PS9

1. $\frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle + \frac{1}{4}|11\rangle$

$= \frac{\sqrt{3}}{2\sqrt{2}}|0\rangle\underbrace{(|0\rangle+|1\rangle)}_{a} + \frac{\sqrt{3}}{4}|1\rangle\underbrace{\left(|0\rangle+\frac{1}{\sqrt{3}}|1\rangle\right)}_{b}$

→ Here, we can't factorize this state more, for the coefficients of a and b are independence.

2. i) Measuring the left qubit, we get
   1. $|0\rangle$ with probability $\frac{4}{8} = \frac{1}{2}$,
   2. $|1\rangle$ with probability $\frac{4}{8} = \frac{1}{2}$.

ii) Collapsed state would be
   1. $\dfrac{\frac{\sqrt{3}}{2\sqrt{2}}|0\rangle + \frac{1}{2\sqrt{2}}|1\rangle}{1/\sqrt{2}} = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$
   2. $\dfrac{\frac{1}{2\sqrt{2}}|0\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|1\rangle}{1/\sqrt{2}} = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$

iii) $\frac{\sqrt{3}}{2\sqrt{2}}|00\rangle + \frac{1}{2\sqrt{2}}|01\rangle + \frac{1}{2\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle$

$= \frac{1}{2\sqrt{2}}|0\rangle\underbrace{(\sqrt{3}|0\rangle+|1\rangle)}_{1} + \frac{1}{2\sqrt{2}}|1\rangle\underbrace{(|0\rangle+\sqrt{3}|1\rangle)}_{2}$, Here coefficients of 1 and 2 are linearly independence

⇒ entangled. (∵ not factorized)

∴ partially entangled, after measuring the left qubit, the state of the right qubit is not determined.

3. i) Measuring left qubit,
   1. $|0\rangle$ with $\frac{1}{2}$ of probability
   2. $|1\rangle$ with $\frac{1}{2}$ of probability

ii) Collapsed state
   1. always $|1\rangle$
   2. always $|0\rangle$

iii) maximally entangled, as the state of the right qubit is determined after measuring the left qubit

4.

a. Let $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|0\rangle$

$$(|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|Z)\cdot(Z\otimes I)\cdot(|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes H)\cdot(X\otimes I)|\psi\rangle$$

b. i) a : $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|0\rangle$

ii) b: $\frac{1}{\sqrt{2}}(|1\rangle+|0\rangle)\otimes|0\rangle = \frac{1}{\sqrt{2}}(|10\rangle+|00\rangle)$

iii) c: $\frac{1}{\sqrt{2}}(|1\rangle(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)+|00\rangle) = \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(|10\rangle+|11\rangle)+|00\rangle) = \frac{1}{2}|10\rangle+\frac{1}{2}|11\rangle+\frac{1}{\sqrt{2}}|00\rangle$

iv) d: $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

v) e: $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$

c. $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{2}|1\rangle(|0\rangle-|1\rangle)$.

Yes, not factorizable.

d. $\frac{1}{\sqrt{2}}|00\rangle$ and $-\frac{1}{2}|10\rangle$.

$$\left|\frac{1}{\sqrt{2}}\right|^2 + \left|-\frac{1}{2}\right|^2 = \frac{1}{2}+\frac{1}{4} = \frac{3}{4}$$

5.

a. i) $(|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X)(H\otimes I)|00\rangle$

$= (|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X)(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|0\rangle$

$= (|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X)(\frac{1}{\sqrt{2}}(|00\rangle+|10\rangle)$

$= \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$ ∴ It is Bell state

ii) $(|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X)(H\otimes I)|01\rangle$

$= (|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X)(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle))|1\rangle = (|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X)(\frac{1}{\sqrt{2}}(|01\rangle+|11\rangle)$

$= \frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)$ ∴ It is Bell state

iii) $(|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X)(H\otimes I)|10\rangle$

$= (|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X)(\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle))|0\rangle$

$= (|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X)(\frac{1}{\sqrt{2}}(|00\rangle-|10\rangle)$

$= \frac{1}{\sqrt{2}}(|00\rangle-|11\rangle)$ ∴ Bell state

iv) $(|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X)(H\otimes I)|11\rangle$

$= (|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X)(\frac{1}{\sqrt{2}}(|01\rangle-|11\rangle)$

$= \frac{1}{\sqrt{2}}(|01\rangle-|10\rangle)$

∴ Bell state

b. i) $(H\otimes I)(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)\left(\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)\right)$

$= (H\otimes I)\ \frac{1}{\sqrt{2}}(|00\rangle+|10\rangle) = (H\otimes I)\ \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|0\rangle$

$= |0\rangle\otimes|0\rangle = |00\rangle$

ii) $(H\otimes I)(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)\left(\frac{1}{\sqrt{2}}(|01\rangle+|10\rangle)\right)$

$= (H\otimes I)\ \frac{1}{\sqrt{2}}(|01\rangle+|11\rangle) = (H\otimes I)\ \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|1\rangle$

$= |01\rangle$

iii) $(H\otimes I)(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)\ \frac{1}{\sqrt{2}}(|00\rangle-|11\rangle)$

$= (H\otimes I)\ \frac{1}{\sqrt{2}}(|00\rangle-|10\rangle) = (H\otimes I)\ \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\otimes|0\rangle$

$= |10\rangle$

iv) $(H\otimes I)(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)\ \frac{1}{\sqrt{2}}(|01\rangle-|10\rangle)$

$= (H\otimes I)\ \frac{1}{\sqrt{2}}(|01\rangle-|11\rangle) = (H\otimes I)\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)|1\rangle$

$= |11\rangle$

c.

i) $(H\otimes I)(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)\ |00\rangle$

$= H|0\rangle\otimes I|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle+|10\rangle)$

ii) No, factorized to $\underbrace{\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right)}_{=|+\rangle}|0\rangle$

6. $(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)(I\otimes|0\rangle\langle 0| + X\otimes|1\rangle\langle 1|)(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)$

i) $(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)(I\otimes|0\rangle\langle 0| + X\otimes|1\rangle\langle 1|)(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)\ |00\rangle$

$= (|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)(I\otimes|0\rangle\langle 0| + X\otimes|1\rangle\langle 1|)\ |00\rangle$

$= (|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)\ |00\rangle = |00\rangle$

ii) $(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)(I\otimes|0\rangle\langle 0| + X\otimes|1\rangle\langle 1|)(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)\ |01\rangle$

$= (|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)(I\otimes|0\rangle\langle 0| + X\otimes|1\rangle\langle 1|)\ |01\rangle$

$= (|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)\ |11\rangle$

$= |10\rangle$

iii) $(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)(I\otimes|0\rangle\langle 0| + X\otimes|1\rangle\langle 1|)(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)\ |10\rangle$

$= (|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)(I\otimes|0\rangle\langle 0| + X\otimes|1\rangle\langle 1|)\ |11\rangle$

$= (|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)\ |01\rangle$

$= |01\rangle$

iv) $(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)(I\otimes|0\rangle\langle 0| + X\otimes|1\rangle\langle 1|)(|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)|11\rangle$

$= (|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)(I\otimes|0\rangle\langle 0| + X\otimes|1\rangle\langle 1|)\ |10\rangle$

$= (|0\rangle\langle 0|\otimes I+|1\rangle\langle 1|\otimes X)|10\rangle$

$= |11\rangle$

for arbitrary quantum state $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$,

This circuit swaps two terms: $|01\rangle$ and $|10\rangle$, so that

$$|\psi\rangle \rightarrow a|00\rangle + b|10\rangle + c|01\rangle + d|11\rangle$$

$\therefore$ Swap gate

7.

a. $(|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X)\,(H\otimes H)\,|01\rangle$

$= (|0\rangle\langle 0|\otimes I + |1\rangle\langle 1|\otimes X)\;|+\rangle\otimes|-\rangle$ ← Control is not $|1\rangle$, Cnot doesn't work

$= |+\rangle\otimes|-\rangle = \underline{\frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)}$

b. · produce state.

· $P(q_0 = |0\rangle) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

· $P(|10\rangle) = \frac{1}{4}$

8.

a.

i) $|\psi_1\rangle = (a|0\rangle + b|1\rangle)\otimes\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(a|001\rangle + a|010\rangle + b|101\rangle + b|110\rangle)$

CNOT

ii) $|\psi_2\rangle = \frac{1}{\sqrt{2}}(a|001\rangle + a|010\rangle + b|111\rangle + b|100\rangle)$

iii) $|\psi_3\rangle = (H\otimes I\otimes I)\,|\psi_2\rangle$

$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes a|01\rangle + \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes a|10\rangle + \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\otimes b|11\rangle + \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\otimes b|00\rangle\right)$

$= \frac{1}{2}(a|001\rangle + a|101\rangle + a|010\rangle + a|110\rangle + b|011\rangle - b|111\rangle + b|000\rangle - b|100\rangle)$

$= \frac{1}{2}\left(|00\rangle(a|1\rangle + b|0\rangle) + |10\rangle(a|1\rangle - b|0\rangle) + |01\rangle(a|0\rangle + b|1\rangle) + |11\rangle(a|0\rangle - b|1\rangle)\right)$

b.

i) $a|1\rangle + b|0\rangle$

ii) $a|0\rangle + b|1\rangle$

iii) $a|1\rangle - b|0\rangle$

iv) $a|0\rangle - b|1\rangle$

c.

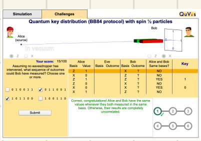
i) $|00\rangle \rightarrow X$ gate

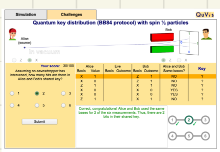
ii) $|01\rangle \rightarrow I$ gate

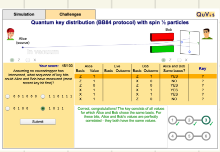
iii) $|10\rangle \rightarrow XZ$ gate

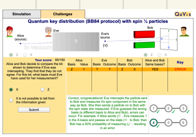
iv) Z gate

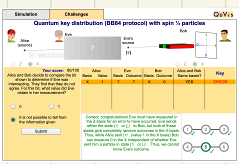
9.

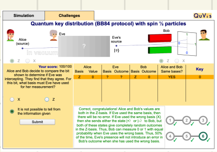
a. 

b. 

c. 

d. 

e. 

f. 

10.

Z	Z	X	Z	X
0	1	1	0	0

a. first: $|0\rangle$ in Z basis

second: $|1\rangle$ in Z basis

fifth: $|0\rangle$ in X basis $= |+\rangle$

b. No,
- third: either $|0\rangle$ or $|1\rangle$ (Alice must have used Z basis)
- fourth: either $|+\rangle$ or $|-\rangle$ (" X basis)

c. As Alice and Bob used the same basis,
if there was an error Eve must have used different basis, which is Z basis.

d. Considering the disagreement,
Alice must have sent $|1\rangle$ in X basis,
which is $|-\rangle$