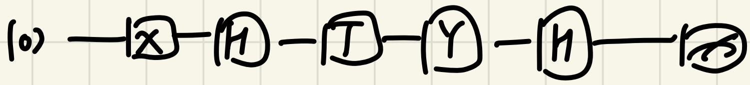
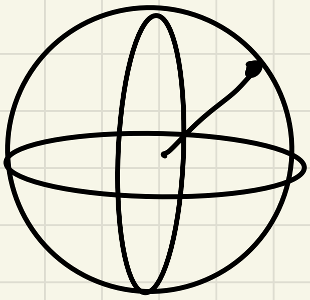
Problem Set 4

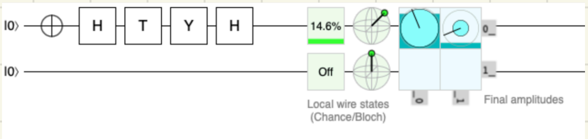
0.

(a)



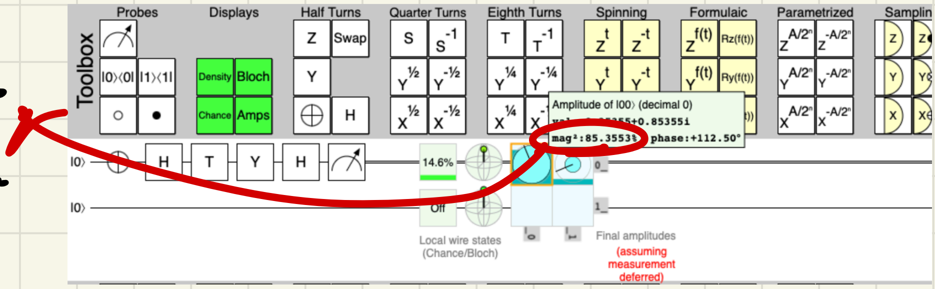
(b)





(c) $|0\rangle$: 85.3553%

$|1\rangle$: 14.6447%



2.

(a) $|0\rangle \Rightarrow \left|\frac{3+i\sqrt{3}}{4}\right|^2 = \frac{9+3}{16} = \frac{3}{4}$

$|1\rangle \Rightarrow \left|-\frac{1}{2}\right|^2 = \frac{1}{4}$

(b) $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

$$\frac{3+i\sqrt{3}}{4}|0\rangle - \frac{1}{2}|1\rangle = \frac{\alpha}{\sqrt{2}}(|0\rangle+|1\rangle) + \frac{\beta}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$\frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle = \frac{3+i\sqrt{3}}{4}|0\rangle - \frac{1}{2}|1\rangle$$

$$\alpha+\beta = \frac{3+i\sqrt{3}}{2\sqrt{2}}, \quad \alpha - \beta = \frac{-1}{\sqrt{2}}$$

$$2\alpha = \frac{1+i\sqrt{3}}{2\sqrt{2}}, \quad \alpha = \frac{1+i\sqrt{3}}{4\sqrt{2}}. \quad \therefore \beta = \frac{5+i\sqrt{3}}{4\sqrt{2}}$$

$$P(|+\rangle) = |\alpha|^2 = \left|\frac{1+i\sqrt{3}}{4\sqrt{2}}\right|^2 = \frac{4}{32} = \frac{1}{8}, \quad P(|-\rangle) = |\beta|^2 = \frac{28}{32} = \frac{7}{8}$$

(c) $|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

$$\frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}i|1\rangle = \frac{3+i\sqrt{3}}{4}|0\rangle - \frac{1}{2}|1\rangle$$

$$\left.\begin{aligned} &\frac{\alpha+\beta}{\sqrt{2}} = \frac{3+i\sqrt{3}}{4}, \quad \alpha+\beta = \frac{3+i\sqrt{3}}{2\sqrt{2}}. \\ &\frac{\alpha-\beta}{\sqrt{2}}i = -\frac{1}{2}, \quad \alpha-\beta = \frac{-1\times i}{\sqrt{2}\,i} = \frac{i}{\sqrt{2}} \end{aligned}\right) \quad 2\alpha = \frac{3+(\sqrt{3}+2)i}{2\sqrt{2}}.$$

$$\alpha = \frac{3+(\sqrt{3}+2)i}{4\sqrt{2}} \Rightarrow \beta = \frac{3+(\sqrt{3}-2)i}{4\sqrt{2}}$$

$$p(|i\rangle) = |\alpha|^2 = \frac{9+7+4\sqrt{3}}{32} = \frac{16+4\sqrt{3}}{32}. = \frac{4+\sqrt{3}}{8}$$

$$p(|-i\rangle) = |\beta^2| = \frac{9+7-4\sqrt{3}}{32} = \frac{16-4\sqrt{3}}{32} = \frac{4-\sqrt{3}}{8}$$

3. $|\psi\rangle = \frac{1}{\sqrt{6}}\left((1-2i)|0\rangle + |1\rangle\right)$.

(a) $\alpha|+\rangle + \beta|-\rangle$.

$$\frac{\alpha+\beta}{\sqrt{2}} = \frac{1-2i}{\sqrt{6}}, \quad \frac{\alpha-\beta}{\sqrt{2}} = \frac{1}{\sqrt{6}}. \Rightarrow \left.\begin{aligned} \alpha+\beta &= \frac{1-2i}{\sqrt{3}} \\ \alpha-\beta &= \frac{1}{\sqrt{3}} \end{aligned}\right) \quad 2\alpha = \frac{2-2i}{\sqrt{3}}, \quad \alpha = \frac{1-i}{\sqrt{3}}, \quad \beta = \frac{-i}{\sqrt{3}}$$

$$\therefore \underline{\frac{1-i}{\sqrt{3}}|+\rangle - \frac{i}{\sqrt{3}}|1\rangle}$$

(b) $\alpha|i\rangle + \beta|-i\rangle$

$$\frac{\alpha+\beta}{\sqrt{2}} = \frac{1-2i}{\sqrt{6}}, \quad \frac{\alpha-\beta}{\sqrt{2}}i = \frac{1}{\sqrt{6}}.$$

$$\alpha+\beta = \frac{1-2i}{\sqrt{3}}, \quad \alpha-\beta = \frac{1}{i\sqrt{3}} = -\frac{i}{\sqrt{3}}$$

$$\Rightarrow 2\alpha = \frac{1-3i}{\sqrt{3}}, \quad \alpha = \frac{1-3i}{2\sqrt{3}}. \quad \beta = \frac{1-i}{2\sqrt{3}}$$

$$\therefore \underline{\frac{1-3i}{2\sqrt{3}}|i\rangle + \frac{1-i}{2\sqrt{3}}|-i\rangle}$$

4.

$$HTU|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{pmatrix}\begin{pmatrix}\frac{\sqrt{2}-i}{2} & \frac{1}{2}\\ -\frac{1}{2} & \frac{\sqrt{2}+i}{2}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{pmatrix}\begin{pmatrix}\sqrt{2}-i & 1\\ -1 & \sqrt{2}+i\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{pmatrix}\begin{pmatrix}\sqrt{2}-i\\ -1\end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\sqrt{2}-i\\ -e^{i\pi/4}\end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\sqrt{2}-i\\ -\frac{(1+i)}{\sqrt{2}}\end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix}\sqrt{2}-i-\frac{1+i}{\sqrt{2}}\\ \sqrt{2}-i+\frac{1+i}{\sqrt{2}}\end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix}\frac{2-i\sqrt{2}-1-i}{\sqrt{2}}\\ \frac{2-i\sqrt{2}+1+i}{\sqrt{2}}\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix}1-i(\sqrt{2}+1)\\ 3-i(\sqrt{2}-1)\end{pmatrix}$$

$$\therefore\ \underline{\frac{1-i(\sqrt{2}+1)}{4}|0\rangle + \frac{3-i(\sqrt{2}-1)}{4}|1\rangle}$$

5.

$$UU^{\dagger} = \frac{1}{2}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} = I.$$

$\therefore$ U is a quantum gate

$$-\ U|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = |i\rangle$$

$$-\ U|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) = |-i\rangle$$

6.

(a) $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}(1\ 0) + \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}(0\ 1) + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}(1\ 0) - \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}(0\ 1)$

$= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&1\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\1&0\end{pmatrix}-\begin{pmatrix}0&0\\0&1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = H$ gate

(b) $H^\dagger H = \frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2&0\\0&2\end{pmatrix} = I.$

$\therefore$ Yes, it's a quantum gate

7. $|0\rangle\langle 0| + |+\rangle\langle +| = \begin{pmatrix}1\\0\end{pmatrix}(1\ 0) + \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}(1/\sqrt{2}\ \ 1/\sqrt{2})$

$= \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}1/2&1/2\\1/2&1/2\end{pmatrix} = \begin{pmatrix}3/2&1/2\\1/2&1/2\end{pmatrix} \neq I$

$\therefore \{|0\rangle, |+\rangle\}$ is not a complete orthonormal basis

8. $|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\cancel{|1\rangle})$, $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\cancel{|1\rangle})$

$|0\rangle = \frac{|i\rangle + |-i\rangle}{\sqrt{2}}$. $\Rightarrow$ after y-basis $\frac{1}{\sqrt{2}}|i\rangle + \frac{1}{\sqrt{2}}|-i\rangle$.

(a) $p(i)\ p(0|i) + p(-i)\,p(0|-i)$

$|\langle 0|i\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2, \ |\langle 0|-i\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 \Rightarrow \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2} = \frac{1}{2}$

(b) $p(i)\,p(1|i) + p(-i)\,p(1|-i) \Rightarrow \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2} = \frac{1}{2}$

$|\langle 1|i\rangle|^2 = \frac{1}{2} \qquad \langle 1|-i\rangle^2 = \frac{1}{2}$

9.

(a) $HZH|0\rangle = HZ\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) = H\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix}0\\ 2\end{pmatrix} = |1\rangle$

$\therefore$ 100% $|1\rangle$, (1000 $|1\rangle$ s)

(b)

(bi) $HZ[Z\text{-m}]\underbrace{H|0\rangle}_{\Rightarrow \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)}$

$\boxed{500\ |0\rangle\text{s and } 500\ |1\rangle\text{s}}$

(bii) States that will enter Z-gate $\left\langle \begin{matrix}|0\rangle\\ |1\rangle\end{matrix} \right. \xrightarrow{Z} \left\langle \begin{matrix}|0\rangle\\ -|1\rangle\end{matrix} \right.$

(biii) $HZ\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right)$

500 $|0\rangle$s and 500 $|1\rangle$s. (still probabilities for both are $\frac{1}{2}$ ($\left|\frac{1}{\sqrt{2}}\right|^2 = \left|\frac{1}{\sqrt{2}}\right|^2$)

(biv) No, after Z-measurement the super position collapses

(bv)

$HZ\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right)$

$|0\rangle \quad -|1\rangle \Rightarrow H|0\rangle = |+\rangle$

$-H|1\rangle = -|-\rangle$

$p(|0\rangle) = p(Z\text{ input} = |0\rangle)(\langle 0|+\rangle) + p(Z\text{ input} = -|1\rangle)(\langle 0|-\rangle)$

$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$

$p(|1\rangle) = p(Z\text{ input} = |0\rangle)(\langle 1|+\rangle) + p(Z\text{ input} = (-|1\rangle))(\langle 1|-\rangle)$

$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$

c) during (b) happens a measurement, which breaks down the coherence of qubits in the middle of calculation. As coherence is lost, following Z and H cannot deterministically do it, as it did in (a).