Notes on Ocean Eddy Diffusivity

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1 The Advection-Diffusion Equation

1.1 Preliminaries

We are concerned with the lateral advection and diffusion of a passive scalar (tracer). The evolution of the tracer concentration θ is governed by

$$\theta_t + \vec{u} \cdot \nabla\theta = \mathcal{S} + \kappa\triangle\theta \,, \tag{1}$$

where $\mathcal{S}$ represents sources, κ is the molecular diffusivity of the tracer θ, and the horizontal laplacian is $\triangle \stackrel{\text{def}}{=} \partial_x^2 + \partial_y^2$. The advection-diffusion equation above is also often written as and (1) can be re-written

$$\theta_t + \mathsf{J}(\psi, \theta)\theta = \mathcal{S} + \kappa\triangle\theta \,, \tag{2}$$

where the horizontal Jacobian is $\mathsf{J}(A, B) = A_x B_y - B_x A_y$, and the streamfunction ψ, associated with the non-divergent flow $\vec{u}$, is defined by

$$u = -\psi_y \,, \qquad \text{and} \qquad v = \psi_x \,. \tag{3}$$

1.2 Zonnaly-averaged Equations

For application in simple periodic domain, it is convenient to introduce zonally-averaged equations. In more complicated geometries, similar decompositions can be achieved given a well-defined average, which in practice is a combination of time and space averaging. Introducing the Reynolds decomposition

$$\theta(x, y, t) = \langle\theta\rangle_x(y, t) + \theta'(x, y, t) \,,$$
$$\psi(x, y, t) = \langle\psi\rangle_x(y, t) + \psi'(x, y, t) \,, \tag{4}$$
$$\tag{5}$$

where

$$\langle f\rangle_x(y, t) = \frac{1}{L_x}\int_0^{L_x} f(x, y, t)\,\mathrm{d}x \,. \tag{6}$$

With periodicity in x, the $x-$averaged $y-$velocity vanishes $\bar{v} = \bar{\psi}_x = 0$. The $x-$averaged tracer equation is then

$$\partial_t\langle\theta\rangle_x + \partial_y\langle v'\theta'\rangle_x = \langle S\rangle_x + \kappa\partial_y^2\langle\theta\rangle_x \,. \tag{7}$$

For completeness, the equation for the perturbation about the $x-$averaged tracer is

$$\theta'_t + \partial_x(u'\theta') + \partial_y(v'\theta') + \partial_x(u'\langle\theta\rangle_x) + \partial_y(v'\langle\theta\rangle_x) - \partial_y\langle v'\theta'\rangle_x = S' + \kappa(\partial_y^2\theta' + \partial_x^2\theta) \,. \tag{8}$$

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1.3 Eddy diffusion

This notes discusses different methods for estimating ocean eddy diffusivities. A common heuristic argument is parameterize eddy transport such as $\overline{v'\theta'}$ with a down-gradient eddy-difussivity

$$\overline{v'\theta'} = -\kappa_e \partial_y \bar{\theta} \,. \tag{9}$$

The introduction of the eddy diffusivity (9) needs to be justified more formally. For now, it suffices to mention that (9) is a good approximation provided there is enough scale separation between the large-scale tracer gradient the eddy scales. Thus the parameterization (9) is used to close the zonally-averaged equation (7):

$$\bar{\theta}_t = \bar{S} + \kappa_{tot} \partial_y^2 \bar{\theta} \,, \tag{10}$$

where the total diffusivity is $\kappa_{tot} \stackrel{\text{def}}{=} \kappa_e + \kappa$. There is significant inconsistency in the literature regarding notation and naming of eddy diffusivities. The term "effective diffusivity" is used to refer to κ_{tot} by some authors. Other investigators refer to κ_e as the "effective diffusivity" likely recognizing that for large Peclet number $\kappa_{tot} \approx \kappa_e$. The term "eddy diffusivity" is equally used to refer to both κ_{tot} or κ_e.

1.4 Integral variance budget

With harmless boundary conditions (e.g., double periodicity or no-flux across the boundaries) the tracer variance equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \tfrac{1}{2}\theta^2 \mathrm{d}A = \int \theta \mathcal{S} \mathrm{d}A - \kappa \int |\nabla \theta|^2 \mathrm{d}A \,. \tag{11}$$

2 Renovated Waves on a Lattice

To begin exploring the accuracy of different methods to estimate ocean eddy diffusivities, we use a simple advection-diffusion model on a lattice first proposed by Pierrehumbert (2000). The idea is to break the advection and difussion in different steps. The advection step is further separated in two sub-steps: advection in the x-direction and y-direction. The advection sub-steps are performed on a lattice. That is, the advection in the x-direction corresponds to a shift in the x-direction, and the advection in the y-direction corresponds to a shift in the y-direction:

$$i_x^{n+1} = i_x^n - \text{int}\left[u(y)\tau/2\right] \,, \tag{12}$$

and

$$i_y^{n+1} = i_y^n - \text{int}\left[v(x)\tau/2\right] \,, \tag{13}$$

where the superscripts represent the iteration and τ is the length of the renovation cycle, typically a eddy-turnover timescale. Figure 1 illustrates the advection sub-steps in a couple of renovation cycles. Clearly, a very complex tracer pattern emerges after a couple of renovation cycles. Notice that this advection scheme exactly conserves the probability density of the tracer if no diffusion is applied (the lattice is just re-combined).

In a slight generalization of Pierrehumbert's model, we represent the non-divergent velocity field as linear combination of a spectrum of waves with random phase:

$$u(y) = C \sum_{j=j_{min}}^{j_{max}} \left(\frac{j}{j_{min}}\right)^{-p/2} \cos\left(\frac{2\pi y}{L_y} j + \phi_n\right) \,, \tag{14}$$

$$v(x) = C \sum_{j=j_{min}}^{j_{max}} \left(\frac{j}{j_{min}}\right)^{-p/2} \cos\left(\frac{2\pi x}{L_x} j + \psi_n\right) \,, \tag{15}$$

where L_x and L_y are the dimensions of the periodic domain, ψ_n and ϕ_n are random phases drawn from a uniform distribution on $[0, 2\pi]$. Because these random phases are changed every iteration, this simple velocity field is termed renovated wave model. Note that in this generalized renovated wave model the kinetic energy spectrum of the flow follows a j^{-p} power-law. Also in (14) and (15), C is a normalization constant, determined so that the root-mean square velocity is prescribed

$$C = u_{rms} \left(\sum_{j=j_{min}}^{j_{max}} \left(\frac{j}{j_{min}} \right)^{-p} \right)^{-1/2} . \tag{16}$$

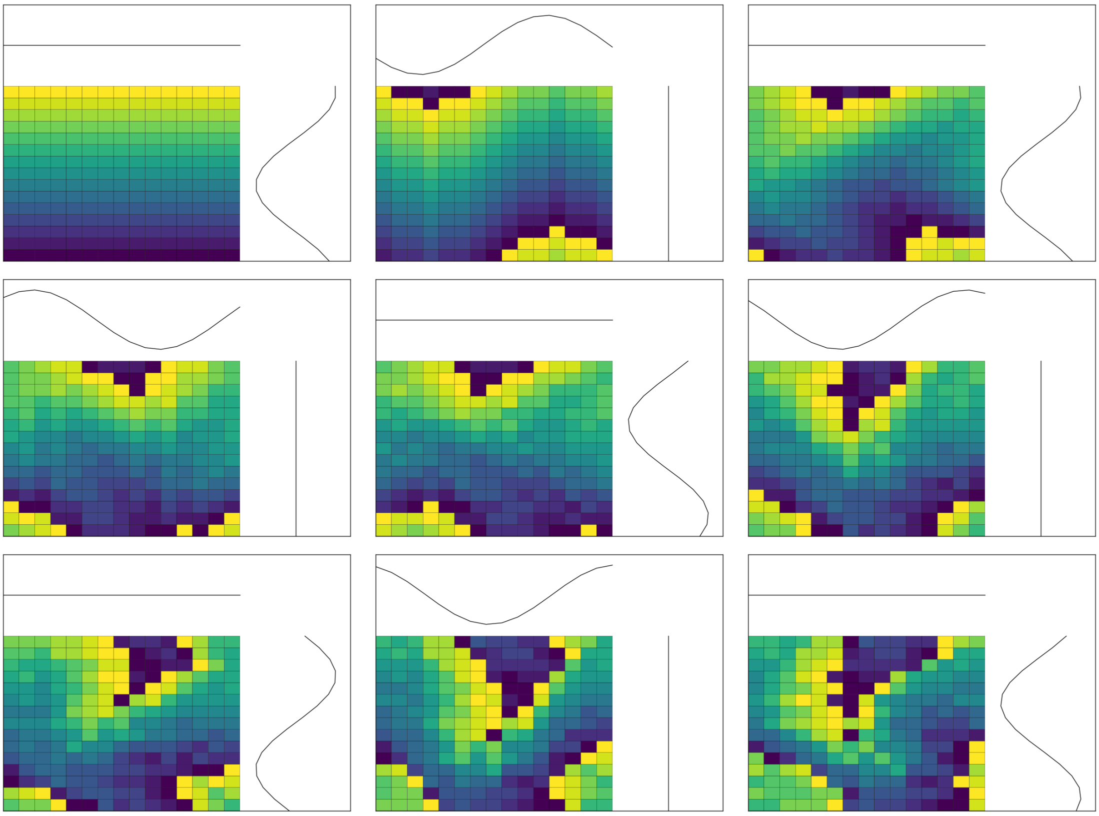

Figure 1: The advection on the two-dimensional periodic lattice. Notice that a complex pattern starts to emerge after a couple of renovation cycles. With a bigger lattice, one no longer sees the pixel fragmentation, and the differential advection results in strong filamentation.

Pierrehumbert performs the diffusion step in physical space, as a simple average of neighboring points, corresponding to a second-order finite-difference approximation for the laplacian operator. Here we take advantage of the periodicity of this model, and perform the diffusion step more accurately in Fourier space:

$$\hat{\theta}_{k,l}^{n+1/2} = \hat{\theta}_{k,l}^{n} \mathrm{e}^{-\kappa(k^2+l^2)\tau/2} . \tag{17}$$

2.1 Advection-diffusion with constant background tracer gradient

Instead of using a large-scale source, we implement a constant background gradient G, so that the tracer concentration equation is

$$\theta_t + \vec{u} \cdot \nabla\theta + Gv = \kappa \triangle \theta . \tag{18}$$

The $x-$averaged variance budget is

$$\partial_t \langle \tfrac{1}{2}\theta \rangle_{x,t} + \partial_y \langle \tfrac{1}{2} v \theta^2 \rangle_{x,t} + G\langle v\theta \rangle_{x,t} = \partial_y^2 \langle \tfrac{1}{2}\theta^2 \rangle_{x,t} - \kappa \langle |\nabla\theta|^2 \rangle_{x,t} . \tag{19}$$

Because the background gradient is constant, the system is statistically homogeneous in y. Thus, in statistical steady state, the $x-$averaged variance budget (19) is exactly given by a balance between variance production and dissipation:

$$G\langle v\theta\rangle_{x,t} = -\kappa\langle|\nabla\theta|^2\rangle_{x,t}\,. \tag{20}$$

2.1.1 The Osborn-Cox method

Cox and Osborn introduced an eddy diffusion closure for the tracer flux,

$$\langle v\theta\rangle_{x,t} = -\kappa_{oc}G\,, \tag{21}$$

to obtain an explicit expression for the eddy diffusivity

$$\kappa_{oc}(y) = \frac{\langle|\nabla\theta|^2\rangle_{x,t}}{G^2}\kappa\,. \tag{22}$$

Note that the Osborn-Cox eddy diffusivity is simply an amplification of the molecular diffusivity, with the amplification factor

$$A_{oc} = \frac{\langle|\nabla\theta|^2\rangle_{x,t}}{G^2}\,. \tag{23}$$

The amplification factor can be thought as a ratio of two length scales. In particular the perturbation of tracer scale as $\theta \sim l_{mix}G$, where l_{mix} is a mixing-length scale. Gradients of θ occur on relatively small scales extending down to the Batchelor scale,

$$l_b = \left(\frac{\kappa}{S}\right)^{1/2}\,, \tag{24}$$

where S is a characteristic scale of the rate of strain of the eddy field. Hence,

$$A_{oc} \sim \frac{l_{mix}^2}{l_b^2} \Longrightarrow \kappa_{oc} \sim l_{mix}^2 S\,. \tag{25}$$

It is pleasing that the eddy diffusivity scaling (25) is independent of the molecular diffusivity κ.

2.1.2 The Nakamura method

A different approach to estimating the eddy diffusivity is the method of Nakamura). In particular, Nakamura showed that averaging the advection-diffusion equation (1) along tracer contours eliminates the advective eddy transport. The transformed equation resemble a diffusion equation in area coordinate with diffusivity (see appendix A for a derivation of the method):

$$K_n(\Theta, t) = L_{eq}^2\kappa\,, \tag{26}$$

where L_{eq} is the equivalent length of the contour. K_N is an instantaneous measure of mixing as a function of each contour Θ; $K_n$it has units of (length)^4 (time^{-1}). The Nakamura eddy diffusivity is then defined as

$$\kappa_n(\Theta) = \langle K_n\rangle_t/L_{min}^2\,, \tag{27}$$

where L_{min} is the minimum length of the contour (L_x in this case). One generally maps the Nakamura diffusivity from contour Θ to physical space $\Theta = \Theta(y)$ either using the structure of the background tracer concentration (Gy) or by constructing a simple one-to-one map based on the distribution of the stirred tracer field.

2.1.3 RW model simulations

We now diagnose eddy diffusivities using both Osborn-Cox and Nakamura methods in reference simulations of the Renovated Waves (RW) model. The RW model implemented with background constant tracer concentration G is a plain-vanilla example of "Statistically Homogeneous Isotropic Transport" setup where the eddy diffusivity is given by Einstein's formula

$$\kappa_{ein} = \frac{u_{rms}^2 \tau}{4} \,. \tag{28}$$

2.2 Advection-diffusion with a simple large-scale source

We employ a simple large-scale source

$$S(y) = \cos\left(\frac{2\pi}{L_y} y\right) \,. \tag{29}$$

Because the source is only a function of y, half of the source is performed after the $x-$advection substep

$$\theta^{n+1/2} = \theta^{n+1/2} + \frac{\Delta t}{2} S(y) \,. \tag{30}$$

The other half of the source could be applied in the same fashion, after the $y-$advection substep

$$\theta^{n+1} = \theta^{n+1} + \frac{\Delta t}{2} S(y) \,. \tag{31}$$

In principle (31) is a brutal way to apply the source since source term is not invariant in y. To apply the source more accurately, we note that in the "y-advection"+"half-source" substeps, we are solving

$$\theta_t + v(x)\theta_y = \cos\left(\frac{2\pi}{L_y} y\right) \,, \tag{32}$$

which can be easily integrated along the characteristics $y = y_0 + v\Delta t/2$, to give

$$\theta(x, y, t + \Delta t) = \theta(x, y, \Delta t) + \frac{\sin\left[k_1(y_0 + v\Delta t/2)\right] - \sin k_1 y_0}{k_1 v} \,. \tag{33}$$

where $k_1 \stackrel{\text{def}}{=} \frac{2\pi}{L_y}$. We anticipate loss of accuracy for in regions where $v \approx 0$. To avoid this inaccuracies, we identify these points numerically, and use the exact value in limit $\Delta t \to 0$. This limit can be calculated using the L'Hospital rule, simply expanding about $v = 0$, to obtain

$$\frac{\sin\left[k_1(y_0 + v\Delta t/2)\right] - \sin k_1 y_0}{k_1 v} \to \frac{\Delta t}{2} \cos k_1 y_0 \qquad \text{as} \qquad v \to 0 \,. \tag{34}$$

In summary, the algorithm for this forced advection-diffusion problem on a lattice, every iteration is composed by the following substeps:

1. $x-$advection
2. half source
3. $y-$advection + half source
4. diffusion

Of course, one could break the diffusion step in two substeps, to be performed after the advection. In practice, there is no significant difference, and we opt to use the single step difusion above for computational efficiency.

2.3 Did Einstein get it right?

For this simple model the $x-$averaged equation is

$$\bar{\theta}_t = \cos k_1 y + D\partial_y^2 \bar{\theta} \,, \tag{35}$$

In statistical steady state, averaging either in time or over ensembles, we obtain

$$D\partial_y^2 \langle \bar{\theta} \rangle = -\cos k_1 y \,, \tag{36}$$

$\langle\rangle$ denotes either time or ensemble average. Thus,

$$\langle \bar{\theta} \rangle = \frac{\cos k_1 y}{D k_1^2} \,. \tag{37}$$

For this simple model, we can calculate the effective diffusivity D exactly (Einstein, 1905)

$$D = \frac{\langle (\Delta x)^2 \rangle}{2\tau} \,, \tag{38}$$

where $\langle (\Delta x)^2 \rangle$ is the mean-square displacement. Intrinsic in the derivation of (38) is the scale separation between the flow that performs the advection and the large-scale gradient. In the generalized RW model, we have

$$\langle (\Delta x)^2 \rangle = \langle u_n^2 \tau^2 \rangle = \frac{u_{rms}^2 (\Delta t)^2}{4} \,, \tag{39}$$

and therefore

$$D = \frac{u_{rms}^2 \tau}{4} \,. \tag{40}$$

Figure 2.5 shows a comparison between theory, using Einstein's effective difusivity (40), and numerical calculation on the lattice model. With scale separation, there is spectacular agreement between theory and numerics (see Figure 2.5 left). Even the $x-$averaged concentration of snapshots is reasonably consistent with the theoretical prediction. Without scale separation, there is still reasonable consistency, but the spread about the time-mean is much larger.

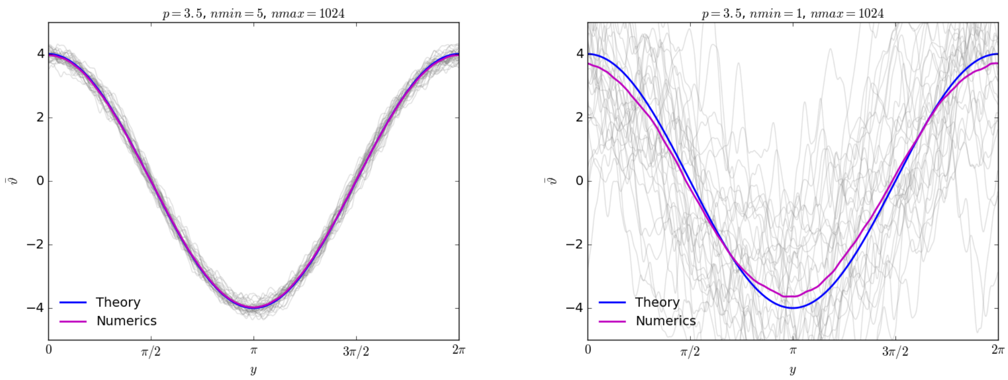

Figure 2: $x-$averaged tracer concentration simulated with $N_x = N_y = 1024$, $L_x = L_y = 2\pi$, $n_{min} = 5$ (left) and $n_{min} = 1$(right), $n_{max} = 1024$, and $p = 1024$. The magenta line represents time-averaged and the thin gray lines represent snapshots. The blue line is the theoretical prediction.

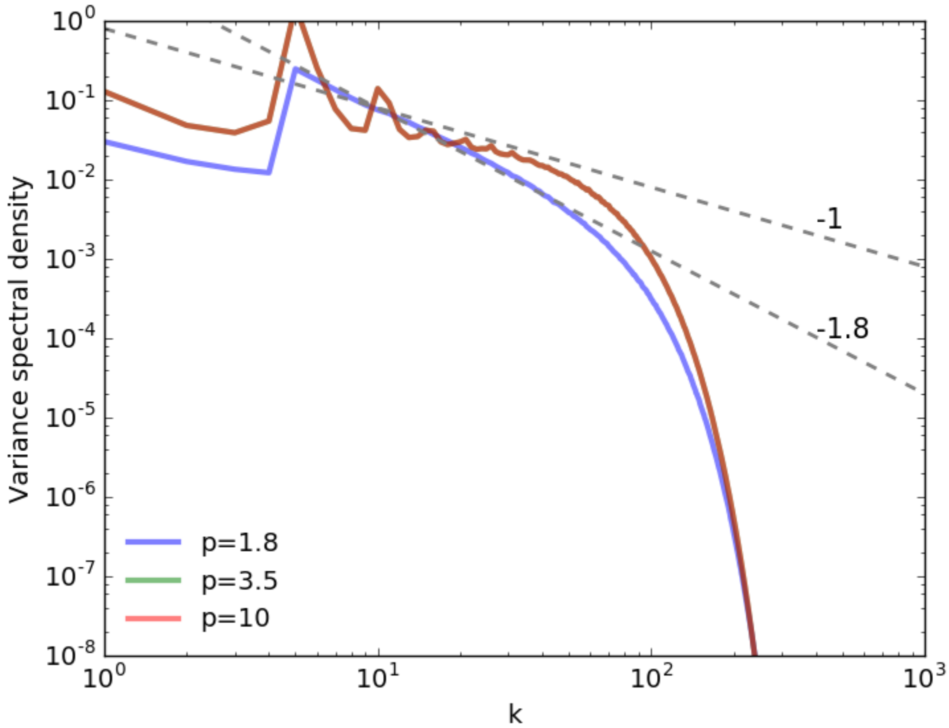

Figure 3: Trace variance spectra as a function of $x-$wavenumber for two runs with $p = 1.8$, $p = 3.5$, $p = 10$, $\kappa = 1.e-3$, $u_{rms} = 1$, n_{min}=5. The $x-$average trace concentration has been removed before calculating the spectrum.

2.4 Tracer variance spectrum

2.5 Dependence on molecular diffusivity

3 Approximate Methods for Effective Diffusivity

These methods have the form

$$K = A^2 \kappa \,, \tag{41}$$

where K is an approximation to the true diffusivity D, κ is the molecular diffusivity, and A^2 is an amplification factor.

3.1 A variance budget approach: Osborn-Cox

Osborn-Cox employs a tracer variance budget approach to estimate the effective diffusivity. Multiplying (8) and $x-$averaging, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\overline{\theta'^2}}{2} + \partial_y \left(\overline{v' \frac{\theta'^2}{2}} \right) + \overline{v'\theta'}\partial_y \bar{\theta} = -\kappa \overline{|\nabla\theta'|^2} + \kappa \partial_y^2 \frac{\overline{\theta'^2}}{2} \,. \tag{42}$$

With the eddy diffusivity hypothesis, i.e.,

$$\overline{v'\theta'} = -K_{oc}\partial_y \bar{\theta} \,, \tag{43}$$

we have, in statistical steady state,

$$+ \partial_y \left(\overline{v' \frac{\theta'^2}{2}} \right) - K_{oc}(\partial_y \bar{\theta})^2 = -\kappa \overline{|\nabla\theta'|^2} + \kappa \partial_y^2 \frac{\overline{\theta'^2}}{2} \,. \tag{44}$$

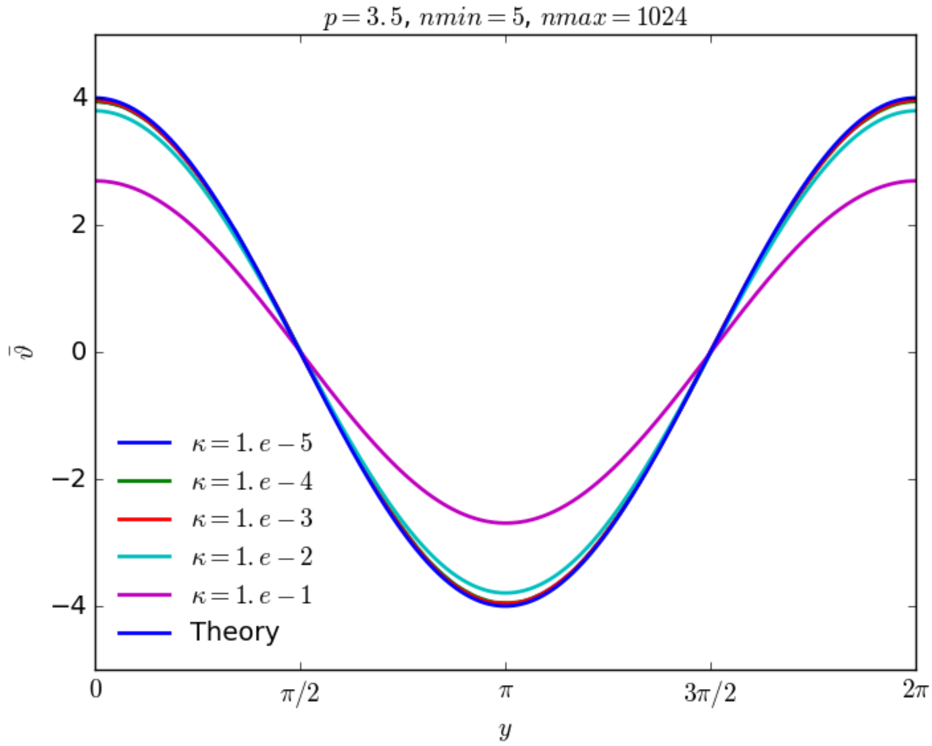

Figure 4: Dependence of $x-$averaged tracer concentration on the numerical value of the molecular diffusivity. As expected, the $x-$averaged tracer concentration is nearly independent unless the molecular diffusivity is too large.

The Osborn-Cox approach ignores the first term on the left and the last term on the right of (44), so that the Osborn-Cox eddy diffusivity is

$$K_{oc} = \frac{\overline{|\nabla\theta'|^2}}{\bar{\theta}_y^2}\kappa\,. \tag{45}$$

3.2 Monitoring the lengthening of tracer contours: Nakamura

See appendix A for a derivation.

$$K_N = \frac{L^2}{L_{min}^2}\kappa\,. \tag{46}$$

4 Diffusion suppression by a mean flow

To study the suppression of eddy diffusivity by a mean flow, we simply add a constant zonal velocity to (14) and (15). The $x-$advection on the lattice is only slightly changed:

$$\begin{aligned}\dot{x} &= U + u(y)\,, [0,\tau/2] : x = x_0 + [U + u(y)]\frac{\tau}{2}\,.\\ \dot{y} &= 0\,, [0,\tau/2] : y = y_0\,.\end{aligned} \tag{47}$$

The y-advection is

$$\begin{aligned}\dot{x} &= U+\,, [\tau/2,\tau] : x = x_0 + U\tau/2\,.\\ \dot{y} &= v(x)\,, [\tau/2,\tau] : \dot{y} = v(x_0 + U\tau/2)\,.\end{aligned} \tag{48}$$

Hence

$$y = y_0 + C \sum_{j=j_{min}}^{j_{max}} \left(\frac{j}{j_{min}} \right)^{-p/2} \frac{1}{Uk} \left[\sin \left(k(x + U\tau)j + \psi_j \right) - \sin \left(k(x + U\tau/2)j + \psi_j \right) \right] \,, \tag{49}$$

The mean square displacement is

$$\langle (\Delta y)^2 \rangle = C^2 \sum_k \left(\frac{j}{j_{min}} \right)^{-p} \left[\frac{1 - \cos \left(Uk\tau/2 \right)}{(Uk)^2} \right] \,. \tag{50}$$

Note that in the limit of small mean velocity, $Uk\tau/2 \to 0$, we obtain the diffusivity for the non-mean-flow case:

$$\langle (\Delta y)^2 \rangle = u_{rms}^2 \frac{\tau^2}{4} \,. \tag{51}$$

A Derivation of Nakamura's Eddy Diffusivity

The Nakamura effective diffusivity arises when transforming the diffusion equation into a area/-contour coordinate system, i.e. $A = A(\Theta, t)$, where A is the area bounded by the contour Θ at time t. Because the area bounded by a contour Θ is a well-defined graph, i.e. $\Theta = \Theta(A, t)$ (see figure A), we have

$$\Theta = \theta[A(\Theta, t), t] \, , \tag{52}$$

so that

$$\frac{\partial \theta}{\partial A} A_t + \theta_t = 0 \qquad \text{and} \qquad \frac{\partial \theta}{\partial A} \frac{\partial A}{\partial \Theta} = 1 \, . \tag{53}$$

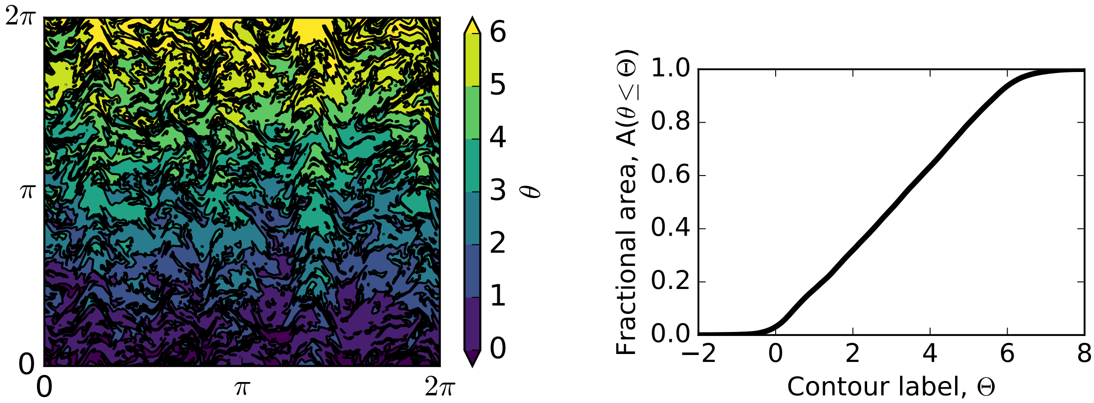

Figure 5: (Left) A snapshot of tracer concentration θ from a simulation of the RW model. (Right) The fractional area of tracer concentration satisfying $\theta \leq \Theta$, that is, the cumulative distribution function of θ.

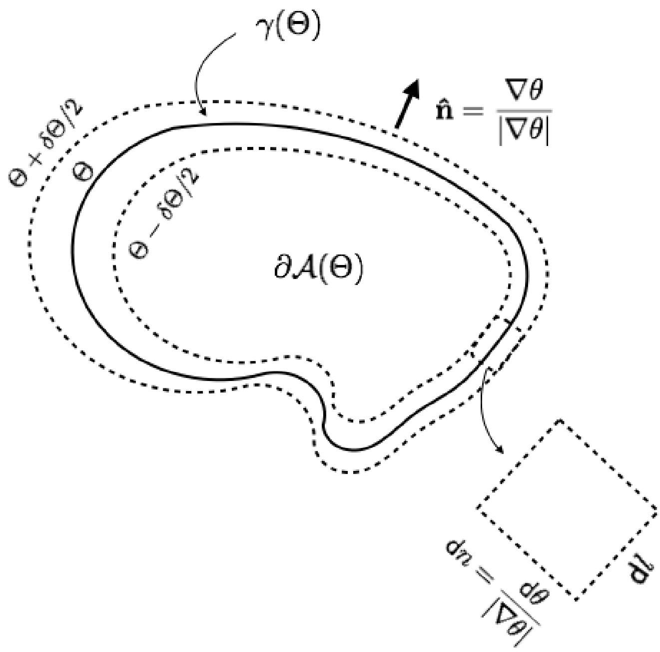

Figure 6: Schematics of the coordinate system defined by the the tracer contour $\theta = \Theta$.

The integral of the scalar function f over the surface inside curves $\gamma(\Theta + \mathrm{d}\Theta/2)$ and $\gamma(\Theta - \mathrm{d}\Theta/2)$ (see figure A) is

$$\delta I_\Theta(f) = \iint_{\partial A^+} f\,\mathrm{d}x\mathrm{d}y - \iint_{\partial A^-} f\,\mathrm{d}x\mathrm{d}y = \delta\theta \oint_\gamma f \frac{\mathrm{d}l}{|\nabla\theta|}\,. \tag{54}$$

where $\partial A^+ = \partial A(\Theta + \delta\Theta/2)$ and $\partial A^- = \partial A(\Theta - \delta\Theta/2)$. Thus in the limit $\delta\Theta \to 0$,

$$\frac{\partial}{\partial\Theta} I_\Theta(f) = \frac{\partial}{\partial\Theta}\iint_{\partial A} f\,\mathrm{d}x\mathrm{d}y = \oint_\gamma f \frac{\mathrm{d}l}{|\nabla\theta|}\,. \tag{55}$$

In particular, the specific case $f = 1$ gives:

$$\frac{\partial}{\partial\Theta} A = \oint_\gamma \frac{\mathrm{d}l}{|\nabla\theta|}\,, \tag{56}$$

where the total area of the surface bounded by Θ is $A = I_\Theta(1)$. The thickness average of f along the $\gamma(\Theta)$ is defined by

$$\langle f\rangle_\Theta \overset{\text{def}}{=} \frac{\oint_\gamma f \frac{\mathrm{d}l}{|\nabla\theta|}}{\oint_\gamma \frac{\mathrm{d}l}{|\nabla\theta|}} = \frac{\partial}{\partial A} I_\Theta(f)\,. \tag{57}$$

With the results above, we are ready to transform the advection-diffusion equation onto a coordinate system defined by the countour Θ. First, we note that the rate of change of the material surface A is given by

$$A_t = \oint_\gamma \mathbf{u}_c \cdot \hat{\mathbf{n}}\,\mathrm{d}l\,, \tag{58}$$

where the velocity of the contour, locally normal to the contour, is defined by

$$\mathbf{u}_c = -\theta_t \frac{\nabla\theta}{|\nabla\theta|^2} = -\frac{\theta_t}{|\nabla\theta|}\hat{\mathbf{n}}\,. \tag{59}$$

Thus, using the advection-diffusion equation (1), we obtain

$$\begin{aligned}\frac{\partial A}{\partial t} &= -\oint_\gamma \theta_t \frac{\mathrm{d}l}{|\nabla\theta|} = \oint_\gamma \nabla\cdot(\mathbf{u}\theta)\frac{\mathrm{d}l}{|\nabla\theta|} - \kappa\oint_\gamma \triangle\theta\frac{\mathrm{d}l}{|\nabla\theta|} - \oint_\gamma S\frac{\mathrm{d}l}{|\nabla\theta|} \\ &= \frac{\partial}{\partial\Theta}\underbrace{\iint \nabla\cdot(\mathbf{u}\theta)\mathrm{d}A}_{=0} - \kappa\frac{\partial}{\partial\Theta}\underbrace{\iint \triangle\theta\mathrm{d}A}_{=\oint |\nabla\theta|^2\frac{\mathrm{d}l}{|\nabla\theta|}} - \frac{\partial}{\partial\Theta}\iint S\mathrm{d}A\,. \end{aligned}\tag{60}$$

Now using (53) we obtain

$$\begin{aligned}\frac{\partial\theta}{\partial t} &= \kappa\frac{\partial}{\partial A}\iint \triangle\theta\mathrm{d}A + \frac{\partial}{\partial A}\iint S\mathrm{d}A \\ &= \frac{\partial}{\partial A}\left(K_e(A,t)\frac{\partial\theta}{\partial A}\right) + \langle S\rangle_\Theta\,,\end{aligned}\tag{61}$$

where the effective diffusivity is

$$K_e(A,t) = \frac{\kappa\oint_\gamma |\nabla\theta|\mathrm{d}l}{\partial\theta/\partial A} = \kappa\underbrace{\frac{\partial A}{\partial\Theta}}_{\oint_\gamma \frac{\mathrm{d}l}{|\nabla\theta|}}\oint_\gamma |\nabla\theta|\mathrm{d}l = \frac{\partial A}{\partial\Theta}\frac{\partial}{\partial A}\iint_{\partial A}\kappa|\nabla\theta|^2\mathrm{d}x\mathrm{d}y = \frac{\langle\kappa|\nabla\theta|^2\rangle_\Theta}{(\partial\theta/\partial A)^2}\,. \tag{62}$$

Note that the effective diffusivity has units of $(\text{length})^4\times(\text{time})^{-1}$. The Nakamura eddy diffusivity is defined by

$$\kappa_N(\Theta,t) = \frac{K_e}{L_{min}^2} = \frac{L_{eq}^2}{L_{min}^2}\kappa\,, \tag{63}$$

where the equivalent length is

$$L_{eq}^2 \stackrel{\text{def}}{=} \oint_\gamma \frac{\mathrm{d}l}{|\nabla\theta|} \oint_\gamma |\nabla\theta| \mathrm{d}l\,, \tag{64}$$

and L_{min} is the minimum length of the contour (without distortion by eddies). In simple domain such as periodic channel, L_{min} is unambiguously defined as the length of the channel. In more complicated realistic geometry and source functions, the definition of L_{min} may not be straightforward. In practice, one can estimate L_{min} by calculating L_{eq} in simulations with large Peclet number ($\kappa_N \to \kappa$). A lower bound on (64) is obtained using the Cauchy-Schwartz inequality[1]:

$$L_{eq}^2 \geq L^2 = \Big(\oint_\gamma \mathrm{d}l \Big)^2 . \tag{65}$$

While the equivalent length notation above provides a useful interpretation, in practice one does not compute line integrals. Instead, the third equality in (62) suggests a straightforward numerical approximation

$$K_e(\Theta, t) \approx \frac{\delta I_\Theta(1)}{(\delta\Theta)^2} \delta I_\Theta(\kappa|\nabla\theta|^2)\,, \tag{66}$$

where $\delta I_\theta(f)$ is the approximate area integral of f within the contours $\Theta + \mathrm{d}\Theta/2$ and $\Theta - \mathrm{d}\Theta/2$ (see figure 5). These approximate integrals are computed simply by counting pixels:

$$\delta I_\Theta(f) = \sum_j f \mathrm{d}x\mathrm{d}y\,, \qquad \Theta - \mathrm{d}\Theta/2 \leq \theta \leq \Theta + \mathrm{d}\Theta/2\,, \tag{67}$$

where the summation on j represents the pixels within the contour.

Finally note that the Nakamura diffusivity is a function of the the contour Θ (or equivalently of the area bounded by the contour). With simple geometry of initial conditions and source function, $\kappa_N(\Theta, t)$ can be mapped to physical space, e.g. as direction of the tracer gradient or the equivalent latitude.

[1] $\oint_\gamma f^2 \mathrm{d}l \oint_\gamma g^2 \mathrm{d}l \geq \Big(\oint_\gamma f\, g\, \mathrm{d}l \Big)^2$.