langle i|M\rangle\delta_{mj} = \langle i|M\rangle = U_{ij} \text{ or } U = \begin{bmatrix} \langle 1|I\rangle & \dots & \langle 1|N\rangle \\ \vdots & \ddots & \vdots \\ \langle n|I\rangle & \dots & \langle n|N\rangle \end{bmatrix} = \begin{bmatrix} I_1 & \dots & N_1 \\ \vdots & \ddots & \vdots \\ I_n & \dots & N_n \end{bmatrix}$$

- $W \equiv U^\dagger$ and $v'_k = \sum_{i=1}^n U_{ki}^\dagger v_i \Rightarrow \mathbf{v}' = U^\dagger \mathbf{v}$ or $\mathbf{v} = U \mathbf{v}'$

Basis Transformation of a Vector Representation

- From the previous page,

$$U = \sum_{m=1}^n |M\rangle\langle m| \text{ and } U_{ij} = \langle i|J\rangle \text{ or } U = \begin{bmatrix} \langle 1|I\rangle & \cdots & \langle 1|N\rangle \\ \vdots & \ddots & \vdots \\ \langle n|I\rangle & \cdots & \langle n|N\rangle \end{bmatrix} = \begin{bmatrix} I_1 & \cdots & N_1 \\ \vdots & \ddots & \vdots \\ I_n & \cdots & N_n \end{bmatrix}$$

- Mapping to a new orthonormal basis

$$\sum_{i=1}^n U_{ik} |i\rangle = \sum_{i=1}^n |i\rangle \langle i|K\rangle = |K\rangle \Leftrightarrow |i\rangle = \left\{ \sum_{j=1}^n |J\rangle \langle J| \right\} |i\rangle = \sum_{j=1}^n |J\rangle \left(\sum_{k=1}^n U_{kj}^* \langle k|i\rangle \right) = \sum_{j=1}^n (U^\dagger)_{ji} |J\rangle$$

- Mapping to another new orthonormal basis

- Define another new matrix $V = U^*$
- Because $V^\dagger = U^T$ and $UU^\dagger = I$, $(UU^\dagger)^* = U^*U^T = VV^\dagger = I \rightarrow V$ is a unitary matrix
- New basis defined as

$$|L'\rangle \equiv \sum_{j=1}^n U_{ij}^\dagger |j\rangle = \sum_{j=1}^n V_{jl} |j\rangle$$

are orthonormal because V is a unitary matrix.

- $|i\rangle$ can be expanded in $|L'\rangle$ basis in the following way

$$|i\rangle = \left\{ \sum_{L=1}^n |L'\rangle \langle L'| \right\} |i\rangle = \sum_{L=1}^n |L'\rangle \left(\sum_{j=1}^n V_{jl}^* \langle j|i\rangle \right) = \sum_{L=1}^n |L'\rangle \left(\sum_{j=1}^n U_{jl} \langle j|i\rangle \right) = \sum_{L=1}^n U_{il} |L'\rangle$$

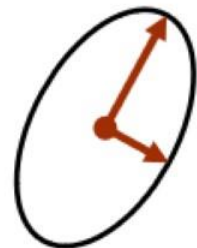
Quantification of Entanglement

- For $|\psi_4\rangle = (2|1_A\rangle|1_B\rangle - |1_A\rangle|2_B\rangle - |2_A\rangle|1_B\rangle + 2|2_A\rangle|2_B\rangle)/\sqrt{10}$
 - $C = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$
 - $\begin{cases} |I_A\rangle = (|1_A\rangle - |2_A\rangle)/\sqrt{2} \\ |II_A\rangle = (|1_A\rangle + |2_A\rangle)/\sqrt{2} \end{cases} \Leftrightarrow \begin{cases} |1_A\rangle = (|I_A\rangle + |II_A\rangle)/\sqrt{2} \\ |2_A\rangle = (-|I_A\rangle + |II_A\rangle)/\sqrt{2} \end{cases}$
 - $\begin{cases} |I'_B\rangle = (|1_B\rangle - |2_B\rangle)/\sqrt{2} \\ |II'_B\rangle = (|1_B\rangle + |2_B\rangle)/\sqrt{2} \end{cases} \Leftrightarrow \begin{cases} |1_B\rangle = (|I'_B\rangle + |II'_B\rangle)/\sqrt{2} \\ |2_B\rangle = (-|I'_B\rangle + |II'_B\rangle)/\sqrt{2} \end{cases}$
 - $|\psi_4\rangle = \frac{1}{\sqrt{10}} (3|I_A\rangle|I'_B\rangle + |II_A\rangle|II'_B\rangle)$
 - Meaning of Hermitian matrix



Quantification of Entanglement

- What about $|\psi\rangle = (|1_A\rangle|1_B\rangle + |1_A\rangle|2_B\rangle + |2_A\rangle|2_B\rangle)/\sqrt{3}$?
 - $c_{ij} \rightarrow \mathbb{C} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
 - c_{ij} is not Hermitian. What can we do?
- In general, we can have two parties that have different dimensions
 - Grouping of multiple qubits to test the separability of the qubits
 - $|\psi\rangle = (|1_A\rangle|1_B\rangle + |1_A\rangle|2_B\rangle + |1_A\rangle|3_B\rangle + |2_A\rangle|1_B\rangle - |2_A\rangle|2_B\rangle)/\sqrt{5}$
 - $\mathbb{C} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$
 - Can we transform it into the following shape?
 - $\blacksquare |I_A\rangle|I_B\rangle + \blacksquare |II_A\rangle|II_B\rangle + \blacksquare |III_A\rangle|III_B\rangle$



Singular Value Decomposition

- Singular Value Decomposition (SVD)

- When a matrix A corresponding to a linear transformation is given, SVD allows us to find an orthogonal basis in the input vector space that will be transformed into another orthogonal basis in the output vector space

- Example

- Assume that the input vector space is \mathbf{R}^3 and the output vector space is \mathbf{R}^2 , and linear transformation is given in terms of matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$.
- Do two orthogonal vectors $(1,0,0)$, $(0,0,1)$ in the input vector space get transformed into orthogonal vectors in the output space?

- $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

→ No!

- What will be the benefit if the orthogonal basis in the input space is transformed into another orthogonal basis in the output space?
 - For a linear transformation, we only need to multiply the corresponding eigenvalues to the coefficients of the basis.
 - Should we give up?

Singular Value Decomposition

- How can we always decompose a given matrix A into $A = UDV^T$ form, where U, V are unitary matrices, D is a diagonal matrix?

$$A^T A = V D^T U^T U D V^T = V D^T D V^T = V \begin{bmatrix} d_1^2 & & \\ & \ddots & \\ & & d_r^2 \end{bmatrix} V^T$$

- Example

- Input space \mathbf{R}^3 , output space \mathbf{R}^2 ,

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

→ positive matrix! Therefore, we can diagonalize and find unitary matrix V . Is this coincidence?

- Eigenvalue & eigenvector: $\lambda = 3, 2, 0$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Singular Value Decomposition

- (Cont'd) Example $A = UDV^T$

- $$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = V(D^T D)V^T$$

- Eigenvalue & eigenvector: $\lambda = 3, 2, 0$, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

- $$V = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}, D^T D = \begin{bmatrix} d_1^2 & & \\ & d_2^2 & \\ & & d_3^2 \end{bmatrix} = \begin{bmatrix} 3 & & \\ & 2 & \\ & & 0 \end{bmatrix}$$

- What should we do if the eigenvalue is negative?

- $$A V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & -\sqrt{2} & 0 \end{bmatrix}$$

Singular Value Decomposition

- (Cont'd) Example $A = UDV^\dagger$

- To find out U , $AA^\dagger = UDV^\dagger VD^\dagger U^\dagger = UDD^\dagger U^\dagger = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

- Eigenvalue & eigenvector: $\lambda = 3, 2, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $U = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, DD^\dagger = \begin{bmatrix} d_1^2 & \\ & d_2^2 \end{bmatrix} = \begin{bmatrix} 3 & \\ & 2 \end{bmatrix}$

- Comparison with $D^\dagger D = \begin{bmatrix} 3 & & \\ & 2 & \\ & & 0 \end{bmatrix} \rightarrow D: 2 \times 3 \text{ matrix}$

- $UDV^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = A$

- SVD works for any types of matrix including the rectangular matrix

- The number of the non-zero diagonal components is equal to the rank of A
 - \rightarrow can be used for the determination of entanglement

Polar decomposition

- Section 2.1.10
- Theorem 2.3 (Polar decomposition): Let A be a linear operator on a vector space V . Then there exists unitary U and positive operator J and K such that

$$A = UJ = KU$$

where the unique positive operator J and K satisfying these equations are defined by $J \equiv \sqrt{A^\dagger A}$ and $K \equiv \sqrt{AA^\dagger}$. Moreover, if A is invertible then U is unique.

- $A = UJ$ is called left polar decomposition of A and $A = KU$ is called the right polar decomposition of A .
- Proof of left polar decomposition
 - $A^\dagger A$ is a positive operator $\rightarrow A^\dagger A$ can be diagonalized $\rightarrow J \equiv \sqrt{A^\dagger A}$ is a positive operator. $\rightarrow J = \sum_i \lambda_i |i\rangle\langle i|$ ($\lambda_i \geq 0$).
 - Define $|\psi_i\rangle \equiv A|i\rangle$. $\rightarrow \langle \psi_i | \psi_i \rangle = \lambda_i^2$
 - Consider for now only those i for which $\lambda_i \neq 0$. \rightarrow define normalized vector $|e_i\rangle \equiv |\psi_i\rangle / \lambda_i$. \rightarrow If $i \neq j$, $\langle e_i | e_j \rangle = \langle i | A^\dagger A | j \rangle / \lambda_i \lambda_j = \langle i | J^2 | j \rangle / \lambda_i \lambda_j = 0$.

Polar decomposition

- Proof of left polar decomposition (continued)
 - From the previous page, $J = \sum_i \lambda_i |i\rangle\langle i|$ ($\lambda_i \geq 0$). $|\psi_i\rangle \equiv A|i\rangle$. $|e_i\rangle \equiv |\psi_i\rangle/\lambda_i$ for $\lambda_i \neq 0$.
 - Using Gram-Schmidt procedure, extend $|i\rangle$ basis (for i with $\lambda_i \neq 0$) to $|j\rangle$ basis (for all j), and find the orthonormal basis set from $|e_i\rangle$ for arbitrary j .
 - Define a unitary operator $U \equiv \sum_i |e_i\rangle\langle i|$.
 - When $\lambda_i \neq 0$, $UJ|i\rangle = \lambda_i |e_i\rangle = A|i\rangle$.
 - When $\lambda_j = 0$, $\langle \psi_j | \psi_j \rangle = \langle j | A^\dagger A | j \rangle = 0$ and $UJ|j\rangle = \lambda_j |e_j\rangle = 0 = |\psi_j\rangle = A|i\rangle$.
 - From the above, the action of UJ and A on $|i\rangle$ are identical.
 $\rightarrow A = UJ$.
 - $A = UJ \rightarrow A^\dagger = J^\dagger U^\dagger \rightarrow A^\dagger A = J^2 \rightarrow J = \sqrt{A^\dagger A}$
 - If A is invertible, J is invertible, so $U = AJ^{-1}$ is uniquely determined.

Singular value decomposition

- Section 2.1.10
- Corollary 2.4 (Singular value decomposition): Let \mathbb{A} be a square matrix. Then there exist unitary matrices \mathbb{U} and \mathbb{V} , and a diagonal matrix \mathbb{D} with non-negative entries such that

$$\mathbb{A} = \mathbb{U}\mathbb{D}\mathbb{V}$$

The diagonal elements of \mathbb{D} are called the singular values of \mathbb{A} .

- Proof
 - By polar decomposition, $\mathbb{A} = \mathbb{S}\mathbb{J}$, for unitary \mathbb{S} , and positive \mathbb{J} . Because \mathbb{J} is a positive matrix and therefore \mathbb{J} is a Hermitian matrix, $\mathbb{J} = \mathbb{T}\mathbb{D}\mathbb{T}^\dagger$, for unitary matrix \mathbb{T} and diagonal matrix \mathbb{D} with non-negative entries. $\mathbb{U} = \mathbb{S}\mathbb{T}$ and $\mathbb{V} = \mathbb{T}^\dagger$.

Schmidt decomposition

- Section 2.5
- Theorem 2.7 (Schmidt decomposition): Suppose $|\psi\rangle$ is a pure state of a composite system, AB . Then there exist orthonormal state $|i_A\rangle$ for system A , and orthonormal state $|i_B\rangle$ for system B such that $|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$, where λ_i are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as Schmidt co-efficient.
- Sketch of proof (for complete proof, refer to page 109)
 - Assume A and B have the same dimension
 - Let $|j\rangle$ and $|k\rangle$ be any fixed orthonormal bases for system A and B , respectively. Then $|\psi\rangle$ can be written as $|\psi\rangle = \sum_j \sum_k a_{jk} |j\rangle |k\rangle$.
 - By the singular value decomposition, $a_{jk} = \mathbb{A} = \mathbb{U}\mathbb{D}\mathbb{V} = \sum_i u_{ji} d_{ii} v_{ik}$,
 $|\psi\rangle = \sum_j \sum_k (\sum_i u_{ji} d_{ii} v_{ik}) |j\rangle |k\rangle$.
 - By defining $|i_A\rangle \equiv \sum_j u_{ji} |j\rangle$, $|i_B\rangle \equiv \sum_k v_{ik} |k\rangle$, and $\lambda_i \equiv d_{ii}$,
 - $|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$

Schmidt decomposition

- $|i_A\rangle, |i_B\rangle$ are called the Schmidt bases
- The number of non-zero values λ_i is called the Schmidt number for the state $|\psi\rangle$.
- Quantification of entanglement (or degree of entanglement)
 - Schmidt number is preserved under unitary transformation
 - The Schmidt number of a product state is 1.
- Purification
 - When a mixed state ρ^A of system A is given, introduce a fictitious system R such that the composite system AR is in a pure state $|AR\rangle$ such that $\rho^A = \text{tr}_R(|AR\rangle\langle AR|)$.
 - If ρ^A can be decomposed into $\rho^A = \sum_i p_i |i^A\rangle\langle i^A|$, then assume that system R has the same dimension as A, with orthonormal basis $|i^R\rangle$, then we can define a pure state $|AR\rangle \equiv \sum_i \sqrt{p_i} |i^A\rangle |i^R\rangle$.