

6 Appendix

6.1 Lotka-Volterra Predator-Prey model

In the domain of biological modeling, a commonly used predator-prey model is the so-called Lotka-Volterra (LV) model. The LV model is a system of coupled differential equations, relating the rate of change in predator and prey populations to the existing population numbers as well as cross-terms. In a general form, we may write the LV system as

$$\begin{aligned}\frac{\partial x(t)}{\partial t} &= \alpha xy - \beta x := f(x, y) \\ \frac{\partial y(t)}{\partial t} &= \gamma y - \delta xy := g(x, y),\end{aligned}\tag{1}$$

where $x(t)$ and $y(t)$ denote the predator and prey population count at time t , respectively, and $\alpha, \beta, \gamma, \delta \geq 0$. Dissecting the functions f and g , we observe that the predator population grows from hunting prey ($+\alpha xy$) and is diminished by natural death ($-\beta x$), whereas the prey population grows by natural reproduction ($+\gamma y$) and is diminished by hunting ($-\delta xy$). This is of course a stark over-generalization of a real dynamic system, since we ignore spatial correlations, aging structure, reproduction cycles, other external influences etc. However, it is possible to analyze this model mathematically and derive some general ideas about extinction and co-existence in predator-prey systems, which motivates the study of LV model. In this project, we focus on a slight adaptation of LV model 1, which incorporates logistic (rather than exponential) growth of the prey population, as seen in System 2. In this system, we introduced the parameter N , which denotes the capacity of prey in the ecological system. The growth rate of y depends on both the current population size as well as the difference to the maximal population size, then.

$$\begin{cases} \frac{\partial x(t)}{\partial t} = \alpha xy - \beta x := f(x, y) \\ \frac{\partial y(t)}{\partial t} = \gamma y(1 - \frac{y}{N}) - \delta xy := g(x, y) \end{cases}\tag{2}$$

The LV model 2 reaches a stable point if $f(x, y) = 0 = g(x, y)$ holds. Trivially, this is the case for

$$(x^*, y^*) = (0, 0) \text{ and } (x^*, y^*) = (0, L).$$

A non-trivial point of stability can be derived by solving the equations separately. Assuming that $x \neq 0$, from the first rate equation, we obtain

$$0 = f(x, y) = x(\alpha y - \beta) \implies y^* = \frac{\beta}{\alpha}.\tag{3}$$

Then assuming that $0 \neq y = \frac{\beta}{\alpha}$, the second rate equation yields

$$\begin{cases} 0 = g(x, y) = \gamma y(1 - \frac{y}{N}) - \delta xy \\ \implies x^* = \frac{\gamma}{\delta}(1 - \frac{y}{N}) = \frac{\gamma}{\delta}(1 - \frac{\beta}{\alpha N}) \end{cases}\tag{4}$$

The stability point

$$(x^*, y^*) = \left(\frac{\gamma}{\delta}(1 - \frac{\beta}{\alpha N}), \frac{\beta}{\alpha} \right)$$

is insofar interesting that it defines a state of *coexistence* between the two species, whereas the trivial points of stability arose from extinction of at least one of the species. In the following, we

will be interested in *bifurcation* points of the system, i.e., values of the parameter N at which the behaviour of the model drastically changes (bifurcates). Therefore, we define the Jacobian matrix of System 2, which reads

$$J(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} \alpha y - \beta & \alpha x \\ -\delta y & \gamma(1 - \frac{2y}{N}) - \delta x \end{bmatrix}. \quad (5)$$

At the coexistence point 6.1, the Jacobian evaluates to

$$J(x^*, y^*) = \begin{bmatrix} 0 & \frac{\alpha\gamma}{\delta}(1 - \frac{\beta}{\alpha N}) \\ -\frac{\delta\beta}{\alpha} & -\frac{\gamma\beta}{N\alpha} \end{bmatrix}. \quad (6)$$

Consequently, we have

$$\text{tr}(J) = -\frac{\gamma\beta}{N\alpha}, \quad \det(J) = \beta\gamma(1 - \frac{\beta}{N\alpha}). \quad (7)$$

Now bifurcations occur at

$$\begin{aligned} \det(J) = 0 &\implies N = \frac{\beta}{\alpha} \text{ and} \\ \det(J) = \frac{\text{tr}^2(J)}{4} &\implies N = \frac{\alpha\beta + \sqrt{\alpha^2\beta^2 + \alpha^2\gamma\beta}}{2\alpha^2}. \end{aligned} \quad (8)$$

For our simulation, we choose

$$\alpha, \beta, \gamma, \delta = 1, 50, 48, 1 \quad (9)$$

and iterate over a prey capacity $N \in [40, 100]$. Then we clearly see the bifurcations at the predicted values, where the system first transitions from extinction to coexistence at $N = \frac{\beta}{\alpha} = 50$, and then from a sink at the point of stability to a spiral sink at $N = \frac{\alpha\beta + \sqrt{\alpha^2\beta^2 + \alpha^2\gamma\beta}}{2\alpha^2} = 60$.