# Basics of Signals and Systems

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## Random Variables and Random Processes

### Joint Probability distribution

Let $X$ and $Y$ be two random variables. The two probability distribution functions - the joint cummulative probability distribution and the joint probability density function can be defined on these two random variables. The joint cummulative probabililty distribution function is given by

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$$

Similarly, the joint probability desity function is given by

$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y}$$

$$\text{also,} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x, y)\, dx\, dy = 1$$

### Marginal Distribution functions

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{x} f_{X,Y}(\varepsilon, \eta)\, d\varepsilon\, d\eta = F_X(x)$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{y} f_{X,Y}(\varepsilon, \eta)\, d\varepsilon\, d\eta = F_Y(y)$$

$F_X(x)$ and $F_Y(y)$ are called marginal distributions functions of $X$ and $Y$.

### Marginal Density functions

$$\int_{-\infty}^{+\infty} f_{X,Y}(x,\eta)\,d\eta = f_X(x)$$

$$\int_{-\infty}^{+\infty} f_{X,Y}(\varepsilon,y)\,d\varepsilon = F_Y(y)$$

$f_X(x)$ and $f_Y(y)$ are called marginal density functions of $X$ and $Y$.

### Conditional probabibility density functions

$$f_Y(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_X(x/y) = \frac{f_{X,Y}(x,y)}{f_y(y)}$$

$$\int_{-\infty}^{+\infty} f_Y(y/x)\,dy = \int_{-\infty}^{+\infty} f_X(x/y)\,dx = 1$$

## Statistical Averages

### Mean

Mean of a random variable $X$ given by $\mu_x$ or expection of $X$, $E[X]$ is computed as

$$\mu_x = E[X] = \int_{-\infty}^{+\infty} x.f_X(x)\,dx$$

If, $Y = g(X)$, then

$$E[Y] = \int_{-\infty}^{+\infty} g(x).f_X(x)\,dx$$

## Moments of a random variable

**Moments about origin:**

$$E[X^n] = \int_{-\infty}^{+\infty} x^n . f_X(x)\, dx$$

**Moments about mean:**

$$E[(X - \mu_X)^n] = \int_{-\infty}^{+\infty} (x - \mu_x)^n . f_X(x)\, dx$$

The second moment about mean is called variance represented as $\sigma_x^2$ or $var[X]$ and defined as

$$\sigma_x^2 = E[(X - \mu_X)^2] = \int_{-\infty}^{+\infty} (x - \mu_x)^2 . f_X(x)\, dx$$

$\sigma_x$ is called the standard deviation of the variable which is the measure of variable's randomness. A relation between variance and mean of a random variable can be developed as follows

$$\begin{aligned} \sigma_x^2 &= E[(X - \mu_X)^2] \\ &= E[X^2] - 2\mu_x E[X] + E[\mu_x^2] \\ &= E[X^2] - 2\mu_x^2 + \mu_x^2 \\ &= E[X^2] - \mu_x^2 \end{aligned}$$

## Correlation of Random Variables

$$\begin{aligned} covariance[X, Y] &= E[(X - E[X])(Y - E[Y])] \\ &= E[XY] - E[X]E[Y] - E[Y]E[X] + E[X]E[Y] \\ &= E[XY] - \mu_x.\mu_y \end{aligned}$$

From the above relation we can see that when the two random variables are statistically independent then $covariance[X, Y]$ becomes zero.