

# Near-inertial waves extract energy from barotropic turbulence: freely-evolving flow

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## 1. Introduction

The closing of the ocean’s energy budget remains poorly understood, largely because geostrophic turbulence transfers energy towards large scales. Existing studies suggest a panoply of ageostrophic mechanisms to account for the required forward flow of energy. These include (but are not limited to) surface and benthic boundary turbulence and wave generation by geostrophic eddies sloshing over bottom topography (see [Nagai \*et al.\* 2015](#), their figure 1, and references therein). But roughly half of the energy flux out of the ocean mesoscales remains unaccounted for.

Recent studies suggest that the interaction of geostrophic flows with externally-forced (windy) internal waves serves as a major sink of mesoscale energy en route to viscous dissipation ([Xie & Vanneste 2015](#); [Taylor & Straub 2016](#); [Wagner & Young 2016](#); [Barkan \*et al.\* 2016](#)). The emphasis on interactions of geostrophic flow with windy internal waves — here denoted stimulated imbalance<sup>†</sup> — contrasts with energy loss via spontaneous emission — a process by which internal waves are emitted during geostrophic adjustment (e.g., [Shakespeare & Hogg 2017](#)). While spontaneous emission is localized at sharp fronts (large Rossby number,  $Ro \gtrsim 1$ ) in the surface boundary layer (e.g., [Shakespeare & Hogg 2017](#)), stimulated imbalance operates at the small Rossby-number ( $Ro \ll 1$ ), quasi-geostrophic regime, both in the upper ocean and in the interior (e.g., [Xie & Vanneste 2015](#)).

We here study the interaction between macroturbulence and internal waves using a simple model that contains stimulated imbalance. This minimal model is a special family of solutions of the [Xie & Vanneste \(2015\)](#) equations, which couple the evolution of near-inertial back-rotated velocity with quasigeostrophic flow. The focus on QG-NIW interactions is justified: geostrophic eddies account for the bulk of the ocean’s eddy kinetic energy (90%, [Ferrari & Wunsch 2009](#)) and near-inertial waves contain most of

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<sup>†</sup> Following [Xie & Vanneste \(2015\)](#) and [Wagner & Young \(2016\)](#) we define *stimulated imbalance* as wave-mean interactions that transfer energy from geostrophic motions (the ‘mean flow’) to existing internal waves. This definition contrasts with the nomenclature recently employed by [Barkan \*et al.\* \(2016\)](#), who term stimulated imbalance a process by which internal waves trigger a forward cascade from mesoscale to submesoscale (but subinertial) flows.

the oceanic high-frequency variability (Alford *et al.* 2016). The minimal QG-NIW coupled model transparently reveals the physics of stimulated imbalance associated with vertical vorticity and lateral strain.

## 2. The physics of NIW energy extraction from a barotropic QG flow

### 2.1. The Xie & Vanneste (2015) minimal QG-NIW model

With barotropic quasigeostrophic flow,  $\psi = \psi(x, y, t)$ , uniform background buoyancy frequency  $N_0$ , and single-mode NIW vertical structure,  $e^{imz}$ , the Xie & Vanneste (2015) QG-NIW coupled model (Appendix A) reduces to

$$\{\text{macroturb}\} \quad q_t + \mathbf{J}(\psi, q) = 0, \quad (2.1)$$

with the NIW-averaged quasigeostrophic potential vorticity (QGPV)

$$\{\text{qgpv}\} \quad q = \Delta\psi + \frac{1}{f_0} \left[ \frac{1}{4} \Delta|\phi|^2 + \frac{i}{2} \mathbf{J}(\phi^*, \phi) \right], \quad (2.2)$$

and the streamfunction is defined so that the geostrophic flow velocity is  $(u_e, v_e) = (-\psi_y, \psi_x)$ ; and

$$\{\text{waves}\} \quad \phi_t + \mathbf{J}(\psi, \phi) + \frac{i}{2} \phi \Delta\psi - \frac{i}{2} f_0 \lambda^2 \Delta\phi = 0, \quad (2.3)$$

with NIW velocity

$$\{\text{w\_velocity}\} \quad u_w + iv_w = e^{i(mz - f_0 t)} \phi(x, y, t). \quad (2.4)$$

Above,  $\phi$  is the back-rotated NIW velocity,  $\Delta \stackrel{\text{def}}{=} \partial_x^2 + \partial_y^2$  is the horizontal laplacian,  $\mathbf{J}(f, g) = f_x g_y - f_y g_x$  is the horizontal Jacobian, and  $\lambda = \frac{N_0}{f_0 m}$  is an intrinsic horizontal scale.

Similarly to the Young & Ben Jelloul (1997) model, the NIW amplitude  $\phi$  evolves through wave dispersion — the last term in (2.3) — and advection and refraction by the QG flow — the second and third terms in (2.3). But the wave equation (2.3) is non-linear owing to the quadratic wave terms in the *inversion* relation (2.2). In other words, the Young & Ben Jelloul (1997) NIW model is purely kinematic: the QG flow evolves as if there were no waves and set an inhomogeneous medium in which NIWs propagate. But the Xie & Vanneste (2015) model is dynamic because both  $q$  and  $\phi$  determine the flow:  $\psi = \psi(x, y, t; q, \phi)$ . Besides advection and refraction by the QG flow, these QG-NIW interactions imply a positive finite-amplitude frequency shift (Appendix A).

The wave equation (2.3) is similar to the reduced-gravity Young & Ben Jelloul model derived by Danioux *et al.* (2015). Without advection, (2.3) is analogous to Schrodinger's equation (e.g., Landau & Lifshitz 2013, pg. 51), with the vorticity playing the role of the potential and the dispersivity mimicking Planck's constant (Danioux *et al.* 2015).

### 2.2. Quadratic invariants and energy conversion

From (2.3), the near-inertial kinetic energy density  $\frac{1}{2}|\phi|^2$  satisfies

$$\{\text{en\_density}\} \quad \partial_t \frac{1}{2}|\phi|^2 + \mathbf{J}(\psi, \frac{1}{2}|\phi|^2) + \underbrace{\nabla \cdot \left[ \frac{i}{4} f_0 \lambda^2 (\phi \nabla \phi^* - \phi^* \nabla \phi) \right]}_{\stackrel{\text{def}}{=} \mathbf{F}_w} = 0. \quad (2.5)$$

Locally,  $\frac{1}{2}|\phi|^2$  changes due to divergences of the geostrophic and wave fluxes of  $\frac{1}{2}|\phi|^2$  — the second and third terms in (2.5). The wave flux  $\mathbf{F}_w$  is analogous to the probability current density of quantum mechanics (e.g., Landau & Lifshitz, pg 57). Using the polar representation  $\phi = |\phi|e^{i\theta}$  (e.g., Klein *et al.* 2004), we can compactly express the wave

flux

$$\{\text{Fw2}\} \quad \mathbf{F}_w = \frac{1}{4} f_0 \lambda^2 (\phi \nabla \phi^* - \phi^* \nabla \phi) = f_0 \lambda^2 \nabla \Theta \times \frac{1}{2} |\phi|^2. \quad (2.6)$$

The expression on the right of (2.6) makes it explicit that  $\mathbf{F}_w$  is a wave flux of kinetic energy density  $\frac{1}{2} |\phi|^2$ , where  $f_0 \lambda^2 \nabla \Theta = \frac{N_0^2}{f_0 \rho m^2} \nabla \Theta$  is the generalized group velocity of hydrostratic NIWs, which can be quickly verified under the plane-wave assumption  $\Theta = kx + ly$ .

With simple boundary conditions, e.g., periodic or no normal flux, the QG-NIW model (2.1)-(2.3) conserves NIW kinetic energy

$$K_w \stackrel{\text{def}}{=} \frac{1}{2} \langle |\phi|^2 \rangle, \quad (2.7) \quad \{\text{action}\}$$

where angle brackets,  $\langle \rangle$ , represent average over the domain of area  $\mathcal{A}$ :

$$\langle f \rangle \stackrel{\text{def}}{=} \frac{1}{\mathcal{A}} \iint_{\mathcal{A}} f \, dx dy. \quad (2.8) \quad \{\text{average}\}$$

Also from (2.3), we form an equation for the wave potential energy:  $\frac{\lambda^4}{4} \langle \Delta \phi^* \times (2.3) + \Delta \phi \times (2.3)^* \rangle$  yields

$$\frac{d}{dt} \underbrace{\langle \frac{\lambda^2}{4} |\nabla \phi|^2 \rangle}_{P_w} = \underbrace{\frac{1}{f_0} \left\langle \frac{1}{2} \Delta \psi \nabla \cdot \mathbf{F}_w \right\rangle}_{\stackrel{\text{def}}{=} \Gamma_r} + \underbrace{\frac{\lambda^2}{2} \left\langle \frac{1}{2} [\Delta \phi^* \mathbf{J}(\psi, \phi) + \Delta \phi \mathbf{J}(\psi, \phi^*)] \right\rangle}_{\stackrel{\text{def}}{=} \Gamma_a}. \quad (2.9) \quad \{\text{Pw}\}$$

Finally, we form an equation for the QG kinetic energy,  $K_e$ :  $\langle \psi \times (2.1) \rangle$  in combination with (2.2), (2.5), and (2.3) gives

$$\frac{d}{dt} \underbrace{\langle \frac{1}{2} |\nabla \psi|^2 \rangle}_{\stackrel{\text{def}}{=} K_e} = -(\Gamma_r + \Gamma_a), \quad (2.10) \quad \{\text{Ke}\}$$

thus proving that the QG-NIW model (2.1)-(2.3) conserves the coupled energy

$$E \stackrel{\text{def}}{=} K_e + P_w. \quad (2.11) \quad \{\text{E}\}$$

In (2.10),  $\Gamma_r + \Gamma_a$  is the energy conversion between QG kinetic energy and wave potential energy. The term  $\Gamma_r$  stems from refraction and is easy to interpret: the convergence,  $\nabla \cdot \mathbf{F}_w < 0$ , of the wave flux of NIW kinetic energy density into anticyclones,  $\Delta \psi < 0$ , is a source of NIW potential energy,  $P_w$ .

The term  $\Gamma_a$  stems from advection and is similar to the source of variance of a trace gradient subject to lateral stirring,

$$c_t + \mathbf{J}(\psi, c) = 0. \quad (2.12)$$

The variance production of passive scalar gradient,  $\langle |\nabla c|^2 \rangle$ , is  $\langle \Delta c \mathbf{J}(\psi, c) \rangle$ . Analogously,  $\Gamma_a$  is the source of NIW potential energy due to geostrophic stirring. After multiple integration by parts, we rewrite this term as

$$\Gamma_a = \left\langle \begin{bmatrix} \phi_x^* & \phi_y^* \end{bmatrix} \mathbf{S} \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} \right\rangle, \quad (2.13) \quad \{\text{gradphi}\}$$

where  $\mathbf{S}$  is the symmetric part of the geostrophic velocity gradient matrix,

$$\mathbf{S} \stackrel{\text{def}}{=} \begin{bmatrix} -\psi_{xy} & \frac{1}{2}(\psi_{xx} - \psi_{yy}) \\ \frac{1}{2}(\psi_{xx} - \psi_{yy}) & \psi_{xy} \end{bmatrix}, \quad (2.14)$$

whose sum of the elements squared (the Frobenius norm) is commonly defined as the square of the QG rate of strain. Geostrophic straining enhances existing gradients of  $\phi$ , thereby generating wave potential energy,  $P_w$ . When the QG flow has a lateral scale much larger than the NIW field, this distortion of  $\phi$  is akin to the wave-capture mechanism of [Bühler & McItyre \(2005\)](#).

### 2.3. Averaged equations and loss of coherence

The domain-averaged potential vorticity  $\langle q \rangle$  is invariant as a consequence of the material invariance of  $q$  — or the invariance of the average of each individual term in (2.2). The spatially-averaged (coherent) NIW amplitude satisfies

$$\{\text{phi\_ave}\} \quad \frac{d}{dt} \langle \phi \rangle + i \left\langle \frac{1}{2} \phi \Delta \psi \right\rangle = 0. \quad (2.15)$$

Introducing the decomposition  $\phi = \langle \phi \rangle + \phi'$ , we have

$$\frac{1}{2} \langle |\phi|^2 \rangle = \underbrace{\frac{1}{2} \langle \langle \phi \rangle \rangle^2}_{\stackrel{\text{def}}{=} K_w^c} + \underbrace{\frac{1}{2} \langle |\phi'|^2 \rangle}_{\stackrel{\text{def}}{=} K_w^i}, \quad (2.16)$$

Thus, the kinetic energy of horizontally incoherent ( $\phi'$ ) and coherent ( $\langle \phi \rangle$ ) NIW amplitude satisfy

$$\{\text{Kiw}\} \quad \dot{K}_w^i = \Pi, \quad (2.17)$$

and

$$\{\text{Kcw}\} \quad \dot{K}_w^c = -\Pi, \quad (2.18)$$

with the kinetic energy transfer

$$\{\text{Pi}\} \quad \Pi = \frac{i}{2} \left[ \langle \frac{1}{2} \phi \Delta \psi \rangle \langle \phi^* \rangle - \langle \frac{1}{2} \phi^* \Delta \psi \rangle \langle \phi \rangle \right]. \quad (2.19)$$

The decay of a laterally coherent near-inertial oscillation due to refraction by geostrophic flow is relevant for mixed-layer slab models used to estimate the work imparted by wind into the NIW field (e.g., [Alford 2001](#)). Without further assumptions, it is unclear if the effect of refraction is sign definite. If the NIWs have initial larger scales than the geostrophic field, then refraction generates eddy-scale perturbations, implying a reduction of lateral coherence ( $\Pi > 0$ ). But it remains unclear if this transfer can be parameterized as a linear damping.

### 2.4. Relevant parameters

Using the scaling

$$\psi \sim U_e k_e^{-1}, \quad \text{and} \quad \phi \sim U_w, \quad (2.20)$$

with characteristic QG and NIW velocity scales  $U_e$  and  $U_w$ , and the characteristic horizontal length scale  $k_e^{-1}$  and time scale  $(U_e k_e)^{-1}$ , reveals that there are two dynamically relevant parameters of the QG-NIW problem described by (2.1)-(2.3). First, the ‘wave amplitude’

$$\{\text{alpha}\} \quad \alpha \stackrel{\text{def}}{=} \underbrace{\frac{U_e k_e}{f_0}}_{\stackrel{\text{def}}{=} Ro} \times \underbrace{\left( \frac{U_w}{U_e} \right)^2}_{\stackrel{\text{def}}{=} \epsilon^2}, \quad (2.21)$$

measures the strenght of the waves compared to the geostrophic flow and scales the contribution of the NIW terms to the QGPV. Second, the ‘wave dispersivity,’

$$\{\text{hslash}\} \quad \hbar \stackrel{\text{def}}{=} f_0 \lambda^2 \times \frac{k_e}{U_e}, \quad (2.22)$$

scales the importance of linear dispersion, which counteracts the unsmoothing effects of advection and refraction.

Also, the two conversion terms,  $\Gamma_r$  and  $\Gamma_a$ , scale with the same non-dimensional parameter  $\alpha \times \hbar$ . But this does not imply that the energy conversion scales linearly with both wave amplitude and dispersivity: the scales delveloped by the NIW amplitude field  $\phi$  and the correlations in  $\Gamma$  depend on  $\alpha$  and  $\hbar$

## 2.5. Summary

We can summarize the physics of stimulated imbalance considering an initial value problem that idealizes the oceanographic ‘post-storm’ context: stormy winds impart momentum into the ocean, thereby generating near-inertial oscillations with a lateral decorrelation scale larger than the mesoscale eddies. The after-storm evolution of an initially large-scale coherent inertial oscillation is described by (2.1)–(2.3) (Xie & Vanneste 2015). Geostrophic refraction focus NIWs in anticyclones. The eddy-scale gradients of  $\phi$  then support geostrophic advection. Advection and refraction reduce the lateral coherence of NIWs — and wave dispersion counteracts this unsmoothing effect.

The explicit expression for energy conversion on the right of (2.9) enlightens the physics of the problem. Refraction causes a convergence of NIW kinetic energy density in anticyclones. And advection strains the NIW field, enhancing the gradients of  $\phi$ . Both processes generate NIW potential energy at the expenses of geostrophic kinetic energy, thus motivating the nomenclature ‘refraction sink’ ( $-\Gamma_r$ ) and ‘advection sink’ ( $-\Gamma_a$ ) of QG kinetic energy.

The remaining of this paper verifies this thought experiment by solving numerically two problems in which an initially perfectly coherent near-inertial oscillation interacts with the Lamb dipole (Section 3) and with a turbulent field emergent from random initial conditions (Section 4).

## 3. The Lamb-Chaplygin dipole

The Lamb-Chaplygin dipole is an exact solution of the Euler equations on an infinite two-dimensional plane where the vorticity is confined to a circle of radius  $R$  (Meleshko & Van Heijst 1994). The solution, steady on a frame moving at uniform zonal velocity  $U$ , is

$$\Delta\psi = \frac{2U\kappa}{J_0(\kappa R)} \begin{cases} J_1(\kappa r) \sin\theta, & \text{if } r \leq R, \\ 0, & \text{if } r \geq R, \end{cases} \quad (3.1) \quad \{\text{lbnd\_q}\}$$

where  $r^2 = (x - x_c)^2 + (y - y_c)^2$  is the radial distance about the dipole’s center  $(x_c, y_c)$ ,  $\tan\theta = (y - y_c)/(x - x_c)$ , and  $J_n$  is the  $n$ ’th order Bessel function of first kind. The matching condition at  $r = R$ ,  $J_1(\kappa R) = 0$ , determines  $\kappa$ . The dipole is the first eigensolution, with eigenvalue  $\kappa R \approx 3.8287$ . If the wave QGPV is zero, then the dipole (3.1) is a solution of (2.1). This is the case when a uniform  $\phi$  is used as initial condition (Sections 3 and 4). Dipole images, an artefact of periodization, cause a small zonal drift of the dipole. This effect is negligible in our examples below because the dipole is confined to a region much smaller than the domain size  $L$ :  $R/L = 0.1$ .

While we perform the calculations with dimensional variables, we non-dimensionalize the results for clarity. For the dipole (3.1), we use  $U_e = U$  and  $k_e = 2\pi/R$ . As for the initial NIW amplitude, we consider a uniform (i.e., perfectly coherent) near-inertial oscillation with speed  $U_w$ :

$$\phi(x, y, t = 0) = \frac{1+i}{\sqrt{2}} U_w. \quad (3.2) \quad \{\text{NIO}\}$$

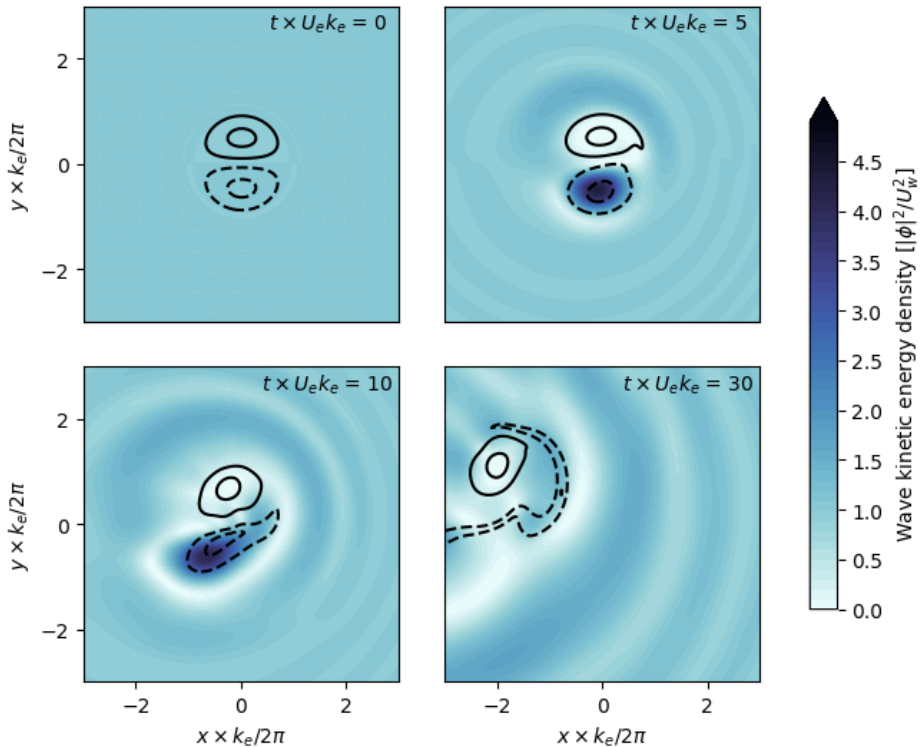


FIGURE 1. Snapshots of the Lamb-Chaplygin dipole solution with  $\hbar \approx 1$  and  $\alpha \approx 3.75$ . Colors represent wave kinetic energy density,  $|\phi'|^2/U_w^2$ . Contours represent potential vorticity,  $q \times U_e k_e = [-1.5, -0.5, 1.5, 0.5]$ , with dashed lines showing negative values. These plots only show the central  $\frac{1}{5} \times \frac{1}{5}$  subdomain.

### 3.1. Solution for $\hbar \approx 1$ and $\alpha \approx 3.75$

The wave concentration in anticyclones is quantified by the correlation coefficient (e.g., [Danioux et al. 2015](#))

$$r \stackrel{\text{def}}{=} \frac{\langle |\phi'|^2 \Delta \psi \rangle}{\langle |\phi'|^4 \rangle^{1/2} \langle (\Delta \psi)^2 \rangle^{1/2}}. \quad (3.3) \quad \{\text{corr\_r}\}$$

## 4. Decaying Macroturbulence

To study the QG-NIW interaction and energy exchange in a turbulent regime relevant to the ocean, we consider a barotropic turbulence flow that emerges from random initial conditions integrated for 20 eddy turnover time units. In other words, we first integrate the initial condition

$$\{\text{psi\_init}\} \quad \psi(x, y, t \times U_e k_e = -20) = \sum_{k,l} |\hat{\psi}| \cos(kx + ly + \chi_{k,l}) \quad (4.1)$$

with waveless QG dynamics before introducing waves at  $t \times U_e k_e = 0$ . In [4.1](#),  $\chi_{k,l}$  is a random phase uniformly distributed on  $[0, 2\pi)$ , and  $|\hat{\psi}|$  is the isotropic spectrum

$$\{\text{psih\_mag}\} \quad |\hat{\psi}| = C \times \{|k| [1 + (|k|/k_e)^4]\}^{-1/2}, \quad (4.2)$$

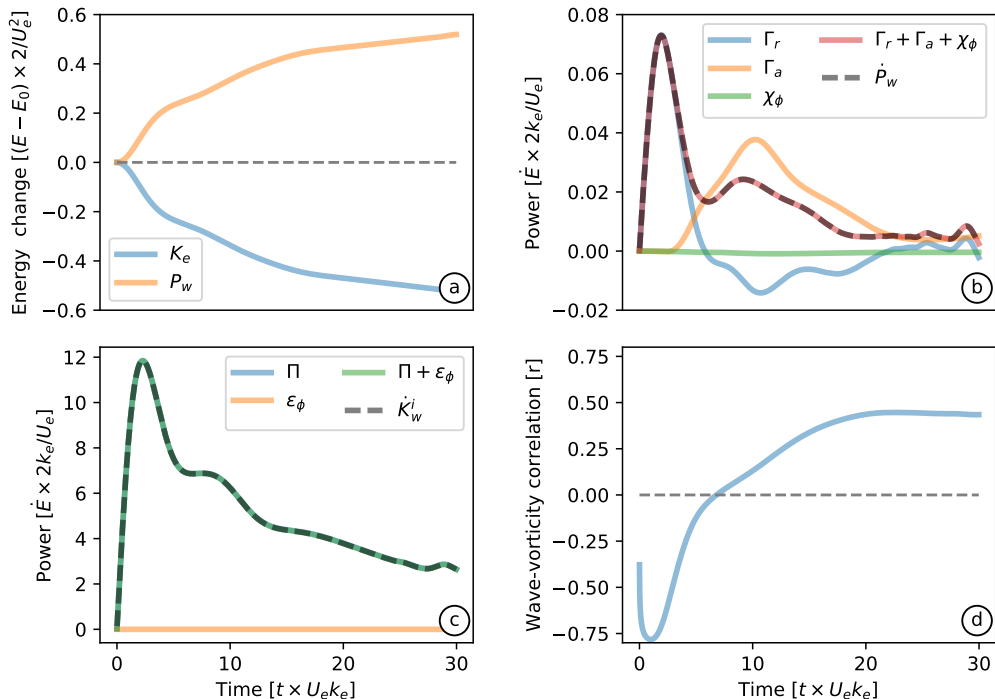


FIGURE 2. Statistics of the Lamb-Chaplygin dipole solution with  $\hbar \approx 1$  and  $\alpha \approx 3.75$ . (a) Energy change about initial condition. (b) Wave potential energy budget (2.9). (c) Incoherent wave kinetic energy budget (2.17). (d) Coefficient of correlation between incoherent wave kinetic energy and relative vorticity (3.3).

with the wavenumber magnitude  $|k|^2 = k^2 + l^2$ . The prescribed initial energy  $U_e^2/2$  determines the constant C:

$$\{\text{ke\_init}\} \quad \sum_{k,l} \underbrace{|k|^2 |\hat{\psi}|^2}_{\stackrel{\text{def}}{=} \mathcal{K}_e} = \frac{1}{2} U_e^2. \quad (4.3)$$

The kinetic energy spectrum  $\mathcal{K}_e$  peaks at the energy-containing scale  $k_e^{-1}$ . At scales smaller than  $k_e^{-1}$ ,  $\mathcal{K}_e$  has a linear dependence in  $|k|$ , whereas it decays as  $|k|^{-3}$  at scales larger than  $k_e^{-1}$ . This red spectrum ensures insignificant energy dissipation by the biharmonic viscosity added to (2.1) to absorb the forward cascade of enstrophy (Appendix B).

The evolution of a random initial condition constrained by the inviscid dynamics (2.1) has been well studied, beginning with [Fornberg \(1977\)](#). Stirring of vorticity  $\triangle\psi$  by the flow  $\psi$  transfers enstrophy towards small scales; energy flows to large scales. Most of enstrophy dissipated within few eddy turnover time units, whereas kinetic energy is nearly conserved. Vorticity concentrates into localized structures: after 20 eddy turnover time units, the vorticity is well-organized into a sea of coherent vortices that form via like-sign vortex merging (e.g., [McWilliams 1984](#)).

We add a perfectly coherent near-inertial oscillation (3.2) to the mature barotropic turbulence at  $t \times U_e k_e = 0$ . For all parameters considered in the remaining of this paper, there are no qualitative long-term differences between the solutions described below and results from introducing the waves at  $t \times U_e k_e = -20$ .

#### 4.1. Parameters for strong mid-latitude macroturbulence

Consider strong mid-latitude macroturbulence, such as the Gulf Stream or Kuroshio Extension. The choice of parameters,  $U_e = 0.1 \text{ m s}^{-1}$ ,  $f_0 = 1 \times 10^{-4} \text{ s}^{-1}$ ,  $2\pi k_e^{-1} = 125 \text{ km}$  gives a Rossby number  $Ro \approx 0.05$ . For reference, the eddy-turnover scale is  $(U_e k_e)^{-1} \approx 5.8 \text{ days}$ . Strong NIW velocity imparted by atmospheric storms,  $U_w \sim \sqrt{2}U_e$ , implies a wave amplitude

$$\alpha \approx 0.1. \quad (4.4)$$

A NIW vertical wavelength of  $\lambda_z = 2\pi/m = 300 \text{ m}$  (e.g., [Alford et al. 2016](#)) on a strongly stratified upper-ocean  $N_0/f_0 = 100$  yields a dispersivity

$$\hbar \approx 1.0. \quad (4.5)$$

#### 4.2. Solutions for $\hbar = 1$ and $\alpha = 0.1$

#### 4.3. Parameter exploration

### 5. Final Remarks

#### Appendix A. Details of the QG-NIW model

##### A.1. The [Xie & Vanneste \(2015\)](#) model

[Xie & Vanneste \(2015\)](#) derive the QG-NIW model using a variational formulation of the generalized Lagrangian mean (GLM) framework with Whitham averaging. Following [Young & Ben Jelloul \(1997\)](#), Xie & Vanneste write the NIW velocity in complex form

$$u_w + iv_w = M_z e^{-if_0 t}. \quad (\text{A } 1)$$

Their derivation recovers the wave equation that governs the evolution of the NIW complex amplitude  $M_z$

$$M_{zzt} + \partial_z J(\psi, M_z) + \frac{i}{2} \left[ \left( \frac{N^2}{f_0} + \psi_{zz} \right) \triangle M - 2\nabla\psi_z \cdot \nabla M_z + M_{zz}(\triangle\psi + 2\beta y) \right] = 0, \quad (\text{A } 2)$$

with the QG streamfunction  $\psi(x, y, z, t)$ . The QG flow evolves through stirring of QGPV

$$q_t + J(\psi, q) = 0. \quad (\text{A } 3)$$

A fundamental difference is that the QGPV in the [Xie & Vanneste \(2015\)](#) model contains quadratic wave terms

$$q = \nabla\psi + \left( \frac{f_0^2}{N^2} \psi_z \right)_z + \beta y + \frac{i}{2f_0} J(M_z^*, M_z) + \frac{1}{4f_0} (2|\nabla M_z|^2 - M_{zz}^* \nabla M - M_{zz} \nabla M^*). \quad (\text{A } 4)$$

Thus the NIWs actively evolve, feedbacking onto the QG flow. The model of [Xie & Vanneste \(2015\)](#), with the potential vorticity that includes wave terms, generalizes early ideas introduced by [Bühler & McIntyre \(1998\)](#) to a setup that avoids spatial scale separation between geostrophic flow and waves, but restricts attention to near-inertial frequencies. Using standard perturbation theory, [Wagner & Young \(2016\)](#) contemporarily recovered and extended this coupled QG-NIW system.

The special family of solutions with barotropic, f-plane QG flow, uniform background stratification, and single vertical mode NIW,  $M_z = e^{imz} \phi(x, y)$ , yields the reduced set of equations (2.1)-(2.3) used in this paper.



## A.2. Wave self-interaction and frequency shift

To gain further insight into the dynamics, we decompose the balanced flow into ‘vorticity-induced’ and ‘wave-induced’ fields,  $\psi = \psi^q + \psi^w$ , where

$$\{q\_qw\} \quad \Delta\psi^q = q, \quad \text{and} \quad \Delta\psi^w = -\frac{1}{4f_0}\Delta|\phi|^2 - \frac{i}{2f_0}J(\phi^*, \phi). \quad (\text{A } 5)$$

With the (A 5) decomposition, the QGPV equation (2.1) becomes

$$\Delta\psi_t^q + J(\psi^q + \psi^w, \Delta\psi^q) = 0. \quad (\text{A } 6) \quad \{\text{lp si q\_t}\}$$

In other words, the relative vorticity of the ‘vorticity-induced’ flow changes due to advection by the total flow  $\psi^q + \psi^w$ . The wave equation satisfies

$$\underbrace{(\partial_t - \frac{i}{2}f_0\lambda^2\Delta)\phi}_{\text{Lin. wave dynamics}} + \underbrace{J(\psi^w, \phi) + \frac{i}{2}\phi\Delta\psi^w}_{\text{Non-lin. wave dynamics}} + \underbrace{J(\psi^q, \phi) + \frac{i}{2}\phi\Delta\psi^q}_{\text{Adv./refrac. by } \psi^q} = 0. \quad (\text{A } 7) \quad \{\text{waves2}\}$$

From (A 5), the ‘wave-induced’ streamfunction  $\psi^w$  is quadratic in  $\phi$ :

$$\psi^w = \frac{1}{4f_0}|\phi|^2 + \frac{i}{2f_0}\Delta^{-1}J(\phi^*, \phi), \quad (\text{A } 8) \quad \{\text{psi w}\}$$

where  $\Delta^{-1}$  is the inverse of the Laplacian:  $\Delta^{-1}\Delta f = f$ .

Together, equations (A 7)-(A 8) show the non-linear nature of the wave equation in this coupled QG-NIW model. In particular, the non-linear wave dynamics is cubic in wave amplitude. Dropping these cubic wave terms yields a quasi-linear (QL) QG-NIW system. This QL system conserves the energy

$$E_{ql} \stackrel{\text{def}}{=} \langle \frac{1}{2}|\nabla\psi^q|^2 \rangle + \langle \nabla\psi^q \cdot \nabla\psi^w \rangle + \langle \frac{\lambda^2}{4}|\nabla\phi|^2 \rangle. \quad (\text{A } 9)$$

Note that  $E = E_{ql} + \langle \frac{1}{2}|\nabla\psi^w|^2 \rangle$ . Thus, the QL approximation is an intermediate model between the uncoupled QG-YBJ model ( $\psi^w = 0$ ) and the non-linear coupled QG-NIW model.

To study the non-linear wave effects, we linearize the QG-NIW equations about the exact solution  $\phi = \Phi = \text{constant}$  and  $\psi = 0$ , so that  $q = 0$ . The QGPV equation (2.1) linearized about this state is trivial:  $q_t = \Delta\psi_t^q = 0$ , thus  $\Delta\psi^q$  remains zero. Also,

$$\Delta\psi^w = -\frac{\Phi}{2f_0}\Delta\frac{1}{2}(\phi + \phi^*), \quad (\text{A } 10) \quad \{\text{lin\_q}\}$$

where  $\phi$  represents the small departure about the constant  $\Phi$ , so that the linearized wave equation becomes

$$\phi_t = \frac{i}{2}\left[\frac{\Phi^2}{2f_0}\Delta\frac{1}{2}(\phi + \phi^*) + f_0\lambda^2\Delta\phi\right]. \quad (\text{A } 11)$$

Recalling that  $u_w + iv_w = \phi e^{i(mz - f_0 t)}$ , we recast the wave equation:

$$\left[\partial_t^2 - \frac{1}{4}f_0\lambda^2\left(f_0\lambda^2 + \frac{\Phi^2}{2f_0}\right)\Delta^2\right]v_w = 0, \quad (\text{A } 12)$$

where  $\Delta^2 = (\partial_x^2 + \partial_y^2)^2$ . Hence, plane-wave solutions yield the dispersion relation

$$\omega = \frac{1}{2}f_0\lambda^2(k^2 + l^2)\left[1 + \frac{1}{2}\frac{\Phi^2}{f_0^2\lambda^2}\right]^{1/2}, \quad (\text{A } 13)$$

and we conclude that the wave self-interaction shifts the frequency by

$$\left[1 + \frac{1}{2}\frac{\Phi^2}{f_0^2\lambda^2}\right]^{1/2} - 1. \quad (\text{A } 14)$$

TABLE 1. Description of parameters of Lamb-Chaplygin simulation.

Parameter	Description	Value
N	Number of modes	512
$L_d$	Domain size	$2\pi \times 200$ km
$2\pi k_e^{-1}$	Dipole radius	$L/15 \approx 84$ km
$U_e$	Dipole strength	$5 \times 10^{-2}$ m s $^{-1}$
$U_w$	NIW speed	$5 \times 10^{-1}$ m s $^{-1}$
$N_0$	Buoyancy frequency	$5 \times 10^{-3}$ s $^{-1}$
$f_0$	Colioris frequency	$10^{-4}$ s $^{-1}$
$2\pi m^{-1}$	NIW vertical wavelength	325 m
$\nu_e$	QGPV biharmonic viscosity	$5 \times 10^7$ m $^4$ s $^{-1}$
$\nu_w$	NIW biharmonic viscosity	$5 \times 10^7$ m $^4$ s $^{-1}$

## Appendix B. Details of the initial value problems

### B.1. Numerical methods

We solve the QG-NIW system (2.1)-(2.3) using a standard collocation Fourier spectral method. In the pseudo-spectral spirit, we evaluate the quadratic non-linearities in physical space, and transform the product into Fourier space. For small-scale dissipation, we use biharmonic viscosity. We find that this choice, originally used in McWilliams (1984), is sufficient to extend the spectral resolution compared to Laplacian viscosity. We make sure the 35% highest modes fall in the dissipation range, so that aliased wavenumbers are strongly damped. We time march the spectral equations using an exponential time differencing method with a fourth order Runge-Kutta scheme (details in Kassam & Trefethen 2005).

### B.2. Parameters

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