

# WKB approximate solutions for standard baroclinic modes

Cesar B Rocha\*

February 5, 2018

In these notes I derive approximate WKB solutions to the standard baroclinic modes of physical oceanography. The elementary textbook example with constant buoyancy frequency is recovered as a special case.

## 1 Pressure modes

The standard baroclinic modes for pressure, here denoted  $\mathbf{p}_n(z)$ , is defined by the regular Sturm-Liouville eigenproblem

$$\mathbf{L}\mathbf{p}_n = -\kappa_n^2 \mathbf{p}_n, \quad (1)$$

with homogeneous Neumann boundary conditions

$$@z = -h, 0 : \quad \mathbf{p}'_n = 0, \quad (2)$$

and the self-adjoint Linear operator

$$\mathbf{L} \stackrel{\text{def}}{=} \frac{d}{dz} \frac{f_0^2}{N^2} \frac{d}{dz}. \quad (3)$$

Hence the eigenmodes,  $\mathbf{p}_n$ , are orthogonal. The eigenvalue  $\kappa_n$  is the deformation wavenumber of the  $n$ 'th mode. It is convenient to normalize the eigenmodes to have unit  $L^2$ -norm:

$$\frac{1}{H} \int_{-h}^0 \mathbf{p}_n \mathbf{p}_m dz = \delta_{mn}, \quad (4)$$

where  $\delta_{mn}$  is the Kronecker delta. Equation (1) can be rewritten as

$$\left(\frac{f_0}{N}\right)^2 \mathbf{p}_n'' + \left[\left(\frac{f_0}{N}\right)^2\right]' \mathbf{p}_n' + \kappa_n^2 \mathbf{p}_n = 0. \quad (5)$$

Introducing the following definitions

$$\epsilon \stackrel{\text{def}}{=} \frac{1}{\kappa_n} \quad \text{and} \quad S^2(z) \stackrel{\text{def}}{=} \left(\frac{N}{f_0}\right)^2. \quad (6)$$

we have the renotated equation

$$\epsilon^2 \mathbf{p}_n'' - \epsilon^2 [\log S^2(z)]' \mathbf{p}_n' + S^2(z) \mathbf{p}_n = 0. \quad (7)$$

In the WKB spirit we assume that  $S^2(z)$  is slowly varying i.e., the buoyancy frequency  $N^2(z)$  does not vary very fast. (This assumption may be problematic near the base of the mixed-layer.)

---

\*Scripps Institution of Oceanography, University of California, San Diego; [crocha@ucsd.edu](mailto:crocha@ucsd.edu)

We also assume that  $\epsilon$  is small; the accuracy of the WKB solution improves with mode number. We now make the exponential approximation (e.g., Bender and Orszag)

$$\mathbf{p}_n^e \stackrel{\text{def}}{=} e^{Q(z)/\epsilon}. \quad (8)$$

Hence

$$\mathbf{p}_n^{e'} = \frac{Q'(z)}{\epsilon} \mathbf{p}_n^e, \quad (9)$$

and

$$\mathbf{p}_n^{e''} = \left[ \left( \frac{Q'(z)}{\epsilon} \right)^2 + \frac{Q''(z)}{\epsilon} \right] \mathbf{p}_n^e, \quad (10)$$

Next we expand  $Q(z)$  in powers of  $\epsilon$

$$Q(z) = Q_0(z) + \epsilon Q_1(z) + \epsilon^2 Q_2(z) + \mathcal{O}(\epsilon^3). \quad (11)$$

Substituting (11) in (7) we obtain, to lowest order,  $\mathcal{O}(\epsilon^0)$ ,

$$Q_0'^2 + S^2(z) = 0. \quad (12)$$

Thus

$$Q_0 = \pm i \int^z S(\xi) d\xi = \pm i \frac{1}{f_0} \int^z N(\xi) d\xi. \quad (13)$$

At next order,  $\mathcal{O}(\epsilon)$ , we have

$$2 Q_0' Q_1' + Q_0'' - Q_0' [\log S^2(z)]' = 0. \quad (14)$$

Hence

$$Q_1 = \frac{1}{2} \log S(z) - \frac{1}{2} \log \pm i S(z) + \text{const}. \quad (15)$$

Notice that the imaginary part in the log in (15) just contributes an irrelevant constant. Thus

$$Q_1 = \log \sqrt{S(z)} + \text{const}. \quad (16)$$

In the most common WKB approximation (a.k.a “physical optics”) we truncate (11) at  $\mathcal{O}(\epsilon)$ . The solution to (7), consistent with the bottom boundary condition (2), is

$$\mathbf{p}_n^{po} = A_n \sqrt{N(z)} \cos \left( \frac{\kappa_n}{f_0} \int_{-h}^z N(\xi) d\xi \right), \quad (17)$$

where  $A_n$  is a constant. By imposing the boundary condition at  $z = 0$  (2), we obtain the eigenvalues  $\kappa_n$ :

$$\kappa_n = \frac{n\pi f_0}{\overline{N} h}, \quad n = 0, 1, 2, \dots, \quad (18)$$

where the mean buoyancy frequency is

$$\overline{N} \stackrel{\text{def}}{=} \frac{1}{h} \int_{-h}^0 N(\xi) d\xi. \quad (19)$$

The constant  $A_n$  is determined by the normalization condition (4). We have

$$A_n^2 \int_{-h}^0 N(z) \cos^2 \left( \frac{\kappa_n}{f_0} \int_{-h}^z N(\xi) d\xi \right) dz = H, \quad n \geq 1. \quad (20)$$

The integral in (20) can be evaluated exactly by making the change of variables

$$\eta \stackrel{\text{def}}{=} \frac{\kappa_n}{f_0} \int_{-h}^z N(\xi) d\xi \quad \Rightarrow \quad d\eta = \frac{\kappa_n}{f_0} N(z) dz, \quad (21)$$

and using the expression for the eigenvalues (18). We obtain

$$A_n = \left(2/\overline{N}\right)^{1/2}, \quad n \geq 1. \quad (22)$$

Thus the WKB approximate solution to the standard pressure modes is

$$\mathbf{p}_n^{po} = \left[ \frac{2N(z)}{\overline{N}} \right]^{1/2} \cos \left( \frac{n\pi}{\overline{N}h} \int_{-h}^z N(\xi) d\xi \right), \quad n \geq 1. \quad (23)$$

The amplitude of the baroclinic modes at the boundaries is independent of the eigenvalue:

$$\mathbf{p}_n^{po}(z=0) = (-1)^n \left[ \frac{2N(0)}{\overline{N}} \right]^{1/2}, \quad (24)$$

and

$$\mathbf{p}_n^{po}(z=-h) = \left[ \frac{2N(-h)}{\overline{N}} \right]^{1/2}. \quad (25)$$

The barotropic mode is not recovered from the WKB solution because  $\kappa_0 = 0$ . From (1) we have that with  $\kappa_0 = 0$ , the barotropic mode is constant, independent of the stratification. With the normalization (4) we obtain  $\mathbf{p}_0 = 1$ .

### Constant buoyancy frequency

With  $N = \text{const.}$  the modes are simple sinusoids. That exact result is recovered as a special case of the WKB solution

$$\mathbf{p}_n^{po} = \sqrt{2} \cos [n\pi(1 + z/h)]. \quad (26)$$

## 2 Density modes

Similarly the baroclinic modes for density, here denoted by  $\mathbf{r}_n$ , are defined via the eigenproblem

$$\mathbf{r}_n'' = -\kappa_n^2 \left( \frac{N}{f_0} \right)^2 \mathbf{r}_n, \quad (27)$$

with homogeneous Dirichlet boundary conditions

$$@z = -h, 0 : \quad \mathbf{r}_n = 0, \quad (28)$$

and normalization

$$\frac{1}{h} \int_{-h}^0 \mathbf{r}_n \mathbf{r}_m dz = \delta_{mn}. \quad (29)$$

Alternatively, we can work on the approximation from the beginning. The WKB approximate solution to (27)-(28), consistent with the bottom boundary conditions (28), is

$$\mathbf{r}_n^{po} = \frac{B_n}{\sqrt{N(z)}} \sin \left( \frac{\kappa_n}{f_0} \int_{-h}^z N(\xi) d\xi \right), \quad (30)$$

The eigenvalues  $\kappa_n$  are the same as before (18). (This should be no surprise because it follows from the definition of  $\mathbf{p}_n$  and  $\mathbf{r}_n$ . Nonetheless, the verification is a good sanity check.) To find  $B_n$  we use the normalization (29)

$$B_n^2 \int_{-h}^0 \frac{1}{N(z)} \sin^2 \left( \frac{\kappa_n}{f_0} \int_{-h}^z N(\xi) d\xi \right) dz = h, \quad n \geq 1. \quad (31)$$

We use a similar trick as above i.e., we change variables with

$$\eta \stackrel{\text{def}}{=} \frac{\kappa_n}{N^2(z)f_0} \int_{-h}^z N(\xi) d\xi \quad \Rightarrow \quad d\eta = \frac{\kappa_n}{N(z)f_0} dz, \quad (32)$$

where, in the WKB spirit, we used the fact that  $N(z)$  is slowly varying when differentiating the relation above. We obtain

$$B_n = \left( 2N^2(0)/\bar{N} \right)^{1/2}. \quad (33)$$

Thus the WKB approximate solution to the density modes is

$$r_n^{po} = \left( \frac{2N^2(0)}{\bar{N}N(z)} \right)^{1/2} \sin \left( \frac{n\pi}{\bar{N}h} \int_{-h}^z N(\xi) d\xi \right), \quad n \geq 1. \quad (34)$$

Finally, note that the modes are simply related

$$\frac{dr_n^{po}}{dz} = N(0) \underbrace{\frac{n\pi}{\bar{N}h}}_{=\kappa_n/f_0} p_n^{po}. \quad (35)$$

### Constant buoyancy frequency

Again we recover the  $N = \text{const.}$  special case from (34):

$$r_n^{po} = \sqrt{2} \sin [n\pi(1 + z/h)]. \quad (36)$$