

Weekly Problem Set #9 – Week 10

PLEASE SHOW ALL YOUR WORK FOR FULL CREDIT. Include intermediate steps in ways that others can understand or writing sentences that help to communicate your assumptions and logic. If you utilize any software tools or apps (e.g. Mathematica, Desmos, ChatGTP, etc.), you must transparently acknowledge your use of them in your HW submission. A subset of these problems will be graded for correctness. The rest of the problems will be graded for effort.

1. (4 pts). **Wong E6.1.** Product states, partially entangled vs. maximally entangled states.
2. (4 pts). **Wong E6.2.** Product states, partially entangled vs. maximally entangled states.
3. (4 pts). **Wong E6.3.** Product states, partially entangled vs. maximally entangled states.
4. (8 pts). **Consider this multi-qubit quantum circuit.**

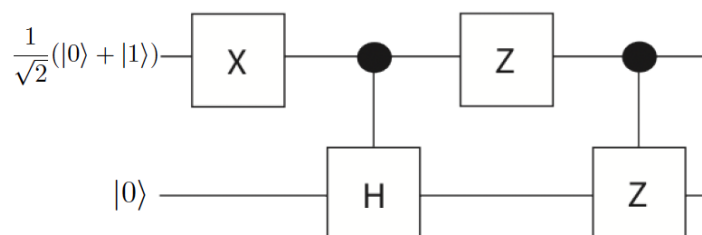


Fig. 1: This figure gives the input states and specifies a series of quantum gates.

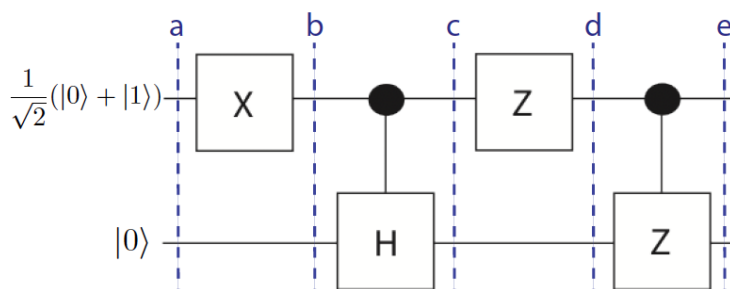
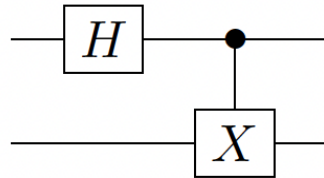


Fig. 2: This figure gives the input states and demarcates intermediate steps that should be specified in your submitted work.

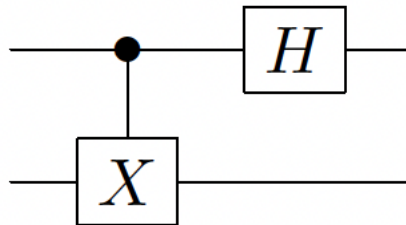
- a) How would you represent this operation in tensor notation?
- b) Find the output state. (Please work out on paper and tell us what the resultant quantum state is at each intermediate step: a, b, c, d, e).
- c) Is the output state entangled?
- d) What is the probability that a measurement on qubit 2 (after it has gone through the circuit) results in a 0?

5. (6 pts) A look at entangling (and disentangling) circuits.

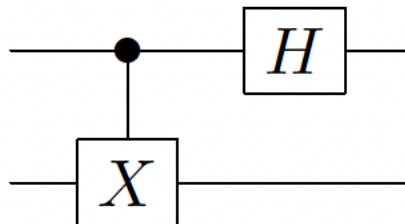
- a. Show that the circuit below takes each of $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$ and transforms it into each of the Bell states.



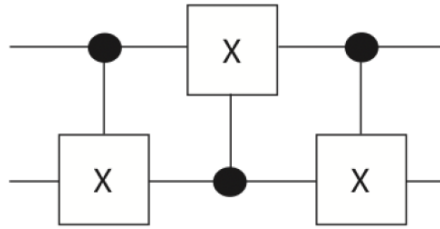
- b. Show that the circuit below (which is the same as the one above, but with the order of the gates reversed), returns the Bell states back to the computational basis states ($|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$). (That is, input the Bell state and show that the output is a computational basis state, thereby disentangling the two qubits.). This is what we call a Bell state measurement device.



- c. Show that the above circuit doesn't always function as a "disentangler." Start with the input state $|00\rangle$ and show what happens when apply this series of gates.



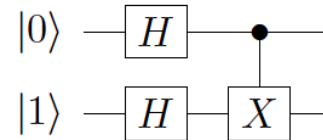
- i. What is the output state of this circuit?
 - ii. Is the output state in (ci) an entangled state? (Y or N) How do you know?
6. (5 pts). **Entanglement & Multi-qubit Gates.** What does the following circuit do to arbitrary two-qubit states (Note: we can do this by showing what happens to each of the computational basis states $|00\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$)? Use this information to propose a name for this circuit.



7. **(6 pts) Entanglement & Multi-qubit Gates.**

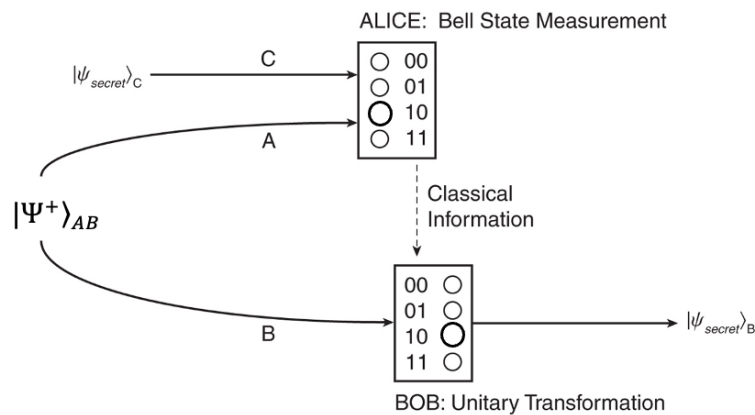
Consider this circuit with the given input state $|10\rangle$:

a. What is the state of the output qubits? (*Show your work.*)

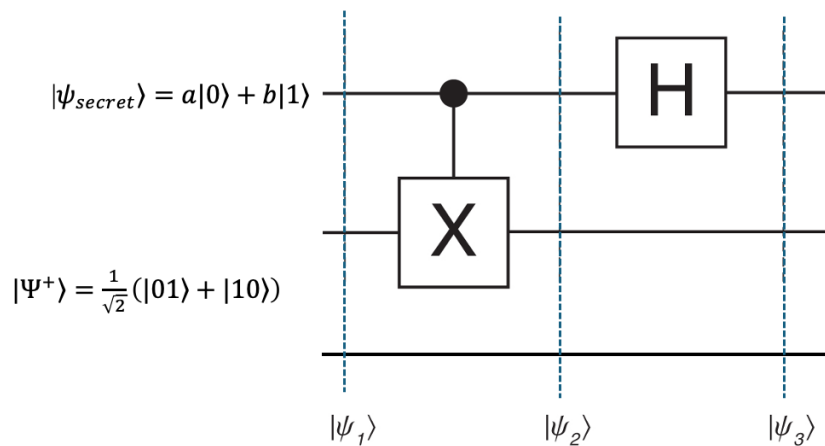


b. Is this output state (given in 6a) entangled? What is the probability that a measurement on the top qubit (after this circuit has acted) results in a 0? What is the probability that a measurement of the entire final state results in $|10\rangle$?

8. **Quantum Teleportation (11 pts).** Let's show that Alice and Bob could have used a different entangled state, say the following Bell State: $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ as the starting point for their experiments.



- a. Recall that Alice's lab starts with an arbitrary "secret state" qubit given by $|\psi_{secret}\rangle = a|0\rangle + b|1\rangle$ (which you can assume has been appropriately normalized). Alice's lab also has one of the two initially entangled qubits in the $|\Psi^+\rangle$ state. (Bob has the other one). Alice makes a Bell State Measurement on the two qubits in her lab, using the following circuit. ****Use the Wong notation that the 2 leftmost qubits are Alice's and the rightmost qubit is Bob's.**



In tensor notation what is the quantum state of the three-qubit system at:

- i. $|\psi_1\rangle$?
 - ii. $|\psi_2\rangle$?
 - iii. $|\psi_3\rangle$?
- b. Then Alice makes a measurement of the two qubits in her lab.
- i. If Alice measures 00, what quantum state does Bob's qubit collapse to?
 - ii. If Alice measures 01, what quantum state does Bob's qubit collapse to?
 - iii. If Alice measures 10, what quantum state does Bob's qubit collapse to?
 - iv. If Alice measures 11, what quantum state does Bob's qubit collapse to?
- c. Alice and Bob also need to have a code system that tells Bob what unitary transformations to make of his qubit to end up with the "secret" state.

If Alice sends...	What unitary transformation should Bob execute to end up with the "secret" state as his output?
00	
01	
10	
11	

9. **(6 pts) Quantum Cryptography.** Visit the QuVis simulations and play with the Quantum Cryptography (BB84 spin) simulation (see link in the ELMs Assignment description). This simulation will introduce you to a slightly different BB84 set-up that is then used in the following HW problem (#8). Work through the challenge problems. There is additional information in the simulation displays that are critical to solving these problems. Tip: you may need to click on the numbers in the bottom right corner to advance to the next question.
- a. (QuVis1) Assuming no eavesdropper has intervened, what sequence of outcomes could Bob have measured? Choose one or more.
 - b. (QuVis2) Assuming no eavesdropper has intervened, how many bits are there in Alice and Bob's shared key?
 - c. (QuVis3) Assuming no eavesdropper has intervened, what sequence of key bits would Alice and Bob have measured (most recent key bit first)?
 - d. (QuVis4) Alice and Bob decide to compare the bit shown to determine if Eve was intercepting. They find that they do not agree. For this bit, what basis must Eve have used for her measurement?
 - e. (QuVis5) Alice and Bob decide to compare the bit shown to determine if Eve was intercepting. They find that they do not agree. For this bit, what value did Eve obtain in her measurement?
 - f. (QuVis6) Alice and Bob decide to compare the bit shown to determine if Eve was intercepting. They find that they do agree. For this bit, what basis must Eve have used for her measurement?
10. **(6 pts) Quantum Cryptography.** Consider the following scenario where Alice and Bob are using the BB84 quantum key distribution protocol that was described in class. Alice sends 5 qubits to Bob. Bob measures in the following bases (in this order): Z Z X Z X. The elements of his key are 0 1 1 0 0. Alice tells Bob that he measured the first, second, and fifth qubits in the "correct" basis.
- a. Based on the above information (and assuming there was no eavesdropper), what is the state of the first, second, and fifth qubits that Alice sent?
 - b. Is it possible to determine the states of the third and fourth qubits Alice sent? If so, what are the states of each qubit? If not, what are the possible states for the third and fourth qubits?

Alice and Bob decide to compare all 5 bits of their key to determine if Eve was intercepting. (They recognize that this will leave them with no key.) They find that they do not agree on the last bit of the key.

- c. What basis must Eve have measured the fifth qubit in?
- d. Based on this new information, what is the state of the fifth qubit that Alice sent?