On WKB approximate solutions for standard baroclinic modes

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March 21, 2015

In this notes I derive approximate WKB solutions to the standard baroclinic modes of physical oceanography. The elementary textbook example with constant buoyancy frequency is recovered as a special case.

1 Pressure modes

The standard baroclinic modes for pressure, here denoted $\mathsf{p}_n(z)$, is defined by the regular Sturm-Liouville eigenproblem

$$\mathsf{L}\mathsf{p}_n = -\kappa_n^2 \mathsf{p}_n\,, \tag{1}$$

with homogeneous Neumann boundary conditions

$$@z = -h,\, 0: \qquad \mathsf{p}_n' = 0\,, \tag{2}$$

where

$$\mathsf{L} \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}z}\frac{f_0^2}{N^2}\frac{\mathrm{d}}{\mathrm{d}z}\,, \tag{3}$$

is a self-adjoint linear operator. Hence the eigenmodes, p_n, are orthogonal. The real egeinvalues, κ_n, are the deformation wavenumber of the n'th mode. It is convenient to normalize the eigenmodes to have the unit L^2-norm

$$\frac{1}{H}\int_{-h}^{0} \mathsf{p}_n \mathsf{p}_m \mathrm{d}z = \delta_{mn}\,, \tag{4}$$

where δ_{mn} is the Dirac delta. Equation (1) can be rewritten as

$$\left(\tfrac{f_0}{N}\right)^2 \mathsf{p}_n'' + \left[\left(\tfrac{f_0}{N}\right)^2\right]' \mathsf{p}_n' + \kappa_n^2\, \mathsf{p}_n = 0\,. \tag{5}$$

Introducing the following definitions

$$\epsilon \stackrel{\text{def}}{=} \frac{1}{\kappa_n} \qquad \text{and} \qquad S^2(z) \stackrel{\text{def}}{=} \left(\tfrac{N}{f_0}\right)^2\,. \tag{6}$$

we have the renotated equation

$$\epsilon^2\, \mathsf{p}_n'' - \epsilon^2 \left[\log S^2(z)\right]' \mathsf{p}_n' + S^2(z)\mathsf{p}_n = 0\,. \tag{7}$$

In the WKB spirit we assume that $S^2(z)$ is slowly varying i.e., the buoyancy frequency $N^2(z)$ does not vary very fast. (This assumption may be problematic near the base of the mixed-layer.)

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We also assume that ϵ is small; the accuracy of the WKB solution improves with mode number. We now make the exponential approximation (e.g., Bender and Orszag)

$$\mathsf{p}_n^e \stackrel{\text{def}}{=} \mathrm{e}^{Q(z)/\epsilon} . \tag{8}$$

Hence

$$\mathsf{p}_n^{e\,\prime} = \frac{Q'(z)}{\epsilon}\mathsf{p}_n^e , \tag{9}$$

and

$$\mathsf{p}_n^{e\,\prime\prime} = \left[\left(\frac{Q'(z)}{\epsilon}\right)^2 + \frac{Q''(z)}{\epsilon}\right]\mathsf{p}_n^e , \tag{10}$$

Next we expand $Q(z)$ in powers of ϵ

$$Q(z) = Q_0(z) + \epsilon\, Q_1(z) + \epsilon^2\, Q_2(z) + \mathcal{O}(\epsilon^3) . \tag{11}$$

Substituting (11) in (7) we obtain, to lowest order, $\mathcal{O}(\epsilon^0)$,

$$Q_0'^2 + S^2(z) = 0 . \tag{12}$$

Thus

$$Q_0 = \pm\mathrm{i}\int^z S(\xi)\,\mathrm{d}\xi = \pm\mathrm{i}\tfrac{1}{f_0}\int^z N(\xi)\,\mathrm{d}\xi . \tag{13}$$

At next order, $\mathcal{O}(\epsilon)$, we have

$$2\,Q_0'Q_1' + Q_0'' - Q_0'\left[\log S^2(z)\right]' = 0 . \tag{14}$$

Hence

$$Q_1 = \frac{1}{2}\log S^2(z) - \frac{1}{2}\log \pm\mathrm{i}S(z) + \text{const} \;. \tag{15}$$

Notice that the imaginary part in the log in (15) just contributes to the irrelevant constant. Thus

$$Q_1 = \log\sqrt{S(z)} + \text{const} \;. \tag{16}$$

In the most common WKB approximation (a.k.a "physical optics") we keep terms to $\mathcal{O}(\epsilon)$ in the expansion (11). The solution to (7), consistent with the bottom boundary condition (2), is

$$\mathsf{p}_n^{po} = A_n\sqrt{N(z)}\,\cos\left(\frac{\kappa_n}{f_0}\int_{-h}^z N(\xi)\mathrm{d}\xi\right) , \tag{17}$$

where A_n is a constant. By imposing the boundary condition at $z = 0$ (2), we obtain the eigenvalues κ_n:

$$\kappa_n = \frac{n\pi\, f_0}{\overline{N}\, h} , \qquad n = 0, 1, 2, \ldots , \tag{18}$$

where

$$\overline{N} = \frac{1}{h}\int_{-h}^0 N(\xi)\mathrm{d}\xi . \tag{19}$$

The constant A_n is determined by the normalization condition (4). We have

$$A_n^2\int_{-h}^0 N(z)\cos\left(\frac{\kappa_n}{f_0}\int_{-h}^z N(\xi)\mathrm{d}\xi\right)\mathrm{d}z = H , \qquad n \geq 1 . \tag{20}$$

The integral in (20) can be evaluated exactly by making the change of variables

$$\eta \stackrel{\text{def}}{=} \frac{\kappa_n}{f_0}\int_{-h}^z N(\xi)\mathrm{d}\xi \qquad \Rightarrow \qquad \mathrm{d}\eta = \frac{\kappa_n}{f_0}N(z)\mathrm{d}z , \tag{21}$$

and using the expression for the eigenvalues (18). We obtain

$$A_n = \left(2/\overline{N}\right)^{1/2}, \qquad n \geq 1\,. \tag{22}$$

Thus the WKB approximate solution to the standard pressure modes is

$$\mathsf{p}_n^{po} = \left[\frac{2\,N(z)}{\overline{N}}\right]^{1/2} \cos\left(\frac{n\pi}{\overline{N}\,h}\int_{-h}^{z} N(\xi)\mathrm{d}\xi\right), \qquad n \geq 1\,. \tag{23}$$

The amplitude of the baroclinic modes at the boundaries are independent of the eigenvalue:

$$\mathsf{p}_n^{po}(z=0) = (-1)^n \left[\frac{2N(0)}{\overline{N}}\right]^{1/2}, \tag{24}$$

and

$$\mathsf{p}_n^{po}(z=-h) = \left[\frac{2N(-h)}{\overline{N}}\right]^{1/2}. \tag{25}$$

The barotropic mode is not recovered from the WKB solution because $\kappa_0 = 0$. From (1) we have that with $\kappa_0 = 0$, the barotropic mode is constant, independent of the stratification. With the normalization (4) we obtain $\mathsf{p}_0 = 1$.

Constant buoyancy frequency

With $N =$ const. the modes are the simple sinusoids. That result is recovered as a special case of the WKB solution

$$\mathsf{p}_n^{po} = \sqrt{2}\cos\left[n\pi(1+z/h)\right]. \tag{26}$$

2 Density modes

Similarly the baroclinic modes for density, here denoted by r_n, are defined via the eigenproblem

$$\mathsf{r}_n'' = -\kappa_n^2\left(\tfrac{N}{f_0}\right)^2 \mathsf{r}_n\,, \tag{27}$$

with homogeneous Dirichlet boundary conditions

$$@z = -h, 0: \qquad \mathsf{r}_n = 0\,, \tag{28}$$

and normalization

$$\frac{1}{h}\int_{-h}^{0} \mathsf{r}_n \mathsf{r}_m \mathrm{d}z = \delta_{mn}\,. \tag{29}$$

Alternatively, we can work on the approximation from the beginning. The WKB approximate solution to (27)-(28), consistent with the bottom boundary conditions (28), is

$$\mathsf{r}_n^{po} = \frac{B_n}{\sqrt{N(z)}} \sin\left(\frac{\kappa_n}{f_0}\int_{-h}^{z} N(\xi)\mathrm{d}\xi\right), \tag{30}$$

The eigenvalues κ_n are the same as before (18). (This should be no surprise because it follow from the definition of p_n and r_n. Nonetheless, the verification is a good sanity check.) To find B_n we use the normalization (29)

$$B_n^2 \int_{-h}^{0} \frac{1}{N(z)} \sin\left(\frac{\kappa_n}{f_0}\int_{-h}^{z} N(\xi)\mathrm{d}\xi\right) \mathrm{d}z = h\,, \qquad n \geq 1\,. \tag{31}$$

We use a similar trick as above i.e., we change variables with

$$\eta \stackrel{\text{def}}{=} \frac{\kappa_n}{N^2(z) f_0} \int_{-h}^{z} N(\xi)\mathrm{d}\xi \qquad \Rightarrow \qquad \mathrm{d}\eta = \frac{\kappa_n}{N(z) f_0}\mathrm{d}z\,, \tag{32}$$

where, in the WKB spirit, we used the fact that $N(z)$ is slowly varying when differentiating the relation above. We obtain

$$B_n = \left(2/\overline{N}\right)^{1/2}. \tag{33}$$

Thus the WKB approximate solution to the density modes is

$$\mathsf{r}_n^{po} = \left(\frac{2}{\overline{N} N(z)}\right) \sin\left(\frac{n\pi}{\overline{N}\,h} \int_{-h}^{z} N(\xi)\mathrm{d}\xi\right)\,, \qquad n \geq 1\,. \tag{34}$$

Finally, note that the modes are simply related

$$\frac{\mathrm{d}\mathsf{r}_n^{po}}{\mathrm{d}z} = \underbrace{\frac{n\pi}{\overline{N}\,h}}_{=\kappa_n/f_0}\ \mathsf{p}_n^{po}\,. \tag{35}$$

Constant buoyancy frequency

Again we recover the $N = $ const. special case from (34):

$$\mathsf{r}_n^{po} = \frac{\sqrt{2}}{\overline{N}} \sin\left[n\pi(1 + z/h)\right]\,. \tag{36}$$