Extraction of energy from barotropic turbulence by near-inertial waves: freely-evolving flow

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Key words:

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1. Introduction

Because geostrophic turbulence encumbers the flow of energy from forcing to dissipative scales, fluxing energy back towards larger scales, poorly constrained ageostrophic motions must play an important role in closing the oceanic energy budget (Ferrari & Wunsch 2009). Existing studies suggest a panoply of mechanisms to account for the required forward flow of energy. These include (but are not limited to) surface and benthic boundary turbulence and wave generation by geostrophic eddies sloshing over bottom topography (see Nagai *et al.* 2015, their figure 1, and references therein). But roughly half of the energy flux out of the ocean mesoscales remains unaccounted for.

Recent studies suggest that the interaction between geostrophic flows with externally-forced (windy) internal waves provides a significant sink of mesoscale eddy energy (Xie & Vanneste 2015; Taylor & Straub 2016; Wagner & Young 2016; Barkan *et al.* 2016). The emphasis on interactions of geostrophic flow with windy internal waves — here denoted stimulated imbalance† — contrasts with energy loss via spontaneous emission — a process by which internal waves are emitted during geostrophic adjustment (e.g., Shakespeare & Hogg 2017). While spontaneous emission is localized at sharp fronts (large Rossby number, Ro $\gtrsim$ 1) in the surface boundary layer (e.g., Shakespeare & Hogg 2017), stimulated imbalance operates at the small Rossby-number (Ro $\ll$ 1), quasi-geostrophic regime, both in the upper ocean and in the interior (e.g., Xie & Vanneste 2015).

We here study the interaction between macroturbulence and internal waves using a simple model that contains stimulated imbalance. This minimal model is a special family of solutions of the Xie & Vanneste (2015) that couples the evolution of the back-rotated velocity of near-inertial waves to quasigeostrophic flow. The focus on QG-NIW

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† Following Xie & Vanneste (2015) and Wagner & Young (2016) we define *stimulated imbalance* as wave-mean interactions that transfer energy from geostrophic motions (the 'mean flow') to existing internal waves. This definition constrasts with the nomenclarute recently employed by Barkan *et al.* (2016), who term stimulated imbalance a process by which internal waves trigger a forward cascade from mesoscale to submesoscale (but subinertial) flows.

interactions is justified: geostrophic eddies account for the bulk of the ocean's eddy kinetic energy (90%, Ferrari & Wunsch 2009) and near-inertial waves contain most of the oceanic high-frequency variability (Alford *et al.* 2016). The use of a minimal QG-NIW coupled model allows us to understand and characterize the physics of stimulated imbalance associated with vertical vorticity and lateral strain.

2. The physics of NIW energy extraction from a barotropic QG flow

2.1. *The Xie & Vanneste (2015) minimal QG-NIW model*

With barotropic quasigeostrophic flow, $\psi = \psi(x, y, t)$, uniform background buoyancy frequency N_0, and single-mode NIW vertical structure, e^{imz}, the Xie & Vanneste (2015) QG-NIW coupled model reduces to

$$q_t + \mathsf{J}(\psi, q) = 0\,, \tag{2.1}$$

with the NIW-averaged quasigeostrophic potential vorticity (QGPV)

$$q = \triangle\psi + \tfrac{1}{f_0}\left[\tfrac{1}{4}\triangle|\phi|^2 + \tfrac{i}{2}\mathsf{J}(\phi^\star, \phi)\right]\,, \tag{2.2}$$

and the streamfunction is defined so that the geostrophic flow velocity is $(u_e, v_e) = (-\psi_y, \psi_x)$; and

$$\phi_t + \mathsf{J}(\psi, \phi) + \tfrac{i}{2}\phi\triangle\psi - \tfrac{i}{2}f_0\lambda^2\triangle\phi = 0\,, \tag{2.3}$$

with NIW velocity

$$u_w + iv_w = \phi(x, y, t)\, e^{i(mz - f_0 t)}\,. \tag{2.4}$$

Above, ϕ is the back-rotated NIW velocity, $\triangle \overset{\text{def}}{=} \partial_x^2 + \partial_y^2$ is the horizontal laplacian, $J(f, g) = f_x g_y - f_y g_x$ is the horizontal Jacobian, and $\lambda = \frac{N_0}{f_0 m}$ is an intrinsic horizontal scale.

Similarly to the Young & Ben Jelloul (1997) model, the NIW amplitude ϕ evolves through wave dispersion — the last term in (2.3) — and advection and refraction by the QG flow — the second and third terms in (2.3). But the wave equation (2.3) is non-linear owing to the quadratic wave terms in the *inversion* relation (2.2). The Young & Ben Jelloul (1997) NIW model is purely kinematic: the QG flow evolves as if there were no waves and set an inhomoneqeous medium in which NIWs propagate. The Xie & Vanneste (2015) model is dynamic because both q and ψ determine the flow: $\psi = \psi(x, y, t; q, \phi)$. Besides advection and refraction by the QG flow, these QG-NIW interactions imply a positive finite-amplitude frequency shift (Appendix A).

The wave equation (2.3) is similar to the $1\frac{1}{2}$ layer, reduced gravity Young & Ben Jelloul model derived by Danioux *et al.* (2015). Without advection, (2.3) is analogous to Schrodinger's equation (e.g., Landau & Lifshitz 2013, pg. 51), with the vorticity mimicking the potential and the dispersivity playing the role of Planck's constant (Danioux *et al.* 2015).

2.2. *Quadratic invariants and energy conversion*

From (2.3), the near-inertial kinetic energy density $\frac{1}{2}|\phi|^2$ satisfies

$$\partial_t \tfrac{1}{2}|\phi|^2 + \mathsf{J}(\psi, \tfrac{1}{2}|\phi|^2) + \boldsymbol{\nabla}\cdot\underbrace{\left[\tfrac{i}{4}f_0\lambda^2\left(\phi\boldsymbol{\nabla}\phi^\star - \phi^\star\boldsymbol{\nabla}\phi\right)\right]}_{\overset{\text{def}}{=}\mathbf{F}_w} = 0\,. \tag{2.5}$$

Locally, $\frac{1}{2}|\phi|^2$ changes due to divergences of the geostrophic and wave fluxes of $\frac{1}{2}|\phi|^2$ — the second and third terms in (2.5). The wave flux $\mathbf{F}_w$ is analogous to the probability

current density of quantum mechanics (e.g., Landau & Lifshitz, pg 57). Using the polar representation $\phi = |\phi|e^{i\Theta}$ (e.g., Klein *et al.* 2004), we can compactly express the wave flux

$$\mathbf{F}_w = \tfrac{i}{4} f_0 \lambda^2 \left(\phi \boldsymbol{\nabla} \phi^\star - \phi^\star \boldsymbol{\nabla} \phi \right) = f_0 \lambda^2 \boldsymbol{\nabla} \Theta \times \tfrac{1}{2} |\phi|^2 \,. \tag{2.6}$$

The expression on the right of (2.6) makes it explicit that $\mathbf{F}_w$ is a wave flux of kinetic energy density $\tfrac{1}{2} |\phi|^2$, where $f_0 \lambda^2 \boldsymbol{\nabla} \Theta = \frac{N_0^2}{f_0 m^2} \boldsymbol{\nabla} \Theta$ is the generalized group velocity of hydrostratic NIWs, which can be quickly verified under the plane-wave assumption $\Theta = kx + ly$.

The unforced and inviscid NIW-QG model (2.1)-(2.3) conserves NIW kinetic energy

$$K_w \overset{\text{def}}{=} \tfrac{1}{2} \langle |\phi|^2 \rangle \,, \tag{2.7}$$

where angle brackets, $\langle \ \rangle$, represent average over the domain of area $\mathcal{A}$:

$$\langle f \rangle \overset{\text{def}}{=} \frac{1}{\mathcal{A}} \iint_{\mathcal{A}} f \, \mathrm{d}x \mathrm{d}y \,. \tag{2.8}$$

Also from (2.3), we form an equation for the NIW potential energy: $\frac{\lambda^4}{4} \langle \triangle \phi^\star \times (2.3) + \triangle \phi \times (2.3)^\star \rangle$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\langle \tfrac{\lambda^2}{4} |\boldsymbol{\nabla} \phi|^2 \rangle}_{P_w} = \underbrace{\tfrac{1}{f_0} \left\langle \tfrac{1}{2} \triangle \psi \boldsymbol{\nabla} \cdot \mathbf{F}_w \right\rangle}_{\overset{\text{def}}{=} \Gamma_r} + \underbrace{\tfrac{\lambda^2}{2} \left\langle \tfrac{1}{2} \left[\triangle \phi^\star \mathsf{J}(\psi, \phi) + \triangle \phi \mathsf{J}(\psi, \phi^\star) \right] \right\rangle}_{\overset{\text{def}}{=} \Gamma_a} \,. \tag{2.9}$$

Finally, we form an equation for the QG kinetic energy, K_e: $\langle \psi \times (2.1) \rangle$ in combination with (2.2), (2.5), and (2.3) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\langle \tfrac{1}{2} |\boldsymbol{\nabla} \psi|^2 \rangle}_{\overset{\text{def}}{=} K_e} = -\Gamma \,, \tag{2.10}$$

thus proving the conservation of energy $E = K_e + P_w$. Above, $\Gamma = \Gamma_r + \Gamma_a$ is the energy conversion between QG kinetic energy and wave potential energy. The term Γ_r stems from refraction and is easy to interpret: the convergence, $\nabla \cdot \mathbf{F}_w < 0$, of the wave flux of NIW kinetic energy density into anticyclones, $\triangle \psi < 0$, is a source of NIW potential energy, P_w.

The term Γ_a stems from advection and is similar to the source of variance of a trace gradient subject to lateral stirring,

$$c_t + \mathsf{J}(\psi, c) = 0. \tag{2.11}$$

The variance production of passive scalar gradient, $\langle |\boldsymbol{\nabla} c|^2 \rangle$, is $\langle \triangle c \mathsf{J}(\psi, c) \rangle$. Analogously, Γ_a is the source of NIW potential energy due to geostrophic stirring. After multiple integration by parts, we rewrite this term:

$$\Gamma_r = \left\langle \begin{bmatrix} \phi_x^\star & \phi_y^\star \end{bmatrix} \mathbf{S} \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} \right\rangle \,, \tag{2.12}$$

where $\mathbf{S}$ is the symmetric part of the geostrophic velocity gradient matrix,

$$\mathbf{S} \overset{\text{def}}{=} \begin{bmatrix} -\psi_{xy} & \tfrac{1}{2}(\psi_{xx} - \psi_{yy}) \\ \tfrac{1}{2}(\psi_{xx} - \psi_{yy}) & \psi_{xy} \end{bmatrix} \,, \tag{2.13}$$

whose sum of the elements squared (the Frobenius norm) is commonly defined as the square of the QG rate of strain. Geostrophic straining enhances existing gradients of ϕ,

thereby generating wave potential energy, P_w. When the QG flow has a scale much larger than the NIW field (the WKB limit), this distortion of ϕ is akin to the wave-capture mechanism of Bühler & McItyre (2005).

2.3. *Averaged equations and loss of coherence*

The domain-averaged potential vorticity $\langle q \rangle$ is invariant as a consequence of the material invariance of q — or the invariance of the average of each individual term in (2.2). The spatially-averaged (coherent) NIW amplitude satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \phi \rangle + \mathrm{i}\left\langle \tfrac{1}{2}\phi \triangle \psi \right\rangle = 0\,. \tag{2.14}$$

Introducing the decomposition $\phi = \langle \phi \rangle + \phi'$, we have $\langle |\phi|^2 \rangle = |\langle \phi \rangle|^2 + \langle |\phi'|^2 \rangle = 2(K_w^c + K_w^i)$. Thus, the kinetic energy of horizontally incoherent (ϕ') and coherent ($\langle \phi \rangle$) NIW amplitude satisfy $\dot{K}_w^i = \Pi$ and $\dot{K}_w^c = -\Pi$, with the kinetic energy transfer

$$\Pi = \tfrac{\mathrm{i}}{2}\left[\left\langle \tfrac{1}{2}\phi \triangle \psi \right\rangle \langle \phi^\star \rangle - \left\langle \tfrac{1}{2}\phi^\star \triangle \psi \right\rangle \langle \phi \rangle\right]\,. \tag{2.15}$$

The decay of a laterally coherent near-inertial oscillation due to refraction by geostrophic flow is relevant for mixed-layer slab models used to estimate the work imparted by wind into the NIW field (e.g., Alford 2001). Without further assumptions, it is unclear if the effect of refraction is sign definite. If the NIWs have initial larger scales than the geostrophic field, then refraction generates eddy-scale perturbations, implying a reduction of lateral coherence ($\Pi > 0$). It remains unclear if this transfer can be parameterized as a linear damping.

2.4. *Relevant parameters*

Using the scaling

$$\psi \sim U_e k_e^{-1}\,, \qquad \text{and} \qquad \phi \sim U_w\,, \tag{2.16}$$

with characteristic QG and NIW velocity scales U_e and U_w, and the characteristic horizontal length scale k_e^{-1} and time scale $(U_e k_e)^{-1}$, reveals that there are two dynamically relevant parameters of the QG-NIW problem described by (2.1)-(2.3). First, the 'wave amplitude'

$$\alpha \overset{\text{def}}{=} \underbrace{\frac{U_e k_e}{f_0}}_{\overset{\text{def}}{=}Ro} \times \underbrace{\left(\frac{U_w}{U_e}\right)^2}_{\overset{\text{def}}{=}\epsilon^2}\,, \tag{2.17}$$

scales the contribution of the NIW terms to the QGPV. Second, the 'wave dispersivity,'

$$\hbar \overset{\text{def}}{=} f_0 \lambda^2 \times \frac{k_e}{U_e}\,, \tag{2.18}$$

scales the importance of linear dispersion, which counteracts the unsmoothing effects of advection and refraction.

Also, the two conversion terms, Γ_r and Γ_a, scale with the same non-dimensional parameter $\alpha \times \hbar$. But this does not imply that the energy conversion scales linearly with both wave amplitude and dispersivity: the scales deloveped by the NIW amplitude field ϕ and the correlations in Γ depend on α and $\hbar$

2.5. *Summary*

We can summarize the physics of stimulated imbalance considering an initial value problem that idealizes the oceanographic 'after-storm' context: stormy winds impart

momentum into the ocean, thereby generating near-inertial oscillations with a lateral decorrelation scale larger than the mesoscale eddies. The after-storm evolution of an initially large-scale coherent inertial oscillation is described by (2.1)–(2.3) (Xie & Vanneste 2015). Geostrophic refraction focus NIWs in anticyclones. The eddy-scale gradients of ϕ then support geostrophic advection. Advection and refraction reduce the lateral coherence of NIWs — and wave dispersion counteracts this unsmoothing effect.

The explicit expression for energy conversion on the right of (2.9) enlightens the physics of the problem. Refraction causes a convergence of NIW kinetic energy density in anticyclones. And advection strains the NIW field, enhancing the gradients of ϕ. Both processes generate NIW potential energy at the expenses of geostrophic kinetic energy, thus motivating the nomenclature 'refraction sink' $(-\Gamma_r)$ and 'advection sink' $(-\Gamma_a)$ of QG kinetic energy.

The remaining of this paper verifies this thought experiment by solving numerically two problems in which an initially perfectly coherent near-inertial oscillation interacts with the Lamb dipole (Section 3) and with a turbulent field emergent from random initial conditions (Section 4).

3. The Lamb-Chaplygin dipole

The Lamb-Chaplygin dipole is an exact solution of the Euler equations on an infinite two-dimensional plane where the vorticity is confined to a circle of radius R (Meleshko & Van Heijst 1994). The solution, steady on a frame moving at uniform zonal velocity U, is

$$\triangle\psi = \frac{2U\kappa}{J_0(\kappa R)} \begin{cases} J_1(\kappa r)\sin\theta\,, & \text{if} \quad r \leqslant R\,, \\ 0\,, & \text{if} \quad r \geqslant R\,, \end{cases} \tag{3.1}$$

where $r^2 = (x - x_c)^2 + (y - y_c)^2$ is the radial distance about the dipole's center (x_c, y_c), $\tan\theta = (y - y_c)/(x - x_c)$, and J_n is the n'th order Bessel function of first kind. The matching condition at $r = R$, $J_1(\kappa R) = 0$, determines κ. The dipole is the first eigensolution, with eigenvalue $\kappa R \approx 3.8287$. If the wave QGPV is zero, then the dipole (3.1) is a solution of (2.1). This is the case when a uniform ϕ is used as initial condition (Sections 3 and 4). Dipole images, an artefact of periodization, cause a small zonal drift of the dipole. This effect is negligible in our examples below because the dipole is confined to a region much smaller than domain size L: $R/L = 0.1$.

While we perform the calculations with dimensional variables, we non-dimensionalize the results for clarity. For the dipole (3.1), we use $U_e = U$ and $k_e = 2\pi/R$. As for the initial NIW amplitude, we consider a uniform (i.e., perfectly coherent) near-inertial oscillation with speed U_w:

$$\phi(x, y, t = 0) = \tfrac{1+\mathrm{i}}{\sqrt{2}}\, U_w\,. \tag{3.2}$$

4. Decaying Macroturbulence

To study the QG-NIW interaction and energy exchange in a turbulent regime relevant to the ocean, we consider a barotropic turbulence flow that emerges from random initial conditions integrated for 20 eddy turnover time units. In other words, we first integrate the initial condition

$$\psi\big(x, y, t \times U_e k_e = -20\big) = \sum_{k,l} |\hat\psi| \cos\left(kx + ly + \chi_{k,l}\right) \tag{4.1}$$

with waveless QG dynamics before introducing waves at $t \times U_e k_e = 0$. In 4.1, χ is a random $[0, 2\pi)$ phase prescribed at every wavevector (k, l) and $|\hat{\psi}|$ is the isotropic spectrum

$$|\hat{\psi}| = C \times \left\{ |k| \left[1 + (|k|/k_e)^4 \right] \right\}^{-1/2}, \qquad (4.2)$$

with the wavenumber magnitude $|k|^2 = k^2 + l^2$. The prescribed initial energy $U_e^2/2$ determines the constant C:

$$\sum_{k,l} \underbrace{|k|^2 |\hat{\psi}|^2}_{\overset{\text{def}}{=} \mathcal{K}_e} = \tfrac{1}{2} U_e^2 . \qquad (4.3)$$

The kinetic energy spectrum $\mathcal{K}_e$ peaks at energy-containing scale k_e^{-1}. At scales smaller than k_e^{-1}, $\mathcal{K}_e$ has a linear dependence in $|k|$, whereas it decays as $|k|^{-3}$ at scales larger than k_e^{-1}. This red spectrum ensures insignificant energy dissipation by the biharmonic viscosity added to (2.1) to absorb the forward cascade of enstrophy (Appendix B).

The evolution of the initial condition (4.1) constrained by the inviscid dynamics (2.1) has been well studied, beginning with Fornberg (1977). Stirring of vorticity $\triangle\psi$ by the flow ψ transfers enstrophy towards small scales; energy flows to large scales. Most of enstrophy dissipated within few eddy turnover time units, whereas kinetic energy is nearly conserved. Vorticity concentrates into localized structures: after 20 eddy turnover time units, the vorticity is well-organized into a sea of coherent vortices that form via like-sign vortex merging (e.g., McWilliams 1984).

We add a perfectly coherent near-inertial oscillation (3.2) to the mature barotropic turbulence at $t \times U_e k_e = 0$. For all parameters considered in the remaing of this paper, there are no qualitative long-term differences between the solutions described below and results from introducing the waves at $t \times U_e k_e = -20$.

4.1. *Parameters for strong mid-latitude macroturbulence*

Consider strong mid-latitude macroturbulence, such as the Gulf Stream or Kuroshio Extension. The choice of parameters, $U_e = 0.1 \text{ m s}^{-1}$, $f_0 = 1 \times 10^{-4} \text{ s}^{-1}$, $2\pi k_e^{-1} = 125 \text{ km}$ gives a Rossby number $Ro \approx 0.05$. For reference, the eddy-turnover scale is $(U_e k_e)^{-1} \approx 5.8$ days. Strong NIW velocity imparted by atmospheric storms, $U_w \sim \sqrt{2} U_e$, implies a wave amplitude

$$\alpha \approx 0.1 . \qquad (4.4)$$

A NIW vertical wavelength of $\lambda_z = 2\pi/m = 300 \text{ m}$ (e.g., Alford *et al.* 2016) on a strongly stratified upper-ocean $N_0/f_0 = 100$ yields a dispersivity

$$\hbar \approx 1.0 . \qquad (4.5)$$

4.2. *Solutions for $\hbar = 1$ and $\alpha = 0.1$*

4.3. *Parameter exploration*

5. Final Remarks

Appendix A. Details of the QG-NIW model

A.1. *The Xie & Vanneste (2015) model*

Xie & Vanneste (2015) derive the NIW-QG model using a variational formulation of the generalized Lagrangian mean (GLM) framework with Whitham averaging. Following Young & Ben Jelloul (1997), Xie & Vanneste write the NIW velocity in complex form

$$u_w + \mathrm{i} v_w = M_z \mathrm{e}^{-\mathrm{i} f_0 t} . \qquad (A\,1)$$

Their derivation recovers the wave equation that governs the evolution of the NIW complex amplitude M_z

$$M_{zzt} + \partial_z \mathsf{J}(\psi, M_z) + \frac{\mathrm{i}}{2}\left[\left(\frac{N^2}{f_0} + \psi_{zz}\right)\triangle M - 2\nabla\psi_z \cdot \nabla M_z + M_{zz}(\triangle\psi + 2\beta y)\right] = 0\,,$$
$$\text{(A\,2)}$$

with the QG streamfunction $\psi(x, y, z, t)$. The QG flow evolves through stirring of QGPV

$$q_t + \mathsf{J}(\psi, q) = 0\,. \tag{A\,3}$$

A fundamental difference is that the QGPV in the Xie & Vanneste (2015) model contains quadratic wave terms

$$q = \nabla\psi + \left(\frac{f_0^2}{N^2}\psi_z\right)_z + \beta y + \frac{\mathrm{i}}{2f_0}\mathsf{J}(M_z^\star, M_z) + \frac{1}{4f_0}\left(2|\nabla M_z|^2 - M_{zz}^\star\nabla M - M_{zz}\nabla M^\star\right)\,. \tag{A\,4}$$

Thus the NIWs actively evolve, feedbacking onto the QG flow. The model of Xie & Vanneste (2015), with the potential vorticy that includes wave terms, generalizes early ideas introduced by Bühler & McIntyre (1998) to a setup that avoids spatial scale separation between geostrophic flow and waves, but restricts attention to near-inertial frequencies. Using standard perturbation theory, Wagner & Young (2016) contemporarily recovered and extended this coupled NIW-QG system.

The special family of barotropic, f-plane QG flow, and uniform background stratification, single vertical mode NIW solutions $M_z = \phi(x, y)\mathrm{e}^{\mathrm{i}mz}$ yields the reduced set of equations (2.1)-(2.3) used in this paper.

A.2. *Wave self-interaction and frequency shift*

To gain further insight into the dynamics, we decompose the balanced flow into 'vorticity-induced' and 'wave-induced' fields, $\psi = \psi^q + \psi^w$, with

$$\triangle\psi^q = q\,, \qquad \text{and} \qquad \triangle\psi^w = -\frac{1}{4f_0}\triangle|\phi|^2 - \frac{\mathrm{i}}{2f_0}\mathsf{J}(\phi^\star, \phi)\,. \tag{A\,5}$$

With the (A\,5) decomposition, the QGPV equation (2.1) becomes

$$\triangle\psi_t^q + \mathsf{J}(\psi^q + \psi^w, \triangle\psi^q) = 0\,. \tag{A\,6}$$

In other words, the relative vorticity of the 'vorticity-induced' flow changes due to advection by the total flow $\psi^q + \psi^w$. The wave amplitude ϕ can be rewritten as

$$\underbrace{\left(\partial_t - \frac{\mathrm{i}}{2}f_0\lambda^2\triangle\right)\phi}_{\text{Lin. wave dynamics}} + \underbrace{\mathsf{J}(\psi^w, \phi) + \frac{\mathrm{i}}{2}\phi\triangle\psi^w}_{\text{Non-lin. wave dynamics}} + \underbrace{\mathsf{J}(\psi^q, \phi) + \frac{\mathrm{i}}{2}\phi\triangle\psi^q}_{\text{Adv./refrac. by }\psi^q} = 0\,. \tag{A\,7}$$

From (A\,5), the 'wave-induced' streamfunction ψ^w is quadratic in ϕ:

$$\psi^w = \frac{1}{4f_0}|\phi|^2 + \frac{\mathrm{i}}{2f_0}\triangle^{-1}\mathsf{J}(\phi^\star, \phi)\,, \tag{A\,8}$$

where $\triangle^{-1}$ is the inverse of the Laplacian: $\triangle^{-1}\triangle f = f$.

Together, equations (A\,7)-(A\,8) explicitly show the non-linear nature of the wave equation in this coupled QG-NIW model. In particular, the non-linear wave dynamics is cubic in wave amplitude. Dropping these cubic wave terms yields a quasi-linear (QL) QG-NIW system. This QL system conserves the energy

$$E_{ql} = \langle\tfrac{1}{2}|\boldsymbol{\nabla}\psi^q|^2\rangle + \langle\boldsymbol{\nabla}\psi^q \cdot \boldsymbol{\nabla}\psi^w\rangle + \langle\tfrac{\lambda^2}{4}|\boldsymbol{\nabla}\phi|^2\rangle\,. \tag{A\,9}$$

Note that $\mathrm{E} = E_{ql} + \langle\tfrac{1}{2}|\boldsymbol{\nabla}\psi^w|^2\rangle$. Thus, the QL approximation yields an intermediate

model between the uncoupled QG-YBJ model ($\psi^w = 0$) and the non-linear coupled QG-NIW model.

To gain insight into the non-linear wave effects, we consider linearized equations about the exact solution: $\phi = \Phi = $ constant and $\psi = 0$, so that $q = 0$. The QGPV equation (2.1) linearized about this state is trivial: $q_t = \triangle\psi^q_t = 0$, hence $\triangle\psi^q$ remains zero. Also,

$$\triangle\psi^w = -\tfrac{\Phi}{2f_0}\triangle\tfrac{1}{2}(\phi + \phi^\star)\,, \tag{A 10}$$

where ϕ represents the small departure about the constant Φ, so that the linearized wave equation becomes

$$\phi_t = \tfrac{i}{2}\left[\tfrac{\Phi^2}{2f_0}\triangle\tfrac{1}{2}(\phi + \phi^\star) + f_0\lambda^2\triangle\phi\right]\,. \tag{A 11}$$

Recalling that $u_w + iv_w = \phi e^{i(mz - f_0 t)}$, we recast the wave equation:

$$\left[\partial_t^2 - \tfrac{1}{4}f_0\lambda^2\left(f_0\lambda^2 + \frac{\Phi^2}{2f_0}\right)\triangle^2\right]v_w = 0\,, \tag{A 12}$$

where $\triangle^2 = (\partial_x^2 + \partial_y^2)^2$. Hence, plane-wave solutions yield the dispersion relation

$$\omega = \tfrac{1}{2}f_0\lambda^2(k^2 + l^2)\left[1 + \tfrac{1}{2}\frac{\Phi^2}{f_0^2\lambda^2}\right]^{1/2}\,, \tag{A 13}$$

and we conclude that the wave self-interaction shifts the frequency by

$$\left[1 + \tfrac{1}{2}\frac{\Phi^2}{f_0^2\lambda^2}\right]^{1/2} - 1\,. \tag{A 14}$$

Appendix B. Details of the initial value problems

B.1. *Numerical methods*

We solve the NIW-QG system (2.1)-(2.3) using a standard collocation Fourier spectral method. In the pseudo-spectral spirit, we evaluate the quadratic non-linearities in physical space, and transform the product into Fourier space. For small-scale dissipation, we use biharmonic viscosity. We find that this choice, originally used in McWilliams (1984), is sufficient to extend the spectral resolution compared to Laplacian viscosity. We make sure the 35% highest modes fall in the dissipation range, so that alised wavenumbers are strongly damped. We time march the spectral equations using an exponential time differencing method with a fourth order Runge-Kutta scheme (details in Kassam & Trefethen 2005).

B.2. *Parameters*

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