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Spectral Methods For PDEs

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Contents

Abstract

The main goal of is to indruce the basic techniques and results of numerical analysis for spectral methods. The required prerequisites to understand this text are basic knowledge of numerical methods, a introductory course on PDEs, on Real and Functional analysis, and on Sobolev spaces. For instance, chapters 1-2 from [**brenner2008mathematical**] covers most of the necessary background on functional analysis and Sobolev spaces. A more thorough treatment of those topics can be found in [**evans2022partial**] or [**brezis2011functional**]. As a refresher, we will include the most important results in the Appendix.
We will study the basics of spectral methods for elliptic boundary value problems and briefly discuss possible extension to more complicated geometries. Every method described will be implemented in MATLAB®.

In finite element methods, the approximation spaces are usually piecewise polynomials with a fixed degree (normally linear or quadratic), and convergence is achieved by refining the mesh, in other words by $h \longrightarrow 0$ and convergence is usually $\mathcal{O}(h^2)$. Moreover the linear systems obtained by the discretizations are usually very large but sparse. The finite element method also has the advantage that it can be used for very complicated geometries.

In spectral methods, convergence is achieved by having one (or more) fixed domain and increasing the degree of polynomial approximation (by taking $N \longrightarrow \infty$). In this case, convergence depends on the regularity of the solution. A typical result is that convergence is $\mathcal{O}(N^{-k})$ where the k depends on the smoothness of the solution. When the solution is C^∞, then convergence is spectrally fast : it is faster than $\mathcal{O}(N^{-k})$ for any k ! Spectral methods are much faster when the solution is known to be very regular. In many applications where the solution is known to be very regular and high precision is needed, spectral methods are very useful. Spectral methods lead to dense matrices but since convergence is fast, good precision can be obtained with small matrices. However, spectral methods are not as efficient when the solutions are not that regular, and the extension to complicated geometries is more complicated to that of finite elements.

Other methods like the Spectral Element or Spectral - hp Finite element methods combine both approaches

Introduction

Some conventions:

- d will always denote the dimension we are working on
- Ω will always denote an bounded Lipschitz domain.
- Every function will be assumed to be (Lebesgue) measurable
- We denote by $\mathcal{P}_N$ the space of polynomials in one variable of degree less or equal than N, and by $\mathcal{P}_N(I)$ the space of said polynomials restricted to an interval I.
- n, m, k, N, i, j always denote positive integers, and s, r nonegative real numbers. C or c always denote positive constants. When we write, for instance, $c = c(s)$, it means that the constant only depends on s. We allow ourselves to constantly abuse notation by not renaming c when scaling the constant.

Our goal: We will restrict our attention to second-order linear elliptic boundary value problems. The prototypical elliptic operator is the Laplacian $L = -\Delta$, where $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$. Our model equation is therefore Poisson's equation with homogeneous boundary conditions:

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{1}$$

Once we have understood the convergence and implementation of the numerical method, one can move on to more general elliptic operators and boundary conditions.

Multiplying the PDE 1 by any function v and integrating over Ω gives

$$-\int_{\Omega} v\Delta u = \int_{\Omega} fv$$

by taking $v \in H_0^1(\Omega)$, supposing momentarily $u \in H^2(\Omega)$ and integrating by parts the left-hand side we obtain

$$\int_{\Omega} \nabla u \nabla v = \int_{\Omega} fv$$

and by taking into account the Dirichlet conditions from 1 we have derived the variational formulation of 1:

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega) \end{cases} \tag{2}$$

where $a(u,v) = \int_{\Omega} \nabla u \nabla v$ and $(f,v) = \int_{\Omega} fv$

Observe that if $f \in L^2(\Omega)$, then the above integrals are well defined and finite. Moreover, note that for point-wise formulation 1 to hold a.e in Ω requires u to be two times (weakly) differentiable. However, the weak formulation 2 only assumes $u \in H^1(\Omega)$. In fact, many elliptic problems don't have solutions in H^2

However, the Lax-Milgram Lemma states that the problem in the variational formulation is unique, and its easy to see that if the solution is $u \in H^2(\Omega)$, then u satisfies $-\Delta u = f$ a.e. in Ω.

The general strategy to approximate the PDE numerically will be to consider a finite dimensional space $V_h \subset H_0^1(\Omega)$ and attempt to solve 2 in V_h, that is

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h,v_h) = (f,v_h) \quad \forall v_h \in V_h \end{cases} \tag{3}$$

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Chapter 1

Legendre Polynomials and applications

1.1 Basic facts about Legendre Polynomials

We denote $I = (-1, 1)$ our reference interval, chosen to obtain nice symmetric properties.

Suppose $\{p_n\}_n$ is a basis for the space of polynomials $\mathcal{P}$. Since $\mathcal{P}$ is dense in $L^2(a, b)$ we can write any function $v \in L^2(a, b)$, as $v = \sum_{n=0}^{\infty} \alpha_n p_n$. A numerical approximation to u may be obtained by truncating the series $v_N = \sum_{n=0}^{N} \alpha_n p_n$. If we take the basis $\{L_n\}_n$ to be orthogonal with respect to the $L^2(I)$ inner product, then the inner product like that ones we need to compute in 3 become easy:

$$v_N = \sum_{n=0}^{N} \alpha_n L_n,\ u_N = \sum_{n=0}^{N} \beta_n L_n \quad \text{then} \quad \int_{-1}^{1} v_N(t)\ u_N(t)\ dt = \sum_{n=0}^{N} \alpha_n \beta_n ||Ln||^2_{L^2(I)}$$

Applications of orthogonal polynomials such as numerical integration or polynomial interpolation that avoids the Runge's phenomenon will be presented.

We recall that a weight is a function $w : (-1, 1) \longrightarrow [0, \infty)$ such that

$$\int_{-1}^{1} p(t)\ w(t)\ dt < \infty$$

for any polynomial p.

For each weight w, $L^2(I, w)$ denotes the inner product with respect to the measure $dw = w(t)dt$. From the Gram-Schmidt process, we can construct a family of orthogonal polynomials with respect to that inner product, and by imposing that leading coefficient is 1, the family is unique. We denote it by $\{q_n^w\}_{n\geq 0}$, where $deg(q_n^w) = n$. The next statements are classic results and hold for a family orthogonal polynomials $\{q_n^w\}_{n=0,1,...}$ with respect to any weight.

- For any n, q_n^w has n distinct zeros in $(-1, 1)$
- If n is even (resp. odd) then q_n^w is an even (resp. odd) function (This is why we take $I = (-1, 1)$)

We now define the Legendre polynomials and describe their properties.

Definition 1.1 (Legendre Polynomials)**.** *The Legendre polynomials $\{L_n\}_n$ is the (unique) family of orthogonal polynomials with respect to the weight $w \equiv 1$ and such that*

$$deg(L_n) = n, L_n(1) = 1, \quad \textit{for all } n = 0,1,\dots$$

We denote the leading coefficient of L_n by c_n

The next formula is fundamental in the theory of Legendre polynomials.

Theorem 1.2 (Fundamental ODE)**.** *For all $n \geq 0$, L_n satisfies*

$$((1-t^2)L_n')' = -n(n+1)L_n \tag{1.1}$$

Proof. First, note that $(1-t^2)L_n')'$ has degree n. If we integrate $((1-t^2)L_n')'$ against any polynomial $p \in \mathcal{P}_{n-1}$, we see that (integrating by parts)

$$\int_{-1}^{1}((1-t^2)L_n')'p(t)\,dt = -\int_{-1}^{1}(1-t^2)L_n'(t)p'(t)\,dt = \int_{-1}^{1}L_n(t)((1-t^2)p')'\,dt = 0$$

Because $((1-t^2)p')'$ has degree $\leq n-1$. So $((1-t^2)L_n')'$ is orthogonal to $\mathcal{P}_{n-1}$ and therefore we must have

$$((1-t^2)L_n')' = \lambda L_n$$

for some $\lambda \in \mathbb{R}$. The leading coefficient of $(1-t^2)L_n')'$ is $-n(n+1)c_n$, so we obtain that $\lambda = -n(n+1)$ $\square$

Some immediate consequences are

Corollary 1.3. *The polynomials $\{L_n'\}_n$ form a family of orthogonal polynomials with respect to the weight $w(t) = 1-t^2$*

$$\int_{-1}^{1}L_n'(t)L_m'(t)(1-t^2)\,dt = n(n+1)\int_{-1}^{1}L_m(t)L_n(t)\,dt \tag{1.2}$$

and $L_n'(1) = \frac{n(n+1)}{2}$

Theorem 1.4 (Rodrigues' Formula)**.**

$$L_n = \frac{(-1)^n}{2^n n!}\cdot\frac{d^n}{dt^n}((1-t^2)^n) \tag{1.3}$$

Remark 1.5. Before proving the formula, we expand $\frac{d^m}{dt^m}((1-t^2)^n)$ for $m \leq n$ using Leibniz's formula.

$$\begin{aligned}\frac{d^m}{dt^m}((1-t^2)^n) &= \frac{d^m}{dt^m}((1-t)^n(1+t)^n) = \sum_{k=0}^{m}\binom{m}{k}\frac{d^k}{dt^k}(1-t)^n\frac{d^{m-k}}{dt^{m-k}}(1+t)^n \\ &= \sum_{k=0}^{m}(-1)^k\frac{(n!)^3}{(k!)^2((n-k)!)^2}(1-t)^{n-k}(1+t)^k\end{aligned} \tag{1.4}$$

So we see that $\frac{d^m}{dt^m}((1-t^2)^n)$ vanishes at ± 1 when $m < n$ and its equal to $(-1)^n 2^n n!$ when $t = 1$ and $m = n$

Proof.
Observe that $\frac{d^n}{dt^n}((1-t^2)^n)$ is a polynomial of degree n. Integrating it against any polynomial $p \in \mathcal{P}_{n-1}$ gives,

$$\int_{-1}^{1} \frac{d^n}{dt^n}((1-t^2)^n)\ p(t)\ dt = (-1)^n \int_{-1}^{1} (1-t^2)^n\ \frac{d^n}{dt^n} p(t)\ dt = 0$$

where we have integrated by parts n times and used that $\frac{d^m}{dt^m}((1-t^2)^n)$ vanishes at ± 1 when $m < n$. Therefore we must have that $\frac{d^n}{dt^n}((1-t^2)^n) = \lambda L_n$ for some $\lambda \in \mathbb{R}$. Computing both sides at $t = 1$ gives the result. $\square$

Corollary 1.6. *From Rodrigues' formula we directly obtain*

$$c_n = \frac{(2n)!}{2^n (n!)^2}$$

Corollary 1.7. *Let $0 \le m \le N$.*
Then $\frac{d^m}{dt^m} L_N$ is orthogonal to $\mathcal{P}_{N-m}$ with respect to the weight $(1-t^2)^m$. So $\{\frac{d^m}{dt^m} L_j\}_{j=m,\dots,N}$ is family of orthogonal polynomials with respect to the weight $(1-t^2)^m$

Proof.
For any $p \in \mathcal{P}_{N-m}$

$$\int_{-1}^{1} \frac{d^{n+m}}{dt^{n+m}}((1-t^2)^n) \cdot p(t)(1-t^2)^m\ dt = (-1)^m \int_{-1}^{1} \frac{d^n}{dt^n}((1-t^2)^n) \cdot \frac{d^m}{dt^m}(p(t)(1-t^2)^m)\ dt = 0$$

where we have used 1.5 and that $\frac{d^m}{dt^m}(p(t)(1-t^2)^m)$ has degree $\le N$ $\square$

We now list some properties about Legendre Orthogonal polynomials which can be proven by elementary techniques and the previous results:

Theorem 1.8 (Properties about Legendre Orthogonal polynomials).

1. *For $n \ge 0$, the L^2 norm is given by*
$$\int_{-1}^{1} L_n(t)^2\ dt = \frac{1}{n+1/2} \tag{1.5}$$
2. *For any $n > 0$*
$$(2n+1)L_n = L'_{n+1} - L'_{n-1} \tag{1.6}$$

3. *Legendre Iduction Formula*

   The family of Legendre polynomials can be computed by

$$\begin{aligned} &L_0(t) = 1, L_1(t) = t \\ &(n+1)L_{n+1}(t) = (2n+1)tL_n(t) - nL_{n-1}(t) \end{aligned} \tag{1.7}$$

4.
$$nL'_{n+1}(t) = (2n+1)tL'_n(t) - (n+1)L'_{n-1}(t) \tag{1.8}$$

5. *Christoffel - Darboux formulas*

   For any $x, y \in I$

$$\sum_{j=0}^{n-1} (2n+1)L_j(x)L_j(y) = n\,\frac{L_n(x)L_{n-1}(y) - L_n(y)L_{n-1}(x)}{x-y} \tag{1.9}$$

   and

$$\sum_{j=1}^{n-1} \frac{2j+1}{j(j+1)} L'_j(x)L'_j(y) = \frac{1}{n}\frac{L'_n(x)L'_{n-1}(y) - L'_n(y)L'_{n-1}(x)}{x-y} \tag{1.10}$$

1.2 Quadrature

The inner products at 3 must be computed to obtain an approximate solution. Since computing integrals analytically is costly and most times not possible, we must integrate numerically.

The goal of this section is to find nodes t_j and weights w_j so that the approximation

$$\int_{-1}^{1} f(t)\;dt = \sum_{j=1}^{N} f(t_j)w_j$$

is as precise as possible. An approximation as above is called a quadrature.

Theorem 1.9. *Let $0 \le m \le N$. There exist*

- *A unique set of nodes ζ_j^m, $1 \le j \le N-m$, in I*
- *A unique set of weights, w_j^m, $1 \le j \le N-m$,*

such that the quadrature formula with is exact for $\mathcal{P}_{2N-2m-1}$ and the weight $(1-t^2)^m$. In other words, if $p \in \mathcal{P}_{2N-2m-1}$ then:

$$\int_{-1}^{1} p(t)(1-t^2)^m\;dt = \sum_{j=1}^{N-m} p(\zeta_j^m)\;w_j^m \tag{1.11}$$

Furthermore, the nodes ζ_j^m, $1 \le j \le N-m$ the $N-m$ zeros of $\frac{d^m}{dt^m}L_N$

Proof.
Denote by ψ_j^m the Lagrange polynomials associated with the nodes ζ_j^m, for $j = 1, \ldots, N-m$. They are of degree $N-m-1$. Plugging into 1.11 gives

$$w_j^m = \int_{-1}^{1} \psi_j^m(t)\ (1-t^2)^m\ dt$$

and so the quadrature formula is exact for $p = \psi_j^m$ which in turn implies that it is exact for $\mathcal{P}_{N-m-1}$. Now, for any $p \in \mathcal{P}_{2N-2m-1}$, we can write it as

$$p(t) = q(t)(t-\xi_1^m)\cdot \ldots \cdot (t-\xi_{N-m}^m) + r(s)$$

with $q, r \in \mathcal{P}_{N-m-1}$. So we have that

$$\sum_{j=1}^{N-m} p(\zeta_j^m)\ w_j^m = \sum_{j=1}^{N-m} r(\zeta_j^m)\ w_j^m = \int_{-1}^{1} r(t)\ (1-t^2)^m\ dt$$

since the quadrature formula is exact for $\mathcal{P}_{N-m-1}$. To be exact for p we need to look for nodes such that

$$\forall q \in \mathcal{P}_{N-m-1} \quad \int_{-1}^{1} q(t)(t-\xi_1^m)\cdot \ldots \cdot (t-\xi_{N-m}^m)\ (1-t^2)^m\ dt = 0$$

This means that $(t-\xi_1^m)\cdot \ldots \cdot (t-\xi_{N-m}^m)$ is orthogonal $\mathcal{P}_{N-m-1}$ with respect to the measure $(1-t^2)^m$. So from 1.7 (or definition 1.1) we have that this nodes $\{\xi_j^m\}_{j=1,\ldots,N-m}$ are the zeros of $\frac{d^m}{dt^m}L_N$ $\square$

Since we are going to use numerical integration to approximate boundary value problems, it makes sense to look for a quadrature formula that involves the endpoints of the interval, namely ± 1.

Theorem 1.10. *Let $0 \le m \le N$. There exist*

- *A unique set of $N-m$ nodes ζ_j^m, $1 \le j \le N-m$*
- *A unique set of $N-m$ real numbers ρ_j^m, $1 \le j \le N-m$*
- *A unique set of $2m$ real numbers $\rho_-^{m,k}$ and $\rho_+^{m,k}$, $0 \le k \le m-1$*

such that for all $p \in \mathcal{P}_{2N-1}$

$$\begin{aligned}\int_{-1}^{1} p(t)\ dt = &\sum_{j=1}^{N-m} p(\zeta_j^m)\rho_j^m \\ &+ \sum_{k=0}^{m-1} \left(\left(\frac{d^k p}{dt^k}\right)(-1)\cdot\rho_-^{m,k} + \left(\frac{d^k p}{dt^k}\right)(1)\cdot\rho_+^{m,k} \right)\end{aligned} \tag{1.12}$$

Moreover, they ζ_j^m , $1 \le j \le N-m$ are given by the zeros of $\frac{d^m}{dt^m}L_N$

Proof.
Every polynomial $p \in \mathcal{P}_{2N-1}$ can be written as

$$p(t) = q(t)(1-t^2)^m + r(s)$$

where $q \in \mathcal{P}_{2N-2m-1}$ and $r \in \mathcal{P}_{2m-1}$.
By applying 1.12 to the first summand, and making use of 1.5, then Theorem 1.9 gives us that necessarily

$$\rho_j^m = (1-(\zeta_j^m)^2)^{-m} w_j^m \quad \forall 1 \leq j \leq N-m \tag{1.13}$$

So we deduce that the quadrature is exact when applied to the first term $q(t)(1-t^2)^m$. Therefore we only have to see that the formula is exact for $\mathcal{P}_{2m-1}$.

Plugging $\{1, x, x^2, \ldots, x^{2m-1}\}$ into 1.12 gives a square $2m \times 2m$ linear system for the $\{\rho_-^{m,k}, \rho_+^{m,k}\}$ with coefficients given by $\left(\frac{d^k}{dt^k} x^j\right)$ (± 1).
If this system was singular, there would exist a linear combination $a_0 + a_1 x + \ldots + a_{2m-1} x^{2m-1}$ for which the derivatives of order 0 to $m-1$ would vanish at ± 1. From 7.3 we now that the only polynomial of degree $2m-1$ to satisfy that is $p = 0$, so we conclude that $a_0 = a_1 = \ldots = a_{2m-1} = 0$ and therefore the linear system is non-singular, which means that the $\{\rho_-^{m,k}, \rho_+^{m,k}\}$ can be determined uniquely. □

Remark 1.11. It's easy to see that the weights w_j^m and therefore the ρ_j^m are stricly positive.

Indeed if we apply 1.11 to the squares $(\psi_j^m)^2$ of the Lagrange polynomials associated with the nodes ζ_j^m, $j = 1, \ldots, N-m$, and use the exactness for $\mathcal{P}_{2N-2m-1}$, then the property easily follows.

1.3 Weights and Nodes

From now on, we are only interested on the cases $m = 0$ of Theorem 1.9 and $m = 1$ of 1.10.

Definition 1.12.
We denote by ζ_j, $j = 1, \ldots, N$ the zeros of L_N and by w_j their associated weights of 1.9. The $\{\zeta_j\}_j$ are called the N-th Gauss-Legendre nodes.
We denote by η_j, $j = 0, \ldots, N$ the zeros of $(1-t^2)L_N'$. We also denote $\rho_j = \rho_j^m$ for $j = 1, \ldots, N-1$ and $\rho_0 = \rho_-^{1,0}$, $\rho_1 = \rho_+^{1,0}$. The $\{\eta_j\}_j$ are called the N-th Gauss-Legendre-Lobatto nodes.

So now the quadrature formulas read as

$$\int_{-1}^{1} \phi(t)\, dt \approx \sum_{j=1}^{N} \phi(\zeta_j) w_j \quad \text{and} \quad \int_{-1}^{1} \phi(t)\, dt \approx \sum_{j=0}^{N} \phi(\eta_j) \rho_j$$

For practical numerical computation we must compute the previous weights and nodes. And for polynomial interpolation we must understand how are these nodes distributed along I.

First we study the case $m = 0$ of the Gauss-Legendre nodes ζ_j From the formula 1.9, and recalling that the ζ_j are the zeros of L_N,

$$L_0(x)L_0(\zeta_j) + \ldots + (2N-1)L_{N-1}(x)L_{N-1}(\zeta_j) = N\frac{L_N(x)L_{N-1}(\zeta_j)}{x-\zeta_j} \tag{1.14}$$

Integrating both sides, and applying the orthogonality relations as well as the exactness of the quadrature

$$2 = NL_{N-1}(\zeta_j)\int_{-1}^{1}\frac{L_N(x)}{x-\zeta_j}\,dx = NL_{N-1}(\zeta_j) = NL_{N-1}(\zeta_j)L_N'(\zeta_j)w_j$$

so we have an explict formula for the weights in term of the nodes

$$w_j = \frac{2}{NL_{N-1}(\zeta_j)L_n'(\zeta_j)} \tag{1.15}$$

By defining $L_n^* = \sqrt{n+1/2}\,L_n$, now the $\{L_n^*\}_n$ form a family of orthonormal polynomials and we can adapt the induction formula to

$$\begin{aligned} tL_n(t) &= \frac{n+1}{\sqrt{(2n+1)(2n+3)}}L_{n+1}^*(t) + \frac{n}{\sqrt{(2n+1)(2n-1)}}L_{n-1}^*(t) \\ &= \alpha_{n+1}L_{n+1}^* + \alpha_n L_{n-1}^* \quad \text{where} \quad \alpha_n = \frac{n}{\sqrt{4n^2-1}} \end{aligned} \tag{1.16}$$

(The reason why we introduced $\{L_n^*\}_n$ will soon follow) We can write the previous equation in matrix form :

$$t\begin{bmatrix} L_0^* \\ L_1^* \\ \vdots \\ \vdots \\ L_{N-2}^* \\ L_{N-1}^* \end{bmatrix} = \begin{bmatrix} 0 & \alpha_1 & 0 & & & & \\ \alpha_1 & 0 & \alpha_2 & & & & \\ 0 & \alpha_2 & 0 & \alpha_3 & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ & & & & \alpha_{N-2} & 0 & \alpha_{N-1} \\ 0 & & & & 0 & \alpha_{N-1} & 0 \end{bmatrix}\begin{bmatrix} L_0^* \\ L_1^* \\ \vdots \\ \vdots \\ L_{N-2}^* \\ L_{N-1}^* \end{bmatrix} + \alpha_N\begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ L_N^* \end{bmatrix}$$

So the ζ_j are the the eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & \alpha_1 & & & \\ \alpha_1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & \alpha_{N-1} \\ & & & \alpha_{N-1} & 0 \end{pmatrix} \tag{1.17}$$

This matrix is tridiagonal with a zero diagonal and symmetric, so its eigenvalues can be computed numerically very fast and precisely with an iterative algorithm. If $x \longrightarrow \zeta_j$ in 1.14,

$$L_0(\zeta_j)^2 + 3L_1(\zeta_j)^2 + \ldots + (2N-1)L_{N-1}(\zeta_j)^2 = NL_{N-1}(\zeta_j)L'_N(\zeta_j)$$

From 1.15 and the L^2 norm of L_n we find that

$$w_j = (L_0^*(\zeta_j)^2 + \ldots + L_{N-1}^*(\zeta_j)^2)^{-1} \tag{1.18}$$

This means we can compute the weight w_j from the components of the eigenvector with eigenvalue ζ_j (with first coordinate $L_0^* = 1/\sqrt{2}$)
To study how the weights behave as $N \longrightarrow \infty$, we derive another expression for w_j.
Using 1.7 we get, $(N+1)L_{N+1}(\zeta_j) = -NL_{N-1}(\zeta_j)$. Integrating both sides of the fundamental ODE 1.1 and of 1.6 we obtain

$$(1-t^2)L'_N(t) = -N(N+1)\int_{-1}^{t} L_N(x)\,dx = -\frac{N(N+1)}{2N+1}(L_{N+1}(t) - L_{N-1}(t)$$

so $(1-\zeta_j^2)L'_N(\zeta_j) = NL_{N-1}(\zeta_j)$ and finally,

$$w_j = \frac{2}{(1-\zeta_j^2)L'_N(\zeta_j)^2} \tag{1.19}$$

We suppose that the ζ_j are in increasing order and we define

$$\theta_j = arccos(\zeta_j)$$

Theorem 1.13 (Location of nodes).
The nodes θ_j are located in:
When $N = 2m$

$$\begin{aligned}
&\frac{(2j-1)\pi}{2N} < \theta_{N-j+1} < \frac{(2j+1)\pi}{2N} \qquad when \quad 1 \le j \le m-1 \\
&\frac{(2j-3)\pi}{2N} < \theta_{N-j+1} < \frac{(2j-1)\pi}{2N} \qquad when \quad m+2 \le j \le N \\
&\frac{(N-1)\pi}{2N} < \theta_{m+1} < \pi/2 < \theta_m < \frac{(N+1)\pi}{2N}
\end{aligned} \tag{1.20}$$

When $N = 2m+1$

$$\begin{aligned}
&\frac{(2j-1)\pi}{2N} < \theta_{N-j+1} < \frac{(2j+1)\pi}{2N} \qquad when \quad 1 \le j \le m-1 \\
&\frac{(2j-3)\pi}{2N} < \theta_{N-j+1} < \frac{(2j-1)\pi}{2N} \qquad when \quad m+3 \le j \le N \\
&\frac{(N-1)\pi}{2N} < \theta_{m+2} < \theta_{m+1} = \pi/2 < \theta_{m-1} < \frac{(N+1)\pi}{2N}
\end{aligned} \tag{1.21}$$

Proof.
Consider the functions

$$\varphi(t) = \left((1-t^2)\right)^{1/2} L_N(t) \quad \text{and} \quad \psi(t) = \left(1-t^2\right)^{1/4} arccos(t)$$

From the Fundamental ODE 1.1, we can check that

$$\varphi'' + \frac{N(N+1)(1-t^2)+1}{(1-t^2)^2}\varphi = 0 \quad \text{and} \quad \psi'' + \frac{N^2(1-t^2)+1/2+1/4t^2}{(1-t^2)^2}\psi = 0$$

So

$$(\varphi\psi' - \varphi'\psi)' = \mu\varphi\psi \tag{1.22}$$

where

$$\mu(t) = \frac{N(1-t^2)+1/2-t^2/4}{(1-t^2)^2} > 0 \quad \text{for} \quad t \in I$$

$cos(Narccos(t))$ is the N-th Chebysev polynomial in it's N zeros are $cos\left(\frac{(2j+1)\pi}{2N}\right)$ for $j = 0, \ldots, N-1$

Let a and b be two consecutive zeros of ψ. Integrating both sides of 1.22 yields

$$\varphi(b)\psi'(b) - \varphi(a)\psi'(b) = \int_a^b \mu(t)\varphi(t)\psi(t)\,dt$$

Now we are going to prove by contradiction that φ necessarily has a zero in (a, b). First, remember that the zeros of ψ are simple.
Suppose φ does not vanish in (a, b); denote by s the sign of φ on $[a, b]$. Then $s \geq 0$ or $s \leq 0$. Since the zeros of ψ are simple, we suppose that $\psi \geq 0$ on $[a, b]$ (the case $\psi \leq 0$ is analogous) . So it follows that $\psi'(b) < 0$, $\psi'(a) > 0$

Then

$$s(-) - s(+) = \int_a^b (+)s(+) \implies (-)s = \int_a^b (+)s$$

where $(+)$ represents a number greater or equal than and $(-)$ a number smaller or equal than zero. Thus we have a contradiction and φ must have a zero in (a, b). So we have located $N-1$ of the zeros of φ, more precisely, we have proven that for all $1 \leq j \leq N-1$ there exists some ζ_i such that

$$cos\left(\frac{(2j+1)\pi}{2N}\right) < \zeta_i < cos\left(\frac{(2j-1)\pi}{2N}\right)$$

To find the missing one, we recall that :

When N is even, $\zeta_{N/2+1}$ and $\zeta_{N/2} = -\zeta_{N/2+1}$ are both in between $\left[\cos\frac{(N+1)\pi}{2N}, \cos\frac{(N-1)\pi}{2N}\right]$

When N is odd, then $\zeta_{(N+1)/2} = 0$. So this zero is the right endpoint of $\left[cos(\frac{(N+1)\pi}{2N}), cos(\pi/2)\right]$ and the left endpoint of $\left[cos\pi/2, cos\frac{(N-1)\pi}{2N}\right]$ □

Observe that the cosine of the nodes are equally distributed along $[0,\pi]$. This means that as $N \longrightarrow \infty$, the nodes cluster along the ends of the interval. This is one of the characterizations of good interpolation nodes, as explained in **[trefethen2000spectral]**.

Theorem 1.14 (Weight estimates)**.**
The weights w_j satisfy the following inequalities for some c independent of N:

$$w_j \leq cN^{-1}(1-\zeta_j^2)^{1/2} \tag{1.23}$$

Proof.
Since the L_N and L_N' are even or odd, and from formula 1.19, we automatically have that

$$w_j = w_{N+1-j} \quad \forall j = 1,\dots,N$$

So it's enough to prove the result for the nonnegative ζ_j. For this purpose, consider the function

$$f(\theta) = (sin\theta)^{1/2} L_N(cos(\theta))$$

From the fundamental ODE 1.1 we can see that

$$f''(\theta) + \mu(\theta)f(\theta) = 0 \quad \text{with} \quad \mu(\theta) = N(N+1) + 1/4 + (sin(\theta))^{-2}/4 \tag{1.24}$$

Observe that from 1.19,

$$(f')^2(\theta_j) = (sin(\theta_j))^3 \; L_N'(cos(\theta_j)) = \frac{2\;(1-\zeta_j^2)^{1/2}}{w_j} \tag{1.25}$$

Take two zeros ζ_i and ζ_j, where $0 \leq \zeta_i < \zeta_j$, owing to 1.24 we get that

$$\begin{aligned}
&(f'(\theta_i))^2 - (f'(\theta_j))^2 \\
&= \int_{\theta_j}^{\theta_i} ((f')^2)'(\theta)\; d\theta = 2\int_{\theta_j}^{\theta_i} f'(\theta)f''(\theta)\; d\theta = -2\int_{\theta_j}^{\theta_i} \mu(\theta)f(\theta)f'(\theta)\; d\theta \\
&= -\int_{\theta_j}^{\theta_i} \mu(\theta)(f^2)'(\theta)\; d\theta = \int_{\theta_j}^{\theta_i} \mu'(\theta)f^2(\theta)\; d\theta
\end{aligned} \tag{1.26}$$

Since $\mu'(\theta) \leq 0$ when $\theta \in (0,\pi)$, we have that $(f')^2(\theta_j) > (f')^2(\theta_i)$, which in turn implies from 1.25 that , for $\zeta_j \geq 0$ the numbers $w_j(1-\zeta_j^2)^{-1/2}$ decrease as j increases.

So if $m = [N/2]+1$ and we check that

$$w_m(1-\zeta^2)^{-1/2}$$

is bounded by cN^{-1} for some $c > 0$ independent of N, then we will have proved the estimate 1.23 for all j. We distinguish cases:

When $N = 2k+1$:

Then $\zeta_m = \zeta_{k+1} = 0$ and from 1.19 we have that $w_m = 2/L_n'(0)^2$. So from the induction formula 1.8 we deduce that

$$L_N'(0) = -\frac{N}{N-1}L_{N-2}'(0) = \ldots = (-1)^{(N-1)/2}\frac{N(N-2)\cdot\ldots\cdot 3}{(N-1)(N-3)\cdot\ldots\cdot 2}L_1'(0) =$$
$$(-1)^{(N-1)/2}\frac{N!}{(N-1)^2(N-3)^2\cdot\ldots\cdot 2^2} = (-1)^{(N-1)/2}\frac{N!}{2^{N-1}(\frac{N-1}{2})^2(\frac{N-3}{2})^2\cdot\ldots\cdot 1}$$
$$= (-1)^{(N-1)/2}\frac{N!}{2^{N-1}((\frac{N-1}{2})!)^2}$$

We now use Stirling's formula $k \sim \sqrt{2\pi}\, e^{-k}\, k^{(k+1/2)}$ So

$$\frac{N!}{2^{N-1}((\frac{N-1}{2})!)^2} \sim \frac{e^{-N}\, N^{N+1/2}}{2^{N-1}\, (\frac{N-1}{2})^N\, e^{-N+1}} \sim \frac{N^{1/2}\, N^N}{2^{N-1}\, (\frac{N-1}{2})^N} \sim N^{1/2}$$

So we deduce that $|L_N'(0)| \geq cN^{1/2}$, which proves the bound.

When $N = 2k$:

Then integrating as before, this time between θ_m and $\pi/2$

$$(f')^2(\pi/2) - (f')^2(\theta_m) = -\int_{\theta_m}^{\pi/2} \mu(\theta)\,(f^2)'(\theta)\,d\theta$$
$$= -\mu(\pi/2)f^2(\pi/2) + \int_{\theta_m}^{\pi/2} \mu'(\theta)f^2(\theta)\,d\theta$$

Since $\mu'(\theta) \leq 0$ when $\theta \in (0,\pi)$, we conclude that

$$\frac{2(1-\zeta_m^2)^{1/2}}{w_m} = (f')^2(\theta_m) \geq \mu(\pi/2)f^2(\pi/2) = (N^2+N+1/2)L_N^2(0)$$

Now apply the induction formula 1.7 as well as Stirling's formula to deduce that $L_N(0)^2 \geq cN^{-1}$ for some c $\square$

It can also be proven that the estimate above is optimal, in the sense that there exists a c' such that $w_j \geq c'N^{-1}(1-\zeta_j^2)^{1/2}$

We now consider the case $m = 1$, which involves the endpoints of the interval, $\eta_0 = -1$ and $\eta = 1$, and the zeros of L_N'

$$\int_{-1}^{1}\phi(t)\,dt \approx \sum_{j=0}^{n}\phi(\eta_j)\rho_j$$

To numerically compute the nodes and weights we are going to proceed analogously as before. Firstly, we are going to define

$$Q_n^* = \sqrt{\frac{n+1/2}{n(n+1)}}L_n'$$

which are orthonormal polynomials with respect to to the weight $(1-t^2)dt = dw$. Using the induction formula 1.8, we have that

$$\begin{aligned} 2t\, Q_n^* &= \sqrt{\frac{n(n+2)}{(n+1/2)(n+3/2)}}\, Q_{n+1}^* + \sqrt{\frac{(n-1)(n+1)}{(n+1/2)(n-1/2)}}\, Q_{n-1}^* \\ &= \gamma_n\, Q_{n+1}^* + \gamma_{n-1}\, Q_{n-1}^* \qquad \gamma_n = \sqrt{\frac{n(n+2)}{(n+1/2)(n+3/2)}} \end{aligned} \tag{1.27}$$

Writing the above expression in matrix form:

$$2t \begin{bmatrix} Q_1^* \\ Q_2^* \\ \vdots \\ \vdots \\ Q_{N-2}^* \\ Q_{N-1}^* \end{bmatrix} = \begin{bmatrix} 0 & \gamma_1 & 0 & & & & \\ \gamma_1 & 0 & \gamma_2 & & & & \\ 0 & \gamma_2 & 0 & \gamma_3 & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ & & & & \gamma_{N-3} & 0 & \gamma_{N-2} \\ 0 & & & & 0 & \gamma_{N-2} & 0 \end{bmatrix} \begin{bmatrix} Q_0^* \\ Q_1^* \\ \vdots \\ \vdots \\ Q_{N-2}^* \\ Q_{N-1}^* \end{bmatrix} + \gamma_{N-1} \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ Q_N^* \end{bmatrix}$$

So the η_j are the eigenvalues of the matrix

$$B = \begin{pmatrix} 0 & \gamma_1 & & & \\ \gamma_1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & \gamma_{N-2} \\ & & & \gamma_{N-2} & 0 \end{pmatrix} \tag{1.28}$$

We recall from 1.13 that the weights ρ_j can be computed in terms of the w_j^1. But we want an expression to compute them in terms of the eigenvectors of B.

Lemma 1.15. *For* $1 \leq \rho_j \leq N-1$

$$w_j^1 = \frac{2N}{L_N''(\eta_j)L_{N-1}'(\eta_j)} \tag{1.29}$$

Proof. Use formula 1.10 with $y = \eta_j$ to obtain

$$\sum_{j=1}^{N-1} \frac{2j+1}{j(j+1)} L_j'(x) L_j'(\eta_j) = \frac{1}{n} \frac{L_N'(x) L_{N-1}'(\eta_j)}{x-\eta_j} \tag{1.30}$$

Integrate both sides against $(1-t^2)$, and use the exactness property of the quadrature

$$2 = \frac{1}{N} L'_{N-1}(\eta_j) L''_N(\eta_j) w_j^1$$

Now from 1.13,

$$\rho_j = w_j^1 (1-\eta_j)^{-1} = \frac{2N}{(1-\eta_j^2) L''_N(\eta_j) L'_N(\eta_j)}$$

$\square$

Now, letting $x \longrightarrow \eta_j$ in 1.29 gives

$$\sum_{k=1}^{N-1} \frac{2k+1}{k(k+1)} L'_k(t)^2 = \frac{1}{N} L'_{N-1}(\eta_j) L''_N(\eta_j) \tag{1.31}$$

and thus

$$\rho_j = (1-\eta_j^2)^{-1} \left(\sum_{k=1}^{N-1} Q_k^*(\eta_j)^2 \right)^{-1} \tag{1.32}$$

So we have a way of computing the weights in terms of the eigenvalues of the matrix B with first component

$$Q_1^* = \sqrt{3}/2$$

By substituting in the formula 1.12, we have that

$$\int_{-1}^{1} L'_N(t)(1-t)\, dt = 2\, \rho_0\, L'_N(-1) \qquad \int_{-1}^{1} (1+t) L'_N(t)\, dt = 2\, \rho_N\, L'_N(1)$$

Integrating by parts, using 1.3 and the parity of orthogonal polynomials

$$\rho_0 = \frac{-2\, L_N(-1)}{2\, \rho_0\, L'_N(-1)} = \frac{2}{N(N+1)} \quad \text{and} \quad \rho_N = \frac{2L_N(1)}{2L'_N(1)} = \frac{2}{N(N+1)}$$

With a little bit more of work we can show that

$$\rho_j = \frac{2}{N(N+1) L_N(\eta_j)^2} \quad 0 \le j \le N \tag{1.33}$$

Since the zeros of L_N are simple, and by Rolle's theorem we easily deduce that

$$\zeta_j < \eta_j < \zeta_{j+1} \quad 0 \le j \le N \tag{1.34}$$

So the location of the Legendre-Gauss-Lobatto nodes is contained in theorem 1.13.

Moreover, using 1.33 we can prove similarly to theorem 1.23 that

Theorem 1.16 (Legendre-Gauss-Lobatto Weight Estimates).

$$\rho_j \leq cN^{-1}\left(1-\eta_j^2\right)^{1/2} \tag{1.35}$$

Remark 1.17. Most of the statements in this chapter can be further generalized (after making the suitable adaptations) to more general classes of orthogonal polynomials, like the Jacobi or Chebysev polynomials. And for example, owing to the Peano Kernel Theorem, the quadrature error can be precisely estimated for sufficiently regular functions (see [**dahlquist2008numerical**]). We have only presented the results for the Legendre polynomials to now focus to our PDE applications and we refer to [**shen2011spectral**] or [**bernardi1997spectral**] for such generalizations.

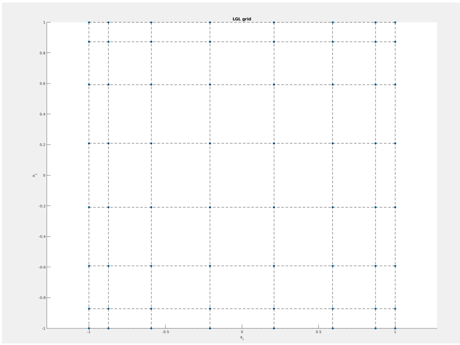

Figure 1.1: Tensor product grid of the LGL points. Observe the "clustering" close to the border

Chapter 2

Orthogonal projections and other operators

Motivated by the results from Proposition 7.9, we first study some orthogonal projections and other operators, which will later be used to estimate the error of a discretization of a PDE. Our finite dimensional "approximation space" will be spaces of polynomials. So we have to find approximation results for such spaces.

2.1 Polynomial Approximations in One Dimension

When trying to understand polynomial approximation on an interval, its natural is to try to understand the orthogonal projection from L^2 to $\mathcal{P}_N$, since it gives the best L^2 polynomial approximation. Denote by π_N the projection from $L^2(I)$ onto $\mathcal{P}_N(I)$

Theorem 2.1. *For $s \geq 0$, there exists $c = c(s) > 0$ such that, for $\varphi \in H^s(I)$,*

$$||\varphi - \pi_N \varphi||_{L^2(I)} \leq cN^{-s}||\varphi||_{H^s(I)} \tag{2.1}$$

We first need a lemma to understand the operator $\mathcal{L}$ from the Fundamental ODE 1.1, which is defined on $H^2(I)$

$$\mathcal{L}v = -\frac{d}{dt}((1-t^2)v')$$

Lemma 2.2. *The operator $\mathcal{L}$, defined in $H^2(I)$ is*

- *Positive with respect to the L^2 inner product* $\big((\mathcal{L}v, v) > 0 \ \forall v \in H^2(\Omega_d)/\{0\}\big)$
- *Self adjoint with respect to the L^2 inner product* $\big((\mathcal{L}u, v) = (u, \mathcal{L}v) \ \forall u, v \in H^2(\Omega_d)\big)$
- *Continuous from* $H^{n+2}(\Omega_d)$ *to* $H^n(\Omega_d)$

Proof. The first two properties are easily derived integrating by parts

By induction we can check that

$$\frac{d^k}{dt^k}(\mathcal{L}v) = -\left(1-t^2\right)\frac{d^{k+2}v}{dt^{k+2}} + 2(k+1)t\,\frac{d^{k+1}v}{dt^{k+1}} + k(k+1)\frac{d^k v}{dt^k}$$

So for any k, $0 \le k \le n$

$$||\frac{d^k(\mathcal{L}v)}{dt^k}||_{L^2(I)} \le C\left(||\frac{d^{k+2}v}{dt^{k+2}}||_{L^2(I)} + ||\frac{d^{k+1}v}{dt^{k+1}}||_{L^2(I)} + ||\frac{d^k v}{dt^k}||_{L^2(I)}\right)$$

where $C = C(n)$ □

Proof. (Of the theorem)
First we prove when $s = 2m$. If $\varphi \in L^2(I)$ then

$$\varphi = \sum_{n=0}^{\infty} \alpha_n L_n \quad \text{and} \quad \pi_N \varphi = \sum_{n=0}^{N} \alpha_n L_n$$

so

$$||\varphi - \pi_N \varphi||^2_{L^2(I)} = \sum_{n=N+1}^{\infty} \alpha_n^2 ||L_n||^2_{L^2(I)}$$

We can compute the coefficients using Theorem 1.1 and that the operator is self-adjoint:

$$\alpha_n = \frac{1}{||L_n||^2_{L^2(I)}}\int_{-1}^{1}\varphi(t)L_n(t)\,dt = \frac{1}{n(n+1)||L_n||^2_{L^2(I)}}\int_{-1}^{1}\varphi(t)(\mathcal{L}L_n)(t)\,dt = \frac{1}{n(n+1)||L_n||^2_{L^2(I)}}\int_{-1}^{1}(\mathcal{L}\varphi)(t)L_n(t)\,dt$$

Again, using the Fundamental ODE and the self adjoint property

$$\frac{1}{n(n+1)||L_n||^2_{L^2(I)}}\int_{-1}^{1}(\mathcal{L}\varphi)(t)L_n(t)\,dt = \frac{1}{(n(n+1)^2||L_n||^2_{L^2(I)}}\int_{-1}^{1}(\mathcal{L}\varphi)(t)(\mathcal{L}L_n)(t)\,dt = \frac{1}{(n(n+1)^2||L_n||^2_{L^2(I)}}\int_{-1}^{1}(\mathcal{L}^2\varphi)(t)L_n(t)\,dt$$

Repeating this procedure and using that $\varphi \in H^{2m}(I)$

$$\alpha_n = \frac{1}{(n(n+1))^m||L_n||^2_{L^2(I)}}\int_{-1}^{1}(\mathcal{L}^m\varphi)(t)L_n(t)\,dt$$

From there, we have

$$||\varphi - \pi_N||^2_{L^2(I)} = \sum_{n=N+1}^{\infty}\frac{1}{(n(n+1))^{2m}}\left(\frac{\int_{-1}^{1}(\mathcal{L}^m\varphi)(t)L_n(t)\,dt}{||L_n||^2_{L^2(I)}}\right)^2 ||L_n||^2_{L^2(I)}$$

And since $n(n+1) > N^2$,

$$||\varphi - \pi_N||^2_{L^2(I)} \leq N^{-4m} \sum_{n=0}^{\infty} \left(\frac{\int_{-1}^{1} (\mathcal{L}^m \varphi)(t) L_n(t)\, dt}{||L_n||^2_{L^2(I)}} \right)^2 ||L_n||^2_{L^2(I)} = N^{-4m} ||\mathcal{L}^m \varphi||^2_{L^2(I)}$$

So

$$||\varphi - \pi_N||^2_{L^2(I)} \leq N^{-4m} ||\mathcal{L}^m \varphi||^2_{L^2(I)}$$

Owing to the last lemma, we have that

$$||\mathcal{L}^m \varphi||_{L^2(I)} \leq C ||\varphi||_{H^{2m}(I)}$$

So $||\varphi - \pi_N||_{L^2(I)} \leq CN^{-2m} ||\varphi||_{H^{2m}(I)}$, with $C = C(s)$.
Now, to prove for general s we use a interpolation argument. This time we explain it in detail.
The operator $id - \pi_N$ is continuous from $L^2(I)$ to $L^2(I)$ with norm 1 :

$$||\varphi - \pi_N||_{L^2(I)} \leq ||\varphi||_{L^2(I)}$$

And $id - \pi_N$ is also continuous from $H^{2m}(I)$ to $L^2(I)$ with norm $\leq CN^{-2m}$

$$||\varphi - \pi_N||_{L^2(I)} \leq CN^{-2m} ||\varphi||_{H^{2m}(I)}$$

By application of the Theorem 7.10, we obtain that $id - \pi_N$ is also continuous from

$$H^{2m(1-\theta)}(I) = [H^{2m}(I), L^2(I)]_\theta \quad \text{to} \quad [L^2(I), L^2(I)]_\theta = L^2(I)$$

with norm

$$\leq C' N^{-2m(1-\theta)}$$

this completes the proof □

However one problem arises with this orthogonal projection, which is that despite that it gives the best L^2 approximation in $\mathcal{P}_N$, it does not give optimal estimates in the H^1 norm. In fact, it may be proven that the best possible estimate when $s \geq r \geq 1$ is

$$||\varphi - \pi_N \varphi||_{H^r(I)} \leq C\, N^{2r-1/2-s}\, ||\varphi||_{H^s(I)}$$

To obtain better estimates in the H^k norms, we introduce other operators.

Definition 2.3. *For any $k \geq 0$, we define*

$$\mathcal{P}_N^{k,0}(I) = H_0^k(I) \cap \mathcal{P}_N(I)$$

So $\mathcal{P}_N^{k,0}(I) = \{p \in \mathcal{P}_N(I) : p^{(m)}(\pm 1) = 0 \quad for \quad m = 0, \ldots, k-1\}$ We also denote $\mathcal{P}_N^0 = \mathcal{P}_N^{1,0}$
And $\pi_N^{k,0}$ is the orthogonal projection operator from $H_0^k(I)$ to $\mathcal{P}_N^{k,0}(I)$ with respect to the inner product $|\cdot|_{H^k(I)}$

Remark 2.4. We recall that, owing to the Sobolev imbedding theorem [**brenner2008mathematical**] (or the trace theorem [**ern2004theory**]), the values and derivatives of order $\leq k-1$ of a function $\varphi \in H^k(I)$ at the endpoints ± 1 are well defined. And $H_0^k(I)$ is the subspace of $H^k(I)$ wich contains the functions whose values and derivatives up to order $\leq k-1$ vanish at ± 1.

Also, the functional

$$|v|_{H^k(\Omega)} = \left(\int_\Omega (v^{(k)}(x))^2 \, dx \right)^{1/2}$$

is a seminorm $H^k(\Omega)$, but it's a norm in $H_0^k(\Omega)$, we for $v \in H_0^k(\Omega)$ have the inequality

$$|v|_{H^k(\Omega)} \leq c||v||_{H^k(\Omega)} \quad (c = c(\Omega, k)) \tag{2.2}$$

And so, $\left(H_0^k(\Omega), (\cdot,\cdot)_{H_0^k(\Omega)}\right)$ is a Hilbert space where the inner product is given by

$$(u, v)_{H_0^k(\Omega)} = \int_\Omega v^{(k)}(x) \, u^{(k)}(x) \, dx$$

For briefness we denote $(u, v)_{H_0^k(\Omega)} = a^k(u, v)$. Then the projection operator $\pi_N^{k,0}$ can be characterized by :

$$\forall \varphi \in H_0^k(I), \quad \pi_N^{k,0}\varphi \in \mathcal{P}_N^{k,0}(I)$$

and

$$\forall \psi_N \in \mathcal{P}_N^{k,0}(I), \quad a^k(\varphi - \pi_N^{k,0}\varphi, \psi_N) = 0 \tag{2.3}$$

Theorem 2.5. *Set $k \geq 1$. For any $0 \leq r \leq k \leq s$, there exist $c = c(r, s, k)$ such that for any $\varphi \in H^s(I) \cap H_0^k(I)$, we have the estimate*

$$||\varphi - \pi_N^{k,0}\varphi||_{H^r(I)} \leq cN^{r-s}||\varphi||_{H^s(I)} \tag{2.4}$$

Proof.
First of all, since this is an asymptotic estimate in terms of the growth of N, we may assume that $N \geq 2k-1$. To prove the theorem we distinguish between the cases $r = k$, $r = 0$ and $0 < r < k$.

Case $r = k$

Let $\pi_N^{0,0} = \pi_N$. We are going to prove the following equality

$$\forall \varphi \in H_0^k(I), \quad (\pi_N^{k,0}\varphi)(t) = \int_{-1}^t (\pi_{N-1}^{k-1,0} \, \varphi')(x) \, dx \tag{2.5}$$

First of all note that since $\varphi' \in H_0^{k-1}(I)$, then both sides of the expression are well defined. To prove that the equality holds, we first see that the relation 2.3 is true for the function in the right-hand side of 2.5. Let $\psi_N \in \mathcal{P}_N^{k,0}$, then

$$\int_{-1}^{1}\left[\frac{d^k\varphi}{dt^k}-\frac{d^k\left(\int_{-1}^{t}\left((\pi_{N-1}^{k-1,0}\varphi')(x)\,dx\right)\right)}{dt^k}\right](t)\cdot\left[\frac{d^k\psi_N}{dt^k}\right](t)\,dt=$$
$$\int_{-1}^{1}\left[\frac{d^{k-1}(\varphi')}{dt^{k-1}}-\frac{d^{k-1}(\pi_{N-1}^{k-1,0}\varphi')}{dt^{k-1}}\right](t)\cdot\left[\frac{d^{k-1}(\psi_N')}{dt^{k-1}}\right](t)\,dt=0$$

where the last equality holds by definition of $\pi_{N-1}^{k-1,0}$. Now we only have to see that the function in the right-hand side of 2.5 belongs to $\mathcal{P}_N^{k,0}(I)$.
First of all its clear that it belongs to $\mathcal{P}_N(I)$, that it has a zero at $t=-1$ and that the first $k-1$ derivatives vanish at ±1. So it only remains to check that it has a zero in $t=1$. We prove this now. Since $N\geq 2K-1$, we may apply the relation 2.3 to $\pi_{N-1}^{k-1,0}$ with $\psi_N=(1-t^2)^{k-1}$:

$$\int_{-1}^{1}\left[\frac{d^{k-1}}{dt^{k-1}}(\pi_{N-1}^{k-1,0}\varphi')\right](t)\left[\frac{d^{k-1}}{dt^{k-1}}\left((1-t^2)^{k-1}\right)\right](t)\,dt=\int_{-1}^{1}\left[\frac{d^{k-1}}{dt^{k-1}}\varphi'\right](t)\left[\frac{d^{k-1}}{dt^{k-1}}\left(1-t^2)^{k-1}\right)\right](t)\,dt$$

Integrate by parts both sides $k-1$ times , use that $\varphi\in H_0^k(I)$ and the definition of $\pi_{N-1}^{k-1,0}$

$$\begin{aligned}(2(k-1))!\int_{-1}^{1}(\pi_{N-1}^{k-1,0}\varphi')(t)\,dt&=(2(k-1))!\int_{-1}^{1}\varphi'(t)\,dt\\&=(2(k-1))!\,(\varphi(1)-\varphi(0))=0\end{aligned}$$

So we have proven 2.5. Now the estimate follows easily from Theorem 2.1

$$|\varphi-\pi_N^{k,0}\varphi|_{H^k(I)}=|\varphi-\pi_{N-1}^{k-1,0}\varphi|_{H^{k-1}(I)}=\ldots=||\frac{d^k\varphi}{dt^k}-\pi_{N-k}\left(\frac{d^k\varphi}{dt^k}\right)||_{L^2(I)}$$
$$\leq c(N-k)^{k-s}\;||\frac{d^k\varphi}{dt^k}||_{H^{s-k}(I)}\leq c'N^{k-s}\;||\varphi||_{H^s(I)}$$

Case $r=0$

We use what is called a duality argument. We know that

$$||\varphi-\pi_N^{k,0}\varphi||_{L^2(I)}=\sup_{g\,\in\,L^2(I)}\frac{\int_{-1}^{1}(\varphi-\pi_N^{k,0})(t)g(t)}{||g||_{L^2(I)}}=\sup_{g\,\in\,C^\infty(\bar{I})}\frac{\int_{-1}^{1}(\varphi-\pi_N^{k,0})(t)g(t)}{||g||_{L^2(I)}}$$

where we have used the density of $C^\infty(\bar{I})$ in $L^2(I)$.

Now, for any $g\in C^\infty(\bar{I})$, consider the problem of finding $u\in H_0^k(I)$ such that

$$\forall\psi\in H_0^k(I),\quad\int_{-1}^{1}\left[\frac{d^ku}{dt^k}\right](t)\left[\frac{d^k\psi}{dt^k}\right](t)\,dt=\int_{-1}^{1}g(t)\psi(t)\,dt\tag{2.6}$$

The existence and uniqueness is guaranteed by the Lax-Milgram Lemma 7.7. Also, it's easy to see that the solution is $u \in C^\infty(\bar{I})$ and that it's given by the ODE

$$\frac{d^{2k}}{dt^{2k}} u = (-1)^k \, g$$

plus imposing the corresponding boundary conditions.

By plugging $\psi = v$ into 2.6, we see that $||u||_{H^k(I)} \leq c|u|_{H^K(I)} \leq ||g||_{L^2(\Omega_d)}$. Also $|u|_{H^{2k}(I)} = ||g||_{L^2(I)}$. Now, using inequality 7.11 repeatedly, we deduce that

$$||u||_{H^{2k}(I)} \leq c||g||_{L^2(I)}$$

So now,

$$\begin{aligned}
\int_{-1}^{1} (\varphi - \pi_N^{k,0}\varphi)(t) g(t)\, dt &= \int_{-1}^{1} \left[\frac{d^k}{dt^k}\left(\varphi - \pi_N^{k,0}\varphi\right)\right](t) \left[\frac{d^k u}{dt^k}\right](t)\, dt \quad \text{(from 2.6)} \\
&= \int_{-1}^{1} \left[\frac{d^k}{dt^k}\left(\varphi - \pi_N^{k,0}\varphi\right)\right](t) \left[\frac{d^k}{dt^k}\left(u - \pi_N^{k,0}u\right)\right](t)\, dt \quad \text{(from 2.3)} \\
&\leq |\varphi - \pi_N^{k,0}\varphi|_{H^k(I)} \, |u - \pi_N^{k,0}u|_{H^k(I)}
\end{aligned}$$

Now, apply the previous case, $r = k$,

$$\int_{-1}^{1} (\varphi - \pi_N^{k,0}\varphi)(t)\, g(t)\, dt \leq c\, N^{k-s}\, ||\varphi||_{H^s(I)}\, N^{-k}\, ||u||_{H^{2k}(I)} \leq cN^{-s}||\varphi||_{H^s(I)}||g||_{L^2(I)}$$

Which finally proves that $||\varphi - \pi_N^{k,0}\varphi|| \leq N^{-s}||\varphi||_{H^s(I)}$

Case $r = 0$

This case is a consequence of the last two cases and the interpolation inequality

$$||\varphi - \pi_N^{k,0}\varphi||_{H^r(I)} \leq ||\varphi - \pi_N^{k,0}\varphi||_{L^2(I)}^{1-r/k}\, ||\varphi - \pi_N^{k,0}\varphi||_{H^k(I)}^{r/k}$$

from Theorem 7.8. □

As a particular case from this theorem, observe that, for $\varphi \in H_0^k(I)$, despite that $||\varphi - \pi_N\varphi||_{L^2(I)} \leq ||\varphi - \pi_N^{k,0}\varphi||_{L^2(I)}$, the asymptotic convergence is equally as fast. However, in the $H^r(I)$ norms, the error of $\varphi - \pi_N^{k,0}\varphi$ is asymptotically faster than the error of $\varphi - \pi_N\varphi$.

We don't just want to approximate functions in H_0^k, so, since we got a good estimate in the previous theorem, we want to adapt that estimate to other boundary conditions, so from any function $\varphi \in H^k(I)$, we define a function $\widetilde{\varphi}_k \in H_0^k$. For this sake, consider we construct the Hermite polynomials $Y_{k,j}$

Definition 2.6. *For any $k \geq 1$, and $0 \leq j \leq k-1$, the polynomials $Y_{k,j}$ are defined by*

$$\begin{cases}
Y_{k,j} \text{ is the unique polynomial in } \mathcal{P}_{2k-1} \text{ that satisfies :} \\
\left(\frac{d^j}{dt^j}Y_{k,j}\right)(-1) = 1 \quad \text{and} \quad \left(\frac{d^m}{dt^m}Y_{k,j}\right)(-1) = 0 \quad \text{for} \quad 0 \leq m \leq k-1, m \neq j \\
\text{and} \\
Y_{k,j}(1) = Y'_{k,j}(1) = \ldots = \left(\frac{d^{k-1}}{dt^{k-1}}Y_{k,j}\right)(1) = 0
\end{cases} \tag{2.7}$$

In other words, $\Upsilon_{k,j}$ is the only polynomial with degree $\leq 2k-1$ whose values and derivatives up to order $k-1$ vanish on ± 1, except for the j-th order derivative at $t=-1$, which is 1.

Now for any function $\varphi \in H^k(I)$, we define $\widetilde{\varphi}_k$ by

$$\widetilde{\varphi}_k(t) = \varphi(t) \; - \; \sum_{j=0}^{k-1} \left(\frac{d^j\varphi}{dt^j}\right)(-1) \cdot \Upsilon_{k,j}(t) \; - \; \sum_{j=0}^{k-1} (-1)^j \left(\frac{d^j\varphi}{dt^j}\right)(1)\Upsilon_{k,j}(-t) \tag{2.8}$$

Observe that $\widetilde{\varphi}_k$ along with its derivatives of order $\leq k-1$ vanish at ± 1, moreover, owing to the Sobolev Embedding theorem

$$\left|\frac{d^m\varphi}{dt^m}(1)\right| + \left|\frac{d^m\varphi}{dt^m}(-1)\right| \leq c||\varphi||_{H^k(I)} \quad \text{for} \quad 0 \leq m \leq k-1$$

so for any $s \geq k$

$$||\widetilde{\varphi}_k||_{H^s(I)} \leq c||\varphi||_{H^s(I)} \quad \text{where} \quad c = c(s) \tag{2.9}$$

Now that we have $\widetilde{\varphi}_k \in H_0^k(I)$, we apply the operator $\pi_N^{k,0}$ to this function.

Definition 2.7. *For any $k \geq 0$, the operator $\widetilde{\pi}_N^k$ acts on $H^k(I)$ and is defined by*

$$\widetilde{\pi}_N^k\varphi \;=\; \left(\pi_N^{k,0}\;\widetilde{\varphi}_k\right) \;+\; \sum_{j=0}^{k-1}\left(\frac{d^j\varphi}{dt^j}\right)(-1)\;\Upsilon_{k,j}(t) \;+\; \sum_{j=0}^{k-1}(-1)^j\left(\frac{d^j\varphi}{dt^j}\right)(1)\;\Upsilon_{k,j}(-t) \tag{2.10}$$

Observation 2.8.
Note that, since $\pi_N^{k,0}\;\widetilde{\varphi}_k \in H_0^k$, the values and derivatives up to order $\leq k-1$ of $\widetilde{\pi}_N^k\varphi$ and φ coincide at ± 1 Also, by definition

$$\varphi \;-\; \widetilde{\pi}_N^k\varphi \;=\; \widetilde{\varphi}_k \;-\; \pi_N^{k,0}\;\widetilde{\varphi} \tag{2.11}$$

Theorem 2.9. *Set $k \geq 1$ and $0 \leq r \leq k \leq s$.*

There exists $c = c(r,s,k)$ such that for all $\varphi \in H^s(I)$

$$||\varphi - \widetilde{\pi}_N^k\varphi||_{H^r(I)} \;\leq\; cN^{r-s}\;||\varphi||_{H^s(I)} \tag{2.12}$$

Proof.
From the last observation, Theorem 2.3, and 2.9

$$||\varphi - \widetilde{\pi}_N^k\varphi||_{H^r(I)} = ||\widetilde{\varphi}_k \;-\; \pi_N^{k,0}\;\widetilde{\varphi}|| \leq cN^{r-s}\;||\widetilde{\varphi}_k||_{H^s(I)} \leq cN^{r-s}\;||\varphi||_{H^s(I)}$$

□

The last result in this section is

Theorem 2.10. *For $k \geq 1$, let π_N^k be the orthogonal projection operator from $H^k(I)$ to $\mathcal{P}_N(I)$ (with respect to the H^k norm). Set $0 \leq r \leq k \leq s$. There exists $c = c(r,s,k)$ such that for any $\varphi \in H^s(I)$,*

$$||\varphi - \pi_N^k \varphi||_{H^r(I)} \leq cN^{r-s}||\varphi||_{H^s(I)} \tag{2.13}$$

Proof.
When $r = k$, this is a consequence of the last theorem

$$||\varphi - \pi_N^k \varphi||_{H^k(I)} \leq ||\varphi - \widetilde{\pi}_N^k \varphi||_{H^k(I)} \leq cN^{k-s}||\varphi||_{H^s(I)}$$

When $r = 0$ we use a duality argument similar to that of the proof of Theorem 2.3
When $0 < r < k$ we use a interpolation inequality. □

2.2 Polynomial Approximations in Hypercubes

We now want to extend the previous results to the domain $\Omega_d = (-1,1)^d$. The main takeaway from this section is that by considering such simple geometries (tensor products of intervals), the approximation results of last section remain true. Since Ω_d is a tensor product of intervals, it is reasonable to take the tensor product of polynomials in one dimension as an approximation space.

Definition 2.11. *For any $n \geq 0$, $\mathbb{Q}_n(\Omega_d)$, is the space of polynomials in d variables and degree $\leq n$ in each variable x_j, restricted to Ω_d. More explicitly:*

$$\mathbb{Q}_n(\Omega_d) = \{v = p|_{\Omega_d} \;:\; p = \sum_{\substack{i_1,\ldots,i_d \leq n \\ i_1,\ldots,i_d \geq 0}} \alpha_{i_1,\ldots,i_d} x_1^{i_1} \cdot \ldots \cdot x_d^{i_d}$$

Naturally, we are going take as a basis of $\mathbb{Q}_N(\Omega_d)$ the tensorized basis

$$\{L_{n_1}(x_1) \cdot \ldots \cdot L_{n_d}(x_d), \quad 0 \leq n_1, \ldots n_d \leq N\}$$

which is also orthogonal in $L^2(\Omega_d)$.
We denote by Π_N the orthogonal projection form $L^2(\Omega_d)$ to $\mathbb{Q}_N(\Omega_d)$. We also denote $\pi_N^{(j)}$ the orthogonal projection applied to the $j-th$ variable, that is

$$\pi_N^{(j)} v(x_1, \ldots, x_d) = \pi_N^{(j)} v_j(x_1, \ldots x_{j-1}, x_{j+1}, \ldots, x_d)(t)$$

where

$$v_j(x_1, \ldots x_{j-1}, x_{j+1}, \ldots, x_d)(t) : \; t \longmapsto v(x_1, \ldots x_{j-1}, t, x_{j+1}, \ldots, x_d)$$

They key observation is that each function $v \in C(\overline{\Omega}_d)$ satisfies for $1 \leq j \leq d$,

$$\int_{-1}^{1} v(x) L_{m_j}(x_j)\, dx_j = \int_{-1}^{1} (\pi_N^{(j)} v) L_{m_j}(x_j)\, dx_j \quad \text{for} \quad 0 \leq m_j \leq N$$

And from here we can see that

$$\int_{-1}^{1} v(x)\ L_{m_1} \cdot \ldots \cdot L_{m_d}\ dx = \int_{-1}^{1} L_{n_1}(x_1)dx_1 \cdot \ldots \cdot \int_{-1}^{1} L_{m_d}(x_d)v(x)\ dx_d =$$
$$\int_{\Omega_d} (\pi_N^{(1)} \circ \ldots \circ \pi_N^{(d)}\ v(x))\ L_{n_1}(x_1) \cdot \ldots \cdot L_{m_d}(x_d)\ dx$$

So we infer that the following holds in $C(\overline{\Omega}_d)$

$$\Pi_N = \pi_N^{(1)} \circ \ldots \circ \pi_N^{(d)} \tag{2.14}$$

and it's clear the the $\pi_N^{(j)}$ commute. Note that the previous expression makes sense and is true in $C(\overline{\Omega}_d)$, but it doesn't make sense to apply $\pi_N^{(j)}$ to a function $v \in L^2(\Omega_d)$

Theorem 2.12. *For any $s \geq 0$, there exists $c = c(s)$ such that for all $v \in H^s(\Omega_d)$, we have the estimate*

$$||v - \Pi_N v||_{L^2(\Omega_d)} \leq cN^{-s}||v||_{H^s(\Omega_d)} \tag{2.15}$$

Proof. We prove it for $s = m \geq 0$. For other values of s it follows by a standard interpolation inequality argument. Since $id - \Pi_N$ is a continuous linear operator from $H^m(\Omega_d)$ to $L^2(\Omega_d)$, and using the density of $C(\overline{\Omega}_d) \cap H^m(\Omega_d)$ in $H^m(\Omega_d)$, we may assume $v \in C(\overline{\Omega}_d) \cap H^m(\Omega_d)$. For simplicity we assume $d = 2$. Then, from 2.14,

$$||v - \Pi_N v||_{L^2(\Omega_d)} \leq ||v - \pi_N^{(1)} v||_{L^2(\Omega_d)} + ||\pi_N^{(1)}(v - \pi_N^{(2)} v)||_{L^2(\Omega_d)}$$

We bound the two terms:

$$\begin{aligned}
||v - \pi_N^{(1)} v||^2_{L^2(\Omega_d)} &= \int_{-1}^{1}\int_{-1}^{1} (v(x,y) - \pi_N^{(1)}(x,y))^2\ dxdy \\
&\leq cN^{-2m} \int_{-1}^{1}\int_{-1}^{1} \sum_{k=0}^{m} \left(\frac{d^k}{dx^k} v(x,y)\right)^2\ dxdy \quad \text{(from Theorem 2.1)} \\
&\leq cN^{-2m}||v||^2_{H^m(\Omega_d)}
\end{aligned}$$

$$\begin{aligned}
||\pi_N^{(1)}(v - \pi_N^{(2)} v)||^2_{L^2(\Omega_d)} &= \int_{-1}^{1}\int_{-1}^{1} \left(\pi_N^{(1)}(v - \pi_N^{(2)})(x,y)\right)^2\ dydx \\
&\leq \int_{-1}^{1}\int_{-1}^{1} \left((v - \pi_N^{(2)})(x,y)\right)^2\ dxdy \quad \text{(using that the operator norm of } \pi_N^{(1)} \text{ is 1)} \\
&\leq cN^{-2m} \int_{-1}^{1}\int_{-1}^{1} \sum_{k=0}^{m} \left(\frac{d^k}{dy^k} v(x,y)\right)^2\ dydx \quad \text{(from Theorem 2.1)} \\
&\leq cN^{-2m}||v||^2_{H^m(\Omega_d)}
\end{aligned}$$

This proves the theorem. □

We now define rest of the analogous operators to those of Section 2. The proofs of the following theorems use similar arguments to the previous proof so we omit them.

Definition 2.13. *For $k > 0$, we define*

$$\mathbb{Q}_N^{k,0}(\Omega_d) = \mathbb{Q}_N(\Omega_d) \cap H_0^k(\Omega_d)$$

and $\Pi_N^{k,0}$ is the orthogonal projection operator from $H_0^k(\Omega_d)$ to $\mathbb{Q}_N^{k,0}(\Omega_d)$ with respect to the norm $|\cdot|_{H^k(\Omega_d)}$

Theorem 2.14. *Set $k \geq 1$ and $s \geq k$.*

There exists $c = c(s)$ such that for all $\varphi \in H^s(\Omega_d) \cap H_0^k(\Omega_d)$, we have

$$||\varphi - \Pi_N^{k,0}\varphi||_{H^k(\Omega_d)} \leq cN^{k-s}||\varphi||_{H^s(\Omega_d)} \tag{2.16}$$

Definition 2.15. *For any $N > 0$, Π_N^k denotes the orthogonal projection from $H^k(\Omega_d)$ to $\mathbb{Q}_N(\Omega_d)$ with respect to the $||\cdot||_{H^k(\Omega_d)}$ norm.*

Theorem 2.16. *Set $k > 0$ and $s \geq k$.*

There exists a constant $c = c(s)$ such that for all $\varphi \in H^k(\Omega_d)$,

$$||\varphi - \Pi_N^k\varphi||_{H^k(\Omega_d)} \leq cN^{k-s}||\varphi||_{H^s(\Omega_d)} \tag{2.17}$$

Chapter 3

Approximation by interpolation

3.1 Technical Lemmas

Before studying the approximation properties of the polynomial interpolation we need some technical lemmas.

Lemma 3.1 (Polynomial Inverse Inequality 1)**.** *For any $N > 0$ and for any $p_N \in \mathcal{P}_N(I)$ we have*

$$\left(\int_{-1}^{1} p_N'(t)^2 \; (1-t^2) \; dt\right)^{1/2} \leq \sqrt{2}N||p_N||_{L^2(I)} \tag{3.1}$$

Proof. Write $p_N = \sum_{m=0}^{N} \alpha_m L_m$, so from 1.3

$$\int_{-1}^{1} p_N'(t)^2 \; (1-t^2) \; dt = \sum_{m=0}^{N}\sum_{j=0}^{N} \alpha_m \alpha_j \int_{-1}^{1} L_m'(t) \; L_n'(t) \; (1-t^2) \; dt = \sum_{m=0}^{N} \alpha_m^2 \; m(m+1) \int_{-1}^{1} L_m(t)^2 \; dt$$

$$\leq 2N^2 \sum_{m=0}^{N} \alpha_m^2 \int_{-1}^{1} L_m(t)^2 \; dt = 2N^2 \; ||p_N||^2$$

□

Lemma 3.2 (Polynomial Inverse Inequality 2)**.**
For any $p_N \in \mathcal{P}_N^0(I)$ the following inequality is true

$$|p_N|_{H^1(I)} \leq \sqrt{2} \; N \left(\int_{-1}^{1} p_N^2(t) \; (1-t^2)^{-1} \; dt\right)^{1/2} \tag{3.2}$$

Proof.
Since p_N has zeros in ± 1, we can write it as

$$p_N = (1-t^2) \sum_{m=1}^{N-1} \beta_m L_m'$$

So now, we simply compute using 1.1 and 1.3.

$$|p_N|^2_{H^1(I)} = \sum_{m=1}^{N-1} \beta_m^2 \; m^2 \; (m+1)^2 \; ||L_m||^2_{L^2(I)}$$

and

$$\int_{-1}^{1} p_N(t)^2 \; (1-t^2)^{-1} \; dt = \sum_{m=1}^{N-1} \; \beta_m^2 \; m \; (m+1) \; ||L_m||^2_{L^2(I)}$$

so

$$\sum_{m=1}^{N-1} \beta_m^2 \; m^2 (m+1)^2 \; ||L_m||^2_{L^2(I)} \leq 2N^2 \sum_{m=1}^{N-1} \beta_m^2 \; m(m+1) \; ||L_m||^2_{L^2(I)}$$

□

Lemma 3.3 (Multiplication by $(1-t^2)^{-1/2}$)**.**
Multiplication by $(1-t^2)^{-1/2}$ *is a continuous operation from* $H_0^1(I)$ *to* $L^2(I)$*. More precisely, there exists a constant C such that for any* $\varphi \in H_0^1(I)$

$$||\varphi \; (1-t^2)^{-1/2}||_{L^2(I)} \leq C||\varphi||_{H^1(I)}$$

Proof. The proof may be consulted in **[lions2012non]** □

Lemma 3.4 (Scaled Sobolev Embedding Inequality)**.**
Let c_1 *be the constant on the Sobolev embedding inequality from* $H^1(I)$ *to* $L^\infty(I)$ *when applied on the interval* $(0,1)$*. Then for any* $\psi \in H^1(a,b)$

$$\max_{a<\theta<b} |\psi(\theta)|^2 \leq c_1 \left(\frac{1}{b-a} ||\psi||^2_{L^2(a,b)} + (b-a)|\psi|^2_{H^1(a,b)} \right) \tag{3.3}$$

Proof.
The estimate is easily derived by applying the Sobolev inequality

$$\max_{0<t<1} |\varphi(\theta)|^2 \leq c_1 \left(||\varphi||^2_{L^2(0,1)} + |\varphi|^2_{H^1(0,1)} \right)$$

to the function $\varphi(t) = \psi(a + t(b-a))$ □

3.2 Interpolation Operator errors

We now study how good a polynomial approximation can we achieved by interpolation with respect to the Gauss-Legendre and Gauss-Legendre-Lobatto nodes. Since pointwise interpolation only makes sense for continuous, and owing to Sobolev's inequality, throughout this chapter it will be assumed that all functions are in $H^r(I)$ for some $r > 1/2$

Definition 3.5 (Gauss-Legendre Interpolant). *The Gauss-Legendre Interpolant operator is defined by*

$$\begin{cases} \textit{For any } \varphi \in C(I)\,, G_N \textit{ is the only polynomial in } \mathcal{P}_{N-1}(I) \textit{ such that} \\ (G_N\ \varphi)(\zeta_j) = \varphi(\zeta_j) \quad \forall\, 1 \le j \le N \end{cases} \tag{3.4}$$

where the ζ_j are the N-th Gauss-Legendre nodes

We now estimate the norm of G_N as an operator from $H^1(I)$ to $L^2(I)$.

Theorem 3.6. *There exists a constant c such that for all $\varphi \in H^1(I)$ we have the estimate*

$$||G_N u||_{L^2(I)} \le c(||u||_{L^2(I)} + N^{-1}||u'\ (1-t^2)^{1/2}||_{L^2(I)}) \tag{3.5}$$

Proof.
Let $t = cos\theta$ and $\widehat{u}(\theta) = u(cos\theta)$ where $\theta \in (0,\pi)$ and $\theta_j = \zeta_j$.

By the exactness property of the quadrature and the weight estimates 1.23,

$$||G_N u||^2_{L^2(I)} = \sum_{j=1}^{N} u(\zeta_j)^2\ w_j \le cN^{-1} \sum_{j=1}^{N} \widehat{u}(\theta_j)^2\ sin\,\theta_j$$

From Theorem 1.13, there exist intervals K_j, $1 \le j \le N$ of length π/N such that $\theta_j \in K_j$ and the intersection between them is empty except possibly the cases $j = i-1$, i, $i+1$. So by changing c to $3c$, we have

$$||G_N u||_{L^2(I)} \le c\ N^{-1/2} \sum_{j=1}^{N} \max_{\theta \in K_j} |\widehat{u}(\theta)(sin\theta)^{1/2}|$$

From Lemma 3.4,

$$||G_N u|| \le c \sum_{j=1}^{N} \left(||\widehat{u} \cdot (sin\theta)^{1/2}||_{L^2(K_j)} + N^{-1}|\widehat{u} \cdot (sin\theta)^{1/2}|_{H^1(K_j)} \right)$$

Now, we note that $\bigcup_j K_j \subset [a_0, a_1] \subset [0,\pi]$ with $a_0 = \frac{2\pi}{N}$ and $a_1 = \pi - \frac{\pi}{2N}$ and since each point of I belongs to at most two K_j,

$$||G_N u|| \le c\ \left(||\widehat{u} \cdot (sin\theta)^{1/2}||_{L^2(0,\pi)} + N^{-1}\ ||\frac{d}{d\theta}\widehat{u} \cdot (sin\theta)^{1/2}||_{L^2(0,\pi)} + N^{-1}\ ||\widehat{u} \cdot \frac{cos\theta}{(sin\theta)^{1/2}}||_{L^2(a_0,a_1)} \right)$$

Change variables again

$$||\widehat{u} \cdot (sin\theta)^{1/2}||_{L^2(0,\pi)} \;=\; \int_0^{\pi} \widehat{u}(\theta)^2\ sin\theta\ d\theta = \int_{-1}^{1} u(x)^2\ dx$$

$$||\frac{d}{d\theta}\widehat{u} \cdot (sin\theta)^{1/2}||_{L^2(0,1)} = \int_0^{\pi} (sin\theta)^3\ u'(cos\theta)^2\ d\theta = \int_{-1}^{1} (1-x^2)u'(x)^2\ dx$$

$$N^{-1}||\widehat{u}\cdot\frac{cos\theta}{(sin\theta)^{1/2}}||_{L^2(a_0,a_1)}=N^{-1}\int_{a_0}^{a_1}\widehat{u}(\theta)^2\,\frac{cos^2\theta}{sin\theta}\,d\theta\le\left(\sup_{a_0<\theta<a_1}\frac{1}{Nsin\theta}\right)\int_{a_0}^{a_1}\widehat{u}(\theta)^2\,cos(\theta)\,d\theta$$

Now use that since $a_0=\mathcal{O}(N^{-1})$ and $a_1=\pi-\mathcal{O}(N^{-1})$ to obtain that,

$$\sup_{a_0\le\theta\le a_1}\frac{1}{Nsin\theta}\le c$$

to conclude that the last term is $\le c||u||_{L^2(I)}$ □

Theorem 3.7. *There exists a constant $c=c(s)$ such that for any $u\in H^s(I)$,*

$$||G_Nu-u||_{L^2(I)}\le cN^{-s}\,||u|||_{H^s(I)} \tag{3.6}$$

Proof.

$$||G_Nu-u||_{L^2(I)}\le||G_Nu-\pi^1_{N-1}u||_{L^2(I)}+||\pi^1_{N-1}u-u||_{L^2(I)}$$

We bound the two terms

$$||\pi^1_{N-1}u-u||_{L^2(I)}\le CN^{-s}||u||_{H^s(I)}\quad\text{(from Theorem 2.1)}$$

and

$$\begin{aligned}||G_Nu-\pi^1_{N-1}u||_{L^2(I)}&=||G_N(u-\pi^1_{N-1}u)||_{L^2(I)}\\&\le c\left(||u-\pi^1_{N-1}u||_{L^2(I)}+N^{-1}\left(\int_1^1(((u-\pi^1_{N-1}u)')^2\,(1-t^2)\,dt\right)^{1/2}\right)\quad\text{(from last theorem)}\\&\le c\left(N^{-s}||u||_{H^s}(I)+N^{-1}||u-\pi_{N-1}u||_{H^1(I)}\right)\quad\text{(from Theorem 2.3 and } 1-t^2\le1)\\&\le cN^{-s}||u||_{H^s}\end{aligned}$$

□

However, and similarly to what occurred in Section 2.1, the fact that G_N does not interpolate the endpoints implies that the H^1 norm of the error is not optimal. In fact it may be proven that for $s\ge r$, $s\ge1$, the best possible estimate is

$$\begin{cases}||\varphi-G_N\varphi||_{H^1(I)}\le CN^{3r/2-s}\quad\text{when}\quad 0\le r\le1\\||\varphi-G_N\varphi||_{H^1(I)}\le CN^{2r-1/2-s}\quad\text{when}\quad r\ge1\end{cases} \tag{3.7}$$

Now we study the polynomial interpolation on the Legendre-Gauss-Lobatto nodes η_j.

Definition 3.8 (Legendre-Gauss-Lobatto Interpolant). *The Legendre-Gauss-Lobatto Interpolant operator is defined by*

$$\begin{cases} \text{For any } \varphi \in C(I)\,, i_N\varphi \text{ is the only polynomial in } \mathcal{P}_N(I) \text{ such that} \\ (i_N\ \varphi)(\eta_j) = \varphi(\eta_j) \quad \forall\, 0 \le j \le N \end{cases} \tag{3.8}$$

where the η_j are the N-th Legendre-Gauss-Lobatto nodes

Lemma 3.9. *For every polynomial $p_N \in \mathcal{P}_N$, we have that*

$$||p_N||^2_{L^2(I)} \ \le\ \sum_{j=0}^{N} p_N(\eta_j)^2\ \rho_j \ \le\ 3\ ||p_N||^2_{L^2(I)} \tag{3.9}$$

Proof.
We may write $p_N = \sum_{m=0}^N \alpha_m L_M$, so $||p_N||_{L^2(I)} = \sum_{m=0}^N \alpha_m^2 ||L_m||^2_{L^2(I)}$ Also,

$$\sum_{j=0}^{N} p_N(\eta_j)^2\ \rho_j = \sum_{j=0}^{N} \left(\sum_{m=0}^{N} \alpha_m L_M(\eta_j)\right)^2 \rho_j \ = \ \sum_{j=0}^{N} \left(\sum_{m=0}^{N-1} \alpha_m L_M(\eta_j)\ +\ \alpha_N L_N(\eta_j)\right)^2 \rho_j$$

$$= \sum_{j=0}^{N} \left(\sum_{m=0}^{N-1} \alpha_m L_M(\eta_j)\right)^2 \rho_j \ \ +\ \ 2\sum_{j=0}^{N} \left(\sum_{m=0}^{N-1} \alpha_m L_M(\eta_j)\right) \alpha_N L_N(\eta_j)\ \rho_j \ \ +\ \ \sum_{j=0}^{N} \alpha_N^2\ L_N^2(\eta_j)\ \rho_j$$

The first term is $\sum_{m=0}^{N-1}\ \alpha_m^2\ ||L_m||^2$ by the exactness property of the quadrature. By the exactness property of the quadrature again, and the orthogonality of the polynomials, the second term is 0 . For the third term,

$$\alpha_N^2 \sum_{j=0}^{N} L_N^2(\eta_j)\ \rho_j = \alpha_N^2\ \sum_{j=0}^{N} \frac{2}{N(N+1)} = \alpha_N^2\ \frac{2}{N} = (2 + \frac{1}{N})\ \alpha_N^2\ ||L_N||^2_{L^2(I)}$$

by 1.33 and 1.5. The result now easily follows. □

Lemma 3.10. *There exists a constant c such that for all $u \in H^1(I)$,*

$$||i_N u||_{L^2(I)} \le C \left(N^{-1}\ |u(1)| + N^{-1}\ |u(-1)| + ||u||_{L^2(I)}\ +\ N^{-1} \left(\int_{-1}^{1} (u'(t)^2 (1-t^2)\ dt\right)^{1/2} \right) \tag{3.10}$$

Proof.
From the last lemma ,

$$||i_N u||^2_{L^2(I)} \le u(-1)^2\rho_0\ +\ \sum_{j=1}^{N-1}\ u^2(\eta_j)\rho_j\ +\ u(1)^2\rho_j$$

Using exactly the same argument as in the last Theorem 3.6 we can see

$$\sum_{j=1}^{N-1}\ u^2(\eta_j)\ \rho_j\ \le\ ||u||^2_{L^2(I)} + N^{-2}\left(\int_{-1}^{1} u'(t)^2(1-t^2)\right)$$

□

The next inequality is key to derive the error for the Lobatto interpolation. Its necessity comes from 3.2.

Lemma 3.11. *There exists a constant C such that for all* $u \in H_0^1(I)$,

$$||(i_N u)\ (1-t^2)^{-1/2}||_{L^2(I)} \leq C\left(||u(t)\ (1-t^2)^{-1/2}||_{L^2(I)} + N^{-1}|u|_{H^1(I)}\right) \tag{3.11}$$

Proof.
Since $(i_N u)\ (1-t^2)^{-1}$ is a polynomial of degree $\leq 2N-2$, from the exactness property of the quadrature, we deduce that

$$\int_{-1}^{1}(i_N u)^2(t)(1-t^2)^{-1}\, dt \ = \ \sum_{j=0}^{N}(i_N u)^2(\eta_j)\ (1-\eta_j^2)^{-1}\ \rho_j \ = \ \sum_{j=1}^{N-1}(i_N u)^2(\eta_j)\ (1-\eta_j^2)^{-1}\ \rho_j$$

Now , using 1.16, we obtain

$$\int_{-1}^{1}(i_N u)^2(t)(1-t^2)^{-1}\, dt \leq cN^{-1}\sum_{j=1}^{N-1}(i_N u)^2(\eta_j)\ (1-\eta_j^2)^{-1/2}\ \sum_{j=1}^{N-1}(u)^2(\eta_j)\ (1-\eta_j^2)^{-1/2}$$

Now, denoting $\widehat{u}(\theta) = u(cos\theta)$ and by similar arguments as before,

$$\int_{-1}^{1}(i_N u)^2(t)(1-t^2)^{-1}\, dt \ = \leq c\left(||\widehat{u}(\theta)\ (sin\theta)^{-1/2}||^2_{L^2(0,\pi)} + N^{-2}\ ||\frac{d}{d\theta}\left(\widehat{u}(\theta)\ (sin\theta)^{-1/2}\right)||^2_{L^2(a_0,a_1)}\right)$$

Reversing the change of variables,

$$||\widehat{u}(\theta)\ (sin\theta)^{-1/2}||^2_{L^2(0,\pi)} = ||u(t)\ (1-t^2)^{-1/2}||_{L^2(I)}$$

$$N^{-2}||\frac{d}{d\theta}(\widehat{u})\ (sin\theta)^{-1/2}||_{L^2(0,\pi)} \ = \ N^{-2}\int_0^{\pi} u'(cos\theta)^2\ cos\theta\ d\theta = N^{-2}|u|_{H^1(I)}$$

$$N^{-2}||\widehat{u}(\theta)\frac{d}{d\theta}((sin\theta)^{-1/2})||_{L^2(a_0,a_1)} = \frac{N^{-2}}{4}\int_{a_0}^{a_1}\widehat{u}(\theta)^2\frac{cos(\theta)}{(sin\theta)^3}\, d\theta \leq c\int_{-1}^{1}u(t)(1-t^2)^{-1/2}$$

□

This is the principal result

Theorem 3.12. *For any real numbers* $s \geq 1 \geq r \geq 0$, *there exists* $c = c(s,r)$ *such that for any* $u \in H^s(I)$,

$$||u - i_N u||_{H^r(I)} \leq cN^{r-s}||u||_{H^s(I)} \tag{3.12}$$

Proof.

Case $r = 0$

$$||u - i_N u||_{L^2(I)} \leq ||u - \widetilde{\pi}_N^{1,0}u||_{L^2(I)} + ||\widetilde{\pi}_N^{1,0}u - i_N u||_{L^2(I)}$$

The first term has the estimate from 2.9. For the second term

$$\begin{aligned}
||\widetilde{\pi}_N^{1,0} u - i_N u||_{L^2(I)} &= ||i_N(\ \widetilde{\pi}_N^{1,0} u - u \ )||_{L^2(I)} \\
&\leq \ c \left(||\widetilde{\pi}_N^{1,0} u - u||_{L^2(I)} + N^{-1} |\widetilde{\pi}_N^{1,0} u - u|_{H^1(I)} \right) \quad \text{(Using the stability theorem 3.10)} \\
&\leq cN^{-s} ||u||_{H^s(I)} \quad \text{(Using 2.9)}
\end{aligned}$$

Case $r = 1$

$$|u - i_N u|_{H^1(I)} \leq |u - \widetilde{\pi}_N^1 u|_{H^1(I)} + |\widetilde{\pi}_N^1 u - i_N u|_{H^1(I)}$$

The first term is estimated using 2.9. We now bound the second term

$$\begin{aligned}
|\widetilde{\pi}_N^1 u - i_N u|_{H^1(I)} &\leq \ cN \ ||(\widetilde{\pi}_N^1 u - i_N u) \ (1 - t^2)^{-1/2}||_{L^2(I)} \quad \text{(Using 3.2 since } (\widetilde{\pi}_N^1 u - i_N u) \in \mathcal{P}_N^0) \\
&= \ cN \ ||i_N(\widetilde{\pi}_N^1 u - u) \ (1 - t^2)^{-1/2}||_{L^2(I)} \\
&\leq cN||(\widetilde{\pi}_N^1 u - u) \ (1 - t^2)^{-1/2}||_{L^2(I)} + |u - \pi_N^1 u|_{H^1(I)} \quad \text{(From last lemma)}
\end{aligned}$$

The estimate now follows by applying 3.3 to the term $||(\widetilde{\pi}_N^1 u - u) \ (1 - t^2)^{-1/2}||_{L^2(I)}$

The case $0 < r < 1$ is derived by an standard interpolation argument. □

The next result is an immediate consequence

Corollary 3.13 (H^1 Stability of the i_N operator)**.** *There exists a constant c such that for $u \in H^1(I)$*

$$||i_N u||_{H^1(I)} < c||u||_{H^1(I)}$$

We can also give a bound on the ∞-norm by applying directly the Sobolev embedding theorem and last Theorem

Corollary 3.14. *For any $s \geq 1$ there exists $c = c(s)$ such that for $u \in H^s(\Omega_d)$*

$$||u - I_N u||_{L^\infty(I)} \ \leq \ cN^{1-s} ||u||_{H^s(I)} \tag{3.13}$$

3.3 Interpolation Operators in Hypercubes

We now extend the results of last section to domains $\Omega_d = (-1, 1)^d$. We only mention the case of Legendre-Gauss-Lobatto nodes, since its the one that concern us in terms of our applications. But the case for Gauss-Legendre is analogous. Since Ω_d is a tensor product of intervals, it is reasonable to take as our interpolation points the grid formed by tensor the tensor products of Legendre-Gauss-Lobatto Nodes

Definition 3.15. *The* N *-th Legendre-Gauss-Lobatto grid (see image 1.1) in* Ω_d *is denoted by* $\Omega_{d,N}$ *and is defined by*

$$\Omega_{d,N} = \{(\eta_{i_1}, \ldots, \eta_{i_d}) \ : \ 0 \leq i_1, \ldots, i_d \leq N\} \tag{3.14}$$

where the η_{i_j} *are the zeros of* $(1-t^2)L'_N(t)$

For any $\alpha = (i_1, \ldots, i_d)$, *we denote* $\eta_\alpha = (\eta_{i_1}, \ldots, \eta_{i_d})$ *and* $\rho_\alpha = \rho_{i_1} \cdot \ldots \cdot \rho_{i_d}$

Definition 3.16. *For any* $\alpha = (i_1, \ldots, i_d)$ *with* $0 \leq i_1, \ldots, i_d \leq N$ *, we define the* α*-th Lagrange polynomial by*

$$l_\alpha(x) = l_{i_1}(x_1) \ldots l_{i_d}(x_d)$$

where l_i *is the* $(1D)$ *Lagrange polynomial defined by* $l_i(\eta_j) = \delta_{ij}$

Its clear that the polynomials $\{l_\alpha\}_\alpha$ define a basis for the space $\mathbb{Q}_N(\Omega_d)$ and that

$$l_\alpha(\eta_{j_1}, \ldots, \eta_{j_d}) = \delta_{i_1 j_1} \cdot \ldots \cdot \delta_{i_1 j_1}$$

Definition 3.17. *The Legendre-Gauss-Lobatto polynomial interpolation operator* I_N^d *is the defined by*

$$\begin{cases} \text{For any } \ u \in C(\overline{\Omega}_d) \ I_N^d u \text{ is defined as the only polynomial in } \mathbb{Q}_N(\Omega_d) \text{ such that} \\ (I_N^d u)(x) = u(x) \quad \text{for all } x \in \Omega_{d,N} \end{cases}$$

Note that we have

$$I_N^d = i_N^{(1)} \circ \ldots \circ i_N^{(d)}$$

Using 3.12, we can also prove, with the same technique to what we did in Theorem 2.15

Theorem 3.18. *Suppose* $s > d/2$ *and* $1 \geq r \geq 0$. *There exists a constant* $c = c(r,s)$ *such that for every* $u \in H^s(\Omega_d)$,

$$||I_N^d u - u||_{H^r(\Omega_d)} \leq c N^{r-s} ||u||_{H^s(\Omega_d)} \tag{3.15}$$

Note that the assumption $s > d/2$ guarantees the continuity of u.

From now on, we assume that we are working on a fixed dimension d and we abuse notation by denoting $I_N = I_N^d$ and $\Omega_N = \Omega_{d,N}$

Chapter 4

Spectral-NI approximation of PDEs

We finally have all the necessary tools to discuss our application to PDEs

4.1 The Poisson equation

We first consider the homogeneous Poisson equation

$$\begin{cases} -\Delta u = f \quad \text{in} \quad \Omega_d \\ u = 0 \quad \text{on} \quad \partial\Omega_d \end{cases} \tag{4.1}$$

The variational formulation is

$$\begin{cases} \text{Find } u \in H_0^1(\Omega_d) \text{ such that} \\ a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega_d) \end{cases} \tag{4.2}$$

Where $a(u,v) = (\nabla u, \nabla v)$
For the discrete problem, we take as our finite dimensional subspace $\mathbb{Q}_N^0(\Omega_d) \subset H_0^1(\Omega_d)$. So we have the following spectral problem

$$\begin{cases} \text{Find } u_N \in \mathbb{Q}_N^0(\Omega_d) \text{ such that} \\ a(u_N, v_N) = (f, v_N) \quad \forall v_N \in \mathbb{Q}_N^0(\Omega_d) \end{cases} \tag{4.3}$$

However, the inner products displayed in the variational formulation may be too complicated or even impossible to compute analytically. For that we define the numerical quadrature on Ω_d by naturally extending the one on one dimension. Let $\varphi, \psi \in C(\overline{\Omega}_d)$. We consider the quadrature defined by

$$\int_{\Omega_d} \varphi(x)\, dx = \sum_{|\alpha|_\infty \leq N} \varphi(\eta_\alpha)\rho_\alpha \tag{4.4}$$

And is clear that the quadrature is exact for $\mathbb{Q}_{2N-1}(\Omega_d)$ .The discrete inner product is defined as

$$(\varphi, \psi)_N = \sum_{|\alpha|_\infty \leq N} \varphi(\eta_\alpha)\psi(\eta_\alpha)\rho_\alpha \tag{4.5}$$

and, similarly,

$$a_N(\varphi,\psi) = (\nabla\varphi, \nabla\psi)_N \tag{4.6}$$

for $\varphi, \psi \in C^1(\overline{\Omega}_d)$

So now we derive the following Spectral - NI (Numerical Integration) formulation

$$\begin{cases} \text{Find } \widetilde{u}_N \in \mathbb{Q}_N^0(\Omega_d) \text{ such that} \\ a_N(\widetilde{u}_N, v_N) = (f, v_N)_N \quad \forall v_N \in \mathbb{Q}_N^0(\Omega_d) \end{cases} \tag{4.7}$$

We begin with with analysis of problems 4.2 and 4.3.

Theorem 4.1 (Existence and uniqueness of the variational formulations)**.** *Suppose that* $f \in L^2(\Omega_d)$ *. Then the problems* 4.2 *and* 4.3 *have each a unique solution*

Proof. This is simply the Lax-Milgram lemma, since $a(\cdot,\cdot)$ is a inner product in $H_0^1(\Omega_d)$ □

From the orthogonal relation 7.4 we deduce that u_N, the solution of problem 4.3, coincides with $\Pi_N^{1,0}$. So the error $u - u_N$ comes from Theorem 2.14.

Theorem 4.2. *Suppose the solution* u *to* 4.2 *is in* $H^s(\Omega_d)$*. Then there exists a constant* $c = c(s)$ *such that, for all* N*,*

$$|u - u_N|_{H^1(\Omega_d)} \leq cN^{1-s}||u||_{H^s(\Omega_d)} \tag{4.8}$$

Now, we give a stability result

Theorem 4.3. *The norm of the discrete solution* u_N *is uniformly controlled by the norm of* f*:*

$$||u_N||_{H^1(\Omega_d)} \leq ||f||_{L^2(\Omega_d)} \tag{4.9}$$

Proof. From $a(u_N, u_N) = (f, v_N)$ the proof is immediate. □

Now, we give an estimate in the L^2 norm.

Theorem 4.4. *Suppose the solution* u *to* 4.2 *is in* $H^s(\Omega_d)$*. Then there exists a constant* $c = c(s)$ *such that, for all* N

$$||u - u_N||_{L^2(\Omega_d)} \leq cN^{-s}||u||_{H^s(\Omega_d)} \tag{4.10}$$

In particular, the method is convergent.

Proof. The proof relies on a duality argument. Consider the problem

$$\begin{cases} -\Delta w = g & \text{in} \quad \Omega_d \\ w = 0 & \text{on} \quad \partial\Omega_d \end{cases}$$

where $g = u - u_N$

The problem (in its variational form) has a unique solution. Moreover differentiating the equation, we can see that $w \in H^2(\Omega_d)$. It can also be proven (**[grisvard2011elliptic]**) that

$$||w||_{H^2(\Omega_d)} \;\leq\; c||g||_{L^2(\Omega_d)}$$

(where the constant c does not depend on w nor on $u - u_N$). From, here, taking $w_N = \Pi_N^{1,0} w$

$$\begin{aligned} ||u - u_N||^2_{L^2(\Omega_d)} &= (u - u_N, u - u_N) = a(w, u - u_N) = a(w - w_N, u - u_N) \\ &\leq ||u - u_N||_{H^1(\Omega_d}||w - w_N||_{H^1(\Omega_d)} \leq cN^{1-s}||u||_{H^s(\Omega_d)}N^{-1}||w||_{H^2(\Omega_d} \\ &\leq cN^{-s}||u||_{H^s(\Omega_d)}||u - u_N||_{L^2(\Omega_d)} \end{aligned}$$

□

We now study the problem 4.7. We will basically have to check that, by employing numerical integration, we are not introducing errors that grow with N. We first check the well-posedness of the problem. From now on, for the quadrature rules to make sense, we assume $f \in C(\overline{\Omega}_d)$.

Lemma 4.5. *We have that, for every $u_N, v_N \in \mathbb{Q}_N(\Omega_d)$*

$$(f, v_N)_N \leq 3^d \; ||I_N f||_{L^2(\Omega_d)}||v_N||_{L^2(\Omega_d)} \tag{4.11}$$

Also, $a_N(\cdot,\cdot)$ is continuous

$$|a(u_N, v_N)| \leq 3^{d-1}|u_N|_{H^1(\Omega_d)}|v_N|_{H^1(\Omega_d)} \tag{4.12}$$

and coercive on $H^1_0(\Omega_d)$

$$a(u_N, v_N) \geq |v_N|^2_{H^1(\Omega_d)} \tag{4.13}$$

Therefore, problem 4.7 has a unique solution. Moreover, the solution is stable

$$||\tilde{u}_N||_{H^1(\Omega_d)} \;\leq\; c||I_N f||_{L^2(\Omega_d)} \tag{4.14}$$

Proof.

We prove the inequalities in order. We assume $d = 2$. The only complication in $d > 2$ is notation.

$$\begin{aligned}
(f, v_N)_N = (I_N f, v_N)_N &= \sum_{j=0}^{N} \sum_{k=0}^{N} I_N f(\eta_j, \eta_k) v_N(\eta_j, \eta_k) \rho_j \rho_k = \sum_{j=0}^{N} \left(\sum_{k=0}^{N} I_N f(\eta_j, \eta_k) v_N(\eta_j, \eta_k) \eta_k \right) \rho_j \\
&\leq \sum_{j=0}^{N} \left(\left(\sum_{k=0}^{N} I_N f(\eta_j, \eta_k)^2 \rho_k \right)^{1/2} \left(\sum_{k=0}^{N} v_N(\eta_j, \eta_k)^2 \rho_k \right)^{1/2} \right) \rho_j \quad \text{(Cauchy - Schwartz inequality)} \\
&\leq 3 \sum_{j=0}^{N} ||I_N f(\eta_j, \cdot)||_{L^2(I)} \; ||v_N(\eta_j, \cdot)||_{L^2(I)} \;\; \rho_j \quad \text{(from 3.9)} \\
&\leq 3^2 ||I_N f||_{L^2(\Omega_d)} ||v_N||_{L^2(\Omega_d)} \quad \text{(Again using Cauchy-Schwarz and 3.9)}
\end{aligned}$$

For the next inequality,consider the first term of $a(u_N, v_N)$, which is $(\partial_x u_N, \partial_x v_N)_N$. Since $\partial_x u$ and $\partial_x v_N$ are of degree $\leq N - 1$ in the variable x, from the exactness property on the variable x we deduce that

$$\begin{aligned}
\sum_{i,j=0}^{N} \partial_x u_N(\eta_i, \eta_j) \; \partial_x v_N(\eta_i, \eta_j) \; \rho_i \rho_j &= \sum_{j=0}^{N} \left(\sum_{i=0}^{N} \partial_x u_N(\eta_i, \eta_j) \; \partial_x v_N(\eta_i, \eta_j) \; \rho_i \right) \rho_j = \sum_{j=0}^{N} \left(\int_{-1}^{1} \partial_x u_N(x, \eta_j) \partial_x v_N(x, \eta_j) \; dx \right) \rho_j \\
&\leq \sum_{j=0}^{N} ||(\partial_x u_N)(\cdot, \eta_j)||_{L^2(I)} \; ||(\partial_x v_N)(\cdot, \eta_j)||_{L^2(I)} \; \rho_j \quad \text{(Holder's Inequality)}
\end{aligned}$$

Now simply apply the Cauchy-Schwarz inequality and then 3.9 like we did in the previous inequality. The term $(\partial_y u_N, \partial_y v_N)_N$ is completely analogous. For the last inequality, consider again only the first term $(\partial_x u_N, \partial_x u_N)_N$. Using again the exactness property

$$\sum_{j=0}^{N} \left(\sum_{i=0}^{N} (\partial_x u_N(\eta_i, \eta_j))^2 \; \rho_i \right) \rho_j = \sum_{j=0}^{N} ||\partial_x u_N(\cdot, \eta_j)||^2_{L^2(I)} \rho_j \geq ||\partial_x u_N||^2_{L^2(\Omega_d)} \quad \text{(from 3.9)}$$

The existence and uniqueness is guaranteed by the Lax-Milgram Lemma. The stability inequality by placing $v_N = \tilde{u}_N$ in 4.7 and the inequalities just proven. □

Note that we are constantly making use of the geometry of Ω_d to compute the integrals. Also, observe that the continuity constant and coercivity constant do not depend on N

We now use an adaptation of Strang's First lemma (see **[ern2004theory]**)

Lemma 4.6 (Strang)**.**
Let u be the solution of the variational problem 7.2. And consider the following discretization of said problem

$$\begin{cases} \text{find} \;\; u_\delta \;\; \text{such that} \\ \forall \, v_\delta \in H_\delta, \quad a_\delta(u_\delta, v_\delta) = F_\delta(v_\delta) \end{cases} \tag{4.15}$$

where $H_\delta \subset H$ is a finite dimensional subspace, and a_δ is a bilinear form defined on H_δ and F_δ is a linear form defined on H_δ. Suppose that a_δ is coercive and continuous on $H_\delta \subset H$

$$\forall u_\delta, v_\delta \in H_\delta \quad a_\delta(u_\delta, u_\delta) \geq \alpha_\delta^2 ||u_\delta||_H^2 \quad \text{and} \quad |a_\delta(u_\delta, v_\delta)| \leq C_\delta ||u_\delta||_H ||v_\delta||_H$$

Then we have the following estimate

$$\begin{aligned} ||u - u_\delta||_H \leq & \frac{1}{\alpha_\delta} \sup_{w_\delta \in H_\delta} \frac{|F(w_\delta) - F_\delta(w_\delta)|}{||w_\delta||_H} \\ & + \inf_{w_\delta \in H_\delta} \left[\left(1 + \frac{C_\delta}{\alpha_\delta}\right) ||u - w_\delta||_H + \frac{1}{\alpha_\delta} \sup_{v_\delta \in H_\delta} \frac{|a(w_\delta, v_\delta) - a_\delta(w_\delta, v_\delta)|}{||v_\delta||_H} \right] \end{aligned} \tag{4.16}$$

Theorem 4.7 (Sepctral - NI convergence). *Suppose $f \in H^\mu(\Omega_d)$ with $\mu > d/2$ and that u, the solution of 4.2 is in $H^s(\Omega_d)$, for $s \geq 1$. Then we have the following estimate*

$$||u - \tilde{u}_N||_{H^1(\Omega_d)} \leq C \left(N^{1-s} ||u||_{H^s(\Omega_d)} + N^{-\mu} ||f||_{H^\mu(\Omega_d)} \right) \tag{4.17}$$

where $C = C(s, \mu)$

Proof. We simply have to make a smart choice of terms in Strang's Lemma. Since the ellipticity and continuity constant from 4.5 are independent of N, we deduce that

$$\begin{aligned} ||u - \tilde{u}_N||_{H^1(\Omega_d)} \leq C \Bigg(& ||u - v_N||_{H^1(\Omega_d)} + \sup_{w_N \in \mathbb{Q}_N^0(\Omega_d)} \frac{|a(v_N, w_N) - a_N(v_N, w_N)|}{||w_N||_{H^1(\Omega_d)}} \\ & + \sup_{w_N \in \mathbb{Q}_N^0(\Omega_d)} \frac{|(f, w_N) - (f, w_N)_N|}{||w_N||_{H^1(\Omega_d)}} \Bigg) \end{aligned}$$

For any $v_N \in \mathbb{Q}_N^0(\Omega_d)$.

Now the key step is to choose $v_N \in \mathbb{Q}_{N-1}^0(\Omega_d)$, since the exactness property of the quadrature implies that

$$\forall w_N \in \mathbb{Q}_N^0(\Omega_d),\ \forall v_N \in \mathbb{Q}_{N-1}^0(\Omega_d) \quad a_N(v_N, w_N) = a_N(v_N, w_N)$$

So the second term vanishes and so we take $v_N = \Pi_{N-1}^{1,0} u$.
We now deal with the third term. Let $w_N \in \mathbb{Q}_N(\Omega_d)$, from a similar argument as before, $(\Pi_{N-1} f, w_N) = (\Pi_{N-1} f, w_N)_N$, so now

$$\begin{aligned} (f, w_N) - (f, w_N)_N &= (f - \Pi_{N-1} f, w_N) + (\Pi_{N-1} f, w_N) - (I_N f, w_N)_N = (f - \Pi_{N-1} f, w_N) - (f - \Pi_{N-1} f, w_N)_N \\ &\leq \left(|||f - \Pi_{N-1} f||_{L^2(\Omega_d)} + 3^d ||I_N f - \Pi_{N-1} f||_{L^2(\Omega_d)} \right) ||w_N||_{L^2(\Omega_d} \quad \text{(Using 3.9)} \end{aligned}$$

Using the triangle inequality again, we obtain that

$$\sup_{w_N \in \mathbb{Q}_N^0(\Omega_d)} \frac{|(f, w_N) - (f, w_N)_N|}{||w_N||_{H^1(\Omega_d)}} \leq C \left(||f - \Pi_{N-1} f||_{L^2(\Omega_d)} + ||f - I_N f||_{L^2(\Omega_d)} \right)$$

So we finally obtain

$$||u - \widetilde{u}_N||_{H^1(\Omega_d)} \leq C \left(|u - \Pi^{1,0}_{N-1} u| + ||f - \Pi_{n-1} f||_{L^2(\Omega_d)} + ||f - I_n f||_{L^2(\Omega_d)} \right)$$

and now simply apply Theorem 2.14, 2.15 and 3.18. □

Chapter 5

Implementation and Numerical examples

We briefly describe some possible implementations. First of all, the weights and nodes $\{\zeta_j, w_j\}$ and $\{\eta_j, \rho_j\}$ can be computed in terms of the eigenvectors and eigenvalues of 1.17 and 1.28, as explained in Section 1.3. Since the matrix is sparse and symmetric, this can be done efficiently with an iterative algorithm.As a first example, observe in Figure 5.1 how the LGL nodes avoid the classic Runge's counterexample.

To implement 4.7 we take into account the boundary condition and write the solution in the Lagrange basis

$$\begin{cases} \sum_{i,j=1}^{N-1} u_N(\eta_i, \eta_j)\; l_i(x_i) l_j(x_j) \quad \text{when} \;\; d = 2 \\ \sum_{i,j,k=1}^{N-1} u_N(\eta_i, \eta_j, \eta_k)\; l_i(x_i)\; l_j(x_j)\; l_k(x_k) \quad \text{when} \;\; d = 3 \end{cases} \tag{5.1}$$

and we solve for $u_N(\eta_i, \eta_j)$ (or $u_N(\eta_i, \eta_j, \eta_k)$). So the unknown is the vector U with $(N-1)^d$ components, and its components are $u(x)$ with $x \in \overset{\circ}{\Omega}_N$. Also, we denote by F the vector whose components are $f(x)$ with $x \in \overset{\circ}{\Omega}_N$. So if let the test function in 4.7 run through the Lagrange basis associated with the interior nodes we obtain that we can express 4.7 equivalently as

$$AU = MF \tag{5.2}$$

The matrix M its diagonal, and its components are $\rho_i \rho_j$ if $d = 2$ and $\rho_i \rho_j \rho_k$ if $d = 3$. A is the stiffness matrix,

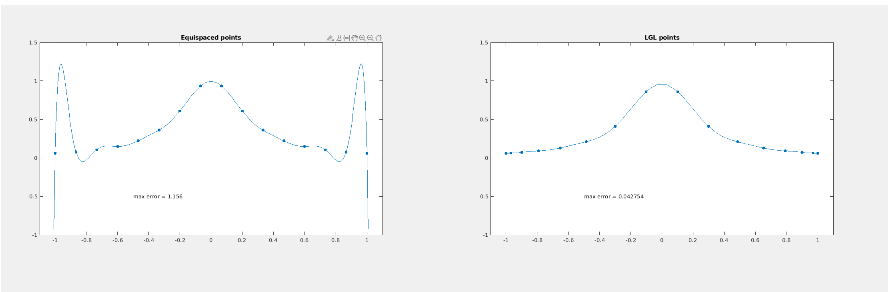

Figure 5.1: Equispaced nodes (left) v.s. LGL nodes (right)

and its components are

$$a_N(\ l_i l_j\ ,\ l_{i'} l_{j'}\ ) \quad \text{if } d = 2 \quad \text{and} \quad a_N(\ l_i l_j l_k\ ,\ l_{i'} l_{j'} l_{k'}\ ) \quad \text{if } d = 3$$

. To compute the mass matrix A we shall make some observations. The first one is how to compute the l_i'. The following lemma is from **[trefethen2000spectral]**.

Lemma 5.1. *If $x_0, \dots, x_m$ are distinct, we know the j-th Lagrange interpolant is*

$$p_j(x) = \frac{1}{a_j} \prod_{k=0,\ k\neq j}^{m} (x - x_k) \quad \textit{where} \quad a_j = \prod_{k=0,\ k\neq j}^{m} (x_j - x_k)$$

Then (taking logarithms and differentiating)

$$p_j'(x_j) = \sum_{k=0,\ k\neq j}^{m} (x_j - x_k)^{-1} \tag{5.3}$$

$$p_j'(x_i) = \frac{a_i}{a_j(x_i - x_j)} \quad \textit{when } i \neq j \tag{5.4}$$

Secondly, the matrix A is symmetric positive definite (from 4.5) so we should use linear solvers that take advantage of this (for example the conjugate gradient method). Moreover, we observe that when $d = 2$

$$a_N(\ l_i l_j\ ,\ l_{i'} l_{j'}\ ) = \alpha_{ii'}\ \delta_{j,j'}\ \rho_j + \alpha_{jj'}\ \delta_{ii'}\ \rho_i$$

where $\quad \alpha_{ii'} = \sum_{k=0}^{N}\ l_i'(\eta_k) l_{i'}'\ (\eta_k)\rho_i$

And if $d = 3$

$$a_N(\ l_i l_j l_k\ ,\ l_{i'} l_{j'} l_{k'}) = \ \alpha_{ii'}\ \delta_{j,j'}\ \delta_{k,k'}\ \rho_j\ \rho_k\ +\ \alpha_{jj'}\ \delta_{ii'}\ \delta_{kk'}\ \rho_i \rho_k\ +\ \alpha_{kk'}\ \delta_{ii'}\ \delta_{jj'}\ \rho_i \rho_j$$

This allows a reduction in the computation of the matrix-vector product Ab

Lemma 5.2. *The product Ab can be computed with $\mathcal{O}(N^{d+1})$ operations instead of $\mathcal{O}(N^{2d})$*

Proof. We show it for $d = 2$. The coefficient ij of Ab is given by

$$\sum_{i',\ j'=1}^{N-1} a_{ij,\ i'j'}\ b_{i'j'} = \sum_{i'=1}^{N-1} \alpha_{ii'}\ \rho_j\ b_{i'j} + \sum_{j'=1}^{N-1} \alpha_{jj'}\ \rho_i\ b_{ij'}$$

and this requires $4(N-1)$ multiplications and $2N-3$ additions. $\square$

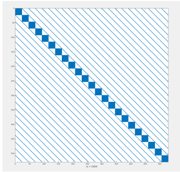

Figure 5.2: Structure of the matrix in the collocation method

Finally, we mention out that the matrix A is dense and somewhat ill-possed. It can be proven that the condition number of A is $\kappa(A) = \mathcal{O}(N^3)$. This could not be an issue since convergence is fast. However, if a preconditioner is used, then the condition number of A can be uniformly bounded with respect to end by a constant, so solving the linear system with this preconditioner is very fast. The preconditioner used is the Finite-element stiffness matrices constructed on piecewise linear affine shape functions centered at LGL nodes. We refer to **[canuto2007spectral2]** for a more detailed explanation.
Another equivalent formulation to 4.7 is the collocation formulation. For simplicity, let $d = 2$.

Take $u_N \in \mathbb{Q}_N(\Omega_d)$ and $v_N \in \mathbb{Q}_N^0(\Omega_d)$, by the exactness property in the x variable and integrating by parts,

$$\sum_{j=0}^{N} \partial_x u_N(\eta_j, \eta_i)\partial_x v_N(\eta_j, \eta_i)\rho_j = \int_{-1}^{1} \partial_x u_N(x, \eta_i)\partial_x v_N(x, \eta_i)\, dx = -\int_{-1}^{1} \partial_x^2 u_N(x, \eta_i) v_N(x, \eta_i)\, dx$$

So we deduce that

$$a_N(u_N, v_N) = (-\Delta u_N, v_N)_N \tag{5.5}$$

So again, by letting v_N run through the Lagrange basis $l_{i_1} l_{i_2}$ with $1 \le i_1, i_d \le N-1$ we obtain that

$$-\Delta u(\eta_{i_1}, \eta_{i_2}) = f(\eta_{i_1}, \eta_{i_2}) \tag{5.6}$$

so $-\Delta u_N$ conicides with f at the $(N-1)^2$ interior nodes $\overset{\circ}{\Omega}$. And, since the polynomials $l_{i_1} l_{i_2}$ with $1 \le i_1, i_d \le N-1$ form a basis of $\mathbb{Q}_N^0(\Omega_d)$, we see that 5.5 and 5.6 imply $a(u_N, v_N)_N = (f, v_N)$ for all $v_N \in \mathbb{Q}_N^0(\Omega_d)$.

Now, if u_N vanishes at the boundary nodes $\partial\Omega_N$, then it means that it vanishes along the $(N+1)$ points on each edge of $\partial\Omega_d$, and since v_N restricted to each edge is a degree N polynomial in one variable, we deduce that $v_N = 0$ along the edges. In other words $v_N(x) = 0 \ \ \forall\, x \in \partial\Omega_N \quad \Longrightarrow \quad v_N(x) = 0 \ \ \forall\, x \in \partial\Omega_d$ So we have deduced the following equivallent "collocation" formulation

$$\begin{cases} \text{find } u_N \in \mathbb{Q}_N(\Omega_d) \text{ such that} \\ -\Delta u_N(x) = f(x) \quad x \in \Omega_N \\ u_N(x) = 0 \quad x \in \partial\Omega_N \end{cases} \tag{5.7}$$

To implement this, we have followed the collocation method implementation explained on **[trefethen2000spectral]** with its use of tensor products of matrices. We have adapted it to LGL nodes and constructed a differentiation matrix based on 5.1.

Chapter 6

Some extensions to Complicated Geometries

To deal with PDEs in domains $\Omega \subset \mathcal{R}^d$ $(d = 2, 3)$ with more complicated geometries, some extensions exist, and we briefly mention two.

The Spectral Element Method :
In the Spectral Element Method we have a mesh $\mathcal{T} = \{\Omega_m\}_m$

$$\Omega = \bigcup_m \overline{\Omega}_m \quad \text{and} \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{when } i \neq j \tag{6.1}$$

We assume that all the Ω_m (which are called elements) are affine images of our reference domain, the hypercube Ω_d. However, the elements can also be affine images of triangles to handle with complicated boundaries. In this method we also have that the partition $\mathcal{T}$ is conforming, that is for any i, j , $\overline{\Omega_i} \cap \overline{\Omega_j}$ is either empty or a whole edge or a whole face. Our approximation space is a space of piecewise polynomials:

$$X_N = \{\varphi \in C(\overline{\Omega}) \ : \ \varphi|_{\Omega_m} \in \mathbb{Q}_N(\Omega_m)\}$$

Now, the Lobatto-Gauss-Legendre nodes in Ω_m are denoted by η_α^m , where $\alpha = (i_1, \ldots, i_d)$ $1 \leq$ with $i_j \leq N$. In X_N we consider the basis formed by the following basis functions:
If η_α^m is a interior node in Ω_m or its a point in the boundary $\partial\Omega$, and $l_\alpha^{(m)}$ is its associated Lagrange polynomial in $\mathbb{Q}_N(\Omega_m)$ then we define

$$\psi = \begin{cases} l_\alpha^{(m)}(x) & \text{if } x \in \overline{\Omega}_m \\ 0 & \text{elsewhere} \end{cases} \tag{6.2}$$

and its clear that $\psi(x) \in X_N$. If $\eta_\alpha^m \in \Omega$ is a node in the interior of a edge (or face if $d = 3$) of Ω_m then since the domain is conformal, η_α^m is also the node in the interior of an edge for another element, say Ω_n and $\eta_\alpha^m = \eta_{\alpha'}^n$. Then we define

$$\psi(x) = \begin{cases} l_\alpha^{(m)}(x) & \text{if } x \in \overline{\Omega}_m \\ l_{\alpha'}^{(n)}(x) & \text{if } x \in \overline{\Omega}_n \\ 0 & \text{elsewhere} \end{cases} \tag{6.3}$$

And from the conformality assumption we deduce that $\psi \in X_N$. Finally, if η_α^m is a node that is shared between many domains, for a example a corner node in $d = 2$ or an edge node in $d = 3$, then the basis function associated with that node is defined analogously, but taking into account all of the domains that share that node. Convergence in the spectral element can be obtained by taking $N \longrightarrow \infty$ or by refining the mesh ($h \longrightarrow 0$), that is, by considering meshes formed by more and smaller elements. The classical reference is **[karniadakis2005spectral]**.

The Mortar Element Method :
Consider the same setting as before were we have a mesh such as 6.1, but this time, we don't assume that it is conformal and we allow for different domains to have different polynomial degrees. To simplify matters, assume $d = 2$. Denote by $\Gamma_{k,l}$, $1 \leq l \leq L_k$ the edges of Ω_k that are not in $\partial\Omega$. We define as $S = \cup_k \partial\Omega_k / \Omega$ the skeleton of the decomposition. We can define S as the union of elementary components called mortars

$$S = \cup_{j=1}^{M} \overline{\gamma_j} \quad \text{with} \quad \gamma_j \cap \gamma_k = \emptyset \ \text{ if } \ k \neq j \tag{6.4}$$

where each mortar γ_j is a whole edge of a specific element denoted by $\Omega_{m(j)}$. And this specific edge is then denoted by $\Gamma_{k(j),m(j)}$. We emphasize that each mortar γ_j is related to a specific edge of a specific element Ω_k. So even if two distinct domains $\Omega_{m(j)}$ and Ω_l share an edge, say e, the mortar γ_j will be associated to the edge "on the side" of $\Omega_{m(j)}$.

On each subdomain Ω_k we look for a discrete solution belongs to $\mathbb{Q}_{N_k}(\Omega_k)$. We also denote by $W_\delta^{(k),(l)} = \mathbb{Q}_{N_k}(\Gamma_{k,l})$ the space of traces of $Y_{k,\delta}$ on $\Gamma_{k,l}$. Finally we define $\widetilde{W}_\delta^{(k),(l)} = \mathbb{Q}_{N_k - 2}(\Gamma_{k,l})$ Now, our approximation space X_δ is defined by the space of functions v_δ such that

- $v_\delta|_{\Omega_k} \in \mathbb{Q}_{N_k}(\Omega_k)$
- they vanish on $\partial\Omega$
- They satisfy the mortar conditions : let φ be the mortar function associated with v_δ , that is, the function that on each $\gamma_j = \Gamma_{k(j),m(j)}$, φ coincides with the restriction to γ_j of $v_\delta|_{\Omega_k}$; then for every $\Gamma_{k,l}$ that is contained in S but is not one of the mortars (so $\Gamma_{k,l} \neq \Gamma_{k(m),l(m)}$ for any m) we have the following matching condition
$$\forall \ \psi \in \widetilde{W}_\delta^{(k),(l)} \qquad \int_{\Gamma_{k,l}} \left(v|_{\Omega_k} - \varphi)\right)(x) \ \psi(x) \ dx = 0$$

The mortar element method can also be coupled with triangular "mortar elements". For many more details, see **[bernardi2005basics]**.

Chapter 7

Appendix

7.1 Polynomial Interpolation

Given a set of distinct nodes $\{\xi_0, \ldots, \xi_N\} \subset \mathbb{R}$, and a set of values $\{f_0, \ldots, f_N\} \subset \mathbb{R}$, our goal is to construct a polynomial p such that

$$p(\xi_i) = f_i \quad \forall i = 1, \ldots, N$$

Definition 7.1 (Lagrange polynomials)**.** *Let $\{\xi_0, \ldots, \xi_N\} \subset \mathbb{R}$. For $1 \leq i \leq N$, we define the i-th Lagrange polynomial associated to the nodes $\{\xi_0, \ldots, \xi_N\}$*

$$l_i(s) = \prod_{\substack{j=1 \\ j \neq i}}^{N} \frac{s - \xi_j}{\xi_i - \xi_j}$$

Observe that l_i has degree N and

$$l_i(\xi_j) = \delta_{ij} \tag{7.1}$$

Theorem 7.2. *Given a set of nodes $\{\xi_0, \ldots, \xi_N\} \subset \mathbb{R}$, and a set of values $\{f_0, \ldots, f_N\} \subset \mathbb{R}$, there exists a unique polynomial $p \in \mathcal{P}_N$ such that*

$$p(\xi_i) = f_i \quad \forall i = 0, 1, \ldots, N$$

We say that p is the interpolant for the given nodes and values.

Proof. Write

$$p(s) = \sum_{i=1}^{N} f_i \cdot l_i(s)$$

Its an immediate computation using 7.1 that $p(\xi_i) = f_i$. Suppose that there exists another polynomial $p^* \in \mathcal{P}_N$ such that $p^*(\xi_i) = f_i$, then $q = p - p^*$ is a polynomial of degree $\leq N$ that has $N+1$ distinct roots $\{\xi_0, \ldots, \xi_N\}$. So $q = 0$ and $p = p^*$ $\square$

Similarly, we can prove

Theorem 7.3 (Hermite Interpolation)**.** *Suppose we have a set of nodes $\{\xi_1, \ldots, \xi_m\}$, a set non-negative integers $\{r_1, \ldots, r_m\}$ such that $n = \sum_{i=1}^m r_i$ and a set of values $\{f_i^{k_i}\}$ for $i = 1, \ldots, m$ and $k_i = 0, \ldots, r_i - 1$. Then there exists a unique polynomial p of degree $\leq n-1$ such that*

$$p^{(k_i)}(\xi_i) = f_i^{k_i} \quad \forall i = 1, \ldots, m \; \forall k_i = 0, \ldots, r_i - 1$$

Where the superscript (k_i) denotes the $k_i - th$ derivative.

7.2 Coercivity and Lax-Milgram lemma

Recall our variational problem 2. We'd like to prove the existence and uniqueness of such problem and a more general class of problems. Four our purposes, we only need the symmetric form of the Lax-Milgram theorem. The general setting is the following :

- $(H, (\cdot, \cdot))$ is a Hilbert space
- V is a closed subspace of H
- $a(\cdot, \cdot)$ is a symmetric bilinear form defined on H
- $F \in V'$ (F is a continuous linear functional on V)

We would like to prove the existence and uniqueness of the problem

$$\begin{cases} \text{find } u \in V \text{ such that :} \\ a(u, v) = F(v) \quad \text{for all } v \in V \end{cases} \tag{7.2}$$

We first need a well-known result.

Lemma 7.4 (Riesz Representation Theorem)**.**
Let L be a continuous linear functional defined on a Hilbert space H.

Then, there exists a unique $u \in V$ such that

$$L(v) = (u, v) \quad \textit{for all } v \in V$$

Furthermore, $||L||_{H'} = ||u||_H$ (where $|| \cdot ||_{H'}$ is the operator norm on H'). So we have an isometry between H and H'

If we could treat $a(\cdot, \cdot)$ as a inner product on V, then we could apply the Riesz Representation theorem on $a(\cdot, \cdot)$ as a way to solve 7.2.

Definition 7.5. *We say that a bilinear form $a(\cdot,\cdot)$ defined on a normed vector space X is continuous if there exists a constant $C < \infty$ such that*

$$|a(u,v)| \leq C||v||_X||u||_X \quad \forall u,v \in X$$

and it is called coercive on $V \subseteq X$ if there exists a constant $\alpha > 0$ such that

$$a(v,v) \geq \alpha||v||_X^2 \quad \forall v \in V$$

C is usually called the continuity constant and α the coercivity constant.

Proposition 7.6. *Let H be a Hilbert space and $a(\cdot,\cdot)$ a symmetric linear form continuous on H and coercive on a closed subspace $V \subseteq H$. Then $(V, a(\cdot,\cdot))$ is a Hilbert space.*

Proof.
First we see that $a(\cdot,\cdot)$ is an inner product because, owing to the coercivity,

$$a(u,u) \geq 0 \quad \text{and} \quad a(u,u) = 0 \Rightarrow ||u||_H = 0 \Rightarrow u = 0$$

Now we see that V with with the associated norm $||v|||_a = \sqrt{a(v,v)}$ is a Banach space. We only need to check that it's complete.

Suppose that $\{v_n\}_n$ is a Cauchy sequence on $(V, ||\cdot||_a)$. By the coercivity assumption, $\{v_n\}_n$ is also a Cauchy sequence on H

$$||v_n - v_m||_a \geq \alpha||v_n - v_m||_H$$

So there exists $v \in H$ such that $v_n \to v$ in H. Since V is closed, $v \in H$. Now, by continuity,

$$||v - v_n||_a \leq C||v - v_n||_H$$

so $v_n \to v$ also in $(V, ||\cdot||_a)$ and therefore this space is complete. □

Theorem 7.7 (Lax-Milgram Lemma)**.** *Suppose that the assumptions from the general setting hold and that the bilinear form $a(\cdot,\cdot)$ is continuous and coercive. Then the problem 7.2 has a unique solution.*

Proof. The last proposition implies that $(V, a(\cdot,\cdot))$ is a Hilbert space, and the coercivity that F is also a continuous linear functional on $(V, a(\cdot,\cdot))$. Now apply the Riesz Representation theorem on $(V, a(\cdot,\cdot))$ to find that 7.2 has a unique solution. □

Keeping the assumptions from the same general setting as before, consider now the discrete problem of finding

$$\begin{cases} \text{find } u_h \in V \text{ such that :} \\ a(u_h, v_h) = F(v_h) \quad \text{for all } v_h \in V_h \end{cases} \tag{7.3}$$

where $V_h \subseteq V$ is a finite-dimensional vector space. The following says that the problem above is well-posed if a is coercive.

Corollary 7.8. *Suppose that $a(\cdot, \cdot)$ is coercive and continuous on V and that the assumptions of the general setting hold. Then the discrete problem 7.3 has a unique solution.*

Proof. V_h being finite dimensional implies that it is also closed. Since $a(\cdot, \cdot)$ is coercive and continuous on V, then it is also coercive on V_h. So we can directly apply the Lax-Milgram lemma on 7.3. □

The general (non-symmetric) Lax-Milgram lemma is not difficult to prove (see **[brenner2008mathematical]**. A more general statement than the Lax-Milgram lemma is the BNB (Banach - Nečas - Babuška) theorem (see **[ern2004theory]**).

Proposition 7.9 (Galerkin orthogonal relation and Cea's Theorem).
Suppose that u is the solution of problem 7.2 and u_h is the solution of the discrete problem 7.3. Then, for every $v_h \in V_h$,

$$a(u - u_h, v_h) = 0 \tag{7.4}$$

And so

$$||u - u_h||_a = \min_{v_h \in V_h} ||u - v_h||_a \tag{7.5}$$

Moreover if $a(\cdot, \cdot)$ is coercive and continuous, then the following estimate holds

$$||u - u_h||_H \leq \frac{C}{\alpha} \min_{v_h \in V_h} ||u - v_h||_H \tag{7.6}$$

Proof.
If u and u_h are, respectively, the solutions of 7.2 and 7.3 then, for all $v_h \in V_h$

$$a(u, v_h) = F(v_h)$$
$$a(u_h, v_h) = F(v_h)$$

From here we deduce that $a(u - u_h, v_h) = 0$

Inequality 7.5 now follows from the orthogonal relation, take $v_h \in V_h$

$$\begin{aligned}||u - u_h||_a^2 &= a(u - u_h, u - u_h) = a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \\ &= a(u - u_h, u - v_h) \leq ||u - u_h||_a ||u - v_h||_a\end{aligned}$$

So $||u - u_h||_a \leq ||u - v_h||_a$.

Now we prove inequality 7.6. Take any $v_h \in V_h$

$$\begin{aligned}\alpha ||u - u_h||_H^2 &\leq a(u - u_h, u - u_h) \\ &= a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h) \\ &= a(u - u_h, u - v_h) \quad (\text{since } v_h - u_h \in V_h) \\ &\leq C||u - u_h||_H ||u - v_h||_H\end{aligned}$$

□

We will make constant use of 7.4.

7.3 Interpolation spaces

Interpolation spaces allow us to define Hilbert spaces " in between " two other Hilbert spaces. We won't be too rigorous defining some concepts and we refer to **[lions2012non]** to a more detailed treatment of this concepts and the proofs. Later on, we will use this results to generalize inequalities to these " in between " spaces.

Suppose X and Y are two Hilbert spaces. We denote by $||\cdot||_X$ and $||\cdot||_Y$ their respective norms. We say that a Hilbert space is separable if it contains a dense countable subset (every Hilbert space that is mentioned in this text is separable). We say that X is continuously or compactly embedded in Y if there exists a constant $C < \infty$ such that $||v||_Y \leq C||v||_X$ for all $v \in X$.

Suppose that X and Y are two separable Hilbert spaces, that X is dense and continuously embedded in Y. Then, for each $0 \leq \theta \leq 1$ its possible to rigorously define an "Interpolation space" denoted $[X,Y]_\theta$ and give it a norm $||\cdot||_{[X,Y]_\theta}$.
We have that, for $0 < \theta < \varphi < 1$

$$X = [X,Y]_0 \subset [X,Y]_\theta \subset [X,Y]_\varphi \subset [X,Y]_1 = Y \tag{7.7}$$

So θ measures " how much is $[X,Y]_\theta$ in between X and Y ". Moreover, for any $0 \leq \theta \leq 1$,

$$\forall v \in X \quad ||v||_{[X,Y]_\theta} \leq ||v||_X^{1-\theta}\,||v||_Y^\theta \tag{7.8}$$

The main results of these section concerning to our applications are 7.10 and 7.12

Theorem 7.10. *Let X and Y (respectively X^* and Y^*) be separable Hilbert spaces such that X (resp. X^*) is continuously embedded and dense in Y (resp. Y^*). If a linear operator L is continuous from X into X^* with norm α and from Y into Y^* with norm β, then L is continuous from $[X,Y]_\theta$ into $[X^*,Y^*]_\theta$ with norm $\leq \alpha^{1-\theta}\beta^\theta$*

For a example application of this results, see the proof of Theorem 2.1 or Corollary 7.13.

7.4 Fractional Order Sobolev spaces

We recall the definition of $H^s(\Omega)$ when s is not an integer.

Definition 7.11 (Fractional order Sobolev spaces). *For $0 < s < 1$, the Sobolev space $H^s(\Omega)$ is defined as*

$$H^s(\Omega) = \{u \in L^2(\Omega) : \frac{u(x)-u(y)}{||x-y||^{s+d/2}} \in L^2(\Omega \times \Omega)\}$$

and we define the norm

$$||u|||_{H^s(\Omega)} = \left(\int_\Omega u^2\,dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\,dxdy\right)^{1/2}$$

When $s = m + \sigma$,m a positive integer and $0 < \sigma < 1$, we define

$$H^s(\Omega) = \{u \in H^m(\Omega) : \partial^\alpha u \in H^\sigma(\Omega) \quad \forall |\alpha| = k\}$$

and

$$||u||_{H^s(\Omega} = \left(||u||^2_{H^m(\Omega)} + \sum_{|\alpha|=k} ||\partial^\alpha u||^2_{H^\sigma(\Omega} \right)^{1/2}$$

We also write $H^0(\Omega) = L^2(\Omega)$

Theorem 7.12. *For any $0 \leq s \leq r$, and for any $0 < \theta < 1$, the following equality is true*

$$[H^r(\Omega), H^s(\Omega)]_\theta = H^{(1-\theta)r+\theta s}(\Omega) \tag{7.9}$$

Moreover, both spaces have equivalent norms.

As an example of an application, we have

Corollary 7.13. *There exists a constant C such that for all $u \in H^2(\Omega)$*

$$||u||_{H^1(\Omega)} \leq C||u||^{1/2}_{H^2(\Omega)}||u||^{1/2}_{L^2(\Omega)} \tag{7.10}$$

And $\forall \epsilon > 0$, there exists $C(\epsilon)$ such that

$$||Du||_{L^2(\Omega)} \leq \epsilon||D^2u||_{L^2(\Omega} + C(\epsilon)||u||_{L^2(\Omega} \tag{7.11}$$

Proof. For the first inequality simply apply 7.10 with $\theta = 1/2$ and 7.12.
For the second one, use the first inequality and Cauchy's inequality (for real numbers) $2ab \leq \frac{a^2}{\epsilon^2} + \epsilon^2 b^2$ □

Bibliography