Problem set 3

1. $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \exp(i\phi)\sin\left(\frac{\theta}{2}\right)|1\rangle$

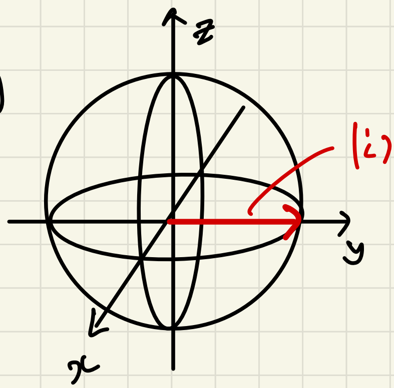
(a) $\cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{2}}$, $\theta = \frac{\pi}{2}$.

$\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\exp(i\phi)\underbrace{\sin\left(\frac{\pi}{4}\right)}_{\cancel{\frac{1}{\sqrt{2}}}} = \frac{i}{\cancel{\sqrt{2}}} \Rightarrow e^{i\phi} = i = e^{i\pi/2}$

$\therefore \phi = \pi/2$

(b)



2. (a) $\alpha = \frac{1-i}{2\sqrt{2}}$, $\beta = \frac{\sqrt{3}}{2}$. $\alpha^2+\beta^2 = \frac{2}{8}+\frac{3}{4} = 1$.

$\underbrace{\cos\left(\frac{\theta}{2}\right) = \alpha = \left|\frac{1-i}{2\sqrt{2}}\right| = \frac{1}{2}}$. $\theta = \frac{2}{3}\pi$.

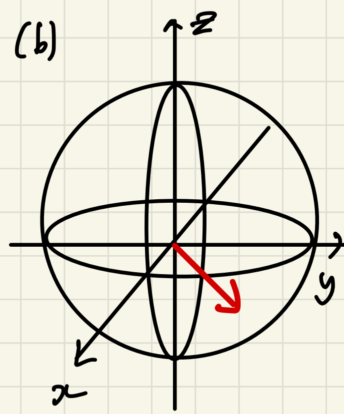
$\underbrace{\arg(z) = \arctan\left(\frac{-1}{1}\right) = -\frac{\pi}{4}}$.

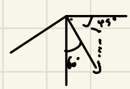
$\Rightarrow |\psi\rangle = e^{-\pi/4}\left(\underbrace{\cos\left(\frac{1}{3}\pi\right)}_{\frac{1}{2}}|0\rangle + e^{i\phi}\underbrace{\sin\left(\frac{1}{3}\pi\right)}_{\frac{\sqrt{3}}{2}}|1\rangle\right)$

$\therefore \phi = \frac{\pi}{4}$

$\left(\frac{2}{3}\pi, \frac{\pi}{4}\right)$

(b)





3. $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, $z = \cos\theta$

(a) $\theta = \phi = \frac{\pi}{2}$, $x = 0$, $y = 1$, $z = 0$

$(0, 1, 0)$

(b) $\theta = \frac{2}{3}\pi$, $\phi = \frac{\pi}{4}$. $x = \sin\frac{2}{3}\pi \cdot \cos\frac{\pi}{4} = \cos(-\frac{\pi}{6}) \cdot \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}}$

$y = \sin\frac{2}{3}\pi \cdot \sin\frac{\pi}{4} = \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}}$

$z = \cos\frac{2}{3}\pi = \sin(-\frac{\pi}{6}) = -\frac{1}{2}$

$\left(\frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2}\right)$

4.

(a) polarization, decoherence (b) electric fields

(c) laser beams (d) molecule, radio-frequency

(e) discrete energy levels, atomic nucleus.

(f) Quantum information, hyperfine

(g) Spin, microwave, optical (h) charge, flux, phase

5.

(a) trapped Ion. (b) atomic energy level (c) spin (d) up

(e) down

6.

(a) $U = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}+i}{4} \\ \frac{\sqrt{3}+i}{4} & \frac{-\sqrt{3}-3i}{4} \end{bmatrix}$

$U|\psi\rangle = \alpha U|0\rangle + \beta U|1\rangle$

$= \left(\frac{\sqrt{3}}{2}\alpha + \frac{\sqrt{3}+i}{4}\beta\right)|0\rangle + \left(\frac{\sqrt{3}+i}{4}\alpha - \frac{\sqrt{3}+3i}{4}\beta\right)|1\rangle$

(b) $\left|\frac{\sqrt{3}}{2}\alpha + \frac{\sqrt{3}+i}{4}\beta\right|^2 + \left|\frac{\sqrt{3}+i}{4}\alpha - \frac{\sqrt{3}+3i}{4}\beta\right|^2$

$= \frac{3}{4}|\alpha|^2 + \frac{4}{16}|\beta|^2 + \frac{4}{16}|\alpha|^2 + \frac{12}{16}|\beta|^2$

$= |\alpha|^2 + |\beta|^2 = 1$ Valid

7.

(a) $|00\rangle \to |01\rangle$
$|01\rangle \to |11\rangle$
$|10\rangle \to |00\rangle$
$|11\rangle \to |10\rangle$
unique = reversible $\Rightarrow$ Valid

(b) $|00\rangle \to |00\rangle$
$|01\rangle \to |00\rangle$
$|10\rangle \to |10\rangle$
$|11\rangle \to |11\rangle$
X unique $\Rightarrow$ invalid

8.

$$Z^{217} X^{101} Y^{50} = Z(Z^2)^{108} X (X^2)^{50} (Y^2)^{25}$$

$$= ZX$$

$$ZX(\alpha|0\rangle + \beta|1\rangle) = Z(\alpha|1\rangle + \beta|0\rangle) = -\alpha|1\rangle + \beta|0\rangle$$

9.

(a) $H|-\rangle = H\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{\sqrt{2}}\left(H|0\rangle - H|1\rangle\right) = \frac{1}{\sqrt{2}}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}} - \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$

$$= \frac{1}{\cancel{\sqrt{2}}} \cdot \frac{\cancel{2}|1\rangle}{\cancel{\sqrt{2}}} = |1\rangle$$

(b) $H|-i\rangle = \frac{1}{\sqrt{2}} H\left(|0\rangle - i|1\rangle\right) = \frac{1}{\sqrt{2}}\left(H|0\rangle - iH|1\rangle\right)$

$$= \frac{1}{\sqrt{2}}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}} - i\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left(\frac{1-i}{\sqrt{2}}|0\rangle + \frac{1+i}{\sqrt{2}}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}|0\rangle + e^{i\pi/4}|1\rangle\right)$$

$$= e^{-i\pi/4}/\sqrt{2}\left(|0\rangle + e^{i\pi/2}|1\rangle\right) = e^{i\pi/4}/\sqrt{2}\left(|0\rangle + i|1\rangle\right)$$

$$= |i\rangle$$

10. $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$

(a) $HTHTH|0\rangle = \frac{1}{\sqrt{2}} HTHT\left(|0\rangle + |1\rangle\right)$

$$= \frac{1}{\sqrt{2}} HTH\left(|0\rangle + e^{i\pi/4}|1\rangle\right)$$

$$= \frac{1}{2} HT\left((1+e^{i\pi/4})|0\rangle + (1-e^{i\pi/4})|1\rangle\right)$$

$$= H\left(\frac{1+e^{i\pi/4}}{2}|0\rangle + \frac{e^{i\pi/4}(1-e^{i\pi/4})}{2}|1\rangle\right)$$

$$= \frac{1+2\cdot e^{i\pi/4} - i}{2\sqrt{2}}|0\rangle + \frac{1+i}{2\sqrt{2}}|1\rangle$$

(b) $p(|0\rangle) = 1 - \frac{1}{4} = \frac{3}{4}$

$p(|1\rangle) = \frac{2}{8} = \frac{1}{4}$