Equilibration of forced barotropic turbulence by stimulated generation of near-inertial waves

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1 The model

In the vertical plane-wave model, waves affect the balanced flow via rectification terms in the quasi-geostrophic potential vorticity q:

$$q = \underbrace{\triangle\psi}_{\stackrel{\text{def}}{=}\zeta} + \underbrace{\frac{1}{f_0}\left[\frac{1}{4}\triangle|\phi|^2 + \frac{\mathrm{i}}{2}J(\phi^\star, \phi)\right]}_{\stackrel{\text{def}}{=}q^w}, \tag{1}$$

`qgpv`

where $\psi(x, y, t)$ is the wave-averaged streamfunction and $\phi(x, y, t)$ is the near-inertial back-rotated velocity of a vertical plane wave with wavenumber m; the total velocity is

$$u + \mathrm{i}v = \phi\mathrm{e}^{\mathrm{i}\varpi} - \psi_y + \mathrm{i}\psi_x\,, \tag{2}$$

`phi`

where $\varpi \stackrel{\text{def}}{=} mz - f_0 t$ is the phase of the near-inertial wave.

The balanced dynamics satisfy the potential vorticity equation

$$q_t + J(\psi, q) = \mathcal{F}_q - \mu\zeta + \mathcal{D}_q\,, \tag{3}$$

`balanced_dy`

where μ is the linear bottom drag. And $\mathcal{F}_q$ is a stochastic forcing that renovates every τ,

$$\mathcal{F}_q = \xi_q(x, y, t)/\tau^{1/2}\,, \tag{4}$$

`F_q`

where ξ_q is a normally-distributed random field with annular horizontal wavenumber spectrum centered at k_f:

$$\mathbb{E}(\hat{\xi}_q^\star\hat{\xi}_q) = A\exp\left\{-[(k^2 + l^2)^{1/2} - k_f]^2/2\Delta_f^2\right\}, \tag{5}$$

`spec_forcin`

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where Δ_f is the width of the spectrum and $\hat{\xi}_q$ is the Fourier transform of ξ_q:

$$\hat{\xi}_q(k,l) = \frac{1}{(2\pi)^2} \iint \xi_q(x,y) \mathrm{e}^{\mathrm{i}(kx+ly)} \mathrm{d}k\mathrm{d}l \,. \tag{6}$$

The normalization condition,

$$\frac{1}{(2\pi)^2} \iint \tfrac{1}{2}(k^2+l^2)\mathbb{E}(\hat{\xi}_q^\star \hat{\xi}_q)\mathrm{d}k\mathrm{d}l = 1 \,, \tag{7}$$

norm_fq

determines the constant A. In the waveless case, this normalization ensures that the expectation of the energy input equals the variance of the forcing σ_q^2:

$$\mathbb{E}[\xi_q(n)\xi_q(m)] = \sigma_q^2 \tau \delta_{mn} \,; \tag{8}$$

see discussion in the next section.

The vertical plane-wave model is completed by the YBJ equation for the evolution of the back-rotated near-inertial velocity ϕ:

$$\phi_t + J(\psi,\phi) + \phi\tfrac{\mathrm{i}}{2}\zeta - \tfrac{\mathrm{i}}{2}\eta\triangle\phi = \mathcal{F}_\phi - \gamma\phi + \mathcal{D}_\phi \,, \tag{9}$$

ybj_dynami

$\eta = f_0\lambda^2$ is the wave dispersivity and γ is a linear damping coefficient, which is related to the vertical viscosity that damps the near-inertial velocity:

$$\nu\partial_z^2(\phi\mathrm{e}^{\mathrm{i}\varpi}) = -\underbrace{m^2\nu}_{\stackrel{\mathrm{def}}{=}\gamma}\phi \,, \tag{10}$$

where $\varpi = mz - f_0 t$. Vertical viscosity parameterizes all processes—including wave-wave interactions—that were neglected by the asymptotic derivation of (9).

Also in (9), $\mathcal{F}_\phi(t)$ is a stochastic forcing that renovates every τ with variance $\mathbb{E}(\mathcal{F}_\phi^\star\mathcal{F}_\phi) = \sigma_\phi^2$; $\mathcal{F}_\phi$ has no spatial structure. The advantage of forcing the waves with this type of stochastic forcing is that the rate of energy input by the forcing is predicted. We experimented with constant and shot-noise forcings. The results from these experiments are qualitatively similar results to the solutions forced by white-noise.

In (3) and (9), $\mathcal{D}_q$ and $\mathcal{D}_\phi$ represent small-scale horizontal dissipation, which are necessary for numerical stability. In practice, we use an exponential spectral filter, which selectively damps aliased wavenumbers. In the solution described bellow, small-scale horizontal dissipation contributes insignificant sinks to the energy budgets.

1.1 Power integrals

The wave action equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\tfrac{1}{2f_0}\langle|\phi|^2\rangle}_{\stackrel{\mathrm{def}}{=}\mathcal{A}} = \tfrac{1}{2f_0}\langle\phi^\star\xi_\phi + \phi\xi_\phi^\star\rangle - \tfrac{1}{f_0}\gamma\langle|\phi|^2\rangle + \tfrac{1}{2f_0}\langle\phi^\star\mathcal{D}_\phi + \phi\mathcal{D}_\phi^\star\rangle \,, \tag{11}$$

A

where $\langle\,\rangle$ represents spatial average. In the white-noise limit, the expectation for the work due the wave forcing is

$$\mathbb{E}\Big(\tfrac{1}{2}\langle\phi^\star\xi_\phi + \phi\xi_\phi^\star\rangle\Big) = \tfrac{1}{2}\sigma_\phi^2 \,. \tag{12}$$

And we obtain a prediction for the equilibrated wave action:

$$\mathbb{E}(\mathcal{A}) = \frac{\sigma_\phi^2}{2\gamma}\,, \tag{13} \qquad \boxed{\texttt{prediced\_A}}$$

provided that small-scale dissipation $\frac{1}{2f_0}\langle\phi^\star\mathcal{D}_\phi + \phi\mathcal{D}_\phi^\star\rangle$ is insiginificant.

The balanced kinetic energy equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\tfrac{1}{2}\langle|\nabla\psi|^2\rangle}_{\stackrel{\text{def}}{=}\mathcal{K}} = -(\Gamma_r + \Gamma_a) + \Xi - \langle\psi\mathcal{F}_q\rangle - \;\mu\langle|\nabla\psi|^2\rangle - \langle\psi\mathcal{D}_q\rangle\,, \tag{14} \qquad \boxed{\texttt{Ke}}$$

where Γ_r and Γ_a are energy conversion terms,

$$\Gamma_r \stackrel{\text{def}}{=} \Big\langle \tfrac{1}{2}\zeta\,\boldsymbol{\nabla\cdot}\boldsymbol{\mathcal{F}}\Big\rangle\,, \tag{15} \qquad \boxed{\texttt{convr}}$$

where the wave action flux is

$$\boldsymbol{\mathcal{F}} \stackrel{\text{def}}{=} \tfrac{\mathrm{i}}{4}\lambda^2\left(\phi\boldsymbol{\nabla}\phi^\star - \phi^\star\boldsymbol{\nabla}\phi\right); \tag{16} \qquad \boxed{\texttt{Fw2}}$$

and

$$\Gamma_a \stackrel{\text{def}}{=} -\tfrac{\lambda^2}{2}\left\langle \begin{bmatrix}\phi_x^\star & \phi_y^\star\end{bmatrix}\begin{bmatrix}-\psi_{xy} & \tfrac{1}{2}(\psi_{xx}-\psi_{yy})\\ \tfrac{1}{2}(\psi_{xx}-\psi_{yy}) & \psi_{xy}\end{bmatrix}\begin{bmatrix}\phi_x\\ \phi_y\end{bmatrix}\right\rangle\,. \tag{17} \qquad \boxed{\texttt{conva}}$$

Also in (14), Ξ is a source of balanced kinetic energy due to wave dissipation:

$$\Xi = -\gamma\left[\langle\mathcal{A}\tfrac{1}{2}\zeta\rangle + \eta^{-1}\langle\mathbf{u}_g\boldsymbol{\cdot}\boldsymbol{\mathcal{F}}\rangle\right] + \tfrac{1}{2}f_0^{-1}\left[\langle(\phi^\star\mathcal{D}_\phi + \phi\mathcal{D}_\phi^\star)\tfrac{1}{2}\zeta\rangle\right] + \mathbf{u}_q\boldsymbol{\cdot}\tfrac{\mathrm{i}}{2}(\mathcal{D}_\phi\boldsymbol{\nabla}\phi^\star - \mathcal{D}_\phi^\star\boldsymbol{\nabla}\phi)\,, \tag{18} \qquad \boxed{\texttt{Xi}}$$

where $\mathbf{u}_g = \hat{\mathbf{z}}\times\boldsymbol{\nabla}\psi$ is the geostrophic velocity. The first two terms in (18) stem from linear dissipation of ϕ. The first terms shows that the dissipation of wave action in anticylones is a source of balanced kinetic energy. The second term shows that the anti-alignment of the geostrophic velocity with the wave action density flux is also a source of balanced kinetic energy. The remanining two terms stem from small-scale dissipation. See RWY for a derivation of Γ_r, Γ_a, and Ξ.

In the waveless case, $\phi(t=0) = 0$ and $\mathcal{F}_\phi = 0$, which implies that $\Gamma_r = \Gamma_a = \Xi = 0$. In the white-noise limit, the waveless expectation for the work delivered by the forcing is

$$\mathbb{E}\big(-\langle\psi\xi_q\rangle\big) = \sigma_q^2\,. \tag{19}$$

Thus, we obtain a prediction for the equilibrated balanced kinetic energy:

$$\mathbb{E}(\mathcal{K}) = \frac{\sigma_q^2}{\mu}\,, \tag{20} \qquad \boxed{\texttt{predicted\_}}$$

Table 1: Details of the reference solution.

Parameter	Description	Value
N	Number of modes	512
L_d	Domain size	$2\pi \times 200$ km
σ_q^2	Balanced-forcing variance	$1.45\ 10^{-8}$ m^2 s^{-3}
σ_w^2	Wave-forcing variance	$5.78\ 10^{-8}$ m^2 s^{-3}
$k_f L_d/2\pi$	Balanced-forcing wavenumber	8
$dk_f L_d/2\pi$	Balanced-forcing width	1
μ	Linear bottom drag coefficient	$5.78\ 10^{-8}$ s^{-1}
γ	Linear wave damping coefficient	$2.31\ 10^{-7}$ s^{-1}
N	Buoyancy frequency	$5\ 10^{-3}$ s^{-1}
f_0	Coriolis frequency	$1\ 10^{-4}$ s^{-1}
$\mathcal{D}_\phi$	Exponential spectral filter	—
$\mathcal{D}_q$	Exponential spectral filter	—

parameters_reference

provided that small-scale dissipation $-\langle \psi \mathcal{D}_q \rangle$ is insignificant.

Finally, the potential energy equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\tfrac{\lambda^2}{4}\langle |\nabla\phi|^2\rangle}_{\stackrel{\mathrm{def}}{=}\mathcal{P}} = \Gamma_r + \Gamma_a - \tfrac{\lambda^2}{2}\gamma\langle|\nabla\phi|^2\rangle - \tfrac{\lambda^2}{2}\langle \triangle\phi^\star\mathcal{D}_\phi + \triangle\phi\mathcal{D}_\phi^\star\rangle \,. \qquad (21) \quad \boxed{\mathrm{P}}$$

Note that there's no external generation of wave potential energy $\mathcal{P}$ because the stochastic forcing has no spatial scale. $\mathcal{P}$ is only created by stimulated generation represented by the conversion terms Γ_r and Γ_a. In statistical steady state, the wave potential energy created by stimilated generation dissipates via $-2\gamma\mathcal{P}$, provided there's insignificant small-scale dissipation. But this isn't a prediction for $\mathcal{P}$ since Γ_a and Γ_r are functions of ϕ.

2 A reference solution

Figure 1 shows snapshots of potential vorticity and wave action density for a solution with $\sigma_w^2 = 2\sigma_q^2$ and $\gamma = 4\mu$; table 1 describes in detail the parameters of this reference solution. The equilibrated potential vorticity in figure 1a resembles the vorticity field of waveless two-dimensional turbulence with its ubiquitous eddies, filaments, and coherent structures. A main difference is that the the potential vorticity of this wave-modified turbulence is more fine-grained (see spectrum?).

The snapshot of wave action density depicts the incoherent nature of the equilibrated wave field, which is being scrambled by the turbulent balanced field (figure 1b). The wave field develops scales smaller than the balanced eddies due to straining by the flow and wave interference. This snapshot resembles the wave field in decaying wave-modified two-dimensional turbulence (RWY).

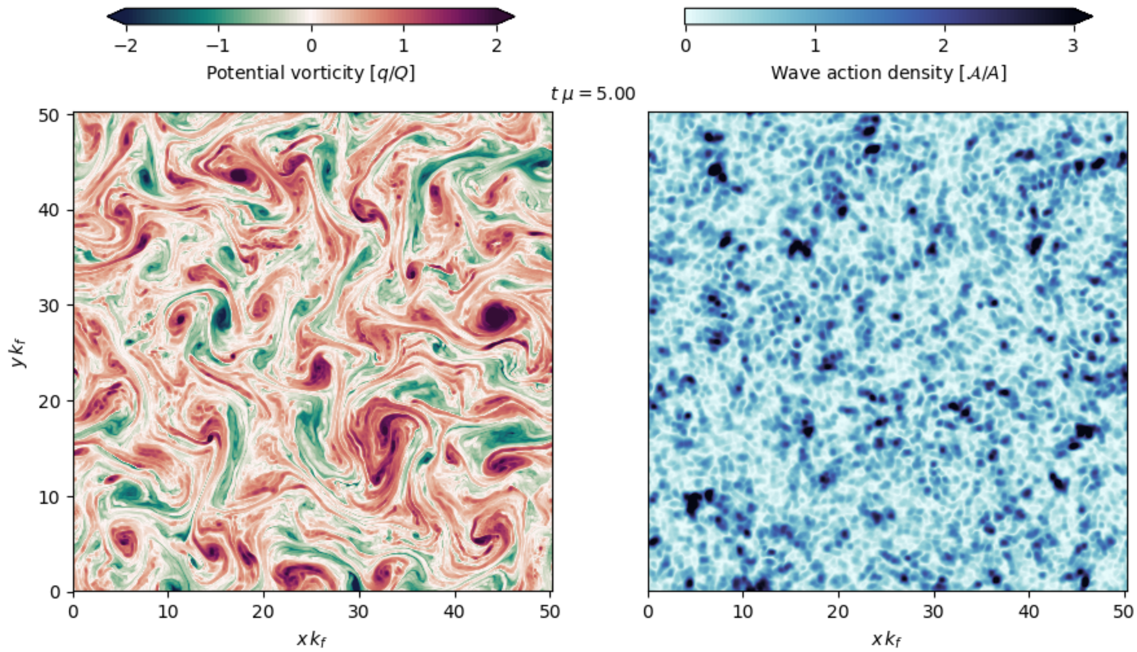

Figure 1: Snapshot of potential vorticity and wave action density for the solution with parameters in table 1. The scale of potential vorticity is $Q = \sigma_q/\mu^{1/2}$ and the scale of wave action density is $A = \sigma_w^2/f_0\gamma$.

snapshots_

Figure (2) shows time series of balanced kinetic energy and wave action and wave potential energy. The system equilibrates after $\sim 1\,\mu^{-1} = 4\gamma^{-1}$. Wave action $\mathcal{A}$ displays large fluctuations (50% of the time-average equilibrated value). Balanced kinetic energy $\mathcal{K}$ and wave potential energy $\mathcal{P}$, on the other hand, show much smaller fluctuations (10% of the equilibrated levels). Interestingly, $\mathcal{K}$ and $\mathcal{P}$ fluctuate largely out of phase.

The wave kinetic energy equilibrate at 60% of theoretical prediction for waveless turbulence forced by white-noise: $E = \sigma_q^2/\mu$. And the wave potential energy equilibrates at about 10% of the wave kinetic energy level, which suggests that stimulated generation plays a crucial role in the equilibration of forced barotropic turbulence. Indeed, figure 3a shows that stimulated generation, $-(\Gamma_r + \Gamma_a)$ in (14), contributes about half of the sink of wave kinetic energy—bottom drag, $-2\mu\mathcal{K}$ in (14), accounts for the other half. Wave streaming, Ξ in (14), is small but significant; Ξ contributes about 5% source of $\mathcal{K}$. See table X1 for details of the budget.

The wave potential energy budget (figure 3b) confirms that linear dissipation $-2\gamma\mathcal{P}$ damps most of $\mathcal{P}$ created via stimulated generation; the residual is smaller than 1% (table X2). Similarly, linear dissipation $-2\gamma\mathcal{A}$ removes virtually all the wave action $\mathcal{A}$ input by the white-noise forcing. While the forcing input is nearly constant, wave action and the linear dissipation are highly interment. Thus, vertical viscosity damps the waves and the details of horizontal small-scale dissipation are irrelevant.

The balanced kinetic energy spectrum has a submesoscale spectrum much smaller the

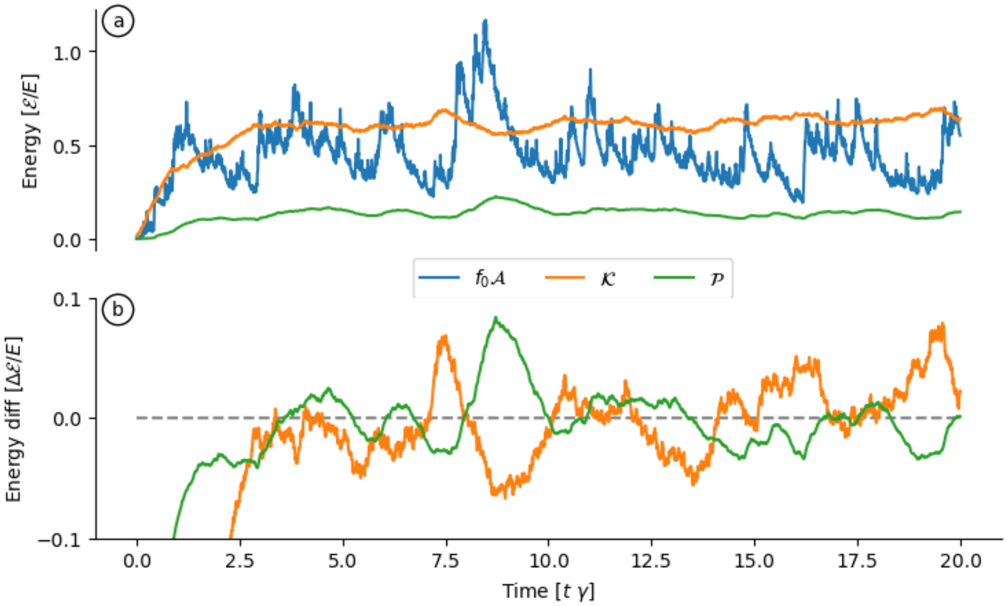

Figure 2: (a) Balanced kinetic energy ($\mathcal{K}$) and wave potential energy ($\mathcal{P}$) and wave kinetic energy ($f_0\mathcal{A}$) for the solution with parameters in table 1. The energy difference, $\Delta\mathcal{K}$ and $\Delta\mathcal{P}$, about a time average after equilibration ($t\,\gamma \geq 5$).

energies_r

one of waveless turbulence. Stimulated generation also appears to slow down the inverse cascade of balanced kinetic energy: there's more energy at large-scales in the spectrum of waveless turbulence. The wave action has a nearly flat spectrum between the forcing scale k_f and the dissipative scale k_d.

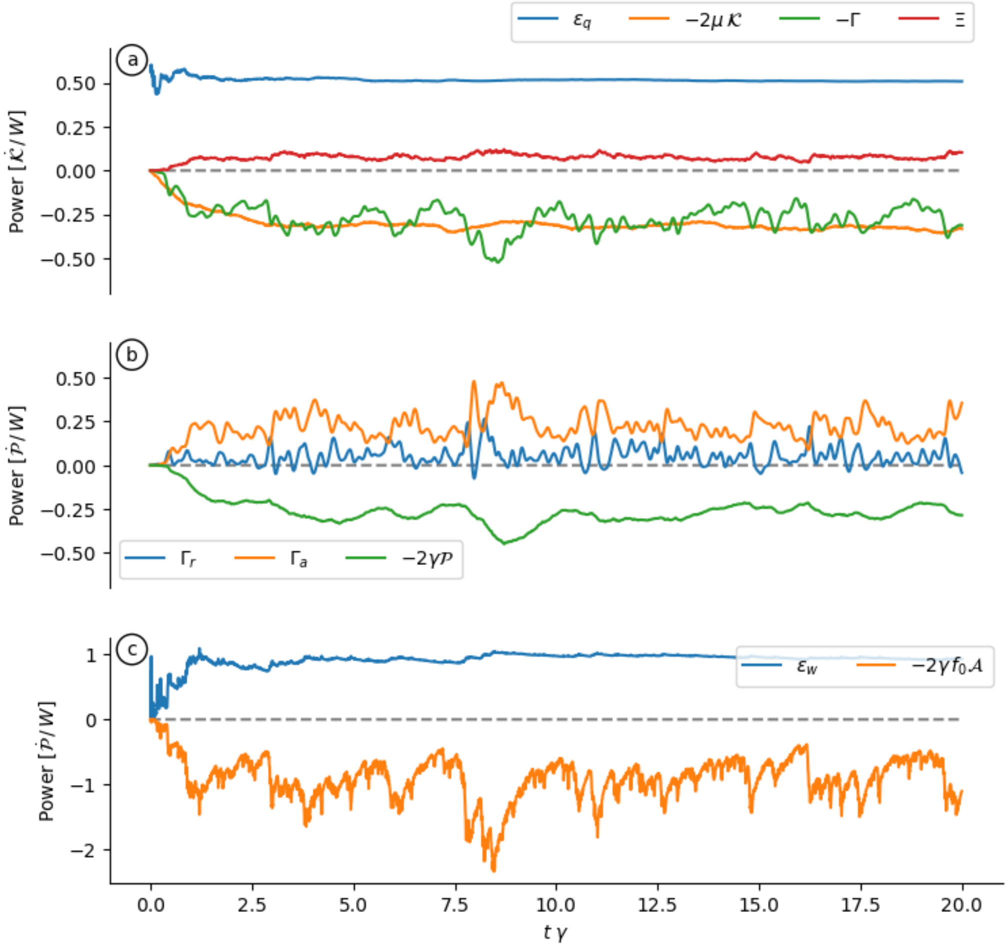

Figure 3: The budget of (a) balanced kinetic energy ($\mathcal{K}$), wave potential energy ($\mathcal{P}$), and (c) wave kinetic energy ($f_0\mathcal{A}$) for the solution with parameters in table 1. The power is scaled by the work due to the wave forcing $W = \sigma_w^2/2$.

energy_bud

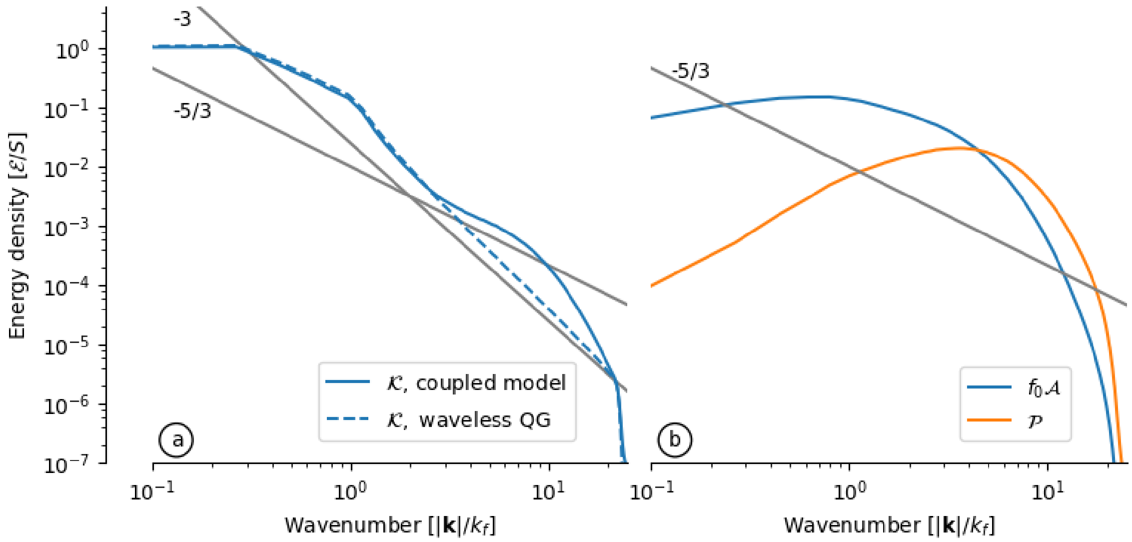

Figure 4: Spectra of balanced kinetic energy (a) and wave kinetic and potential energies (b) calculated after equilibration ($t\gamma \geq 5$). The dashed line in (a) shows the balanced kinetic energy spectrum from a reference waveless simulation.

spectra_re