Equilibration of forced barotropic turbulence by stimulated generation of near-inertial waves

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1 The model

In the vertical plane-wave model, waves affect the balanced flow via rectification terms in the quasi-geostrophic potential vorticity q:

$$q = \underbrace{\triangle\psi}_{\stackrel{\text{def}}{=}\zeta} + \underbrace{\frac{1}{f_0}\left[\frac{1}{4}\triangle|\phi|^2 + \frac{\mathrm{i}}{2}J(\phi^\star,\phi)\right]}_{\stackrel{\text{def}}{=}q^w}, \qquad (1)$$ qgpv

where $\psi(x,y,t)$ is the wave-averaged streamfunction and $\phi(x,y,t)$ is the near-inertial back-rotated velocity of a vertical plane wave with wavenumber m; the total horizontal velocity is

$$u + \mathrm{i}v = \phi \mathrm{e}^{\mathrm{i}\varpi} - \psi_y + \mathrm{i}\psi_x\,, \qquad (2)$$ phi

where $\varpi \stackrel{\text{def}}{=} mz - f_0 t$ is the phase of the near-inertial wave.

1.1 The forced potential vorticity equation: balanced dynamics

The balanced dynamics satisfy the potential vorticity equation

$$q_t + J(\psi, q) = \mathcal{F}_q - \mu\zeta + \mathcal{D}_q\,, \qquad (3)$$ balanced_dy

where μ is the linear bottom drag. And $\mathcal{F}_q$ is a stochastic forcing that renovates every τ,

$$\mathcal{F}_q = \xi_q(x,y,t)/\tau^{1/2}\,, \qquad (4)$$ F_q

where ξ_q is a normally-distributed random field with annular horizontal wavenumber spectrum centered at k_f:

$$\mathbb{E}(\hat{\xi}_q^\star \hat{\xi}_q) = A \exp\left\{-[(k^2+l^2)^{1/2} - k_f]^2/2\Delta_f^2\right\}; \qquad (5)$$ spec_forci

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Δ_f is the width of the spectrum; and $\hat{\xi_q}$ is the Fourier transform of ξ_q:

$$\hat{\xi_q}(k,l) = \frac{1}{(2\pi)^2} \iint \xi_q(x,y) \mathrm{e}^{\mathrm{i}(kx+ly)} \mathrm{d}k\mathrm{d}l\,. \tag{6}$$

The constant A is determined by the normalization condition,

$$\frac{1}{(2\pi)^2} \iint \tfrac{1}{2}(k^2+l^2)^{-1} \mathbb{E}(\hat{\xi_q}^{\star}\hat{\xi_q}) \mathrm{d}k\mathrm{d}l = \sigma_q^2\,. \tag{7}$$ norm_fq

In the waveless case, this normalization ensures that the expectation of the energy input equals the variance of the forcing σ_q^2:

$$\mathbb{E}[\xi_q(n)\xi_q(m)] = \sigma_q^2 \tau \delta_{mn}\,; \tag{8}$$

see discussion in the next section.

The white-noise forcing requires a renovation time scale smaller than the numerical time step, $\tau \ll \Delta t$. For numerical implementation, however, we are forced choose $\tau = \Delta t$. The stochastic forcing $\mathcal{F}_q$ is thus only an approximation to white noise. The white-noise approximation is accurature provided $\Delta t \ll (k_f U)^{-1}$, where U is the root-mean-square velocity of the turbulence. This requirements is satisfied in all solutions discussed below. We thus loosely refer to $\mathcal{F}_q$, and the wave forcing $\mathcal{F}_\phi$ defined below, as white-noise throughout.

1.2 The vertical plane-wave YBJ equation: wave dynamics

The vertical plane-wave model is completed by the YBJ equation for the evolution of the back-rotated near-inertial velocity ϕ:

$$\phi_t + J(\psi,\phi) + \phi \tfrac{\mathrm{i}}{2}\zeta - \tfrac{\mathrm{i}}{2}\eta\triangle\phi = \mathcal{F}_\phi - \gamma\phi + \mathcal{D}_\phi\,, \tag{9}$$ ybj_dynami

$\eta = f_0\lambda^2$ is the wave dispersivity and γ is a linear damping coefficient, which is related to the vertical viscosity that damps the near-inertial velocity:

$$\nu\partial_z^2(\phi \mathrm{e}^{\mathrm{i}\varpi}) = -\underbrace{m^2\nu}_{\stackrel{\mathrm{def}}{=}\gamma}\phi\,, \tag{10}$$

where $\varpi = mz - f_0 t$. Vertical viscosity parameterizes all processes—including wave-wave interactions—that were neglected by the asymptotic derivation of (9).

Also in (9), $\mathcal{F}_\phi(t)$ is a stochastic forcing that renovates every τ with variance $\mathbb{E}(\mathcal{F}_\phi^\star \mathcal{F}_\phi) = \sigma_\phi^2$; $\mathcal{F}_\phi$ has no spatial structure. The advantage of forcing the waves with this type of stochastic forcing is that the rate of energy input by the forcing is predicted. We experimented with constant and shot-noise forcings. The results from these experiments are qualitatively similar results to the solutions forced by white-noise.

In (3) and (9), $\mathcal{D}_q$ and $\mathcal{D}_\phi$ represent small-scale horizontal dissipation, which are necessary for numerical stability. In practice, we use an exponential spectral filter, which selectively damps aliased wavenumbers. In the solution described bellow, small-scale horizontal dissipation contributes insignificant sinks to the energy budgets.

1.3 Power integrals

The wave action equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\tfrac{1}{2f_0}\langle|\phi|^2\rangle}_{\stackrel{\text{def}}{=}\mathcal{A}} = \tfrac{1}{2f_0}\langle\phi^\star\xi_\phi + \phi\xi_\phi^\star\rangle - \tfrac{1}{f_0}\gamma\langle|\phi|^2\rangle + \tfrac{1}{2f_0}\langle\phi^\star\mathcal{D}_\phi + \phi\mathcal{D}_\phi^\star\rangle\,, \qquad \boxed{\texttt{A}} \tag{11}$$

where $\langle\,\rangle$ represents spatial average. In the white-noise limit, the expectation for the work due the wave forcing is

$$\mathbb{E}\Big(\tfrac{1}{2}\langle\phi^\star\xi_\phi + \phi\xi_\phi^\star\rangle\Big) = \tfrac{1}{2}\sigma_\phi^2\,. \tag{12}$$

And we obtain a prediction for the equilibrated wave action:

$$\mathbb{E}(\mathcal{A}) = \frac{\sigma_\phi^2}{2f_0\gamma}\,, \qquad \boxed{\texttt{prediced\_A}} \tag{13}$$

provided that small-scale dissipation $\frac{1}{2f_0}\langle\phi^\star\mathcal{D}_\phi + \phi\mathcal{D}_\phi^\star\rangle$ is insiginificant.

The balanced kinetic energy equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\tfrac{1}{2}\langle|\nabla\psi|^2\rangle}_{\stackrel{\text{def}}{=}\mathcal{K}} = -(\Gamma_r + \Gamma_a) + \Xi - \langle\psi\mathcal{F}_q\rangle - \mu\langle|\nabla\psi|^2\rangle - \langle\psi\mathcal{D}_q\rangle\,. \qquad \boxed{\texttt{Ke}} \tag{14}$$

Above, Γ_r and Γ_a are energy conversion terms,

$$\Gamma_r \stackrel{\text{def}}{=} \Big\langle \tfrac{1}{2}\zeta\, \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\mathcal{F}} \Big\rangle\,, \qquad \boxed{\texttt{convr}} \tag{15}$$

where the wave action flux is

$$\boldsymbol{\mathcal{F}} \stackrel{\text{def}}{=} \tfrac{\mathrm{i}}{4}\lambda^2\left(\phi\boldsymbol{\nabla}\phi^\star - \phi^\star\boldsymbol{\nabla}\phi\right); \qquad \boxed{\texttt{Fw2}} \tag{16}$$

and

$$\Gamma_a \stackrel{\text{def}}{=} -\tfrac{\lambda^2}{2}\left\langle \begin{bmatrix}\phi_x^\star & \phi_y^\star\end{bmatrix} \begin{bmatrix} -\psi_{xy} & \tfrac{1}{2}(\psi_{xx}-\psi_{yy}) \\ \tfrac{1}{2}(\psi_{xx}-\psi_{yy}) & \psi_{xy}\end{bmatrix} \begin{bmatrix}\phi_x \\ \phi_y\end{bmatrix} \right\rangle\,. \qquad \boxed{\texttt{conva}} \tag{17}$$

Also in (14), Ξ is a source of balanced kinetic energy due to wave dissipation:

$$\Xi = -\gamma\left[\big\langle\mathcal{A}\tfrac{1}{2}\zeta\big\rangle + \eta^{-1}\langle\mathbf{u}_g\boldsymbol{\cdot}\boldsymbol{\mathcal{F}}\rangle\right] + \tfrac{1}{2}f_0^{-1}\left[\big\langle(\phi^\star\mathcal{D}_\phi + \phi\mathcal{D}_\phi^\star)\tfrac{1}{2}\zeta\big\rangle\right] + \mathbf{u}_q\boldsymbol{\cdot}\tfrac{\mathrm{i}}{2}(\mathcal{D}_\phi\boldsymbol{\nabla}\phi^\star - \mathcal{D}_\phi^\star\boldsymbol{\nabla}\phi)\,, \qquad \boxed{\texttt{Xi}} \tag{18}$$

where $\mathbf{u}_g = \hat{\mathbf{z}}\times\boldsymbol{\nabla}\psi$ is the geostrophic velocity; Ξ is a form of "wave streaming." The first two terms in (18) stem from linear dissipation of ϕ. The first terms shows that the dissipation of wave action in anticylones is a source of balanced kinetic energy. The second term shows that the alignment of the geostrophic velocity with the wave action density flux is a sink of balanced kinetic energy. The remanining two terms stem from small-scale dissipation. See [1] for a derivation of Γ_r, Γ_a, and Ξ.

Table 1: Details of the reference solution.

parameters_reference

Parameter	Description	Value
N	Number of modes	512
L_d	Domain size	$2\pi \times 200$ km
σ_q^2	Balanced-forcing variance	$3.62\ 10^{-9}$ m^2 s^{-3}
σ_w^2	Wave-forcing variance	$5.78\ 10^{-8}$ m^2 s^{-3}
$k_f L_d/2\pi$	Balanced-forcing wavenumber	8
$dk_f L_d/2\pi$	Balanced-forcing width	1
μ	Linear bottom drag coefficient	$5.78\ 10^{-8}$ s^{-1}
γ	Linear wave damping coefficient	$2.31\ 10^{-7}$ s^{-1}
N	Buoyancy frequency	$5\ 10^{-3}$ s^{-1}
f_0	Coriolis frequency	$1\ 10^{-4}$ s^{-1}
$2\pi\, m^{-1}$	Vertical wavelength	800 m
$\mathcal{D}_\phi$	Exponential spectral filter	—
$\mathcal{D}_q$	Exponential spectral filter	—

In the waveless case, $\phi(t=0) = 0$ and $\mathcal{F}_\phi = 0$, which implies that $\Gamma_r = \Gamma_a = \Xi = 0$. In the white-noise limit, the waveless expectation for the work delivered by the forcing is

$$\mathbb{E}\big(- \langle \psi \xi_q \rangle \big) = \sigma_q^2 \,. \tag{19}$$

We thus obtain a prediction for the equilibrated balanced kinetic energy in the absence of waves:

$$\mathbb{E}(\mathcal{K}) = \frac{\sigma_q^2}{\mu} \,, \tag{20}$$ predicted_

provided that small-scale dissipation $-\langle \psi \mathcal{D}_q \rangle$ is insignificant.

Finally, the potential energy equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\tfrac{\lambda^2}{4} \langle |\nabla \phi|^2 \rangle}_{\stackrel{\text{def}}{=} \mathcal{P}} = \Gamma_r + \Gamma_a - \tfrac{\lambda^2}{2} \gamma \langle |\nabla \phi|^2 \rangle - \tfrac{\lambda^2}{2} \langle \triangle \phi^\star \mathcal{D}_\phi + \triangle \phi \mathcal{D}_\phi^\star \rangle \,. \tag{21}$$ P

Note that there's no external generation of wave potential energy $\mathcal{P}$ because the stochastic forcing has no spatial scale. $\mathcal{P}$ is only created by stimulated generation represented by the conversion terms Γ_r and Γ_a. In statistical steady state, the wave potential energy created by stimilated generation dissipates via $-2\gamma\mathcal{P}$, provided there's insignificant small-scale dissipation. But this isn't a prediction for $\mathcal{P}$ since Γ_a and Γ_r are functions of ϕ.

2 A reference solution

Figure 1 shows snapshots of potential vorticity and wave action density for a solution with $\sigma_w = 4\sigma_q$ and $\gamma = 4\mu$; table 1 describes in detail the parameters of this reference solution.

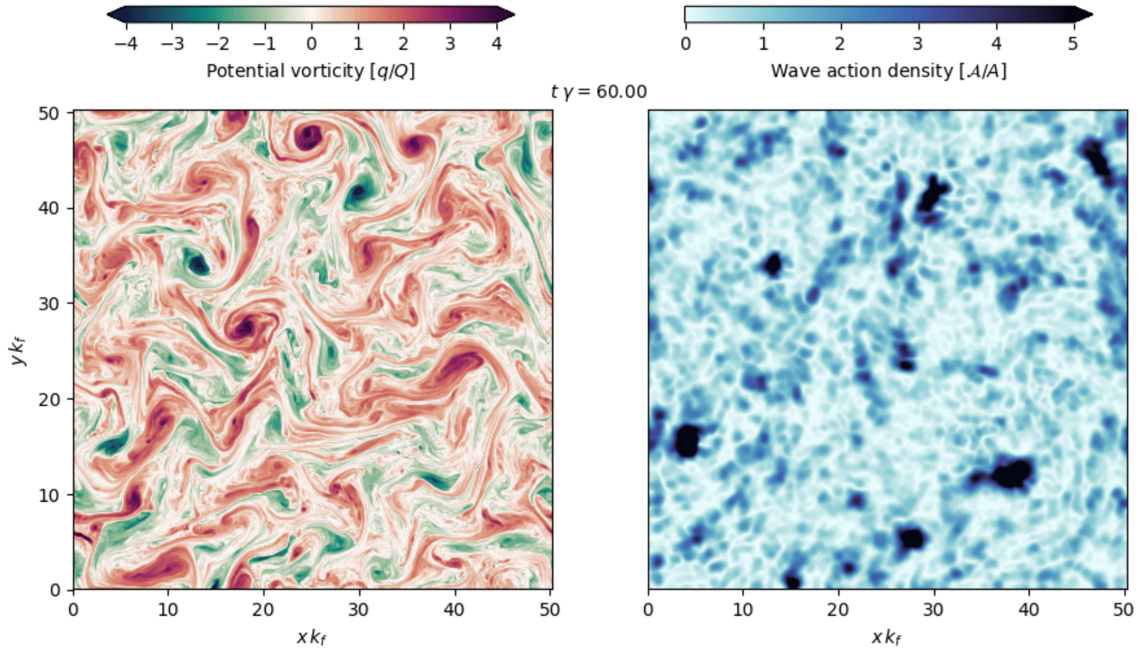

Figure 1: Snapshot of potential vorticity and wave action density for the solution with parameters in table 1. The scale of potential vorticity is $Q = \sigma_q/\mu^{1/2}$ and the scale of wave action density is $A = \sigma_w^2/f_0\gamma$.

snapshots_

Table 2: Balanced kinetic energy budget of the reference and no-drag and no-wave solutions. Each term is normalized by the predicted amount of power delivered by the random force σ_q^2.

K-budgets

	Reference	No-drag	No-wave
Work	1.00	1.00	1.00
Streamming	0.24	0.60	–
Stimulated generation	-0.64	-1.77	–
Linear drag	-0.67	–	-0.99
Residual.	-0.08	0.17	-0.01

Table 3: Wave kinetic energy budget of the reference and no-drag solutions. Values represent averages after equilibration, normalized by the work delivered the random force.

A-budgets

	Reference	No-drag
Work	1.00	1.00
Linear diss.	-1.06	–
Residual	0.04	0.06

Table 4: Wave potential energy budget of the reference and no-drag solutions. Values represent averages after equilibration, normalized by the production of wave potential energy via stimulated generation.

P-budgets

	Reference	No-drag
Refractive conversion	0.32	0.37
Advective conversion	0.68	0.63
Linear dissipation	-0.99	-0.97
Residual	-0.01	-0.03

(The parameters yield a solution in the asymptotic regime of valid of the model.) The equilibrated potential vorticity in figure 1a resembles the vorticity field of waveless two-dimensional turbulence with its ubiquitous eddies, filaments, and coherent structures. A main difference is that the the potential vorticity of this wave-modified turbulence is more fine-grained.

The snapshot of wave action density depicts the incoherent nature of the equilibrated wave field, which is being scrambled by the turbulent balanced field (figure 1b). The wave field develops scales smaller than the balanced eddies due to straining by the flow and wave interference. The snapshot in figure 1b resembles the wave field in decaying wave-modified two-dimensional turbulence [1].

Figure (2) shows time series of balanced kinetic energy and wave action and wave potential energy. The system equilibrates after $\sim 1\,\mu^{-1} = 4\gamma^{-1}$. Wave action $\mathcal{A}$ displays large fluctuations (50% of the time-average equilibrated value). Balanced kinetic energy $\mathcal{K}$ and wave potential energy $\mathcal{P}$, on the other hand, show much smaller fluctuations (10% of the equilibrated levels). Interestingly, $\mathcal{K}$ and $\mathcal{P}$ fluctuate largely out of phase.

The wave kinetic energy equilibrate at 60% of theoretical prediction for waveless turbulence forced by white-noise: $\mathbb{E}(\mathcal{K}) = \sigma_q^2/\mu$. And the wave potential energy equilibrates at about 10% of the wave kinetic energy level, which suggests that stimulated generation plays a crucial role in the equilibration of forced barotropic turbulence. Indeed, figure 3a shows that stimulated generation, $-(\Gamma_r + \Gamma_a)$ in (14), contributes about half of the sink of wave kinetic energy—bottom drag, $-2\mu\mathcal{K}$ in (14), accounts for the other half. Wave streaming, Ξ in (14), is small but significant: Ξ contributes about 5% source of $\mathcal{K}$. See table X1 for details of the budget.

The wave potential energy budget (figure 3b) confirms that linear dissipation $-2\gamma\mathcal{P}$ damps most of $\mathcal{P}$ created via stimulated generation; the residual is smaller than 1% (table X2). Similarly, linear dissipation $-2\gamma\mathcal{A}$ removes virtually all the wave action $\mathcal{A}$ input by the white-noise forcing. While the forcing input is nearly constant, wave action and the linear dissipation are highly intermitent. Thus, vertical viscosity damps the waves and the details of small-scale horizontal dissipation are irrelevant for the energy budget.

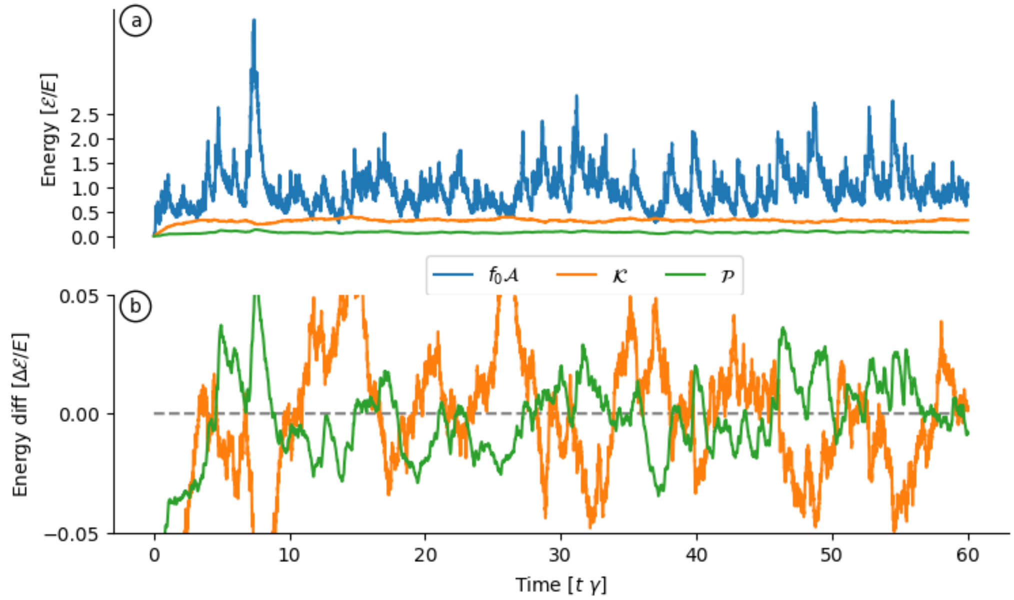

Figure 2: (a) Balanced kinetic energy ($\mathcal{K}$) and wave potential energy ($\mathcal{P}$) and wave kinetic energy ($f_0\mathcal{A}$) for the solution with parameters in table 1. The energy difference, $\Delta\mathcal{K}$ and $\Delta\mathcal{P}$, about a time average after equilibration ($t\,\gamma \geq 5$). energies_r

3 Solution with $\mu = 0$

A natural question is whether the system equilibrates without bottom drag. We thus solve the vertical plane-wave model with $\mu = 0$; all other parameters are the same as in the reference solution above. The balanced kinetic energy did not equilibrate over course of 80 wave dissipative time units (see figure 5). But the waves do equilibrate.

The snapshot of potential vorticity at $t\gamma = 60$ (figure 4) shows weaker large-scale vortices. These vortices are bigger than the the vortices of the reference solution with $\mu \neq 0$ but also less coherent. Thus, the balanced kinetic energy has cascade towards larger scales. The wave action density is concentrated near the big vortices and has much finer scales than the $\mu \neq 0$ solution above.

By $t\gamma = 60$ the balanced kinetic energy $\mathcal{K}$ is about the same times larger than the equilibrated wave kinetic energy $f_0\mathcal{A}$. Wave potential energy $\mathcal{P}$ appears to equilibrate at 10% of $f_0\mathcal{A}$.

The wave action budget remains the same: the action delivered by the the stochastic forcing is removed by linear dissipation of ϕ. The wave potential energy budget is also as simple as before: $\mathcal{P}$ is created by stimulated generation and removed by linear dissipation. And geostrophic straining accounts for most of stimulated generation.

The balanced kinetic energy budget differs markedly from the $\mu \neq 0$ solution. Wave streaming accounts for about 30-40% of the generation of $\mathcal{K}$; all the input energy is removed by stimulated generation.

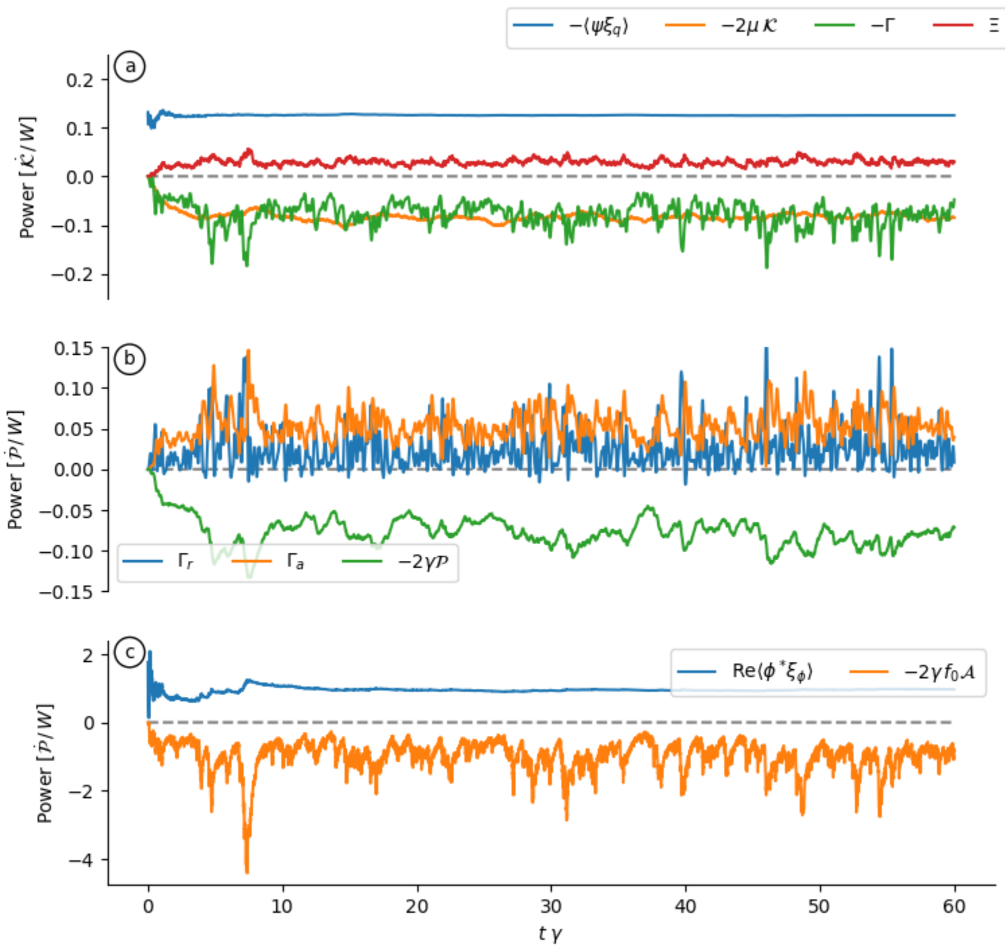

Figure 3: The budget of (a) balanced kinetic energy ($\mathcal{K}$), wave potential energy ($\mathcal{P}$), and (c) wave kinetic energy ($f_0\mathcal{A}$) for the solution with parameters in table 1. The power is scaled by the work due to the wave forcing $W = \sigma_w^2/2$.

energy_budg

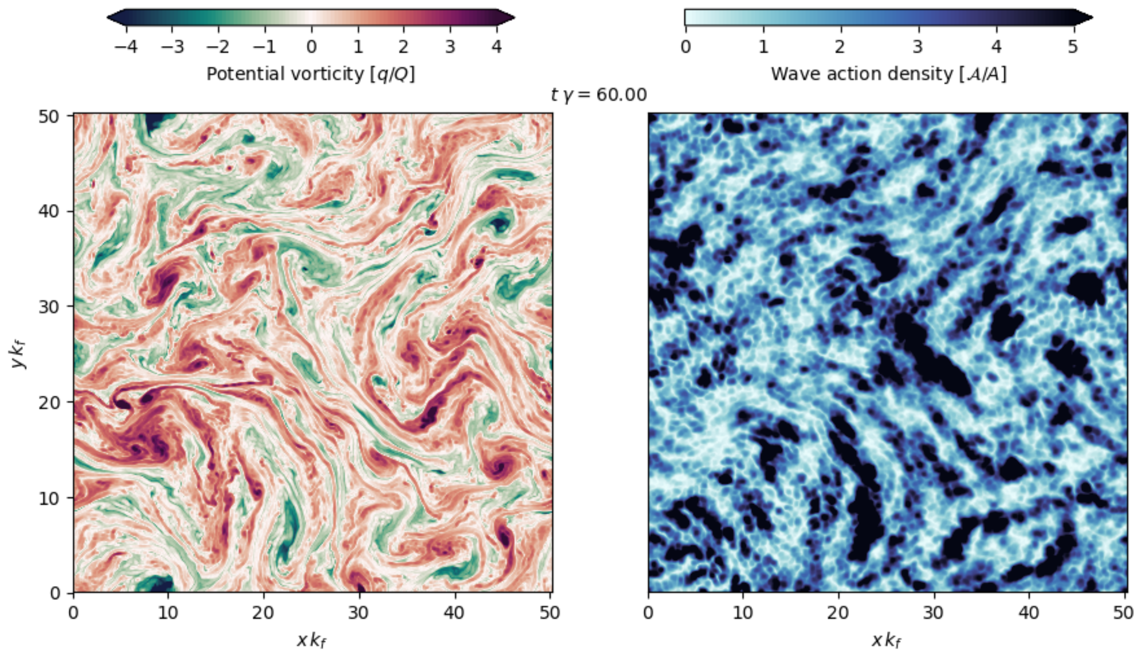

Figure 4: Snapshot of potential vorticity and wave action density for the solution with parameters in table 1 but $\mu = 0$. The scale of potential vorticity and action are the same as in figure 1.

snapshots_

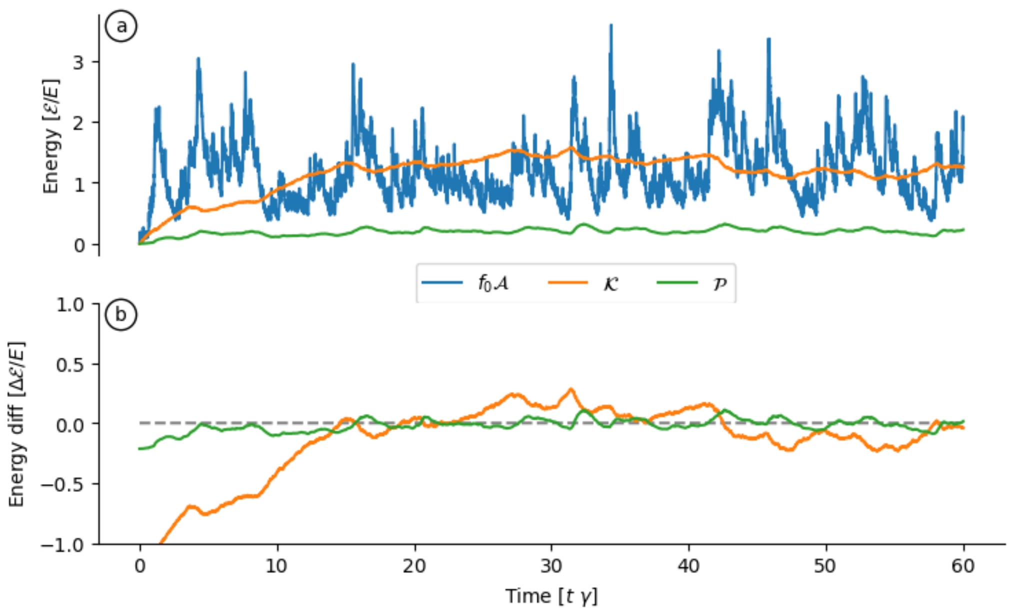

Figure 5: (a) Balanced kinetic energy ($\mathcal{K}$) and wave potential energy ($\mathcal{P}$) and wave kinetic energy ($f_0\mathcal{A}$) for the solution with parameters in table 1. The energy difference, $\Delta\mathcal{K}$ and $\Delta\mathcal{P}$, about a time average after equilibration ($t\,\gamma \geq 5$).

energies_n

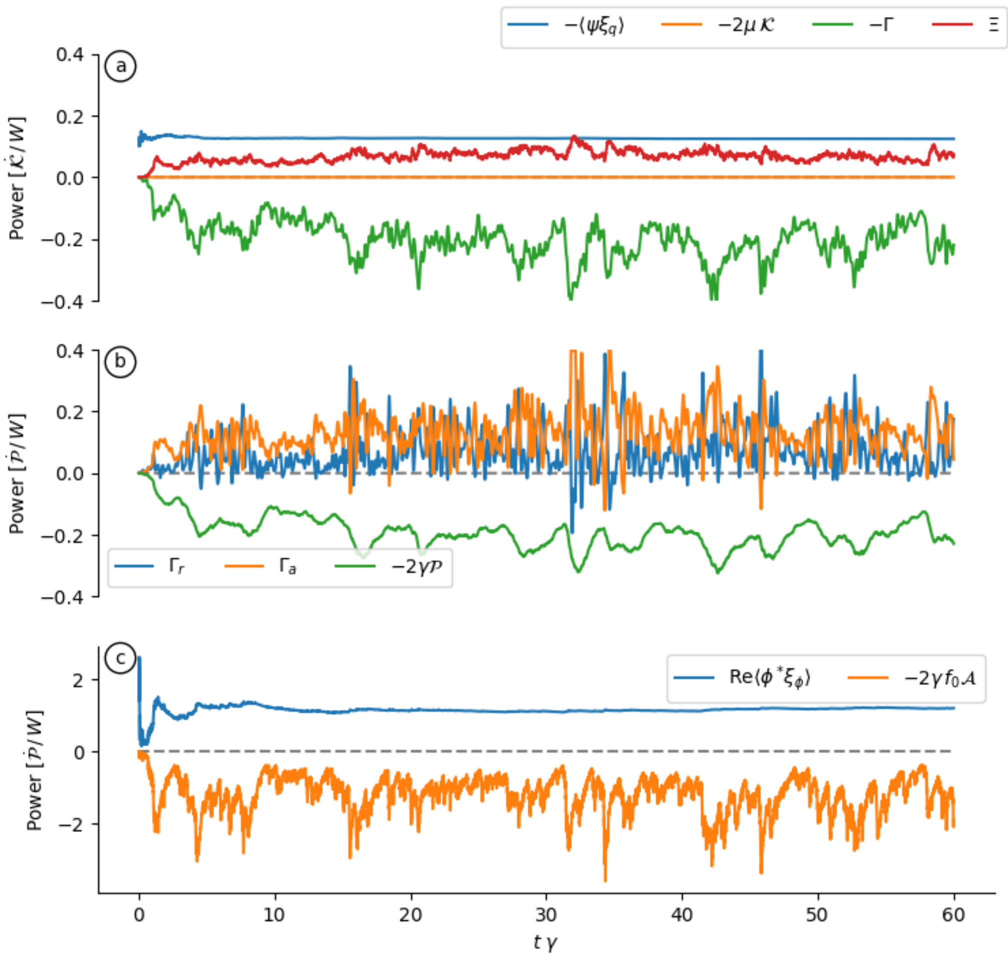

Figure 6: The budget of (a) balanced kinetic energy ($\mathcal{K}$), wave potential energy ($\mathcal{P}$), and (c) wave kinetic energy ($f_0\mathcal{A}$) for the solution with parameters in table 1. The power is scaled by the work due to the wave forcing $W = \sigma_w^2/2$.

energy_budg

References

a_etal2017 [1] Cesar B Rocha, Gregory L. Wagner, and William R. Young. Stimulated generation: extraction of energy from balanced flow by near-inertial waves. *Submitted, Journal of Fluid Mechanics.*