PS 8

1.

(a) $P(\uparrow_z) + P(\downarrow_z)$ should be 1.

$|\psi_{in}\rangle = \alpha|\uparrow_z\rangle + \beta|\downarrow_z\rangle$,

$\alpha = 6, \beta = -i$, $|\alpha|^2 + |\beta|^2 = 37$.

$\therefore$ normalization constant $= \frac{1}{\sqrt{37}}$,

normalized state vector $= \frac{1}{\sqrt{37}}(6|\uparrow_z\rangle - i|\downarrow_z\rangle)$

(b) $P(S_z = +\frac{\hbar}{2}) = \left|\frac{6}{\sqrt{37}}\right|^2 = \frac{36}{37}$

$P(S_z = -\frac{\hbar}{2}) = \left|\frac{-i}{\sqrt{37}}\right|^2 = \frac{1}{37}$

(c) $\frac{36}{37} + \frac{1}{37} = \frac{37}{37}$

2.

(a) The classical prediction

: In a field gradient one would get one broadened continuous streak, not separate beams.

(b) Empirical result.

: The beam actually splits into two discrete spots with 50:50 ratio. We assume the spin S_z is quantized, not being continuous.

3.

$|+_\theta\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle$

$|-_\theta\rangle = -\sin\frac{\theta}{2}|\uparrow\rangle + \cos\frac{\theta}{2}|\downarrow\rangle$

(a) (i) $\langle +_{\theta_2} | +_{\theta_1} \rangle = \left(\cos\frac{\theta_2}{2}\langle\uparrow| + \sin\frac{\theta_2}{2}\langle\downarrow|\right)\left(\cos\frac{\theta_1}{2}|\uparrow\rangle + \sin\frac{\theta_1}{2}|\downarrow\rangle\right)$

$$= \cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2}\langle\uparrow|\uparrow\rangle + \cancel{\cos\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\langle\uparrow|\downarrow\rangle}$$
$$+ \sin\frac{\theta_2}{2}\cancel{\cos\frac{\theta_1}{2}\langle\downarrow|\uparrow\rangle} + \sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2}\langle\downarrow|\downarrow\rangle$$
$$= \cos\frac{\theta_2}{2}\cdot\cos\frac{\theta_1}{2} + \sin\frac{\theta_2}{2}\cdot\sin\frac{\theta_1}{2}$$
$$= \cos\left(\frac{\theta_1}{2} - \frac{\theta_2}{2}\right) \ldots$$

$\to \cos(A-B) = \mathbb{R}(e^{i(A-B)})$
$= \mathbb{R}(e^{iA}e^{-iB})$
$= \mathbb{R}((\cos A + i\sin A)(\cos B - i\sin B))$
$= \cos A\cos B + \sin A\sin B$

(hint using GPT)

Here $\theta_1 = -15°$,
$\theta_2 = 35°$.

$$\Rightarrow \cos\left(\frac{\theta_1}{2} - \frac{\theta_2}{2}\right)$$
$$= \cos\left(\frac{\theta_2}{2} - \frac{\theta_1}{2}\right) = \cos\frac{50°}{2}$$
$$= \underline{\cos 25°}.$$

(ii) $\langle +\theta_3 | +\theta_2 \rangle$

with same logic, with formula $\cos\left(\frac{\theta_3}{2} - \frac{\theta_2}{2}\right)$,

$$\Rightarrow \cos\frac{55°}{2} = \cos 27.5°$$

$$P(+_{\theta_3}|+_{\theta_1}) = |\langle +_{\theta_2}|+_{\theta_1}\rangle|^2 |\langle +_{\theta_3}|+_{\theta_2}\rangle|^2$$
$$= \cos^2 25° \times \cos^2 27.5° \simeq 0.646$$

(b) $P(-_{\theta_3}|+_{\theta_1}) = \underbrace{|\langle +_{\theta_2}|+_{\theta_1}\rangle|^2}_{\cos^2 25°} \underbrace{|\langle -_{\theta_3}|+_{\theta_2}\rangle|^2}_{\sin^2 27.5} \Rightarrow \cos^2 25 \times \sin^2 27.5$

$\simeq 0.175$

$$\left|\left(-\sin\frac{\theta_3}{2}\langle\uparrow| + \cos\frac{\theta_3}{2}\langle\downarrow|\right)\left(\cos\frac{\theta_2}{2}|\uparrow\rangle + \sin\frac{\theta_2}{2}|\downarrow\rangle\right)\right|^2$$
$$= \left|-\sin\frac{\theta_3}{2}\cos\frac{\theta_2}{2} + \cos\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\right|^2$$
$$= \left|\sin\left(\frac{\theta_2}{2} - \frac{\theta_3}{2}\right)\right|^2 \ldots \to$$
$$= \underline{\sin^2 27.5}$$

$\sin(A-B) = \mathrm{Im}(e^{i(A-B)}) = \mathrm{Im}(e^{iA}\cdot e^{-iB})$
$\mathrm{Im}((\cos A + i\sin A)(\cos B - i\sin B))$
$= -\cos A\sin B + \sin A\cos B$

(c) It is because the particles that are $|-_{\theta_2}\rangle$ are neglected.

$|-\theta_1\rangle$ is neglectable as $|+_{\theta_1}\rangle$ is given as the starting state, but once the beam reaches the second analyzer it neglects the particles that go to $|-\theta_2\rangle$ and the rest goes on to the third analyzer.

4.

$$P(+_3|+_1) = \underbrace{P(+_2|+_1)}_{①}\underbrace{P(+_3|+_2)}_{②} = \frac{3}{8}$$

① $|\langle+_{\theta_2}|+_{\theta_1}\rangle|^2$, $\theta_2 = 90^\circ$, $\theta_1 = 30^\circ$

$= \left|\cos\left(\frac{\theta_2-\theta_1}{2}\right)\right|^2 = |\cos 30^\circ|^2 = \frac{3}{4}$.

② $|\langle+\theta_3|+\theta_2\rangle|^2$, $\theta_3 = 0^\circ$, $\theta_2 = 90^\circ$.

$= \left|\cos\left(\frac{\theta_2-\theta_3}{2}\right)\right|^2 = |\cos 45^\circ|^2 = \frac{1}{2}$.

$\frac{3}{8}$

5.

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\uparrow_z\rangle + |\downarrow_z\downarrow_z\rangle\right)$$

(a) $|\uparrow_{120^\circ}\rangle$, 100%. (= probability : 1)

Spin is in an entangled state and we already measured $|\uparrow_{120^\circ}\rangle$ for A.

(b) $|\uparrow_{120^\circ}\rangle$, for the entanglement, after z-measurement, $P(\uparrow) = \frac{1}{4}$

$|\uparrow_{120^\circ}\rangle = \cos 60^\circ|\uparrow_z\rangle + \sin 60^\circ|\downarrow_z\rangle$

$= \frac{1}{2}|\uparrow_z\rangle + \frac{\sqrt{3}}{2}|\downarrow_z\rangle$

(c) When both analyzers are aligned, entangled state brings perfect correlation.
But when there is misalignment by angle θ, the transition probability is $\cos^2\left(\frac{\theta}{2}\right)$

(I could not find the slide we looked through last week on Canvas.)