Homotopy Languages

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Abstract

We attach to each weak model category $\mathcal{M}$ a class of first order formulas about the fibrant objects of $\mathcal{M}$ whose validity is invariant under homotopies and weak equivalences. This is a generalization of the classical Blanc-Freyd Language of categories - which involves formula avoiding equality on objects and which are invariant under isomorphism and equivalences of categories. In particular, we obtain similar homotopy invariant languages for 2-categories, bicategories, chain complexes, Kan complexes, quasi-categories, Segal spaces, and so on...

Terminology: Acyclic vs trivial

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1 Introduction

It is a well-known result in category theory ([Fre76], [Bla78]) that any property of a category or of objects and morphisms in a category that does not use equality between objects is automatically invariant under equivalence of categories, and by replacing the objects and morphisms involved by (consistently) isomorphic.

For example, because the notion of limit in a category can be formulated without using equality between objects we automatically deduce that equivalence of categories preserves limits, or that if two diagrams are isomorphic then a limit for one is also a limit for the other.

To be precise, the above refers to first-order formula in which we can have quantifiers over all objects of the category, or over all morphisms in a given hom-set "hom(X,Y)", and we can use equality between two terms of type hom(X,Y), but not between two terms of type "objects", or two terms that are in different hom-set.

For example, the property of an object X to be a terminal object, which can be written as

$$\texttt{isTerminal}(X) \coloneqq \forall y \in \mathrm{Ob}, (\exists v \in \mathrm{Hom}(y,X) \text{ and } \forall u,w \in \mathrm{Hom}(y,X), u = w)$$

is such a formula, but the following formula

$$\begin{aligned}\forall X, Y \in \mathrm{Ob}, \forall f \in \mathrm{Hom}(X,Y), \forall g \in \mathrm{Hom}(Y,X),&\\ (f \circ g = \mathsf{id}_Y \text{ and } g \circ f = \mathsf{id}_X \Rightarrow X = Y)&\end{aligned}$$

which say that the category we are working with is skeletal, or the formula

$$\begin{aligned}\forall X, Y \in \mathrm{Ob}, \forall f \in \mathrm{Hom}(X,Y), \forall g \in \mathrm{Hom}(Y,X),&\\ (f \circ g = \mathsf{id}_Y \text{ and } g \circ f = \mathsf{id}_X \Rightarrow f = \mathsf{id}_X)&\end{aligned}$$

are not of this form, and they are indeed not invariant under equivalence of categories.

Note that in order for this to make sense, it is key to use a notion of "dependent type". Indeed, we need to be able to formulate the idea that a morphism f is in $\mathrm{Hom}(X,Y)$, without being able to say that $s(f) = X$ and $t(f) = Y$ as this would involve using equality between objects. So, given two objects X and Y, we need to be able to consider the type of arrows from X to Y as a primitive notion.

Now, it is natural to expect that similar results can be generalized to higher categories. For example, we expect (and it can be shown) that a property of 2-categories or bicategories that does not use equality between objects or between 1-arrows will also be invariant under biequivalence. One can also expect it can be generalized to other sort of higher structures, for example a result about multicategories not using equality between objects should also have similar good invariance properties.

The main goal of this paper is, informally, to establish a version of this result for essentially any kind of higher structures, independently of the type of structure or the "categoricity level". The only requirement is that the sort of higher structure we are considering must be organised into the fibrant objects of a model category (or semi-model category, or weak model category).

That is, we will attach to every (semi/weak) model category a "first-order language", whose formulas are statement about objects of the cateory (possibly with parameters) such that

- Replacing the value of the parameters by homotopically equivalent parameters does not change the validity of a formula.
- Two weakly equivalent fibrant objects satisfies the same formula.

We call these two results respectively the first and second invariance theorem, and their precise statement is given as 2.43.

We will now go into a little more detail of how this language is defined, and explain the role of the different of the paper.

As mentioned above, we need to use dependent types. So our starting point is a "Generalised algebraic theory" T in the sense of Cartmell ([Car78]) as our basis–if we compare to traditional model theory–T plays a role similar to a signature.

We then build on top of T the associated first order language in section 2.1. The idea is that for each formula, the (free) variables are taken from a context of the theory T, and there can be no equality at all. We will see through example how in some cases, equality can be recovered indirectly using the fact that the theory T can include equality axioms (it can be any Generalized algebraic theory). Because we want to be able to do infinitary logic, we use everywhere an infinitary generalization of the notion of Generalized algebraic theory that is introduced in appendix A, however a reader familiar with Generalized algebraic theory can probably guess how it works.

In section 2.2 we review quickly some important properties of the category of models of a Generalized algebraic theory, or equivalently of the category of models of a "clan" (in the sense of Joyal), most notably their canonical weak factorization system. In section 2.3 we explain how the language defined in section 2.1 can be given an alternative categorical definition that can be applied to any clan. Note that every clan can be shown to be the syntactic category of a generalized algebraic theory (and we prove more generally that in our infinitary setting any "κ-clan" is the syntactic category of a generalized κ-algebraic theory, this is in appendix B.4,) and the category theoretic definition of the language of the clan is equivalent to the syntactic definition of the language of any such Generalized algebraic theory.

This reinterpretation is the key to associate a language to any model category: Given a (weak) model category $\mathcal{M}$ we take the category $\mathcal{M}^{\mathrm{Cof}}$ of cofibrant objects and cofibration between them. This category constitutes a co-clan (the opposite of a clan) and we can take the language assocaited to it. This is what we call the language of the model category $\mathcal{M}$. We review briefly the general theory of weak model category in appendix C.1 and in section 2.4 we explain in details how this language of $\mathcal{M}$ actually talk about the objects of $\mathcal{M}$ and prove the first two invariances theorem mentioned above.

To give a general picture of how this language works, if $\mathcal{M}$ is our model category, each formula in the language has a "context" C, which informally can be thought of as the list of free variables that can appear in the formula as well as their types. This "context" C is concretely just a cofibrant object of $\mathcal{M}$. An interpretation of the context C into an object $X \in \mathcal{M}$ is just a map $v : C \to X$. And given ϕ a formula in context C and $v : C \to X$ a map, $\phi(v)$ can be either true or false. We write

$$M \vdash \phi(v)$$

if it is true. The first invariance theorem asserts that if X is fibrant and $v : C \to X$ is homotopic to $v' : C \to X$ then $M \vdash \phi(v) \Leftrightarrow M \vdash \phi(v')$. The second invariance theorem states that if $F : X \to Y$ is a weak equivalence between fibrant objects then $X \vdash \phi(v) \Leftrightarrow Y \vdash \phi(f(v))$.

For example, if $\mathcal{M}$ is the canonical or folk model structure on categories, the language we obtain is essentially the language of categories we mentioned at the begining. The formula

$$\forall Z \in \mathrm{Ob}, \forall g, h \in \mathrm{Hom}(Y, Z), g \circ f = h \circ f \Rightarrow g = h$$

is a formula in context $X, Y \in \mathrm{Ob}, f \in \mathrm{Hom}(X, Y)$ which corresponds to the (cofibrant) object X which has two objects (say X and Y) and a unique non-identity arrow $f : X \to Y$. A map from C to another category $\mathcal{D}$ is the choice of an arrow f in $\mathcal{D}$ and $\phi(f)$ is true if and only if f is an epimorphism. The second invariance theorem says (in this special case) that equivalence of categories preserves epimorphisms, and the first invariance theorem that if f is isomorphic to another arrow then one is an epimorphism if and only if the other is.

In section 3 we show how these notions specialize to many classical model structures, and we also discuss briefly some general tools to construct this language explicitly for any model structure.

Finally, section 4 is devoted to statement and proof of the third invariance theorem, which essentially says that two Quillen equivalent model categories have the same homotopy language.

2 The homotopy invariant language

{fol-gat}

2.1 Syntactic approach: The first-order language of a generalized algebraic theory

{sec:syntactic-approac

In this section we give a very classical syntactical approach to the language we consider in this paper. We start from a generalized algebraic theory and we build its first-order language on top of it.

Since we aim to do infinitary logic, we enhance Cartmell's notion of generalized algebraic theory to what we call *generalized κ-algebraic theory* for κ a regular cardinal, which we develop in detail in appendix A. Nevertheless, this generalization is very straightforward and a reader familiar with Cartmell formalism should be able to guess how it works and read this section directly.

We fix two regular cardinals κ and λ, and T a generalized κ-algebraic theory. We will define the first-order language of T denoted $\mathcal{L}^T_\lambda$ or $\mathcal{L}^T_{\lambda,\kappa}$.

More precisely, for each context Γ of T, we will define a set $\mathcal{L}^T_\lambda(\Gamma)$ of "T-formulas in context Γ". Essentially these are first-order formulae with λ-small conjunctions and disjunctions whose free variables are the variables of the context Γ, in particular, they have less than κ-variables.

{folgat}

Definition 2.1. The sets $\mathcal{L}^T_\lambda(\Gamma)$ of T-formulas in context Γ are defined inductively using the following rules:

1. For each context Γ, the true formula $\top$ and false formula $\bot$ are in $\mathcal{L}^T_\lambda(\Gamma)$.
2. If $\Phi \in \mathcal{L}^T_\lambda(\Gamma)$ then $\neg\Phi \in \mathcal{L}^T_\lambda(\Gamma)$.
3. For each collection of formulas $\Phi_i \in \mathcal{L}^T_\lambda(\Gamma)$, indexed by a λ-small set I, the conjunction and disjunction
$$\bigvee_{i\in I} \Phi_i \qquad \bigwedge_{i\in I} \Phi_i$$
are in $\mathcal{L}^T_\lambda(\Gamma)$.
4. Given two ordinals $\gamma < \alpha < \kappa$: If $\Gamma' = \{x_\beta : \Gamma_\beta\}_{\beta<\alpha}$ is a context of length α, and $\Gamma = \{x_\beta : \Gamma_\beta\}_{\beta<\gamma}$ is the subcontext of length γ, then for any formula $\Phi \in \mathcal{L}^T_\lambda(\Gamma')$ we have formulas
$$\exists\{x_\beta : \Gamma_\beta\}_{\gamma\leqslant\beta<\alpha}\Phi \qquad \forall\{x_\beta : \Gamma_\beta\}_{\gamma\leqslant\beta<\alpha}\Phi$$
in $\mathcal{L}^T_\lambda(\Gamma)$.

The collection of all formulas $\{\mathcal{L}_\lambda^T(\Gamma)\}_{\Gamma \in T}$ is what we call *the language of* T. Often, we will simply refer to it by $\mathcal{L}_\lambda^T$.

Remark 2.2. The key point in definition 2.1 is that we are not including atomic formulas other than $\top$ and $\bot$. In particular, the language *does not include any equality.* At this point it might be unclear how we get non-trivial formulae in this language as it seems that applying quantifiers, conjunction or disjunction to formulas that are either $\bot$ or $\top$ will never produce any formulas that are not immediately interpreted as $\bot$ or $\top$. Or even, on how we might obtain formulas with free variables. The central idea is that free variables appear thanks to the fact we quantify over dependent types, that is types in which free variables can appear. The following examples will demonstrate this phenomena.

Example 2.3. Let Cat be the generalized ω-algebraic theory of categories as introduced in example A.7. Then in the context $(x : \mathsf{Ob})$ we can form the formula

$$\phi(x) \coloneqq (\forall y : \mathsf{Ob},\ \exists f : \mathsf{Hom}(x, y), \top)$$

which expresses that for any object y there is an arrow from x to y. This simply means that x is a weakly initial object. Indeed, $\top$ is a formula in context $(x : \mathsf{Ob},\ y : \mathsf{Ob},\ f : \mathsf{Hom}(x, y))$, so that $\exists f : \mathsf{Hom}(x, y), \top$ is a formula in context $(x : \mathsf{Ob},\ y : \mathsf{Ob})$, and $\forall y : \mathsf{Ob}, \exists f : \mathsf{Hom}(x, y), \top$ is a formula in context $(x : \mathsf{Ob})$.

The logic is still not strong enough to express many of the interesting category theoretic notions. For example, without any kind of equality predicate on morphisms there is no way to write down formula for an initial object, or a limit. In the next example, we show how modifying the theory Cat allows to recover equality on morphisms:

{ex:cat=}

Example 2.4. We consider the theory $Cat_=$ obtained by adding to the theory Cat the following:

$$x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y) \vdash \mathsf{Eq}(f, g)\mathsf{Type}$$

$$x, y : \mathsf{Ob}, f : \mathsf{Hom}(x, y) \vdash r_f : \mathsf{Eq}(f, f)$$

$$x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y), a : \mathsf{Eq}(f, g) \vdash f \equiv g$$

$$x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y), a : \mathsf{Eq}(f, g) \vdash a \equiv r_f$$

One easily see that a model of $Cat_=$ is just a category, with the type $\mathsf{Eq}(f, g)$ being empty if $f \neq g$ and $\{r_f\}$ if $f = g$.

In this new theory, we can now form a formula "$f = g$" in context $(x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y))$ which is defined as

$$(f = g) \coloneqq (\exists v : \mathsf{Eq}(f, g), \top).$$

Therefore, in the language $\mathcal{L}_{\omega}^{Cat_{=}}$ we can form formulas involving equality between parallel morphisms. Then we recover the "language of categories" as studied in [Bla78] and [Fre76]. For example, we can form the formula "x is initial" in context $(x : \mathsf{Ob})$ as

$$\mathsf{isInitial}(x) \coloneqq \forall y : \mathsf{Ob}, (\exists f : \mathsf{Hom}(x, y)) \wedge (\forall f, g : \mathsf{Hom}(x, y), f = g)$$

{cstr:f*}

Construction 2.5. If $f : \Delta \to \Gamma$ is a context morphism and $\phi \in \mathcal{L}_{\lambda}^{T}(\Gamma)$, then we can define its pullback $f^*\phi$. This pullback is obtained by substituting the free variables of ϕ by the components of f. Formally, this is defined inductively as:

1. $f^*\top = \top$ and $f^*\bot = \bot$.
2. $f^*(\neg\Phi) = \neg f^*\Phi$
3. $f^*\left(\bigvee_{i\in I} \Phi_i\right) = \bigvee_{i\in I} f^*\Phi_i$ and $f^*\left(\bigwedge_{i\in I} \Phi_i\right) = \bigwedge_{i\in I} f^*\Phi_i$.
4. If $\Gamma' = (\Gamma, x_1 \in X_1, \dots, x_\alpha \in X_\alpha)$ then

$$f^*\left(\exists(x_1 \in X_1, \dots, x_\alpha \in X_\alpha)\Phi\right) = \exists(x_1 \in f^*X_1, \dots, x_\alpha \in f^*X_\alpha) f^*\Phi$$

$$f^*\left(\forall(x_1 \in X_1, \dots, x_\alpha \in X_\alpha)\Phi\right) = \forall(x_1 \in f^*X_1, \dots, x_\alpha \in f^*X_\alpha) f^*\Phi$$

Where f^*X_i denotes the pullback of types, obtained by substitution, that is, the types appearing in the canonical pullback of the generalized display map:

$$\begin{tikzcd}
(\Delta, f^*X_1, \dots, f^*X_\alpha) \arrow[r] \arrow[d, two heads] & (\Gamma, X_1, \dots, X_\alpha) \arrow[d, two heads] \\
\Delta \arrow[r] & \Gamma.
\end{tikzcd}$$

{def:vdash_relation}

Definition 2.6. For each context Γ in T we define the relation $\vdash_\Gamma$ on $\mathcal{L}_{\lambda}^{T}(\Gamma)$ as the smallest family of relations such that:

{def:vdash_relation:re

1. $\vdash_\Gamma$ is a transitive and reflexive relation on $\mathcal{L}_{\lambda}^{T}(\Gamma)$.

{def:vdash_relation:bot

2. $\forall \Phi \in \mathcal{L}_{\lambda}^{T}(\Gamma)$, $\Phi \vdash_\Gamma \top$ and $\bot \vdash_\Gamma \Phi$.

{def:vdash_relation:neg

3. $\forall \Phi \in \mathcal{L}_\lambda^T(\Gamma), \Phi \wedge \neg\Phi \vdash \bot$ and $\top \vdash \Phi \vee \neg\Phi$.

{def:vdash_relation:vee

4. For any λ-small family $(\Phi_i)_{i\in I} \in \mathcal{L}_\lambda^T(\Gamma)$ we have

$$\bigvee_{i\in I} \Phi_i \vdash_\Gamma \Psi \Leftrightarrow \forall i, (\Phi_i \vdash_\Gamma \Psi)$$

$$\Psi \vdash \bigwedge_{i\in I} \Phi_i \Leftrightarrow \forall i, (\Psi \vdash_\Gamma \Phi_i)$$

{def:vdash_relation:Qua

5. For $\Gamma' = \left(\Gamma, \left\{x_\beta : \Gamma'_\beta\right\}_{\gamma\leqslant\beta<\alpha}\right)$ a context extension, with $p : \Gamma' \twoheadrightarrow \Gamma$ the corresponding generalized display map, $\Psi \in \mathcal{L}_\lambda^T(\Gamma')$ and $\Phi \in \mathcal{L}_\lambda^T(\Gamma)$ we have

$$\exists\{x_\beta : \Gamma_\beta\}_{\gamma\leqslant\beta<\alpha}\Psi \vdash_\Gamma \Phi \Leftrightarrow \Psi \vdash_{\Gamma'} p^*\Phi$$

$$\Phi \vdash_\Gamma \forall\{x_\beta : \Gamma_\beta\}_{\gamma\leqslant\beta<\alpha}\Psi \Leftrightarrow p^*\Phi \vdash_{\Gamma'} \Psi$$

While we have not included the following in the definition we can show that:

{prop:vdash_structural_

Proposition 2.7. *If $f : \Delta \to \Gamma$ is a context morphism in T, and $\Phi \vdash_\Gamma \Psi$ then $f^*\Phi \vdash_\Delta f^*\Psi$.*

Proof. We can show that if we define the relations $\Phi \vdash'_\Gamma \Delta$ to be "For all $f : \Delta \to \Gamma$, we have $f^*\Phi \vdash_\Delta f^*\Psi$" then it satisfies all the conditions from definition 2.6. Which shows that $\vdash \Rightarrow \vdash'$ and hence conclude the proof. □

Definition 2.8. A *model* of a generalized κ-algebraic theory T is simply a contextual functor $X : \mathbb{C}_T \to \mathbf{Set}$. We will usually write $X : T \to \mathbf{Set}$.

{cstr:validity_of_formu

Construction 2.9. Given a model X of our theory T, Γ a context, $x \in X(\Gamma)$ and $\Phi \in \mathcal{L}_\lambda^T(\Gamma)$, we can interpret $\Phi(x)$ as a proposition *i.e.,* true or false in the obvious way by substituting the components of x into ϕ and interpreting all the logic symbols in the usual way. Formally we have:

1. If $\Phi = \top$, then $\Phi(x)$ is true and if $\Phi = \bot$ then $\Phi(x)$ is false.

2. If $\Phi = \neg\Psi$, then $\Phi(x)$ is true if and only if $\Psi(x)$ is false.

3. If $\Phi = \bigvee \Phi_i$, then $\Phi(x)$ is true if and only if $\Phi_i(x)$ is true for some i.

4. If $\Phi = \bigwedge \Phi_i$, then $\Phi(x)$ is true if and only $\Phi_i(x)$ is true for all i.

5. $\Phi = \exists \{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} \Psi$ for $\Gamma' = \left(\Gamma, \left\{x_\beta : \Gamma'_\beta\right\}_{\gamma \leqslant \beta < \alpha}\right)$ a context extension, with $p : \Gamma' \twoheadrightarrow \Gamma$ the corresponding generalized display map, then $\Phi(x)$ is true if there exists a $y \in X(\Gamma')$ such that $p(y) = x$ and $\Psi(y)$.

6. If $\Phi = \forall \{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} \Psi$ in the same situation as above, then $\Phi(x)$ is true if for any $y \in X(\Gamma')$ such that $p(y) = x$ we have $\Psi(y)$.

The following lemma is immediate by induction, the proof is left to the reader.

Lemma 2.10. *Let X be a model of a generalized κ-algebraic theory T.*

1. *For $\Phi, \Psi \in \mathcal{L}^T_\lambda(\Gamma)$ and $x \in X(\Gamma)$, then if $\Psi \vdash_\Gamma \Phi$ and $\Psi(x)$ then $\Phi(x)$.*

2. *If $f : \Gamma \to \Delta$ is any context morphism and $\Phi = f^* \Psi$ and $x \in X(\Gamma)$ then $\Phi(x) \Leftrightarrow \Psi(f(x))$.*

{def:Lb}

Definition 2.11. We write $\Psi \dashv\vdash_\Gamma \Phi$ to mean both $\Psi \vdash_\Gamma \Phi$ and $\Phi \vdash_\Gamma \Psi$. We denote

$$\mathbb{L}^T_\lambda(\Gamma) \coloneqq \mathcal{L}^T_\lambda(\Gamma)/(\dashv\vdash_\Gamma)$$

the quotient.

Note that $(\dashv\vdash_\Gamma)$ is indeed an equivalence relation as $\vdash_\Gamma$ is transitive and reflexive.

{rk:logical_operation_c

Remark 2.12. It follows from proposition 2.7 that for a context morphism $f : \Delta \to \Gamma$ the f^* operation from $\mathcal{L}^T_\lambda(\Gamma) \to \mathcal{L}^T_\lambda(\Delta)$ is compatible to the relation $\dashv\vdash$, and hence it descent to an operation

$$f^* : \mathbb{L}^T_\lambda(\Gamma) \to \mathbb{L}^T_\lambda(\Delta)$$

It is also easy to see from definition 2.6 that the relation $\vdash$ is compatible to all the logical operations on $\mathcal{L}^T_\lambda$, that is $\neg, \bigvee, \bigwedge, \exists, \forall$ in the sense that for example if $\Phi_i \vdash \Psi_i$ for all $i \in I$ then $\bigvee_{i \in I} \Phi_i \vdash \bigvee_{i \in I} \Psi_i$ and hence they all descent into operation on L^T_λ.

Construction 2.13. At the beginning of the section, we have briefly called than language $\mathcal{L}^T_{\lambda,\kappa}$ before dropping the κ from the notation as it can be read of from the fact that T is a generalized κ-algebraic theory. However, we can consider $\mathcal{L}^T_{\lambda,\kappa'}$ for any $\kappa' \geqslant \kappa$. Indeed, given T a generalized κ-algebraic theory we can define a generalized κ'-algebraic theory $T_{\kappa'}$ by taking a set of axioms for T and seeing them as axioms for a generalized κ'-algebraic theory. A model of $T_{\kappa'}$ is the same as a model of T. We then define

$$\mathcal{L}^{T}_{\lambda,\kappa'} \coloneqq \mathcal{L}^{T_{\kappa'}}_{\lambda,\kappa'} = \mathcal{L}^{T_{\kappa'}}_{\lambda},$$

as well as its quotient

$$\mathbb{L}^{T}_{\lambda,\kappa'} \coloneqq \mathbb{L}^{T_{\kappa'}}_{\lambda,\kappa'} = \mathbb{L}^{T_{\kappa'}}_{\lambda},$$

Example 2.14. Let Σ be a signature in the sense of traditional model theory. Then we can consider the generalized algebraic theory $T_{\Sigma,=}$, which has one sort in empty context of each type symbol X, each of these type has an equality predicate as the one constructed in 2.4, a term for each function symbol, and for each relation symbol $R \subset X_1, \ldots, X_n$ an additional type axiom

$$x_1 : X_1, \ldots, x_n : X_n \vdash R(x_1, \ldots, x_n)\,\mathsf{Type}$$

with the additional axiom

$$x_1 : X_1, \ldots, x_n : X_n, t_1, t_2 : R(x_1, \ldots, x_n) \vdash t_1 = t_2$$

Models of this theory are exactly Σ structures, and elements of $\mathbb{L}^{T_{\Sigma,=}}_{\omega,\omega}$ are essentially the same as usual first order formula in this signature. Elements of $\mathbb{L}^{T_{\Sigma,=}}_{\lambda,\kappa}$ corresponds to infinitary first-order formulas using λ-small conjunction and disjunction and where $\exists$ and $\forall$ quantifier can quantify over κ-small set of variables.

2.2 Models of Clans and their weak factorization system

{sec:models_of_clans}

We recall that:

Definition 2.15. A *clan*, or *ω-clan*, is a category $\mathcal{C}$ endowed with a class of maps called *fibrations* such that

- $\mathcal{C}$ has an initial object 1, and for every $X \in \mathcal{C}$ the unique map $X \twoheadrightarrow 1$ is a fibration.
- Isomorphisms are fibrations, the composite of two fibrations is a fibrations.
- Pullback of fibrations exists and are fibrations.

For κ a regular cardinal, a κ-clan is a clan which further satisfies:

- For any ordinal $\lambda < \kappa$, if $A_\bullet : \lambda^{\mathtt{op}} \to \mathcal{C}$ is a diagram in which all the transition maps $A_\beta \twoheadrightarrow A_\alpha$ for $\alpha < \beta$ are fibrations, then the limits

$$\mathtt{Lim}_{\alpha<\lambda} A_\alpha$$

exists, and all the projection maps $\pi_\beta : \mathtt{Lim}_{\alpha<\lambda} A_\alpha \twoheadrightarrow A_\beta$ are fibrations. We refer to these as *limits of κ-small chains of fibrations.*

A *morphism of clans* is a functor that send fibrations to fibrations, preserve the initial object and pullback of fibrations. A *morphism of κ-clans* is in addition required to preserves the limits of κ-small chains of fibrations.

Fibrations will be denoted with a double-headed arrow $\twoheadrightarrow$.

Remark 2.16. We define *coclans* and *κ-coclans* dually, as the category $\mathcal{C}$ endowed with a class of *cofibrations* whose opposite category are clans and κ-clans.

Definition 2.17. If $\mathcal{C}$ is a κ-clan, a *model* X of $\mathcal{C}$ is a functor $X : \mathcal{C} \to \mathbf{Set}$ that preserves the terminal object, pullback of fibrations and limits of κ-small chains of fibrations. The category $\mathrm{Mod}(\mathcal{C})$ of models of $\mathcal{C}$ is defined as a full subcategory of the category $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ of all functors.

Remark 2.18. A key observation is of course that if T is a generalized κ-algebraic theory and $\mathcal{C}_T$ is its contextual category, then $\mathcal{C}_T$ can be seen as a κ-clan where fibration are the map that are isomorphic to generalized display maps. Moreover, the models of T are exactly the models of this clan $\mathrm{Mod}(T) = \mathrm{Mod}(\mathcal{C}_T)$, so that models of generalized algebraic theories are a special cases of models of clans.

Also note that:

- By **??** every κ-clan $\mathcal{C}$ is equivalent to a κ-contextual category.
- By theorem B.46 every κ-contextual category is isomorphic to the contextual category $\mathcal{C}_T$ of a generalized κ-algebraic theory.

Combining these two results, every κ-clan is equivalent to one of the form $\mathcal{C}_T$ for T a generalized κ-algebraic theory. Hence there is no fundamental difference between the models of a clans and the models of a generalized κ-algebraic theory.

Construction 2.19. Let $\mathcal{C}$ be a κ-clan and $よ_\bullet : \mathcal{C}^{\mathtt{op}} \to Fun(\mathcal{C}, \mathbf{Set})$ be the contravariant Yoneda embedding. Note that for every $A \in \mathcal{C}^{\mathtt{op}}$ the functor $よ_A : \mathcal{C} \to \mathbf{Set}$ preserves all limits, so in particular it is a model. Hence we have an embedding $\text{よ}_\bullet : \mathcal{C}^{\mathsf{op}} \to \mathrm{Mod}(\mathcal{C})$. Note that by the Yoneda lemma we have a natural isomorphism

$$\mathrm{Hom}(\text{よ}_A, X) \simeq X(A)$$

for $X \in \mathrm{Mod}(\mathcal{C})$ and $A \in \mathcal{C}$.

{is-presentable-mod:rml

Remark 2.20. The category of models of a κ-clan $\mathcal{C}$ is characterized by preservation of certain κ-small limits. This implies, by general category theoretic results that, for a small κ-clan $\mathcal{C}$:

- The category $\mathrm{Mod}(\mathcal{C})$ is locally κ-presentable.
- The representable models よ_A for $A \in \mathcal{C}$ are κ-presentable objects.

Indeed, the category $\mathrm{Mod}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ is closed under κ-filtered colimits because κ-filtered colimits commutes to κ-small limits, which because of the isomorphism $\mathrm{Hom}(\text{よ}_A, X) \simeq X(A)$ implies that the object よ_A are κ-presentable in $\mathrm{Mod}(\mathcal{C})$. Moreover, since every $X \in \mathrm{Mod}(\mathcal{C})$ can be written as $X = \mathtt{Colim}_{\text{よ}_A \to X} \text{よ}_A$ this implies that the category $\mathrm{Mod}(\mathcal{C})$ is locally κ-accessible, and hence locally κ-presentable as it is also closed under small limits.

Remark 2.21. More generally, any κ-presentable category $\mathcal{C}$ is equivalent to the category of functors $\mathcal{C}_\kappa^{\mathsf{op}} \to \mathbf{Set}$ that preserves κ-small limits, where $\mathcal{C}_\kappa$ is the (essentially small) category of κ-presentable objects of $\mathcal{C}$. In particular, every κ-presentable category is the category of models of a κ-clan: One can take the category $\mathcal{C}_\kappa^{\mathsf{op}}$, with all maps being fibrations. However, the category $\mathrm{Mod}(\mathcal{C})$ of models of a κ-clans comes with an additional structure that is more specific:

{wfs-models}

Definition 2.22. Given a κ-clan $\mathcal{C}$, we consider the weak factorization on the category $\mathrm{Mod}(\mathcal{C})$ which is cofibrantly generated by the maps

$$\text{よ}_A \hookrightarrow \text{よ}_B$$

where $B \twoheadrightarrow A$ is a fibration in $\mathcal{C}$. The element of the left class will be called *cofibrations* and the element of right class *trivial fibrations*.

trivial fibration lifting property

{cofibrant-objects:wfs-

Remark 2.23. In the special case $\kappa = \omega$, this weak factorization was defined in [Hen16, Definition 2.4.2] and extensively studied in [Fre23]. In particular, Jonas Frey gave in [Fre23] a complete characterization of which pairs of a category and a weak factorization can be obtained in this way from an ω-clan. The methods used by Frey can be extended to the κ-case to obtain a similar characterization. Frey also shows that (in the $\kappa = \omega$ case) the ω-presentable cofibrant object in $\mathrm{Mod}(\mathcal{C})$ are exactly the retracts of representable models. The same proof generalizes to the κ-case to show that when $\mathcal{C}$ is a κ-clan the κ-presentable cofibrant objects are exactly the retracts of representables. We will not directly use these results, but we might occasionally make some side comments that assumes these results hold for arbitrary κ.

trivial fibration lifting property {lem:trivial_fibration_

Lemma 2.24. *Given $\mathcal{C}$ a clan, a morphisms $f : M \to N$ of $\mathcal{C}$-models is a trivial fibration if and only if for every fibration $p : X \twoheadrightarrow Y$ in $\mathcal{C}$, the naturality square:*

$$\begin{tikzcd} M(X) \arrow[r] \arrow[d] & M(Y) \arrow[d] \\ N(X) \arrow[r] & N(Y) \end{tikzcd}$$

is a weak pullback square*, that is if the induced map $M(X) \to N(X) \times_{N(Y)} M(Y)$ is a surjection.*

Proof. By the Yoneda lemma, there is a one-to-one correspondence between elements of $M(X)$ and morphisms of models $よ_X \to M$. The map $M(X) \to M(Y)$ is obtained as the composite $よ_Y \to よ_X \to M$ and the map $M(X) \to N(X)$ as the composite $よ_X \to M \to N$. An element of $N(X) \times_{N(Y)} M(Y)$ is hence the data of maps $よ_X \to N$ and $よ_Y \to M$ such that the composite $よ_Y \to M \to N$ and $よ_Y \to よ_X \to N$ coincide. This is exactly a commutative square:

$$\begin{tikzcd} よ_Y \arrow[r] \arrow[d, "よ_p"'] & M \arrow[d, "f"] \\ よ_X \arrow[r] & N \end{tikzcd}$$

An element of $M(X)$ whose image in $N(X) \times_{N(Y)} M(Y)$ is the square above is then exactly a dotted diagonal filling:

$$\begin{tikzcd} よ_Y \arrow[r] \arrow[d, "よ_p"'] & M \arrow[d, "f"] \\ よ_X \arrow[r] \arrow[ur, dotted] & N \end{tikzcd}$$

Hence the surjectivity of this map is equivalent to the fact that f has the right lifting property against $よ_Y \to よ_X$ for all fibrations $X \twoheadrightarrow Y$, which concludes the proof. □

2.3 The Category theoretic approach: The first order language of a κ-clans

{sec:category_approach}

In this section we present another equivalent approach to the definition of the language, which is more categorical in spirit, and strongly inspired from Lawvere's theory of Hyperdoctrines ([Law69], [Law70]). This approach, while much more abstract, has several advantages over the syntactic one. Mainly it allows to work directly with the more general notion of a clan (see appendix B.4), instead of a generalized κ-algebraic theory. This will be useful latter on to define the language of a model category without having to build explicitly a syntax for it.

As before, we fix λ a regular cardinal. A λ-boolean algebra is a boolean algebra which admits joins (and hence intersections) of λ-small families. We denote by $\mathbf{Bool}_\lambda$ the category whose objects are λ-boolean algebras and whose morphisms are boolean algebra morphisms preserving λ-small joins (and hence intersections).

We introduce the notion of λ-boolean algebra over a clan $\mathcal{C}$ which can be thought of as an axiomatization of the structure that the $\mathbb{L}_\lambda^T$ from section 2.1 have over the contextual category of T.

{def:Boolan_alg_over_c

Definition 2.25. Given $\mathcal{C}$ a clan and λ a regular cardinal, a λ-*boolean algebra* over $\mathcal{C}$ is a functor

$$\mathcal{B} : \mathcal{C}^{op} \to \mathbf{Bool}_\lambda$$

such that

1. For each fibration $\pi : Z \twoheadrightarrow X$ in $\mathcal{C}$, $\pi^* : \mathcal{B}(X) \to \mathcal{B}(Z)$ has a left adjoint:
$$\exists_\pi : \mathcal{B}(Z) \leftrightarrows \mathcal{B}(X) : \pi^*.$$

2. The Beck-Chevalley condition holds for each pullback square along a fibration. Given any pullback square:

$$\begin{tikzcd} Z' \arrow[r, "f'"] \arrow[d, two heads, "\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, two heads, "\pi"] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$

we have $f^*\exists_\pi = \exists_{\pi'} f'^*$.

Morphisms of λ-boolean algebras over $\mathcal{C}$ are natural transformations that commute with the $\exists_\pi$. We call weak morphisms the general natural transformations.

Remark 2.26. If $\mathcal{B}$ a λ-boolean algebra over $\mathcal{C}$, then for each $X \in \mathcal{C}$, the negation $\neg : \mathcal{B}(X) \to \mathcal{B}(X)^{op}$ is a contravariant equivalence. Therefore, if $\pi : Z \to X$ is a fibration, then the map $\pi^* : \mathcal{B}(X) \to \mathcal{B}(Z)$ also has a right adjoint defined by:

$$\forall_\pi(\phi) \coloneqq \neg(\exists_\pi \neg\phi)$$

We immediately have from this definition the other Beck-Chevalley condition $f^*(\forall_\pi) = \forall_\pi f^*$ and the fact that morphisms of boolean algebras over $\mathcal{C}$ are also compatible to $\forall_\pi$, simply because both the f^* and morphism are compatible to both $\exists_\pi$ and the negation.

Remark 2.27. Definition 2.25 will in practice be applied to $\mathcal{C}$ a κ-clan (and not just a clan), the only reason it is stated like that is because the definition actually does not explicitly involves κ. This is related to the fact that the dependencies in κ of the language defined in the previous subsection is only through the choice of which context can our variables (including bound variables) be taken from: taking a larger κ mean we can quantify over more variable at the same time. Similarly, the dependency on κ is hidden in the dependency on $\mathcal{C}$, as $\mathcal{C}$ is playing the role of the category of κ-contexts.

Let us start with our main example of such boolean algebra over a clan, which is the motivating example for the notion:

{th:Lb_is_initial}

Theorem 2.28. *Let T be a generalized κ-algebraic theory and $\mathcal{C}_T$ is the corresponding κ-contextual category, seen as a clan. Then the construction $X \mapsto \mathbb{L}^T_\lambda(X)$ from definition 2.11 (see also definition 2.1 and 2.6) is a λ-boolean algebra over $\mathcal{C}_T$. In fact, it is an initial object in the category of λ-boolean algebra over $\mathcal{C}_T$.*

Proof. We first check that $\mathcal{L}^T_\lambda$ is a λ-boolean algebra over $\mathcal{C}_T$. We have mentioned in remark 2.12 that all the logical operations $\vee, \wedge, \neg, \exists$ and so on are compatible with the equivalence relation $\dashv\vdash$. Therefore, they all induce operations on the quotient $\mathbb{L}^T_\lambda$. The first four points of definition 2.6 immediately shows that each $\mathbb{L}^T_\lambda(X)$ is a boolean algebra whose order relation is given by $\vdash$, and with λ-small unions. By construction 2.5 the map $f^* : \mathcal{L}^T_\lambda(X) \to \mathcal{L}^T_\lambda(Y)$ is compatible with all the logical operation, so it gives rise to a morphism of boolean algebra $\mathbb{L}^T_\lambda(X) \to \mathbb{L}^T_\lambda(Y)$. We get a functor

$\mathcal{C}_T \to \mathbf{Bool}_\lambda$, the conditions $(g \circ f)^*(\phi) = f^* g^*(\phi)$ and $id^*(\phi) = \phi$ follow immediately by induction. Next, the last two conditions of definition 2.6 shows that $\exists$ and $\forall$ defines left and right adjoint to π^*. Finally, the Beck-Chevalley condition follows from how f^* is defined on formulas starting with a $\exists$ quantifier:

$$f^* \left(\exists \{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} \Phi\right) = \exists \{x_\beta : f^* \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} f^* \Phi$$

which (after passing to the quotient $\mathcal{L} \to \mathbb{L}$) exactly says that $f^* \exists_\pi = \exists \pi f^*$ where π is the generalized display map corresponding to forgetting the variables $\{x_\beta\}_{\gamma \leqslant \beta < \alpha} \in X_\alpha$.

We now check that it is an initial object in the category of λ-boolean algebras over $\mathcal{C}_T$. Let $\mathcal{B}$ be any λ-boolean algebra over $\mathcal{C}$. Any morphism $v : \mathbb{L}^T_\lambda \to \mathcal{B}$ has to satisfy:

1. $v(\bot) = \bot_\mathcal{B}$ and $v(\top) = \top_\mathcal{B}$.
2. $v(\neg \Phi) = \neg v(\Phi)$.
3. $v(\bigvee\limits_{i \in I} \Phi_i) = \bigvee\limits_{i \in I} v(\Phi_i)$ and $v(\bigwedge\limits_{i \in I} \Phi_i) = \bigwedge\limits_{i \in I} v(\Phi_i)$.
4. $$v(\exists \{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} \Phi) = \exists \{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} v(\Phi)$$
   and
   $$v(\forall \{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} \Phi) = \forall \{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} v(\Phi).$$

These form an inductive definition for a function $\mathcal{L}^T_\lambda \to \mathcal{B}$. So there is a unique such function $v : \mathcal{L}^T_\lambda \to \mathcal{B}$. To conclude we only need to check that this function v descent to a function $\mathbb{L}^T_\lambda \to \mathcal{B}$ and is a morphism of λ-boolean algebra over $\mathcal{C}$. But this is rather immediate: We first observe, by induction over definition 2.6, that if $\Phi \vdash \Psi$ then $v(\Phi) \leqslant v(\Psi)$. This implies that if $\Phi \dashv\vdash \Psi$ then $v(\Phi) = v(\Psi)$, so v does define a function $\mathbb{L}^T_\lambda \to \mathcal{B}$. The naturality condition

$$v(f^*(\Phi)) = f^*(v(\Phi))$$

can be proved by induction on the formula Φ, and the compatibility of v with all the boolean algebra operations and the quantifiers follows immediately from the definition of v. $\square$

Proposition 2.29. *Given any (small) clan $\mathcal{C}$ and λ a regular cardinal, there is an initial λ-boolean algebra over $\mathcal{C}$, which we denote by $\mathbb{L}^\mathcal{C}_\lambda$.*

Note that by theorem 2.28, if T is a generalized κ-algebraic theory, with $\mathcal{C}_T$ its κ-contextual category them

$$\mathbb{L}_\lambda^{\mathcal{C}_T} = \mathbb{L}_\lambda^T$$

This provides a way to define (or at least to characterize) the first order language of any clan without having to explicitly give a syntactic description of the clan.

Proof. We can either remark that the λ-boolean algebras over $\mathcal{C}$ are (by their definition) the models of a multisorted λ-algebraic theory (with one sort for each object $c \in \mathcal{C}$) and hence there is an initial objects by usual results on algebraic theories. Alternatively, we can use (see appendix B.4) that every clan is equivalent to the contextual category of a generalized algebraic theory and use theorem 2.28 to conclude. $\square$

Next, we mention a few more examples:

{ex:Power_Set_boolean_a

Example 2.30.

1. Let **Set** be the category of sets, considered as a clan where every arrow is a fibration. The contravariant power-set functor $\mathcal{P} : \mathbf{Set}^{op} \to \mathbf{Bool}_\lambda$ is a λ-Boolean algebra over **Set**. The Beck-Chevalley condition follow from lemma 2.31 below.

2. Given $F : \mathcal{C} \to \mathcal{D}$ a morphism of clans, if $\mathcal{B}$ is a λ-boolean algebra over $\mathcal{D}$, then $F^*\mathcal{B}$ defined by $F^*\mathcal{B}(\Gamma) = \mathcal{B}(F(\Gamma))$ is a λ-boolean algebra over $\mathcal{C}$.

3. Combining the two observations above, given any model M of a clan $\mathcal{C}$, that is a morphism of clans $M : \mathcal{C} \to \mathbf{Set}$, one has a boolean algebra $\mathcal{P}(M)$ over $\mathcal{C}$ given by pulling back example 1 along the morphism $M : \mathcal{C} \to \mathbf{Set}$. More explicitly:

$$\begin{array}{rccc} \mathcal{P}(M): & \mathcal{C}^{op} & \to & \mathbf{Set} \\ & \Gamma & \mapsto & \mathcal{P}(M(\Gamma)) \end{array}$$

{lem:BC_Set}

Lemma 2.31. *Given a square of sets:*

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "h"] \\ Y \arrow[r, "k"] & Z \end{tikzcd}$$

Then the power set functor satisfies the Beck-Chevalley condition on this square, i.e. $k^*\exists_h = \exists_g f^*$ *as maps* $\mathcal{P}(X) \to \mathcal{P}(Y)$ *if and only if the square is a weak pullback square i.e., if and only if the cartesian gap map* $W \to Y \times_Z X$ *is surjective.*

Proof. Given a subset $P \subset X$ one has:

$$k^* h_! P = \{y \in Y | k(y) = h(p) \text{ for some } p \in P\}$$

$$g_! f^* P = \{g(w) | f(w) \in P\}$$

Surjectivity of the map $W \to Y \times_Z X$ gives a canonical way to make any element of $k^* h_! P$ into an element of $g_! f^* P$, and conversely, applying the equality to $P = \{p\}$ produces the surjectivity of $W \to Y \times_Z X$. □

In this new setting with just a clan $\mathcal{C}$, one can still define the set of formulas $\mathbb{L}_\lambda^{\mathcal{C}}$ as the initial λ-boolean algebra over $\mathcal{C}$. We now explain what it means for formulas defined this way to be "true" or "false" in given a model and an interpretation of its variables in the model.

Construction 2.32. Given a clan $\mathcal{C}$ and a model of $M : \mathcal{C} \to \mathbf{Set}$ we have, as explained in example 2.30, a λ-boolean algebra over $\mathcal{C}$ defined by $c \mapsto \mathcal{P}(M(c))$. By initiality of the κ-boolean algebra $\mathbb{L}_\lambda^{\mathcal{C}}$ there exists a unique morphism of λ-boolean algebras over $\mathcal{C}$: {cstr:validity_formulas

$$| - |_M : \mathbb{L}_\lambda^{\mathcal{C}} \to \mathcal{P}(M).$$

This morphism associates each formula ϕ in context Γ to a subset $|\phi|_M \subseteq M(\Gamma)$. An element $x \in M(\Gamma)$ is said to satisfy ϕ if $x \in |\phi|_M$, with some abuse of notation we say that "$\phi(x)$ is true" in this case. We also write

$$M \vdash \phi(x)$$

when we want to insist on which model we are talking about.

When Γ is the terminal object of $\mathcal{C}$ *i.e.,* ϕ is a closed formula, then $M(\Gamma) = \{*\}$. Therefore, $\mathcal{P}(M(\Gamma)) = \{\bot, \top\}$ so that $|\phi|_M$ is simply a proposition. One then says that M satisfies ϕ, and we write $M \vdash \phi$.

Lemma 2.33. *When* $\mathcal{C} = \mathcal{C}_T$ *is the* κ*-contextual category of a* κ*-generalized algebraic theory, then through the identification* $\mathbb{L}_\lambda^T = \mathbb{L}_\lambda^{\mathcal{C}}$*, the two definitions of validity of a formula on elements of a model given by construction 2.9 and construction 2.32 are equivalent.*

Proof. Defining the validity of formulas as in construction 2.32 it is immediate to verify all the explicit conditions of the inductive definition given in construction 2.9 simply because the map $\mathbb{L}_\lambda^{\mathcal{C}} \to \mathcal{P}(M)$ is a morphism of λ-boolean algebra. Hence, it immediately follows by induction on formulas that the two definitions are equivalents. □

{action-functor:formula

Construction 2.34. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of clans. And let $\mathbb{L}_\lambda^{\mathcal{C}}$ and $\mathbb{L}_\lambda^{\mathcal{D}}$ be their respective initial λ-boolean algebras. From the fact that $\mathbb{L}_\lambda^{\mathcal{C}}$ is initial there is a morphisms of λ-boolean algebras

$$\alpha^F : \mathbb{L}_\lambda^{\mathcal{C}} \to F^* \left(\mathbb{L}_\lambda^{\mathcal{D}}\right).$$

For any $\Gamma \in \mathcal{C}$ and any formula $\Phi \in \mathbb{L}_\lambda^{\mathcal{C}}(\Gamma)$ we denote $F(\Phi) \coloneqq \alpha_\Gamma^F(\Phi)$ which is a formula in context $F(\Gamma)$ i.e an element of $\mathbb{L}_\lambda^{\mathcal{D}}(F(\Gamma))$. The following is immediate from the definition above:

{pullback-model:prop}

Proposition 2.35. *Let $M : \mathcal{D} \to$ **Set** a model of the clan $\mathcal{D}$, $\Phi \in \mathbb{L}_\lambda^{\mathcal{C}}(\Gamma)$ a formula in context Γ and $x \in M(F(\Gamma))$. Then, $M \vdash \alpha_F(\Phi)(x)$ if and only if $F^*M \vdash \Phi(x)$.*

Finally, we finish this section by showing the key property of invariance of formulas along trivial fibrations. A invariance property will be established in the next section assuming we are working with a model category, but this first invariance property is purely algebraic. This is also the key observation in Makkai FOLDS [Mak95] and it is directly inspired from it.

We start with the following observation: Let $\mathcal{C}$ be a clan and $f : M \to N$ a morphisms of two $\mathcal{C}$-models, then we have an obvious map $f^* : P(N) \to P(M)$ which sends a subset $A \subset N(c)$ for $c \in \mathcal{C}$ to

$$f_c^{-1}(A) \subset M(c)$$

this map is easily seen to be a *weak* morphism of boolean algebra over $\mathcal{C}$. It is compatible with the boolean algebra operations and the ordinary contravariant functoriality, but it does not have to be compatible with the covariant functoriality $\exists_\pi$ along fibrations. However, one has:

{morphism-bol-alg-iff-t

Lemma 2.36. *Let $\mathcal{C}$ be a clan and let $f : M \to N$ be a morphism between two $\mathcal{C}$-models. Then f is a trivial fibration if and only if $f^* : P(N) \to P(M)$ is a morphism of λ-boolean algebras.*

Proof. We only need to show that for every fibration $p : X \to Y$ the following square

$$\begin{tikzcd}
P(N(X)) \arrow[r, "f_X^*"] \arrow[d, "\exists"] & P(M(X)) \arrow[d, "\exists"] \\
P(N(Y)) \arrow[r, "f_Y^*"] & P(M(Y)).
\end{tikzcd}$$

commutes. From lemma 2.31 this is equivalent to say that the dotted map in

$$\begin{tikzcd}
M(X) \arrow[dr, dashed] \arrow[drr, bend left, "f_X"] \arrow[ddr, bend right, "\pi_*"'] & & \\
& P \arrow[r] \arrow[d] & N(X) \arrow[d, "\pi_*"] \\
& M(Y) \arrow[r, "f_Y"] & N(Y)
\end{tikzcd}$$

is surjective. But this is exactly the characterization of trivial fibrations given in lemma 2.24. ☐

This allows us to deduce the key result of invariance of formulae along trivial fibrations of models. Basically, the validity of formulae is preserved by trivial fibrations of models:

{cor:Invariance_triv_f

Corollary 2.37. *Let $\mathcal{C}$ be a clan and let $f : M \twoheadrightarrow N$ be a trivial fibration between two $\mathcal{C}$-models. For $c \in \mathcal{C}$, let $x \in M(c)$ and $\phi \in \mathbb{L}_\lambda^{\mathcal{C}}$ be any formula. Then*

$$M \vdash \phi(x) \Leftrightarrow N \vdash \phi(f(x))$$

Proof. As $f : M \to N$ is a trivial fibration, it follows from lemma 2.36 that the map $f^* : \mathcal{P}(N) \to \mathcal{P}(M)$ is a morphism of boolean algebra over $\mathcal{C}$. Hence, by initiality of $\mathbb{L}_\lambda^{\mathcal{C}}$, the unique morphism $|\_|_M : \mathbb{L}_\lambda^{\mathcal{C}} \to \mathcal{P}(M)$ is obtained as a composite

$$\mathbb{L}_\lambda^{\mathcal{C}} \xrightarrow{|\_|_M} \mathcal{P}(M) \xrightarrow{f^*} \mathcal{P}(N).$$

By definition, $M \vdash \phi(x)$ means that $x \in |\phi|_M$ while $N \vdash \phi(f(x))$ means that $x \in f^*|\phi|_N$, hence the result immediately follows. ☐

2.4 The language of a weak model category and two invariance theorems

{language-model-language-construction}

Construction 2.38. Given $\mathcal{M}$ a weak model category, the category $\mathcal{M}^{\text{COF}}$ of cofibrant objects with cofibrations between them forms a coclan. We define the language of $\mathcal{M}$ to be the language of the coclan $\mathcal{M}^{\text{COF}}$. For any regular cardinal λ, we denote by $\mathbb{L}_\lambda^{\mathcal{M}}$ the λ-boolean algebra $\mathbb{L}_\lambda^{\mathcal{M}^{\text{COF}}}$ over $\mathcal{M}^{\text{COF}}$.

Note that we have for each *cofibrant* object $X \in \mathcal{M}$ a set (or possibly a class if $\mathcal{M}$ is large) of formulas $\mathbb{L}_\lambda^{\mathcal{M}}(X)$.

Remark 2.39. There is a size issue to be mentioned here. In most practical examples, $\mathcal{M}^{\text{COF}}$ is a large category while the construction of $\mathbb{L}_\lambda^{\mathcal{M}^{\text{COF}}}$ developed in section 2.3 assume it is a small category. We can deal with this by invoking a larger Grothendieck universe, but this has a practical consequence: The set of formulas $\mathbb{L}_\lambda^{\mathcal{M}}(X)$ might not be a small set. Indeed, it lives in the same Grothendieck universe as the one in which $\mathcal{M}^{\text{COF}}$ is small.

{cstr:object_to_model_y

Construction 2.40. If $X \in \mathcal{M}$ then we can define a model of the coclan $\mathcal{M}^{\text{COF}}$ using the restricted Yoneda embedding:

$$\text{よ}_X : \begin{array}{ccc} (\mathcal{M}^{\text{COF}})^{\text{op}} & \to & \textbf{Set} \\ c & \mapsto & \text{Hom}(c, X) \end{array}$$

Which defines a functor $\text{よ} : \mathcal{M} \to \text{Mod}(\mathcal{M}^{\text{COF}})$.

{def:wms_language}

Definition 2.41. Let $\mathcal{M}$ be a weak model category. For $c \in \mathcal{M}$ a cofibrant, and $X \in \mathcal{M}$ any object, $v : c \to X$ and $\phi \in \mathbb{L}_\lambda^{\mathcal{M}}(c)$ we write

$$X \vdash \phi(v)$$

to mean

$$\text{よ}_X \vdash \phi(v)$$

where v is seen as an element of $\text{よ}_X(c) = \text{Hom}(c, X)$.

Remark 2.42. In the special case where $\mathcal{M} = \text{Mod}(T)$ is the category of models of a generalized κ-algebraic theory (or more generally of a κ-coclan), then $\mathbb{L}_\lambda^{\mathcal{M}}$ is the initial λ-boolean algebra over the coclan of all cofibrant objects of $\mathcal{M}$, while the syntactic category of T is equivalent to a full sub-κ-coclan of that. In particular there is a morphism of λ-boolean algebra over the syntactic category $\mathcal{C}_T$

$$\mathbb{L}_\lambda^T(X) \to \mathbb{L}_\lambda^{\mathcal{M}}(X) \qquad (\text{For } X \in \mathcal{C}_T)$$

If we denote this map by i then for X any model of T we can easily check that

$$X \vdash \phi(v) \Leftrightarrow X \vdash i(\phi)(v)$$

for any $c \in \mathcal{C}_T$ and $\phi \in \mathbb{L}^T_\lambda(c)$, where the left hand side is interpreted in the sense of definition 2.1 while the right hand side is in terms of definition 2.41.

Note that we do expect these to be the same. Informally, $\mathbb{L}^T_\lambda$ corresponds to an $\mathcal{L}_{\kappa,\lambda}$ logic, in the sense that quantifier can only be applied to formulas in κ-small context (so, to less that κ-many variables at the same time), while $\mathbb{L}^{\mathcal{M}}_\lambda$ corresponds to an $\mathcal{L}_{\infty,\lambda}$ logic, where quantifiers can be applied to arbitrarily many formulas at the same time.

{invariance-theorems}

Theorem 2.43. *Let $\mathcal{M}$ be a weak model category, $c \in \mathcal{M}$ a cofibrant object,X, Y two fibrant objects, $v : c \to X$ a maps, and $\phi \in \mathbb{L}^{\mathcal{M}}_\lambda(c)$ then:*

- ***First invariance theorem:*** *Let $v_1, v_2 : c \to X$ be two homotopically equivalent maps with X fibrant. Then*

$$X \vdash \phi(v_1) \quad \Leftrightarrow \quad X \vdash \phi(v_2)$$

- ***Second invariance theorem:*** *Let $f : X \to Y$ be a weak equivalence between two fibrant objects and $v : c \to X$ any maps then*

$$X \vdash \phi(v) \quad \Leftrightarrow \quad Y \vdash \phi(fv)$$

Proof. We start by first observing that the second invariance theorem in the special case where f is a trivial fibration immediately follows from corollary 2.37 as a trivial fibration f has the right lifting property against all core cofibrations and hence is sent to a trivial fibration in $\mathrm{Mod}(\mathcal{M}^{\mathrm{Cof}})$ by the functor from construction 2.40.

We use this to prove the first invariance theorem: If $v_1, v_2 : c \to X$ are homotopic then there exists a map h:

$$\begin{tikzcd} & X \\ c \arrow[ur, "v_2"] \arrow[r, dotted, "h"] \arrow[dr, "v_1"'] & PX \arrow[u, "p_2"'] \arrow[d, "p_1"] \\ & X \end{tikzcd}$$

The two maps $p_1, p_2 : PX \to X$ are trivial fibrations (they are both fibrations and weak equivalences), and $v_1 = p_1 \circ h$ and $v_2 = p_2 \circ h$. By the observation above we have:

$$\begin{array}{ccccc} & X & \vdash & \phi(v_1) \\ \Leftrightarrow & X & \vdash & \phi(p_1 h) \\ \Leftrightarrow & PX & \vdash & \phi(h) \\ \Leftrightarrow & X & \vdash & \phi(p_2 h) \\ \Leftrightarrow & X & \vdash & \phi(v_2) \end{array}$$

This concludes the proof of the first invariance theorem.

Next, we observe it is enough to prove the second invariance theorem when X and Y are both bifibrant. Indeed, starting from $f : X \to Y$ a weak equivalence between fibrant objects, $v : c \to X$ and $\phi \in \mathbb{L}_\lambda^{\mathcal{M}}(c)$ as in the theorem. We can replace both X and Y by bifibrant objects

$$\begin{tikzcd} X^{\mathrm{Cof}} \arrow[r, "\sim", "f'"'] \arrow[d, two heads, "\sim"] & Y^{\mathrm{Cof}} \arrow[d, two heads, "\sim"] \\ X \arrow[r, "\sim", "f"'] & Y \end{tikzcd}$$

by first replacing X by a cofibrant object X^{Cof} and then factoring the map $X^{\mathrm{Cof}} \to Y$, which is a weak equivalence, as a trivial cofibration followed by a trivial fibration. The map $v : C \to X$, can be lifted to map $v' : c \to X^{\mathrm{Cof}}$. As we can already apply the second isomorphisms theorem to trivial fibrations, we have that

$$X \vdash \phi(v) \Leftrightarrow X^{\mathrm{Cof}} \vdash \phi(v')$$

$$Y \vdash \phi(fv) \Leftrightarrow Y^{\mathrm{Cof}} \vdash \phi(f'v')$$

so that it is enough to show the second isomorphism for bifibrant objects.

This last step is achieved essentially using a “Brown factorization”: any weak equivalence between bifibrant objects can be factored as a section of a trivial fibration followed by a trivial fibration. Indeed, if $f : X \to Y$ is a map between bifibrant objects we can form the pullbacks:

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "e'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "e"] \\
X \times_Y PY \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & PY \arrow[d, two heads] \\
X \times Y \arrow[r] \arrow[d, two heads, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times Y \arrow[d, two heads, "\pi_1"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}
$$

Note that because the fibrations $PY \to Y$ are trivial fibration, the map $X \times_Y PY \to X$ in the diagram above is also a trivial fibration. The total vertical maps are both the identity. Which gives us a diagram:

$$
\begin{tikzcd}
& X \arrow[dl, "id_X"'] \arrow[d, "e'"] \arrow[dr, "f"] & \\
X & X \times_Y PY \arrow[l, two heads, "\sim", "q"'] \arrow[r, two heads, "p"'] & Y
\end{tikzcd}
$$

Where p is the map $X \times_Y PY \twoheadrightarrow X \times Y \overset{\pi_2}{\twoheadrightarrow} Y$. Note that all maps in this diagram are weak equivalences due to the 2-out-of-3 condition. We can now prove the theorem, we have

$$X \vdash \phi(v) \Leftrightarrow X \times_Y PY \vdash \phi(e'v)$$

because $v = qe'v$ and q is a trivial fibration, and

$$X \times_Y PY \vdash \phi(e'v) \Leftrightarrow Y \vdash \phi(fv)$$

because p is a trivial fibration and $fv = pe'v$. Hence, combining the two

$$X \vdash \phi(v) \Leftrightarrow Y \vdash \phi(fv)$$

□

Finally, we quickly mention how Quillen adjunctions act on formulas. A Quillen adjunction between two weak model categories is an adjunction

$$L : \mathcal{C} \leftrightarrows \mathcal{D} : R$$

where the left adjunction L sends cofibrations to cofibrations and the right adjoint R sends fibrations to fibrations.

Remark 2.44. There is also a more general notion called "weak Quillen functors" introduced in [Hen20] which is sometimes more convenient. The functor L is only defined on cofibrant objects and R on fibrant objects, and they are only required to preserve core (co)cofibrations –all results in this section, as well as the third invariance theorem from section 4 apply to weak Quillen adjunctions too. We restrict ourselves to Quillen adjunctions in the paper for simplicity, and because this already cover most of the applications.

Such a Quillen adjunction (or weak Quillen adjunction) induce in particular a coclan morphism $L : \mathcal{C}^{\text{Cof}} \to \mathcal{D}^{\text{Cof}}$, which following construction 2.34 we have a (unique) comparison map

$$\alpha_L : \mathbb{L}^{\mathcal{C}}_{\lambda} \to L^* \mathbb{L}^{\mathcal{D}}_{\lambda}$$

As before, if $\phi \in \mathbb{L}^{\mathcal{C}}_{\lambda}(C)$ we often write $L(\phi)$ instead of $\alpha_L(\Phi)$. Note that $L(\phi) \in \mathbb{L}^{\mathcal{D}}_{\lambda}(L(C))$.

Finally, exactly as in construction 2.34, for any (fibrant) object $X \in \mathcal{D}$, and cofibrant object $C \in \mathcal{C}$, any map $v : C \to R(X)$ corresponding to $\tilde{v} : LC \to X$, and $\phi \in \mathbb{L}^{\mathcal{C}}_{\lambda}$ we have

$$R(X) \vdash \phi(v) \Leftrightarrow X \vdash L(\phi)(\tilde{v})$$

The third invariance theorem that we will establish in section 4 show that for a Quillen equivalence, this construction gives an equivalence between the language of $\mathcal{C}$ and of $\mathcal{D}$ in an appropriate sense.

3 Examples of languages of model categories

{sec:examples}

In this section we examine some examples of the language associated to a model category by applying the construction as described in section 2. We include examples we believe to be of interest. Firstly, we start with some general considerations.

{cofibrations:order}

Definition 3.1. Let $\mathcal{C}$ be model category and $\text{Cof}(\mathcal{C})$ the class of cofibrations. Assume that the cofibrations are generated by a set I. We say that the set of generating cofibrations is *well-founded* if there exists a well-founded relation $<$ on I such that for all $i \in I$, the map $\emptyset \to Dom(i)$ can be written as a κ-composite of pushouts of maps $j \in I$ with $j < i$.

{combinatorial:well-fou

Example 3.2. Let $\mathcal{C}$ be combinatorial (accessible) weak model category [Hen23]. Then $\mathcal{C}$ is κ-combinatorial for some regular cardinal κ. Consider I to be the set (category) of arrows between κ-presentable objects of generating cofibrations. [Hen23, B.5 Theorem] shows that the weak factorization on $\mathcal{C}$ is generated by Garner's small object argument using the set I. Then the class (category) of cofibrations of $\mathcal{C}$ is ordered as in definition 3.1.

Moreover, [Hen23, 4.7 Lemma] shows that one can get a combinatorial (accessible) weak factorization system such that the domain of each cofibration is cofibrant. The way it works in our setting is as follows: We start with a combinatorial (accessible) weak factorization system (L, R). The cited result produces another combinatorial (accessible) weak factorization system (L', R') such that R' are exactly the maps with the right lifting property against maps $l \in L$ such that the map $\emptyset \to Dom(l) \in L$.

Explicitly, if L is the class (category) of cofibrations of a combinatorial (accessible) weak model category then we can get a well-founded class of cofibrations by setting

$$L' \coloneqq \{\emptyset \to c | c \in L\} \cup L.$$

{cofibrations-models:we

Example 3.3. Let $\mathcal{C}$ be a κ-clan. The class of cofibrations in the weak factorization system of section 2.2 defined on the category of models $\mathrm{Mod}(\mathcal{C})$ admits a well-founded relation. Recall from definition 2.22 that the weak factorization system on $\mathrm{Mod}(\mathcal{C})$ defined there is cofibrantly generated by $\text{よ}_A \hookrightarrow \text{よ}_B$ where $B \twoheadrightarrow A$ is a fibration of $\mathcal{C}$, denote this set by I and $(L_{\mathcal{C}}, R_{\mathcal{C}})$ the weak factorization system. Furthermore, Frey [Fre23, Corollary 5.5] proves that the models an ω-clan can be realized as a filtered colimit of representable models. This result can be generalized to any κ-clan, also see remark 2.23.

We can induce a well-founded order on I using that any κ-clan is equivalent to κ-contextual category as shown in corollary B.55. Since the objects in a κ-contextual category are graded, this give us the well-founded order. This means that $\text{よ}_A < \text{よ}_B$ if and only if $ht(A) < ht(B)$ where $ht(A), ht(B)$ are the heights of A and B, respectively. The order on generalized display maps is, and hence on the generating cofibrations, simply given by comparing the length.

{syntatic:well-founded}

Example 3.4. The previous example 3.3 is even more illustrative when specialized to the κ-clan $\mathbb{C}_T$ obtained as the syntatic κ-contextual category of a generalized κ-algebraic theory T.

In this case, we have a more explicit description of generalized display maps $B \twoheadrightarrow \Delta$ in proposition A.39, and hence of the natural well-founded order on the set $I = \{ よ_\Delta \hookrightarrow よ_B | B \twoheadrightarrow \Delta \in \mathbb{C}_T \}$. This says that a generalized display map is the limit of a tower of display maps, each of which is obtained as a pullback a display map $\Gamma.A \twoheadrightarrow \Gamma$ associated to a the type axiom $\Gamma \vdash A\ \mathsf{Type}$ of the theory.

Therefore, the dependency level of a type in the theory T is reflected into the well-foundedness of the order on the set I.

The previous examples show that starting with a κ-clan, one can get a cofibrantly generated weak factorization system on the category of models $\mathrm{Mod}(\mathcal{C})$ such that the generating set of cofibrations is well-founded. We can reverse this process in the sense that if we are given a weak factorization system with a well-founded set of generating cofibrations, then we can produce a generalized algebraic theory from it.

{theory-wfs:constructi

Construction 3.5. Let $\mathcal{C}$ be a κ-clan. Assume that $\mathcal{C}$ has a weak factorization system that is cofibrantly generated by a set I with a well-founded relation. Recall that this means that for a cofibration $i : A \hookrightarrow B$ the map $\emptyset \to A$ is a κ-composite of maps $j \in I$ with $j < i$. Therefore, we can recursively introduce a type axiom:

$$\vdash \overline{A}\ \mathsf{Type}$$

whenever $\emptyset \hookrightarrow A \in I$ and

$$\overline{A} \vdash \overline{B}\ \mathsf{Type}$$

for $A \hookrightarrow B \in I$.

We can think of this construction as similar to the functor $U : \kappa\text{-CON} \to \kappa\text{-GAT}$ from appendix B.3.2 which produces a generalized κ-algebraic theory $U(\mathcal{C})$ from a κ-contextual category $\mathcal{C}$. In particular, for a display map $B_{\lambda+1} \twoheadrightarrow B_\lambda \in \mathcal{C}$ it gives a type axiom $\overline{B_\lambda} \vdash \overline{B_{\lambda+1}}\ \mathsf{Type}$. Of course, the main point of construction 3.5 is that it applies as soon as we have a well-behaved weak factorization system..

Remark 3.6. For each of the examples below, we start with a Quillen model category $\mathcal{M}$ and apply construction 3.5 to obtain a theory $T_{\mathcal{M}}$. In general, this is the guiding principle that will allows to identify the statements, and the language, to which the invariance theorems apply.

Furthermore, using the theory $T_{\mathcal{M}}$ we can consider the category $\mathrm{Mod}(T_{\mathcal{M}})$ and use definition 2.22 to obtain a weak factorization system. Through this process, the cofibrations and trivial fibrations we obtain coincide with the ones from the Quillen model category we start with.

3.1 Categories

Let us summarize our construction on this prime example we have been using throughout the paper. Recall that $\mathbf{0}$ is the empty category, $\mathbf{1} \coloneqq \{0\}$ is the category with a single object, $\mathbf{2} \coloneqq \{0 \to 1\}$ the arrow category and $P \coloneqq \{0 \rightrightarrows 1\}$ the category with two parallel arrows. Finally, $\mathcal{J} \coloneqq \{0 \leftrightarrows 1\}$ denotes the walking isomorphism category. The following result appears in [Rez96].

{folkmodel:categories}

Theorem 3.7. *There is Quillen model structure on the category* **Cat** *such that:*

1. *Weak equivalences are the equivalences of categories,*

2. *Cofibrations are the functors injective on objects,*

3. *Fibrations are the isofibrations.*

Furthermore, this models structure is cofibrantly generated. The sets

$$I \coloneqq \{\mathbf{0} \xrightarrow{u} \mathbf{1},\ \{0\} \sqcup \{1\} \xrightarrow{v} \mathbf{2},\ P \xrightarrow{w} \mathbf{2}\} \text{ and } J \coloneqq \{\mathbf{1} \to \mathcal{J}\}$$

are the generating cofibration and trivial cofibrations respectively.

In this model structure all objects are cofibrant. We can immediately associate for each generator in I a sort in the following way:

$$\begin{aligned} \mathbf{0} \to \mathbf{1} &\longmapsto \vdash \mathsf{Ob}\ \mathsf{Type} \\ \{0\} \sqcup \{1\} \to \mathbf{2} &\longmapsto x, y : \mathsf{Ob} \vdash \mathsf{Hom}(x,y)\ \mathsf{Type} \\ P &\longmapsto x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x,y) \vdash \mathsf{Eq}(f,g)\ \mathsf{Type} \end{aligned}$$

Note that while the type Ob has no dependencies, the type $\mathsf{Hom}(x,y)$ depends on two elements of type Ob, which is encoded in the cofibration $\{0\} \sqcup \{1\} \to \mathbf{2}$. The same situation applies with the type Eq which furthermore has dependencies on the types Ob and Hom, now the cofibration $P \hookrightarrow \mathbf{2}$ expresses this.

The resulting theory is what we introduced earlier $Cat_{=}$ which by convenience we recall here. This is defined as:

1. Type of objects: $\vdash$ Ob Type.

2. Type of morphisms: $x : \mathsf{Ob}$, $y : \mathsf{Ob} \vdash$ $\mathsf{Hom}(x,y)$ Type.

3. Equality type: $x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y) \vdash \mathsf{Eq}(f, g)\mathsf{Type}$

4. Composition operation: $x : \mathsf{Ob}$, $y : \mathsf{Ob}$, $z : \mathsf{Ob}$, $f : \mathsf{Hom}(x, y)$, $g : \mathsf{Hom}(y, z) \vdash g \circ f : \mathsf{Hom}(x, z)$.

5. Identity operator: $x : \mathsf{Ob} \vdash \mathsf{id}_x : \mathsf{Hom}(x, x)$.

Subject to the following axioms:

- $x : \mathsf{Ob}$, $y : \mathsf{Ob}$, $f : \mathsf{Hom}(x, y) \vdash \mathsf{id}_y \circ f \equiv f$.
- $x : \mathsf{Ob}$, $y : \mathsf{Ob}, f : \mathsf{Hom}(x, y) \vdash f \circ \mathsf{id}_x \equiv f$.
- $x : \mathsf{Ob}$, $y :: \mathsf{Ob}$, $z : \mathsf{Ob}$, $w : \mathsf{Ob}$, $f : \mathsf{Hom}(x, y), g : \mathsf{Hom}(y, z), h : \mathsf{Hom}(z, w) \vdash (h \circ g) \circ f \equiv h \circ (g \circ f)$.
- $x, y : \mathsf{Ob}, f : \mathsf{Hom}(x, y) \vdash r_f : \mathsf{Eq}(f, f)$.
- $x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y), a : \mathsf{Eq}(f, g) \vdash f \equiv g$.
- $x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y), a : \mathsf{Eq}(f, g) \vdash a \equiv r_f$.

As remarked in example 2.4 the language we obtain is the same as the one given by [Bla78] and [Fre76].

3.2 2-categories and Bicategories

In this section we examine the language associated to the canonical model structures on the categories **2-Cat** and $\mathbf{Bicat_s}$, respectively. The model structure for these two categories was defined in [Lac02] and [Lac04].

Given a category C its suspension $\sum C$, is defined as the 2-category with two objects X, Y, the hom categories are $\sum C(X, X) = \sigma C(Y, Y) = \sum C(Y, X) = \emptyset$ and $\sum C(X, Y) = C$. Furthermore, each bicategory $\mathscr{B} \in \mathbf{Bicat_s}$ has an underlying **Cat-graph**, in the sense of [Wol74]. This induces a functor $U : \mathbf{Bicat_s} \to \mathbf{Cat\text{-}graph}$ which has left adjoint F, this gives us the free bicategory generated by a **Cat**-graph. The suspension of a category C can be seen as a **Cat**-graph associated to C. The free bicategory generated by the suspension of a category is denoted by $\sum \mathscr{C}$. Moreover, this construction is functorial.

[Lac04, Theorem 3] constructs a model structure for the category of bicategories. This model structure is cofibrantly generated with generating cofibrations given by the suspension of the generating cofibrations of the canonical model structure on **Cat** and an additional functor we specify below. Finally, $\mathscr{E}$ is the "free-living adjoint equivalence " is the bicategory with objects x, y, freely generated by 1-cells $f : x \to y$ and $g : y \to x$, and two invertible 2-cells $\eta : 1_x \Rightarrow gf$, $\varepsilon : fg \Rightarrow 1_y$ satisfying the familiar triangle identities.

{model:bicategories}

Theorem 3.8. *There is a model structure on the category* $\mathbf{Bicat_s}$ *of bicategories and strict bifunctors such that:*

1. *Weak equivalences are the biequivalences,*
2. *Fibrations are the strict bifunctors with the equivalence lifting property.*

Furthermore, the model structure is cofibrantly generated by the sets

$$I \coloneqq \{\mathbb{0} \to \mathbb{1}, \Sigma u, \Sigma v, \Sigma w\} \text{ and } J \coloneqq \{\mathbb{1} \to \mathscr{E}\}$$

where $\mathbb{0}$ *is the empty bicategory,* $\mathbb{1}$ *is the bicategory with a single object and no non-identity 2-cells, the functors* u, v, w *come from theorem 3.7, and the bifunctor in* J *picks the object* x.

When we analyze the set of generating cofibrations I we rediscover the generalized algebraic theory of bicategories $Bicat_{=}$.

- $\mathbb{0} \to \mathbb{1} \longmapsto\ \vdash \mathsf{Ob}\ \mathsf{Type}$
- $\{x\} \sqcup \{y\} \xrightarrow{\sum u} \{x \to y\} \mapsto x, y : \mathsf{Ob} \vdash \mathsf{Hom}(x, y)$
- $\begin{tikzcd} x \arrow[r, bend left, "0"] \arrow[r, bend right, "1"'] & y \end{tikzcd} \xrightarrow{\sum v} \begin{tikzcd} x \arrow[r, bend left, "0"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow"] & y \end{tikzcd} \mapsto x, y : \mathsf{Ob},\ f, g : \mathsf{Hom}(x, y) \vdash \mathsf{Hom}(f, g)\ \mathsf{Type}$
- $\begin{tikzcd} x \arrow[r, bend left, "0"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow\Downarrow"] & y \end{tikzcd} \xrightarrow{\sum w} \begin{tikzcd} x \arrow[r, bend left, "0"] \arrow[r, bend right, "1"'] \arrow[r, phantom, "\Downarrow"] & y \end{tikzcd} \mapsto \begin{cases} x, y : \mathsf{Ob},\ f, g : \mathsf{Hom}(x, y), \\ \alpha, \beta : \mathsf{Hom}(f, g) \vdash \mathsf{Eq}(\alpha, \beta)\ \mathsf{Type} \end{cases}$

Moreover, we can also introduce the composition and identity operations for arrows and cells:

- Composition operation for arrows: $x : \mathsf{Ob},\ y : \mathsf{Ob},\ z : \mathsf{Ob},\ f : \mathsf{Hom}(x, y),\ g : \mathsf{Hom}(y, z) \vdash g \circ f : \mathsf{Hom}(x, z)$.
- Identity operator for arrows: $x : \mathsf{Ob} \vdash \mathsf{id}_x : \mathsf{Hom}(x, x)$.

- Vertical composition of cells: $x, y : \mathsf{Ob}, f, g, h : \mathsf{Hom}(x, y), \alpha : \mathsf{Hom}(f, g), \beta : \mathsf{Hom}(g, h) \vdash \beta \circ \alpha : \mathsf{Hom}(f, h)$.

- Horizontal composition of cells: $x, y, z : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y), h, k : \mathsf{Hom}(y, z), \alpha : \mathsf{Hom}(f, g), \beta : \mathsf{Hom}(h, k) \vdash \alpha * \beta : \mathsf{Hom}(h \circ f, k \circ g)$.

- Identity operator for cells: $x, y : \mathsf{Ob}, f : \mathsf{Hom}(x, y) \vdash \mathsf{id}_f : \mathsf{Hom}(f, f)$.

We also include the axioms for Eq (same as for categories) that give us the expected behaviour. One can also attempt to list all the axioms that the above theory ought to satisfy, with the risk of running out of space. We simply exemplify this by writing:

- The associator: $w, x, y, z : \mathsf{Ob}, f : \mathsf{Hom}(w, x), g : \mathsf{Hom}(x, y), h : \mathsf{Hom}(y, z), \alpha : \mathsf{Hom}((h \circ g) \circ f, h \circ (g \circ f)), \beta : \mathsf{Hom}((h \circ (g \circ f), h \circ g) \circ f) \vdash r : \mathsf{Eq}(\alpha \circ \beta, \mathsf{id}_{(h \circ (g \circ f))}) \wedge s : \mathsf{Eq}(\beta \circ \alpha, \mathsf{id}_{(h(\circ g) \circ f})$.

By forgetting the bicategorical structure in the set of generators in theorem 3.8 we obtain generating sets for the model structure in **2-Cat**. If we now try to obtain the associated theory $2Cat_{=}$ we see that this has the same types and operations as the theory $Bicat_{=}$ of bicategories. We can distinguish these theories by means of their axioms. Or by looking at the languages $\mathcal{L}_{\omega}^{Bicat_{=}}$, the formulas are meaningful only for bicategories (or 2-categories).

3.3 Bounded below chain complexes

In this section examine the language of the projective model structure on bounded below chain complexes $Ch(R)$ over a commutative ring R. We start by recalling some facts about this model structure. The detailed proofs can be found elsewhere e.g. [Hov99].

Given an R-module M for each $n \in \mathbb{Z}$ define $S^n(M) \in Ch(R)$ by

$$S^n(M)_k := \begin{cases} M, \; k = n \\ 0, \; k \neq n. \end{cases}$$

Similarly, $D^n(M) \in Ch(R)$ is defined as

$$D^n(M)_k := \begin{cases} M, \; k = n-1, \; n \\ 0, \;\; \text{otherwise}. \end{cases}$$

where the only non-trivial differential $d_n : M \to M$ is the identity. Obviously, we get an inclusion $S^{n-1}(M) \to D^n(M)$.

This constructions induce functors $S^n : R\text{-}Mod \to Ch(R)$ and $D^n : R\text{-}Mod \to Ch(R)$ for each $n \in \mathbb{Z}$. Both functors have left adjoint $Z_n : Ch(R) \to R\text{-}Mod$ and $Ev_n : Ch(R) \to R\text{-}Mod$, respectively, where $Z_n X \coloneqq Ker(d_n)$ and $Ev_n X \coloneqq X_n$.

In particular, when $M = R$ the chains above are denoted by S^n and D^n, respectively. We can define the sets

$$I \coloneqq \{S^{n-1} \to D^n | n \in \mathbb{Z}\} \text{ and } J \coloneqq \{0 \to D^n | n \in \mathbb{Z}\}.$$

All constructions above work on unbounded chain complexes too. In the next result we restrict to bounded below chains i.e. $n \geq 0$, where by definition $(D^0)_{-1} = 0$, so that $S^0 = D^0$. With this information, what we need to know about the projective model structure is summarized in the following:

{projective-model:bound

Theorem 3.9. *The category of chain complexes $Ch(R)$ admits a model structure were:*

1. *Weak equivalences are the quasi-isomorphisms*
2. *Fibrations are the degree-wise epimorphisms.*
3. *Cofibrations are the degree-wise monomorphims with projective cokernel.*

Furthermore, this model structure is proper, cofibrantly generated and combinatorial. Cofibrations and trivial cofibrations are generated by I and J respectively.

The cofibrant objects in the mode structure from theorem 3.9 are complexes such that each R-module is projective. However, this is not the case for unbounded chain complexes where not every chain complex with projective modules is cofibrant. Nevertheless, in both cases all objects are fibrant.

Using the adjunction $Z_n \dashv S^n$, any chain complex X a map $S^n \to X$ is simply a map $R \to Z_n X$ of R-modules. And from $Ev_n \dashv D^n$, a map $D^n \to X$ corresponds to $y \in X_n, z \in X_{n-1}$ such that $d_n y = z$, we denote this as (y, z). Therefore, a commutative square

$$\begin{tikzcd} S^{n-1} \arrow[r, "x"] \arrow[d, "i_n"'] & X \arrow[d, "f"] \\ D^n \arrow[r, "(y,z)"'] & Y \end{tikzcd}$$

means that $x \in Z_{n-1}X \subseteq X_{n-1}$ i.e. $d_{n-1}x = 0$ and that $d_n f(x) = d_n y = z \in Y_{n-1}$. Therefore taking a pushout simply means we freely add $(n-1)$-cycles to X_{n-1} with a specified boundary.

The first element of the set I is the cofibration

$$\begin{tikzcd} 0 \arrow[d, "i_0"'] & 0 \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ D^0 & 0 & R \arrow[l] & 0 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

For any $n \geq 1$ we have cofibrations i_n

$$\begin{tikzcd} S^{n-1} \arrow[d, "i_n"'] & 0 \arrow[d] & \cdots \arrow[l] & R \arrow[l] \arrow[d, dashed, "1_R"] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ D^n & 0 & \cdots \arrow[l] & R \arrow[l] & R \arrow[l, dashed, "1_R"'] & 0 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

The ω-generalized algebraic theory we obtain can be written as follows:

$$\vdash 0\text{-}\mathsf{Chain}\ \mathsf{Type}$$
$$x : (n-1)\text{-}\mathsf{Chain} \vdash \{y|dy = x\}\ \mathsf{Type}$$

for $n \geq 1$.

3.4 Unbounded chain complexes

When we work with unbounded chain complexes, with the obvious modifications, theorem 3.9 becomes:

{projective-model:chai

Theorem 3.10. *The category of chain complexes $Ch(R)$ admits a model structure were:*

1. *Weak equivalences are the quasi-isomorphisms*
2. *Fibrations are the degree-wise epimorphisms.*
3. *Cofibrations are the retracts of monomorphims with projective cokernel.*

Furthermore, this model structure is proper, cofibrantly generated and combinatorial. Cofibrations and trivial cofibrations are generated by I and J, respectively.

Unlike the case for bounded chains, the cofibrations, or I, is not well-founded. However, we can obtain a new generating set of cofibrations following example 3.2. We consider the new set $I' \coloneqq I \cup \{0 \to S^{n-1} | n \in \mathbb{Z}\}$. Note that since $0 \to S^{n-1}$ is a cofibration, we are not changing the model structure. Therefore, the resulting theory can be written as follows:

$$\vdash n\text{-}\mathsf{Chain}\ \mathsf{Type}$$
$$x : (n-1)\text{-}\mathsf{Chain} \vdash \{y | dy = x\}\ \mathsf{Type}$$

for $n \in \mathbb{Z}$.

3.5 Topological spaces

Here we recall the Quillen model structure on the category of topological spaces **Top** [Qui06]. Recall that a map $f : X \to Y \in \mathbf{Top}$ is a *weak homotopy equivalence* if for all $x \in X$ and $n \geq 1$ the induced map $f_* : \pi_n(X,x) \to \pi_n(Y, f(x))$ is an isomorphism of groups and for $n = 0$ is a bijection. Additionally, the map f is a *Serre fibration* if for any CW-complex W the following square has a diagonal filler:

$$\begin{tikzcd}
A \times \{0\} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
A \times [0,1] \arrow[r] \arrow[ur, dashed] & Y.
\end{tikzcd}$$

Theorem 3.11. *The category* **Top** *has a model category structure such that:*

1. *Weak equivalences are the weak homotopy equivalences.*

2. *fibrations are the Serre fibrations.*

3. *Cofibrations are the maps with the left lifting property against trivial fibrations.*

Moreover, this model structure is cofibrantly generated. The generating cofibrations is the set of boundary inclusions $\{S^{n-1} \to D^n | n \in \mathbb{N}\}$. The set $\{D^n \to D^n \times [0,1] | n \in \mathbb{N}\}$ generates trivial cofibrations.

We can immediately write some of the relevant type axiom of the resulting theory:

- $\vdash 0\text{-}\mathsf{CW}\ \mathsf{Type}$.

- $x, y : 0\text{-}\mathsf{CW} \vdash 1\text{-}\mathsf{CW}(x,y)\ \mathsf{Type}$.

- $x : 0\text{-}\mathsf{CW}, \gamma : 1\text{-}\mathsf{CW}(x, x) \vdash 2\text{-}\mathsf{CW}(x, \gamma)$ Type.

- $\vdots$

It is well-known that there is no (algebraic) axiomatization for topological spaces. A common resort is to instead axiomatize the open or closed subsets. However, this is indirect and in general unsatisfactory. Note that the language associated to the model structure allows to express properties of topological spaces without reliying on a specific set of axioms.

3.6 Kan complexes and quasicategories

In this section we analyze two very well-known models structure on the category of simplicial sets **sSet**, the Kan–Quillen and the Joyal model structures. One interesting feature is that we obtain the same theory for both models, but under the light of theorem 2.43 meaningful statements is delimited by the fibrant objects. In the first model we are interested in Kan complexes. While in the second model in the quasicategories. The first model appears in [Qui06] and the second in [Joy08]. These are the first references one can find, but the literature is ample for both models.

Recall that a map $f : X \to Y$ between simplicial sets is a *Kan fibration* if it has the right lifting property for all horn inclusions *i.e.,* the solid diagram below a diagonal filler

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

for all $0 \leq k \leq n \in \mathbb{N}$. The simplicial set X is a *Kan complex* if the unique map t the terminal presheaf is a Kan fibration. This is the result from [Qui06]:

Theorem 3.12. *The category of simplicial sets* **sSet** *carries a model structure in which:*

1. *Weak equivalences are maps $f : X \to Y$ whose geometric realization $|f| : |X| \to |Y|$ is a weak homotopy equivalence in the category of topological spaces* **Top**. *These are called Kan equivalences.*

2. *Fibrations are the Kan fibrations.*

3. *Cofibrations are the monomorphisms*

The class of cofibrations is generated by $I := \{\partial^n \hookrightarrow \Delta[n] | n \in \mathbb{N}\}$ *and acyclic cofibrations are generated by* $J := \{\Lambda^k[n] \to \Delta[n] | n \in \mathbb{N}$ *and* $0 \leq k \leq n\}$.

Similarly, a map $f : X \to Y$ between simplicial sets is a *inner Kan fibration* if it has the right lifting property for all inner horn inclusions *i.e.,* the solid diagram below a diagonal filler

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

for all $0 < k < n \in \mathbb{N}$. The simplicial set X is a *quasi-category* if the unique map t the terminal presheaf is an inner Kan fibration. This is the result from [Joy08]:

Theorem 3.13. *The category of simplicial sets* **sSet** *carries a model structure in which:*

1. *Weak equivalences are the weak categorical equivalences.*
2. *Fibrations are the inner Kan fibrations.*
3. *Cofibrations are the monomorphisms*

The class of cofibrations is generated by $I := \{\partial\Delta[n] \hookrightarrow \Delta[n] | n \in \mathbb{N}\}$, *the set of boundary inclusions.*

Notice that both model structures have the same class of generating cofibrations. Hence, we expect that they have the same theories. We get a type for each cofibration in I. The list is as follows:

- $\vdash \mathsf{0\text{-}simplex}\ \mathsf{Type}$.
- $\sigma^0_0, \sigma^0_1 : \mathsf{0\text{-}simplex} \vdash \mathsf{1\text{-}simplex}(\sigma^0_0, \sigma^0_1)\ \mathsf{Type}$.
- $\sigma^0_0, \sigma^0_1, \sigma^0_2 : \mathsf{0\text{-}simplex}, \quad \sigma^1_0 : \mathsf{1\text{-}simplex}(\sigma^0_0, \sigma^0_1), \quad \sigma^1_1 : \mathsf{1\text{-}simplex}(\sigma^0_1, \sigma^0_2), \quad \sigma^1_2 : \mathsf{1\text{-}simplex}(\sigma^0_0, \sigma^0_2) \vdash \mathsf{2\text{-}simplex}(\sigma^0_0, \sigma^0_1, \sigma^0_2, \sigma^1_0, \sigma^1_1, \sigma^1_2)\ \mathsf{Type}$.

If we leave the dependencies implicitly, we can write succinct judgments that give us the sorts of this theory:

- $\vdash \mathsf{0\text{-}simplex}\ \mathsf{Type}$.
- $\sigma^0_0, \sigma^0_1 : \mathsf{0\text{-}simplex} \vdash \mathsf{1\text{-}simplex}(\sigma^0_0, \sigma^0_1)\ \mathsf{Type}$.

- $\coprod_{i=1}^{n}\{\binom{n}{i}\sigma^{n-i} : (n-i)\text{-}\mathsf{simplex}\} \vdash (n+1)\text{-}\mathsf{simplex}(\sigma_0^0, \sigma_1^0)\ \mathsf{Type}$, for $n \geq 1$.

One can introduce face and degeneracy operations as dependent types. As we anticipated, the only way to tell apart which formula are meaningful is through the fibrant objects, quasi–categories and Kan complexes, respectively.

3.7 Reedy languages

{reedy-languages:sec}

The pourpose of this subsection is to provide languages for a presheaf category over a Reedy category. This encompases some of the the previous examples and opens the door to further applications.

Recall that if $\mathcal{M}$ is a cofibrantly generated model category whose cofibrations are generated by a well-founded set of cofibrations I then for each cofibration $A \hookrightarrow B \in I$ we can associate a type introduction axiom $\bar{A} \vdash \bar{B}\ \mathsf{Type}$, where $\bar{A}$ is well-formed context previously constructed.

Let K be a Reedy category with degree function $\deg : R \to \omega$. This restriction is artificial since we could consider more general Reedy categories, however, for the examples this construction is aimed at this is enough. The objects of K have well-founded relation induced by the degree function.

Construction 3.14. Let $\partial\text{よ}_k$ be the latching object of the representable functor よ_k and $d_k : \partial\text{よ}_k \to \text{よ}_k$ the induced map. There is a bifunctor

$$\otimes : \mathbf{Set}^{K^{\mathrm{op}}} \times M \to M^{K^{\mathrm{op}}}$$

defined by $(A \otimes X)_k \coloneqq \coprod_{A_k} X$. Let I as above, given $i : X \to Y \in I$ and $k \in K$ we apply the usual Leibniz construction and obtain the dashed arrow below

$$\begin{tikzcd}
\partial\text{よ}_k \otimes X \arrow[r] \arrow[d] & \text{よ}_k \otimes X \arrow[d] \arrow[ddr, bend left] & \\
\partial\text{よ}_k \otimes Y \arrow[r] \arrow[drr, bend right] & \partial\text{よ}_k \otimes Y \coprod_{\partial\text{よ}_k \otimes X} \text{よ}_k \otimes X \arrow[dr, dashed, "d_k \hat{\otimes} i"] & \\
 & & \text{よ}_k \otimes Y.
\end{tikzcd}$$

We now consider the set of maps $K\hat{\otimes}I \coloneqq \{i\hat{\otimes}d_k | i \in I,\, k \in K\}$. By identifying each map $d_k\hat{\otimes}i \in K\hat{\otimes}I$ with a pair (k, i), we see that $K\hat{\otimes}I$ is also a well-founded relation which we denote by $\leq_{\otimes}$.

The previous construction is further justified by [Bar19, Proposition 2.3.22] for the premodel categories, but a similar description is abundant appear in the literature for Quillen model categories.

Proposition 3.15. *The Reedy weak factorization system on $\mathcal{M}^{K^{op}}$ is generated by $K\hat{\otimes}I$, and therefore the Reedy model category structure on it is combinatorial whenever $\mathcal{M}$ is combinatorial.*

Remark 3.16. The previous construction allows to identify the language associated to $\mathcal{M}^{K^{\mathrm{op}}}$. In particular, if $d_k\hat{\otimes}i \in K\hat{\otimes}I$ then

$$(\partial よ_k \otimes Y \coprod_{\partial よ_k \otimes X} よ_k \otimes X) \leq_{\otimes} よ_k \otimes Y.$$

Therefore, inductively obe has a well-formed context

$$\overline{\partial よ_k \otimes Y \coprod_{\partial よ_k \otimes X} よ_k \otimes X}$$

over which one constructs a new type $\overline{よ_k \otimes Y}$. This is what we conceive as the Reedy language of $\mathcal{M}^{K^{\mathrm{op}}}$.

Observation 3.17. Many models for higher categories are build starting with presheaves over a Reedy category. Then to obtain the desired model one takes a left Bousfield localization for an appropriate class of maps. Importantly, this localization does not change the generating cofibrations. This is just to say that the language of $\mathcal{M}^{K^{\mathrm{op}}}$ remains the same after localization.

We demonstrate this construction for Segal spaces. However, the construction applies to any other model category constructed in a similar fashion.

3.8 Segal spaces

We denote $\mathbf{ssSet} \coloneqq [\Delta^{\mathrm{op}}, \mathbf{sSet}]$ the category of simplicial spaces. This category has two model structures that are obtained as left Bousfield localizations of the Reedy model structure. For both of these localizations we use the Kan-Quillen model structure from the previous section. Recall that this model structure is cofibrantly generated. The set of generating cofibrations are the boundary inclusions. We will use the following facts and notation.

- There is an adjunction of two variables $\square : \mathbf{sSet} \times \mathbf{sSet} \to \mathbf{ssSet}$ defined as $(X \square Y)_{mn} \coloneqq X_m \times Y_n$ for each $m, n \in \mathbb{N}$. This is called the box product.

- **sSet** can be seen as vertically embedded into **ssSet**. If $X \in$ **sSet** then we it can bee seen as a simplicial space $X \square \Delta[0]$. There is also a horizontal embedding by setting $\Delta[0] \square X$.
- For $[m] \in \Delta$ we write $F(n) := \Delta[n] \square \Delta[0]$ and $\partial F(n) := \partial \Delta[n] \square \Delta[0]$.
- The simplicial spaces $F(n)$ represent the n-th mapping space functors, respectively $Map(F(n), X) = X_n$.

There is map $\iota : F(1) \coprod_{F(0)} \cdots \coprod_{F(0)} F(1) \to F(n)$, where the colimit on left has n factors. The following two model category structures were constructed by Rezk [Rez01].

Theorem 3.18. *The category admits a unique simplicial model category structure such that:*

1. *The cofibrations are the monomorphisms.*
2. *Fibrant objects are simplicial spaces X such that the map*
$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$
induced by ι is a Kan equivalence. These objects are called Segal spaces.
3. *The weak equivalences are the maps $f : X \to Y \in$* **ssSet** *such that*
$$Map(f, W) : Map(Y, W) \to Map(X, W)$$
is a Kan equivalence for every Segal space W.
4. *A map $f : X \to Y$ between Segal spaces is a fibration (weak equivalence) if and only if is a Reedy fibration (Reedy weak equivalence).*

Recall that $\mathcal{J}$ denotes the category with two objects and two arrows that are mutually inverses. It is usual to denote by $E(1)$ to the Segal space which is obtained by considering the nerve $N\mathcal{J}$ as a discrete simplicial space. This produces a map $F(1) \to E(1)$.

Theorem 3.19. *The category admits a unique simplicial model category structure such that:*

1. *The cofibrations are the monomorphisms.*
2. *Fibrant objects are Segal spaces X such that the map*
$$Map(E(1), X) \to Map(F(0), X)$$
is a Kan equivalence. These objects are called complete Segal spaces.

3. *The weak equivalences are the maps $f : X \to Y \in$ **ssSet** such that*

$$Map(f, W) : Map(Y, W) \to Map(X, W)$$

is a Kan equivalence for every complete Segal space W.

4. *A map $f : X \to Y$ between complete Segal spaces is a fibration (weak equivalence) if and only if is a Reedy fibration (Reedy weak equivalence).*

These models are cofibrantly generated. The set of generating cofibrations can be described using the box product [JT07, Proposition 2.2]. This set is given by $\hat{I} := \{d_m \hat{\Box} d_n | m, n \in \mathbb{N}\}$. Explicitly a map in $\hat{I}$ is of the form

$$d_m \hat{\Box} d_n : \partial\Delta[m]\Box\Delta[n] \coprod_{\partial\Delta[m]\Box\partial\Delta[n]} \Delta[m]\Box\partial\Delta[n] \to \Delta[m]\Box\Delta[n]$$

We can obtain the generalized algebraic theory for (complete) Segal space. The domains of these maps provide the context in which a new type is formed. For example, when $n = 0$ the resulting subset of maps are of the the form

$$d_m \hat{\Box}\Delta[0] : \partial\Delta[m]\Box\Delta[0] \to \Delta[m]\Box\Delta[0],$$

this is simply the map $\partial F(m) \to F(m)$. In this setting, for each $m \in \mathbb{N}$ we can write the folowing types:

- $\vdash \mathsf{0\text{-}space}\ \mathsf{Type}$.
- $x, y : \mathsf{0\text{-}space} \vdash \mathsf{1\text{-}space}(x, y)\ \mathsf{Type}$.
- $x, y, z : \mathsf{0\text{-}space}, f : \mathsf{1\text{-}space}(x, y), g : \mathsf{1\text{-}space}(y, z), h : \mathsf{1\text{-}space}(x, z) \vdash \mathsf{2\text{-}space}(x, y, z, f, g, h)$.
- $\vdots$

In general, the description of the context for a new type can be involved.

3.9 Functors and Isofibrations

We denote $[1] := \{0 \to 1\}$ the category with two objects and single non-identity arrow. This category can be viewed as a Reedy category in two ways. The first one respects the direction of the arrow, so we take $[1]_+$ to be the non-identity map. While for the second we take the same map to be in $[1]_-$, we denote this by $[1]^{\mathsf{op}}$.

Given a model category $\mathcal{C}$ we denote $\mathcal{C}^{[1]}$ and $\mathcal{C}^{[1]^{\mathsf{op}}}$ with the corresponding Reedy model structures.

Proposition 3.20. *The Reedy model structure on $\mathcal{C}^{[1]}$ coincides with the projective model structure. In particular, weak equivalences and fibrations are the level-wise weak equivalences and fibrations in $\mathcal{C}$. The cofibrations in this model can be described as those maps $A \to B \in \mathcal{C}^{[1]}$ such that $A_0 \to B_0$ and $B_0 \sqcup_{A_0} A_1 \to B_1$ are cofibrations of $\mathcal{C}$.*

We are interested in the particular case of $\mathcal{C} = \mathbf{Cat}$. Since fibrations are level-wise, the fibrant objects in $\mathbf{Cat}^{[1]}$ are diagrams X where the unique map to the terminal diagram $\mathbf{1}$ is a fibration *i.e.,* each functor $X_0 \to \mathbf{1}$, $X_1 \to \mathbf{1}$ are isofibrations in $\mathbf{Cat}$, but no further condition on the functor $X_0 \to X_1$. Therefore, the language for this model structure is the one for all functors.

Proposition 3.21. *The Reedy model structure on $\mathcal{C}^{\leftarrow}$ coincides with the injective model structure. In particular, weak equivalences and cofibrations are the level-wise weak equivalences and cofibrations in $\mathcal{C}$. The fibrations in this model can be described as those maps $X \to Y \in \mathcal{C}^{[1]}$ such that $X_1 \to Y_1$ and $X_0 \to X_1 \times_{Y_1} Y_0$ are fibrations of $\mathcal{C}$.*

We find that fibrant objects in $X \in \mathbf{Cat}^{[1]}$ are those such that $X_0 \to X_1$ is an isofibration. The language associated to the model structure on $\mathbf{Cat}^{[1]}$ is the language for isofibrations.

4 Language invariance under Quillen equivalences

{sec:invariance}

The main goal of this section is to show that the language associated to a weak model category from section 2.4 is invariant under weak Quillen equivalences between weak model categories, more precisely, in section 4.3 we show:

{third-invariance:thm}

Theorem 4.1 (Third invariance theorem)**.** *Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ a Quillen equivalence between weak model categories. Then for any cofibrant object $A \in \mathcal{M}$, the induced map*

$$h\mathbb{L}F_A : h\mathbb{L}^{\mathcal{M}}_{\lambda}(A) \to h\mathbb{L}^{\mathcal{N}}_{\lambda}(FA)$$

is an isomorphism.

Reid Barton [Bar19] constructs a model 2-category structure on the 2-category of model categories. In this construction there is a class of left Quillen equivalences satisfying an additional property, see definition 4.10, which play the role of trivial fibrations in this model 2-category. In this section, we call those functors *Barton trivial fibrations*. We use these functors to prove our result.

Firstly, in section 4.1 we show that the result holds for Barton trivial fibrations. What we do next is a reduction step to the case of Barton trivial trivial fibrations. We use section 4.2 to prove in theorem 4.38 that any left Quillen functor, part of a Quillen equivalence between weak model categories can be factorized as a section of a Barton trivial fibration followed by a barton trivial fibration. After this is done, the proof of theorem 4.1 is straightforward.

4.1 Invariance along Barton trivial fibrations

{invariance-along-triv

Recall from construction 2.38 that given $\mathcal{M}$ a weak model category, the language of $\mathcal{M}$ is the language of the clan $(\mathcal{M}^{\mathrm{Cof}})^{\mathrm{op}}$ *i.e.,* for any regular cardinal λ, it is the λ-boolean algebra $\mathbb{L}_\lambda^{(\mathcal{M}^{\mathrm{Cof}})^{\mathrm{op}}}$ over $(\mathcal{M}^{\mathrm{Cof}})^{\mathrm{op}}$. Which is simply denoted by $\mathbb{L}_\lambda^{\mathcal{M}}$.

Remark 4.2. The fibrations in the clan $(\mathcal{M}^{\mathrm{Cof}})^{\mathrm{op}}$ are precisely the core cofibrations of $\mathcal{M}$. Therefore, the first condition of definition 2.25 for the language can be read as: For any core cofibration $f : \Gamma \hookrightarrow \Gamma'$ in $\mathcal{M}$, the induced map $f^* : \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma) \to \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma')$ has a left adjoint $\exists_\pi : \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma') \to \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$. Similarly, the Beck-Chevalley condition involves core cofibrations only.

{left-quillen:map-langu

Construction 4.3. We apply construction 2.34 when we have a left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ between weak model categories. We can consider the underlying morphism of coclans $F : \mathcal{M}^{\mathrm{Cof}} \to \mathcal{N}^{\mathrm{Cof}}$ and see this as a morphism between clans $F : (\mathcal{M}^{\mathrm{Cof}})^{\mathrm{op}} \to (\mathcal{N}^{\mathrm{Cof}})^{\mathrm{op}}$ in the obvious way. And let $\mathbb{L}_\lambda^{\mathcal{M}}$ and $\mathbb{L}_\lambda^{\mathcal{N}}$ their respective initial λ-boolean algebras as in construction 2.38. From the fact that $\mathbb{L}_\lambda^{\mathcal{M}}$ is initial there is a morphisms of λ-boolean algebras

$$\mathbb{L}F : \mathbb{L}_\lambda^{\mathcal{M}} \to F^*\left(\mathbb{L}_\lambda^{\mathcal{N}}\right).$$

For any $\Gamma \in \mathcal{M}$ and any formula $\phi \in \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ we denote $F(\phi) \coloneqq \mathbb{L}F_\Gamma(\phi)$ which is a formula in context $F(\Gamma)$ *i.e.,* an element of $\mathbb{L}_\lambda^{\mathcal{N}}(F(\Gamma))$.

{formula-adjuntion:vali

Observation 4.4. We can reinterpret proposition 2.35 in the presence of $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$ a Quillen adjunction between weak model categories. Let $\Gamma \in \mathcal{M}$ be a cofibrant object and $X \in \mathcal{N}$ be a fibrant object. Then for any $v : F(\Gamma) \to X$ and $v' : \Gamma \to G(X)$ its adjoint transpose we have

$$G(X) \vdash \phi(v') \Leftrightarrow X \vdash F(\phi)(v).$$

In definition 2.11 we defined the the relation "$\dashv\vdash_\Gamma$" between formulas in context Γ to encode when formulas are "syntactically equivalent", or that two formulas are mutually provable. Now we turn our attention to a more semantical relation on formulas. This relation will allows to construct boolean algebras over the homotopy category of $\mathcal{M}$.

Definition 4.5. Let Γ be a cofibrant object of $\mathcal{M}$. Two formulas $\phi, \psi \in \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ are said to be *semantically equivalent* if for all fibrant objects $X \in \mathcal{M}$ we have $|\phi|_X = |\psi|_X$. In this situation we write $\phi \approx \psi$.

{semantic-rel:equivalen

Observation 4.6. For any cofibrant object $\Gamma \in \mathcal{M}$, the relation $\approx$ on $\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ is an equivalence relation. Furthermore, $\approx$ is compatible with the λ-boolean algebra structure of $\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$.

For any Γ cofibrant we get a boolean algebra $h\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ whose elements are equivalence classes of formula in $\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ under the relation $\approx$. For each formula $\phi \in \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ we write $[\phi]$ for its equivalence class, these are exactly the elements of $h\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$.

It is obvious, but worth mentioning, that if $\phi, \psi \in \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ are such that $\phi \approx \psi$, then

$$X \vdash \phi(v) \Leftrightarrow X \vdash \psi(v)$$

for any fibrant object $X \in \mathcal{M}$ and map $v : \Gamma \to X$. {map-hlang}

Remark 4.7. If $F : \mathcal{M} \to \mathcal{N}$ is a left Quillen functor then we can use construction 4.3 to obtain a map between the respective languages

$$\mathbb{L}F : \mathbb{L}_\lambda^{\mathcal{M}} \to F^*\left(\mathbb{L}_\lambda^{\mathcal{N}}\right).$$

Now, observation 4.6 implies that this map descends to a map between λ-boolean algebras

$$h\mathbb{L}F : h\mathbb{L}_\lambda^{\mathcal{M}} \to F^*\left(h\mathbb{L}_\lambda^{\mathcal{N}}\right).$$

By definition $h\mathbb{L}F_\Gamma([\phi]) := [F(\phi)]$, and we write $F[\phi]$ for this. To see that this is well-defined we observe that given formulas $\phi, \psi \in \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ such that $\phi \approx \psi$ then $F(\phi) \approx F(\psi)$. Indeed, take $\Gamma \in \mathcal{M}$ cofibrant, $X \in \mathcal{N}$ fibrant, $v : F(\Gamma) \to X$ and $v' : \Gamma \to G(X)$ its adjoint transpose, then

$$X \vdash F(\phi)(v) \Leftrightarrow G(X) \vdash \phi(v') \Leftrightarrow G(X) \vdash \psi(v') \Leftrightarrow X \vdash F(\psi)(v)$$

{induced-map:homotopy-l

Proposition 4.8. *Let $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$ be a Quillen adjunction between weak model categories. For any cofibrant object $\Gamma \in \mathcal{M}$, fibrant object $X \in \mathcal{N}$, morphism $v : F(\Gamma) \to X$ with adjoint transpose $v' : \Gamma \to G(X)$ and $[\phi] \in h\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$, we have*

$$G(X) \vdash [\phi](v') \Leftrightarrow X \vdash F([\phi])(v).$$

Proof. This follows immediately from remark 4.7 and observation 4.4. $\square$

Correct this

{induced-functor:inject

Lemma 4.9. *Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ a Quillen equivalence. Then for any cofibrant object $\Gamma \in \mathcal{M}$, the induced map $h\mathbb{L}F_\Gamma : h\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma) \to h\mathbb{L}_\lambda^{\mathcal{N}}(F\Gamma)$ is injective.*

Proof. Let ϕ and ψ be formulas in $\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ such that $F(\phi) \approx F(\psi)$ *i.e.*, $F(\phi)$ and $F(\psi)$ are equal in $h\mathbb{L}_\lambda^{\mathcal{N}}(F\Gamma)$. We must show that $\psi \approx \phi$. Let X be a fibrant object of $\mathcal{M}$. The Quillen equivalence induces an equivalence between homotopy categories $Ho(G) : Ho(\mathcal{N}^{\mathrm{FIB}}) \to Ho(\mathcal{M}^{\mathrm{FIB}})$. Hence, there is a fibrant object $Y \in \mathcal{N}$ such that GY is isomorphic to X in $Ho(\mathcal{M}^{\mathrm{FIB}})$. Given any $x : \Gamma \to X$, denote by $y : \Gamma \to GY$ any map such that the following triangle

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[dr, "y"'] & X \arrow[d, "\cong"] \\ & GY \end{tikzcd}$$

commutes in $Ho(\mathcal{M}^{\mathrm{FIB}})$. Lastly, let $y' : F\Gamma \to Y$ the transpose of y via the Quillen adjunction. It follows from the first invariance theorem theorem 2.43 that $X \vdash \phi(x)$ if and only if $GY \vdash \phi(y)$. From proposition 4.8 this is equivalent to $Y \vdash F(\psi)(y')$. By assumption $F(\phi) \approx F(\psi)$, so $Y \vdash F(\psi)(y')$. Again, this is $GY \vdash \psi(y)$ and $X \vdash \psi(x)$. This establishes the equality $|\phi|_X = |\psi|_X$ for all $X \in \mathcal{M}$ fibrant, which proves the statement. $\square$

{extensive-morphism:de

Definition 4.10. Let $F : \mathcal{C} \to \mathcal{D}$ a morphism between κ-coclans. We say that F is *extensive* if for every object in $X \in \mathcal{C}$ and for any cofibration $g : FX \hookrightarrow Y \in \mathcal{D}$ there exists $f : X \to Z$ and an isomorphism $\theta : F(Z) \cong Y$ making the obvious triangle commutative.

Dually, $F : \mathcal{C} \to \mathcal{D}$ a morphism between κ-clans is *extensive* if the induce map of κ-coclans $F^{\mathsf{op}} : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}^{\mathsf{op}}$.

In our setting, a functor $F : \mathcal{M} \to \mathcal{N}$ between weak model categories will be called extensive if the morphism of coclans $F : \mathcal{M}^{\mathrm{COF}} \to \mathcal{N}^{\mathrm{COF}}$ is extensive.

The terminology *extensive* in the definition above for both clans and coclans, instead of “extensive” and “co-extensive”, is simply because it is always clear whether refers to fibrations or cofibrations. This is because, for example, when considering a morphism between clans the relevant structure that ought to be preserved is that related to fibrations. The name extensive from definition 4.10 is adapted from Reid Barton’s PhD thesis [Bar19, Definition 8.3.1] where is used in the context of premodel categories.

{trivial-fibration:sur

Lemma 4.11. *Let $F : \mathcal{M} \to \mathcal{N}$ be an extensive morphism between κ-clans and $\Gamma \in \mathcal{M}$. Then, any formula $\Phi \in \mathbb{L}^{\mathcal{N}}_{\lambda}(F\Gamma)$ is the image by F of a formula $\Phi_0 \in \mathbb{L}^{\mathcal{M}}_{\lambda}(\Gamma)$.*

Proof. Since every κ-clan is of the form $\mathbb{C}_T$ for some T generalized κ-algebraic theory it is enough to show the result is valid for the syntactic definition of language as in definition 2.1. We prove by induction on formulas $\Phi \in \mathbb{L}^{\mathcal{N}}_{\lambda}(\Delta)$ that given any context Γ such that $f : \Delta \cong F(\Gamma)$ there is a formula $\Phi_0 \in \mathbb{L}^{\mathcal{M}}_{\lambda}(\Gamma)$ such that $f^*(F\Phi_0) = \Phi$.

1. When $\Phi = \top$ or $\Phi = \bot$, then this can clearly be lifted to $\top$ and $\bot$.

2. If $\Phi = \neg\Psi$ or $\Phi = \bigvee_{i\in I} \Psi_i$ or $\Phi = \bigwedge_{i\in I} \Psi_i$ then it is also clear that Φ can be lifted. Indeed, we can simply use the inductive hypothesis to lift each Ψ_i and then use the boolean algebra structure to conclude.

3. Suppose that Φ is of the form $\exists_\pi \Psi$ or $\forall_\pi \Psi$ for some fibration $\pi : \Gamma' \twoheadrightarrow F(\Gamma)$. Here the formula $\Psi \in \mathbb{L}^{\mathcal{N}}_{\lambda}(\Gamma')$, so $\Phi \in \mathbb{L}^{\mathcal{N}}_{\lambda}(F\Gamma)$. Furthermore, we assume that Ψ can be lifted. Since F is a trivial fibration, there is a lift $\bar{\pi} : \bar{\Gamma}' \to \Gamma \in \mathcal{M}$ of $\pi : \Gamma' \twoheadrightarrow F(\Gamma)$, which comes with an isomorphism $g : \Gamma' \cong F(\bar{\Gamma}')$ such that the following triangle commutes

$$\begin{tikzcd} \Gamma' \arrow[r, two heads, "\pi"] \arrow[d, "g", "\cong"'] & F(\Gamma) \\ F(\bar{\Gamma}'). \arrow[ur, two heads, "F(\bar{\pi})"'] & \end{tikzcd}$$

   Therefore, we get a commutative square as in the left, and at the level of languages as on the right

$$\begin{tikzcd} \Gamma' \arrow[r, two heads, "\pi'"] \arrow[d, "g", "\cong"'] & \Delta \arrow[d, "f"', "\cong"] \\ F(\bar{\Gamma}') \arrow[r, two heads, "F(\bar{\pi})"'] & F(\Gamma) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{L}^{\mathcal{N}}_{\lambda}(F(\bar{\Gamma}')) \arrow[r, "\exists_{\pi'}"] \arrow[d, "g^*"'] & \mathbb{L}^{\mathcal{N}}_{\lambda}(F(\Gamma)) \arrow[d, "f^*"] \\ \mathbb{L}^{\mathcal{N}}_{\lambda}(\Gamma') \arrow[r, "\exists_{F(\bar{\pi})}"'] & \mathbb{L}^{\mathcal{N}}_{\lambda}(\Delta). \end{tikzcd}$$

   By assumption $\psi \in \mathbb{L}^{\mathcal{N}}_{\lambda}(\Gamma')$ can be lifted. Hence, there is a formula $\Psi_0 \in \mathbb{L}^{\mathcal{M}}_{\lambda}(\bar{\Gamma}')$ such that $g^*(F\Psi_0) = \Psi$. Using the right hand square above, one can see that $\exists_{\bar{\pi}}\Psi_0$ is a lift for Φ.

□

There is an immediate consequence of lemma 4.11, namely when the map $F : \mathcal{M} \to \mathcal{N}$ is a *trivial fibration* (which we make precise below) between weak model categories.

Definition 4.12. Let $F : \mathcal{M} \to \mathcal{N}$ a left Quillen equivalence between weak model categories. We say that F is a *Barton trivial fibration* if it is extensive as a morphism between of the coclans $\mathcal{M}^{\text{Cof}}$ and $\mathcal{N}^{\text{Cof}}$.

Barton trivial fibrations are exactly the trivial fibrations in [Bar19] in the model 2-category of model categories. As the reader might anticipate, the notion of fibration between model categories exists as well, but we make no use of these.

{trivial-fibration:inva

Theorem 4.13. *Let $F : \mathcal{M} \to \mathcal{N}$ be a Barton trivial fibration between weak model categories. Then for any cofibrant $\Gamma \in \mathcal{M}$ the induced map $h\mathbb{L}F_A : h\mathbb{L}^{\mathcal{M}}_{\lambda}(\Gamma) \to h\mathbb{L}^{\mathcal{N}}_{\lambda}(F\Gamma)$ is an isomorphism.*

Proof. By the previous lemma 4.9 we know that $h\mathbb{L}F_\Gamma : h\mathbb{L}^{\mathcal{M}}_{\lambda}(\Gamma) \to h\mathbb{L}^{\mathcal{N}}_{\lambda}(\Gamma)$ is injective. Next we can use lemma 4.11 by observing that this surjectivity also descends at the level of $h\mathbb{L}F_\Gamma : h\mathbb{L}^{\mathcal{M}}_{\lambda}(\Gamma) \to h\mathbb{L}^{\mathcal{N}}_{\lambda}(\Gamma)$. $\square$

Since our goal is to prove the third invariance theorem, with theorem 4.13 at hand, we simply need to reduce our problem to the case in which the we have Barton trivial fibrations. The constructions to come are essentially the necessary steps for this reduction process.

4.2 Path objects for weak model categories

{path-objects}

The constructions are somewhat convoluted, so we give a high level overview of what we do. In a nutshell, given a weak model category $\mathcal{M}$ and two categories I and J (to be specified) we obtain a commutative diagram of weak model categories

$$\begin{tikzcd} \mathcal{M}^J \arrow[r] \arrow[d, two heads, "\sim"'] & \mathcal{M}^I \arrow[d] \\ \mathcal{M} \arrow[r] & \mathcal{M} \times \mathcal{M} \end{tikzcd}$$

were the arrow on the left and the two maps $\mathcal{M}^I \to \mathcal{M}$ induced by the projections are Barton trivial fibrations. More precisely, the construction we do takes as input a left Quillen equivalence $F : \mathcal{M} \to \mathcal{N}$ between weak model categories and produces a diagram

$$\begin{tikzcd}
\mathcal{M}^J \arrow[r] \arrow[d, two heads, "\sim"'] & \mathcal{N}^I_F \arrow[d] \\
\mathcal{M} \arrow[r] & \mathcal{N} \times \mathcal{N}
\end{tikzcd}$$

were again the arrow on the left and the two maps $\mathcal{N}^I_F \to \mathcal{N}$ induced by the projections are Barton trivial fibrations. Hence, the first diagram is a particular case when $F = Id_{\mathcal{M}}$.

The bulk of the work lies in endowing the model categories with the correct weak model structure. This can be summarized as follows: We start with the Reedy weak model structure on the category $\mathcal{M}^J$, or $\mathcal{N}^I$, and perform a "right Bousfield localization" to obtain our desired models.

Remark 4.14. The weak model structure we obtain on $\mathcal{M}^J$ encodes objects in M with a cylinder object. On the other hand the resulting weak model structure on $\mathcal{N}^I$ encodes pair of objects in $\mathcal{N}$ with a correspondence between them.

4.2.1 Construction of path objects

We start by fixing a weak model category $\mathcal{M}$ and let J be the category

$$a \overset{i}{\underset{j}{\rightrightarrows}} b \xrightarrow{k} c$$

such that $ki = kj$. Consider the degree function making J into a direct category, $\deg(a) = 0,\ \deg(b) = 1,\ \deg(c) = 2$. Our first goal is to prove:

{model-paths:wms}

Theorem 4.15. *The category of diagrams $\mathcal{M}^J$ with the Reedy weak model structure has another a weak model structure were*

1. *A map between diagrams $X \to Y$ is a cofibration if*
   - *(a) It is a Reedy cofibration,*
   - *(b) $Y_a \sqcup_{X_a} X_c \overset{\sim}{\hookrightarrow} Y_c$ and $Y_b \sqcup_{X_b} X_c \overset{\sim}{\hookrightarrow} Y_c$ are trivial cofibrations in $\mathcal{M}$.*
2. *Fibrations are level-wise fibrations.*

Notation 4.16. For the sake of clarity, we denote by $\mathcal{M}^J_{Reedy}$ when refering to the Reedy weak model structure and $\mathcal{M}^J_{Loc}$ for the weak model structure of theorem 4.15. Of course, *a priori* we have yet to prove that the last is indeed a weak model structure. Therefore, whenever we say, for example, that a map $f : X \to Y$ is a cofibration we just mean that f satisfies the corresponding condition of theorem 4.15.

We will justify that the following construction, which is simply the conditions of the theorem, is the correct one.

{cofibrations-wms-cyli

Construction 4.17. Consider the category of diagrams $\mathcal{M}^J$ with the Reedy weak model structure from theorem C.11. Additionally, we say that a map of diagrams $f : X \to Y$ is a *cofibration* if is a Reedy cofibration and if the induced maps

$$Y_a \sqcup_{X_a} X_c \overset{\sim}{\hookrightarrow} Y_c \text{ and } Y_b \sqcup_{X_b} X_c \overset{\sim}{\hookrightarrow} Y_c$$

are trivial cofibrations in $\mathcal{M}$. The class of *fibrations* are Reedy fibrations, which are level-wise fibrations since the category J is directed.

{trivialcofibrations-be

Observation 4.18. One can verify that in this new model structure the core fibrations and core trivial cofibrations coincide with the ones in the Reedy weak model structure (see lemma 4.20).

The reader might suspect that this is not a fortuitous coincidence, this suspicions is well justified. As we mentioned, what we have done is a right Bousfield localization of a Reedy weak model structure on $\mathcal{M}^J$. Such localizations are studied in [Hen23] in the case when $\mathcal{M}$ is a combinatorial (accessible) weak model category. Due to the lack of a general theorem that justifies the existence of these localizations indeed produce a weak model category, we verify all required conditions by hand.

We examine the class of cofibrations. For a diagram $X \in \mathcal{M}^J$ the latching objects are $L_aX = \emptyset$, $L_bX = X_a \sqcup X_a$ and $L_cX = X_b \sqcup_{X_a} X_b$. These are cofibrant in $\mathcal{M}$. Then a map $f : X \to Y$ being a cofibration means that $X_a \hookrightarrow Y_a$,

$$X_b \sqcup_{X_a \sqcup X_a} (Y_a \sqcup Y_a) \hookrightarrow Y_b \text{ and } X_c \sqcup_{(X_b \sqcup_{X_a} X_b)} (Y_b \sqcup_{Y_a} Y_b) \hookrightarrow Y_c$$

are cofibrations in $\mathcal{M}$, and additionally $Y_a \sqcup_{X_a} X_c \overset{\sim}{\hookrightarrow} Y_c$ and $Y_b \sqcup_{X_b} X_c \overset{\sim}{\hookrightarrow} Y_c$ are trivial cofibrations in $\mathcal{M}$.

Therefore, a diagram $Y \in \mathcal{M}^J$ is *cofibrant* if Y_a is a cofibrant object in $\mathcal{M}$,

$$Y_a \sqcup Y_a \hookrightarrow Y_b \text{ and } Y_b \sqcup_{Y_a} Y_b \hookrightarrow Y_c$$

are cofibrations, and additionally $Y_a \overset{\sim}{\hookrightarrow} Y_c$ and $Y_b \overset{\sim}{\hookrightarrow} Y_c$ are trivial cofibrations. Spelling out the second Reedy condition give us the following commutative diagram:

$$\begin{tikzcd}
\emptyset \arrow[r, hook] \arrow[d, hook] & Y_a \arrow[d, hook] \arrow[ddr, bend left] & \\
Y_a \arrow[r, hook] \arrow[drr, bend right] & Y_a \sqcup Y_a \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, hook] & \\
& & Y_b
\end{tikzcd}$$

This says that both maps $Y_a \overset{Yi}{\underset{Yj}{\rightrightarrows}} Y_b$ are cofibrations. We can use this on the following diagram

$$\begin{tikzcd}
Y_a \arrow[r, hook] \arrow[d, hook] & Y_b \arrow[d, hook] \arrow[ddr, bend left] & \\
Y_b \arrow[r, hook] \arrow[drr, bend right] & Y_b \sqcup_{Y_a} Y_b \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, hook] & \\
& & Y_c
\end{tikzcd}$$

to conclude that $Y_b \hookrightarrow Y_c$ is a cofibration. Of course this is in principle not necessary since we also have $Y_b \overset{\sim}{\hookrightarrow} Y_c$ is a trivial cofibration, the novel aspect is that this follows only from Reedy cofibrancy. We also have a trivial cofibration $Y_a \overset{\sim}{\hookrightarrow} Y_c$, by the two-out-of-three property the maps $Y_a \overset{Yi}{\underset{Yj}{\rightrightarrows}} Y_b$ are trivial cofibrations. We collect the above in the following:

Remark 4.19. If Y is cofibrant then we obtain the following diagram:

$$\begin{tikzcd}
Y_a \sqcup Y_a \arrow[r, "\triangledown"] \arrow[d, hook] & Y_a \arrow[d, hook, "\sim"] \\
Y_b \arrow[r, hook, "\sim"] & Y_c.
\end{tikzcd}$$

This is just to say that cofibrant diagrams of $\mathcal{M}^J_{Loc}$ encode objects of $\mathcal{M}$ for which a weak cylinder exists in the sense of construction C.6.

We reiterate that our goal is to show that the category of diagrams $\mathcal{M}^J_{Loc}$ has a weak model structure on it where the cofibrations are the ones from construction 4.17. We begin by showing:

{localized-triv-cof:lem

Lemma 4.20. *Let $X \in \mathcal{M}^J_{Loc}$ cofibrant and $X \to Z \in \mathcal{M}^J_{Reedy}$ a Reedy trivial cofibration. Then Z is cofibrant in $\mathcal{M}^J_{Loc}$. Furthermore, $X \to Z$ is a trivial cofibration in $\mathcal{M}^J_{Loc}$.*

Proof. Since $X \overset{\sim}{\hookrightarrow} Z$ is a Reedy trivial cofibration then $X_a \overset{\sim}{\hookrightarrow} Z_a$, $X_b \sqcup_{X_a \sqcup X_a} (Z_a \sqcup Z_a) \overset{\sim}{\hookrightarrow} Z_b$ and $X_c \sqcup_{(X_b \sqcup_{X_a} X_b)} (Z_b \sqcup_{Z_a} Z_b) \overset{\sim}{\hookrightarrow} Z_c$ are trivial cofibrations. We then obtain the following diagram:

$$\begin{tikzcd}
X_a \sqcup X_a \arrow[r] \arrow[d, hook, "\sim"] & X_b \arrow[d, hook, "\sim"] \arrow[ddr, bend left] & \\
Z_a \sqcup Z_a \arrow[r] & \bullet \arrow[dr, hook, "\sim"] & \\
& & Z_b
\end{tikzcd}$$

This shows that $X_b \overset{\sim}{\hookrightarrow} Z_b$ is a trivial cofibration. Since X is cofibrant then all the maps in the diagram

$$\begin{tikzcd}
X_a \arrow[r, shift left] \arrow[r, shift right] & X_b \arrow[r] & X_c
\end{tikzcd}$$

are trivial cofibrations. Consider the commutative diagram where the back and front faces are pushouts

$$\begin{tikzcd}
X_a \arrow[rr, hook, "\sim"] \arrow[dr, hook, "\sim"] \arrow[dd, hook, "\sim"] & & X_b \arrow[dr, hook, "\sim"] \arrow[dd, hook, "\sim"] & \\
& Z_a \arrow[rr, hook, "\sim"] \arrow[dd, hook, "\sim"] & & Z_b \arrow[dd, hook, "\sim"] \\
X_b \arrow[rr, hook, "\sim"] \arrow[dr, hook, "\sim"] & & X_b \sqcup_{X_a} X_b \arrow[dr] & \\
& Z_b \arrow[rr, hook, "\sim"] & & Z_b \sqcup_{Z_a} Z_b,
\end{tikzcd}$$

which, by the two-out-of-three, shows that $X_b \sqcup_{X_a} X_b \overset{\sim}{\hookrightarrow} Z_b \sqcup_{Z_a} Z_b$ is a trivial cofibration. Remains to prove that $Z_b \overset{\sim}{\hookrightarrow} Z_c$ is a trivial cofibration.

The pushout

$$\begin{tikzcd}
X_b \sqcup_{X_a} X_b \arrow[r] \arrow[d, hook, "\sim"] & X_c \arrow[d, hook, "\sim"] \arrow[ddr, bend left] & \\
Z_b \sqcup_{Z_a} Z_b \arrow[r] & \bullet \arrow[ul, phantom, very near start, "\ulcorner"] \arrow[dr, hook, "\sim"] & \\
& & Z_c
\end{tikzcd}$$

shows that $X_c \overset{\sim}{\hookrightarrow} Z_c$ is a trivial cofibration. Note that Z is Reedy cofibrant, hence $Z_b \hookrightarrow Z_c$ is a cofibration. By the two-out-of-three property we can conclude that $Z_b \overset{\sim}{\hookrightarrow} Z_c$ is indeed an acyclic cofibration. The above says that Z is cofibrant.

The second part follows as $X \to Z$ is a level-wise weak equivalence. $\square$

{factorization-trivial-

Corollary 4.21. *Any map between diagrams $f : X \to Y$, where X is a cofibrant diagram X and Y is a fibrant diagram in $\mathcal{M}^J_{Loc}$, can be factored as a trivial cofibration followed by a fibration.*

Proof. We factor $f : X \to Y$ in $\mathcal{M}^J_{Reedy}$ to obtain $X \overset{\sim}{\hookrightarrow} Z \twoheadrightarrow Y$. $Z \twoheadrightarrow Y$ is also a fibration in $\mathcal{M}^J_{Loc}$ as is it is level-wise. Finally, $X \overset{\sim}{\hookrightarrow} Z \in \mathcal{M}^J_{Loc}$ by the previous lemma 4.20. $\square$

For the factorization of a diagram map $f : X \to Y$ in $\mathcal{M}^J$, with X cofibrant and Y fibrant, into a cofibration followed by a trivial fibration we will need an auxiliary class of diagrams.

Construction 4.22. Denote by K the category J with the opposite Reedy structure given above (the degree function reversed). We endow $\mathcal{M}^K$ with the Reedy model structure. Then a diagram $Y \in \mathcal{M}^K_{Reedy}$ is fibrant if $Y_c \twoheadrightarrow 1$, $Y_b \twoheadrightarrow Y_c$ and $Y_a \twoheadrightarrow Y_b \times_{Y_c} Y_b$ are fibrations in $\mathcal{M}$. In this situation Y_b is also fibrant.

The limit of a diagram $Y \in \mathcal{M}^K$ is simply the equalizer $Eq(Y_i, Y_j)$. Note that the following pullback also computes the limit of Y:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & Y_a \arrow[d, two heads] \\
Y_b \arrow[r] & Y_b \times_{Y_c} Y_b.
\end{tikzcd}$$

From this we conclude that $\mathtt{Lim}\, Y$ is a fibrant object of $\mathcal{M}$ if $Y \in \mathcal{M}^K_{Reedy}$ is fibrant, and letting Z to denote the constant diagram at $\mathtt{Lim}\, Y$ then this comes with a diagram map $Z \to Y$ of the following form

$$\begin{tikzcd}
\mathtt{Lim}\, Y \arrow[r, shift left] \arrow[r, shift right] \arrow[d, two heads] & \mathtt{Lim}\, Y \arrow[r] \arrow[d, two heads] & \mathtt{Lim}\, Y \arrow[d, two heads] \\
Y_a \arrow[r, shift left] \arrow[r, shift right] & Y_b \arrow[r] & Y_c
\end{tikzcd}$$

where all top arrows are identities. Finally, note that Y being fibrant in $\mathcal{M}^K_{Reedy}$ implies that both maps $Y_a \rightrightarrows Y_b$ are fibrations. This can be deduced from the following diagram:

$$\begin{tikzcd}
Y_a \arrow[dr, two heads] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Y_b \times_{Y_c} Y_b \arrow[r, two heads] \arrow[d, two heads] & Y_b \arrow[d, two heads] \\
& Y_b \arrow[r, two heads] & Y_c
\end{tikzcd}$$

Observation 4.23. Recall that the fibrations in $\mathcal{M}^J_{Loc}$ are the level-wise fibrations. Since $Z \in \mathcal{M}^K$ is point-wise fibrant then it is Reedy fibrant in $\mathcal{M}^J_{Loc}$. Similarly, Y is Reedy fibrant in $\mathcal{M}^K_{Reedy}$, in particular, implies that is object-wise fibrant, so it is fibrant in $\mathcal{M}^J_{Loc}$.

{cofacycfib1}

Lemma 4.24. *The map $Z \to Y$ from above has the right lifting property with respect to any core cofibration $A \hookrightarrow B$ in $\mathcal{M}^J_{Loc}$.*

This is just to say that the map $Z \to Y$ is a trivial fibration in $\mathcal{M}^J_{Loc}$.

Proof. First, assume that $A = \emptyset$, B is a cofibrant object in $\mathcal{M}^J_{Loc}$ and Y a fibrant diagram in $\mathcal{M}^K_{Reedy}$. We consider the lifting problem in $\mathcal{M}^J_{Loc}$:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, hook] & Z \arrow[d] \\
B \arrow[r] & Y
\end{tikzcd}$$

From the discussion above we obtain the following commutative diagram:

$$\begin{tikzcd}
B_a \arrow[r, hook, shift left, "\sim"] \arrow[r, hook, shift right, "\sim"'] \arrow[d] & B_b \arrow[r, hook, "\sim"] \arrow[d] & B_c \arrow[d] \\
Y_a \arrow[r, two heads, shift left] \arrow[r, two heads, shift right] & Y_b \arrow[r, two heads] & Y_c
\end{tikzcd}$$

Thus, we obtain the following lifts:

$$\begin{tikzcd} B_a \arrow[r] \arrow[d, hook, "Bi"', "\sim"] & Y_a \arrow[d, two heads, "Yi"] \\ B_b \arrow[r] \arrow[ur, dashed, "l_i"] & Y_b \end{tikzcd} \qquad \begin{tikzcd} B_a \arrow[r] \arrow[d, hook, "Bj"', "\sim"] & Y_a \arrow[d, two heads, "Yj"] \\ B_b \arrow[r] \arrow[ur, dashed, "l_j"] & Y_b \end{tikzcd} \qquad \begin{tikzcd} B_b \arrow[r] \arrow[d, hook, "Bk"', "\sim"] & Y_b \arrow[d, two heads, "Yk"] \\ B_c \arrow[r] \arrow[ur, dashed, "l_k"] & Y_c \end{tikzcd}$$

Using this we can construct the following commutative diagram:

$$\begin{tikzcd} B_a \arrow[r, hook, "\sim"] \arrow[d, hook, "\sim"] & B_b \arrow[d, hook, "\sim"] \arrow[dr, "l_j"] & & \\ B_b \arrow[r, hook, "\sim"] \arrow[dr, "B_k"'] \arrow[ur, phantom, "\ulcorner"] & B_b \sqcup_{B_a} B_b \arrow[r] \arrow[d, hook, "\sim"] & Y_a \arrow[d, two heads] \arrow[dr, "Y_j"] & \\ & B_c \arrow[r] \arrow[dr, "l_k"'] & Y_b \sqcup_{Y_c} Y_b \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & Y_b \arrow[d, two heads] \\ & & Y_b \arrow[r, two heads] & Y_c \end{tikzcd}$$

where the middle trivial cofibration and fibration come from B being cofibrant in $\mathcal{M}^J_{Loc}$ and Y being fibrant in $\mathcal{M}^K_{Reedy}$ respectively. Then there exist a map $B_c \xrightarrow{r} Y_a$ that fits in the diagram. Furthermore, we readily see from the diagram that $Y_j r = l_k = Y_i r$. Therefore, there is a unique arrow $B_c \xrightarrow{t} Eq(Y_i, Y_j) = \mathtt{Lim}\, Y$ making the obvious triangle commutative. By taking the appropriate compositions with the map t we can construct a diagram map $B \to Z$ such that is a solution to the lifting problem.
For the general case

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & Z \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

one can play the same game, the only change is that the diagram is a bit more involved. □

{acycfib2}

Lemma 4.25. *If $Y \in \mathcal{M}^K_{Reedy}$ is fibrant then there exists a trivial fibration $W \twoheadrightarrow Y \in \mathcal{M}^J_{Loc}$.*

Proof. Since Y is fibrant in $\mathcal{M}^K_{Reedy}$, then it is fibrant in $\mathcal{M}^J_{Loc}$ as these are point-wise fibrant. Similarly, Z is fibrant in $\mathcal{M}^J_{Loc}$. We can take a Reedy cofibrant replacement in $\mathcal{M}^J_{Reedy}$, $Z' \overset{\sim}{\twoheadrightarrow} Z$. From corollary 4.21 we can factor this as $Z' \overset{\sim}{\hookrightarrow} W \twoheadrightarrow Y$. Let be $A \hookrightarrow B$ a cofibration between cofibrant objects in $\mathcal{M}^J_{Loc}$ and consider the lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & W \arrow[d, two heads] \\
B \arrow[r] & Y.
\end{tikzcd}$$

In the diagrams

$$\begin{tikzcd}
0 \arrow[rr, hook] \arrow[d, hook] & & Z' \arrow[d, hook, "\sim"] \\
 & & W \arrow[d, two heads] \\
A \arrow[r, hook] \arrow[urru, dashed] & B \arrow[r] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[dd, hook] & Z' \arrow[d, hook, "\sim"] \\
 & W \arrow[d, two heads] \\
B \arrow[r] \arrow[uur, dashed] & Y
\end{tikzcd}$$

we obtain the dashed arrow on the left because A is cofibrant and the vertical on the right is the fibration $Z' \overset{\sim}{\twoheadrightarrow} Z \overset{\sim}{\twoheadrightarrow} Y$, where the last arrow is a trivial fibration by lemma 4.24. Similarly, for the problem on the right, the top horizontal arrow is now the diagonal on the left. We can assemble this arrows to give a solution to the original problem through Z'

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd, hook] \arrow[dr, dashed] & & W \arrow[dd, two heads] \\
 & Z' \arrow[ur, hook, "\sim"] \arrow[dr] & \\
B \arrow[rr] \arrow[ur, dashed] & & Y.
\end{tikzcd}$$

□

Before giving the factorization, we need a technical result that follows from the next lemma.

{slice:wms}

Remark 4.26. From [Hen20, 2.1.11 Proposition], if $A \in \mathcal{M}$ is cofibrant then the coslice category $A/\mathcal{M}$ inherits a weak model structure from $\mathcal{M}$ where a map in $A/\mathcal{M}$ is cofibration, fibration and weak equivalences if is one in $\mathcal{M}$. Dually, one induces a weak model structure on the slice $\mathcal{M}/Y$ if Y is fibrant.

{double-slice:construct

Construction 4.27. We start with a core cofibration $A \hookrightarrow B$ and a core fibration $X \twoheadrightarrow Y$ and a map $f : A \to Y$. Consider $A/\mathcal{M}$ with the natural weak model described in the previous remark 4.26.

Since Y is fibrant in $\mathcal{M}$, it is also fibrant as an object in $A/\mathcal{M}$. We can take the slice $(A/\mathcal{M})/Y$ where Y is seen as an object under A via $f : A \to Y$. Objects of $(A/\mathcal{M})/Y$ are factorizations of the form

$$\begin{tikzcd} A \arrow[d] \arrow[dr, "f"] & \\ W \arrow[r] & Y. \end{tikzcd}$$

Let two objects in this category

$$\begin{tikzcd} A \arrow[d] \arrow[dr, "f"] & \\ B \arrow[r] & Y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \arrow[r] \arrow[dr, "f"'] & X \arrow[d] \\ & Y \end{tikzcd}$$

which we refer to as B and X. A map from B to X is a diagonal filler of the resulting commutative square:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

A cofibrant object in $(A/\mathcal{M})/Y$ is one in which the first map is a cofibration in $\mathcal{M}$, and a fibrant object when the last map is a fibration *i.e.,*

$$\begin{tikzcd} A \arrow[d, hook] \arrow[dr, "f"] & \\ B \arrow[r] & Y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \arrow[r] \arrow[dr, "f"'] & X \arrow[d, two heads] \\ & Y \end{tikzcd}$$

respectively. Note that the category $(A/\mathcal{M})/Y$ coincides with $A/(\mathcal{M}/Y)$. {ho-adjunction:weak}

Observation 4.28. [Hen20, 2.4.3 Proposition] observed that the Quillen adjunction descends to the homotopy categories: If $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a Quillen pair, the we obtain a natural isomorphism

$$\mathrm{Ho}(\mathcal{C}^{bif})(W, G(Z)) \cong \mathrm{Ho}(\mathcal{D}^{bif})(F(W), Z)$$

of the homotopy categories.

The category $\mathrm{Ho}(\mathcal{C}^{bif})$ is the localization of the subcategory of bifibrant objects at acyclic (cofibrations) fibrations. This is the content of [Hen20, 2.2.6 Theorem], which also proves that there are equivalences

$$\mathrm{Ho}(\mathcal{C}^{\mathrm{Cof}}) \cong \mathrm{Ho}(\mathcal{C}^{bif}) \cong \mathrm{Ho}(\mathcal{C}^{\mathrm{Fib}})$$

where the first category is the localization of $\mathcal{C}^{\text{Cof}}$ at acyclic cofibrations, and the second is the localization of $\mathcal{C}^{\text{Fib}}$ at acyclic fibrations. Therefore, up to these equivalences of categories we say that $\text{Ho}(F) : \text{Ho}(\mathcal{C}^{\text{Cof}}) \to \text{Ho}(\mathcal{D}^{\text{Cof}})$ and $\text{Ho}(G) : \text{Ho}(\mathcal{D}^{\text{Fib}}) \to \text{Ho}(\mathcal{C}^{\text{Fib}})$ are "adjoint".

Lemma 4.29. *For all $i : A \hookrightarrow B$ and $i' : A' \hookrightarrow B'$ cofibrations between cofibrant objects, for all $p : X \twoheadrightarrow Y$ fibration between fibrant objects, if there is a commutative diagram:* {liftequiv}

$$\begin{tikzcd} A \arrow[r, "\sim", "k"'] \arrow[d, hook, "i"'] & A' \arrow[d, hook, "i'"] \\ B \arrow[r, "\sim", "l"'] & B' \end{tikzcd}$$

then $i \pitchfork p$ if and only if $i' \pitchfork p$. The dual statement also holds: For all $i : A \hookrightarrow B$ core cofibrations, for all $p : X \twoheadrightarrow Y$ and $p' : X' \twoheadrightarrow Y'$ fibrations between fibrant objects, if there is a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\sim", "m"'] \arrow[d, two heads, "p"'] & X' \arrow[d, two heads, "p'"] \\ Y \arrow[r, "\sim", "n"'] & Y' \end{tikzcd}$$

then $i \pitchfork p$ if and only if $i \pitchfork p'$.

Proof. We prove the first part of the lemma, the second part is dual. We have the following commutative squares

$$\begin{tikzcd} A \arrow[r, "k", "\sim"'] \arrow[d, hook, "i"'] & A' \arrow[d, hook, "i'"] \\ B \arrow[r, "\sim", "l"'] & B' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d, two heads, "p"] \\ B \arrow[r, "g"'] & Y \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, hook, "i'"'] & X \arrow[d, two heads, "p"] \\ B' \arrow[r, "g'"'] & Y \end{tikzcd}$$

We start with the induced weak model structure on the slice $\mathcal{M}/Y$. Note that from [Hen20, 2.4.2 Example] the weak equivalence $k : A \to A'$ induces a weak Quillen equivalence $P_k : A/(\mathcal{M}/Y) \leftrightarrows A'/(\mathcal{M}/Y) : U_k$. Observe that B, B' are cofibrant and Y is fibrant. In what follows we leave Y implicit as we work in the slice $(A/\mathcal{M})/Y$, here we use that $(A/\mathcal{M})/Y = A/(\mathcal{M}/Y)$ from construction 4.27.

The functor P_k takes a cofibration $A \hookrightarrow C$ along $k : A \to A'$, while U_k precomposes with k. Using the following diagram, since $P_k B$ is cofibrant, by the two-out-of-three property

$$\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, hook, "i"'] & A' \arrow[d, hook] \arrow[ddr] & \\
B \arrow[r, "\sim"'] \arrow[rrd, bend right, "\sim"'] & P_k B \arrow[ul, phantom, "\lrcorner" very near start] \arrow[dr, dashed] & \\
& & B'
\end{tikzcd}$$

we see that there is a weak equivalence $P_k B \xrightarrow{\sim} B'$, this implies they are isomorphic in $\mathrm{Ho}(A'/(\mathcal{M}/Y))$. We have:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Ho}(A'/(\mathcal{M}/Y))}(B', X) &\cong \mathrm{Hom}_{\mathrm{Ho}(A'/(\mathcal{M}/Y))}(P_k(B), X) \\
&\cong \mathrm{Hom}_{\mathrm{Ho}(A/(\mathcal{M}/Y))}(B, U_k(X)) \\
&\cong \mathrm{Hom}_{\mathrm{Ho}(A/(\mathcal{M}/Y))}(B, X).
\end{aligned}$$

The first isomorphism follows from $B' \cong P_k(B)$ in $\mathrm{Ho}(A'/(\mathcal{M}/Y))$, the second is the weak Quillen adjunction $P_k \dashv U_k$ applied to the cofibrant object $B \in (A/\mathcal{M})/Y$ and the fibrant object $X \in (A'/\mathcal{M})/Y$. We crucially use observation 4.28, so the second isomorphism is really up some equivalence of categories.

Now we use $\mathrm{Hom}_{\mathrm{Ho}(A'/(\mathcal{M}/Y))}(B', X) \cong \mathrm{Hom}_{\mathrm{Ho}(A/(\mathcal{M}/Y))}(B, X)$ to conclude. First recall that a diagonal filler of

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & X \arrow[d, two heads] \\
B \arrow[r] & Y
\end{tikzcd}$$

is the same as a map $B \to X$ in $A/\mathcal{M}/Y$, and similarly for B' and X. Assume that $i \pitchfork p$, this give us a map $B \to X$ in $\mathrm{Ho}(A/\mathcal{M}/Y)$. Using the isomorphism we a map $B' \to X$ in $\mathrm{Ho}(A'/\mathcal{M}/Y)$, from which we can select a representative of the homotopy class, which implies that $i' \pitchfork p$. Similarly, we get that $i' \pitchfork p$ implies $i \pitchfork p$. $\square$

{factorization-cofibra

Lemma 4.30. *Let $X \to Y$ be a map in $\mathcal{M}^J$ with X cofibrant and Y fibrant. Then such a map can be factored as a cofibration followed by a trivial fibration.*

Proof. Observe first that Y can be assumed to be Reedy cofibrant in $\mathcal{M}^J$. Indeed, we can simply take a Reedy cofibrant replacement $Y' \xrightarrow{\sim} Y$, and instead use the dashed arrow

$$\begin{tikzcd} 0 \arrow[r, hook] \arrow[d, hook] & Y' \arrow[d, two heads, "\sim"] \\ X \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

Under this assumption, Y is point-wise cofibrant, whence Reedy cofibrant in $\mathcal{M}^K$. Therefore, we can take a fibrant replacement in $\mathcal{M}^K$, $Y \overset{\sim}{\hookrightarrow} Y'$. Using [Hen20, Corollary 2.4.4] equivalences are preserved under pullbacks along fibrations, so we get the pullback square

$$\begin{tikzcd} LY \arrow[r, "\sim"] \arrow[d, two heads] & W \arrow[d, two heads] \\ Y \arrow[r, hook, "\sim"] & Y'. \end{tikzcd}$$

Furthermore, we know from lemma 4.25 that $W \twoheadrightarrow Y'$ is a trivial fibration in $\mathcal{M}^J$. Therefore, it has the right lifting property against any cofibration between cofibrant objects in $\mathcal{M}^J$. We can use lemma 4.29 to conclude that $LY \twoheadrightarrow Y$ satisfies the same property *i.e.,* it is a trivial fibration in $\mathcal{M}^J$. Since X is cofibrant, we obtain a lift

$$\begin{tikzcd} 0 \arrow[r, hook] \arrow[d, hook] & LY \arrow[d, two heads, "\sim"] \\ X \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

The map $X \to LY$ can be factored in the Reedy model structure $\mathcal{M}^J$ as $X \hookrightarrow X' \overset{\sim}{\twoheadrightarrow} LY$. The diagram X' is cofibrant in $\mathcal{M}^J$ since is equivalent the cofibrant diagram LY, and X is cofibrant by assumption. Therefore, it follows from observation 4.18 that the Reedy cofibration $X \hookrightarrow X'$ is a cofibration in the model $\mathcal{M}^J$. This give us the desired factorization in $\mathcal{M}^J$, $X \hookrightarrow X' \overset{\sim}{\twoheadrightarrow} Y$. □

All the previous work can be summarized in the following proof of theorem 4.15. This proves that the category of diagrams $\mathcal{M}^J$ has a weak model structure with the specified cofibrations and fibrations, which as explained above, encodes objects with a weak cylinder object. Again we verify all the conditions of definition C.1

Proof. (theorem 4.15)

1. The existence of initial and terminal diagrams is clear.

2. The fact that it is a Reedy cofibration was established in theorem C.11. One can use the fact that in the weak model category $\mathcal{M}$ the acyclic cofibrations are pushout stable to conclude that the map obtained by the pushout in $\mathcal{M}^J$ is a cofibration in the new sense.

3. This is dual to the previous condition.

4. In addition to the property be true in the Reedy weak model structure on $\mathcal{M}^J$ we need to use repeatedly that maps isomorphic to weak equivalences are also weak equivalences, this is simply because the new condition we added involves the requirement that certain maps are weak equivalences.

5. The factorization of a map $f : X \to Y$, where X is cofibrant and Y is fibrant, into a cofibration followed by a trivial fibration is the content of lemma 4.30.

6. The factorization of a map $f : X \to Y$, where X is cofibrant and Y is fibrant, into a trivial cofibration followed by a fibration is the content of corollary 4.21.

7. Solutions to lifting problems come from the Reedy weak model structure.

8. The 2-out-of-3 property is immediate.

9. This is also clear.

$\square$

4.2.2 Weak model on correspondences

Next, we consider another diagram category I:

$$0 \to 2 \leftarrow 1$$

Where $\deg(0) = \deg(1) = 0$ and $\deg(2) = 1$. Similarly to the previous section, we construct a "right Bousfield localization" of the Reedy weak model structure on $\mathcal{N}^I$.

Theorem 4.31. *There is a weak model structure $\mathcal{N}^I_{Loc}$ on the category of diagrams $\mathcal{N}^I$ obtained from the Reedy weak model structure $\mathcal{N}^I_{Reedy}$, were:*

1. *A map between diagrams $X \to Y$ is a cofibration if*
   1. *(a) It is a Reedy cofibration,*
   2. *(b) $X_2 \sqcup_{X\ 1} Y_1 \overset{\sim}{\hookrightarrow} Y_2$ and $X_2 \sqcup_{X_0} Y_0 \overset{\sim}{\hookrightarrow} Y_2$ are trivial cofibrations in $\mathcal{M}$.*
2. *Fibrations are level-wise fibrations.*

The proof the theorem is completely analogous to theorem 4.15. {cylinders:wms}

Observation 4.32. Unwinding the definitions, a diagram $X \in \mathcal{N}^I_{Loc}$ is cofibrant if both maps $X_0 \overset{\sim}{\hookrightarrow} X_2$ and $X_1 \overset{\sim}{\hookrightarrow} X_2$ are trivial cofibrations.

{first-projection:triv

Lemma 4.33. *The functor $\mathcal{N}^I \to \mathcal{N}$ such that $A \to B \leftarrow C \in \mathcal{N}^I \mapsto A \in \mathcal{N}$, is a Barton trivial fibration. Also the functor $\mathcal{N}^I \to \mathcal{N}$ such that $A \to B \leftarrow C \in \mathcal{N}^I \mapsto C \in \mathcal{N}$, is a Barton trivial fibration.*

Proof. Let $A \coloneqq a \overset{\sim}{\hookrightarrow} b \overset{\sim}{\hookleftarrow} c \in \mathcal{N}^I_{Loc}$ be a cofibrant diagram and $x \in \mathcal{N}^{\mathrm{Cof}}$ a cofibrant object. We take the fibrant replacement of x and consider the pushout as indicated below, and we obtain a solution to the lifting problem on the right:

$$\begin{tikzcd}
 & a \arrow[r, hook] \arrow[d, hook, "\sim"'] & x \arrow[r, hook, "\sim"] \arrow[d, hook, "\sim"] & x^{fib} \\
c \arrow[r, hook, "\sim"] & b \arrow[r] & b \sqcup_a x \arrow[ur, dashed] &
\end{tikzcd}$$

The resulting map $c \to x^{fib}$ can be factored as $c \hookrightarrow z \overset{\sim}{\twoheadrightarrow} x^{fib}$. We can take further pushouts

$$\begin{tikzcd}
 & a \arrow[r, hook] \arrow[d, hook, "\sim"'] & x \arrow[r, hook, "\sim"] \arrow[d, hook, "\sim"] & x^{fib} \\
c \arrow[r, hook, "\sim"] \arrow[d, hook] & b \arrow[r, hook] \arrow[d, hook] & b \sqcup_a x \arrow[ur] \arrow[d, hook] & \\
z \arrow[r, hook, "\sim"'] \arrow[rrruu, two heads, bend right, "\sim"'] & z \sqcup_c b \arrow[r, hook] & P. &
\end{tikzcd}$$

There is a map $P \to x^{fib}$ which we can factor as $P \hookrightarrow y \overset{\sim}{\twoheadrightarrow} x^{fib}$, and the resulting diagram we get

$$\begin{tikzcd}
& a \arrow[r, hook] \arrow[d, hook, "\sim"'] & x \arrow[r, hook, "\sim"] \arrow[d, hook, "\sim"] \arrow[dl, phantom, "\lrcorner", very near end] & x^{fib} \\
c \arrow[r, hook, "\sim"] \arrow[d, hook] & b \arrow[r, hook] \arrow[d, hook] \arrow[dl, phantom, "\lrcorner", very near end] & b \sqcup_a x \arrow[ur] \arrow[d, hook] \arrow[dl, phantom, "\lrcorner", very near end] & y \arrow[u, two heads, "\sim"'] \\
z \arrow[r, hook, "\sim"'] & z \sqcup_c b \arrow[r, hook] & P \arrow[ur, hook] &
\end{tikzcd}$$

Furthermore, there is a map $b \sqcup_a x \to y$ which is a cofibration as it is the composite of the two cofibrations. Using the 2-out-of-3 property repeatedly one concludes that the map $z \sqcup_c b \to y$ is a trivial cofibration. Thus, we have constructed the cofibrant object $X := z \overset{\sim}{\hookrightarrow} y \overset{\sim}{\hookleftarrow} x \in \mathcal{N}^I_{Loc}$. The induced map $A \to X$ is a level-wise cofibration. The maps $b \sqcup_a x \to y$ and $b \sqcup_a z \to y$ are trivial cofibrations.

Remains to show that $A \to X$ is a Reedy cofibration. We already have that $a \to x$ and $c \to z$ are cofibrations. We now need to show that the induced map

$$\begin{tikzcd}
a \sqcup c \arrow[r] \arrow[d, hook] & b \arrow[d, hook] \arrow[ddr, bend left] & \\
x \sqcup z \arrow[r] \arrow[drr, bend right] & (x \sqcup z) \sqcup_{a \sqcup c} b \arrow[dr, dashed] & \\
& & y
\end{tikzcd}$$

is a cofibration. By diagram chasing one can show that the diagram

$$\begin{tikzcd}
a \sqcup c \arrow[r] \arrow[d] & b \arrow[d] \\
x \sqcup z \arrow[r] & (z \sqcup_c b) \sqcup_b (b \sqcup_a x)
\end{tikzcd}$$

commutes. One shows that the bottom right corner computes the pushout of the span. Using that the map $P \hookrightarrow y$ is a cofibration one concludes that $(x \sqcup) \sqcup_{a \sqcup c} b \to y$ is also a cofibration. This concludes the proof that $A \to X$ is a Reedy core cofibration in $\mathcal{N}^I$. Therefore, it must a cofibration. We summarize our construction with the following diagram:

$$
\begin{tikzcd}
c \arrow[rr, hook] \arrow[d, hook, "\sim"] & & z \arrow[d, hook, "\sim"] \\
b \arrow[rr, hook] & & y \\
a \arrow[rr, hook] \arrow[u, hook, "\sim"] & {} \arrow[d] & x \arrow[u, hook, "\sim"] \\
a \arrow[rr, hook] & {} & x
\end{tikzcd}
$$

This cofibration is a (strict) lift of $a \hookrightarrow x$, showing that the functor $\mathcal{N}^I \to N$ is a trivial fibration. Of course, the second part of the lemma is analogous. $\square$

We now want to see that any left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ part of a Quillen equivalence between weak model categories factors a section of a trivial fibration followed by a trivial fibration. To this end, consider the following:

{fcylinders:wms}

Construction 4.34. We define the category of diagrams

$$\mathcal{N}^I_F := \{Fa \to b \leftarrow c | a \in \mathcal{M}^{\mathrm{Cof}},\, b, c \in \mathcal{N}\}.$$

The weak model structure on this category is similar to that of $\mathcal{N}^I$, the only difference is that $X \to Y$ is a cofibration if $X_b \sqcup_{FX_a} FY_a \to Y_b$ is a trivial fibration.

When F is the identity functor we recover $\mathcal{N}^I$ from observation 4.32. A cofibrant object in $\mathcal{N}^I_F$ is a diagram of the form

$$Fa \xhookrightarrow{\sim} b \xhookleftarrow{\sim} c.$$

Observation 4.35. With the set up above, it follows immediately from lemma 4.33 that the projection $\pi_1 : \mathcal{N}^I_F \to \mathcal{M}$, sending each diagram $Fa \to b \leftarrow c$ to a, is a trivial fibration.

To show that the projection from $\pi_2 : \mathcal{N}^I_F \to \mathcal{N}$ sending each diagram $Fa \to b \leftarrow c$ to $c \in \mathcal{N}$ is a trivial fibration we make use of the following:

{transpose:factorizatio

Lemma 4.36. *Let $F : \mathcal{M} \to \mathcal{N}$ be a left Quillen equivalence between weak model categories. For any objects $x \in \mathcal{M}^{\mathrm{Cof}}$, $y \in \mathcal{N}^{\mathrm{Fib}}$ and a map $f : Fx \to$*

y there exists an object $z \in \mathcal{M}^{\mathrm{Cof}}$ such that f factors as

$$\begin{tikzcd}
Fx \arrow[rr, "f"] \arrow[dr, hook] & & y \\
& Fz. \arrow[ur, "\sim"'] &
\end{tikzcd}$$

Proof. We know that there is an isomorphism

$$\varphi : \mathrm{Hom}_{\mathcal{N}}(Fx, y) \simeq \mathrm{Hom}_{\mathcal{M}}(x, Gy) : \varphi^{-1}$$

given by the Quillen adjunction, natural in $x \in \mathcal{M}^{\mathrm{Cof}}$ and $y \in \mathcal{N}^{\mathrm{Fib}}$. Recall from [Hen20, 2.4.3 Proposition] that $F : \mathcal{M}^{\mathrm{Cof}} \to \mathcal{N}^{\mathrm{Cof}}$ and $G : \mathcal{N}^{\mathrm{Fib}} \to \mathcal{M}^{\mathrm{Fib}}$ preserve equivalences. Take φf the adjoint transpose of f. We can take a factorization

$$\begin{tikzcd}
& z \arrow[dr, two heads, "s", "\sim"'] & \\
x \arrow[ur, hook, "r"] \arrow[rr, "\varphi f"'] & & Gy
\end{tikzcd}$$

By naturality one checks that $f = \varphi^{-1} s F r$ where Fr is a cofibration. Since the Quillen pair is an equivalence we deduce from [Hen20, 2.4.5 Proposition (i)] that $\varphi^{-1}s$ is an equivalence. $\square$

{second-projection:triv

Corollary 4.37. *Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a Quillen equivalence. Then the projection $\pi_2 : \mathcal{N}^I_F \to \mathcal{N}$ sending each diagram $Fa \to b \leftarrow c$ to $c \in \mathcal{N}$ is a trivial fibration.*

Proof. We show that in a situation as in the diagram

$$\begin{tikzcd}
Fa \arrow[d, hook, "\sim"] & \\
b & \\
c \arrow[u, hook, "\sim"'] \arrow[d, mapsto] & \\
c \arrow[r, hook] & z
\end{tikzcd}$$

there is a cofibrant object over z that projects onto $c \hookrightarrow z$. By taking a fibrant replacement, we can assume that the diagram is point-wise fibrant. From [Hen20, 2.2.3 Proposition] there exists a homotopy inverse of $c \overset{\sim}{\hookrightarrow} b$, this give us a map $Fa \to c$. Using lemma 4.36 this last map can be factored as $Fa \hookrightarrow Fx \overset{\sim}{\to} c$. The rest of the proof continues as in lemma 4.33. $\square$

{quillen-equivalence:fa

Theorem 4.38. *Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a Quillen equivalence between weak model categories. Then, $F : \mathcal{M} \to \mathcal{N}$ admits a factorization $F = \pi_2 W_F$ where π_2 is a Barton trivial fibration and W_F is the section of a Barton trivial fibration.*

Proof. The functor $F : \mathcal{M} \to \mathcal{N}$ can be factored as $\mathcal{M} \overset{W_F}{\to} \mathcal{N}^I_F \overset{\pi_2}{\to} \mathcal{N}$ where the functor $W_F : \mathcal{M} \to \mathcal{N}^I_F$ sends each $x \in \mathcal{M}$ to the constant diagram at Fx, this is a section of the projection π_1. We have shown in lemma 4.33 and corollary 4.37 that both projections are Barton trivial fibrations. $\square$

4.3 Proof of main theorem

Theorem 4.39. *Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ a Quillen equivalence. Then for any cofibrant object $A \in \mathcal{M}$. The induced map $h\mathbb{L}F_A : h\mathbb{L}^{\mathcal{M}}_\lambda(A) \to h\mathbb{L}^{\mathcal{N}}_\lambda(FA)$ is an isomorphism.*

Proof. Using theorem 4.38 we obtain a factorization of $F = P_F W_F$ where W_F is a section of a trivial fibration. There is a span of trivial fibrations

$$\begin{tikzcd} & \mathcal{J} \arrow[dl, two heads, "\sim"'] \arrow[dr, two heads, "\sim"] & \\ \mathcal{M} \arrow[rr, "F"'] & & \mathcal{N} \end{tikzcd}$$

We can apply theorem 4.13 to conclude that $h\mathbb{L}F_A : h\mathbb{L}^{\mathcal{M}}_\lambda(A) \to h\mathbb{L}^{\mathcal{N}}_\lambda(FA)$ is an isomorphism. This is because the factorization of F also induces a factorization of $h\mathbb{L}F_A$ as in the diagram

$$\begin{tikzcd} & h\mathbb{L}^{\mathcal{M}}_\lambda(A) \arrow[dr, "h\mathbb{L}P_F"] & \\ h\mathbb{L}^{\mathcal{M}}_\lambda(A) \arrow[ur, "h\mathbb{L}W_F"] \arrow[rr, "h\mathbb{L}F_A"'] & & h\mathbb{L}^{\mathcal{M}}_\lambda(A) \end{tikzcd}$$

$\square$

It is an immediate from **??** that

Corollary 4.40. *For any Quillen equivalence $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$. The functors $Ho(F) \circ h\mathbb{L}^{\mathcal{M}}_\lambda$ and $h\mathbb{L}^{\mathcal{N}}_\lambda : Ho(\mathcal{N}) \to \mathbf{Bool}_\lambda$ are naturally isomorphic via $h\mathbb{L}F$.*

A Infinitary Cartmell theories

{appendix-a}

We introduce a generalization of *Cartmell theories*, also known as *generalized algebraic theories*, Cartmell [Car78]. This is straightforward and most of the proofs will be omitted since they are similar to those in [Car78], in very few cases we will need to provide new proofs. We claim no originality other than the generalization itself. We begin by recalling some definitions given in *Ibidem.* We assume to have a set of variables V whose size is $\aleph_0$ and an alphabet A. Informally, a *Cartmell generalized algebraic theory* consists of:

i) A set S, called the set of *sort symbols*,

ii) A set O, called the set of *operation symbols*,

iii) An introductory rule for each sort symbol,

iv) An introductory rule for each operation symbol,

v) A set of axioms.

To understand our generalization let us examine the previous definition in more detail, for this we need some preliminary notions. An *expression* is a finite sequence of $A \cup V \cup \{(\} \cup \{)\} \cup \{,\}$, inductively:

i) Elements of V and A are expressions,

ii) If $f \in A$ and $e_1, e_2, ..., e_n$ are expressions then $f(e_1, e_2, ..., e_n)$ is an expression.

The set of expression is denoted by E. This is simply to say that an expression is a finite string taken from the set $A \cup V \cup \{(\} \cup \{)\} \cup \{,\}$. A *premise* is a finite (possibly empty) sequence of $V \times E$. A *conclusion* will be an n-tuple of expressions i.e. any element of E^n for some $n \in \mathbb{N}$. Finally, a *rule* is given by a premise P and a conclusion C. Rules are written as: $P \vdash C$. This intends to convey the idea that under the premise P the conclusion C is a valid expression. Whenever P is a premise we will write $x_1 : \Delta_1, x_2 : \Delta_2, ..., x_n : \Delta_n$. For a conclusion this is slightly more involved since we differentiate depending on the size of the tuple. For example if we have a 1-tuple Δ then we write: $\Delta\mathsf{Type}$. We favour the notation ":" from type theory instead of the set theoretic one "ϵ" used by Cartmell. Furthermore, we will take advantage of conventions and notation from type theory.

The most important definition we will need to change is that of a *context*. In a Cartmell theory, a *context* is the premise such that a rule

$$x_1 : \Delta_1,\, x_2 : \Delta_2(x_1), ..., x_n : \Delta_n(x_1, x_2, \cdots, x_{n-1}) \vdash \Delta(x_1, x_2, \cdots, x_n)\ \mathsf{Type}$$

is a *derived rule.*

The only difference between Cartmell theories and infinitary Cartmell theories is that in the contexts we allow infinitely many variables. Just as any Cartmell theory gives rise to a contextual category, the same is true for the infinitary case with the appropriate generalized version of a contextual category.

A.1 Generalized algebraic theories

In this section we give the formal definition of an infinitary Cartmell theory. We follow Cartmell [Car78] to develop the theory, there will be some instances were a change has to be made. We could say that by changing in the definition every instance of "finite" by "size strictly less than κ" we get the correct notion, this is indeed the case. We carve out the definition with a fair amount of details since the applications we have in mind benefit from having an explicit syntax. The technicalities and motivations for introducing a generalized algebraic in the following way are presented in Cartmell [Car78].

From now on we fix a regular cardinal κ, unless otherwise stated, all other ordinals mentioned will be strictly smaller than κ.
Let V be a set such that $|V| = \kappa$, this set will be called the set of *variables*. We make an additional assumption on this set: Its elements have *canonical names*, this is $V = \{x_\alpha\}_{\alpha<\kappa}$, This also known as an *enumeration*. This is a minor assumption that allows to change variables. Otherwise we would need to proof a result similar [Car78, Corollary, pp 1.32][1]. Let A be any set which as before is called *alphabet*. Following [Car78] we define inductively the collection of *expressions* A^* over the alphabet A. An expression any λ-sequence of $A \cup V \cup \{(\} \cup \{)\} \cup \{,\}$ subject to:

i) If $x_\alpha \in V$ then $x_\alpha \in A^*$,

ii) If $F \in A$ then $F \in A^*$,

iii) If $F \in A$ and $\{e_\alpha\}_{\alpha<\lambda} \subseteq A^*$ then $F(e_\alpha)_{\alpha<\lambda} \in A^*$.

[1] This result states that under the substitution property the derived rules are stable under substitution of variables by another variables

A *premise* is any λ-sequence of $V \times A^*$. We will usually write premises as $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$, where x_α are variables and Δ_α are expressions for $\alpha < \lambda$. Suppose we have a premise Γ, or later a *context*, and we need an extra premise (or *context*), according to our variable numeration we formally must to write $\Gamma, \{x_\alpha : \Delta_\alpha\}_{\lambda\leq\alpha<\mu}$ where λ represent the number of variables in Γ. This is clearly a problem when the expression complexity increases. In order to avoid overloading the notation we choose to reset the variable counting to only essential variables in use. Under this convention we will write $\Gamma, \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ instead. We will freely assume that Γ is a premise unless otherwise specified.

Definition A.1. A *judgment* is an expression over the alphabet A that has one of the following forms: {judgment1}

1. Type judgment: $\Gamma \vdash \Delta\ \mathsf{Type}$.
2. Element judgment: $\Gamma \vdash t : \Delta$.
3. Type equality judgment: $\Gamma \vdash \Delta \equiv \Delta'$.
4. Term equality judgment: $\Gamma \vdash t \equiv_\Delta t'$.

where Γ is a premise.

Given a premise Γ, $\{e_\alpha\}_{\alpha<\lambda}$ expression and $\{x_\alpha\}_{\alpha<\lambda}$ variables then the new expression

$$\Gamma[e_\alpha|x_\alpha]_{\alpha<\lambda}$$

it is obtained by simultaneously changing the variables in Γ by the expressions. This process, unsurprisingly, is called *substitution* of variables. Along with the infinitary substitutions we will also allow operations to have possibly infinite arity. This is made explicit:

Definition A.2. A κ-*pretheory* T consist of the following data:

i) A set S, called the set of *sort symbols*,

ii) A set O, called the set of *operation symbols*,

iii) For each sort symbol B, a judgment of the form:

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash B(x_\alpha)_{\alpha<\lambda}\ \mathsf{Type}$$

where λ is some ordinal strictly smaller than κ,

iv) For each operator symbol F, a judgment:

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash F(x_\alpha)_{\alpha<\lambda} : \Delta$$

where λ is some ordinal strictly smaller than κ,

v) A set of judgments, each of which is either a type equality judgment or term equality judgment listed in definition A.1. This is the set of *axioms* of the κ-pretheory.

The following definitions are of inductive nature:

{contextdefinition}

Definition A.3. 1. A premise $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a *context* if the judgment

$$\{x_\beta : \Delta_\beta\}_{\beta<\alpha} \vdash \Delta_\alpha \ \mathsf{Type}$$

is a *derived judgment* of T for every $\alpha < \lambda$. Whenever we want to specify that a premise Γ is a context we will write $\vdash \Gamma \ \mathsf{Ctxt}$.

2. The judgment

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \ \mathsf{Type}$$

is a *well-formed judgment* of T if and only if $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a context.

3. The judgment

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta$$

is *well-formed* if and only if

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \ \mathsf{Type}$$

is a *derived judgment* of T.

{derivedrules}

Definition A.4. Let T be a κ-pretheory. The set of *derived judgments* of T are the ones that can be derived from the following list:

1.
$$\frac{\Gamma \vdash A \ \mathsf{Type}}{\Gamma \vdash A \equiv A}$$

{drule1}

2.
$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t \equiv_A t}$$

{drule2}

3.

$$\frac{\Gamma \vdash A_1 \equiv A_2}{\Gamma \vdash A_2 \equiv A_1}$$

{drule3}

4.

$$\frac{\Gamma \vdash t_1 \equiv_A t_2}{\Gamma \vdash t_2 \equiv_A t_1}$$

{drule4}

5.

$$\frac{\Gamma \vdash A_1 \equiv A_2 \qquad \Gamma \vdash A_2 \equiv A_3}{\Gamma \vdash A_1 \equiv A_3}$$

{drule5}

6.

$$\frac{\Gamma \vdash t_1 \equiv_A t_2 \qquad \Gamma \vdash t_2 \equiv_A t_3}{\Gamma \vdash t_1 \equiv_A t_3}$$

{drule6}

7.

$$\frac{\Gamma \vdash A_1 \equiv A_2 \qquad \Gamma \vdash t_1 \equiv_{A_1} t_2}{\Gamma \vdash t_2 \equiv_{A_2} t_1}$$

{drule8}

8.

$$\frac{\Gamma \vdash A_1 \equiv A_2 \qquad \Gamma \vdash t : A_1}{\Gamma \vdash t : A_2}$$

{drule9}

9.

$$\frac{\Gamma,\, \{x_\delta : A_\delta\}_{\delta<\beta<\lambda} \vdash A_\beta \ \mathsf{Type}}{\Gamma,\, \{x_\alpha : A_\alpha\}_{\alpha<\lambda} \vdash x_\alpha : A_\alpha}$$

{drule10}

10.

$$\frac{\{x_\alpha : A_\alpha\}_{\alpha<\lambda} \vdash B(x_\lambda) \ \mathsf{Type}, \quad \vdash \Gamma \ \mathsf{Ctxt}, \quad \Gamma \vdash t_\alpha : B[t_\alpha | x_\alpha]}{\Gamma \vdash B(t_\lambda) \ \mathsf{Type}}$$

This is true for any B sort symbol with a well-formed introduction type judgment. {drule11}

11.
$$\frac{\Gamma,\ \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash F(x_\lambda) : \Delta, \quad \Gamma \vdash t_\alpha : \Delta_\alpha[t_\alpha|x_\alpha]}{\Gamma,\ \{t_\alpha : \Delta_\alpha[t_\alpha|x_\alpha]\}_{\alpha<\lambda} \vdash F(t_\lambda) : \Delta[t_\lambda \mid x_\lambda]}$$

{drule12}

This is true for any F operator symbol with a well-formed introduction type element judgment.

12.
$$\frac{\begin{array}{c}\vdash \Gamma\ \mathsf{Ctxt} \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \equiv \Delta' \\ \Gamma,\ t_\alpha : \Delta_\alpha[t_\beta \mid x_\beta]_{\beta<\alpha},\ t'_\alpha : \Delta'_\alpha[t'_\beta \mid x_\beta]_{\beta<\alpha} \vdash t_\alpha \equiv_{\Delta_\alpha[t_\beta|x_\beta]_{\beta<\alpha}} t'_\alpha\end{array}}{\begin{array}{c}\Gamma,\ \{t_\alpha : \Delta_\alpha[t_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda},\ \{t'_\alpha : \Delta'_\alpha[t'_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda} \\ \vdash \Delta[t_\alpha|x_\alpha]_{\alpha<\lambda} \equiv \Delta'[t'_\alpha|x_\alpha]_{\alpha<\lambda}\end{array}}$$

{drule13}

13.
$$\frac{\begin{array}{c}\vdash \Gamma\ \mathsf{Ctxt} \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t' \\ \Gamma,\ s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha},\ s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha} \vdash s_\alpha \equiv_{\Delta_\alpha[s'_\beta|x_\beta]_{\beta<\alpha}} s'_\alpha\end{array}}{\begin{array}{c}\Gamma,\ \{s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda},\ \{s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda} \\ \vdash t[s_\alpha \mid x_\alpha]_{\alpha<\lambda} \equiv_{\Delta[s_\alpha|x_\alpha]_{\alpha<\lambda}} t'[s'_\alpha \mid x_\alpha]_{\alpha<\lambda}\end{array}}$$

{drule14}

14. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \equiv \Delta'$ is an axiom then

$$\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta\ \mathsf{Type} \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta'\ \mathsf{Type},}{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \equiv \Delta'}$$

15. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t'$ is an axiom then

$$\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t' : \Delta}{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t'}$$

We are now ready for the following:

Definition A.5. A κ-pretheory T is *well-formed* if all its rules are well-formed. A *κ-generalized algebraic theory* or *κ-Cartmell theory* is a well-formed κ-pretheory.

Remark A.6. Observe that a generalized algebraic theory as defined by Cartmell [Car78] is the same as an ω-generalized algebraic theory in our sense.

We introduce an important example of κ-algebraic theories.

{gatcategories}

Example A.7. Let *Cat* denote the ω-algebraic theory defined in the following way:

1. Type of objects: $\vdash \mathsf{Ob}\,\mathsf{Type}$.
2. Type of morphisms: $x : \mathsf{Ob},\ y : \mathsf{Ob} \vdash \mathsf{Hom}(x, y)\,\mathsf{Type}$.
3. Composition operation: $x : \mathsf{Ob},\ y : \mathsf{Ob},\ z : \mathsf{Ob},\ f : \mathsf{Hom}(x, y),\ g : \mathsf{Hom}(y, z) \vdash g \circ f : \mathsf{Hom}(x, z)$.
4. Identity operator: $x : \mathsf{Ob} \vdash \mathsf{id}_x : \mathsf{Hom}(x, x)$.

Subject to the following axioms:

$$\frac{x : \mathsf{Ob},\ y : \mathsf{Ob},\ f : \mathsf{Hom}(x, y)}{\mathsf{id}_y \circ f \equiv f} \qquad \frac{x : \mathsf{Ob},\ y : \mathsf{Ob}, f : \mathsf{Hom}(x, y)}{f \circ \mathsf{id}_x \equiv f}$$

$$\frac{x : \mathsf{Ob},\ y :: \mathsf{Ob},\ z : \mathsf{Ob},\ w : \mathsf{Ob},\ f : \mathsf{Hom}(x, y), g : \mathsf{Hom}(y, z), h : \mathsf{Hom}(z, w)}{(h \circ g) \circ f \equiv h \circ (g \circ f)}$$

A.2 Substitution property

Let T be a κ-Cartmell theory. Recall that given Δ, $\{t_\alpha\}_{\alpha<\lambda}$ expressions and $\{x_\alpha\}_{\alpha<\lambda}$ variables then the new expression $\Delta[e_\alpha|x_\alpha]_{\alpha<\lambda}$ denotes the substitution of variables by the expressions.

Definition A.8. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta$ be a derived judgment of T. We say that this judgment has the *substitution property* if for every $\vdash \Gamma\,\mathsf{Ctxt}$ and expressions $\{t_\alpha\}_{\alpha<\lambda}$, such that for all $\alpha < \lambda$

$$\Gamma,\ \{t_\beta : \Delta_\beta[t_\gamma|x_\gamma]_{\gamma<\beta}\}_{\beta<\alpha} \vdash t_\alpha : \Delta_\alpha[t_\beta|x_\beta]_{\beta<\alpha}$$

are derived rules then

$$\Gamma \vdash \Delta[t_\alpha|x_\alpha]_{\alpha<\lambda}$$

is a derived rule of T.

In [Car78] it is proven that all derived judgment of a generalized algebraic theory satisfy the substitution property. This is done through a series of results that can be generalized to our setting. The proofs are omitted since they are the same as in the original reference.

Lemma A.9. *If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta$ is a derived judgment of T then the variables that appear in Δ is a subset of $\{x_\alpha\}_{\alpha<\lambda}$*

Proof. See [Car78, Lemma 1, Section 1.7]. $\square$

Lemma A.10. *1. The premise of a derived judgment is a context.*

2. If $\vdash \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}\mathsf{Ctxt}$ then for $\alpha < \lambda$, we have

$$\{x_\beta : \Delta_\beta\}_{\beta<\alpha} \vdash \Delta_\alpha \ \mathsf{Type}$$

Proof. See [Car78, Lemma 2, Section 1.7]. $\square$

Theorem A.11. *Every derived judgment of a κ-Cartmell theory has the substitution property.*

Proof. The same as proof as in [Car78, 1.7] applies. This goes by proving that each judgment has the substitution property. For the last two judgments in definition A.1 this is part of definition A.4. While for the first two it is done by induction on the derivations. It is shown that each derivation rule of definition A.4 preserve the substitution property. $\square$

This result has similar consequences to those in [Car78]. The proofs are analogous or the same. For us is only relevant to know that our κ-Cartmell theories are well defined. Meaning:

Proposition A.12. *The derived judgments of a κ-Cartmell theory are well-formed.*

Proof. Again, by induction on the derivations [Car78, pp. 1.33]. $\square$

Both the statement and proof of the next lemma are the same as The Derivation Lemma [Car78, pp. 1.34]. The proof does not rely on the context size.

{derivationlemma}

Lemma A.13. *1. Every derived type judgment of T is of the form*

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash A(t_\alpha)_{\alpha<\lambda}$$

for some type symbol A with introductory rule

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash A(x_\alpha)_{\alpha<\lambda} \ \mathsf{Type}$$

and $\{t_\alpha\}_{\alpha<\lambda}$ are expressions such that for all $\alpha < \lambda$ the rule

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\alpha : \Delta_\alpha[t_\delta \mid x_\delta]_{\delta<\alpha}.$$

2. *Every type element judgment of T is of the form*

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash x_\beta : \Omega$$

for some x_β and such that $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega_\beta \equiv \Omega$, or is of the form

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash f(t_\alpha)_{\alpha<\lambda} : \Omega$$

for some operator symbol f of T with introductory judgment of the form

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash f(x_\alpha)_{\alpha<\lambda} : \Delta$$

such that for each $\alpha < \lambda$ the rules

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\alpha : \Delta_\alpha[t_\delta \mid x_\delta]_{\delta<\alpha}$$

and

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Delta[t_\alpha \mid x_\alpha]_{\alpha<\lambda} \equiv \Omega$$

are derived rules of T.

Proof. This is follows from definition A.4 (10) and (11). □

A.3 Equivalence relation on judgments

Trough out this section we work in an κ-Cartmell theory. We first introduce a relation that allows us to identify context which express the same meaning, but differ on the variables that are used in it [Car78, 1.13].

There is a relation defined on the judgments of the κ-Cartmell theory T.

Definition A.14. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta_\lambda\ \mathsf{Type}$, $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega_\mu\ \mathsf{Type}$ be two type judgments of T. We say that

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta_\lambda\ \mathsf{Type} \approx \{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega_\mu\ \mathsf{Type}$$

if either:

1. Both ordinals are successors such that $\lambda = \mu = \nu + 1$ and for all $\alpha \leq \nu$ we have

$$\{x_\delta : \Delta_\delta\}_{\delta<\alpha} \vdash \Delta_\alpha \equiv \Omega_\alpha$$

   is a derived rule of T.

2. Both ordinals are limits with $\lambda = \mu$ and for any successor ordinal $\nu + 1 < \lambda$ we have

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\nu} \vdash \Delta_\nu\ \mathsf{Type} \approx \{x_\beta : \Omega_\beta\}_{\beta<\nu} \vdash \Omega_\nu\ \mathsf{Type}.$$

{equivalenceofcontexts}

Lemma A.15. *The relation $\approx$ is an equivalence relation on type judgments of the theory T.*

Proof. x This is an immediate result since we have assumed canonical names for variables. Otherwise we could repeat the argument as in [Car78, 1.13]. □

Definition A.16. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ and $\{x_\beta : \Omega_\beta\}_{\beta<\mu}$ be two contexts. We say that

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x_\beta : \Omega_\beta\}_{\beta<\mu}$$

if and only if $\lambda = \mu$ and for all $\alpha < \lambda$

$$\{x_\delta : \Delta_\delta\}_{\delta<\alpha} \vdash \Delta_\alpha \,\mathsf{Type} \approx \{x_\gamma : \Omega_\gamma\}_{\gamma<\alpha} \vdash \Omega_\alpha \,\mathsf{Type}$$

It follows that this induces an equivalence relation on contexts.

Definition A.17. We say that

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta \approx \{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash s : \Omega$$

if and only if $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \,\mathsf{Type} \approx \{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega \,\mathsf{Type}$ and $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv s$.

{remarktransitivityofpr

Remark A.18. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ and $\{x_\beta : \Omega_\beta\}_{\beta<\mu}$ be two contexts. Assume further that

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x_\beta : \Omega_\beta\}_{\beta<\mu}.$$

Then for all derived rules

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega$$

the rule

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Omega$$

is also a derived rule.

Regardless of its simplicity this remark is useful in the next:

Corollary A.19. *The relation $\approx$ is an equivalence relation on judgments of the form $\{x_\beta : \Delta_\beta\}_{\beta<\mu} \vdash t : \Delta$.*

Proof. Reflexivity is a consequence of 2 from definition A.4. Assume that $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta \approx \{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda} \vdash s : \Omega$. Hence the contexts satisfy $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda}$. Applying the symmetry of the relation $\approx$ on contexts and using remark A.18 we see that that $\{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda} \vdash t \equiv s$. Then we must have $\{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda} \vdash s : \Delta$ and $\{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda} \vdash \Omega \equiv \Delta$. We can apply 4 from definition A.4 to conclude that $\{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda} \vdash s \equiv t$, thus proving symmetry. Transitivity is a straightforward application of remark A.18. $\square$

Definition A.20. A *morphism* between contexts

$$\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$$

is μ-sequence of terms $\{t_\beta\}_{\beta<\mu}$ such that for all $\beta < \mu$ we have

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t_\beta : \Omega_\beta[t_\gamma | x_\gamma]_{\gamma<\beta}.$$

Just as in the finite case, with the substitution as composition and the obvious identity, it can be shown that contexts form a category with morphism as defined above. This is called the *category of realizations* of the theory T. The composition of

$$\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$$

and

$$\langle s_\delta \rangle_{\delta<\nu} : \{x_\beta : \Omega_\beta\}_{\beta<\mu} \to \{x_\delta : \Omega'_\delta\}_{\delta<\nu}$$

is the map

$$\langle s_\delta \rangle_{\delta<\nu} \circ \langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\delta : \Omega'_\delta\}_{\delta<\nu}$$

defined as the sequence $\langle s_\delta[\langle t_\beta | x_\beta \rangle_{\beta<\mu}] \rangle_{\delta<\nu}$.

Using the previous relation $\approx$ on contexts and rules we induce one on morphisms between contexts. If we have morphisms

$\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ and $\langle t'_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta'_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega'_\beta\}_{\beta<\mu}$

Then

$$\langle t_\beta \rangle_{\beta<\mu} \approx \langle t'_\beta \rangle_{\beta<\mu}$$

if and only if

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \approx \{x'_\beta : \Omega'_\beta\}_{\beta<\mu}$$

and for all $\gamma < \mu$

$$\{x_\beta : \Delta_\beta\}_{\beta<\mu} \vdash t_\gamma : \Omega_\gamma[t_{\gamma'}|x_{\gamma'}]_{\gamma'<\gamma} \approx \{x_\beta : \Delta'_\beta\}_{\beta<\mu} \vdash t'_\gamma : \Omega'_\gamma[t'_{\gamma'}|x_{\gamma'}]_{\gamma'<\gamma}.$$

Unfolding the definition this means that

$$\{x_\beta : \Delta_\beta\}_{\beta<\mu} \vdash \Omega_\gamma[t_{\gamma'}|x_{\gamma'}]_{\gamma'<\gamma}\ \mathsf{Type} \approx \{x_\beta : \Delta'_\beta\}_{\beta<\mu} \vdash \Omega'_\gamma[t'_{\gamma'}|x_{\gamma'}]_{\gamma'<\gamma}\ \mathsf{Type}$$

and that $\{x_\beta : \Delta_\beta\}_{\beta<\mu} \vdash t_\gamma \equiv t'_\gamma$ for all $\gamma < \mu$.

The following remarks are results from [Car78] whose proofs are completely similar. However, it is important to make them explicit since they imply that we can define a composition operation of equivalence classes of morphisms between contexts.

{lemma3}

Remark A.21. Let $\langle t_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ and $\langle t'_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega'_\beta\}_{\beta<\mu}$ two morphisms between contexts with $\langle t_\beta\rangle_{\beta<\mu} \approx \langle t'_\beta\rangle_{\beta<\mu}$.

1. If $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega\ \mathsf{Type}$ and $\{x_\beta : \Omega'_\beta\}_{\beta<\mu} \vdash \Omega'\ \mathsf{Type}$ are derived judgment of the theory such that

$$\{x_\beta : \Omega_\beta,\, x_\mu : \Omega\}_{\beta<\mu} \approx \{x_\beta : \Omega'_\beta,\, x_\mu : \Omega'\}_{\beta<\mu}$$

then

$$\{x_\alpha : \Delta_\alpha,\, x_\mu : \Omega[t_\beta|x_\beta]_{\beta<\mu}\}_{\alpha<\lambda} \approx \{x_\alpha : \Delta'_\alpha,\, x_\mu : \Omega'[t'_\beta|x'_\beta]_{\beta<\mu}\}_{\alpha<\lambda}$$

This follows by unwinding the relation $\approx$ and applying the principle 12 from definition A.4. This simply means that we can extend contexts by a fresh variable. Moreover, there is a more general result:

For all $\varepsilon > 0$, if $\{x_\beta : \Omega_\beta\}_{\beta<\mu+\varepsilon}$ and $\{x_\beta : \Omega'_\beta\}_{\beta<\mu+\varepsilon}$ are contexts then

$$\{x_\alpha : \Delta_\alpha,\, x_\beta : \Omega_\beta[t_\gamma|x_\gamma]_{\gamma<\beta}\}_{\substack{\alpha<\lambda,\\ \mu\le\beta<\mu+\varepsilon}} \approx \{x_\alpha : \Delta'_\alpha,\, x_\beta : \Omega'_\beta[t'_\gamma|x_\gamma]_{\gamma<\beta}\}_{\substack{\alpha<\lambda,\\ \mu\le\beta<\mu+\varepsilon}}$$

{item2}

2. If $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash s : \Omega$ and $\{x_\beta : \Omega'_\beta\}_{\beta<\mu} \vdash s' : \Omega'$ are derived judgment such that

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash s \equiv_\Omega s'.$$

Then

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash s[t_\beta|x_\beta]_{\beta<\mu} \equiv_{\Omega[t_\beta|x_\beta]_{\beta<\mu}} s'[t'_\beta|x_\beta]_{\beta<\mu}.$$

Observe that the principle 13 from definition A.4 implies this result.

Remark A.22. {lemma4}

1. Let $\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ be a morphism between two contexts. If

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x'_\alpha : \Delta'_\alpha\}_{\alpha<\lambda} \text{ and } \{x_\beta : \Omega_\beta\}_{\beta<\mu} \approx \{x'_\beta : \Omega'_\beta\}_{\beta<\mu}$$

   then $\langle t_\beta \rangle_{\beta<\mu} : \{x'_\alpha : \Delta'_\alpha\}_{\alpha<\lambda} \to \{x'_\beta : \Omega'_\beta\}_{\beta<\mu}$ is also a morphisms between these contexts.

2. If we have a context $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+1}$ and $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x'_\alpha : \Delta'_\alpha\}_{\alpha<\lambda}$ then we can extend the context $\{x'_\alpha : \Delta'_\alpha\}_{\alpha<\lambda}$ to $\{x'_\alpha : \Delta'_\alpha\}_{\alpha<\lambda+1}$ such that $x'_\alpha : \Delta'_\alpha$ is $x_\lambda : \Delta_\lambda$.

Remark A.23. Let $\langle t_\beta \rangle_{\beta<\mu+1} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu+1}$ and $\langle s_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ be morphisms between contexts. Then we have a morphism {lemma5}

$$\langle s_\beta \rangle_{\beta<\mu+1} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu+1}$$

where $s_\mu \equiv t_\mu$, and such that $\{s_\beta\}_{\beta<\mu+1} \approx \{t_\beta\}_{\beta<\mu+1}$.

A.4 The category of κ-Cartmell theories

We construct a category where the objects are κ-Cartmell theories with maps *interpretations*. This is analogous the category that Cartmell constructs in [Car78, 1.11], all the results can be copied from there to our setting. Since we work with different theories the alphabets, expressions and rules are marked accordingly. If T is a theory then these sets are denoted $Alp(T)$, $Exp(T)$, $Rul(T)$ respectively.

Let T and T' two κ-Cartmell theories. Let any function $I : Alp(T) \to Exp(T')$. Using this function we can define a *preinterpretation* $\widetilde{I} : Exp(T) \to Exp(T')$ by induction on the construction of expressions:

1. If $x \in V$
$$\widetilde{I}(x) := x,$$

2. If $F \in Alp(T)$
$$\widetilde{I}(F) := I(F),$$

3. If $L \in Alp(T)$ alphabet symbol and $\{t_\alpha\}_{\alpha<\lambda}$ are expressions
$$\widetilde{I}(L(t_\alpha)_{\alpha<\lambda}) := I(L)(\widetilde{I}(t_\alpha))_{\alpha<\lambda}.$$

{interpretation}

Definition A.24. Given a preinterpretation $\widetilde{I}$ we define a new function $\widehat{I} : Rul(T) \to Rul(T')$.

1. $\widehat{I}(\Gamma \vdash \Delta\ \mathsf{Type}) := \widetilde{I}(\Gamma) \vdash \widetilde{I}(\Delta)\ \mathsf{Type}$
2. $\widehat{I}(\Delta \vdash t : \Delta) := \widetilde{I}(\Delta) \vdash \widetilde{I}(t) : \widetilde{I}(\Delta)$
3. $\widehat{I}(\Delta,\ \Delta' \vdash \Delta \equiv \Delta') := \widetilde{I}(\Delta),\ \widetilde{I}(\Delta') \vdash \widetilde{I}(\Delta) \equiv \widetilde{I}(\Delta')$.
4. $\widehat{I}(\Delta,\ t, t' : \Delta \vdash t \equiv_{\Delta} t') := \widetilde{I}(\Delta),\ \widetilde{I}(t), \widetilde{I}(t') : \widetilde{I}(\Delta) \vdash \widetilde{I}(t) \equiv_{\widetilde{I}(\Delta)} \widetilde{I}(t')$.

This function is an *interpretation* from T into T' if all introductory judgment and axioms of T are sent to introductory judgment and axioms of T', we will simply denote this as $I : T \to T'$.

Just as in [Car78] it is possible to prove that:

{lemma1112}

Lemma A.25. *If I is an interpretation from T to T' then it preserves derived judgment of the theory T.*

Proof. From Lemma 2 [Car78, pp 1.52]. To illustrate how this is done we show that the derived judgment definition A.4 (13) it is preserved by I. Consider the derived judgment

$$\frac{\begin{array}{c}\vdash \Gamma\ \mathsf{Ctxt} \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t' \\ \Gamma,\ s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha},\ s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha} \vdash s_\alpha \equiv_{\Delta_\alpha[s'_\beta|x_\beta]_{\beta<\alpha}} s'_\alpha\end{array}}{\begin{array}{c}\Gamma,\ \{s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda},\ \{s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda} \\ \vdash t[s_\alpha \mid x_\alpha]_{\alpha<\lambda} \equiv_{\Delta[s_\alpha|x_\alpha]_{\alpha<\lambda}} t'[s'_\alpha \mid x_\alpha]_{\alpha<\lambda}\end{array}}$$

in the theory T. We may assume that the context Γ is of the form $\{x_\beta : \Omega_\beta\}_{\beta<\mu}$, so we get

$$\frac{\begin{array}{c}\vdash \{x_\beta : \Omega_\beta\}_{\beta<\mu}\ \mathsf{Ctxt} \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t' \\ \{x_\beta : \Omega_\beta\}_{\beta<\mu},\ s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha},\ s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha} \vdash s_\alpha \equiv_{\Delta_\alpha[s'_\beta|x_\beta]_{\beta<\alpha}} s'_\alpha\end{array}}{\begin{array}{c}\{x_\beta : \Omega_\beta\}_{\beta<\mu},\ \{s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda},\ \{s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda} \\ \vdash t[s_\alpha \mid x_\alpha]_{\alpha<\lambda} \equiv_{\Delta[s_\alpha|x_\alpha]_{\alpha<\lambda}} t'[s'_\alpha \mid x_\alpha]_{\alpha<\lambda}\end{array}}$$

Applying the I to the hypothesis and by lemma A.26 we obtain the following derivations in T'.

- $\vdash \{x_\beta : \widetilde{I}(\Omega_\beta)\}_{\beta<\mu}\ \mathsf{Ctxt}$,
- $\{x_\alpha : \widetilde{I}(\Delta_\alpha)\}_{\alpha<\lambda} \vdash \widetilde{I}(t) \equiv_\Delta \widetilde{I}(t')$,

- $\{x_\beta : \widetilde{I}(\Omega_\beta)\}_{\beta<\mu},\ s_\alpha : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s_\beta) \mid x_\beta]_{\beta<\alpha},\ \widetilde{I}(s'_\alpha) : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s'_\beta) \mid x_\beta]_{\beta<\alpha} \vdash \widetilde{I}(s_\alpha) \equiv_{\widetilde{I}(\Delta_\alpha)[\widetilde{I}(s'_\beta)|x_\beta]_{\beta<\alpha}} \widetilde{I}(s'_\alpha)$.

We have all the requirements to use definition A.4 (13) for the theory T'. Thus

$$\frac{\begin{array}{c}\vdash \{x_\beta : \widetilde{I}(\Omega_\beta)\}_{\beta<\mu}\,\mathsf{Ctxt} \qquad \{x_\alpha : \widetilde{I}(\Delta_\alpha)\}_{\alpha<\lambda} \vdash \widetilde{I}(t) \equiv_\Delta \widetilde{I}(t') \\ \{x_\beta : \widetilde{I}(\Omega_\beta)\}_{\beta<\mu},\ s_\alpha : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s_\beta) \mid x_\beta]_{\beta<\alpha},\ \widetilde{I}(s'_\alpha) : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s'_\beta) \mid x_\beta]_{\beta<\alpha} \\ \vdash \widetilde{I}(s_\alpha) \equiv_{\widetilde{I}(\Delta_\alpha)[\widetilde{I}(s'_\beta)|x_\beta]_{\beta<\alpha}} \widetilde{I}(s'_\alpha)\end{array}}{\begin{array}{c}\{x_\beta : \widetilde{I}(\Omega_\beta)\}_{\beta<\mu}, \{\widetilde{I}(s_\alpha) : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s_\beta) \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda}, \{\widetilde{I}(s'_\alpha) : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s'_\beta) \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda} \\ \vdash \widetilde{I}(t)[\widetilde{I}(s_\alpha) \mid x_\alpha]_{\alpha<\lambda} \equiv_{\widetilde{I}(\Delta)[\widetilde{I}(s_\alpha)|x_\alpha]_{\alpha<\lambda}} \widetilde{I}(t')[\widetilde{I}(s'_\alpha) \mid x_\alpha]_{\alpha<\lambda}\end{array}}$$

is a derived rule of T'. Therefore, the rule is preserved by the interpretation I.

□

The following lemma fills the gap:

{lemma1111}

Lemma A.26. *If I is an interpretation of T into T' and we have expressions f and $\{t_\alpha\}_{\alpha<\lambda}$ on the alphabet A_T then*

$$\widetilde{I}(f[t_\alpha \mid x_\alpha]_{\alpha<\lambda}) = \widetilde{I}(f)[\widetilde{I}(t_\alpha) \mid x_\alpha]_{\alpha<\lambda}.$$

Proof. This is done by induction on the length of f in [Car78, Lemma 1, pp. 1.52]. The interesting case is when $f = F(e_\beta)_{\beta<\mu}$ for some F in the alphabet and expressions $\{e_\beta\}_{\beta<\mu}$. We assume inductively the result true for the expressions $\{e_\beta\}_{\beta<\mu}$. Then we have:

$$\begin{aligned}\widetilde{I}(f[t_\alpha \mid x_\alpha]_{\alpha<\lambda}) &= \widetilde{I}(F(e_\beta[t_\alpha \mid x_\alpha]_{\alpha<\lambda})_{\beta<\mu}) \\ &= I(F)(\widetilde{I}(e_\beta[t_\alpha \mid x_\alpha]_{\alpha<\lambda}))_{\beta<\mu} \\ &= I(F)(\widetilde{I}(e_\beta)[\widetilde{I}(t_\alpha) \mid x_\alpha]_{\alpha<\lambda})_{\beta<\mu}, \text{ by induction hypothesis} \\ &= I(F)(\widetilde{I}(e_\beta))_{\beta<\mu}[\widetilde{I}(t_\alpha) \mid x_\alpha]_{\alpha<\lambda} \\ &= \widetilde{I}(F(e_\beta)_{\beta<\mu})[\widetilde{I}(t_\alpha) \mid x_\alpha]_{\alpha<\lambda} \\ &= \widetilde{I}(f)[\widetilde{I}(t_\alpha) \mid x_\alpha]_{\alpha<\lambda}\end{aligned}$$

□

There is also a notion of composition of interpretations: If $I : S \to T$ and $J : T \to U$ are interpretations then there is an interpretation $J \circ I : S \to U$ that is defined in the obvious way. It is also easy to infer what is the identity for this composition. A crucial result to define this compositions is:

Lemma A.27. *If $I : S \to T$ and $J : T \to U$ are interpretations then* $\widetilde{J \circ I}(e) = \widetilde{J}(\widetilde{I}(e))$

Proof. This is by induction of the expression e see [Car78, Lemma 3, pp. 1.55]. □

We can define the category of κ-GAT of κ-generalized algebraic theories. There is an equivalence relation on interpretations between two theories T and T'. If I, $J : T \to T'$ are two interpretations then $I \approx J$ if an only if for every rule $r \in R_U$ we have $I(r) \approx J(r)$ in the theory T'.

{lemma11114}

Lemma A.28. *If I and J are interpretations from T to T' such that $I \approx J$ then for all type and element judgment $\mathcal{J}$ of U, $\widehat{I}(\mathcal{J}) \approx \widehat{J}(\mathcal{J})$ in T'.*

Proof. See [Car78, Lemma 1, Section 1.14]. □

Then lemma A.28 implies that the compositions as given is well-defined. Finally, in order to get the correct morphisms we need to know that the equivalence relation on interpretations is compatible with the composition. Another advantageous consequence is that this it give us a criteria to establish whether two interpretations are equivalent.

{corollary2114}

Corollary A.29. *If I and J are interpretations from T to T' then $I \approx J$ if and only if for any type element judgment r, $\widehat{I}(r) \approx \widehat{J}(r)$.*

Proof. This follows from lemma A.28 and (3) of definition A.3. □

Corollary A.30. *If I and J are interpretations from T to T' and I' and J' are interpretations from T' to T'' then from $I \approx J$ and $I' \approx J'$ we conclude that $I' \circ I \approx J' \circ J$.*

Proof. [Car78, pp. 1.72]. □

The category κ-GAT has morphisms equivalence classes of interpretations [Car78, pp. 1.72].

A.5 Construction and properties of the category $\mathbb{C}_T$

{syntacticcat}

Let be T an κ-Cartmell theory. The category $\mathbb{C}_T$ has the following data:

- Objects: Equivalence classes of contexts under the relation $\approx$. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a context then the object in $\mathbb{C}_T$ is denoted $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$.
- Morphisms: A morphism between $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$ and $[\{x_\beta : \Omega_\mu\}_{\beta<\mu}]$ it is the equivalence class of a map

$$\langle t_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$$

  induced by the relation $\approx$. We denote this set by

$$\hom_{\mathbb{C}_T}([\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}], [\{x_\beta : \Omega_\mu\}_{\beta<\mu}]).$$

- Composition: This is induced by the composition of maps between contexts. This is again well-defined in view of 2 of remark A.21.
- Identity: For a context $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ its identity is the equivalence class $[\{x_\alpha\}_{\alpha<\lambda}]$.

Remark A.31. The category $\mathbb{C}_T$ has a unique object $1 := [\emptyset]$ the equivalence class of the empty context. Note that this is a terminal object.

{remarkdisplay}

Remark A.32. Let $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$ an object of $\mathbb{C}_T$. Then for any $\mu < \lambda$ we get a morphism $[\langle x_\beta\rangle_{\beta<\mu}] : [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : \Delta_\beta\}_{\beta<\mu}]$. Indeed, since $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a context then for any $\beta < \lambda$ we have $\{x_\delta : \Delta_\delta\}_{\delta<\delta} \vdash \Delta_\beta \ \mathsf{Type}$. Therefore, it follows from (definition A.4, 9) that $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash x_\alpha : \Delta_\alpha$ for all $\alpha < \lambda$. In particular this is true for all $\beta < \mu$, this gives the morphism above.

Following the same argument if $\nu < \mu$ then we also we a map $[\langle x_\gamma\rangle_{\gamma<\nu}] : [\{x_\beta : \Delta_\beta\}_{\beta<\mu}] \to [\{x_\gamma : \Delta_\gamma\}_{\gamma<\nu}]$. Furthermore, we get a commutative diagram:

$$\begin{tikzcd}
{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \arrow[dr, "{[\langle x_\gamma\rangle_{\gamma<\nu}]}"'] & {[\{x_\beta : \Delta_\beta\}_{\beta<\mu}]} \arrow[d, "{[\langle x_\gamma\rangle_{\gamma<\nu}]}"] \\
 & {[\{x_\gamma : \Delta_\gamma\}_{\gamma<\nu}]}
\end{tikzcd}$$

{display:generalized}

Remark A.33. Since this morphisms are somewhat canonical we will use the notation " $\twoheadrightarrow$ ", and whenever we use this arrow for a morphism it must be assumed that such map is of this form. These morphisms are called display, which is Cartmell's terminology. In contrast, our we our 'display' maps can be of arbitrary length, which we will often refer as *generalized display* maps.

Suppose there is a context $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+\varepsilon}]$ with $\varepsilon \geq 0$. Then we can consider an ε-indexed sequence of display morphisms:

$$\cdots \quad [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+2}] \longrightarrow\!\!\!\!\!\rightarrow [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+1}] \longrightarrow\!\!\!\!\!\rightarrow [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$$

Also, there is a display map $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+\varepsilon}] \twoheadrightarrow [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$. This display morphism will be by definition the composition for the sequence. If $\varepsilon = 0$ then this maps is simply the identity. We also get a factorization of the map $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \twoheadrightarrow 1$ via display maps for any $\lambda \geq 0$.

{limitcontext}

Observation A.34. From the previous remark A.32 we can observe that if λ is a limit ordinal then $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$ is the limit of the sequence

$$\cdots \quad [\{x_1 : \Delta_1, x_2 : \Delta_2\}] \longrightarrow\!\!\!\!\!\rightarrow [\{x_1 : \Delta_1\}] \longrightarrow\!\!\!\!\!\rightarrow 1.$$

If there is another context $[\{x_\delta : \Gamma_\delta\}_{\delta<\gamma}]$ and maps

$$[\langle t_\beta\rangle_{\beta<\alpha}] : [\{x_\delta : \Gamma_\delta\}_{\delta<\gamma}] \to [\{x_\beta : \Delta_\beta\}_{\beta<\alpha}]$$

for all $\alpha < \lambda$ then we can simply take the map

$$[\langle t_\alpha\rangle_{\alpha<\lambda}] : [\{x_\delta : \Gamma_\delta\}_{\delta<\gamma}] \to [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}].$$

This can be shown the cone map (which is unique). This verifies our claim.

Using remark A.32 we can define a function:

$$\nu : Ob(\mathbb{C}_T) \longrightarrow \kappa$$

as $\nu([\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]) := \lambda$. We call this the *length function*. We can use ν to construct a filtration on the objects of $\mathbb{C}_T$: we define

$$Ob_\lambda(\mathbb{C}_T) := \nu^{-1}(\lambda)$$

then $Ob(\mathbb{C}_T) = \coprod_{\lambda<\kappa} Ob_\lambda(\mathbb{C}_T)$, and so if $\alpha \leq \beta$ then $Ob_\alpha(\mathbb{C}_T) \subseteq Ob_\beta(\mathbb{C}_T)$. Furthermore, if $p : A \twoheadrightarrow B$ is a display morphism then $\nu(B) \leq \nu(A)$.

For $\alpha < \beta$ there are functions

$$\pi_\beta : Ob_\beta(\mathbb{C}_T) \to Ob_\alpha(\mathbb{C}_T)$$

that are defined in the obvious way. Additionally, $1 \in Ob_0(\mathbb{C}_T)$ is unique.

The proof of the following lemma is the same as in [Car78].

{pullbacksyntactic}

Lemma A.35. *The pullback of a display map along arbitrary morphisms in $\mathbb{C}_T$ exists and it is also display.*

Proof. We use induction over the context length. Assume we have the following diagram in $\mathbb{C}_T$:

$$\begin{tikzcd} & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu+1}]} \arrow[d, two heads, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \\ {[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, "{[\langle t_\beta\rangle_{\beta<\mu}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu}]} \end{tikzcd}$$

Then the pullback is given using remark A.21, the context is

$$[\{x_\alpha : \Delta_\alpha,\, x_\mu : \Omega_\mu[t_\beta \mid x_\beta]_{\beta<\mu}\}_{\alpha<\lambda}].$$

Therefore we have a commutative square

$$\begin{tikzcd} {[\{x_\alpha : \Delta_\alpha,\, x_\mu : \Omega_\mu[t_\beta \mid x_\beta]_{\beta<\mu}\}_{\alpha<\lambda}]} \arrow[r, "{[\langle t_\beta, x_\mu\rangle_{\beta<\mu}]}"] \arrow[d, two heads, "{[\langle x_\alpha\rangle_{\alpha<\lambda}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu+1}]} \arrow[d, two heads, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \\ {[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, "{[\langle t_\beta\rangle_{\beta<\mu}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu}]} \end{tikzcd} \tag{1}$$

{{pullback1}}

Note that by definition the left vertical morphism is also display. If there is another commutative square

$$\begin{tikzcd} {[\{x_\zeta : \Gamma_\zeta\}_{\zeta<\xi}]} \arrow[r, "{[\langle g_\beta\rangle_{\beta<\mu+1}]}"] \arrow[d, "{[\langle f_\alpha\rangle_{\alpha<\lambda}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu+1}]} \arrow[d, two heads, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \\ {[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, "{[\langle t_\beta\rangle_{\beta<\mu}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu}]} \end{tikzcd}$$

The map

$$[\langle f_\alpha, g_\mu\rangle_{\alpha<\lambda}] : [\{x_\zeta : \Gamma_\zeta\}_{\zeta<\xi}] \to [\{x_\alpha : \Delta_\alpha,\, x_\mu : \Omega_\mu[t_\beta \mid x_\beta]_{\beta<\mu}\}_{\alpha<\lambda}]$$

shows that the square (1) is the pullback.
Next, assume that we have a diagram

$$\begin{tikzcd} & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu}]} \arrow[d, two heads, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \\ {[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, "{[\langle t_\beta\rangle_{\beta<\nu}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\nu}]} \end{tikzcd}$$

where μ is a limit ordinal strictly larger than ν. We simplify the notation as follows:

$$\begin{tikzcd} & B_\mu \arrow[d, two heads] \\ A_\lambda \arrow[r, "\langle t_\beta \rangle_{\beta<\nu}"'] & B_\nu \end{tikzcd}$$

Assume that factorization of the map $B_\mu \twoheadrightarrow B_\nu$ is of the form

$$\ldots \twoheadrightarrow B_{\nu+2} \twoheadrightarrow B_{\nu+1} \twoheadrightarrow B_\nu$$

and therefore B_μ is the limit (obtained in a similar way as in observation A.34 and remark A.32). Then we can take the successive pullback

$$\begin{tikzcd} f^*B_\mu \arrow[rr, "q(f,B_\mu)"] \arrow[dr, phantom, "\lrcorner", very near start] & & B_\mu \\ \vdots & & \vdots \\ q(f,B_{\nu+1})^*B_{\nu+2} \arrow[rr, "q(q(f,B_{\nu+1}),B_{\nu+2})"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & B_{\nu+2} \arrow[d, two heads] \\ f^*B_{\nu+1} \arrow[rr, "q(f,B_{\nu+1})"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & & B_{\nu+1} \arrow[d, two heads] \\ A_\lambda \arrow[rr, "f"'] & & B_\nu \end{tikzcd} \tag{2}$$

{{limitpullback}}

where at each successor stage it is given as before, $f := \langle t_\beta \rangle_{\beta<\nu}$, the context

$$f^*B_\mu := [\{x_\alpha : \Delta_\alpha,\, x_\beta : \Omega_\beta[t_\delta \mid x_\delta]_{\delta<\beta}\}_{\substack{\alpha<\lambda \\ \nu<\beta<\mu}}]$$

is the limits of the sequence on the left hand side, with the obvious display maps to each object in the sequence, and

$$q(f, B_\mu) := [\langle t_\beta,\, x_\gamma \rangle_{\beta<\nu<\gamma<\mu}].$$

This makes the outer rectangle in (2) commutative. Moreover, the map $q(f, B_\mu)$ is the unique cone map induced by the family of maps

$$\{[\langle t_\beta,\, x_\gamma \rangle_{\beta<\nu<\gamma<\delta}] : f^*B_\mu \to B_\delta\}_{\nu<\delta<\mu}.$$

$\square$

Using the same notation as in the lemma above we have

Remark A.36. 1. If $f = Id_{B_\nu}$ then $(Id_{B_\nu})^* B_\mu = B_\mu$ and $q(Id_{B_\nu}, B_\mu) = Id_{B_\mu}$.

2. For a diagram

$$\begin{tikzcd} & & A \arrow[d, two heads, "p"] \\ D \arrow[r, "g"] & C \arrow[r, "f"] & B \end{tikzcd}$$

we have that $g^*(f^*(A)) = (fg)^*(A)$ and $q(fg, A) = q(f, A)(g, f^*A)$.

We will refer the category $\mathbb{C}_T$ as the *syntactic category* associated to the κ-Cartmell theory T.

{explicitpullbacks}

Observation A.37. We note that lemma A.35 give us an explicit construction of pullbacks in $\mathbb{C}_T$, as well the pullback of the maps and an explicit description of $q(f, B_\mu)$.

We finish this section by characterizing the display maps in the category $\mathbb{C}_T$. This result says that display maps are somehow generic. We start with a preparatory result.

{display:pullback-axiom

Lemma A.38. *Let T a κ-Cartmell theory and $\mathbb{C}_T$ its syntactic κ-contextual category. Assume that there is a $f : \Delta \to \Gamma$, then any display map $B \twoheadrightarrow \Delta$ of length 1 can be obtained as a pullback of the form*

$$\begin{tikzcd} B \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma' \arrow[d, two heads] \\ \Delta \arrow[r, "f"'] & \Gamma \end{tikzcd}$$

where $\Gamma' \twoheadrightarrow \Gamma$ is of length 1.

Proof. This simply a reformulation of lemma A.13. Assume that

$$f = [\langle t_\beta \rangle_{\beta<\mu}] : [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : \Gamma_\beta\}_{\beta<\mu}].$$

Therefore, when the display map is of the form

$$[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+1}] \twoheadrightarrow [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}].$$

We can construct the square

$$\begin{tikzcd} {[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+1}]} \arrow[r, "\langle t_\beta, x_\lambda \rangle_{\beta<\mu}"] \arrow[d, two heads] & {[\{x : \Gamma_\beta, x_\lambda : \Delta_\lambda\}_{\beta<\mu}]} \arrow[d, two heads] \\ {[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, "\langle t_\beta \rangle_{\beta<\mu}"'] & {[\{x : \Gamma_\beta\}_{\beta<\mu}].} \end{tikzcd}$$

Since for all $\beta < \mu$, x_β does not occur in Δ_λ we have that $\Delta_\lambda[t_\beta|x_\beta]_{\beta<\mu} \equiv \Delta_\lambda$. Hence, it follows from the construction of pullbacks in $\mathbb{C}_T$ (lemma A.35) that the square above is indeed a pullback diagram. □

We are ready to give the full description of display maps.

{generalized-display:li

Proposition A.39. *Every Display map $B \twoheadrightarrow \Delta$ in $\mathbb{C}_T$ is a limit of a κ-small tower $V : \lambda \to \mathbb{C}_T$ where for each limit ordinal $\beta < \lambda$*

$$V(\beta) = \mathit{Lim}_{\alpha<\beta} V(\alpha)$$

and the map $V(\alpha+1) \to V(\alpha)$ is a pullback of a length one display map of the form $(\Gamma, A) \twoheadrightarrow \Gamma$ where $\Gamma \vdash A\,\mathsf{Type}$ is a type axiom of the theory T.

Proof. Each display map in $\mathbb{C}_T$ has a length λ. Just as in remark A.32 it admits a decomposition into display maps. It will be enough to prove the second claim, but this follows by an inductive argument in conjunction with the previous lemma A.38. The inductive step provide us with the required map $f : V(\alpha) \to \Gamma$ in lemma A.38. □

B Contextual categories and Cartmell theories

This section is the most relevant part. We will show that from the syntax of a κ-Cartmell theory we can construct a category, called κ-*Contextual category*, which we now introduce.

B.1 κ-contextual categories

The discussion in appendix A.5 on the properties of the syntactic category $\mathbb{C}_T$ can be summarized with the next definition which is the natural generalization of Cartmell's [Car78] or [KL18]. We present our definition in the same way as in the later. Recall that κ is a regular cardinal.

{contextualcat}

Definition B.1. A category $\mathcal{C}$ is said to be a κ-*contextual category* if:

1. The objects of $\mathcal{C}$ have grading $Ob(\mathcal{C}) = \coprod_{\lambda<\kappa} Ob_\lambda(\mathcal{C})$. This grading determines the *height* of any object $B \in \mathcal{C}$, which we write as $ht(B)$.

2. There is a terminal object $1 \in \mathcal{C}$ and it is unique up to equality with height 0.

3. There is a wide subcategory $Dis(\mathcal{C})$ with distinguished maps " $\twoheadrightarrow$ " called *display morphisms*,

4. The subcategory $Dis(\mathcal{C})$ is closed under transfinite compositions: If we have

$$\cdots \twoheadrightarrow B_3 \twoheadrightarrow B_2 \twoheadrightarrow B_1 \twoheadrightarrow B_0$$

a λ-sequence of display maps then there a unique object B in $Dis(\mathcal{C})$ with height λ and for each $\mu \leq \lambda$ a display map $B \twoheadrightarrow B_\mu$ such that for any $\alpha < \lambda$ we have a factorization

$$\begin{tikzcd} B \arrow[rr, two heads] \arrow[dr, two heads] & & B_0 \\ & B_\alpha \arrow[ur, two heads] & \end{tikzcd}$$

5. The inclusion functor preserve $i : Dis(\mathcal{C}) \hookrightarrow \mathcal{C}$ transfinite compositions.

6. If $A \twoheadrightarrow B$ is an arrow in $Dis(\mathcal{C})$ then $B \in Ob_\mu(\mathcal{C})$ and $A \in Ob_\lambda(\mathcal{C})$ for some ordinals λ, μ with $\mu \leq \lambda$.

7. For any object $A \in Ob_\lambda(\mathcal{C})$ and any $\mu \leq \lambda$ there exists a unique object $B \in Ob_\mu(\mathcal{C})$ and a unique display map $A \twoheadrightarrow B$. The *length* of this display map is the unique ordinal α such that $\lambda = \mu + \alpha$, is such situation we write $lt(p)$.

8. For any $A \in Ob_\lambda(\mathcal{C})$, a map $A \twoheadrightarrow B$ and any map $f : C \to B$ there is a pullback square

$$\begin{tikzcd} f^*A \arrow[r, "q(f,A)"] \arrow[d, two heads, "f^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "p"] \\ C \arrow[r, "f"'] & B \end{tikzcd}$$

called *canonical pullback* of A along f, and we require $lt(f^*p) = lt(p)$.

9. Canonical pullbacks are strictly functorial: for ordinals with $\mu \leq \lambda$, $A \in Ob_\lambda(\mathcal{C})$

   (a) If $f = id_B$ then $id_B^* A = A$ and $q(id_B, A) = id_A$.

   (b) For a diagram

$$\begin{tikzcd} & & A \arrow[d, two heads, "p"] \\ D \arrow[r, "g"] & C \arrow[r, "f"] & B \end{tikzcd}$$

   we have that $g^*(f^*(A)) = (fg)^*(A)$ and $q(fg, A) = q(f, A)(g, f^*A)$.

10. Given display maps $p : A \twoheadrightarrow B$ and $q : B \to C$ and any $f : X \to C$, in the diagram

$$\begin{tikzcd}
q(f,B)^*A \arrow[r, "q(q(f,B),A)"] \arrow[d, two heads, "q(f,B)^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "p"] \\
f^*B \arrow[r, "q(f,B)"'] \arrow[d, two heads, "f^*r"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "r"] \\
X \arrow[r, "f"'] & C
\end{tikzcd}$$

We have that $f^*r \circ (q(f,B)^*p) = f^*(r \circ p)$ and $q(q(f,B),A) = q(f,A)$.

Remark B.2. We use the term "display map" in rather different way to Cartmell. For us, a display map can have any height and it is only bounded by the regular cardinal κ.

We have already seen one example of such category.

{syntactic:contextual}

Corollary B.3. *For any κ-Cartmell theory T the syntactic category $\mathbb{C}_T$ is a κ-contextual category.*

Proof. This is done throughout appendix A.5. □

{convention1}

Remark B.4. It follows from definition B.1 that for any object $B \in \mathcal{C}$ the map $B \twoheadrightarrow 1$ can be decomposed as a transfinite composition of display maps

$$B_\lambda \twoheadrightarrow \ldots \twoheadrightarrow B_1 \twoheadrightarrow 1.$$

The length of decomposition above is given by the degree of B. This is what [Car78] calls the tree structure of the category. Whenever we refer to objects in a κ-contextual category as above, we will emphasize its height by writing B_λ. Likewise, we will denote the display maps as $p_\alpha : B_\lambda \twoheadrightarrow B_\alpha$ for each $\alpha < \lambda$.

The following lemma is a consequence of definition B.1 and remark B.4.

Lemma B.5. *Let $B \in Ob_\lambda(\mathcal{C})$ such that λ is a limit ordinal. Then B itself is a limit object in $\mathcal{C}$.*

Proof. From remark A.32 we obtain a sequence

$$\cdots \longrightarrow\!\!\!\!\!\rightarrow B_3 \longrightarrow\!\!\!\!\!\rightarrow B_2 \longrightarrow\!\!\!\!\!\rightarrow B_1 \longrightarrow\!\!\!\!\!\rightarrow 1$$

It follows from Axiom 4 of definition B.1 that B must be limit of the sequence. Finally, we use that the inclusion $Dis(\mathcal{C}) \to \mathcal{C}$ preserve limits. □

Definition B.6. Let $\mathcal{C}$, $\mathcal{D}$ contextual categories. A functor $F : \mathcal{C} \to \mathcal{D}$ it is called *contextual functor* if it satisfies the following conditions:

1. $F(Ob_\lambda(\mathcal{C})) \subseteq Ob_\lambda(\mathcal{D})$ for all $\lambda < \kappa$,

2. F restricts to a functor $Dis(\mathcal{C}) \to Dis(\mathcal{D})$,

3. F preserve canonical pullbacks up to equality, meaning that for any square in $\mathcal{C}$

$$\begin{tikzcd} f^*A \arrow[r, "q(f,A)"] \arrow[d, two heads, "f^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "p"] \\ C \arrow[r, "f"'] & B \end{tikzcd}$$

   we have $F(f^*A) = (Ff)^*(FA)$ and $F(q(f,A)) = q(Ff, FA)$.

Since the degree of each object is preserved by a κ-contextual functor, it makes sense to denote $F(A_\lambda) := F(A)_\lambda$ for $A_\lambda \in \mathcal{C}$. Another piece of notation we can introduce is from the functor $F : Dis(\mathcal{C}) \to Dis(\mathcal{D})$; since any display map $p_\alpha : A_\lambda \twoheadrightarrow A_\alpha$ is sent to a display map $F(p_\alpha) : F(A)_\lambda \twoheadrightarrow F(A)_\alpha$ and the degrees are preserved, we agree to omit F on this maps.
Contextual functors are the morphisms of the category of κ-contextual categories, we will denote it as κ-CON.

B.2 Interlude: categorical facts

{interlude}

We collect and recall some categorical facts about general κ-contextual categories.

Proposition B.7 (The slice κ-contextual category)**.** *Let $\mathcal{C}$ be a κ-contextual category. For any object $B \in Ob_\mu(\mathcal{C})$ there is a κ-contextual category which is a full subcategory of the slice $\mathcal{C}_{/B}$ which has objects display maps $A \twoheadrightarrow B$ where $A \in Ob_\lambda(\mathcal{C})$ with $\lambda \geq \mu$.*

Since we will rarely use categories other than κ-contextual categories, we will employ the slice notation $\mathcal{C}_{/B}$ for the category from the previous proposition.

Proof. The proof is purely completely formal. The important fact to remember is that the pullback of a display map is also display. $\square$

It is a well known fact that the pasting of two pullbacks give us a pullback, in our case consider the following diagram:

$$\begin{tikzcd}
f^*B_\mu \arrow[r, "q(f,B_\mu)"] \arrow[dr, phantom, "\lrcorner", very near start] & B_\mu \\
\vdots & \vdots \\
q(f,B_{\nu+1})^*B_{\nu+2} \arrow[r, "q(q(f,B_{\nu+1}),B_{\nu+2})"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\nu+2} \arrow[d, two heads] \\
f^*B_{\nu+1} \arrow[r, "q(f,B_{\nu+1})"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\nu+1} \arrow[d, two heads] \\
A_\lambda \arrow[r, "f"'] & B_\nu
\end{tikzcd}$$

Then if μ is a limit ordinal, the object B_μ is the limit of the sequence on the right hand side. Thus f^*B_μ is the limit of the sequence on the left hand side. Note that pairwise we have $q(f,B_{\nu+1})^*B_{\nu+2} = f^*B_{\nu+2}$ and $q(f,B_{\mu+2}) = q(q(f,B_{\mu+1}),B_{\mu+2})$.
If $f : A_\lambda \to B_\nu$ and $p_\nu : B_\mu \twoheadrightarrow B_\nu$ is a display map with $\mu = \nu+1$, using the universal property of the pullback we can construct the following diagram

$$\begin{tikzcd}
A_\lambda \arrow[dr, dashed, "\delta^\nu_f"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "Id_{A_\lambda}"'] & & \\
& (p_\nu f)^*B_\mu \arrow[r] \arrow[d, two heads, "(p_\nu f)^*p_\nu"'] \arrow[dr, phantom, "\lrcorner", very near start] & B_\mu \arrow[d, two heads, "p_\nu"] \\
& A_\lambda \arrow[r, "p_\nu f"'] & B_\nu .
\end{tikzcd}$$

The map δ^ν_f makes both triangles commutative. We will focus on the fact that $((f_\nu)^*p_\nu)\delta^\nu_f = Id_{A_\lambda}$, where $f_\nu = p_\nu f$. Assume that we have a map $p : B_\mu \twoheadrightarrow B_\nu$ with μ a limit ordinal, in particular the length of p is a limit ordinal. Then a map $f : A_\lambda \to B_\mu$ is determinate by a family of maps

$\{f_\gamma : A_\lambda \to B_\gamma\}$. Then we obtain:

$$
\begin{tikzcd}
A_\lambda \arrow[drr, bend left, "f"] \arrow[dr, dashed, "\delta^\nu_f"] \arrow[dddddr, bend right, "Id"'] & & \\
& f^*B_\mu \arrow[r, "q(f,B_\mu)"] \arrow[dr, phantom, "\lrcorner", very near start] & B_\mu \arrow[dddd, bend left, "p"] \\
& \vdots & \vdots \\
& q(f,B_{\nu+1})^*B_{\nu+2} \arrow[r, "q(q(f,B_{\nu+1}),B_{\nu+2})"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\nu+2} \arrow[d, two heads] \\
& f^*B_{\nu+1} \arrow[r, "q(f,B_{\nu+1})"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\nu+1} \arrow[d, two heads] \\
& A_\lambda \arrow[r, "f_\nu"'] & B_\nu
\end{tikzcd}
$$

where the map δ^ν_f is given a the family of maps $(\delta^\nu_f)_\gamma$ each given by an intermediate pullback square in the diagram above.

Notation B.8. If the situation above, for $f : A_\lambda \to B_\mu$ we denote

$$\Gamma(B^\mu_\nu) := \{h : A_\lambda \to (p_\nu f)^*B_\mu \mid ((p_\nu f)^* p_\nu)h = Id_{A_\lambda}\}.$$

We can consider a more general case, if $A_\lambda \in Ob_\lambda(\mathcal{C})$ and $B_\mu \in Ob_\mu(\mathcal{C})$ with $\lambda < \mu$, then there is a unique display map $p : B_\mu \twoheadrightarrow A_\lambda$. We set

$$\Gamma(B^\mu_\lambda) := \{s : A_\lambda \to B_\mu \mid ps = Id_{A_\lambda}\}$$

for this situation as well, since the object A_λ will be inferred from the context.

If the contextual category is $\mathbb{C}_T$ then, recalling lemma A.35, we can give an explicit description of the map δ^ν_f.

{descriptiondelta}

Lemma B.9. *Assume that* $f := [\langle t_\beta \rangle_{\beta<\nu}] : [\{x_\alpha : A_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : B_\beta\}_{\beta<\nu}]$ *and there is a display map* $p : [\{x_\beta : B_\beta\}_{\beta<\mu}] \twoheadrightarrow [\{x_\beta : B_\beta\}_{\beta<\nu}]$ *then* $\delta^\nu_f = [\langle x_\alpha, t_\beta \rangle_{\substack{\alpha<\lambda \\ \nu<\beta<\mu}}]$.

Proof. This follows by induction on μ and the explicit construction of pullbacks from lemma A.35. ☐

In certain situations, the property above characterizes the map δ^ν_f.

Lemma B.10. *If* $[\{x_\beta : B_\beta\}_{\beta<\mu}]$ *is an object of* $\mathbb{C}_T$ *and* $\nu < \mu$ *then* $f \in \Gamma(B^\mu_\nu)$ *if and only if* $f = [\langle x_\beta, t_\gamma\rangle_{\beta<\nu<\gamma<\mu}]$, *where for all* $\nu < \gamma < \mu$ *the rule* $\{x_\beta : B_\beta\}_{\beta<\nu}, \{t_{\gamma'} : B_{\gamma'}\}_{\gamma'<\gamma} \vdash t_\gamma : B_\gamma$ *is a derived rule.*

The next result follows from the previous lemmas and it is used in observation B.41.

{lemma3220}

Lemma B.11. *Let* A_λ, B_μ *objects of* $\mathcal{C}$ *and for each* $\beta < \mu$ *we have maps* $r_{\beta+1} \in \Gamma(r^*_\beta \cdots r^*_1 p^* B_{\beta+1})$ *then there exists a unique sequence of maps* $\{g_\beta : A_\lambda \to B_\beta\}_{\beta<\mu}$ *such that for all* $\beta < \mu$ *we have* $p_\beta g_{\beta+1} = g_\beta$ *such that* $\delta_{g_\beta} = r_\beta$.

Some words about the previous lemma are in order. The expression $r^*_\beta \cdots r^*_1 p^* B_{\beta+1}$ can be illustrated by the first two steps:

$$
\begin{tikzcd}
p^*B_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_2 \arrow[d, two heads] \\
p^*B_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \arrow[d, two heads] \\
A_\lambda \arrow[r, "p"'] \arrow[u, bend left, "r_1"] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
r^*_1 p^* B_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & p^*B_2 \arrow[d] \\
A_\lambda \arrow[r, "r_1"'] \arrow[u, bend left, "r_2"] & p^*B_1
\end{tikzcd}
$$

B.3 The equivalence between κ-GAT and κ-CON

B.3.1 The functor $\mathbb{C} : \kappa\text{-GAT} \to \kappa\text{-CON}$

To establish this equivalence of categories we first define a functor $\mathbb{C} : \kappa\text{-GAT} \to \kappa\text{-CON}$ using the construction of appendix A.5. The proof again comes from ([Car78], section 2.4.1). We register all preliminary results needed to define this functor, however again we omit the proofs since they are similar to the original ones given by Cartmell.

On objects $\mathbb{C} : \kappa\text{-GAT} \to \kappa\text{-CON}$ is defined as $\mathbb{C}_T$ for T a κ-Cartmell theory. For a map $[I] : T \to T'$ between theories we need functor $\mathbb{C}(I) : \mathbb{C}_T \to \mathbb{C}_{T'}$:

1. On objects; $\mathbb{C}(I)([\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]) := [\{x_\alpha : \widetilde{I}(\Delta_\alpha)\}_{\alpha<\lambda}]$,
2. On morphisms: If $[\langle t_\beta\rangle_{\beta<\mu}] : [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : \Delta_\beta\}_{\beta<\mu}]$ then $\mathbb{C}(I)([\langle t_\beta\rangle_{\beta<\mu}]) := [\langle \widetilde{I}(\langle t_\beta\rangle_{\beta<\mu})]$.

If we there is an interpretation J in the equivalence class $[I]$ then by lemma A.28 any rule r of T we get $\widehat{I}(r) \approx \widehat{J}(r)$. Therefore, the definition of $\mathbb{C}(I)$ does not depend on the representative of $[I]$.

Remains to verify that $\mathbb{C}(I)$ is indeed a contextual functor. Firstly, it is primordial to verify it is well-defined.

Lemma B.12. *Let $[I] : T \to T'$ be a map in κ-GAT then the following hold:*

1. *The interpretation I preserves contexts: If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a context in the theory T then $\{x_\alpha : \overline{I}(\Delta_\alpha)\}_{\alpha<\lambda}$ is a context in the theory T'.*

2. *The interpretation I preserves the equivalence relation $\approx$ between contexts: If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ and $\{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda}$ are contexts in the theory U with $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda}$ then $\{x_\alpha : \overline{I}(\Delta_\alpha)\}_{\alpha<\lambda} \approx \{x_\alpha : \overline{I}(\Omega_\alpha)\}_{\alpha<\lambda}$.*

3. *The interpretation I preserves morphisms between contexts: If $\langle t_\beta\rangle_{\beta<\mu}\} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ is a morphism between contexts in the theory T then $\langle \overline{I}(t_\beta)\rangle_{\beta<\mu}\} : \{x_\alpha : \overline{I}(\Delta_\alpha)\}_{\alpha<\lambda} \to \{x_\beta : \overline{I}(\Omega_\beta)\}_{\beta<\mu}$ is a morphism between contexts in the theory T'.* {itemiv}

4. *The interpretation I preserves the equivalence relation $\approx$ between morphisms of contexts: If $\langle s_\beta\rangle_{\beta<\mu}, \langle t_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ are morphisms between contexts in the theory T with $\langle s_\beta\rangle_{\beta<\mu} \approx \langle t_\beta\rangle_{\beta<\mu}$ then $\langle \overline{I}(s_\beta)\rangle_{\beta<\mu} \approx \langle \overline{I}(t_\beta)\rangle_{\beta<\mu}$.*

Proof. The proof of each statement is consequence of lemma A.26 or lemma A.25. Our enumeration of variables give us a notation simplification of the proof given by [Car78].
For example to prove 4; we have by assumption that $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t_\gamma \equiv_{\Omega_\gamma[t_\beta|x_\beta]_{\beta<\gamma}} s_\gamma$ for all $0 < \gamma \leq \mu$. Therefore, since the interpretation preserves this rule oT we get that $\{x_\alpha : \overline{I}(\Delta_\alpha)\}_{\alpha<\lambda} \vdash \overline{I}(t_\gamma) \equiv_{\overline{I}(\Omega_\gamma)[\overline{I}(t_\beta)|x_\beta]_{\beta<\gamma}} \overline{I}(s_\gamma)$ for all $0 < \gamma \leq \mu$. This exactly establishes $\langle \overline{I}(s_\beta)\rangle_{\beta<\mu} \approx \langle \overline{I}(t_\beta)\rangle_{\beta<\mu}$. □

We have seen that the definition of $\mathbb{C}(I)$ give us the correct objects and morphisms. Now we show that it is indeed a contextual functor.

Lemma B.13. *Let $I : T \to T'$ be a morphism in κ-GAT. Then the map $\mathbb{C}(I) : \mathbb{C}_T \to \mathbb{C}_{T'}$ is a contextual functor.*

Proof. The map is functor trivially. That it preserves the grading and restricts to a functor between the display subcategories $Dis(\mathbb{C}_T) \to Dis(\mathbb{C}_{T'})$ it is also immediate. To prove it preserves canonical pullbacks consider the following pullback square in the category $\mathbb{C}_T$:

$$\begin{tikzcd}
{[\{x_\alpha : \Delta_\alpha,\, x_\gamma : \Omega_\gamma[t_\beta \mid x_\beta]_{\beta<\mu}\}_{\substack{\alpha<\kappa,\\ \mu\leq\gamma<\mu+\varepsilon}}]} \arrow[r, "{[\langle t_\beta, x_\gamma\rangle_{\substack{\beta<\mu,\\ \mu\leq\gamma<\mu+\varepsilon}}]}"] \arrow[d, "{[\langle x_\alpha\rangle_{\alpha<\kappa}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu+\varepsilon}]} \arrow[d, two heads, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \\
{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\kappa}]} \arrow[r, "{[\langle t_\beta\rangle_{\beta<\mu}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu}]}
\end{tikzcd}$$

Then a straightforward computation, using the definition of $\mathbb{C}(I)$ shows that this is send to a pullback square in the category $\mathbb{C}_{T'}$. $\square$

Corollary B.14. *There is a functor $\mathbb{C} : \kappa$-GAT $\to \kappa$-CON.*

B.3.2 The functor $U : \kappa$-CON $\to \kappa$-GAT

{functor-contextual-to-

We now turn to construct a functor that to each κ-contextual category $\mathcal{C}$ associates a κ-generalized algebraic theory $U(\mathcal{C})$, this is part of [Car78, Section 2.4]. We will use the notation introduced in remark B.4. This means we identify each object by its height, say B_λ, and write display maps as $p_\alpha : B_\lambda \twoheadrightarrow B_\alpha$ if $\lambda > 0$ and $\alpha < \lambda$. If $\alpha = 0$ then $B_0 = 1$ the terminal object. A morphism $f : A_\lambda \to B_\mu$ is trivial when B_μ is trivial i.e $\mu = 0$.

{definitiongatfromcat}

Definition B.15. We define $U(\mathcal{C}) \in \kappa$-GAT as:

1. For each non-trivial object B_μ with $\mu = \lambda + 1$ a type symbol $\overline{B_\mu}$ with introductory rule: $\{x_\beta : \overline{B_\beta}\}_{\beta<\mu} \vdash \overline{B_\mu}(x_\beta)_{\beta<\mu}\,\mathsf{Type}$. The notation emphasizes the fact that $\overline{B_\mu}$ depends on the indicated variables.

2. If $f : A_\lambda \to B_\mu$ is morphism of $\mathcal{C}$ with $\mu = \nu + 1$ we get an operator symbol $\overline{f}$. It has introductory rule;

   - If $f : A_\lambda \to B_{\mu+1}$, denote by $\rho_\mu : B_{\mu+1} \twoheadrightarrow B_\mu$. Then the operator symbol has introductory rule

   $$\{x_\alpha : \overline{A}_\alpha\}_{\alpha<\lambda} \vdash \overline{f}(x_\alpha)_{\alpha<\lambda} : \overline{(\rho_\mu f)^* B_{\mu+1}}(x_\alpha)_{\alpha<\lambda}.$$

   This does not clash with the notation from the previous point since it always refer to an object of $\mathcal{C}$ and in this case refers to map.

Subject to the following axioms in $U(\mathcal{C})$:

1. Let A_λ, B_μ, $C_{\nu+1}$ be objects of $\mathcal{C}$ and maps $f : A_\lambda \to B_\mu$, $g : B_\mu \to C_{\nu+1}$:

$$\{x_\alpha : \overline{A}_\alpha\}_{\alpha<\lambda} \vdash \overline{gf}(x_\alpha)_{\alpha<\lambda} \equiv_{\overline{(p_\nu gf)^* C_{\nu+1}}(x_\alpha)_{\alpha<\lambda}} \overline{g}\big(\overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\big)_{\beta<\mu}.$$

2. Let B_μ be a non-trivial object of $\mathcal{C}$. For each $\delta < \mu$ we have

$$\{x_\beta : \overline{B}_\beta\}_{\beta<\mu} \vdash \overline{p_\delta}(x_\beta)_{\beta<\mu} \equiv_{\overline{B}_\delta(x_\beta)_{\beta<\delta}} x_\delta.$$

3. Let A_λ, $B_{\mu+1}$ objects of $\mathcal{C}$ and a map $f : A_\lambda \to B_\mu$ then

$$\{x_\alpha : \overline{A}_\alpha\}_{\alpha<\lambda} \vdash \overline{f^* B_{\mu+1}}(x_\alpha)_{\alpha<\lambda} \equiv \overline{B_{\mu+1}}\big(\overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\big)_{\beta<\mu}$$

and

$$\{x_\alpha : \overline{A}_\alpha,\, x_\delta : \overline{f^* B_{\mu+1}}(x_\alpha)_{\alpha<\lambda}\}_{\alpha<\lambda} \vdash \overline{q(f, B_{\mu+1})}(x_\alpha, x_\delta)_{\alpha<\lambda} \equiv_{\overline{f^* B_\mu}(x_\alpha)_{\alpha<\lambda}} x_\delta.$$

Observation B.16. It is immediate to observe that $U(\mathcal{C})$ as defined is a κ-pretheory. We have sort symbol and operator symbols introduced by type judgment and type element judgments respectively. Note that the list of axioms we provided are well-formed rules. This is because the premise of each axiom is by definition a context.

Remark B.17. If $f : A_\lambda \to B_\mu$ is a map in $\mathcal{C}$, where μ is a limit ordinal i.e B_μ is a limit object, then we get a family of maps $\{f_\nu : A_\lambda \to B_\nu\}_{\nu<\mu}$. Therefore, the associated operator $\overline{f}$ is uniquely determined by the operators $\overline{f_\nu}$ for which in this case we can assume that ν is a successor ordinal.

If $F : \mathcal{C} \to \mathcal{D}$ is a functor between κ-contextual categories then we need an interpretation $U(F) : U(\mathcal{C}) \to U(\mathcal{D})$;

1. For an object A_λ, the interpretation is defined as

$$U(F)(\overline{A_\lambda}) := \overline{FA_\lambda}(x_\alpha)_{\alpha<\lambda}.$$

2. For a morphism $f : A_\lambda \to B_{\mu+1}$, the operator $\overline{f}$ is interpreted as

$$U(F)(\overline{f}) := \overline{F(f)}(x_\alpha)_{\alpha<\lambda}.$$

The next step is to prove that this is indeed an map between the κ-Cartmell theories, this is done in [Car78, pp 2.29]. For this, it is enough to show that rules and axioms of $U(\mathcal{C})$ are send to rules of $U(\mathcal{D})$. The functoriality of $U : \kappa\text{-CON} \to \kappa\text{-GAT}$ is also immediate from its definition. This is tested on each type and operator symbol. It is then enough to take the equivalence class $[U(F)]$.

B.3.3 The natural isomorphism $U \circ \mathbb{C} \cong Id_{\kappa\text{-}GAT}$

For each $T \in \kappa$-GAT we want to define an interpretation $[\varphi_T] : T \to U(\mathbb{C}_T)$, we do this by defining a preinterpretation $\varphi_T : Exp(T) \to Exp(U(\mathbb{C}_T))$:

1. If Δ is a type symbol of T with introduction rule

$$\{x_\alpha : \Delta_\beta\}_{\beta<\mu} \vdash \Delta(x_\beta)_{\beta<\mu}\ \mathsf{Type}$$

   then
$$\varphi_T(\Delta) := \overline{[\{x_\beta : \Delta_\beta,\, x_\delta : \Delta(x_\beta)_{\beta<\mu}\}_{\beta<\mu}]}(x_\beta)_{\beta<\mu}$$

2. If f is an operator symbol with introductory rule

$$\{x_\alpha : \Delta_\beta\}_{\beta<\mu} \vdash f(x_\beta)_{\beta<\mu} : \Delta$$

   then
$$\varphi_T(f) := \overline{[\langle x_\beta, f(x_\beta)_{\beta<\mu}\rangle_{\beta<\mu}]}(x_\beta)_{\beta<\mu}$$

   where $\langle x_\beta, f(x_\beta)_{\beta<\mu}\rangle_{\beta<\mu}$ is the morphism $\{x_\alpha : \Delta_\beta\}_{\beta<\mu} \to \{x_\alpha : \Delta_\beta, x_\delta : \Delta\}_{\beta<\mu}$.

We proceed to verify that as defined $\varphi_T : T \to U(\mathbb{C}_T)$ is an interpretation. This a crucial point in the proof so we spell out some details in corollary B.26. The results before it are technical steps towards it.

{lemma1231}

Lemma B.18. *If $\mathcal{C}$ is a contextual category, objects A_λ, B_μ and $f : A_\lambda \to B_\mu$ is map with $\mu = \nu + 1$ (in particular it is non-trivial) then the rule*

$$\{x_\alpha : \overline{A}_\alpha(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{f}(x_\alpha)_{\alpha<\lambda} : \overline{B_\mu}\big(\overline{p_\beta \circ f}(x_\alpha)_{\alpha<\lambda}\big)_{\beta<\mu}$$

is a derived rule of $U(\mathcal{C})$.

Proof. We have the axiom

$$\{x_\alpha : \overline{A}_\alpha\}_{\alpha<\lambda} \vdash \overline{f^* B_\mu}(x_\alpha)_{\alpha<\lambda} \equiv \overline{B_\mu}\big(\overline{p_\beta \circ f}(x_\alpha)_{\alpha<\lambda}\big)_{\beta<\mu}$$

for $U(\mathcal{C})$ and the derivation rule for κ-GAT

$$\frac{\Gamma \vdash A_1 \equiv A_2 \qquad t : A_1}{\Gamma \vdash t : A_2}.$$

These put together give us the result. ☐

Lemma B.19. *Let $\mathcal{C}$ a κ-contextual category, objects $\{A_\alpha\}_{\alpha<\lambda}$, $\{B_\beta\}_{\beta<\mu+1}$, $\{C_\gamma\}_{\gamma<\varepsilon}$ and a commutative diagram*

$$\begin{tikzcd} C_\varepsilon \arrow[r, "l"] \arrow[d, "k"'] & B_{\mu+1} \arrow[d, two heads, "p"] \\ A_\lambda \arrow[r, "f"'] & B_\mu . \end{tikzcd}$$

If $h : C_\varepsilon \to f^* B_{\mu+1}$ is the unique map given by the pullback, then the rule

$$\{x_\gamma : \overline{C_\gamma}(x_\delta)_{\delta<\gamma}\}_{\gamma<\varepsilon} \vdash \overline{h}(x_\gamma)_{\gamma<\varepsilon} \equiv_{\overline{(fk)^* B_{\mu+1}}(x_\gamma)_{\gamma<\varepsilon}} \overline{l}(x_\gamma)_{\gamma<\varepsilon}$$

is a derived rule of $U(\mathcal{C})$.

Proof. The proof is the same as [Car78, Lemma 2 pp. 2.32] using lemma B.18. □

{lemma2232}

Lemma B.20. *Let $\mathcal{C}$ a κ-contextual category, objects $\{A_\alpha\}_{\alpha<\lambda}$, $\{B_\beta\}_{\beta<\mu}$, $\{C_\gamma\}_{\gamma<\varepsilon}$ and for $0 < \nu < \mu$ a commutative diagram*

$$\begin{tikzcd} C_\varepsilon \arrow[r, "l_\nu"] \arrow[d, "k_\nu"'] & B_\mu \arrow[d, two heads, "p_\nu"] \\ A_\lambda \arrow[r, "f"'] & B_\nu . \end{tikzcd}$$

If $h_\nu : C_\varepsilon \to f^* B_\mu$ is the unique map given by the pullback, then the rule

$$\{x_\gamma : \overline{C_\gamma}(x_\delta)_{\delta<\gamma}\}_{\gamma<\varepsilon} \vdash \overline{h_\nu}(x_\gamma)_{\gamma<\varepsilon} \equiv_{\overline{(fk_\nu)^* B_\mu}(x_\gamma)_{\gamma<\varepsilon}} \overline{l_\nu}(x_\gamma)_{\gamma<\varepsilon}$$

is a derived rule of $U(\mathcal{C})$.

Proof. This by induction on the height of p_ν. When it is a successor ordinal, this is the previous lemma B.20. When it is a limit ordinal B_μ is a limit object, therefore the result reduces to the inductive hypothesis, which is the successor case again. □

Recall from appendix B.2 we defined the set of maps $\Gamma(B)$. It follows from the previous result that

{corollary3234}

Corollary B.21. *If $\mathcal{C}$ is a κ-contextual category and $f : A_\lambda \to B_\mu$ is a map in $\mathcal{C}$, then for all $\nu < \mu$*

$$\{x_\alpha : A_\alpha(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{\delta_f^\nu}(x_\alpha)_{\alpha<\lambda} \equiv \overline{f}(x_\alpha)_{\alpha<\lambda}.$$

is a derived rule of $U(\mathcal{C})$.

If we specialize corollary B.21 to the syntactic κ-contextual category of a κ-Cartmell theory T, then

{corollary4234}

Corollary B.22. *Assume that* $\{x_\beta : B_\beta\}_{\beta<\mu}$ *is a context,* $\nu < \mu$ *and*

$$f_\nu := [\langle t_\beta \rangle_{\beta<\nu}] : [\{x_\alpha : A_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : B_\beta\}_{\beta<\nu}]$$

a map in $\mathbb{C}_T$ *then*

$$\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{[\langle x_\alpha, t_\varepsilon \rangle_{\substack{\alpha<\lambda \\ \nu\leq\varepsilon<\mu}}]} \equiv \overline{[\langle t_\beta, t_\varepsilon \rangle_{\beta<\nu\leq\varepsilon<\mu}]}.$$

is a derived rule of $U(\mathbb{C}_T)$*.*

Proof. This follows from corollary B.21 and the explicit description of $\delta^\nu_{f_\nu}$ given in lemma B.9. $\square$

{lemma5234}

Lemma B.23. *If* A_λ, B_μ *are objects and* $f_\nu : A_\lambda \to B_\nu$*, with* $\nu < \mu$*, is a map in a* κ*-contextual category* $\mathcal{C}$*, then:*

1. *The rule*

$$\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{f^*_\nu B_\mu}(x_\alpha)_{\alpha<\lambda} \equiv \overline{B}(\delta^\gamma_{(p_\gamma f)}(x_\alpha)_{\alpha<\lambda})_{\gamma<\nu}$$

   is a derived rule of $U(\mathcal{C})$*.*

2. *If* $g : \Gamma(B^\mu_\nu)$ *then the rule*

$$\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{\delta^\nu_{gf}}(x_\alpha)_{\alpha<\lambda} \equiv \overline{\delta^\nu_g}(\overline{\delta^\gamma_{p_\gamma f}}(x_\alpha)_{\alpha<\lambda})_{\gamma<\nu}$$

   is a derived rule of $U(\mathcal{C})$*.*

{corollary6235}

Corollary B.24. *If* T *is a* κ*-Cartmell theory,* $\{x_\beta : B_\beta\}_{\beta<\mu}$ *is a context,* $\nu < \mu$ *and*

$$f_\nu := [\langle t_\beta \rangle_{\beta<\nu}] : [\{x_\alpha : A_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : B_\beta\}_{\beta<\nu}]$$

is a map in $\mathbb{C}_T$ *then;*

1.

$$\frac{\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda}}{\overline{[\{x_\alpha, x_\gamma : B_\gamma[t_\delta|x_\delta]_{\delta<\gamma}\}_{\substack{\alpha<\lambda \\ \nu\leq\gamma<\mu}}]}(x_\alpha)_{\alpha<\lambda} \equiv \overline{[\{x_\beta : B_\beta\}_{\beta<\nu}]}(\overline{g_\beta}(x_\alpha)_{\alpha<\lambda})_{\beta<\nu}}$$

   where for each $\beta < \nu$ *the map* $g_\beta := [\langle x_\alpha, t_\beta \rangle_{\alpha<\lambda}]$*.*

2. *If for all γ, with $\nu < \gamma < \mu$, the rule*

$$\{x_\beta : B_\beta\}_{\beta<\nu}, \{t_{\gamma'} : B_{\gamma'}\}_{\gamma'<\gamma} \vdash t_\gamma : B_\gamma$$

is a derived rule then

$$\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{[\langle x_\alpha, t_\gamma[t_{\gamma'} \mid x_{\gamma'}]_{\gamma'<\gamma}\rangle_{\substack{\alpha<\lambda \\ \nu<\gamma<\mu}}]} \equiv \overline{h}(\overline{g_\beta}(x_\alpha)_{\alpha<\lambda})_{\beta<\nu}$$

where g_β is defined as in the previous point and $h := [\langle x_\beta, t_\gamma\rangle_{\substack{\beta<\nu \\ \nu<\gamma<\mu}}]$.

Proof. This is a direct application of lemma B.23. We remark that the assumption of point (2) simply give us an element of $\Gamma(B_\nu^\mu)$ and the map on the left depend on variables that according to our convention we leave implicit. ☐

The following lemma is key to prove that we have an interpretation $\varphi_T : T \to U(\mathbb{C}_T)$, the results above are used to prove:

{lemma7236}

Lemma B.25. *If T is a κ-Cartmell theory then:*

1. *If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta$ Type is a type judgment of T, then the rule*

$$\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{A}(x_\alpha)_{\alpha<\lambda+1} \equiv \widetilde{\varphi_T}(\Delta)$$

is a derived rule of $U(\mathbb{C}_T)$ where $A := \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+1}$ and $A_\alpha := \{x_\delta : \Delta_\delta\}_{\delta\leq\alpha}$.

2. *If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta$ is a type element judgment of T, then the rule*

$$\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{\langle x_\alpha, t\rangle_{\alpha<\lambda}}(x_\alpha)_{\alpha<\lambda+1} \equiv_{\overline{A}(x_\alpha)_{\alpha<\lambda}} \widetilde{\varphi_T}(t)$$

is a derived rule of $U(\mathbb{C}_T)$.

Proof. The proof is by induction on the derivations, by showing that rule derivation preserves the properties above. ☐

The important result of this section is the following.

{isinterpreation}

Corollary B.26. *For every κ-Cartmell theory T, the map $\varphi_T : U \to U(\mathbb{C}_T)$ is an interpretation.*

Proof. We see that the function $\widehat{\varphi_T} : Rul(T) \to Rul(U(\mathbb{C}_T))$ is well defined. We start with a rule $\mathcal{J}$ of T and show that $\widehat{\varphi_T}(\mathcal{J})$ is a rule of $U(\mathbb{C}_T)$

1. Type judgment: Assume that $\mathcal{J} := \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta\ \mathsf{Type}$ is a rule of T, from definition A.24 it follows that

$$\widehat{\varphi_T}(\mathcal{J}) = \{x_\alpha : \widetilde{\varphi}(\Delta_\alpha)\}_{\alpha<\lambda} \vdash \widetilde{\varphi_T}(\Delta)\ \mathsf{Type}.$$

From lemma B.25 we have for any $\gamma < \lambda + 1$ the rule

$$\{x_\alpha : \overline{\Delta_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{A_{\gamma+1}}(x_\alpha)_{\alpha<\gamma+1} \equiv \widetilde{\varphi_T}(\Delta_\gamma)$$

is a derived rule of $U(\mathbb{C}_T)$. Thus, so it is

$$\{x_\alpha : \widetilde{\varphi_T}(\Delta_\alpha)(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{A_{\gamma+1}}(x_\alpha)_{\alpha<\lambda+1} \equiv \widetilde{\varphi_T}(\Delta).$$

Then it must be the case that $\{x_\alpha : \widetilde{\varphi}(\Delta_\alpha)\}_{\alpha<\lambda} \vdash \widetilde{\varphi_T}(\Delta)\ \mathsf{Type}$ is a rule of $U(\mathbb{C}_T)$.

2. Element judgment: $\Gamma \vdash t : \Delta$. This very similar the previous rule.

3. Type equality judgment: $\Gamma \vdash \Delta \equiv \Delta'$. Also follows from lemma B.25.

4. Term equality judgment: $\Gamma \vdash t \equiv_\Delta t'$. The same argument works.

□

Corollary B.27. *For every κ-Cartmell theory T, the map $[\varphi_T] : U \to U(\mathbb{C}_T)$ is morphism in the category κ-GAT.*

Next, we will show that $[\varphi_-] : Id_{\kappa\text{-GAT}} \Rightarrow U \circ \mathbb{C}$ is a natural transformation.

Lemma B.28. *Let T, T' two κ-Cartmell theories and $I : T \to T'$ an interpretation between them. Then, we have a commutative diagram*

$$\begin{array}{ccc} T & \xrightarrow{[\varphi_T]} & U(\mathbb{C}_T) \\ {\scriptstyle [I]}\downarrow & & \downarrow{\scriptstyle U(\mathbb{C}(I))} \\ T' & \xrightarrow[{[\varphi_{T'}]}]{} & U(\mathbb{C}_{T'}). \end{array}$$

Proof. We use corollary A.29. Therefore, it will be enough to test the commutativity of the diagram on type element judgments. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta_\lambda$ a type element judgment of T. For any $\alpha \leq \lambda$ we denote $A_\alpha := [\{x_\delta : \Delta_\delta\}_{\delta\leq\alpha}]$. It follows from lemma B.25 that

$$\widehat{\varphi_T}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right) \approx \frac{\{x_\alpha : \overline{A_\alpha}\}_{\alpha<\lambda}}{\overline{[\langle x_\alpha, t\rangle_{\alpha<\lambda}]} : \overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}}.$$

We conclude that

$$U(\mathbb{C}(I))\left(\widehat{\varphi_T}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right)\right) \approx \frac{\{x_\alpha : \overline{\mathbb{C}(I)(A_\alpha)}\}_{\alpha<\lambda}}{\overline{\mathbb{C}(I)([\langle x_\alpha, t\rangle_{\alpha<\lambda}])} : \overline{\mathbb{C}(I)(A_\lambda)}(x_\alpha)_{\alpha<\lambda}}.$$

Looking at the other composition: we get

$$\widehat{I}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right) = \frac{\{x_\alpha : \widetilde{I}(\Delta_\alpha)\}_{\alpha<\lambda}}{\widetilde{I}(t) : \widetilde{I}(\Delta_\lambda)}.$$

A second use of lemma B.25 give us that

$$\widehat{\varphi_{T'}}\left(\widehat{I}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right)\right) \approx \frac{\{x_\alpha : \overline{B_\alpha}\}_{\alpha<\lambda}}{\overline{[\langle x_\alpha, \widetilde{I}(t)\rangle_{\alpha<\lambda}]} : \overline{B_\lambda}(x_\alpha)_{\alpha<\lambda}}$$

where for $\alpha \leq \lambda$, $B_\alpha := [\{x_\delta : \widetilde{I}(\Delta_\delta)\}_{\delta\leq\alpha}]$. However, by definition we have $\mathbb{C}(I)(A_\alpha) = B_\alpha$ for $\alpha \leq \lambda$. This completes our verification. □

Remains to show that $[\varphi_T]$ is an isomorphism and natural natural in T. We proceed to give an inverse $\psi_T : U(\mathbb{C}_T) \to T$. Recall that a type symbol of $U(\mathbb{C}_T)$ is of the form $\overline{A_\lambda} = \overline{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]}$. If $\lambda = \nu + 1$ then by choosing a representative of this equivalence class of the context we can define $\psi_T(\overline{A_\lambda}) := \Delta_\nu$.
If λ is a limit ordinal once we chose a representative $\Delta_\lambda = \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$. Then we know that $[\Delta_\lambda] = \lim_{\alpha<\lambda}[\Delta_\alpha]$ in $\mathbb{C}_T$, and this limit is unique. In this case the value of ψ_T is determined by non-limit ordinals $\alpha < \lambda$, which are $\psi_T(\overline{\Delta_\alpha}) = \Delta_\alpha$. Therefore we define $\psi_T(\overline{[\Delta_\lambda]}) := \Delta_\lambda$ for some choice of a representative of the equivalence class. However, note that the successor case determinate the limit case.
Operator symbols of $U(\mathbb{C}_T)$ come from morphisms of $\mathbb{C}_T$. Therefore, for a morphism $\overline{f} := [\langle t_\beta\rangle_{\beta<\mu}] : [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : \Omega_\beta\}_{\beta<\mu}]$ in order to define ψ_T on the associated operator, it is enough to assume that μ is a successor ordinal. First of all, we need to make choices for the contexts and morphism. However, the definition does not depend on this choices because of (1) from remark A.22. This allows to define ψ_T as

$$\psi_T(\overline{f}) := t_\mu$$

where $t_\mu : \Omega_\mu[t_\beta|x_\beta]_{\beta<\mu}$.

{lemma10244}

Lemma B.29. *The function ψ_T is an interpretation from $U(\mathbb{C}_T) \to T$.*

Proof. We need to check that rules and axioms are preserved by ψ_T. It will be enough to deal with the case where $\lambda = \nu + 1$. Suppose that $\overline{A_\lambda}$ has

$$\frac{\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\nu}}{\overline{A_\nu}(x_\alpha)_{\alpha<\nu}\ \mathsf{Type}}$$

Furthermore, we assume that $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is such that $A_\lambda = [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$. By definition

$$\widehat{\psi_T}\left(\frac{\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\nu}}{\overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}\ \mathsf{Type}}\right) = \frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\nu}}{\Delta_\nu\ \mathsf{Type}}.$$

This is obviously a derived rule of T. Preservation of the rule for operator symbols are is straightforward too. $\square$

Lemma B.30. *For any κ-Cartmell theory T we have $\psi_T \circ \varphi_T \approx Id_T$.*

Proof. From corollary A.29 it is enough to verify the statement on type element judgments. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta_\lambda$ a type element judgment. For any $\alpha \leq \lambda$ we denote $A_\alpha := [\{x_\delta : \Delta_\delta\}_{\delta\leq\alpha}]$. It follows from lemma B.25 that

$$\widehat{\varphi_T}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right) \approx \frac{\{x_\alpha : \overline{A_\alpha}\}_{\alpha<\lambda}}{[\langle x_\alpha, t\rangle_{\alpha<\lambda}] : \overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}}.$$

Hence

$$\widehat{\psi_T}\left(\widehat{\varphi_T}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right)\right) \approx \widehat{\psi_T}\left(\frac{\{x_\alpha : \overline{A_\alpha}\}_{\alpha<\lambda}}{[\langle x_\alpha, t\rangle_{\alpha<\lambda}] : \overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}}\right) = \frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}.$$

$\square$

Lemma B.31. *For any κ-Cartmell theory T we have $\psi_T \circ \varphi_T \approx Id_T$.*

Proof. The proof is similar to the previous lemma. All the definitions and technical results have been established, specially lemma B.25. $\square$

Corollary B.32. *There is a natural isomorphism $Id_{\kappa\text{-}GAT} \Rightarrow U \circ \mathbb{C}$.*

Proof. We have constructed $[\varphi_-] : Id_{\kappa\text{-GAT}} \Rightarrow U \circ \mathbb{C}$. $\square$

B.3.4 The natural isomorphism $\mathbb{C} \circ U \cong Id_{\kappa\text{-}CON}$

In this section we aim to construct a natural isomorphism $\eta : Id_{\kappa\text{-CON}} \Rightarrow \mathbb{C} \circ U$. Let $\mathcal{C}$ be a κ-contextual category. For this, we first construct a κ-contextual functor $\eta_{\mathcal{C}} : \mathcal{C} \to \mathbb{C}_{U(\mathcal{C})}$. Recall that if A_λ is an object in $\mathcal{C}$ then for any $\alpha \leq \lambda$ we denoted $p_\alpha : A_\lambda \twoheadrightarrow A_\alpha$ to the canonical display map that exists. Then we can make the following definition:

1. For $\eta_{\mathcal{C}}(1) := 1$.
2. If A_μ is an object with $\mu = \lambda + 1$ then
$$\eta_{\mathcal{C}}(A_\mu) := [\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha\leq\mu}].$$
3. For an object A_λ we define $\eta_{\mathcal{C}}(p_0) := \eta_{\mathcal{C}}(p)_0$ where $\eta_{\mathcal{C}}(p)_0 : \eta_{\mathcal{C}}(A) \twoheadrightarrow 1$.
4. If A_λ, B_μ are non-trivial objects, with μ a successor ordinal, and $f : A_\lambda \to B_\mu$ is a morphism in $\mathcal{C}$ then
$$\eta_{\mathcal{C}}(f) := [\langle \overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\rangle_{\beta\leq\mu}].$$

Again we observe that if μ is a limit ordinal then any map $f : A_\lambda \to B_\mu$ is determined by a family of maps $\{f_\nu : A_\lambda \to B_\nu\}_{\nu<\mu}$. Thus, in order to define η on such map it is enough to do it on ordinals $\nu < \mu$ which we can assume to be successor ordinals. The map $\eta(f)$ is the map induced by the family of maps $\{\eta(f_\nu) : \eta(A_\lambda) \to \eta(B_\nu)\}_{\nu<\mu}$. In conclusion, we simply need to prove properties of η for successor ordinals. The property for limit ordinals follows using the universal property of the limit object.

{etaisfunctor}

Lemma B.33. *For any $\mathcal{C}$, $\eta_{\mathcal{C}} : \mathcal{C} \to \mathbb{C}_{U(\mathcal{C})}$ is a κ-contextual functor.*

Proof. First we verify that it is a functor. Since for any $\alpha < \lambda$ we have $\overline{p_\alpha}(x_\alpha)_{\alpha<\lambda} = x_\alpha$, then it is immediate to see that $\eta_{\mathcal{C}}$ preserves the identities. Assume we have non-trivial morphisms $f : A_\lambda \to B_\mu$ and $g : B_\mu \to C_\nu$ then

$$\eta_{\mathcal{C}}(gf) = [\langle \overline{p_\gamma g f}(x_\alpha)_{\alpha<\lambda}\rangle_{\beta\leq\nu}]$$

From the first axiom in definition B.15 $U(\mathcal{C})$ it follows that the above must be $\eta_{\mathcal{C}}(g)\eta_{\mathcal{C}}(f)$ whenever μ and ν are successor ordinals. When we have limits Now we must verify that it preserves display maps and canonical pullbacks. Both statements are direct consequences from the definitions. Furthermore,

the proof from [Car78] works without mayor changes.
For the preservation of pullbacks: We let $f : A_\lambda \to B_{\mu+1}$ then

$$\begin{aligned}
\eta_{\mathcal{C}}(f^*B) &= [\langle x_\alpha : \overline{A_\delta}(x_\gamma)_{\gamma<\alpha},\, x_\epsilon : \overline{f^*B_{\mu+1}}(x_\alpha)_{\alpha<\lambda}\rangle_{\alpha<\lambda}] \\
&= [\langle x_\alpha : \overline{A_\delta}(x_\gamma)_{\gamma<\alpha},\, x_\epsilon : \overline{B_{\mu+1}}\big(\overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\big)_{\beta<\mu}\rangle_{\alpha<\lambda}] \\
&= [\langle\overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\rangle_{\beta\le\mu}]^*[\langle x_\beta : \overline{B_\beta}(x_\gamma)_{\gamma<\beta}\rangle_{\beta\le\mu}] \\
&= \eta_{\mathcal{C}}(f)^*\eta_{\mathcal{C}}(B).
\end{aligned}$$

For a display map of $p_\nu : B_\mu \twoheadrightarrow B_\nu$ with successor ordinal as height the same argument shows that the pullback along $f_\nu : A_\lambda \to B_\nu$ is preserved. When the height is a limit ordinal we combine the previous case and the fact that in any κ-contextual category canonical pullbacks are unique. $\square$

Lemma B.34. *Let $\mathcal{C}$, $\mathcal{C}'$ be κ-contextual categories and a contextual functor $F : \mathcal{C} \to \mathcal{C}'$. Then the following diagram is commutative:*

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\eta_{\mathcal{C}}} & \mathbb{C}_{U(\mathcal{C})} \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle \mathbb{C}(U(F))} \\
\mathcal{C}' & \xrightarrow[\eta_{\mathcal{C}'}]{} & \mathbb{C}_{U(\mathcal{C}')}.
\end{array}$$

Proof. If $f : A_\lambda \to B_\mu$ is a map in $\mathcal{C}$ then

$$\begin{aligned}
\mathbb{C}(U(F))(\eta_{\mathcal{C}}(f)) &= \mathbb{C}(U(F))([\langle\overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\rangle_{\beta\le\mu}]) \\
&= [\langle\overline{F(p_\beta f)}(x_\alpha)_{\alpha<\lambda}\rangle_{\beta\le\mu}] \\
&= [\langle\overline{p_\beta F(f)}(x_\alpha)_{\alpha<\lambda}\rangle_{\beta\le\mu}] \\
&= \eta_{\mathcal{C}'}(f).
\end{aligned}$$

$\square$

Corollary B.35. *There is a natural transformation $Id_{\kappa\text{-}CON} \Rightarrow \mathbb{C} \circ U$.*

Remains to show that this natural transformation is an isomorphism. For each κ-contextual category $\mathcal{C}$ we construct a κ-contextual functor

$$\xi_{\mathcal{C}} : \mathbb{C}_{U(\mathcal{C})} \to \mathcal{C}$$

which is a two-sided inverse to $\eta_{\mathcal{C}}$. From lemma A.13 we see that:

1. Every derived type judgment of $U(\mathcal{C})$ is of the form

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \overline{A_\lambda}(t_\alpha)_{\alpha<\lambda}\ \mathsf{Type}$$

for some object A_λ of $\mathcal{C}$ where for $\alpha \leq \lambda$ the rule

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\alpha : \overline{A_\alpha}[t_\delta \mid x_\delta]_{\delta<\alpha}$$

is a derived rule of $U(\mathcal{C})$.

2. Every type element judgment of T is of the form

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash x_\beta : \Omega_\beta$$

for some $\beta < \mu$, or is of the form

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \overline{f}(t_\alpha)_{\alpha<\lambda} : \Omega$$

for some map $f : A_\lambda \to B_\mu$ of $\mathcal{C}$ such that for each $\alpha < \lambda$ the rules

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\alpha : \overline{A_\alpha}[t_\delta \mid x_\delta]_{\delta<\alpha}$$

and

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \overline{B_\mu}(t_\beta)_{\beta<\mu} \equiv \Omega$$

are derived rules of $U(\mathcal{C})$.

We may assume that $\mu = \nu + 1$, the limit case will follow induction. Let $\mathcal{R}_\mathcal{C}$ be the set of type and element type judgments of $U(\mathcal{C})$. Next, we define $\mathcal{J} : \mathcal{R}_\mathcal{C} \to \mathcal{C}$ inductively. First we get maps:

1. A rule $r_{\Omega_\mu} := \{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega_\mu$ is sent an object $\mathcal{J}(r_{\Omega_\mu}) \in \mathcal{C}$.
2. For any $\alpha < \lambda$ the judgment $r_{t_\alpha} := \{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\alpha : \overline{A_\alpha}[t_\delta|x_\delta]_{\delta<\alpha}$ is sent to a map $\mathcal{J}(r_{t_\alpha})$.

The we can make the following definitions:

1. $\mathcal{J}(r_{A_\mu}) := (\mathcal{J}(t_\alpha)_{\alpha<\lambda})^* A_\mu$.
   Where $\mathcal{J}(t_\alpha)_{\alpha<\lambda}$ denotes the pullbacks as in lemma B.11.
2. $\mathcal{J}(\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \overline{f}(t_\alpha)_{\alpha<\lambda} : \Omega) := (\mathcal{J}(t_\alpha)_{\alpha<\lambda})^* \delta^\nu_f$.
3. $\mathcal{J}(\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash x_\beta : \Omega) := \delta^\beta_{p_\beta}$ where $p_\beta : \mathcal{J}(r_{\Omega_\mu}) \to \mathcal{J}(r_{\Omega_\beta})$.

The burden of the proof falls into showing that the function $\mathcal{J}$ is well-defined. The proof is by induction on the derived rules of $U(\mathcal{C})$. We will focus on writing down the inductive hypothesis H as in [Car78] for this induction.

- For rules r_{Ω_μ} of the form $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega_\mu \,\mathsf{Type}$ then $H(r_{\Omega_\mu})$ is either:

  1. If the premise of r_{Ω_μ} is a non-empty context then $H(r_{\Omega_\beta})$ for all $\beta < \mu$.
  2. If r_{Ω_μ} is the rule $\vdash \Delta \,\mathsf{Type}$ then $ht(\mathcal{J}(r_{\Omega_\mu})) = 1$. Otherwise for all $\beta < \mu$ we have $ht(\mathcal{J}(r_{\Omega_\beta})) < ht(\mathcal{J}(r_{\Omega_\mu}))$.
  3. For a map $\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$. If for each $\beta + 1 < \mu$ we have $\mathcal{J}(r_{t_{\beta+1}}) \in \Gamma(\mathcal{J}(r_{\Omega_{\beta+1}[t_\gamma|x_\gamma]_{\gamma\leq\beta}}))$ where $r_{\Omega_{\beta+1}[t_\gamma|x_\gamma]_{\gamma\leq\beta}}$ is the rule $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Omega_{\beta+1}[t_\gamma|x_\gamma]_{\gamma\leq\beta} \,\mathsf{Type}$ then
  $$\mathcal{J}(r_{\Omega_\mu[t_\beta|x_\beta]_{\beta<\mu}}) = (\mathcal{J}(t_\beta)_{\beta<\mu})^* \mathcal{J}(r_{\Omega_\mu})$$

- For rules r_{t_μ} of the form $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\mu : \Omega_\mu$ then $H(r_{t_\mu})$ is either:

  1. $H(r_{\Omega_\mu})$.
  2. $\mathcal{J}(r_{t_\mu}) \in \Gamma(\mathcal{J}(r_{\Omega_\mu}))$.
  3. For a map $\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$. If for each $\beta + 1 < \mu$ we have $\mathcal{J}(r_{t_{\beta+1}}) \in \Gamma(\mathcal{J}(r_{\Omega_{\beta+1}[t_\gamma|x_\gamma]_{\gamma\leq\beta}}))$ then
  $$\mathcal{J}(r_{t_\mu[t_\beta|x_\beta]_{\beta<\mu}}) = (\mathcal{J}(t_\beta)_{\beta<\mu})^* \mathcal{J}(r_{t_\mu})$$
  where $r_{t_\mu[t_\beta|x_\beta]_{\beta<\mu}}$ is the rule $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t_\mu[t_\beta|x_\beta]_{\beta<\mu} : \Omega_\mu[t_\beta|x_\beta]_{\beta<\mu}$.

- For rules $r_\equiv$ or the form $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \equiv \Delta'$ the hypothesis $H(r_\equiv)$ is either:

  1. $H(r_{\Delta'})$ and $\mathcal{J}(r_\Delta) = \mathcal{J}(r_{\Delta'})$.
  2. $H(r_\Delta)$ and $\mathcal{J}(r_\Delta) = \mathcal{J}(r_{\Delta'})$.

- For rules r_ϵ or the form $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t'$ the hypothesis $H(r_\epsilon)$ is either:

  1. $H(r_t)$ and $\mathcal{J}(r_t) = \mathcal{J}(r_{t'})$.
  2. $H(r_{t'})$ and $\mathcal{J}(r_t) = \mathcal{J}(r_{t'})$.

{lemma11256}

Lemma B.36. *Let $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega$ a rule such that H is satisfied. If $\langle t_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ is a map such that $H(r_{t_\beta})$ for all $\beta < \mu$ then $H(\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega[t_\beta|x_\beta]_{\beta<\mu})$*

Proof. By induction on μ and treating all different cases for H. The proof in [Car78, Lemma 11 pp.2.56] works here too. □

{lemma12263}

Lemma B.37. *1. For any object $A_\lambda \in \mathcal{C}$, we have:*

(a) $A\lambda = \mathcal{J}(\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}\ \mathsf{Type})$.

(b) For all $\alpha < \lambda$, $\delta_{p^\lambda_\alpha} = \mathcal{J}(\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha})$ where $p^\lambda_\alpha : A_\lambda \twoheadrightarrow A_\alpha$.

2. For any non-trivial object A_λ and $f : A_\lambda \to B_{\mu+1}$, $\delta_f = \mathcal{J}(\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{f}(x_\alpha)_{\alpha<\lambda}\overline{(p_\mu f)^*B}(x_\alpha)_{\alpha<\lambda})$ where $p_\mu : B_{\mu+1} \twoheadrightarrow B_\mu$.

Proof. This is [Car78, Lemma 12 pp.263]. □

{lemma265}

Lemma B.38. *Every derived rule of $U(\mathcal{C})$ satisfies the hypothesis H*

Proof. This is by induction on derived rules of $U(\mathcal{C})$. Indeed, [Car78, Lemma pp.2.65] shows that every derivation from definition A.4 preserves H. □

Corollary B.39. *1. For any type symbol $\overline{A_\lambda}$ of the theory $U(\mathcal{C})$ we have $H(\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}\ \mathsf{Type})$.*

2. For every operator symbol $\overline{f}$ in $U(\mathcal{C})$ where $f : A_\lambda \to B_{\mu+1}$ we have $H(\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{f}(x_\alpha)_{\alpha<\lambda}\overline{(p_\mu f)^*B}(x_\alpha)_{\alpha<\lambda})$.

The foremost important result which summarizes the above is:

{corollary14272}

Corollary B.40. *1. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a context of the theory then for any $\alpha < \delta < \lambda$ we have $ht(r_{\Delta_\alpha}) < ht(r_{\Delta_\beta})$.*

2. If there is a map $\langle t_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ then for each $\beta < \mu$ we have $\mathcal{J}(r_{t_\beta}) \in \Gamma(\mathcal{J}(r_{\Omega_\beta[t_\gamma|x_\gamma]_{\gamma<\beta}}))$ where $r_{\Omega_\beta[t_\gamma|x_\gamma]_{\gamma<\beta}}$ is the rule $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Omega_\beta[t_\gamma|x_\gamma]_{\gamma<\beta}\ \mathsf{Type}$.

3. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \equiv \{x_\alpha : \Delta'_\alpha\}_{\alpha<\lambda}$ then $\mathcal{J}(r_{\Delta_\lambda}) = \mathcal{J}(r_{\Delta'_\lambda})$.

4. If $\langle t_\alpha\rangle_{\alpha<\lambda} \equiv \langle t'_\alpha\rangle_{\alpha<\lambda}$ then for each $\beta < \mu$, $\mathcal{J}(r_{t_\beta}) = \mathcal{J}(r_{t'_\beta})$.

We are almost ready to define a contextual functor $\xi_\mathcal{C} : \mathcal{C}_{U(\mathcal{C})} \to \mathcal{C}$. We only need the next:

{definitionfunctorepsi

Observation B.41. Let a map $\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ then there are maps $\{g_\beta : \mathcal{J}(r_{\Delta_\lambda}) \to \mathcal{J}(r_{\Omega_\beta})\}_{\beta<\mu}$ with $\delta_{g_\beta} = \mathcal{J}(r_{t_beta})$ and $pg_{\beta+1} = g_\beta$. This is a consequence of corollary B.40 and lemma B.11. Therefore, there exists a unique $g : \mathcal{J}(r_{\Delta_\lambda}) \to \mathcal{J}(r_{\Omega_\mu})$ such that for all $\beta < \mu$ we have $\delta_{pg} = \mathcal{J}(r_{t_\beta})$ where $p : \mathcal{J}(r_{\Delta_\lambda}) \to \mathcal{J}(r_{\Omega_\beta})$.

Definition B.42. We define a function

$$\xi_{\mathcal{C}} : \mathcal{C}_{U(\mathcal{C})} \to \mathcal{C}$$

by:

1. For an object $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \in \mathcal{C}_{U(\mathcal{C})}$,

$$\xi([\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]) := \mathcal{J}(r_{\Delta_\lambda}).$$

2. For an morphism $[\langle t_\beta \rangle_{\beta<\mu}] : [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : \Omega_\beta\}_{\beta<\mu}]$

$$\xi([\langle t_\beta \rangle_{\beta<\mu}]) := g$$

where $g : \mathcal{J}(r_{\Delta_\lambda}) \to \mathcal{J}(r_{\Omega_\mu})$ is the unique map from observation B.41.

{lemma15274}

Lemma B.43. *1. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta_\lambda$ Type is a derived rule of $U(\mathcal{C})$ then for all $\alpha \leq \lambda$, $\{x_\gamma : \Delta_\gamma\}_{\gamma<\lambda} \vdash \Delta_\alpha \equiv \mathcal{J}(r_{\Delta_\alpha})(x_\gamma)_{\gamma<\alpha}$ is a derived rule of $U(\mathcal{C})$.*

2. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t_\lambda : \Delta_\lambda$ is a derived rule of $U(\mathcal{C})$ then $\{x_\gamma : \Delta_\gamma\}_{\gamma<\lambda} \vdash t \equiv \mathcal{J}(r_{t_\lambda})(x_\alpha)_{\alpha<\lambda}$ is a derived rule of $U(\mathcal{C})$.

Proof. See [Car78, Lemma 15 pp. 2.74]. □

Corollary B.44. *As functions we have $\eta_{\mathcal{C}}\xi_{\mathcal{C}} = id_{\mathcal{C}_{U(\mathcal{C})}}$ and $\xi_{\mathcal{C}}\eta_{\mathcal{C}} = Id_{\mathcal{C}}$*

The results needed for this have been introduced throughout the section. Using that we have a bijection and that $\eta_{\mathcal{C}}$ is already a functor it follows:

Corollary B.45. *The function $\xi_{\mathcal{C}} : \mathcal{C}_{U(\mathcal{C})} \to \mathcal{C}$ is a contextual functor.*

The main result that is of our interest is:

{synactic-forgetful-ide

Theorem B.46. *There is a natural isomorphism $\mathcal{C}_{\_} \circ U \cong Id_{\kappa\text{-}CON}$.*

Finally,

Corollary B.47. *The categories κ-CON of κ-contextual categories and κ-GAT of κ-algebraic theories are equivalent.*

B.4 Coclans and contextual categories

{appendix-c}

In this section we use prove that every κ-contextual category can be obtained by strictification of a κ-clan. Clans were introduced in [Joy17], a related definition appears in [Hen20] under the name category with fibrations.

Definition B.48. We say that a category $\mathcal{C}$ is a κ-*coclan* if it has a a collection of maps $\text{Cof}(\mathcal{C})$ satisfying the following conditions:

1. $\mathcal{C}$ has initial object 0.
2. For any $X \in \mathcal{C}$, the map $0 \to X \in \text{Cof}(\mathcal{C})$.
3. Any isomorphism is an element of $\text{Cof}(\mathcal{C})$.
4. $\text{Cof}(\mathcal{C})$ is closed under compositions.
5. $\text{Cof}(\mathcal{C})$ is closed under pushouts: If $f : A \to C$ is a morphism in $\mathcal{C}$ and $A \to B \in \text{Cof}(\mathcal{C})$ then the map $C \to C \coprod_A B \in \text{Cof}(\mathcal{C})$.
6. $\text{Cof}(\mathcal{C})$ is closed under transfinite compositions: for any $\lambda < \kappa$ and any λ-diagram of maps in $\text{Cof}(\mathcal{C})$

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

$\mathtt{Colim}_\lambda A_\alpha$ exists and the map $A_0 \to \mathtt{Colim}_\lambda A_\alpha$ belongs to $\text{Cof}(\mathcal{C})$.

As is usual, maps in $\text{Cof}(\mathcal{C})$ are called *cofibrations* and they are indicated by arrows " $\rightarrowtail$ ".

Dually, a category $\mathcal{C}$ is κ-*clan* if $\mathcal{C}^{op}$ is a κ-coclan. The distinguished maps are called *fibrations* and they are denoted by $\text{Fib}(\mathcal{C})$. The fibrations are indicated by arrows " $\twoheadrightarrow$ ". When working with κ-clans we keep the terminology "transfinite compositions" from κ-coclans as there is no risk of confusion.

{contextual:clan}

Observation B.49. The κ-contextual category $\mathbb{C}_T$ associated to a κ-generalized algebraic theory T has a natural κ-clan structure. Indeed, we can take $\text{Fib}(\mathbb{C}_T)$ as the display maps. All the axioms are easily verified. Moreover, this is true for any κ-contextual category not only for $\mathbb{C}_T$.

Recall that a *comprehension category* consists of a category $\mathcal{C}$, a fibration $p : \mathcal{E} \to \mathcal{C}$ and a functor $F : \mathcal{E} \to \mathcal{C}^{\to}$ such that:

1. $\partial_0 F = p$.

2. If f is a cartesian arrow in $\mathcal{E}$ then Ff is a pullback in $\mathcal{C}$, equivalently Ff is a cartesian arrow with respect to the codomain functor $\partial_0 : \mathcal{C}^{\to} \to \mathcal{C}$.

The fibration p is *cloven* if it comes with a choice of cartesian lifts. The comprehension category is said to be *split* is p is a split fibration. We also say that is *full* if F is fully faithful, the notation $(\mathcal{C}, \mathcal{E}, p, F)$

The following example appears in [Jac93, Example 4.5], we rewrite it in our setting of κ-clans. Let us fix a κ-clan $\mathcal{C}$, then the inclusion functor $\iota : \textsc{Fib}(\mathcal{C}) \hookrightarrow \mathcal{C}^{\to}$ and $P = \partial_0 \iota$ form a full comprehension category. More precisely: $\textsc{Fib}(\mathcal{C})$ has objects fibrations in $\mathcal{C}$ and arrows between two fibrations $\alpha : f \to g$ are commutative squares of the form

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, two heads, "f"'] & B \arrow[d, two heads, "g"] \\ \Delta \arrow[r, "l"'] & \Gamma. \end{tikzcd}$$

Whence an object in $\textsc{Fib}(\mathcal{C})_\Gamma$ over $\Gamma \in \mathcal{C}$ is a fibration $A \twoheadrightarrow \Gamma$. Observe that an arrow $\alpha : f \to g$ as above is cartesian if and only if it is a pullback square in $\mathcal{C}$. In conclusion, for an arrow $l : \Delta \to \Gamma$ and $B \twoheadrightarrow \Gamma \in \textsc{Fib}(\mathcal{C})_\Gamma$, a cartesian lift in $\textsc{Fib}(\mathcal{C})$ is a pullback square

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, two heads, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, two heads, "g"] \\ \Delta \arrow[r, "l"'] & \Gamma. \end{tikzcd}$$

This comprehension category is not necessarily split, reflecting the fact that taking pullbacks is not strictly functorial. Nevertheless, we can replace it by a split one via the functor

$$(-)_! : \mathbf{CompCat}(\mathcal{C}) \to \mathbf{SplCompCat}(\mathcal{C})$$

from the category of comprehension categories over $\mathcal{C}$ to the category of split comprehension categories over $\mathcal{C}$, the description of this functor appears in [LW15, 3.1] which we now recall. This produces a split comprehension category $(\mathcal{C}_!, \textsc{Fib}(\mathcal{C})_!, p_!, F_!)$ which is equivalent to the one we started with. Unfolding the result, we take the $\mathcal{C}_!$ to be simply $\mathcal{C}$.

The category $\textsc{Fib}(\mathcal{C})_!$ has:

- Objects: for each $\Gamma \in \mathcal{C}$ is a tuple $A \coloneqq (V_A, E_A, f_A)$ where $V_A \in \mathcal{C}$, $E_A \twoheadrightarrow V_A \in \textsc{Fib}(\mathcal{C})_{V_A}$ and $f_A : \Gamma \to V_A \in \mathcal{C}$. We also employ the notation $[A] \coloneqq f_A^* E_A$ given by taking the pullback of $E_A \twoheadrightarrow V_A$ along f_A, so we get a fibration $[A] \twoheadrightarrow \Gamma$. In addition, we write $(E_A)_{f_A}$ for the arrow $[A] \to E_A$. Thus, an object over Γ is a diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} & E_A \arrow[d, two heads] \\ \Gamma \arrow[r, "f_A"'] & V_A. \end{tikzcd}$$

- Morphisms: A map between $(V_B, E_B, f_B) \to (V_A, E_A, f_A)$ over $\sigma : \Delta \to \Gamma$ is a map in $\mathcal{E}$ between $[B] \twoheadrightarrow \Delta$ and $[A] \twoheadrightarrow \Gamma$ i.e. a commutative square

$$\begin{tikzcd} {[B]} \arrow[r] \arrow[d, two heads] & {[A]} \arrow[d, two heads] \\ \Delta \arrow[r, "\sigma"'] & \Gamma. \end{tikzcd}$$

- Composition is induced by the composition in $\mathcal{E}$, consequently, given by pasting commutative squares.

- The identity for (V_A, E_A, f_A) is the identity of $[A] \twoheadrightarrow \Gamma$ as an object in $\mathcal{C}^{\to}$.

We now unpack the cartesian lifts for the induced functor $p_! : \textsc{Fib}(\mathcal{C})_! \to \mathcal{C}_!$. Let $\sigma : \Delta \to \Gamma$ and $(V_A, E_A, f_A) \in \textsc{Fib}(\mathcal{C})_!$ over Γ. Set $A[\sigma] \coloneqq (V_A, E_A, f_A\sigma)$, pulling back along $f_A\sigma$ we obtain the commutative outer rectangle below

$$\begin{tikzcd} {[A[\sigma]]} \arrow[r, dashed] \arrow[d, two heads] \arrow[rr, bend left] & {[A]} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E_A \arrow[d, two heads] \\ \Delta \arrow[r, "\sigma"'] & \Gamma \arrow[r, "f_A"'] & V_A \end{tikzcd}$$

The universal property of the pullback on the right give us the unique map $A_\sigma : [A[\sigma]] \to [A]$. Therefore, a lift for σ is given by the evident map $A_\sigma : (V_A, E_A, f_A\sigma) \to (V_A, E_A, f_A)$. From the definition of A_σ the square

$$\begin{tikzcd} {[A[\sigma]]} \arrow[r, "A_\sigma"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & {[A]} \arrow[d, two heads] \\ \Delta \arrow[r, "\sigma"'] & \Gamma \end{tikzcd}$$

is a pullback, this implies that the square as a map in $\textsc{Fib}(\mathcal{C})_!$ is a cartesian lift of σ for $p_!$. Most importantly, this lift is uniquely determined by the composition $f_A\sigma$. Note that the transfinite composition of fibrations play no role in the construction. We summarize the discussion above in the following:

{split-comprehension-c

Theorem B.50. *For any κ-clan $\mathcal{C}$ there exist a full split comprehension category $(\mathcal{C}', \mathcal{E}, p_!, \iota_!)$ equivalent to $(\mathcal{C}, \textsc{Fib}(\mathcal{C}), p, \iota)$.*

Proof. We apply the previous construction, this give us $(\mathcal{C}_!, \textsc{Fib}(\mathcal{C})_!, p_!)$. Since the putative cartesian map is uniquely determined by the composition $f_A\sigma$ we can use a slight abuse of notation and write $A_\sigma := f_A\sigma$. Thus, if $\chi : \Xi \to \Delta$ is another map then $f(\sigma\chi) = (f\sigma)\chi$. This shows that the fibration $p_! : \textsc{Fib}(\mathcal{C})_! \to \mathcal{C}_!$ is split. The functor $\iota_! : \textsc{Fib}(\mathcal{C})_! \to \mathcal{C}^{\to}$ is defined as $\iota_!(V_A, E_A, f_A) := \iota([A] \twoheadrightarrow \Gamma) = [A] \twoheadrightarrow \Gamma$, similarly for arrows. The comprehension category $(\mathcal{C}_!, \textsc{Fib}(\mathcal{C})_!, p_!, \iota_!)$ is full since $(\mathcal{C}, \textsc{Fib}(\mathcal{C}), p, \iota)$ is full. $\square$

A *category with attributes* is a comprehension category $(\mathcal{C}, \mathcal{E}, p, F)$ such that p is a discrete fibration. Equivalently, a category with attributes can be defined as:

1. A category $\mathcal{C}$ with a terminal object 1,
2. A presheaf $\mathsf{Ty} : \mathcal{C}^{op} \to \mathbf{Set}$,
3. A function that assigns to each object $A \in \mathsf{Ty}(\Gamma)$, an object $\Gamma.A \in \mathcal{C}$ together with a map $\Gamma.A \to \Gamma$,
4. For each $A \in \mathsf{Ty}(\Gamma)$ and $\sigma : \Delta \to \Gamma$, a pullback square

$$\begin{tikzcd} \sigma^*\Gamma.A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d] \\ \Delta \arrow[r, "\sigma"'] & \Gamma \end{tikzcd}$$

{category-attributes-c

Corollary B.51. *For any κ-clan $\mathcal{C}$ there exist a category with attributes equivalent to $\mathcal{C}$.*

Proof. theorem B.50 give us a full split comprehension category $(\mathcal{C}_!, \textsc{Fib}(\mathcal{C})_!, p_!, \iota_!)$. We take the category to be $\mathcal{C}_! = \mathcal{C}$. The additional data is given in the obvious way. Defining $\mathsf{Ty}(\Gamma) := (\textsc{Fib}(\mathcal{C})_!)_\Gamma$, for each $A \in \mathsf{Ty}(\Gamma)$, we get $[A] \twoheadrightarrow \Gamma$ as described above. The required pullbacks are given by the cartesian lifts of $p_!$. Furthermore, these pullbacks are computed strictly along compositions since $p_!$ is a split fibration. $\square$

Our next goal is from the category with attributes given by corollary B.51 define a κ-contextual equivalent to $\mathcal{C}$. In particular, for each object $\Gamma \in \mathcal{C}$ we get a κ-contextual category $\mathcal{C}(\Gamma)$. We start with the following observation:

{category-structure-c-g

Definition B.52. The category structure is given by the following data:

- **Objects**: For each ordinal $\mu < \kappa$ we define the set $Ob_\mu(\mathcal{C}(\Gamma))$ inductively over μ;
  - If $\mu = \lambda + 1$ then we define $Ob_\mu(\mathcal{C}(\Gamma)) \coloneqq \mathsf{Ty}([A_\lambda])$. More explicitly, an object $A_\mu \in Ob_\mu(\mathcal{C}(\Gamma))$ can be represented as the sequence
$$A_\mu \twoheadrightarrow A_\lambda \twoheadrightarrow \cdots \twoheadrightarrow \Gamma$$
    and comes with a fibration $A_\mu \twoheadrightarrow \Gamma$.
  - If μ is a limit ordinal then $Ob_\mu(\mathcal{C}(\Gamma))$ is the collection of objects of the form $A_\mu \coloneqq \mathtt{Lim}_{\lambda<\mu} A_\lambda$ obtained as the transfinite composition of a sequence
$$\cdots \twoheadrightarrow A_\lambda \twoheadrightarrow \cdots \twoheadrightarrow \Gamma.$$
    Each object comes with a fibration $A_\mu \twoheadrightarrow \Gamma$. This is given by the transfinite composition axiom of $\mathcal{C}$.
- **Morphisms**: For ordinals $\mu \leq \lambda < \kappa$ and objects $B_\lambda \in Ob_\lambda(\mathcal{C}(\Gamma))$, $A_\mu \in Ob_\mu(\mathcal{C}(\Gamma))$ we set
$$Hom_{\mathcal{C}(\Gamma)}(B_\lambda, A_\mu) \coloneqq Hom_{\mathcal{C}/\Gamma}(B_\lambda, A_\mu).$$
- The rest of the structure of $\mathcal{C}(\Gamma)$ is induced by $\mathcal{C}/\Gamma$, in particular the transfinite composition is that of $\mathcal{C}/\Gamma$.

Before proving that this gives us a κ-contextual category, let us explain the objects of this category. Recall that for $A \in \mathsf{Ty}(\Gamma)$ means we have a diagram of the form

$$\begin{tikzcd} & E_A \arrow[d, two heads] \\ \Gamma \arrow[r, "f_A"'] & V_A. \end{tikzcd}$$

When identify this object with $[A]$, then $\mathsf{Ty}([A])$ is the set of objects of the form

$$\begin{tikzcd} & E_B \arrow[d, two heads] \\ {[A]} \arrow[r, "(E_A)_{f_A}"'] & E_A. \end{tikzcd}$$

Each of such objects give $(V_A, f_A, E_B) \in \mathsf{Ty}(\Gamma)$ where $E_B \twoheadrightarrow V_A$ is the composition $E_B \twoheadrightarrow E_A \twoheadrightarrow V_A$. Equivalently, this is the composition $[B] \twoheadrightarrow [A] \twoheadrightarrow \Gamma$. Furthermore, if we write $\Gamma.A \coloneqq [A]$ then we can rewrite this in a more familiar fashion $\Gamma.A.B \twoheadrightarrow \Gamma.A \twoheadrightarrow \Gamma$. This illustrates the general procedure for successor ordinals. A related construction appears in [KL18, Definition 4.3].

{contextual-clan}

Lemma B.53. *For any κ-clan $\mathcal{C}$ and any $\Gamma \in \mathcal{C}$, the category $\mathcal{C}(\Gamma)$ is a κ-contextual category.*

Each axiom follows more or less immediately. We start with the category with attributes we obtained in corollary B.51 and the construction from definition B.52.

Proof.

1. The objects of $\mathcal{C}(\Gamma)$ have grading $Ob(\mathcal{C}(\Gamma)) = \coprod_{\mu<\kappa} Ob_\mu(\mathcal{C}(\Gamma))$ as in definition B.52. This grading determines the height of each object.
2. The terminal object is Γ.
3. Given ordinals $\mu \leq \lambda < \kappa$ and objects $A_\lambda, A_\mu \in \mathcal{C}(\Gamma)$, the display maps between them are the maps in $Hom_{\mathcal{C}(\Gamma)}(A_\lambda, A_\mu)$ which are also fibrations of $\mathcal{C}$. We group these maps and objects in $Dis(\mathcal{C}(\Gamma))$, which is easily seen to be a subcategory.
4. $Dis(\mathcal{C}(\Gamma))$ is closed under transfinite compositions since $\mathcal{C}$ is itself closed under such compositions.
5. The inclusion functor $i : Dis(\mathcal{C}(\Gamma)) \hookrightarrow \mathcal{C}(\Gamma)$ preserve transfinite compositions: ??.
6. If $A \twoheadrightarrow B$ is an arrow in $Dis(\mathcal{C}(\Gamma))$ then $B \in Ob_\mu(\mathcal{C}(\Gamma))$ and $A \in Ob_\lambda(\mathcal{C}(\Gamma))$ for some ordinals λ, μ with $\mu \leq \lambda$: This follows directly by definition of the objects of $\mathcal{C}(\Gamma)$

7. For any object $A \in Ob_\lambda(\mathcal{C}(\Gamma))$ and any $\mu \leq \lambda$ there exists a unique object $B \in Ob_\mu(\mathcal{C}(\Gamma))$ and a unique display map $A \twoheadrightarrow B$: We can easily obtain this by induction on λ and verify that the map has the correct length

8. Canonical pullbacks: This is given by the category with attributes structure on $\mathcal{C}$ as explained in corollary B.51.

9. Canonical pullbacks are strictly functorial: This is exactly what corollary B.51 achieves.

10. It follows from the description of objects given above.

□

Before we can state our main result, we first need state the appropriate notion of equivalence between κ-clans. We borrow the definitions from [Joy17] adapted to our setting. Let $\mathcal{C}$ and $\mathcal{E}$ be two κ-coclans. We say that a functor $F : \mathcal{C} \to \mathcal{E}$ is a *morphism of κ-coclans* if

1. sends initial objects to initial objects,

2. preserves cofibrations,

3. preserves pushouts of cofibrations along any map

4. preserves transfinite compositions.

Furthermore, a morphism between κ-coclans $F : \mathcal{C} \to \mathcal{E}$ is an *equivalence of κ-coclans* if there exists another morphism of κ-coclans $G : \mathcal{E} \to \mathcal{C}$ and natural isomorphisms $GF \cong Id_\mathcal{C}$ and $FG \cong Id_\mathcal{E}$.

Similarly, $F : \mathcal{C} \to \mathcal{E}$ is a *morphism of κ-clans* simply if $F^{op} : \mathcal{C}^{op} \to \mathcal{E}^{op}$ morphism of κ-coclans, and an *equivalence of κ-clans* if $F^{op} : \mathcal{C}^{op} \to \mathcal{E}^{op}$ is an equivalence κ-coclans.

Proposition B.54. *A morphism of clans $F : \mathcal{C} \to \mathcal{E}$ is equivalence of clans if and only if F reflects fibrations and transfinite compositions in $Dis(\mathcal{E})$, this is; if $F(\mathtt{Lim}_\lambda A_\alpha) \twoheadrightarrow F(A_0)$ is the transfinite composition of the sequence*

$$F(\mathtt{Lim}_\lambda A_\alpha) \cdots \twoheadrightarrow FA_2 \twoheadrightarrow FA_1 \twoheadrightarrow FA_0$$

then $\mathtt{Lim}_\lambda A_\alpha \twoheadrightarrow A_0$ is the transfinite composition of the sequence

$$\cdots \twoheadrightarrow A_2 \twoheadrightarrow A_1 \twoheadrightarrow A_0.$$

The equivalence of theorem B.50 give us an equivalence between clans.

{clan-equivalent-conte

Corollary B.55. *For any κ-coclan $\mathcal{C}$ there exists a κ-contextual category equivalent to it.*

Proof. Let us take the κ-clan given by $\mathcal{D} := \mathcal{C}^{op}$. We can then observe that $\mathcal{D} \cong \mathcal{D}(1)$ where $\mathcal{D}(1)$ is the κ-contextual category obtained from lemma B.53. We can take the opposites again to get $\mathcal{C}$. $\square$

C Weak model categories

The most general setting in which we will show good homotopy theoretic properties of the language introduced in section 2 is for the weak model categories introduced in [Hen20], which we will briefly recall here. In practice this extra-generality compared to Quillen model structure is not extremely useful - all the examples we will consider in section 3 are Quillen model structures, so it would not be unreasonable to skip the present subsection. There are two reasons we need weak model categories:

- A key construction towards the proof of the third invariance theorem in section 4 is in general only a weak model structure, and we need to use its language as an intermediate tool.
- Future applications to left and right semi-model structure: actual weak model structure that are not left or right semi-model structures are fairly uncommon, but the weak model categories which include both left and right semi-model structure at the same time, are considerably more common.

C.1 Review

{def:wms:intro}

Definition C.1. A *weak model category* is a category $\mathcal{M}$ with three classes of maps, *cofibrations*, *fibrations* and *weak equivalences* satisfying the following conditions:

{def:wms:initial-termi

1. $\mathcal{M}$ has an initial object 0 and a terminal object 1, the identity of 0 is a cofibration, the identity of 1 is a fibration.

{def:wms:composite-fib

2. A composite of cofibrations with cofibrant domain is a cofibration. A composite of fibrations with fibrant codomain is a fibration.

{def:wms:2-out-of-3}

3. Given two composable arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$ where each one of X,Y and Z are fibrant or cofibrant, if two of f, g, $g \circ f$ are weak equivalences, then the third also is.

{def:wms:iso-are-we}

4. Every isomorphism between objects that are either fibrant or cofibrant is a weak equivalence.

{def:wms:po-cof}

5. Given a solid diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"] & B \arrow[d, tail, dotted, "j"] \\ C \arrow[r, dotted] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

   Where i is a cofibration and A and B are cofibrant, then the pushout j exists and is a cofibration.

{def:wms:pb-fib}

6. The dual of condition 5 holds for fibrations between fibrant objects.

{def:wms:iso-closed}

7. Every arrow isomorphic to a fibration, cofibration, or weak equivalence is also one.

{def:wms:cof-trivfib-fa

8. Every arrow from a cofibrant to a fibrant object can be factored as a cofibration followed by a trivial fibration.

{def:wms:trivcof-fib-fa

9. Every arrow from a cofibrant to a fibrant object can be factored as a trivial cofibration followed by a fibration.

{def:wms:lifting-prop}

10. Given a solid square:

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"] & X \arrow[d, two heads, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

    Where A and B are cofibrant, i is a cofibration, X and Y are fibrant, p is a fibration and either p or i is a weak equivalence, then there exists a dotted map that makes the diagram to commute.

Remark C.2. In definition C.1 we use the usual conventions: a *cofibrant object* is an object such that the unique map $0 \to X$ is a cofibration, and a *fibrant object* is an object such that the unique map $X \to 1$ is a fibration. A trivial (co)fibration is a map which is both an equivalence and a (co)fibration. We will also use the term *core cofibrations* to mean "cofibration between cofibrant objects" and *core fibrations* to mean "fibration between fibrant objects".

{rk:only_core_matter}

Remark C.3. It is crucial to observe that definition C.1 only involve the core cofibrations, core fibrations and weak equivalences between objects that are either fibrant or cofibrant. By that we mean that if given $\mathcal{M}$ a category with these three class of maps, then ($\mathcal{M}$,cofibrations, fibrations, weak equivalences) is a weak model structure if and only if ($\mathcal{M}$, core cofibrations, core fibrations, weak equivalences between objects that are either fibrant or cofibrant) is a model structure.

For this reason, we generally consider that only core cofibrations, core fibrations and weak equivalence between objects that are either fibrant or cofibrant are to be treated as relevant notions. Nothing we will do here depends on the three class of maps outside of these restrictions. In [Hen20] it was even considered that the words cofibrations, fibrations and weak equivalences to mean "core cofibrations", "core fibrations" and "weak equivalences between fibrant or cofibrant objects".

Remark C.4. The definition of weak model structure in [Hen20] is different from definition C.1, but it is equivalent. It is stated without reference to the class of weak equivalence and using notion of (weak relative) path object and cylinder object. It is easy to show that a weak model structure in the sense of definition C.1 is a weak model structure in the sense of [Hen20] by constructing the cylinder and path objects as factorization of the codiagonal and diagonal maps (see C.5 below). Conversely, it is shown in [Hen20] that given a weak model structure, it admits a (unique[2]) class of weak equivalences such that all conditions of definition C.1 are satisfied.

It is shown in [Hen20] that most of the basic theory of Quillen model categories carries over to weak model categories, with only some additional care taken - mostly replacing objects by fibrant and cofibrant replacement of objects before applying the usual construction. The main significant difference is that the homotopy category (defined in terms of homotopy class of maps between bifibrant objects as we will recall below) is no longer equivalent to $\mathcal{M}[W^{-1}]$ - the localization of $\mathcal{M}$ at weak equivalence, but only to $\mathcal{M}^{\mathrm{cof}\vee\mathrm{fib}}[W^{-1}]$ the localization the full subcategory of objects that are either fibrant or cofibrant at the weak equivalences. The problem is that the axioms of a weak model category allows to take a fibrant replacement of a cofibrant object C as a (trivial cofibration/fibration) factorization of $C \to 1$, and similarly we can take a cofibrant replacement of a fibrant objects, but there is no way to do similar replacement with an object which is neither fibrant nor cofibrant.

We now quickly go over some aspects of the construction of the homotopy

[2]Keeping in mind remark C.3. Only the class of weak equivalence between fibrant or cofibrant objects is uniquely defined, outside of this, there no restriction whatsoever on weak equivalence from definition C.1.

category of a weak model category, the result mentioned below are all proved in section 2.1 and 2.2 of [Hen20].

{cstr:path_and_cylinder

Construction C.5. If X is a bifibrant object (i.e. fibrant and cofibrant), we can form a *cylinder objects* IX for X as a (cofibration,trivial fibration) factorization:

$$X \coprod X \hookrightarrow IX \overset{\sim}{\twoheadrightarrow} X$$

and a path objects for X as a (trivial cofibration, fibration) factorization

$$X \overset{\sim}{\hookrightarrow} PX \twoheadrightarrow X \times X.$$

Given a pair of maps $f, g : X \rightrightarrows Y$ between bifibrant objects, we say they are homotopic if there is a dotted map h making the diagram below commutative:

$$\begin{tikzcd} X \arrow[d] \arrow[dr, "f"] & \\ IX \arrow[r, dotted, "h"] & Y \\ X \arrow[u] \arrow[ur, "g"'] & \end{tikzcd}$$

or equivalently a map h

$$\begin{tikzcd} & Y \\ X \arrow[ur, "g"] \arrow[r, dotted, "h"] \arrow[dr, "f"'] & PY \arrow[u] \arrow[d] \\ & Y. \end{tikzcd}$$

This is an equivalence relation, and the homotopy category $\mathrm{Ho}(\mathcal{M})$ of $\mathcal{M}$ can be defined as the category of bifibrant objects with homotopy class of maps between them. Moreover this category is equivalent to the formal localization $\mathcal{M}^{\mathrm{cof} \vee \mathrm{fib}}[W^{-1}]$.

{cstr:weak_path_and_cyl

Construction C.6. Note that if an object $C \in \mathcal{M}$ is only cofibrant and not fibrant we cannot define a cylinder object in the same was as above, as the factorization axiom does not allows to factor the maps $X \coprod X \to X$ if X is not fibrant. In place of this, we can consider a fibrant replacement $X \overset{\sim}{\hookrightarrow} X^{\mathrm{FIB}} \twoheadrightarrow 1$, and then form a factorization:

$$\begin{tikzcd} X \coprod X \arrow[r, hook] \arrow[d, "\nabla"] & IX \arrow[d, two heads, "\sim"] \\ X \arrow[r, hook, "\sim"] & X^{\mathrm{FIB}}. \end{tikzcd}$$

This object IX, and more generally any object fitting into a diagram:

$$\begin{tikzcd} X \coprod X \arrow[r, hook] \arrow[d, "\nabla"] & IX \arrow[d, "\sim"] \\ X \arrow[r, hook, "\sim"] & DX \end{tikzcd}$$

is called a weak cylinder object. Dually, if Y is fibrant we define a weak path object of Y as any object PY that fits into a diagram:

$$\begin{tikzcd} TX \arrow[r, "\sim"] \arrow[d, two heads, "\sim"] & PX \arrow[d, two heads] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

We can then show that for a pair of maps $X \rightrightarrows Y$ from a cofibrant object X to a fibrant object Y the following are equivalent:

- f is homotopic to g in terms of a weak cylinder object for X.
- f is homotopic to g in terms of a weak path objects for Y.
- f and g are equal in the localization $\mathcal{M}^{\mathrm{cof}\vee\mathrm{fib}}[W^{-1}]$.

Moreover any arrow $X \to Y$ in the localization $\mathcal{M}^{\mathrm{cof}\vee\mathrm{fib}}[W^{-1}]$ comes from an arrow $X \to Y$ in $\mathcal{M}$.

C.2 Weak Reedy model structure

Before doing all the constructions, we need to set up the formalism needed for such. In this section we study Reedy weak model categories. These are, as the name suggests, the counterpart of Reedy model categories. Most of the proofs are straightforward adaptation of the classical ones, so they are omitted.

Definition C.7. A *Reedy category* is a category R together with two wide subcategories R_+ and R_- and a functor $deg : R \to \alpha$, where α is an ordinal, such that:

1. For every $a \to b \in R_+$ a non-identity arrow, $\deg(a) < \deg(b)$.
2. For every $a \to b \in R_-$ a non-identity arrow, $\deg(b) < \deg(a)$.

3. Every arrow in R factors uniquely as an arrow in R_- followed by an arrow in R_+.

When the subcategory R_- consists of identity arrows only, then R is called *direct category*. Similarly, when the subcategory R_+ consists of identity arrows only, then R is called *inverse category*

Let R be a Reedy category and $\mathcal{M}$ be a weak model category. Consider $\mathcal{M}^R$ the category of R-shaped diagram in $\mathcal{M}$. Given $X : R \to \mathcal{M}$ such a diagram and $r \in R$ any object. The *latching object* at r is the colimit (if it exists)

$$L_rX \coloneqq \mathtt{Colim}_{s\in(R_+/r)-\{Id_r\}} X_s.$$

Dually, the *matching object* at r is the limit (if it exists)

$$M_rX \coloneqq \mathtt{Lim}_{s\in(r/R_-)-\{Id_r\}} X_s.$$

{def:ReedyMap}

Definition C.8. A map $f : X \to Y$ in $\mathcal{M}^R$ is said to be an *(acyclic) Reedy cofibration* at $r \in R$ if the colimit $L_rY \sqcup_{L_rX} X_r$ exists and the induced dotted map in the diagram below

$$\begin{tikzcd}
L_rX \arrow[r] \arrow[d] & X_r \arrow[d] \arrow[ddr, bend left] & \\
L_rY \arrow[r] \arrow[drr, bend right] & L_rY \sqcup_{L_rX} X_r \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\
& & Y_r
\end{tikzcd}$$

is an (acyclic) cofibration in $\mathcal{M}$.

Dually, $f : X \to Y$ in $\mathcal{M}^R$ is said to be an *(acyclic) Reedy fibration* at $r \in R$ if the limit $M_rX \times_{M_rY} Y_r$ exists and the induced dotted map in the diagram below

$$\begin{tikzcd}
X_r \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& M_rX \times_{M_rY} Y_r \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_r \arrow[d] \\
& M_rX \arrow[r] & M_rY
\end{tikzcd}$$

exists and is an (acyclic) fibration in $\mathcal{M}$.

A map is said to be an (acyclic) Reedy (co)fibration if it is one at each $r \in R$.

Remark C.9. We want to clarify that in definition C.8 the colimit $L_rY \sqcup_{L_rX} X_r$ is considered as a single colimit not as a pushout using the object L_rX and L_rY. It is possible that $L_rY \sqcup_{L_rX} X_r$ exists without the colimit L_rY or L_rX existing. Explicitly, it is the colimits of all the X_i for $i \in R^+/r$ and of the Y_i for $i \in R^+/r - \{id_r\}$. with all the maps coming from the functoriality in i and the natural map $X_i \to Y_i$. We apply the same logic to the limit $M_rX \times_{M_rY} Y_r$.

Definition C.10. A Reedy category is said to be *locally finite* if for any object $X \in R$ the categories (R_+/X) and (R_-/X) are finite.

It is a classical result that for any Quillen model category $\mathcal{M}$ and a Reedy category R that the category of functors $\mathcal{M}^R$ carries a model structure in which the weak equivalences are the level-wise weak equivalences, the (acyclic) (co)fibrations are precisely the Reedy (acyclic) (co)fibrations. The same result can be obtained if we simply assume that the base category carries a weak model structure.

{reedy-model:theorem}

Theorem C.11. *Assume that $\mathcal{M}$ is a weak model category and that R is a locally finite Reedy category. Then there is a weak model structure on $\mathcal{M}^R$ such that a map $f : X \to Y$ it is:*

1. *A weak equivalence if and only if $f_r : X_r \to Y_r$ is a weak equivalence for all $r \in R$.*

2. *An (acyclic) cofibration if it is an (acyclic) Reedy cofibration.*

3. *An (acyclic) fibration if it is an (acyclic) Reedy fibration.*

{Rk:latching_map_as_si

Remark C.12. When the Reedy category is directed this model structure coincides with the projective weak model structure. It is straightforward to define this last weak model category. In this weak model, the weak equivalences and the fibrations are the level-wise weak equivalences and fibrations respectively. Similarly, when the Reedy category is an inverse category, then the Reedy weak model structure is Quillen equivalent to the injective model structure. In this other case, weak equivalences and cofibrations are given level-wise.

We now prove the theorem:

{lem:Direct_colimit}

Lemma C.13. *Let I be a direct category and $X : I \to \mathcal{M}$ be a diagram. Let $U \subset V \subset I$ be two sieves*[3] *of I, such that $V - U$ has a finite number of objects. Assume that the colimit*

$$X(U) \coloneqq \mathit{Colim}_{u \in U} X(u)$$

exists and is cofibrant, and that for each $v \in V - U$. The latching object $L_v X$ exists and is cofibrant, and the map $L_v X \to X(v)$ is a cofibration. Then $X(V)$ exists and the comparison map $X(U) \to X(V)$ is a cofibration. If $L_v X \to X(v)$ is actually and acyclic cofibration for every $v \in V - U$ then $X(U) \to X(V)$ is an acyclic cofibration.

Proof. This is immediate by induction on the number of objects of $V - U$. If it only has one objects then $X(U) \to X(V)$ can be seen to be a pushout of the core cofibration $L_v X \to X_v$ to the cofibrant object $X(U)$. If $V - U$ has several object we iterate this process once for each object of $V - U$. □

{cor:Latching_are_cofil

Corollary C.14. *Let R be a locally finite Reedy category, $X : R \to \mathcal{M}$ be a diagram and let $k \in R$ an object. Assume that X is Reedy cofibrant at every r such that $\deg(r) < \deg(k)$. Then the latching object $L_k(X)$ exists and is cofibrant.*

Proof. Using a proof by induction on $\deg(x)$ we can freely assume that all the latching object $L_r(X)$ are cofibrant for all r such that $\deg(r) < \deg(x)$. We can then just apply the lemma C.13 to the finite direct category $I = R^+/x$ and $U = \varnothing$, $V = I$. □

{cor:Direct_colimit}

Corollary C.15. *Let I be a finite direct category, and let $X : I \to \mathcal{M}$ be a Reedy cofibrant diagram and $U \subset I$ be a sieve. Then $\mathit{Colim}_I X$ and $\mathit{Colim}_U X$ exists, are cofibrant and the obvious comparison map $\mathit{Colim}_U X \to \mathit{Colim}_I X$ is a cofibration.*

If furthermore the latching map $L_r X \to X(r)$ is an acyclic cofibration for each $r \in I - U$ then the map $\mathit{Colim}_U X \to \mathit{Colim}_I X$ is an acyclic cofibrations.

Proof. By corollary C.14 all the latching objects of X are cofibrant, so we can simply apply lemma C.13 and conclude. □

{cor:core_cof_are_level

Corollary C.16. *Let R be a locally finite Reedy category.*

[3] That is subcategories with the properties that if there is an arrow $x \to x'$ and $x' \in V$ then $x \in V$.

- *Any core (acyclic) Reedy cofibration $X \to Y$ in $\mathcal{M}^R$ is in particular a levelwise (acyclic) cofibration. That is the map $X(r) \to Y(r)$ are (acyclic) cofibrations for any $r \in R$.*

- *A map $X \to Y$ in $\mathcal{M}^R$ which is both a core Reedy cofibration and a level-wise weak equivalence is an acyclic Reedy cofibration.*

Dually, the same is true for fibrations and acyclic fibrations.

Proof. As both statement only depends on the restriction to the subcategory R^+, we can freely assume that R is a (locally finite) direct category. In both cases, we consider the natural transformation $X \to Y$ as a diagram $T : R \times \{0 < 1\} \to \mathcal{M}$. We then observe that the latching map of T at an object $(r, 0)$ is just $L_r X \to X$, and the latching map of T at $(r, 1)$ is

$$L_r Y \sqcup_{L_r X} X(r) \to Y(r)$$

Hence the assumption that $X \to Y$ is a core Reedy cofibration translate into the fact that T is Reedy cofibrant. For any object $r \in R$, the composite $R \times \{0 < 1\}/(r, 1) \to R \times \{0 < 1\} \to \mathcal{M}$ is immediately seen to be Reedy cofibrant as well and we can then apply corollary C.15 to the Sieve $U = R/r \times \{0\}$ to conclude that $X(r) \to Y(r)$ is a cofibration.

If $X \to Y$ is further assumed to be acyclic, then the latching map of T at all objects of the form $(r, 1)$ are acyclic, and hence using the acyclic case of corollary C.15 we conclude that $X(r) \to Y(r)$ is acyclic.

If instead we assume that $X(r) \to Y(r)$ is a weak equivalence for all r, then we proceed by strong induction on $\deg(r)$. Assume that we already know that at all k such that $\deg(k) < \deg(r)$.

If $\deg(r) = 0$, then the latching map is just $X(r) \to Y(r)$ itself so it is an acyclic cofibration as it is a cofibration and a weak equivalence. Assume now that we already know that all the latching maps

$$L_r Y \sqcup_{L_r X} X(r) \to Y(r)$$

are acyclic cofibration for any r such that $\deg(r) < \deg(k)$. We can then deduce by the same argument as above that the map $L_k(X) \to L_k(Y)$ is a core acyclic cofibration, which shows that the map $X(r) \to L_r Y \sqcup_{L_r X} X(r)$ is an acyclic cofibration, hence an equivalence, and hence by 2-out-of-3 for equivalences, the map $L_r Y \sqcup_{L_r X} X(r) \to Y(r)$, is both an equivalence and a core cofibration, so it is a weak equivalence. □

Note that we have also proved that:

{lem:LatchingDomain_are

Lemma C.17. *Let R be a locally finite Reedy category, and $i : X \to Y$ be a core Reedy cofibration in $\mathcal{M}^R$. Then the domain of the latching map $L_rY \sqcup_{L_rX} X(r)$ is cofibrant.*

Proof. At the beginning of the proof of corollary C.16 we observed that it could be written as a latching object $L_{(r,1)}T$ of a cofibrant Reedy diagram T. Hence, the result follows from corollary C.14. □

{prop:composite_fib}

Proposition C.18. *For any locally finite Reedy category R, in $\mathcal{M}^R$, the composite of two Reedy core cofibrations is a Reedy core cofibrations.*

Proof. We use a strategy very similar to the proof of corollary C.16. Here again, the result only depends on the restriction to R^+ so we can freely assume that R is a direct category. Let $X \to Y \to Z$ be two composable Reedy core cofibrations in $\mathcal{M}^R$. We consider this as a diagram $T : R \times \{0 < 1 < 2\} \to \mathcal{M}$. As in the proof of corollary C.16. We observe that the latching map at an element of the form $(r, 0)$ is the latching map $L_rX \to X$ of X hence is a cofibration as X is Reedy cofibrant. The latching map at an element $(r, 1)$ is the map

$$L_rY \sqcup_{L_rX} X(r) \to Y(r)$$

which is a cofibration as $X \to Y$ is assumed to be a Reedy cofibration. And finally, the latching map at $(r, 2)$ is the map

$$L_rZ \sqcup_{L_rY} Y(r) \to Z(r)$$

which is also a cofibration. So this diagram $R \times \{0 < 1 < 2\} \to \mathcal{M}$ is Reedy cofibrant. It immediately follows that, for any $r \in R$ the composite $R \times \{0 < 1 < 2\}/(r, 2) \to R^- \times \{0 < 1 < 2\} \to \mathcal{M}$ is a Reedy cofibrant diagram. Hence applying corollary C.15, we can deduce that the map

$$\mathtt{Colim}_U T \to Z(r)$$

is a cofibration, where $U \subset R \times \{0 < 1 < 2\}/(r, 2)$ is the sieve containing all the objects except $(r, 1)$ and $(r, 2)$. But this map can be seen to be exactly

$$L_rZ \sqcup_{L_rX} X(r) \to Z(r)$$

by remark C.12. This concludes the proof as this can be applied to any object $r \in R$. □

{prop:po-Reedy-cofibra

Proposition C.19. *Consider a cospan $Y \leftarrow X \rightarrow Z$ of diagram $R \rightarrow \mathcal{M}$, such that X, Y, Z are all Reedy cofibrant and the arrow $X \rightarrow Y$ is a Reedy cofibration. Then the (level-wise) pushout $Y \sqcup_X Z$ exists in $\mathcal{M}^R$ and the natural transformation $Z \rightarrow Y \sqcup_X Z$ is a Reedy cofibration.*

Proof. It follows from corollary C.16 that for each $r \in R$ the three objects in the diagram $Y(r) \leftarrow X(r) \rightarrow Z(r)$ are cofibrant and the map $X(r) \rightarrow Y(r)$ is a cofibration, so the levelwise pushout $Y(r) \sqcup_{X(r)} Z(r)$ exists and by general category theoretic results is functorial in r and is a pushout in the category of diagrams $\mathcal{M}^R$. We only need to check that the map $Z(r) \rightarrow Y(r) \sqcup_{X(r)} Z(r)$ is a Reedy cofibration. For this observe that as colimits commutes with colimits we have:

$$L_r(Y \sqcup_X Z) = \mathtt{Colim}_{r' \rightarrow r \in R^+} Y(r') \sqcup_{X(r')} Z(r') = L_r Y \sqcup_{L_r X} L_r Z$$

So that in the latching map

$$L_r(Y \sqcup_X Z) \sqcup_{L_r Z} Z \rightarrow Y \sqcup_X Z$$

the domain can be identified with

$$(L_r Y \sqcup_{L_r X} L_r Z) \sqcup_{L_r Z} Z = L_r Y \sqcup_{L_r X} Z = (L_r Y \sqcup_{L_r X} X) \sqcup_X Z$$

so the latching map is

$$(L_r Y \sqcup_{L_r X} X) \sqcup_X Z \rightarrow Y \sqcup_X Z$$

which is a pushout of the latching map $L_r Y \sqcup_{L_r X} X \rightarrow Y$, which is itself a core cofibration as $X \rightarrow Y$ is a core Reedy cofibration. Hence this concludes the proof. □

We are now ready to prove theorem C.11:

Proof. We go over all the conditions of definition C.1. The validity of conditions 1, 3, 7 and 4 is trivial. Condition 2 is proposition C.18 together with its dual. Condition 5 is proposition C.19, and condition 6 is the dual statement.

The proof of conditions 10 is essentially the same as the proof for ordinary model categories, as for example in Chapter 15 of [Hir03] or in Chapter 5.2 of [Hov99]. The key step in the proof is that in order to construct a diagonal lift in a square:

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"] & X \arrow[d, two heads, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

where say i is a core cofibration and p is a core fibration, one of them being a (level-wise) weak equivalence. Then we proceed by induction as in the usual proof, at each step we need to produce a diagonal lift in a square of the form

$$\begin{tikzcd} A(r) \sqcup_{L_r A} L_r(B) \arrow[r] \arrow[d, tail] & X(r) \arrow[d, two heads] \\ B(r) \arrow[r] \arrow[ur, dotted] & Y(r) \times_{M_r Y} M_r X \end{tikzcd}$$

Now by lemma C.17 (and its dual) the object $A(r) \sqcup_{L_r A} L_r(B)$ is cofibrant and $Y(r) \times_{M_r Y} M_r X$ is fibrant, by definition of Reedy cofibration and fibration, the left vertical map is a cofibration and the right vertical is a fibration, and if one of i or p (say i) is a weak equivalence, then the second point of corollary C.16 show that the left vertical map is an acyclic cofibration, hence the square admit a diagonal lift, which concludes the proof.

The proof of condition 8 and (dually of condition 9), also follows very closely the classical proof as in Chapter 15 of [Hir03] or in Chapter 5.2 of [Hov99]. Given $A \to X$ a map from a Reedy cofibrant diagram to a Reedy fibrant diagram, that we want to factor as a core acyclic Reedy cofibration followed and a core Reedy fibration, $A \to B \to X$, we also proceed by induction, constructing the diagram the object $B(r)$ and the maps $A(r) \to B(r) \to X(r)$ gradually by induction on the degree of r. Following the classical proof, at each stage, we need to construct a factorization of a map in $\mathcal{M}$:

$$A(r) \sqcup_{L_r A} L_r B \to X(r) \times_{M_r X} M_r B$$

as an acyclic cofibration followed by a fibration. But as observed above, the domain is cofibrant and the target is fibrant, so this is indeed possible in $\mathcal{M}$. The case of condition 9 is done in the exact same way, but factoring the map above as a cofibration followed by an acyclic fibration.

□

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