Investigating population dynamics with Cellular Automata

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Outline

1. Introduction
2. Lotka-Volterra (LV) model: Analytical solution of exponential + logistic model
3. Cellular Automata (CA) simulation: Energy-based model
4. Mapping from exponential LV model to probabilistic CA
5. Conclusion

1. Introduction

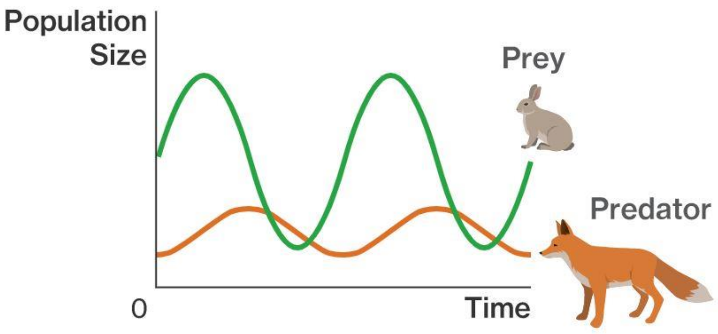
Ecological systems are complex, but insight into their dynamics is crucial

- Weather forecasting and climate change
- Protection of natural diversity
- Economic considerations

Source: *NOAA Fisheries, 2022*
Available at: https://www.fisheries.noaa.gov/feature-story/new-indicators-could-help-manage-global-overfishing

Goal: Understand dynamics of simple system via two different perspectives

- Analytical models using differential equations
- Numerical models simulating species interaction



Source: *Predator-prey models to model users*, Medium, 2020.
Available at: https://medium.com/hello-cdo/predator-prey-models-to-model-users-9ed717fa548f

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2.0 Lotka-Volterra (LV) model

Pair of first-order (nonlinear) differential equations to describe population dynamics in predator-prey systems

$$\begin{cases} \dfrac{\partial x(t)}{\partial t} = \underbrace{\alpha x y}_{\text{Predator growth}} - \underbrace{\beta x}_{\text{Predator death}} := f(x,y) \\ \dfrac{\partial y(t)}{\partial t} = \underbrace{\gamma y}_{\text{Prey growth}} - \underbrace{\delta x y}_{\text{Prey death}} := g(x,y) \end{cases}$$

Multiple variations exists, e.g., featuring exponential or logistic prey growth

[1] Alfred J Lotka. *Elements of Physical Biology*. Williams and Wilkins Company, 1925.

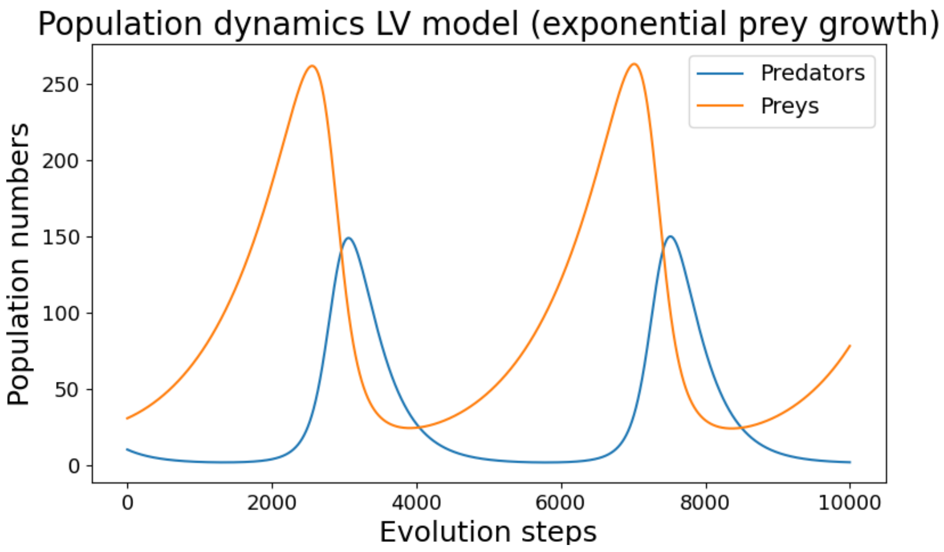
[2] Vito Volterra. "Fluctuations in the Abundance of a Species considered Mathematically". In: *Nature* 118 (1926), pp. 558–560.

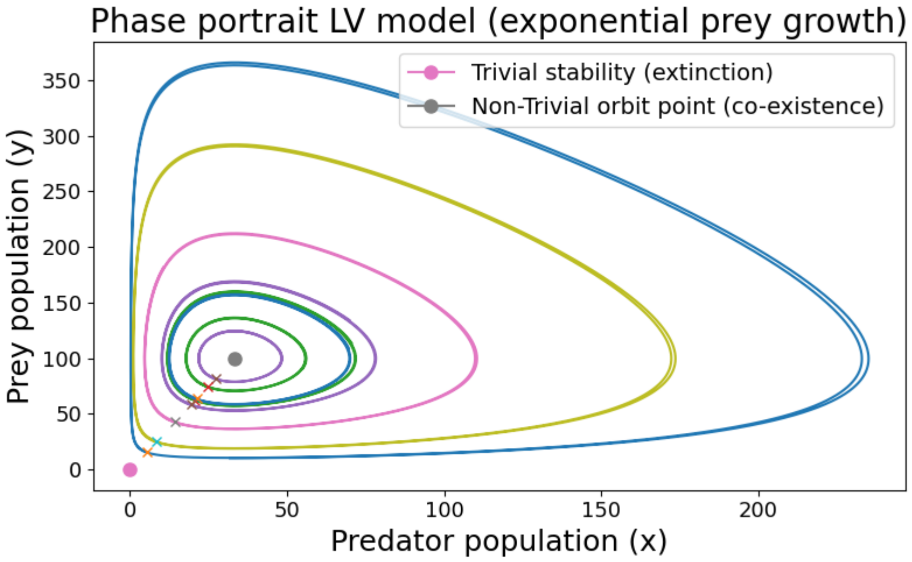
2.1 Exponential LV model

$$\begin{cases} \dfrac{\partial x(t)}{\partial t} = \underbrace{\alpha xy}_{\text{Predator growth}} - \underbrace{\beta x}_{\text{Predator death}} := f(x,y) \\ \dfrac{\partial y(t)}{\partial t} = \underbrace{\gamma y}_{\text{Prey growth}} - \underbrace{\delta xy}_{\text{Prey death}} := g(x,y) \end{cases}$$

Simple model with exponential prey growth, exhibits non-trivial stable point at $(x^*, y^*) = \left(\frac{\gamma}{\delta}, \frac{\beta}{\alpha}\right)$

For $\alpha = \delta = 0.003,\ \beta = 0.3,\ \gamma = 0.1, dt = 0.01$:



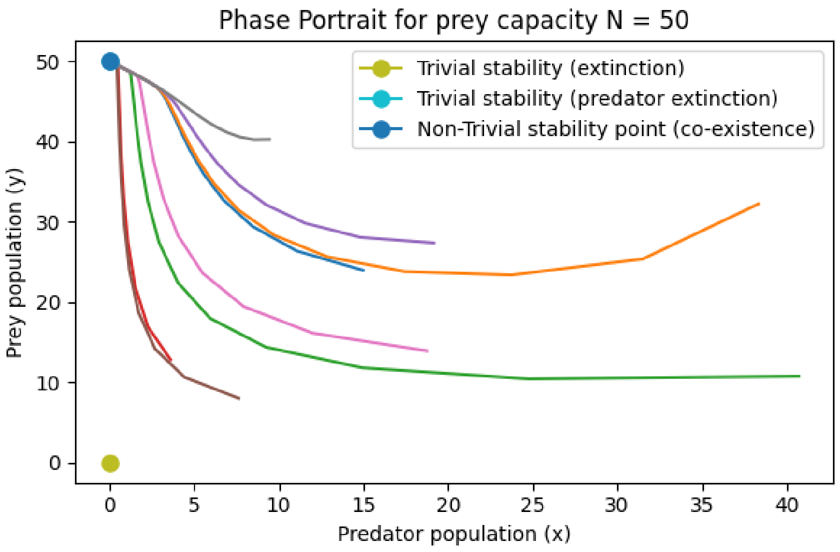


$$\begin{cases} \frac{\partial x(t)}{\partial t} = \alpha x y - \beta x & := f(x,y) \\ \frac{\partial y(t)}{\partial t} = \gamma y(1 - \frac{y}{N}) - \delta x y & := g(x,y) \end{cases}$$

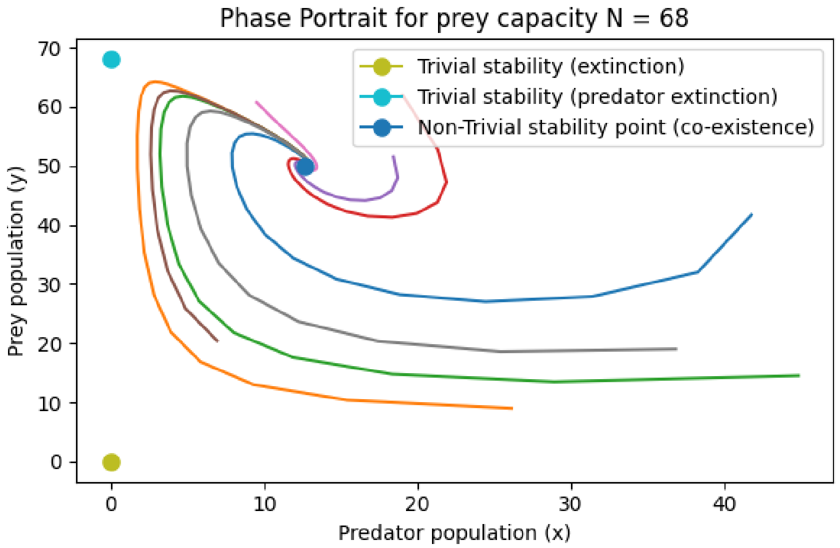
2.2 Logistic LV model

Logistic prey growth (capacity N) with non-trivial stable point $(x^*, y^*) = \left(\frac{\gamma}{\delta}(1 - \frac{\beta}{\alpha N}), \frac{\beta}{\alpha}\right)$ and bifurcation at capacity $N = \frac{\alpha\beta + \sqrt{\alpha^2\beta^2 + \alpha^2\gamma\beta}}{2\alpha^2}$.

For $\alpha = \delta = 1,\ \beta = 50,\ \gamma = 48,\ dt = 0.01$:



bifurcation at $N = 60$



[3] Virginia W. Noonburg. *Ordinary Differential Equations: From Calculus to Dynamical Systems.* MAA Press, American Mathematical Society, 2014.

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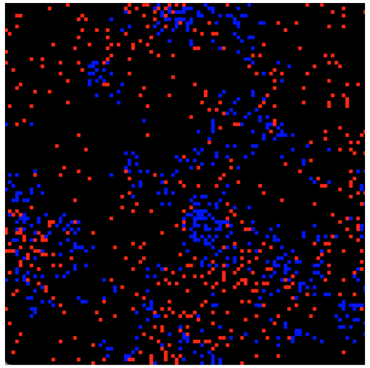
2.3 Population dynamics: Beyond the LV model



Numerical simulation replicates analytical model, but is restricted in expressivity

We want a more realistic simulation: **Cellular Automata (CA)**

CAs are computational models to simulate the behaviour of complex systems:

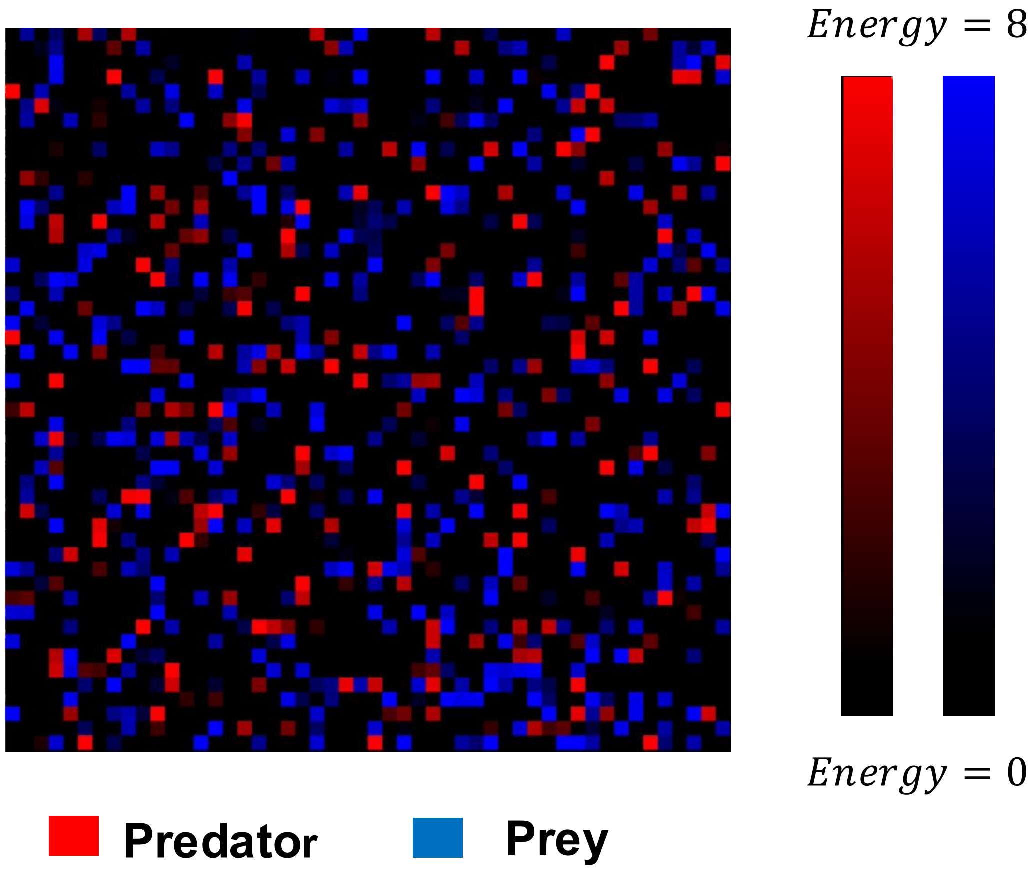
- Grid of cells with finite number of states
- Discrete update rules (can be deterministic or probabilistic) to define system evolution
- Locality due to neighbor interactions

Useful for our goal to simulate realistic predator-prey systems

3.1. Energy-based CA: Set-up and initialization

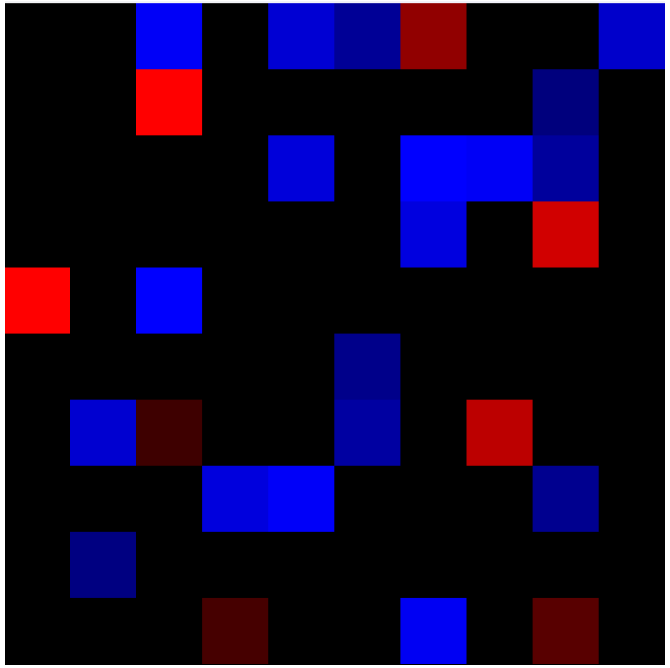
- **Set-up**
  - We use an OOP approach
  - Habitat is represented by a NxN lattice
  - Each cell can be empty, contain a predator or a prey
  - It also has an associated energy ⟶ health
- **Initialization**
  - Random allocation of predators and preys
  - Energies sampled from $\mathcal{N}(\mu_E = 5, \sigma_E = 3)$

70% empty cells, 10% predators, 20% preys



3.2. Energy-based CA: Evolution rules

- At each evolution step, the order in which cells act is randomized
- Animals act based on the current state of the lattice, which is updated immediately after each individual acts.
- Periodic boundary conditions are used to determine each cell's neighbors.
- Implement a set of action rules to simulate real ecosystem interactions
- Separate action rules for predators and prey



3.2. Energy-based CA: Evolution rules

Predators:

- Search for preys in its neighbouring cells
- If available, a random prey target is selected
  - If the predator has higher energy, it kills prey

    If energy (*4.7*) is sufficient, it reproduces; otherwise, its energy is restored to $\mu_E = 5$ as a consequence of eating
  - Else: Predator loses energy due to failed hunt

  Else: Predator searches for empty spaces
  - If there are, predator makes random move and loses energy (*2.5*)
  - If not, predator remains stationary
- Predator dies if energy ≤ 0

3.2. Energy-based CA: Evolution rules

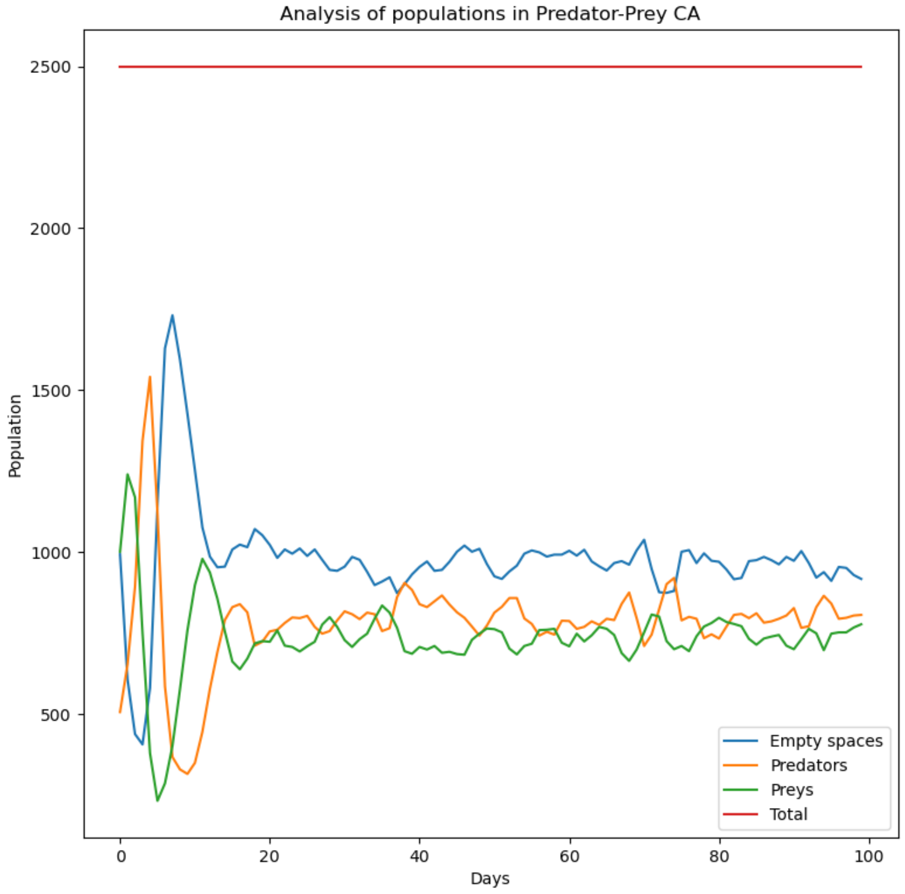
Preys:

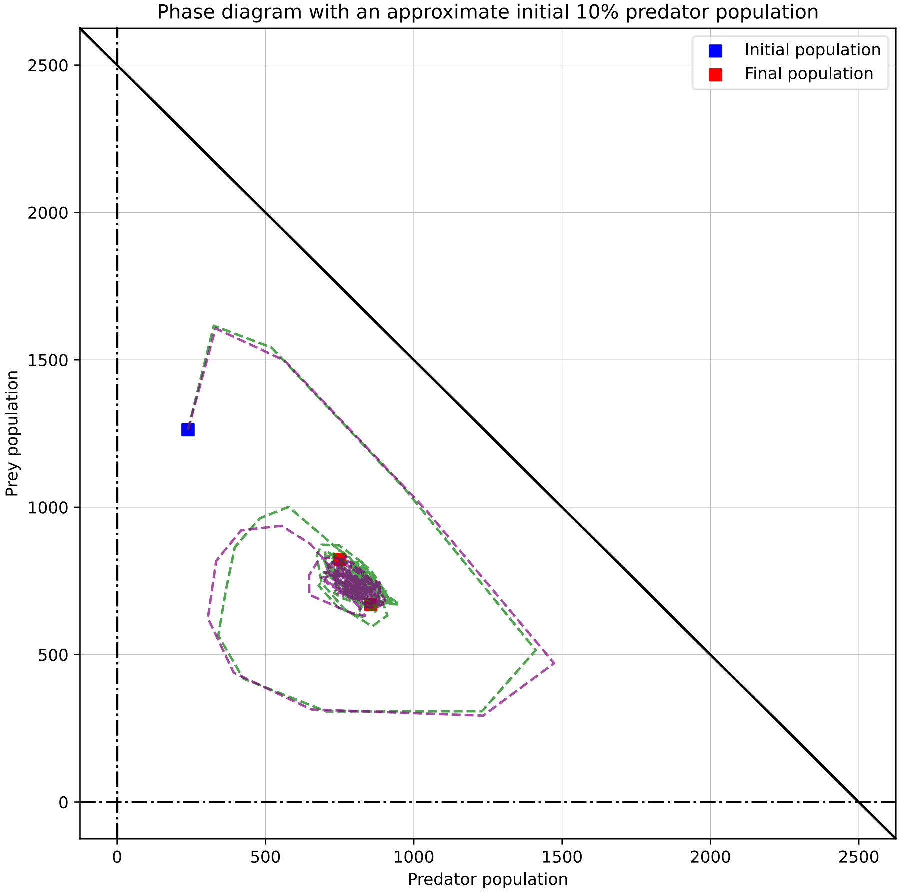
- Check overpopulation in Moore neighbourhood: if more than three preys, it loses energy (1.7)
- Look for neigbouring empty cells
- If available:
  - If energy is sufficient (3.9), prey reproduces and loses energy (1.2)

    Else: prey moves randomly and gains energy (0.8) due to foraging

  Else: prey remains stationary, maintaining its energy
- Prey dies if energy ≤ 0

3.3 Balanced evolution (No seasons, no prey defence)





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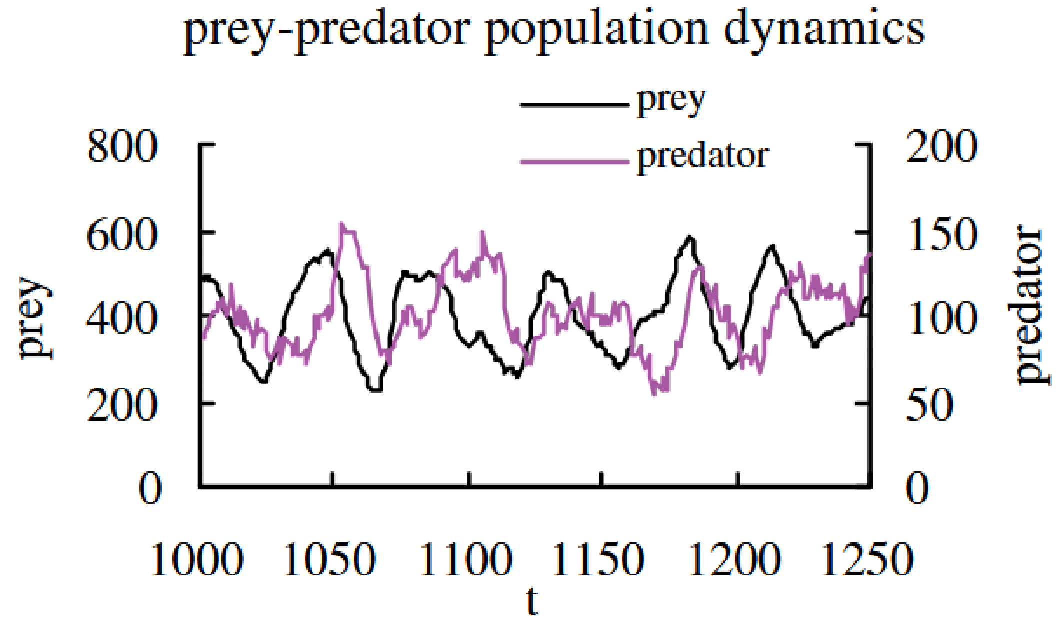


Fig. 6. Cyclic behaviour of prey–predator populations.

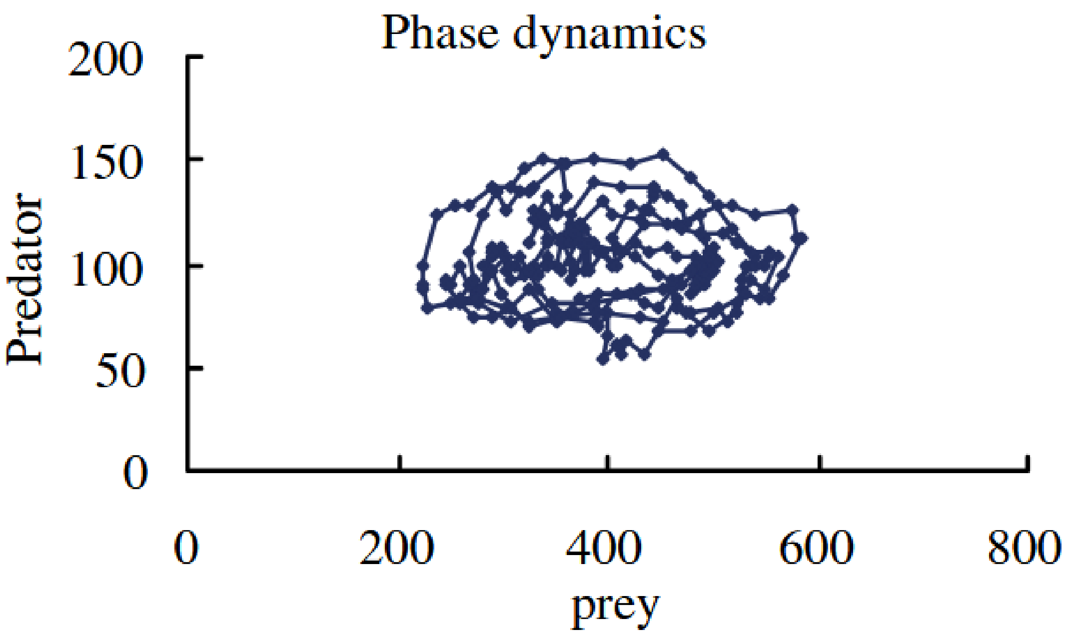
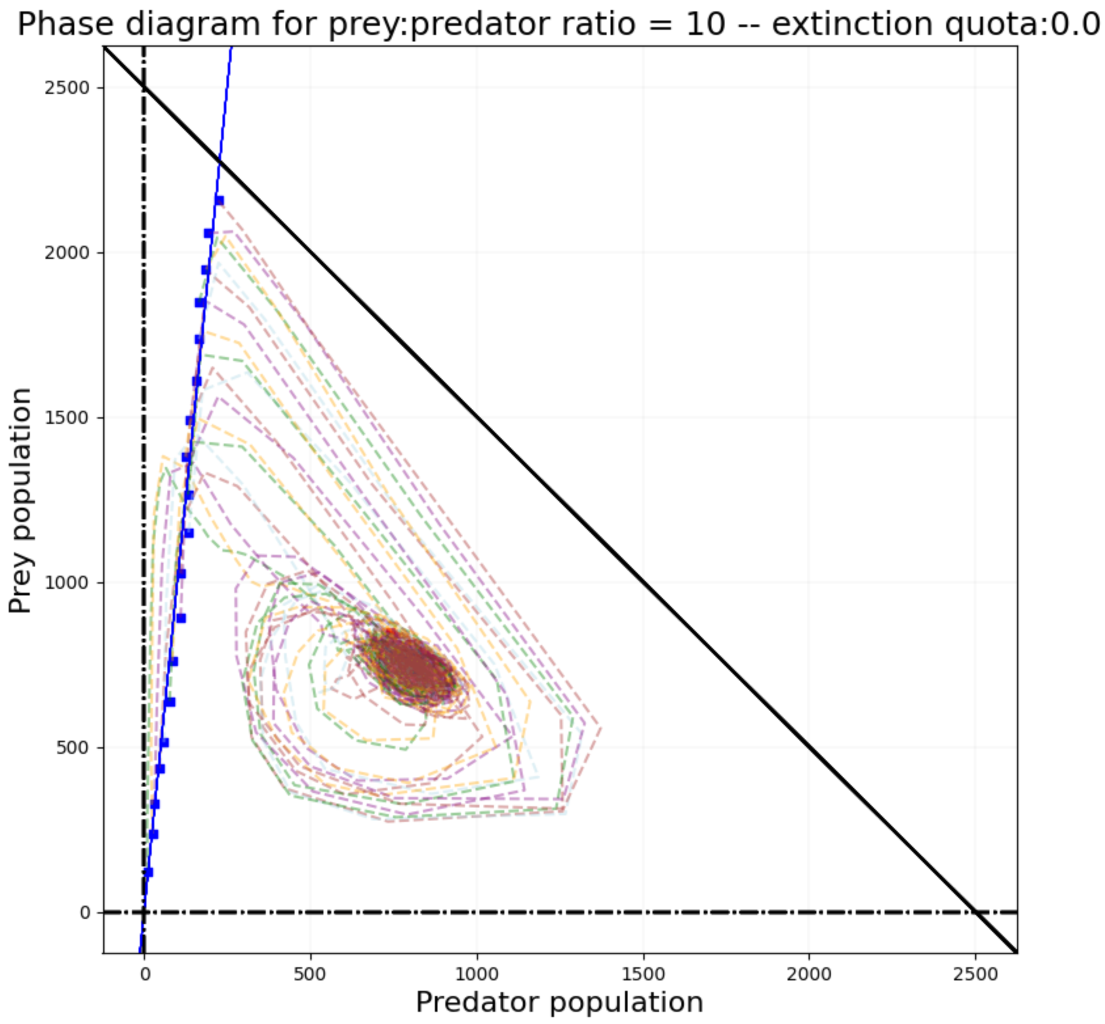


Fig. 7. Phase dynamics.

[4] Q. Chen and A.E. Mynett. "Effects of cell size and configuration in cellular automata based prey–predator modelling". In: Simulation Modelling Practice and Theory 11.7 (2003), pp. 609– 625. issn: 1569-190X. doi: https://doi.org/10.1016/j.simpat.2003.08.006.

[5] Q. et al. Chen. "Stability Analysis of Harvesting Strategies in a Cellular Automata Based Predator-Prey Model". In: Cellular Automata. Springer Berlin, Heidelberg, 2006. isbn: 978- 3-540-40929-8.

3.4 Stability analysis



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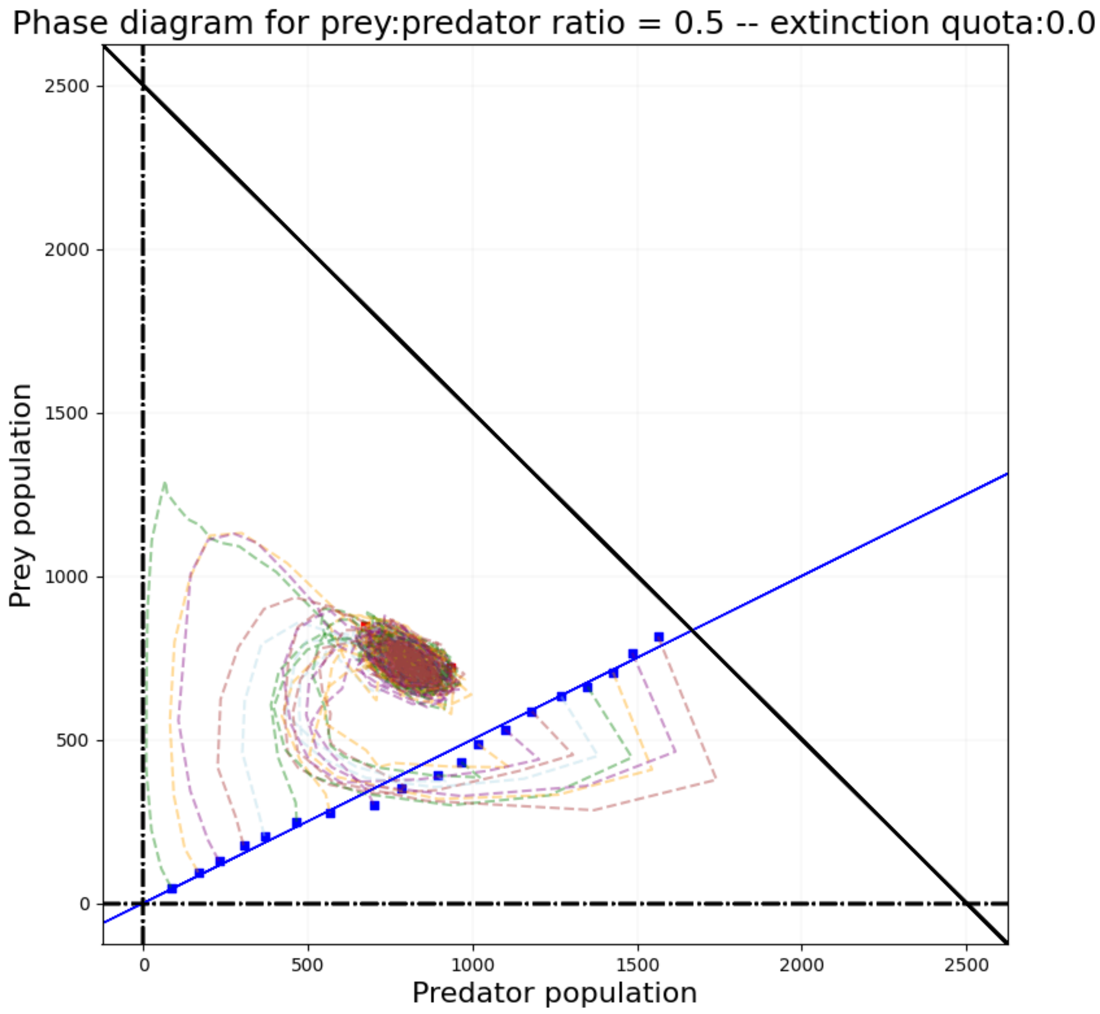
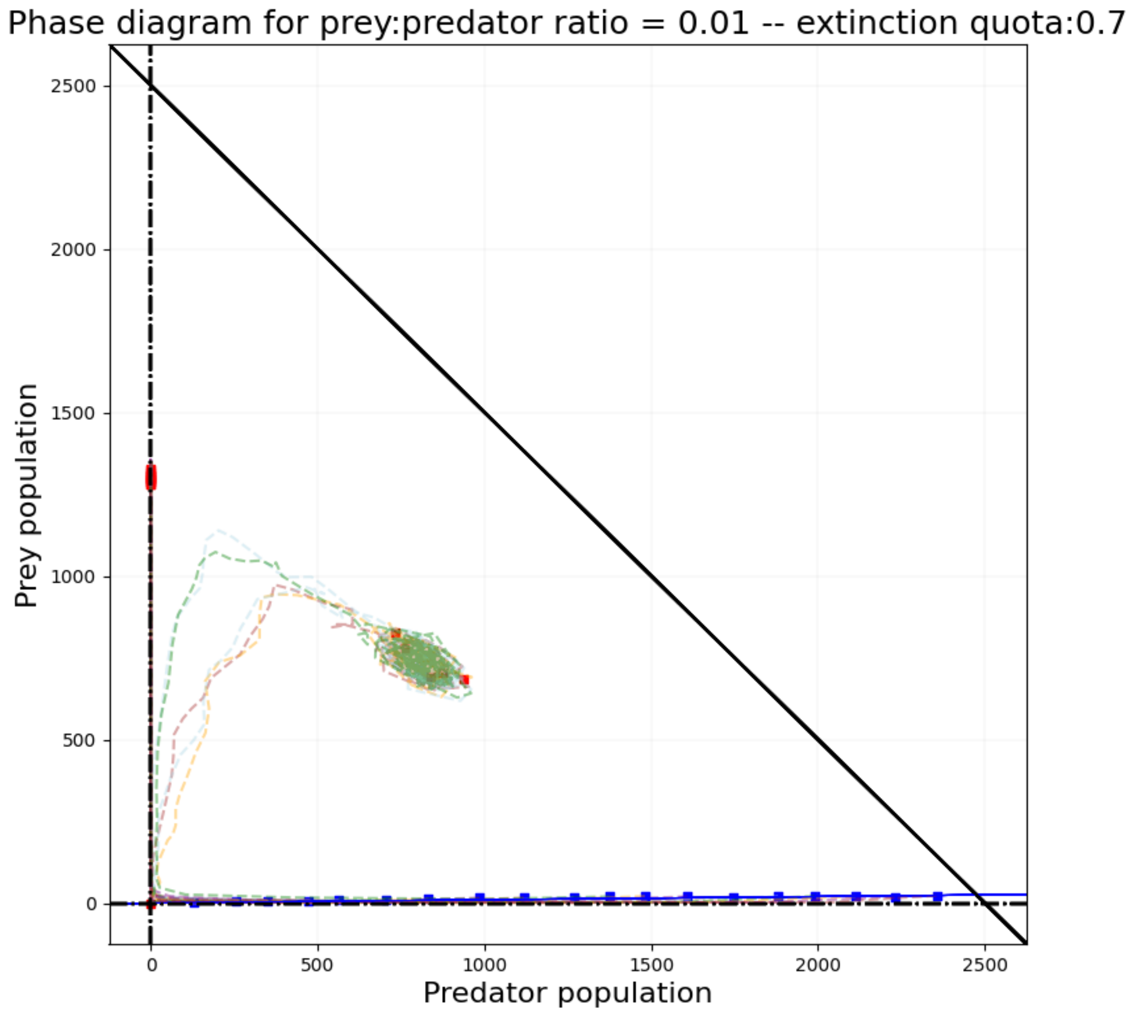


3.4 Stability analysis



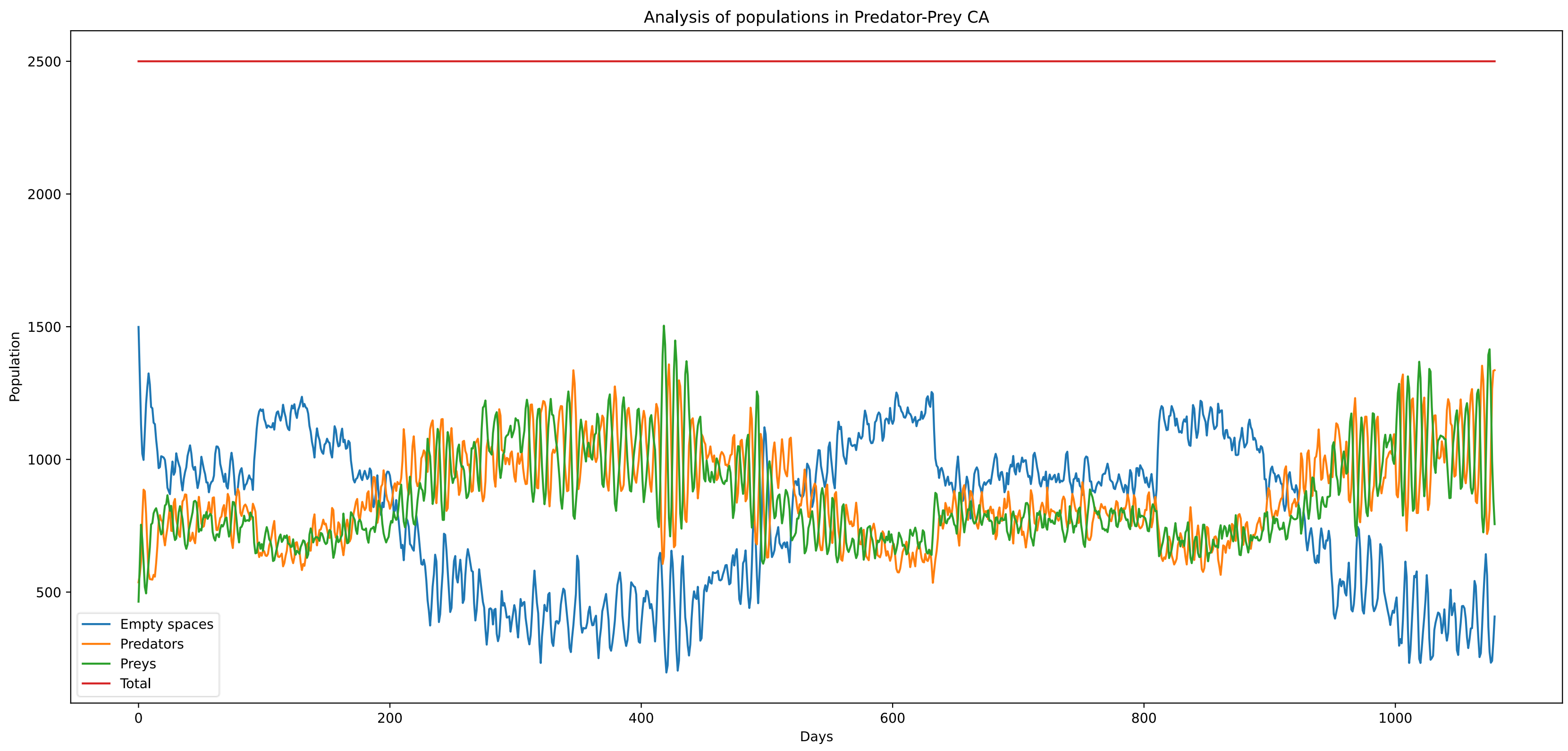
70% of the cases are ending in extinction

3.5 Fishing extinction?

- Energy-based CA model can potentially simulate species extinction by human fishing
- But our model is not well fitted for fishing:
    - Humans can hunt or fish more than one prey
    - Humans are not dependent on a single energy source
    - Displacement beyond neighbouring cells

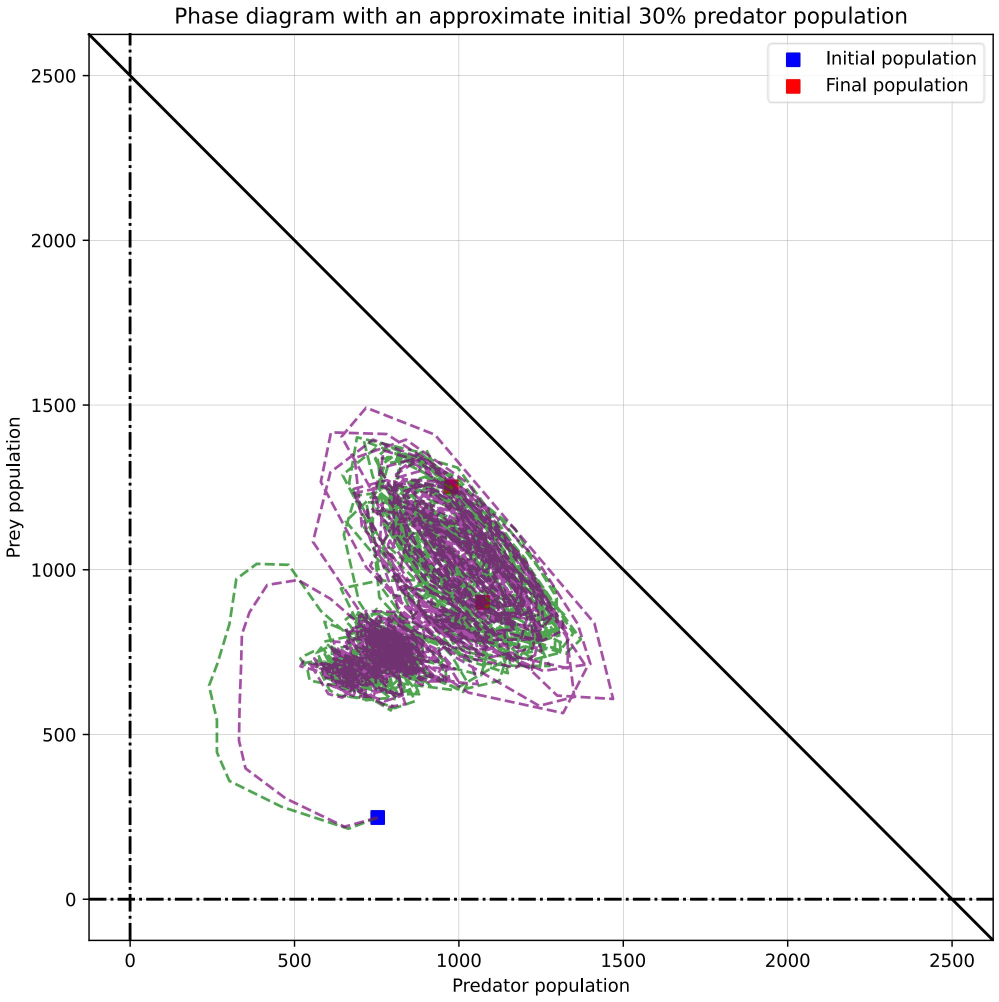
[6] Milner B. Schaefer. “Some aspects of the dynamics of populations important to the management of the commercial marine fisheries”. In: Bulletin of Mathematical Biology 53.1 (1991), pp. 253–279. issn: 0092-8240. doi: https://doi.org/10.1016/S0092-8240(05)80049-7. url: https://www.sciencedirect.com/science/article/pii/S0092824005800497.

3.6 Seasonal oscillations (No defence)



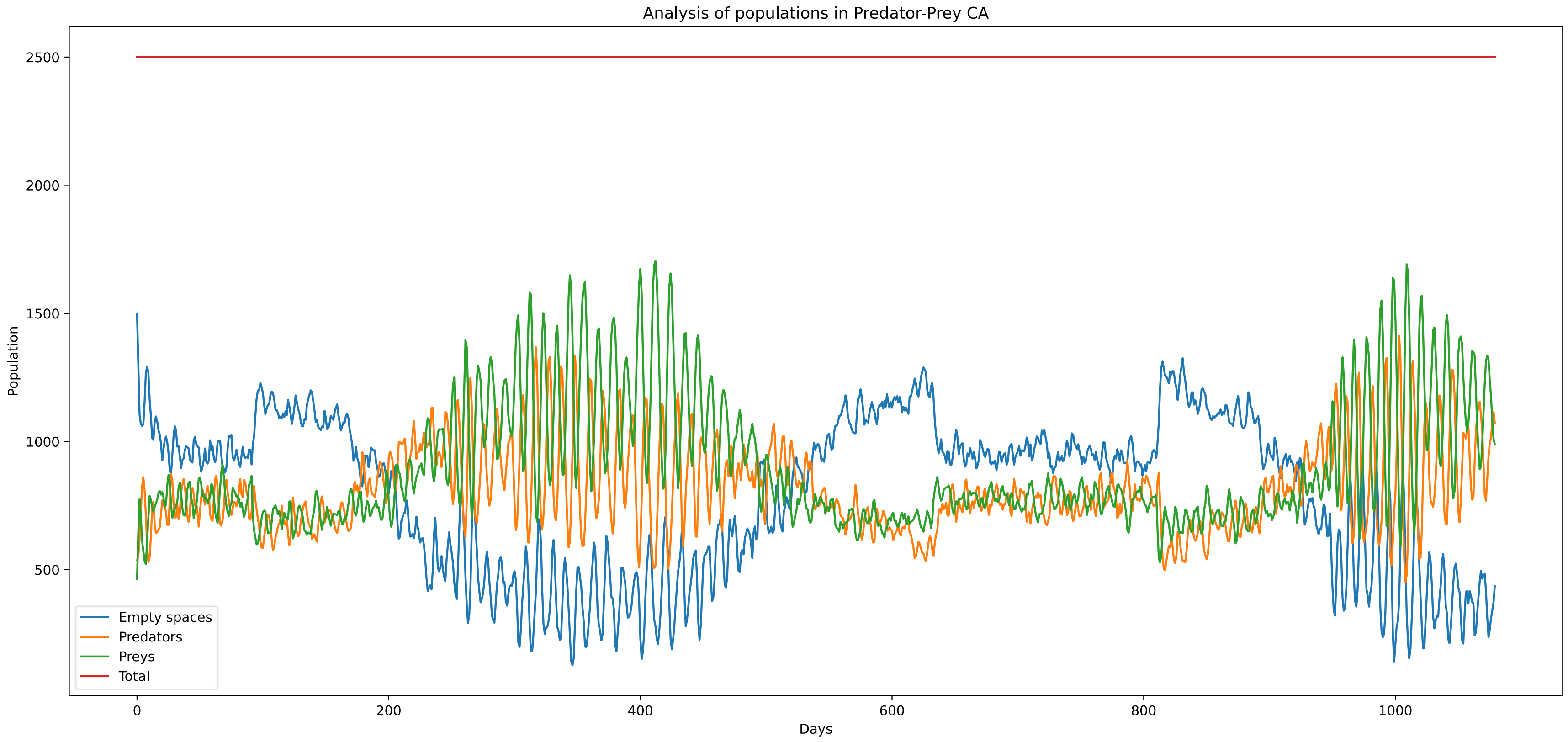
Seasons are modelled with a cosine: energy gain and losses vary in time

3.6 Seasonal oscillations (No defence)



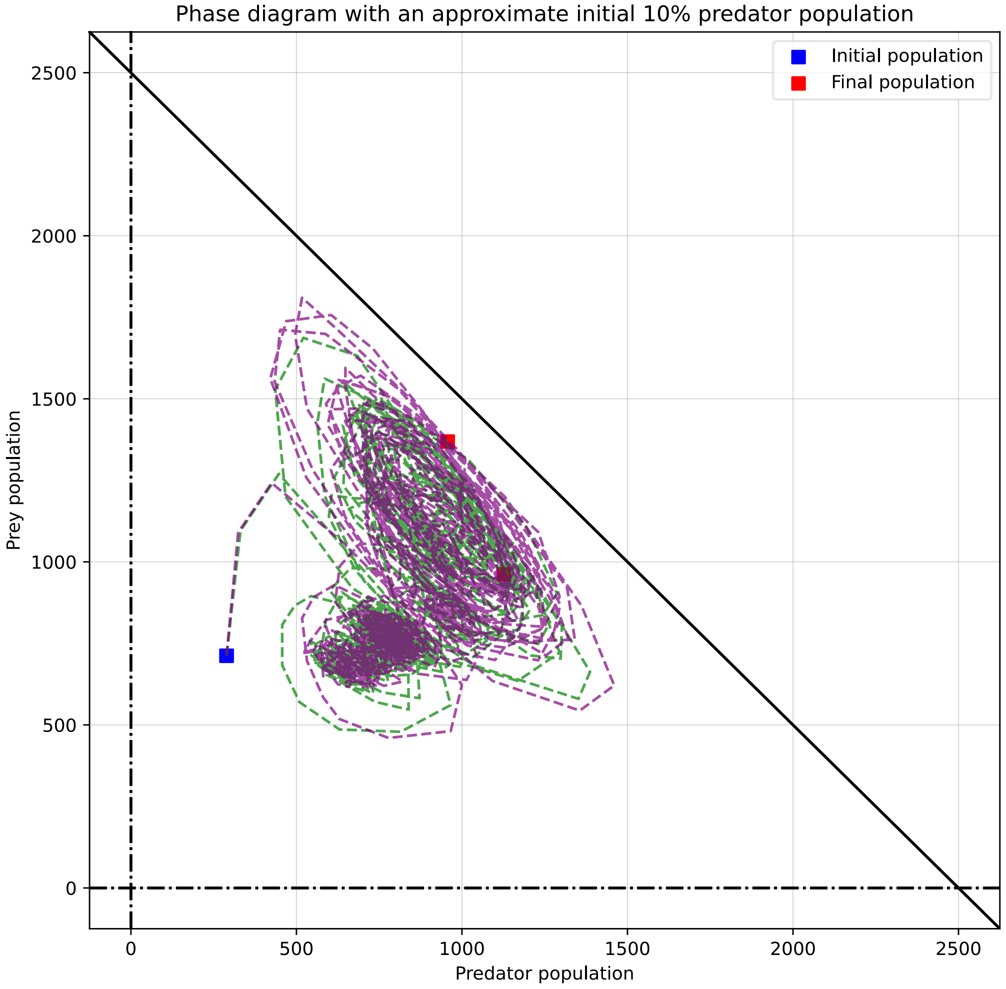
Formation of two stable points

3.7 Prey defence (with seasons)

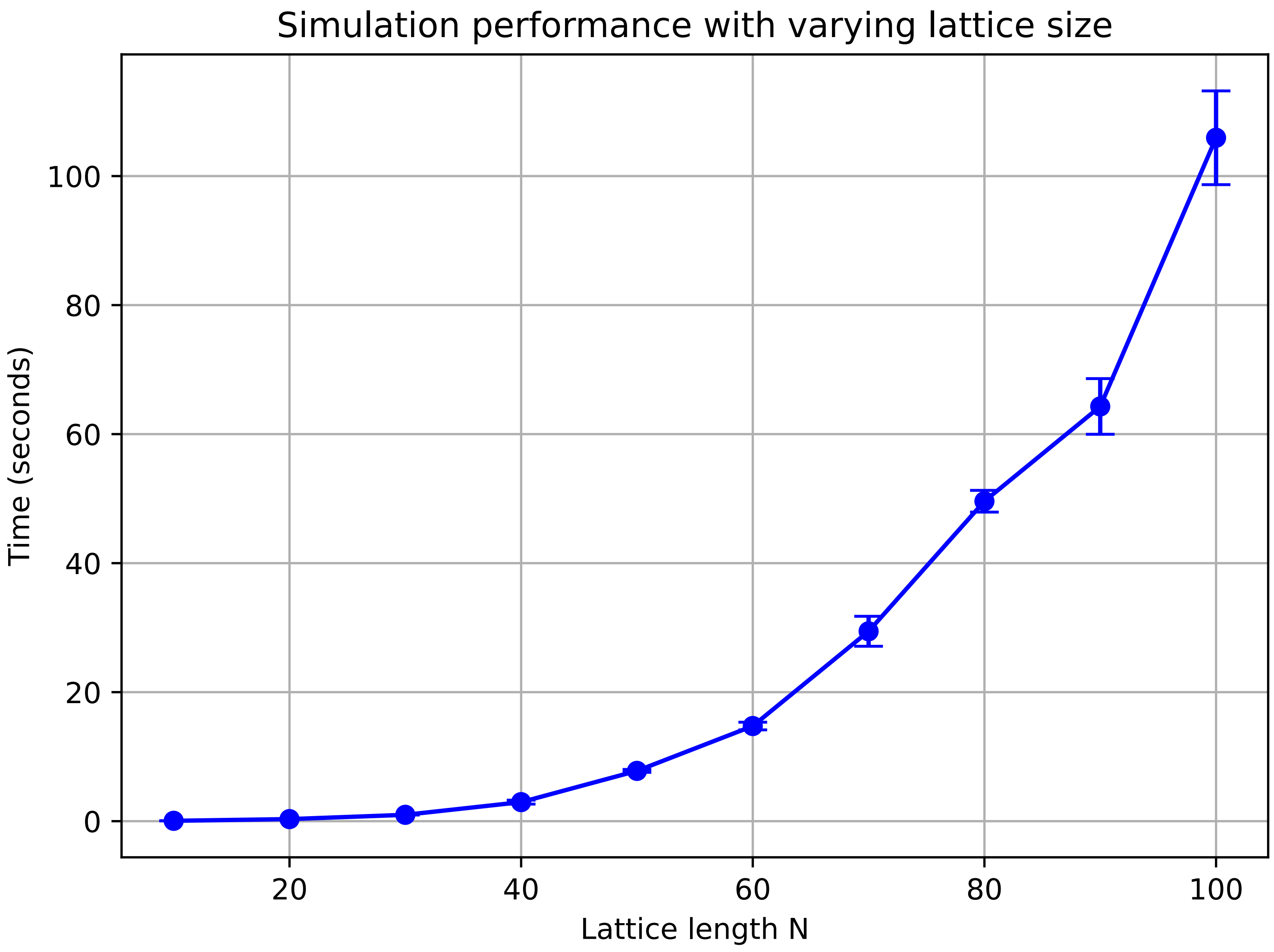


If predator has at least six prey in its surrounding, it cannot hunt

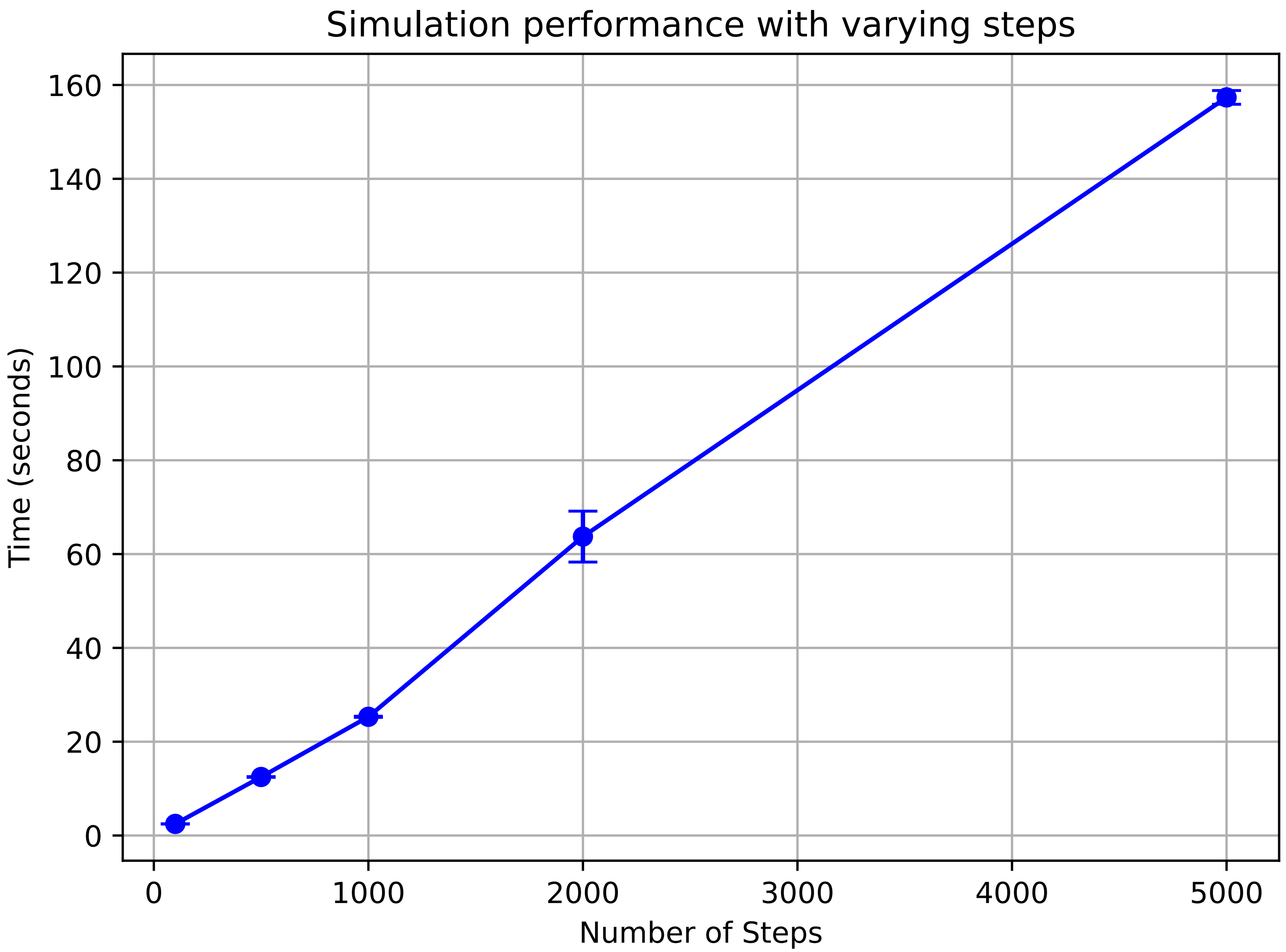
3.7 Prey defence (with seasons)



3.8 Performance: varying lattice sites



3.8 Performance: varying time steps



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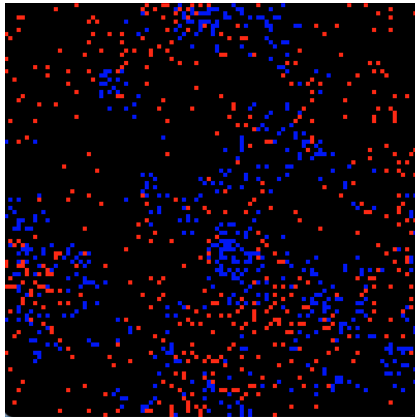
4.0 A probabilistic CA for predator-prey model

To reconcile the LV model with our CA approach, we investigate a probabilistic CA.



Parameters of CA:

- Predator death rate
- Prey reproduction rate
- Prey hunting success probability
- Prey migration rate

Natural question: Can we map the LV parameters to the CA parameters?

4.1 Parameter mapping LV > CA

$$\begin{cases} \frac{\partial x(t)}{\partial t} = \underbrace{\alpha x y}_{\text{Predator growth}} - \underbrace{\beta x}_{\text{Predator death}} := f(x,y) \\ \frac{\partial y(t)}{\partial t} = \underbrace{\gamma y}_{\text{Prey growth}} - \underbrace{\delta x y}_{\text{Prey death}} := g(x,y) \end{cases}$$

Predator death rate (β'):

In LV model, $x' = -\beta x \mapsto x(t) = x_0 \ \exp(-\beta t) \approx x_0(1 - \frac{\beta t}{n})^n$, which becomes better for large n.

$\mapsto$ set $t = n$ ('make simulation time large'), and $\beta' = \beta$

Prey birth rate (γ'):

Likewise, $y' = \gamma x \mapsto y(t) = y_0 \ \exp(\gamma t)$ in CA model $\mapsto$ set $\gamma' = \gamma$.

Predation (α', δ'):

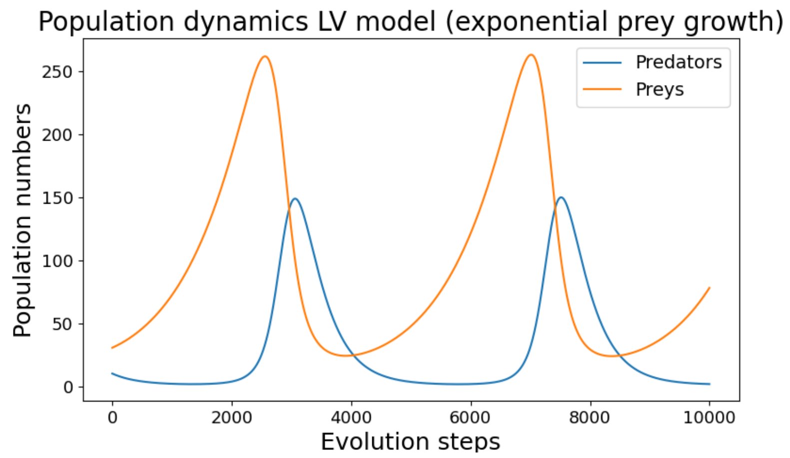
Treat cross-terms qualitatively: $x' = \alpha x y$ means 'predator growth $\propto x, y$'

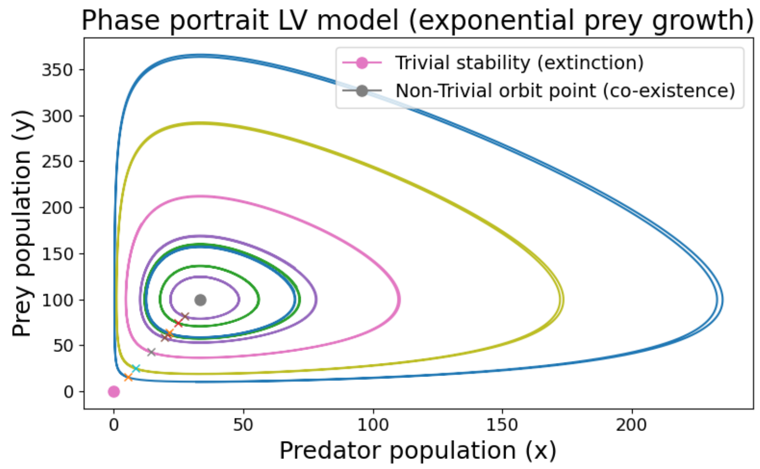
CA model is local/short-sighted $\mapsto$ set $\delta' = \alpha' = \frac{S_l}{S_n}\alpha$, for S_l lattice size, S_n neighbourhood size

4.1 Parameter mapping LV > CA

$$\begin{cases} \dfrac{\partial x(t)}{\partial t} = \underbrace{\alpha x y}_{\text{Predator growth}} - \underbrace{\beta x}_{\text{Predator death}} := f(x,y) \\ \dfrac{\partial y(t)}{\partial t} = \underbrace{\gamma y}_{\text{Prey growth}} - \underbrace{\delta x y}_{\text{Prey death}} := g(x,y) \end{cases}$$

Recall from earlier: $\alpha = \delta = 0.003,\ \beta = 0.3,\ \gamma = 0.1.$



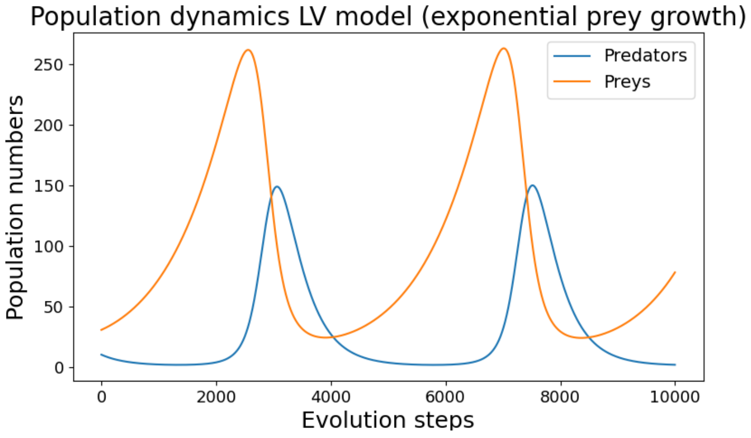


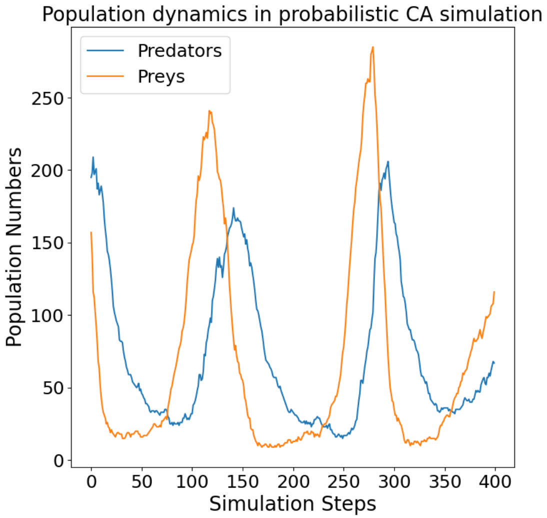
Define dimension = (30,30), $S_l = 900$, $S_n = 9, t = 400.$

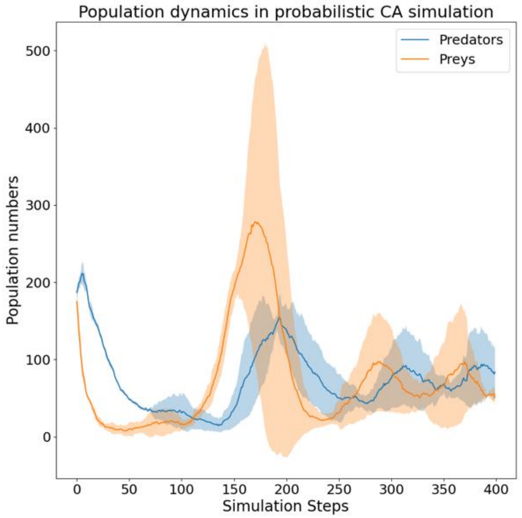
$\mapsto \alpha' = \delta' = \frac{S_l}{S_n}\alpha = 0.3$, $\beta' = \beta = 0.3, \gamma' = \gamma = 0.1$

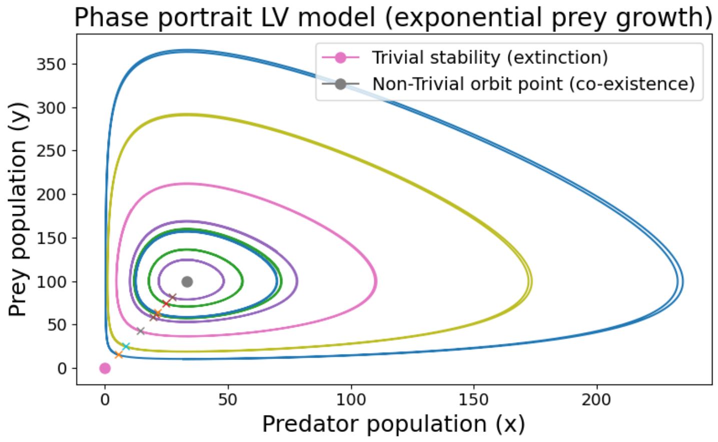
Initialize 10% predators, 30% predators and 60% empty cells

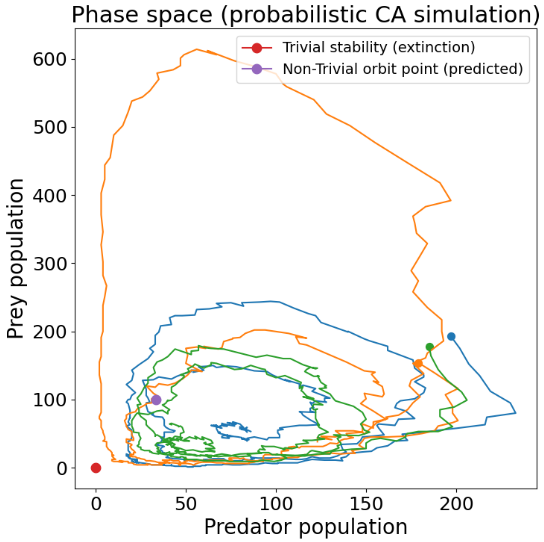
4.2 Comparison exp. LV and prob. CA model











Parameter mapping is possible!

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5. Conclusion & Outlook

- Both analytical and numerical methods give insight into population dynamics
- Analytical methods give explainable insight into stability and bifurcations
- Cellular Automata can reasonably approximate LV model, but also explore behaviours that are difficult to formulate analytically
- CA simulations give insight into dynamics of predator-prey systems, and allow for evidence-based policies (environmental protection, economic viability, etc.)

What's next?

- Adding more species to the simulation
- Defining more complex food chain hierarchies
- Adapting more realistic reproduction behaviour

References

[1] Alfred J Lotka. *Elements of Physical Biology*. Williams and Wilkins Company, 1925.

[2] Vito Volterra. "Fluctuations in the Abundance of a Species considered Mathematically". In: *Nature* 118 (1926), pp. 558–560.

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Thank you!

Supplementary Slides

Exponential LV model

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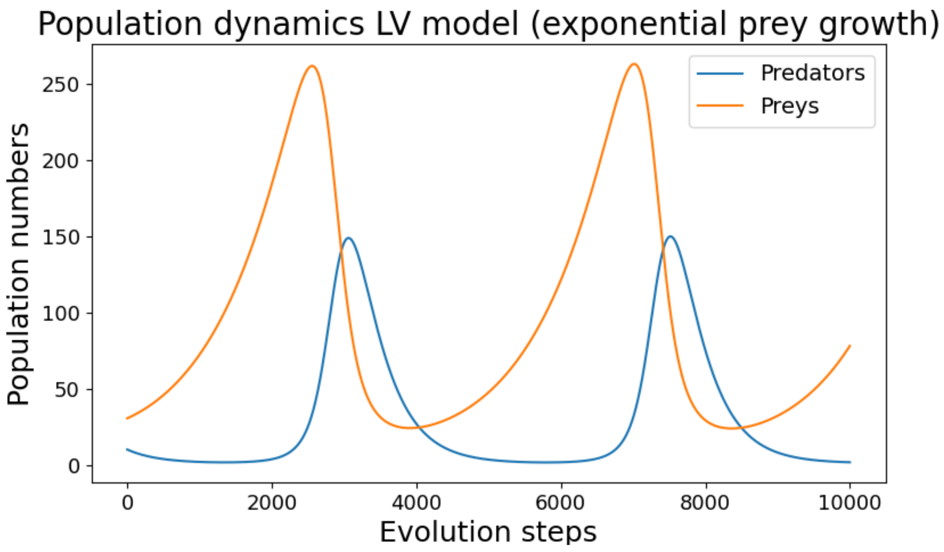
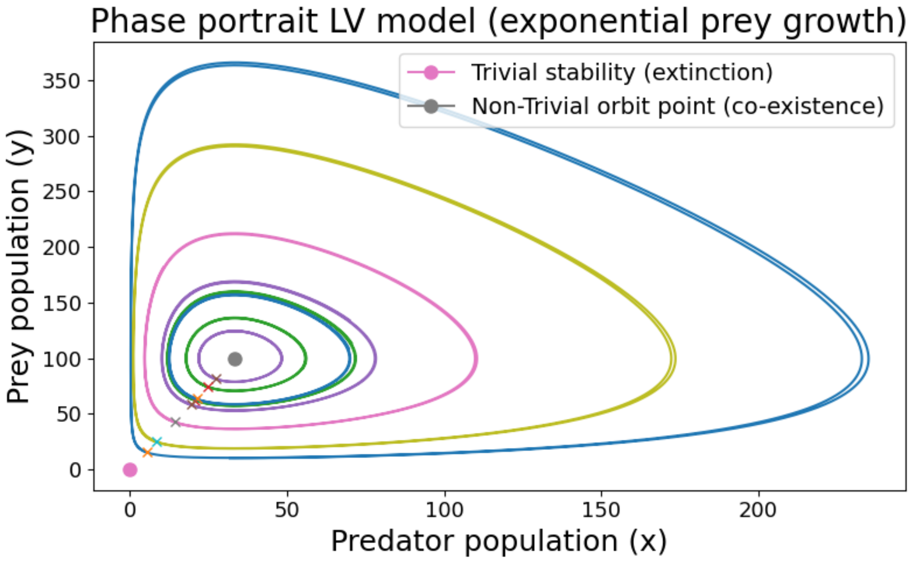
Simple model; stable points can be determined analytically:

$$\begin{cases} 0 = f(x,y) = x(\alpha y - \beta) \implies y^* = \frac{\beta}{\alpha} \\ 0 = g(x,y) = x(\alpha y - \beta) \implies x^* = \frac{\gamma}{\delta} \end{cases}$$

For $\alpha = \delta = 0.003,\ \beta = 0.3,\ \gamma = 0.1, dt = 0.01$:





$$\begin{cases} \dfrac{\partial x(t)}{\partial t} = \alpha xy - \beta x & := f(x,y) \\ \dfrac{\partial y(t)}{\partial t} = \gamma y(1 - \dfrac{y}{N}) - \delta xy & := g(x,y) \end{cases}$$

Logistic LV model

Stable points are $(x^*, y^*) \in \left\{ \Big(0,0\Big), \Big(0,L\Big), \Big(\frac{\gamma}{\delta}(1 - \frac{\beta}{\alpha N}), \frac{\beta}{\alpha}\Big) \right\}$

Bifurcation Analysis on non-trivial stable point:

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} \alpha y - \beta & \alpha x \\ -\delta y & \gamma(1 - \frac{2y}{N}) - \delta x \end{bmatrix} \longrightarrow J(x^*, y^*) = \begin{bmatrix} 0 & \frac{\alpha\gamma}{\delta}(1 - \frac{\beta}{\alpha N}) \\ -\frac{\delta\beta}{\alpha} & -\frac{\gamma\beta}{N\alpha} \end{bmatrix}$$

Considering eigenvalues of Jacobian

$$\det(J) = 0 \implies N = \frac{\beta}{\alpha} \text{ and}$$

$$\det(J) = \frac{\mathrm{tr}^2(J)}{4} \implies N = \frac{\alpha\beta + \sqrt{\alpha^2\beta^2 + \alpha^2\gamma\beta}}{2\alpha^2}$$

Logistic LV model

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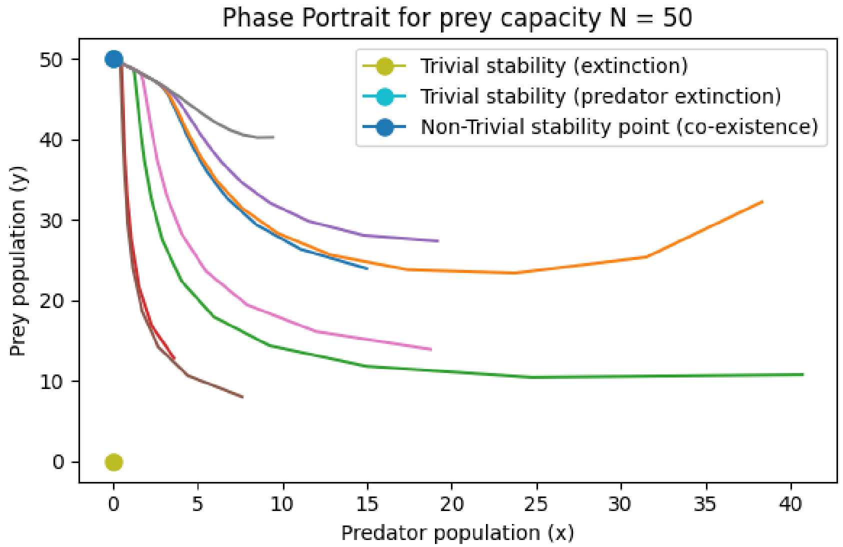
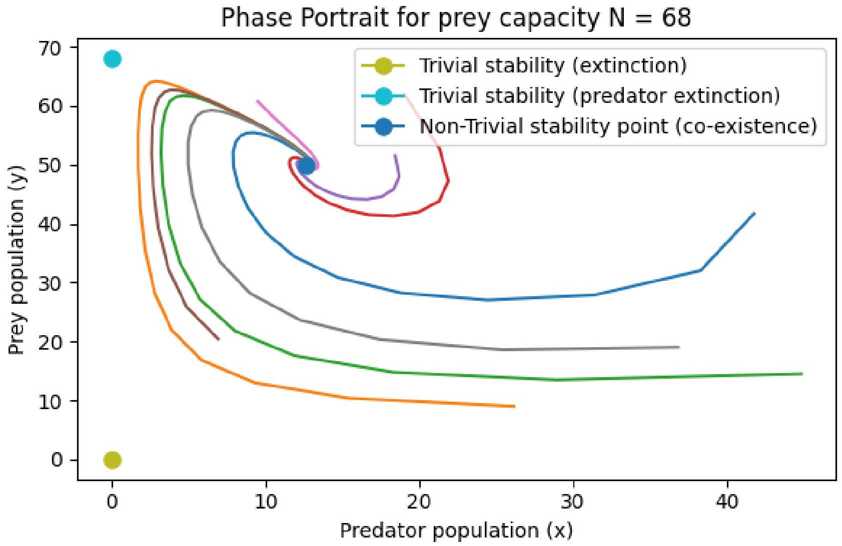
$$\det(J) = 0 \implies N = \frac{\beta}{\alpha} \text{ and}$$

$$\det(J) = \frac{\mathrm{tr}^2(J)}{4} \implies N = \frac{\alpha\beta + \sqrt{\alpha^2\beta^2 + \alpha^2\gamma\beta}}{2\alpha^2}$$

For $\alpha = \delta = 1,\ \beta = 50,\ \gamma = 48,\ dt = 0.01$:

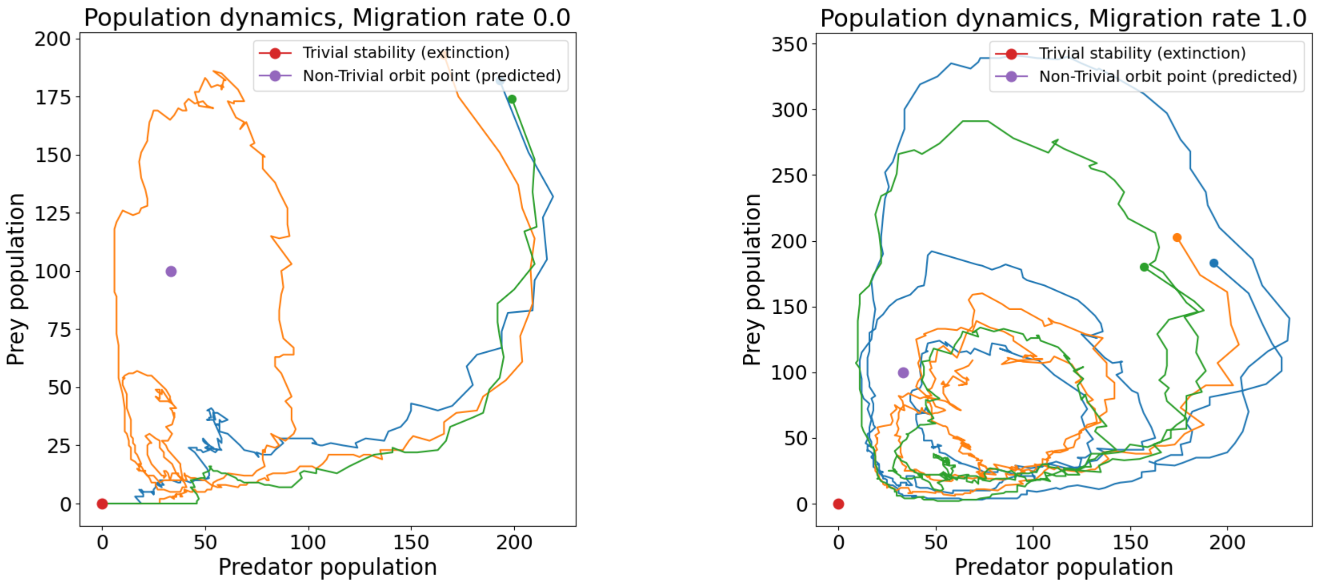
Bifurcation at $N = 60$ (towards spiral sink coexistence)





Probabilistic CA: migration analysis

Migration as another phenomenon that resembles reality (respecting spatio-temporal relation) which is difficult to capture analytically



Migration must be nonzero in order to observe predicted behaviour; hypothesis: we need reasonable mobility to ensure mixing and not getting stuck in local extinction, which can propagate to the entire system

Parameters for balanced evolution

- Overpopulation: 3; Loss prey in overpopulation: 1.7
- Predator's loss for moving: 2.5; Prey's gain for moving: 0.8
- Loss when hunting fails: 3.2
- Prey's loss when reproducing: 1.2
- Minimum energy for prey's reproduction: 3.9
- Minimum energy for predator's reproduction: 4.7