Abstract

Equations of motion for a gyrostat comprised of two rigid bodies (a carrier with a lateral plane of symmetry and an axially symmetric a rotor) are derived in two different ways. In the first approach, each rigid body is treated separately as each having their own mass and inertia. In the second approach, the symmetry of the rotor is used to combine the inertias in such a way that simplifies the derivation of the equations of motion. It is shown that the two approaches are equivalent.

Model Description

Consider a rigid body A, of mass m_a, along with a dextral set of unit vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, fixed to A, with $\boldsymbol{a}_3 \triangleq \boldsymbol{a}_1 \times \boldsymbol{a}_2$. Let A^* denote the center of mass of A. Let the inertia dyadic of A relative to its mass center A^* be:

$$\boldsymbol{I}^{A/A^*} = I_{11}\boldsymbol{a}_1\boldsymbol{a}_1 + I_{22}\boldsymbol{a}_2\boldsymbol{a}_2 + I_{33}\boldsymbol{a}_3\boldsymbol{a}_3 + I_{13}\boldsymbol{a}_1\boldsymbol{a}_3 + I_{13}\boldsymbol{a}_3\boldsymbol{a}_1$$

The plane spanned by the $\boldsymbol{a}_1$ and $\boldsymbol{a}_3$ unit vectors is a plane of symmetry.

Consider a second rigid body B, of mass m_b, along with a dextral set of unit vectors $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$, fixed to B, with $\boldsymbol{b}_3 \triangleq \boldsymbol{b}_1 \times \boldsymbol{b}_2$. Let B^* denote the center of mass of B. Let the inertia dyadic of B relative to its mass center B^* be:

$$\boldsymbol{I}^{B/B^*} = I\boldsymbol{b}_1\boldsymbol{b}_1 + J\boldsymbol{b}_2\boldsymbol{b}_2 + I\boldsymbol{b}_3\boldsymbol{b}_3$$

B is axially symmetric about the $\boldsymbol{b}_2$ axis.

Let B be rigidly fixed to A such that the mass center B^* is located in the plane of symmetry of A so that:

$$\boldsymbol{r}^{B^*/A^*} = l_1\boldsymbol{a}_1 + l_3\boldsymbol{a}_3$$

where l_1, l_3 are real constants.

Denote the center of mass of the system as AB^*. The position of A^* and B^* relative to the system mass center AB^* is:

$$\begin{aligned}
\boldsymbol{r}^{A^*/AB^*} &= -\frac{m_b}{m_a + m_b}(l_1\boldsymbol{a}_1 + l_3\boldsymbol{a}_3) \\
\boldsymbol{r}^{B^*/AB^*} &= \frac{m_a}{m_a + m_b}(l_1\boldsymbol{a}_1 + l_3\boldsymbol{a}_3)
\end{aligned}$$

B is oriented relative to A by means of a revolute joint whose axis is aligned with $\boldsymbol{a}_2$ and passes through B^*. Because of the inertial symmetry of the rotor, the central inertia dyadic of B can be expressed equivalently in the A frame:

$$\boldsymbol{I}^{B/B^*} = I\boldsymbol{a}_1\boldsymbol{a}_1 + J\boldsymbol{a}_2\boldsymbol{a}_2 + I\boldsymbol{a}_3\boldsymbol{a}_3$$

A couple of torque T is applied between A and B about the axis of rotation $\boldsymbol{a}_2$. If A were held fixed, a positive torque T would cause B to rotate relative to A in a counterclockwise direction about the $\boldsymbol{a}_2$ axis.

Equations of motion

Kinematics

The system is completely described by seven generalized coordinates: three which locate the system in inertial space, three which orient A in inertial space, and one which orients B relative to A.

The orientation of A relative to the inertial frame N is given by a sequence of rotations (q_1, q_2, q_3) 3-1-2 Euler angles. The direction cosine matrix relating the unit vectors fixed in N and A is shown in Table 1.

	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$
$\boldsymbol{n}_1$	$c_1c_3 - s_1s_2s_3$	$-s_1c_2$	$c_1s_3 + c_3s_1s_2$
$\boldsymbol{n}_2$	$s_1c_3 + c_1s_2s_3$	c_1c_2	$s_1s_3 - c_1c_3s_2$
$\boldsymbol{n}_3$	$-c_2s_3$	s_2	c_2c_3

Table 1: Direction cosine matrix relating unit vectors of A and N

B is oriented relative to A by first aligning $\boldsymbol{b}_i$ with $\boldsymbol{a}_i$ $(i = 1, 2, 3)$ then performing a right handed rotation of B by an angle q_4 about the $\boldsymbol{a}_2$ axis.

Let the mass center AB^* be located relative to the inertial origin N^* by:

$$\boldsymbol{r}^{AB^*/N^*} = q_5\boldsymbol{n}_1 + q_6\boldsymbol{n}_2 + q_7\boldsymbol{n}_3$$

For sake of symbolic brevity, introduce the following terms:

$$l_{1a} = -\frac{l_1 m_b}{m_a + m_b} \qquad l_{3a} = -\frac{l_3 m_b}{m_a + m_b}$$
$$l_{1b} = \frac{l_1 m_a}{m_a + m_b} \qquad l_{3b} = \frac{l_3 m_a}{m_a + m_b}$$

Let $\boldsymbol{\omega}^A$ and $\boldsymbol{\omega}^B$ denote the angular velocity of A and B relative to the inertial frame, respectively. Let $\boldsymbol{v}^{A^*}$, $\boldsymbol{v}^{B^*}$, and $\boldsymbol{v}^{AB^*}$ denote the velocity of A^*, B^*, and AB^* relative to the inertial frame, respectively. Define the following generalized speeds:

$$u_i \triangleq \boldsymbol{\omega}^A \cdot \boldsymbol{a}_i \qquad (i = 1, 2, 3) \tag{1}$$
$$u_4 \triangleq \boldsymbol{\omega}^B \cdot \boldsymbol{a}_2 \tag{2}$$
$$u_i \triangleq \boldsymbol{v}^{AB^*} \cdot \boldsymbol{n}_{i-4} \qquad (i = 5, 6, 7) \tag{3}$$

The above generalized speeds result in the following angular velocities and velocities:

$$\begin{aligned}
\boldsymbol{\omega}^A &= u_1\boldsymbol{a}_1 + u_2\boldsymbol{a}_2 + u_3\boldsymbol{a}_3 \\
\boldsymbol{\omega}^B &= u_1\boldsymbol{a}_1 + u_4\boldsymbol{a}_2 + u_3\boldsymbol{a}_3 \\
\boldsymbol{v}^{AB^*} &= u_5\boldsymbol{n}_1 + u_6\boldsymbol{n}_2 + u_7\boldsymbol{n}_3 \\
\boldsymbol{v}^{A^*} &= \boldsymbol{v}^{AB^*} + \boldsymbol{\omega}^A \times \boldsymbol{r}^{A^*/AB^*} \\
&= u_5\boldsymbol{n}_1 + u_6\boldsymbol{n}_2 + u_7\boldsymbol{n}_3 + l_{3a}u_2\boldsymbol{a}_1 + (l_{1a}u_3 - l_{3a}u_1)\boldsymbol{a}_2 - l_{1a}u_2\boldsymbol{a}_3 \\
\boldsymbol{v}^{B^*} &= \boldsymbol{v}^{AB^*} + \boldsymbol{\omega}^A \times \boldsymbol{r}^{B^*/AB^*} \\
&= u_5\boldsymbol{n}_1 + u_6\boldsymbol{n}_2 + u_7\boldsymbol{n}_3 + l_{3b}u_2\boldsymbol{a}_1 + (l_{1b}u_3 - l_{3b}u_1)\boldsymbol{a}_2 - l_{1b}u_2\boldsymbol{a}_3
\end{aligned}$$

All of the above velocities are relative to the inertial frame, and of particular interest here is that the angular velocity expressions for A and B only in their $\boldsymbol{a}_2$ component, which is the axis of rotation of B relative to A.

By inspection of the angular velocity expressions for A and B, and the velocity expressions for A^* and B^*, the partial angular velocities and partial velocities can be determined. They are simply the vector coefficients of the generalized speeds, which appear linearly in the velocity and angular velocity expressions. They are shown in Table 2.

$\boldsymbol{\omega}_1^A = \boldsymbol{a}_1$	$\boldsymbol{\omega}_2^A = \boldsymbol{a}_2$	$\boldsymbol{\omega}_3^A = \boldsymbol{a}_3$	$\boldsymbol{\omega}_4^A = \boldsymbol{0}$	$\boldsymbol{\omega}_5^A = \boldsymbol{0}$	$\boldsymbol{\omega}_6^A = \boldsymbol{0}$	$\boldsymbol{\omega}_7^A = \boldsymbol{0}$
$\boldsymbol{\omega}_1^B = \boldsymbol{a}_1$	$\boldsymbol{\omega}_2^B = \boldsymbol{0}$	$\boldsymbol{\omega}_3^B = \boldsymbol{a}_3$	$\boldsymbol{\omega}_4^B = \boldsymbol{a}_2$	$\boldsymbol{\omega}_5^B = \boldsymbol{0}$	$\boldsymbol{\omega}_6^B = \boldsymbol{0}$	$\boldsymbol{\omega}_7^B = \boldsymbol{0}$
$\boldsymbol{v}_1^{AB^*} = \boldsymbol{0}$	$\boldsymbol{v}_2^{AB^*} = \boldsymbol{0}$	$\boldsymbol{v}_3^{AB^*} = \boldsymbol{0}$	$\boldsymbol{v}_4^{AB^*} = \boldsymbol{0}$	$\boldsymbol{v}_5^{AB^*} = \boldsymbol{n}_1$	$\boldsymbol{v}_6^{AB^*} = \boldsymbol{n}_2$	$\boldsymbol{v}_7^{AB^*} = \boldsymbol{n}_3$
$\boldsymbol{v}_1^{A^*} = -l_{3a}\boldsymbol{a}_2$	$\boldsymbol{v}_2^{A^*} = l_{3a}\boldsymbol{a}_1 - l_{1a}\boldsymbol{a}_3$	$\boldsymbol{v}_3^{A^*} = l_{1a}\boldsymbol{a}_2$	$\boldsymbol{v}_4^{A^*} = \boldsymbol{0}$	$\boldsymbol{v}_5^{A^*} = \boldsymbol{n}_1$	$\boldsymbol{v}_6^{A^*} = \boldsymbol{n}_2$	$\boldsymbol{v}_7^{A^*} = \boldsymbol{n}_3$
$\boldsymbol{v}_1^{B^*} = -l_{3b}\boldsymbol{a}_2$	$\boldsymbol{v}_2^{B^*} = l_{3b}\boldsymbol{a}_1 - l_{1b}\boldsymbol{a}_3$	$\boldsymbol{v}_3^{B^*} = l_{1b}\boldsymbol{a}_2$	$\boldsymbol{v}_4^{B^*} = \boldsymbol{0}$	$\boldsymbol{v}_5^{B^*} = \boldsymbol{n}_1$	$\boldsymbol{v}_6^{B^*} = \boldsymbol{n}_2$	$\boldsymbol{v}_7^{B^*} = \boldsymbol{n}_3$

Table 2: Partial angular velocities and partial velocities

These definitions, along with (1, 2, 3) yield the following kinematic differential equations:

$$
\begin{aligned}
\dot{q}_1 &= -s_3u_1/c_2 + c_3u_3/c_2 \\
\dot{q}_2 &= c_3u_1 + s_3u_3 \\
\dot{q}_3 &= t_2s_3u_1 + u_2 - t_2c_3u_3 \\
\dot{q}_4 &= -u_2 + u_4 \\
\dot{q}_5 &= u_5 \\
\dot{q}_6 &= u_6 \\
\dot{q}_7 &= u_7
\end{aligned}
$$

While these equations are necessary for kinematics, they aren't too interesting because all the coordinates are ignorable with respect to the dynamic differential equations.

Dynamics

There are two contributing forces and torques acting on the system, the first arising from gravity, the second arising from the applied torque between the two bodies.

$$
\begin{aligned}
\boldsymbol{G}_{A^*} &= m_a g \boldsymbol{n}_3 \\
\boldsymbol{G}_{B^*} &= m_b g \boldsymbol{n}_3 \\
\boldsymbol{T}_A &= -T\boldsymbol{a}_2 \\
\boldsymbol{T}_B &= T\boldsymbol{a}_2
\end{aligned}
$$

To compute the inertia forces of the system, we need to form the angular accelerations of A and B as well as accelerations of the points AB^*, A^*, and B^*. The angular accelerations relative to the inertial frame are:

$$
\begin{aligned}
\boldsymbol{\alpha}^A &= \dot{u}_1\boldsymbol{a}_1 + \dot{u}_2\boldsymbol{a}_2 + \dot{u}_3\boldsymbol{a}_3 \\
\boldsymbol{\alpha}^B &= (u_2u_3 - u_3u_4 + \dot{u}_1)\boldsymbol{a}_1 + \dot{u}_4\boldsymbol{a}_2 + (u_1u_4 - u_1u_2 + \dot{u}_3)\boldsymbol{a}_3
\end{aligned}
$$

The accelerations of AB^*, A^*, and B^*, relative to the inertial origin are:

$$\begin{aligned}
\boldsymbol{a}^{AB^*} &= \dot{u}_5\boldsymbol{n}_1 + \dot{u}_6\boldsymbol{n}_2 + \dot{u}_7\boldsymbol{n}_3 \\
\boldsymbol{a}^{A^*} &= \dot{u}_5\boldsymbol{n}_1 + \dot{u}_6\boldsymbol{n}_2 + \dot{u}_7\boldsymbol{n}_3 + (l_{3a}(\dot{u}_2 + u_1u_3) - l_{1a}(u_2^2 + u_3^2))\boldsymbol{a}_1 \\
&\quad + (l_{1a}(\dot{u}_3 + u_1u_2) + l_{3a}(u_2u_3 - \dot{u}_1))\boldsymbol{a}_2 - (l_{1a}(\dot{u}_2 + u_1u_3) + l_{3a}(u_1^2 + u_2^2))\boldsymbol{a}_3 \\
\boldsymbol{a}^{B^*} &= \dot{u}_5\boldsymbol{n}_1 + \dot{u}_6\boldsymbol{n}_2 + \dot{u}_7\boldsymbol{n}_3 + (l_{3b}(\dot{u}_2 + u_1u_3) - l_{1b}(u_2^2 + u_3^2))\boldsymbol{a}_1 \\
&\quad + (l_{1b}(\dot{u}_3 + u_1u_2) + l_{3b}(u_2u_3 - \dot{u}_1))\boldsymbol{a}_2 - (l_{1b}(\dot{u}_2 + u_1u_3) + l_{3b}(u_1^2 + u_2^2))\boldsymbol{a}_3
\end{aligned}$$

The inertia force and inertia torque for the rigid bodies A and B are:

$$\begin{aligned}
\boldsymbol{R}_A^* &= -\, m_a\boldsymbol{a}^{A^*} \\
\boldsymbol{T}_A^* &= -\, \boldsymbol{\alpha}^A \cdot \boldsymbol{I}^{A/A^*} - \boldsymbol{\omega}^A \times \boldsymbol{I}^{A/A^*} \cdot \boldsymbol{\omega}^A \\
&= -\, (I_{11}\dot{u}_1 + I_{13}\dot{u}_3)\boldsymbol{a}_1 - I_{22}\dot{u}_2\boldsymbol{a}_2 - (I_{13}\dot{u}_1 + I_{33}\dot{u}_3)\boldsymbol{a}_3 \\
&\quad + (I_{13}u_1u_2 + (I_{33} - I_{22})u_2u_3)\boldsymbol{a}_1 + (I_{13}u_1^2 + (I_{33} - I_{11})u_1u_3 - I_{13}u_3^2)\boldsymbol{a}_2 \\
&\quad + ((I_{11} - I_{22})u_1u_2 + I_{13}u_2u_3)\boldsymbol{a}_3 \\
\boldsymbol{R}_B^* &= -\, m_b\boldsymbol{a}^{B^*} \\
\boldsymbol{T}_B^* &= -\, \boldsymbol{\alpha}^B \cdot \boldsymbol{I}^{B/B^*} - \boldsymbol{\omega}^B \times \boldsymbol{I}^{B/B^*} \cdot \boldsymbol{\omega}^B \\
&= -\, I(u_2u_3 - u_3u_4 + \dot{u}_1)\boldsymbol{a}_1 - J\dot{u}_4\boldsymbol{a}_2 - I(u_1u_4 - u_1u_2 + \dot{u}_3)\boldsymbol{a}_3 \\
&\quad + (Ju_3u_4 - Iu_3u_4)\boldsymbol{a}_1 + (Iu_1u_4 - Ju_1u_4)\boldsymbol{a}_3 \\
&= -\, (I\dot{u}_1 - Ju_3u_4 + Iu_2u_3)\boldsymbol{a}_1 - J\dot{u}_4\boldsymbol{a}_2 - (I\dot{u}_3 + Ju_1u_4 - Iu_1u_2)\boldsymbol{a}_3
\end{aligned}$$

Kane's equations of motion can now be written concisely as:

$$\boldsymbol{v}_r^{A^*} \cdot (\boldsymbol{G}_{A^*} + \boldsymbol{R}_A^*) + \boldsymbol{v}_r^{B^*} \cdot (\boldsymbol{G}_{B^*} + \boldsymbol{R}_B^*) + \boldsymbol{\omega}_r^A \cdot (\boldsymbol{T}_A + \boldsymbol{T}_A^*) + \boldsymbol{\omega}_r^B \cdot (\boldsymbol{T}_B + \boldsymbol{T}_B^*) = 0$$

where $r = 1, \ldots, 7$.

Explicitly, the first four (orientation related) dynamic differential equations are:

$$\begin{aligned}
-\left(I_{11} + m_a l_{3a}^2 + I + m_b l_{3b}^2\right)\dot{u}_1 - (I_{13} - m_a l_{1a} l_{3a} - m_b l_{1b} l_{3b})\,\dot{u}_3 &= (I_{13} - m_a l_{1a} l_{3a} - m_b l_{1b} l_{3b})u_1u_2 \\
&\quad + (I_{33} + I - I_{22} - m_a l_{3a}^2 - m_b l_{3b}^2)u_2u_3 - Ju_3u_4 \\
-(I_{22} + m_a(l_{1a}^2 + l_{3a}^2) + m_b(l_{1b}^2 + l_{3b}^2))\dot{u}_2 &= T - (I_{13} - m_a l_{1a} l_{3a} - m_b l_{1b} l_{3b})u_1^2 \\
+(I_{13} - m_a l_{1a} l_{3a} - m_b l_{1b} l_{3b})u_3^2 &+ (I_{11} - I_{33} + m_a(l_{3a}^2 - l_{1a}^2) + m_b(l_{3b} - l_{1b}^2))u_1u_3 \\
-(I_{13} - m_a l_{1a} l_{3a} - m_b l_{1b} l_{3b})\dot{u}_1 - (I_{33} + m_a l_{1a}^2 + I + m_b l_{1b}^2)\dot{u}_3 &= Ju_1u_4 \\
+(I_{22} - I_{11} + m_a l_{1a}^2 - I + m_b l_{1b}^2)u_1u_2 &- (I_{13} - m_a l_{1a} l_{3a} - m_b l_{1b} l_{3b})u_2u_3 \\
-J\dot{u}_4 &= -T
\end{aligned}$$

If not for the introduction of the variables l_{1a}, l_{1b}, l_{3a}, and l_{3b}, these equations would be significantly longer. As it is, there are a substantial number of multiply and add operations that need only be done once, but if the above symbolic expressions were coded (or auto generated) in a numerical routine verbatim (i.e., for numerical integration), they would be performed at every time step (and intermediate time steps). This is a computationally wasteful approach, and further, the structure of the equations is clouded by excessively long symbolic expressions.

The three equations of motion associated with translation are: $\dot{u}_5 = 0$, $\dot{u}_6 = 0$, $\dot{u}_7 = g$. Although these equations are trivial and intuitive, the symbolic manipulations required in order for this simple form to be apparent are substantial when each mass center is treated individually; when computed symbolically on a computer, these necessary symbolic cancellations may or may not happen without additional user input.

Equivalent formulation

Consider now $\tilde{A}$ as a rigid body of mass M, whose mass center $\tilde{A}^*$ is coincident with AB^*, and additionally has the same angular orientation as A. Let the central inertia dyadic of $\tilde{A}$ be:

$$\boldsymbol{I}^{\tilde{A}/\tilde{A}^*} = I_{\tilde{A}11}\boldsymbol{a}_1\boldsymbol{a}_1 + I_{\tilde{A}22}\boldsymbol{a}_2\boldsymbol{a}_2 + I_{\tilde{A}33}\boldsymbol{a}_3\boldsymbol{a}_3 + I_{\tilde{A}13}\boldsymbol{a}_1\boldsymbol{a}_3 + I_{\tilde{A}13}\boldsymbol{a}_3\boldsymbol{a}_1$$

Consider a second rigid body $\tilde{B}$ as massless, yet having a central inertia dyadic of

$$\boldsymbol{I}^{\tilde{B}/\tilde{B}^*} = \tilde{J}\boldsymbol{a}_2\boldsymbol{a}_2$$

Additionally, consider the angular orientation of $\tilde{B}$ to be identical to that of B in the original system.

The active force and torques are:

$$\begin{aligned}
\boldsymbol{G}_{\tilde{A}^*} &= Mg\boldsymbol{n}_3 \\
\boldsymbol{T}_{\tilde{A}} &= -T\boldsymbol{a}_2 \\
\boldsymbol{T}_{\tilde{B}} &= T\boldsymbol{a}_2
\end{aligned}$$

Only one inertia force needs to be computed:

$$\begin{aligned}
\boldsymbol{R}^*_{\tilde{A}} &= -M\boldsymbol{a}^{AB^*} \\
&= -M\left(\dot{u}_5\boldsymbol{n}_1 + \dot{u}_6\boldsymbol{n}_2 + \dot{u}_7\boldsymbol{n}_3\right)
\end{aligned}$$

The two inertia torques are:

$$\begin{aligned}
\boldsymbol{T}^*_{\tilde{A}} &= -\boldsymbol{\alpha}^A \cdot \boldsymbol{I}^{\tilde{A}/\tilde{A}^*} - \boldsymbol{\omega}^A \times \boldsymbol{I}^{\tilde{A}/\tilde{A}^*} \cdot \boldsymbol{\omega}^A \\
&= (-I_{\tilde{A}11}\dot{u}_1 - I_{\tilde{A}13}\dot{u}_3 - I_{\tilde{A}13}u_1u_2 + (I_{\tilde{A}22} - I_{\tilde{A}33})u_2u_3)\boldsymbol{a}_1 \\
&\quad - (I_{\tilde{A}22}\dot{u}_2 + (I_{\tilde{A}11} - I_{\tilde{A}33})u_1u_3 + I_{\tilde{A}13}u_1^2 - I_{\tilde{A}13}u_3^2)\boldsymbol{a}_2 \\
&\quad - (I_{\tilde{A}13}\dot{u}_1 + I_{\tilde{A}33}\dot{u}_3 + (I_{\tilde{A}22} - I_{\tilde{A}11})u_1u_2 - I_{\tilde{A}13}u_2u_3)\boldsymbol{a}_3 \\
\boldsymbol{T}^*_{\tilde{B}} &= -\boldsymbol{\alpha}^B \cdot \boldsymbol{I}^{\tilde{B}/\tilde{B}^*} - \boldsymbol{\omega}^B \times \boldsymbol{I}^{\tilde{B}/\tilde{B}^*} \cdot \boldsymbol{\omega}^B \\
&= \tilde{J}(u_3u_4\boldsymbol{a}_1 - \dot{u}_4\boldsymbol{a}_2 - u_1u_4)\boldsymbol{a}_3
\end{aligned}$$

Kane's equations of motions are:

$$\boldsymbol{v}_r^{AB^*} \cdot (\boldsymbol{G}_{\tilde{A}} + \boldsymbol{R}^*_{\tilde{A}}) + \boldsymbol{\omega}_r^A \cdot (\boldsymbol{T}_{\tilde{A}} + \boldsymbol{T}^*_{\tilde{A}}) + \boldsymbol{\omega}_r^B \cdot (\boldsymbol{T}_{\tilde{B}} + \boldsymbol{T}^*_{\tilde{B}}) = \boldsymbol{0} \qquad r = 1, \ldots, 7$$

More explicitly, the first four of these equations are:

$$\begin{aligned}
-I_{\tilde{A}11}\dot{u}_1 - I_{\tilde{A}13}\dot{u}_3 &= I_{\tilde{A}13}u_1u_2 + (I_{\tilde{A}33} - I_{\tilde{A}22})u_2u_3 - \tilde{J}u_3u_4 \\
-I_{\tilde{A}22}\dot{u}_2 &= T - I_{\tilde{A}13}u_1^2 + I_{\tilde{A}13}u_3^2 + (I_{\tilde{A}11} - I_{\tilde{A}33})u_1u_3 \\
-I_{\tilde{A}13}\dot{u}_1 - I_{\tilde{A}33}\dot{u}_3 &= \tilde{J}u_1u_4 + (I_{\tilde{A}22} - I_{\tilde{A}11})u_1u_2 - I_{\tilde{A}13}u_2u_3 \\
-\tilde{J}\dot{u}_4 &= -T
\end{aligned}$$

It is important to note that the form of these equations are identical to the previous formulation, and that the only difference is the coefficients of the $\dot{u}_i$ terms and the u_iu_j terms. From a symbolic computation point of view, these equations are *much* easier to obtain because all of the dot products necessary for Kane's equations are between unit vectors in the *same* frame. This means that the direction cosine matrix is never needed to compute the dynamic equations of motion. This is not the case in the first formulation, specifically because the velocities of A^* and B^* have terms in both the inertial frame N and the body frame A.

To see the relationship between these two sets of equations, we compute the inertia dyadic of fictitious particles of mass m_a and m_b, located at A^* and B^*, respectively, relative to the system mass center AB^*. They are:

$$\begin{aligned}
\boldsymbol{I}^{A^*/AB^*} &= m_a l_{3a}^2 \boldsymbol{a}_1\boldsymbol{a}_1 + m_a(l_{1a}^2 + l_{3a}^2)\boldsymbol{a}_2\boldsymbol{a}_2 + m_a l_{1a}^2 \boldsymbol{a}_3\boldsymbol{a}_3 - m_a l_{1a} l_{3a} \boldsymbol{a}_1\boldsymbol{a}_3 - m_a l_{1a} l_{3a} \boldsymbol{a}_3\boldsymbol{a}_1 \\
\boldsymbol{I}^{B^*/AB^*} &= m_b l_{3b}^2 \boldsymbol{a}_1\boldsymbol{a}_1 + m_b(l_{1b}^2 + l_{3b}^2)\boldsymbol{a}_2\boldsymbol{a}_2 + m_b l_{1b}^2 \boldsymbol{a}_3\boldsymbol{a}_3 - m_b l_{1b} l_{3b} \boldsymbol{a}_1\boldsymbol{a}_3 - m_b l_{1b} l_{3b} \boldsymbol{a}_3\boldsymbol{a}_1
\end{aligned}$$

Additionally, consider the inertia dyadic of B as being comprised of two components, one in the plane of symmetry ($\boldsymbol{I}_p^{B/B^*}$) and one about the axis of symmetry ($\boldsymbol{I}_a^{B/B^*}$):

$$\begin{aligned}
\boldsymbol{I}^{B/B^*} &= \boldsymbol{I}_p^{B/B^*} + \boldsymbol{I}_a^{B/B^*} \\
&= (I\boldsymbol{a}_1\boldsymbol{a}_1 + I\boldsymbol{a}_3\boldsymbol{a}_3) + J\boldsymbol{a}_2\boldsymbol{a}_2
\end{aligned}$$

Next, consider the inertia dyadic of the whole system, about the point AB^*, *except* the out of plane component of the symmetric rotor B:

$$\begin{aligned}
\boldsymbol{I}^{AB/AB^*} &= \boldsymbol{I}^{A/A^*} + \boldsymbol{I}^{A^*/AB^*} + \boldsymbol{I}_p^{B/B^*} + \boldsymbol{I}^{B^*/AB^*} \\
&= (I_{11} + m_a l_{3a}^2 + I + m_b l_{3b}^2)\boldsymbol{a}_1\boldsymbol{a}_1 \\
&+ (I_{22} + m_a(l_{1a}^2 + l_{3a}^2) + m_b(l_{1b}^2 + l_{3b}^2))\boldsymbol{a}_2\boldsymbol{a}_2 \\
&+ (I_{33} + m_a l_{1a}^2 + I + m_b l_{1b}^2)\boldsymbol{a}_3\boldsymbol{a}_3 \\
&+ (I_{13} - m_a l_{1a} l_{3a} - m_b l_{1b} l_{3b})\boldsymbol{a}_1\boldsymbol{a}_3 + (I_{13} - m_a l_{1a} l_{3a} - m_b l_{1b} l_{3b})\boldsymbol{a}_3\boldsymbol{a}_1
\end{aligned}$$

Define the following inertia scalars:

$$\begin{aligned}
I_{\tilde{A}11} &= I_{11} + m_a l_{3a}^2 + I + m_b l_{3b}^2 \\
I_{\tilde{A}22} &= I_{22} + m_a(l_{1a}^2 + l_{3a}^2) + m_b(l_{1b}^2 + l_{3b}^2) \\
I_{\tilde{A}33} &= I_{33} + m_a l_{1a}^2 + I + m_b l_{1b}^2 \\
I_{\tilde{A}13} &= I_{13} - m_a l_{1a} l_{3a} - m_b l_{1b} l_{3b} \\
\tilde{J} &= J
\end{aligned}$$

Using these definitions, and substituting them into the equations of motion obtained in the alternative formulation, we obtain identical equations of motion as in the first formulation where each rigid body was treated separately.

Conclusions

The fundamental concept that allows for this lumping of rigid bodies is that the inertia of the system not change with configuration (as generalized coordinates change, the system inertia remains constant). While it should be clear that under these circumstances, inertia forces are identical for both formulations, it may not be so clear that active forces are also the same. In the case of gravitational forces (as in this example), one can probably be convinced that they may be treated as acting at the system center of mass without much effort. But what about contact forces, or systems with linear motion constraints (nonholonomic constraints)? The formulation of the generalized active forces as advocated by Kane will be identical in either case – therefore the final equations of motion will also be the same. Specifically, linear motion constraints will require the solution of constraint equations (i.e., solving a linear system), so that nonholonomic partial velocities and partial angular velocities can be formed. These are independent of inertial considerations and therefore will not affect the equations of motion.