Notes on Modal Projection and non-zero vertical shear at the boundary, etc.

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March 2, 2016

We want to calculate the mean quasigeostrophic potential vorticity gradients:

$$Q_y = \beta - \mathsf{L}V(z)\,, \qquad \text{and} \qquad Q_x = \mathsf{L}U(z)\,, \tag{1}$$

where $\mathsf{L} \stackrel{\text{def}}{=} \partial_z (f_0/N)^2 \partial_z$ is the stretching operator. Calculation of $\mathsf{L}U$ and $\mathsf{L}V$ from data is typically noisy. To reduce the noise, we want to project the basic state velocity onto the familiar vertical modes p_n that satisfy

$$\mathsf{L}\mathsf{p}_n = -\kappa_n^2 \mathsf{p}_n\,, \tag{2}$$

with $\mathsf{p}_n' = 0\,,\ z = -h, 0$. However, with nonzero vertical shear at the boundaries we cannot differentiate the series for $U(z)$ and $V(z)$. So, instead of the tradition series, we write

$$U(z) \approx U_s(z) + U_i(z)\,, \tag{3}$$

where $U_i(z)$ is an "interior", boundary shearless, velocity component:

$$U_i(z) = \sum_{n=0}^{\mathrm{N}} \breve{U}_n \mathsf{p}_n\,, \qquad \breve{U}_n = \tfrac{1}{h}\int_{-h}^{0} \mathsf{p}_n \left[U(z) - U_s(z)\right]\mathrm{d}z\,, \tag{4}$$

and $U_s(z)$ satisfies $U'(z^-) = U_s'(z^-)$ and $U'(z^+) = U_s'(z^+)$. Also in (3) the approximate sign stems from the truncated nature of the series (4). A simple choice is

$$\frac{\mathrm{d}U_s}{\mathrm{d}z} = \frac{N^2(z)}{N^2(0)}\frac{z+h}{h}U'(0) - \frac{N^2(z)}{N^2(-h)}\frac{z}{h}U'(-h)\,. \tag{5}$$

The reason we include the N^2 factors above, e.g. $N^2(z)/N^2(0)$ is that we want to calculate $\mathsf{L}U_s$ as smoothly as possible, and hence independently of derivatives of $N^2(z)$. With this choice we have

$$U_s(z) = \frac{U'(0)}{N^2(0)h}\int_0^z N^2(z')(z'+h)\mathrm{d}z' - \frac{U'(-h)}{N^2(-h)h}\int_0^z N^2(z')z'\mathrm{d}z + A\,, \tag{6}$$

where A is determined by imposing a no net transport condition

$$\int_{-h}^{0} U_s'(z)\mathrm{d}z = 0\,. \tag{7}$$

We obtain

$$A = -\frac{U'(0)}{N^2(0)h^2}\int_{-h}^{0}\mathrm{d}z\int_0^z N^2(z')(z'+h)\mathrm{d}z' + \frac{U'(-h)}{N^2(-h)h^2}\int_{-h}^{0}\mathrm{d}z\int_0^z N^2(z')z'\mathrm{d}z \tag{8}$$

The integrals above are calculated numerically for an arbitrary buoyancy profile $N^2(z)$.

In summary, knowledge of the shear at the boundaries determines the "surface" velocity component $U_s(z)$ that can be then subtract from the total velocity to obtain a "interior" velocity component $U_i(z)$. Because $U_i(z)$ is shearless at the boundaries, we can accurately calculate $\mathsf{L}U$

$$\mathsf{L}U \approx \mathsf{L}U_s + \mathsf{L}U_i = \frac{f_0^2}{N^2(0)}\frac{U'(0)}{h} - \frac{f_0^2}{N^2(-h)}\frac{U'(-h)}{h} - \sum_{n=1}^{\mathrm{N}} \kappa_n^2 \breve{U}_n \mathsf{p}_n\,. \tag{9}$$