Problem Set 5

1. (a) no coefficien for $|01\rangle$ $\Rightarrow$ $|\psi\rangle = \begin{bmatrix} 1/2 \\ 0 \\ i/\sqrt{2} \\ (\sqrt{3}+i)/4 \end{bmatrix}$

(b) $\langle\psi| = \frac{1}{2}\langle 00| - \frac{i}{\sqrt{2}}\langle 10| + \frac{\sqrt{3}-i}{4}\langle 11|$

$\Rightarrow \begin{bmatrix} \frac{1}{2} & 0 & -\frac{i}{\sqrt{2}} & \frac{\sqrt{3}-i}{4} \end{bmatrix}$

2. $|00\rangle = |0\rangle\otimes|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, $|01\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

$|10\rangle = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $|11\rangle = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

$|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|$

$= \begin{pmatrix}1\\0\\0\\0\end{pmatrix}(1\,0\,0\,0) + \begin{pmatrix}0\\1\\0\\0\end{pmatrix}(0\,1\,0\,0) + \begin{pmatrix}0\\0\\1\\0\end{pmatrix}(0\,0\,1\,0) + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}(0\,0\,0\,1)$

$= \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}$

$= \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix} = I.$

3.

$P(00) = \left|\frac{i}{\sqrt{10}}\right|^2 = \frac{1}{10}$, $P(10) = \left|\frac{e^{i\pi/16}}{\sqrt{10}}\right|^2 = \frac{1}{10}$

$P(01) = \left|\frac{1-2i}{\sqrt{10}}\right|^2 = \frac{5}{10}$, $P(11) = \left(\frac{\sqrt{3}}{\sqrt{10}}\right)^2 = \frac{3}{10}$

4.

$P(00) = \frac{1}{4}$, $P(01) = 1$, $P(10) = 2$, $P(11) = 1$.

$\Rightarrow \frac{1}{4} + 1 + 2 + 1 = \frac{17}{4}$

for the sum of the probabilities to be 1, $\frac{4}{17}$ should be multiplied to all the terms.

$\therefore$ $A = \frac{4}{17}$, $|\psi\rangle = \frac{4}{17}\left(\frac{1}{2}|00\rangle + i|01\rangle + \sqrt{2}|10\rangle - |11\rangle\right)$

5.

i) both left & right are $|0\rangle \Rightarrow \frac{1}{6}|000\rangle + \frac{1}{\sqrt{6}}|010\rangle$

$\frac{1}{36} + \frac{1}{6} = \frac{7}{36}$

ii) left: $|0\rangle$, right: $|1\rangle \Rightarrow \frac{1}{3\sqrt{2}}|001\rangle + \frac{1}{2}|011\rangle$

$\frac{1}{18} + \frac{1}{4} = \frac{11}{36}$

iii) left: $|1\rangle$, right: $|0\rangle \Rightarrow \frac{1}{6}|100\rangle + \frac{1}{6}|110\rangle$

$\frac{1}{3}$

iv) left & right: $|1\rangle \Rightarrow \frac{1}{3}|101\rangle + \frac{1}{\sqrt{3}}|111\rangle$

$\frac{1}{9} + \frac{1}{3} = \frac{4}{9}$

6.

(a) entangled, not factorized.

(b) $\frac{1}{\sqrt{2}}\{|1\rangle\otimes(|0\rangle+i|1\rangle)\} = |1\rangle\otimes\frac{|0\rangle+i|1\rangle}{\sqrt{2}}$

$\therefore$ product state

7.

(a) $\alpha\gamma=3,\ \alpha\delta=-\sqrt{3},\ \beta\gamma=\sqrt{3},\ \beta\delta=-1.$

$\alpha=\gamma=\sqrt{3}.\quad \delta=-1,\ \beta=1$

$\underline{\frac{1}{2}(\sqrt{3}|0\rangle+|1\rangle)\otimes\frac{1}{2}(\sqrt{3}|0\rangle-|1\rangle)}$ product state

(b) $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle\pm|1\rangle)$

$\frac{1}{\sqrt{3}}|0\rangle|+\rangle + \sqrt{\frac{2}{3}}|1\rangle|-\rangle$

$= \frac{1}{\sqrt{3}}|0\rangle\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right) + \sqrt{\frac{2}{3}}|1\rangle\left(\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right)$

$= \frac{1}{\sqrt{6}}(|00\rangle+|01\rangle) + \frac{1}{\sqrt{3}}(|10\rangle-|11\rangle)$

should be + to be factorized

not factorized

$\therefore$ entangled state

8.

(a) $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$H \otimes X = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & -1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

(b)

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1/4 \\ 1/2 \\ 1/\sqrt{2} \\ \sqrt{3}/4 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1/2 + \sqrt{3}/4 \\ 1/4 + 1/\sqrt{2} \\ 1/2 - \sqrt{3}/4 \\ 1/4 - 1/\sqrt{2} \end{pmatrix}$$

$$= \left(\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{4\sqrt{2}}\right)|00\rangle + \left(\frac{1}{4\sqrt{2}} + \frac{1}{2}\right)|01\rangle + \left(\frac{1}{2\sqrt{2}} - \frac{3}{4\sqrt{2}}\right)|10\rangle$$
$$+ \left(\frac{1}{4\sqrt{2}} - \frac{1}{2}\right)|11\rangle$$

$$= \frac{\sqrt{2}(2+\sqrt{3})}{8}|00\rangle + \frac{\sqrt{2}+4}{8}|01\rangle + \frac{\sqrt{2}(2-\sqrt{3})}{8}|10\rangle + \frac{\sqrt{2}-4}{8}|11\rangle$$

9.

(a)

A	B	C	$\vert\psi_1\rangle$	A'	B'	C'
0	0	0	000	0	0	0
0	0	1	001	0	0	1
0	1	0	010	0	1	1
0	1	1	011	0	1	0
1	0	0	101	1	0	1
1	0	1	100	1	0	0
1	1	0	111	1	1	0
1	1	1	110	1	1	1

(b) $c = (C \oplus A) \oplus B$

Same (with reversible one)

10.

i) $|+-\rangle$

$$\mathrm{CNOT}\,|+\rangle|-\rangle = (H\otimes H)\,\mathrm{CNOT}_{01}\,(H\otimes H)\,|+\rangle|-\rangle$$
$$= (H\otimes H)\,\mathrm{CNOT}_{01}\,|0\rangle|1\rangle$$
$$= (H\otimes H)\,|1\rangle|1\rangle = |-\rangle|-\rangle$$

ii) $|-\rangle|+\rangle$

$$\mathrm{CNOT}\,|-\rangle|+\rangle = (H\otimes H)\,\mathrm{CNOT}_{01}\,(H\otimes H)\,|-\rangle|+\rangle$$
$$= (H\otimes H)\,\mathrm{CNOT}_{01}\,|1\rangle|0\rangle$$
$$= (H\otimes H)\,|1\rangle|0\rangle = |-\rangle|+\rangle$$

iii) $|-\rangle|-\rangle$

$$(H\otimes H)\,\mathrm{CNOT}_{01}\,(H\otimes H)(|-\rangle|-\rangle)$$
$$= (H\otimes H)\,\mathrm{CNOT}_{01}\,|1\rangle|1\rangle$$
$$= (H\otimes H)\,|0\rangle|1\rangle$$
$$= |+\rangle|-\rangle$$

11.

(a)
$$MS = \frac{1}{\sqrt{2}} \begin{array}{c|c} & |00\rangle\ \ |01\rangle\ \ |10\rangle\ \ |11\rangle \\ \begin{array}{c} 00\rightarrow \\ 01\rightarrow \\ 10\rightarrow \\ 11\rightarrow \end{array} & \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

(b) Mathematica

```
X  = PauliMatrix[1];
UMS[θ_] := MatrixExp[-I (θ/2) KroneckerProduct[X, X]];

u = UMS[Pi/2];

MatrixPower[u, 8] // Simplify
```