

Summary of previous lecture

- Density matrix

- When an ensemble of pure states $\{p_i, |\psi_i\rangle\}$ is given,

$$\rho \equiv \sum_i p_i |\psi_i\rangle \langle \psi_i|$$

- Reformulation of postulate

- Postulate 2: $\rho \xrightarrow{U} \sum_i p_i (U|\psi_i\rangle \langle \psi_i|U^\dagger) = U\rho U^\dagger$

- Postulate 3

- $\rho \rightarrow p(m) = \sum_i p_i \text{tr}(M_m^\dagger M_m |\psi_i\rangle \langle \psi_i|) = \text{tr}(M_m^\dagger M_m \rho)$

- After measurement of m , $\rho \rightarrow \rho_m = \frac{M_m \rho M_m^\dagger}{\text{tr}(M_m^\dagger M_m \rho)}$

- Properties of density matrix

- $\text{tr}(\rho^2) = \begin{cases} 1 & \Rightarrow \text{Pure state} \\ < 1 & \Rightarrow \text{Mixed state} \end{cases}$

- When mixture of mixed states $\{p_i, \rho_i\}$ are given,

$$\rho = \sum_{i=1}^n p_i \rho_i$$

Ensembles of quantum states

- Assumption
 - The initial state of some quantum system is given as ρ_{init} .
 - Someone already measured this system, but the measurement result is not available to us.
- What is the best way to describe the quantum state of this quantum system?
 - If the measurement outcome is m , the probability of such case is $p(m) = \text{tr}(M_m^\dagger M_m \rho_{\text{init}})$ and the density matrix of final state is $\rho_m = \frac{M_m \rho_{\text{init}} M_m^\dagger}{\text{tr}(M_m^\dagger M_m \rho_{\text{init}})} = \frac{M_m \rho_{\text{init}} M_m^\dagger}{p(m)}$.
 - We don't know what m is. Therefore we need to consider for all possible m 's.
 - Density matrix: $\sum_m p(m) \rho_m = \sum_m p(m) \frac{M_m \rho_{\text{init}} M_m^\dagger}{p(m)} = \sum_m M_m \rho_{\text{init}} M_m^\dagger$

Properties of density matrix

- Theorem 2.5: (Characterization of density operators) An operator ρ is the density operator associated to some ensemble $\{p_i, |\psi_i\rangle\}$ if and only if it satisfies the conditions:
 - (1) (Trace condition) $\text{tr}(\rho) = 1$
 - (2) (Positivity condition) ρ is a positive operator
- Proof
 - $\text{tr}(\rho) = \text{tr}(\sum_i p_i |\psi_i\rangle\langle\psi_i|) = \sum_i p_i \text{tr}(|\psi_i\rangle\langle\psi_i|) = \sum_i p_i = 1$
 - For arbitrary $|v\rangle$, $\langle v|\rho|v\rangle = \langle v|(\sum_i p_i |\psi_i\rangle\langle\psi_i|)|v\rangle = \sum_i p_i \langle v|\psi_i\rangle\langle\psi_i|v\rangle = \sum_i p_i |\langle v|\psi_i\rangle|^2 \geq 0$
 - Conversely, assume ρ is any operator satisfying trace condition (1) and positivity condition (2).

Due to positivity of ρ , $\rho = \sum_j \lambda_j |j\rangle\langle j|$ (spectral decomposition) ↪
Any positive operator is Hermitian (Exercise 2.24))

$|j\rangle$: orthonormal basis.

λ_j are real, non-negative eigenvalues of ρ .

$\sum_j \lambda_j = 1$ due to trace condition.

Therefore, $\{\lambda_i, |j\rangle\}$ forms good ensemble for density operator.

Summary of QM in terms of DM

- Reformulation of quantum mechanics from page 102 of the textbook.
 - Postulate 1
 - Associated to any isolated physical system is a complex vector space with inner product (that is, a Hilbert space) known as the *state space* of the system. The system is completely described by its *density operator*, which is a positive operator ρ with trace one, acting on the state space of the system. If a quantum system is in the state ρ_i with probability p_i , then the density operator for the system is
$$\sum_i p_i \rho_i$$
 - Postulate 2
 - The evolution of a *closed* quantum system is described by a *unitary transformation*. That is, the state ρ of the system at time t_1 is related to the state ρ' of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2 ,
$$\rho' = U\rho U^\dagger$$

Summary of QM in terms of DM

- (Cont'd) Reformulation of quantum mechanics from page 102 of the textbook.

- Postulate 3

- Quantum measurements are described by a collection $\{M_m\}$ of *measurement operators*. These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is ρ immediately before the measurement then the probability that result m occurs is given by

$$p(m) = \text{tr}(M_m^\dagger M_m \rho)$$

- The state of the system after the measurement is

$$\frac{M_m \rho M_m^\dagger}{\text{tr}(M_m^\dagger M_m \rho)}$$

- The measurement operators satisfy the *completeness equation*,

$$\sum_m M_m^\dagger M_m = I$$

- Postulate 4

- The state space of a composite physical system is the tensor product of the state spaces of component physical systems. Moreover, if we have systems numbered 1 through n , and system number i is prepared in the state ρ_i , then the joint state of the total system is $\rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$.

Decomposition of DM

- Does the eigenvalues and eigenvectors of a density matrix have some special significance with regard to the ensemble of quantum states represented by the given density matrix?
 - The answer is **NO**.
- Example

- $\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$

- One decomposition

- $|0\rangle$ with probability of $\frac{3}{4}$ and $|1\rangle$ with probability of $\frac{1}{4}$

- Another decomposition

- $|a\rangle \equiv \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle$ with probability of $\frac{1}{2}$

- $|b\rangle \equiv \sqrt{\frac{3}{4}}|0\rangle - \sqrt{\frac{1}{4}}|1\rangle$ with probability of $\frac{1}{2}$

- $\rho = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b| = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$

Decomposition of DM

- What class of ensembles gives rise to a particular density matrix?
 - Define un-normalized vector $|\tilde{\psi}_i\rangle = \sqrt{p_i}|\psi_i\rangle$ so that $\rho = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|$
 - When two sets of vectors $|\tilde{\psi}_i\rangle$ and $|\tilde{\phi}_j\rangle$ are connected by unitary matrix, they generate the same operator ρ .
- Example

- $$\begin{cases} |\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) & \text{with probability of } 1/3 \\ |\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \alpha|1\rangle) & \text{with probability of } 1/3 \\ |\psi_3\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \alpha^*|1\rangle) & \text{with probability of } 1/3 \end{cases}$$
 - $\alpha = e^{i\frac{2\pi}{3}} \rightarrow \alpha^2 + \alpha + 1 = 0$ or $\alpha^* + \alpha + 1 = 0$ and $\alpha^* \cdot \alpha = 1$
 - $\rho = \frac{1}{3}|\psi_1\rangle\langle\psi_1| + \frac{1}{3}|\psi_2\rangle\langle\psi_2| + \frac{1}{3}|\psi_3\rangle\langle\psi_3| = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$
 - $$\begin{cases} |\phi_1\rangle = |0\rangle & \text{with probability of } 1/2 \\ |\phi_2\rangle = |1\rangle & \text{with probability of } 1/2 \end{cases}$$
 - Can we connect the set $\{p_i, |\psi_i\rangle\}$ to the set $\{p_j, |\phi_j\rangle\}$ with unitary transformation?

Decomposition of DM

- Example (continued)
 - $|\tilde{\psi}_i\rangle$ set: $|\tilde{\psi}_1\rangle = \sqrt{p_1}|\psi_1\rangle = \frac{1}{\sqrt{6}}(|0\rangle + |1\rangle)$, $|\tilde{\psi}_2\rangle = \frac{1}{\sqrt{6}}(|0\rangle + \alpha|1\rangle)$, $|\tilde{\psi}_3\rangle = \frac{1}{\sqrt{6}}(|0\rangle + \alpha^*|1\rangle)$
 - $|\tilde{\phi}_j\rangle$ set: $|\tilde{\phi}_1\rangle = \frac{1}{\sqrt{2}}|0\rangle$, $|\tilde{\phi}_2\rangle = \frac{1}{\sqrt{2}}|1\rangle$
Number of vector elements in each set do not match
→ Pad dummy vectors to the smaller set. That is, add $|\tilde{\phi}_3\rangle = \sqrt{0}|v\rangle$ to $|\tilde{\phi}_j\rangle$ set
 - $u_{ij} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^* \\ 1 & \alpha^* & \alpha \end{bmatrix}$ where $\alpha = e^{i\frac{2\pi}{3}}$
 - $|\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\phi}_j\rangle$
- Theorem 2.6: (Unitary freedom in the ensemble for density matrix)
The sets $|\tilde{\psi}_i\rangle$ and $|\tilde{\phi}_j\rangle$ generate the same density matrix if and only if
$$|\tilde{\psi}_i\rangle = \sum_j u_{ij} |\tilde{\phi}_j\rangle,$$
where u_{ij} is a unitary matrix of complex numbers, with indices i and j , and we 'pad' whichever set of vectors $|\tilde{\psi}_i\rangle$ or $|\tilde{\phi}_j\rangle$ is smaller with additional vectors 0 so that the two sets have the same number of elements. → Proof in page 104

Reduced density matrix

- Section 2.4.3
- Suppose we have physical systems A and B, whose state is described by a density operator ρ^{AB} .
 - Example: $|0_A\rangle|0_B\rangle$ with probability of $3/4$, $|0_A\rangle|1_B\rangle$ with probability of $1/4$.
 - $\rho^{AB} = \frac{3}{4}|0_A\rangle\langle 0_A| \otimes |0_B\rangle\langle 0_B| + \frac{1}{4}|0_A\rangle\langle 0_A| \otimes |1_B\rangle\langle 1_B|$
- Reduced density operator for system A is defined by

$$\rho^A \equiv \text{tr}_B(\rho^{AB})$$

where tr_B is a map of operators known as the *partial trace* over system B.

▫ Example (not mathematically sound, but gives intuition):

$$\begin{aligned}\rho^A &= \text{tr}_B(\rho^{AB}) = \sum_{j=1}^2 \langle j_B | \rho^{AB} | j_B \rangle \\ &= \langle 0_B | \rho^{AB} | 0_B \rangle + \langle 1_B | \rho^{AB} | 1_B \rangle = \frac{3}{4}|0_A\rangle\langle 0_A| + \frac{1}{4}|0_A\rangle\langle 0_A| = |0_A\rangle\langle 0_A|\end{aligned}$$

- Formally, the partial trace is defined by

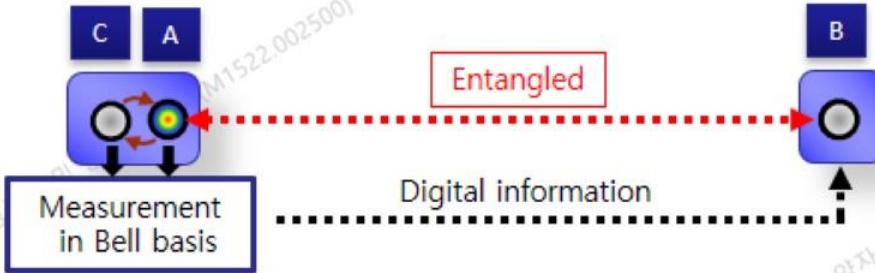
$$\begin{aligned}\text{tr}_B(|a_{1A}\rangle\langle a_{2A}| \otimes |b_{1B}\rangle\langle b_{2B}|) &\equiv |a_{1A}\rangle\langle a_{2A}| \otimes \text{tr}(|b_{1B}\rangle\langle b_{2B}|) \\ &= |a_{1A}\rangle\langle a_{2A}| (\langle b_{2B}|b_{1B}\rangle)\end{aligned}$$

and it should be linear in its input.

Reduced density matrix

- Why are we interested in partial trace and reduced density matrix?
- Entangled state $|\psi^-\rangle = [|0_A\rangle|1_B\rangle - |1_A\rangle|0_B\rangle]/\sqrt{2}$
 - $\rho^{AB} = \left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right) \left(\frac{\langle 01| - \langle 10|}{\sqrt{2}}\right) = \frac{|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|}{2}$
- Partial trace on A:
 - $\rho^B = \text{tr}_A(\rho^{AB}) = \frac{\text{tr}_A(|0_A 1_B\rangle\langle 0_A 1_B|) - \text{tr}_A(|0_A 1_B\rangle\langle 1_A 0_B|) - \text{tr}_A(|1_A 0_B\rangle\langle 0_A 1_B|) + \text{tr}_A(|1_A 0_B\rangle\langle 1_A 0_B|)}{2}$ $= \frac{(|1_B\rangle\langle 1_B|)\langle 0_A|0_A\rangle - (|1_B\rangle\langle 0_B|)\langle 1_A|0_A\rangle - (|0_B\rangle\langle 1_B|)\langle 0_A|1_A\rangle + (|0_B\rangle\langle 0_B|)\langle 1_A|1_A\rangle}{2}$ $= \frac{|1_B\rangle\langle 1_B| + |0_B\rangle\langle 0_B|}{2} = \frac{I_B}{2}$
- The same result as we obtained assuming actual measurements
- Note
 - The reduced density matrix represents mixed state because $\text{tr}((\rho^B)^2) = 1/2 < 1$.
 - Consider that the entire state $|\psi^-\rangle$ is pure state, but if we only look at the partial system A, it appears as mixed state.
 - → Signature of entanglement
 - Using partial trace, can you explain why communication faster than light with entangled state is impossible?

Review of quantum teleportation



- $|\Psi_C\rangle = (\alpha|0_C\rangle + \beta|1_C\rangle)$: arbitrary quantum state to teleport

$$|\Psi_C\rangle|\psi_{AB}^-\rangle = (\alpha|0_C\rangle + \beta|1_C\rangle)(|0_A\rangle|1_B\rangle - |1_A\rangle|0_B\rangle)/\sqrt{2}$$

$$= [\alpha|0_C\rangle|0_A\rangle|1_B\rangle - \alpha|0_C\rangle|1_A\rangle|0_B\rangle + \beta|1_C\rangle|0_A\rangle|1_B\rangle - \beta|1_C\rangle|1_A\rangle|0_B\rangle]/\sqrt{2}$$

$$= [\alpha(|\phi_{CA}^+\rangle + |\phi_{CA}^-\rangle)|1_B\rangle - \alpha(|\psi_{CA}^+\rangle + |\psi_{CA}^-\rangle)|0_B\rangle$$

$$+ \beta(|\psi_{CA}^+\rangle - |\psi_{CA}^-\rangle)|1_B\rangle + \beta(|\phi_{CA}^+\rangle - |\phi_{CA}^-\rangle)|0_B\rangle]/2$$

$$= [|\psi_{CA}^+\rangle(-\alpha|0_B\rangle + \beta|1_B\rangle) - |\psi_{CA}^-\rangle(\alpha|0_B\rangle + \beta|1_B\rangle)$$

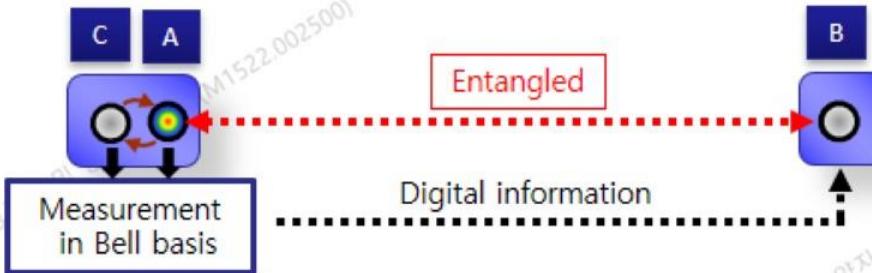
$$+ |\phi_{CA}^+\rangle(-\alpha|1_B\rangle + \beta|0_B\rangle) + |\phi_{CA}^-\rangle(\alpha|1_B\rangle - \beta|0_B\rangle)]/2$$

$$\begin{cases} |\psi_{CA}^+\rangle = [|0_C\rangle|1_A\rangle + |1_C\rangle|0_A\rangle]/\sqrt{2} \\ |\psi_{CA}^-\rangle = [|0_C\rangle|1_A\rangle - |1_C\rangle|0_A\rangle]/\sqrt{2} \\ |\phi_{CA}^+\rangle = [|0_C\rangle|0_A\rangle + |1_C\rangle|1_A\rangle]/\sqrt{2} \\ |\phi_{CA}^-\rangle = [|0_C\rangle|0_A\rangle - |1_C\rangle|1_A\rangle]/\sqrt{2} \end{cases}$$



$$\begin{cases} |0_C\rangle|1_A\rangle = [|\psi_{CA}^+\rangle + |\psi_{CA}^-\rangle]/\sqrt{2} \\ |1_C\rangle|0_A\rangle = [|\psi_{CA}^+\rangle - |\psi_{CA}^-\rangle]/\sqrt{2} \\ |0_C\rangle|0_A\rangle = [|\phi_{CA}^+\rangle + |\phi_{CA}^-\rangle]/\sqrt{2} \\ |1_C\rangle|1_A\rangle = [|\phi_{CA}^+\rangle - |\phi_{CA}^-\rangle]/\sqrt{2} \end{cases}$$

Quantum teleportation with reduced DM



- $|\Psi_C\rangle = (\alpha|0_C\rangle + \beta|1_C\rangle)$: arbitrary quantum state to teleport
- $|\Psi_C\rangle|\psi_{AB}^-\rangle = [|\psi_{CA}^+\rangle(-\alpha|0_B\rangle + \beta|1_B\rangle) - |\psi_{CA}^-\rangle(\alpha|0_B\rangle + \beta|1_B\rangle)$
 $+ |\phi_{CA}^+\rangle(-\alpha|1_B\rangle + \beta|0_B\rangle) + |\phi_{CA}^-\rangle(\alpha|1_B\rangle - \beta|0_B\rangle)]/2$
- Density matrix of $|\Psi_C\rangle|\psi_{AB}^-\rangle$
 - $\rho^{CAB} = \frac{1}{4} [|\psi_{CA}^+\rangle(-\alpha|0_B\rangle + \beta|1_B\rangle) - |\psi_{CA}^-\rangle(\alpha|0_B\rangle + \beta|1_B\rangle) + |\phi_{CA}^+\rangle(\beta|0_B\rangle + \alpha|1_B\rangle) + |\phi_{CA}^-\rangle(-\beta|0_B\rangle + \alpha|1_B\rangle)] [\langle\psi_{CA}^+|(-\alpha^*\langle 0_B | + \beta^*\langle 1_B |) - \langle\psi_{CA}^-|(\alpha^*\langle 0_B | + \beta^*\langle 1_B |) + \langle\phi_{CA}^+|(\beta^*\langle 0_B | + \alpha^*\langle 1_B |) + \langle\phi_{CA}^-|(-\beta^*\langle 0_B | + \alpha^*\langle 1_B |)]$
- Reduced density matrix at B
 - $\rho^B = \text{tr}_{CA}(\rho^{CAB})$
 - $= \frac{1}{4} [(-\alpha|0_B\rangle + \beta|1_B\rangle)(-\alpha^*\langle 0_B | + \beta^*\langle 1_B |) + (\alpha|0_B\rangle + \beta|1_B\rangle)(\alpha^*\langle 0_B | + \beta^*\langle 1_B |)$
 - $+ (\beta|0_B\rangle + \alpha|1_B\rangle)(\beta^*\langle 0_B | + \alpha^*\langle 1_B |) + (-\beta|0_B\rangle + \alpha|1_B\rangle)(-\beta^*\langle 0_B | + \alpha^*\langle 1_B |)]$
 - $= \frac{1}{4} [(2\alpha\alpha^* + 2\beta\beta^*)|0_B\rangle\langle 0_B | + (2\beta\beta^* + 2\alpha\alpha^*)|1_B\rangle\langle 1_B |] = I_B/2$