

# Summary of previous lecture

- Density matrix

- When an ensemble of pure states  $\{p_i, |\psi_i\rangle\}$  is given,

$$\rho \equiv \sum_i p_i |\psi_i\rangle \langle \psi_i|$$

- Reformulation of postulate

- Postulate 2:  $\rho \xrightarrow{U} \sum_i p_i (U|\psi_i\rangle \langle \psi_i| U^\dagger) = U\rho U^\dagger$

- Postulate 3

- $\rho \rightarrow p(m) = \sum_i p_i \text{tr}(M_m^\dagger M_m |\psi_i\rangle \langle \psi_i|) = \text{tr}(M_m^\dagger M_m \rho)$

- After measurement of  $m$ ,  $\rho \rightarrow \rho_m = \frac{M_m \rho M_m^\dagger}{\text{tr}(M_m^\dagger M_m \rho)}$

- Properties of density matrix

- $\text{tr}(\rho^2) = \begin{cases} 1 & \Rightarrow \text{Pure state} \\ < 1 & \Rightarrow \text{Mixed state} \end{cases}$

- When mixture of mixed states  $\{p_i, \rho_i\}$  are given,

$$\rho = \sum_{i=1}^n p_i \rho_i$$

# Ensembles of quantum states

- Assumption

- The initial state of some quantum system is given as  $\rho_{\text{init}}$ .
- Someone already measured this system, but the measurement result is not available to us.

- What is the best way to describe the quantum state of this quantum system?

- If the measurement outcome is  $m$ , the probability of such case is  $p(m) = \text{tr}(M_m^\dagger M_m \rho_{\text{init}})$  and the density matrix of final state is  $\rho_m = \frac{M_m \rho_{\text{init}} M_m^\dagger}{\text{tr}(M_m^\dagger M_m \rho_{\text{init}})} = \frac{M_m \rho_{\text{init}} M_m^\dagger}{p(m)}$ .
- We don't know what  $m$  is. Therefore we need to consider for all possible  $m$ 's.
- Density matrix:  $\sum_m p(m) \rho_m = \sum_m p(m) \frac{M_m \rho_{\text{init}} M_m^\dagger}{p(m)} = \sum_m M_m \rho_{\text{init}} M_m^\dagger$

# Properties of density matrix

- Theorem 2.5: (Characterization of density operators) An operator  $\rho$  is the density operator associated to some ensemble  $\{p_i, |\psi_i\rangle\}$  if and only if it satisfies the conditions:

(1) (Trace condition)  $\text{tr}(\rho) = 1$

(2) (Positivity condition)  $\rho$  is a positive operator

- Proof

- $\text{tr}(\rho) = \text{tr}(\sum_i p_i |\psi_i\rangle\langle\psi_i|) = \sum_i p_i \text{tr}(|\psi_i\rangle\langle\psi_i|) = \sum_i p_i = 1$

- For arbitrary  $|v\rangle$ ,  $\langle v|\rho|v\rangle = \langle v|(\sum_i p_i |\psi_i\rangle\langle\psi_i|)|v\rangle = \sum_i p_i \langle v|\psi_i\rangle\langle\psi_i|v\rangle = \sum_i p_i |\langle v|\psi_i\rangle|^2 \geq 0$

- Conversely, assume  $\rho$  is any operator satisfying trace condition (1) and positivity condition (2).

Due to positivity of  $\rho$ ,  $\rho = \sum_j \lambda_j |j\rangle\langle j|$  (spectral decomposition) ←

Any positive operator is Hermitian (Exercise 2.24))

$|j\rangle$ : orthonormal basis.

$\lambda_j$  are real, non-negative eigenvalues of  $\rho$ .

$\sum_j \lambda_j = 1$  due to trace condition.

Therefore,  $\{\lambda_i, |j\rangle\}$  forms good ensemble for density operator.



# Summary of QM in terms of DM

- Reformulation of quantum mechanics from page 102 of the textbook.

- Postulate 1

- Associated to any isolated physical system is a complex vector space with inner product (that is, a Hilbert space) known as the *state space* of the system. The system is completely described by its *density operator*, which is a positive operator  $\rho$  with trace one, acting on the state space of the system. If a quantum system is in the state  $\rho_i$  with probability  $p_i$ , then the density operator for the system is

$$\sum_i p_i \rho_i$$

- Postulate 2

- The evolution of a *closed* quantum system is described by a *unitary transformation*. That is, the state  $\rho$  of the system at time  $t_1$  is related to the state  $\rho'$  of the system at time  $t_2$  by a unitary operator  $U$  which depends only on the times  $t_1$  and  $t_2$ ,

$$\rho' = U\rho U^\dagger$$

# Summary of QM in terms of DM

- (Cont'd) Reformulation of quantum mechanics from page 102 of the textbook.

- Postulate 3

- Quantum measurements are described by a collection  $\{M_m\}$  of *measurement operators*. These are operators acting on the state space of the system being measured. The index  $m$  refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is  $\rho$  immediately before the measurement then the probability that result  $m$  occurs is given by

$$p(m) = \text{tr}(M_m^\dagger M_m \rho)$$

- The state of the system after the measurement is

$$\frac{M_m \rho M_m^\dagger}{\text{tr}(M_m^\dagger M_m \rho)}$$

- The measurement operators satisfy the *completeness equation*,

$$\sum_m M_m^\dagger M_m = I$$

- Postulate 4

- The state space of a composite physical system is the tensor product of the state spaces of component physical systems. Moreover, if we have systems numbered 1 through  $n$ , and system number  $i$  is prepared in the state  $\rho_i$ , then the joint state of the total system is  $\rho_1 \otimes \rho_2 \otimes \cdots \rho_n$ .

# Decomposition of DM

- Does the eigenvalues and eigenvectors of a density matrix have some special significance with regard to the ensemble of quantum states represented by the given density matrix?

- The answer is **NO**.

- Example

- $$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

- One decomposition

- $|0\rangle$  with probability of  $\frac{3}{4}$  and  $|1\rangle$  with probability of  $\frac{1}{4}$

- Another decomposition

- $|a\rangle \equiv \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle$  with probability of  $\frac{1}{2}$

- $|b\rangle \equiv \sqrt{\frac{3}{4}}|0\rangle - \sqrt{\frac{1}{4}}|1\rangle$  with probability of  $\frac{1}{2}$

- $$\rho = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b| = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$$



# Decomposition of DM

- What class of ensembles gives rise to a particular density matrix?
  - Define un-normalized vector  $|\widetilde{\psi}_i\rangle = \sqrt{p_i}|\psi_i\rangle$  so that  $\rho = \sum_i |\widetilde{\psi}_i\rangle\langle\widetilde{\psi}_i|$
  - When two sets of vectors  $|\widetilde{\psi}_i\rangle$  and  $|\widetilde{\phi}_j\rangle$  are connected by unitary matrix, they generate the same operator  $\rho$ .
- Example

- $$\begin{cases} |\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) & \text{with probability of } 1/3 \\ |\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \alpha|1\rangle) & \text{with probability of } 1/3 \\ |\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \alpha^*|1\rangle) & \text{with probability of } 1/3 \end{cases}$$
- $\alpha = e^{i\frac{2\pi}{3}} \rightarrow \alpha^2 + \alpha + 1 = 0$  or  $\alpha^* + \alpha + 1 = 0$  and  $\alpha^* \cdot \alpha = 1$
- $\rho = \frac{1}{3}|\psi_1\rangle\langle\psi_1| + \frac{1}{3}|\psi_2\rangle\langle\psi_2| + \frac{1}{3}|\psi_3\rangle\langle\psi_3| = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$
- $$\begin{cases} |\phi_1\rangle = |0\rangle & \text{with probability of } 1/2 \\ |\phi_2\rangle = |1\rangle & \text{with probability of } 1/2 \end{cases}$$
- Can we connect the set  $\{p_i, |\psi_i\rangle\}$  to the set  $\{p_j, |\phi_j\rangle\}$  with unitary transformation?

# Decomposition of DM

- Example (continued)

- $|\widetilde{\psi}_i\rangle$  set:  $|\widetilde{\psi}_1\rangle = \sqrt{p_1}|\psi_1\rangle = \frac{1}{\sqrt{6}}(|0\rangle + |1\rangle)$ ,  $|\widetilde{\psi}_2\rangle = \frac{1}{\sqrt{6}}(|0\rangle + \alpha|1\rangle)$ ,  $|\widetilde{\psi}_3\rangle = \frac{1}{\sqrt{6}}(|0\rangle + \alpha^*|1\rangle)$

- $|\widetilde{\phi}_j\rangle$  set:  $|\widetilde{\phi}_1\rangle = \frac{1}{\sqrt{2}}|0\rangle$ ,  $|\widetilde{\phi}_2\rangle = \frac{1}{\sqrt{2}}|1\rangle$

- Number of vector elements in each set do not match

→ Pad dummy vectors to the smaller set. That is, add  $|\widetilde{\phi}_3\rangle = \sqrt{0}|v\rangle$  to  $|\widetilde{\phi}_j\rangle$  set

- $u_{ij} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^* \\ 1 & \alpha^* & \alpha \end{bmatrix}$  where  $\alpha = e^{i\frac{2\pi}{3}}$

- $|\widetilde{\psi}_i\rangle = \sum_j u_{ij}|\widetilde{\phi}_j\rangle$

- Theorem 2.6: (Unitary freedom in the ensemble for density matrix)  
The sets  $|\widetilde{\psi}_i\rangle$  and  $|\widetilde{\phi}_j\rangle$  generate the same density matrix if and only if

$$|\widetilde{\psi}_i\rangle = \sum_j u_{ij}|\widetilde{\phi}_j\rangle ,$$

where  $u_{ij}$  is a unitary matrix of complex numbers, with indices  $i$  and  $j$ , and we 'pad' whichever set of vectors  $|\widetilde{\psi}_i\rangle$  or  $|\widetilde{\phi}_j\rangle$  is smaller with additional vectors 0 so that the two sets have the same number of elements. → Proof in page 104



# Reduced density matrix

- Section 2.4.3
- Suppose we have physical systems A and B, whose state is described by a density operator  $\rho^{AB}$ .
  - Example:  $|0_A\rangle|0_B\rangle$  with probability of  $3/4$ ,  $|0_A\rangle|1_B\rangle$  with probability of  $1/4$ .
  - $\rho^{AB} = \frac{3}{4}|0_A\rangle\langle 0_A| \otimes |0_B\rangle\langle 0_B| + \frac{1}{4}|0_A\rangle\langle 0_A| \otimes |1_B\rangle\langle 1_B|$
- Reduced density operator for system A is defined by
$$\rho^A \equiv \text{tr}_B(\rho^{AB})$$

where  $\text{tr}_B$  is a map of operators known as the *partial trace* over system B.

- Example (not mathematically sound, but gives intuition):

$$\begin{aligned}\rho^A &= \text{tr}_B(\rho^{AB}) = \sum_{j=1}^2 \langle j_B | \rho^{AB} | j_B \rangle \\ &= \langle 0_B | \rho^{AB} | 0_B \rangle + \langle 1_B | \rho^{AB} | 1_B \rangle = \frac{3}{4}|0_A\rangle\langle 0_A| + \frac{1}{4}|0_A\rangle\langle 0_A| = |0_A\rangle\langle 0_A|\end{aligned}$$

- Formally, the partial trace is defined by
$$\begin{aligned}\text{tr}_B(|a_{1A}\rangle\langle a_{2A}| \otimes |b_{1B}\rangle\langle b_{2B}|) &\equiv |a_{1A}\rangle\langle a_{2A}| \otimes \text{tr}(|b_{1B}\rangle\langle b_{2B}|) \\ &= |a_{1A}\rangle\langle a_{2A}| (\langle b_{2B} | b_{1B} \rangle)\end{aligned}$$

and it should be linear in its input.

# Reduced density matrix

- Why are we interested in partial trace and reduced density matrix?

- Entangled state  $|\psi^-\rangle = [ |0_A\rangle|1_B\rangle - |1_A\rangle|0_B\rangle ] / \sqrt{2}$

$$\rho^{AB} = \left( \frac{|01\rangle - |10\rangle}{\sqrt{2}} \right) \left( \frac{\langle 01| - \langle 10|}{\sqrt{2}} \right) = \frac{|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|}{2}$$

- Partial trace on A:

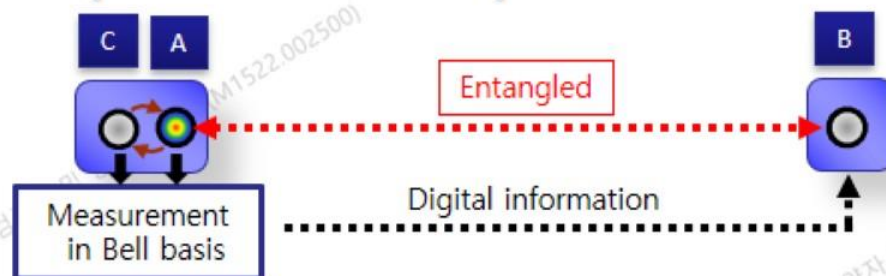
$$\begin{aligned} \rho^B &= \text{tr}_A(\rho^{AB}) = \frac{\text{tr}_A(|0_A 1_B\rangle\langle 0_A 1_B|) - \text{tr}_A(|0_A 1_B\rangle\langle 1_A 0_B|) - \text{tr}_A(|1_A 0_B\rangle\langle 0_A 1_B|) + \text{tr}_A(|1_A 0_B\rangle\langle 1_A 0_B|)}{2} \\ &= \frac{(|1_B\rangle\langle 1_B|)\langle 0_A|0_A\rangle - (|1_B\rangle\langle 0_B|)\langle 1_A|0_A\rangle - (|0_B\rangle\langle 1_B|)\langle 0_A|1_A\rangle + (|0_B\rangle\langle 0_B|)\langle 1_A|1_A\rangle}{2} \\ &= \frac{|1_B\rangle\langle 1_B| + |0_B\rangle\langle 0_B|}{2} = \frac{I_B}{2} \end{aligned}$$

- The same result as we obtained assuming actual measurements

- Note

- The reduced density matrix represents mixed state because  $\text{tr}((\rho^B)^2) = 1/2 < 1$ .
- Consider that the entire state  $|\psi^-\rangle$  is pure state, but if we only look at the partial system A, it appears as mixed state.
- Signature of entanglement
- Using partial trace, can you explain why communication faster than light with entangled state is impossible?

# Review of quantum teleportation



- $|\Psi_C\rangle = (\alpha|0_C\rangle + \beta|1_C\rangle)$  : arbitrary quantum state to teleport

- $|\Psi_C\rangle|\psi_{AB}^-\rangle = (\alpha|0_C\rangle + \beta|1_C\rangle)(|0_A\rangle|1_B\rangle - |1_A\rangle|0_B\rangle)/\sqrt{2}$

$$= [\alpha|0_C\rangle|0_A\rangle|1_B\rangle - \alpha|0_C\rangle|1_A\rangle|0_B\rangle + \beta|1_C\rangle|0_A\rangle|1_B\rangle - \beta|1_C\rangle|1_A\rangle|0_B\rangle]/\sqrt{2}$$

$$= [\alpha(|\phi_{CA}^+\rangle + |\phi_{CA}^-\rangle)|1_B\rangle - \alpha(|\psi_{CA}^+\rangle + |\psi_{CA}^-\rangle)|0_B\rangle + \beta(|\psi_{CA}^+\rangle - |\psi_{CA}^-\rangle)|1_B\rangle + \beta(|\phi_{CA}^+\rangle - |\phi_{CA}^-\rangle)|0_B\rangle]/2$$

$$= [|\psi_{CA}^+\rangle(-\alpha|0_B\rangle + \beta|1_B\rangle) - |\psi_{CA}^-\rangle(\alpha|0_B\rangle + \beta|1_B\rangle) + |\phi_{CA}^+\rangle(\alpha|1_B\rangle + \beta|0_B\rangle) + |\phi_{CA}^-\rangle(\alpha|1_B\rangle - \beta|0_B\rangle)]/2$$

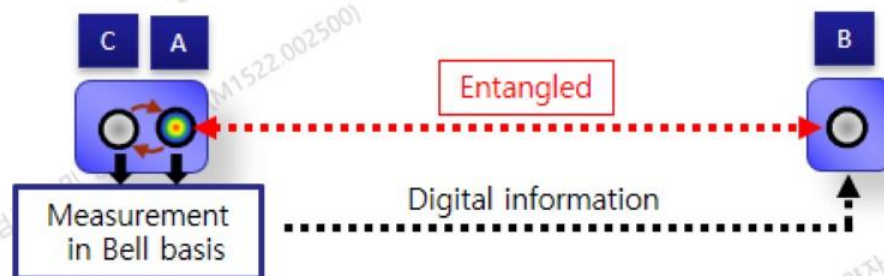
$$\begin{cases} |\psi_{CA}^+\rangle = [|0_C\rangle|1_A\rangle + |1_C\rangle|0_A\rangle]/\sqrt{2} \\ |\psi_{CA}^-\rangle = [|0_C\rangle|1_A\rangle - |1_C\rangle|0_A\rangle]/\sqrt{2} \\ |\phi_{CA}^+\rangle = [|0_C\rangle|0_A\rangle + |1_C\rangle|1_A\rangle]/\sqrt{2} \\ |\phi_{CA}^-\rangle = [|0_C\rangle|0_A\rangle - |1_C\rangle|1_A\rangle]/\sqrt{2} \end{cases}$$



$$\begin{cases} |0_C\rangle|1_A\rangle = [|\psi_{CA}^+\rangle + |\psi_{CA}^-\rangle]/\sqrt{2} \\ |1_C\rangle|0_A\rangle = [|\psi_{CA}^+\rangle - |\psi_{CA}^-\rangle]/\sqrt{2} \\ |0_C\rangle|0_A\rangle = [|\phi_{CA}^+\rangle + |\phi_{CA}^-\rangle]/\sqrt{2} \\ |1_C\rangle|1_A\rangle = [|\phi_{CA}^+\rangle - |\phi_{CA}^-\rangle]/\sqrt{2} \end{cases}$$



# Quantum teleportation with reduced DM



- $|\Psi_C\rangle = (\alpha|0_C\rangle + \beta|1_C\rangle)$  : arbitrary quantum state to teleport

$$|\Psi_C\rangle|\psi_{AB}^-\rangle = [|\psi_{CA}^+\rangle(-\alpha|0_B\rangle + \beta|1_B\rangle) - |\psi_{CA}^-\rangle(\alpha|0_B\rangle + \beta|1_B\rangle) + |\phi_{CA}^+\rangle(\alpha|1_B\rangle + \beta|0_B\rangle) + |\phi_{CA}^-\rangle(\alpha|1_B\rangle - \beta|0_B\rangle)]/2$$

- Density matrix of  $|\Psi_C\rangle|\psi_{AB}^-\rangle$

$$\rho^{CAB} = \frac{1}{4} [|\psi_{CA}^+\rangle(-\alpha|0_B\rangle + \beta|1_B\rangle) - |\psi_{CA}^-\rangle(\alpha|0_B\rangle + \beta|1_B\rangle) + |\phi_{CA}^+\rangle(\beta|0_B\rangle + \alpha|1_B\rangle) + |\phi_{CA}^-\rangle(-\beta|0_B\rangle + \alpha|1_B\rangle)] [\langle\psi_{CA}^+|(-\alpha^*\langle 0_B| + \beta^*\langle 1_B|) - \langle\psi_{CA}^-|(\alpha^*\langle 0_B| + \beta^*\langle 1_B|) + \langle\phi_{CA}^+|(\beta^*\langle 0_B| + \alpha^*\langle 1_B|) + \langle\phi_{CA}^-|(-\beta^*\langle 0_B| + \alpha^*\langle 1_B|)]$$

- Reduced density matrix at B

$$\begin{aligned} \rho^B &= \text{tr}_{CA}(\rho^{CAB}) \\ &= \frac{1}{4} [(-\alpha|0_B\rangle + \beta|1_B\rangle)(-\alpha^*\langle 0_B| + \beta^*\langle 1_B|) + (\alpha|0_B\rangle + \beta|1_B\rangle)(\alpha^*\langle 0_B| + \beta^*\langle 1_B|) \\ &\quad + (\beta|0_B\rangle + \alpha|1_B\rangle)(\beta^*\langle 0_B| + \alpha^*\langle 1_B|) + (-\beta|0_B\rangle + \alpha|1_B\rangle)(-\beta^*\langle 0_B| + \alpha^*\langle 1_B|)] \\ &= \frac{1}{4} [(2\alpha\alpha^* + 2\beta\beta^*)|0_B\rangle\langle 0_B| + (2\beta\beta^* + 2\alpha\alpha^*)|1_B\rangle\langle 1_B|] = I_B/2 \end{aligned}$$