Homotopy Languages

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Abstract

We attach to each weak model category $\mathcal{M}$ a class of first order formulas about the fibrant objects of $\mathcal{M}$ whose validity is invariant under homotopies and weak equivalences. This is a generalization of the classical Blanc-Freyd Language of categories - which involves formula avoiding equality on objects and which are invariant under isomorphism and equivalences of categories. In particular, we obtain similar homotopy invariant languages for 2-categories, bicategories, chain complexes, Kan complexes, quasi-categories, Segal spaces, and so on...

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1 Introduction

It is a well known result in category theory ([Fre76], [Bla78]) that any property of a category or of objects and morphisms in a category that does not use equality between objects is automatically invariant by equivalence of categories and by replacing the object and morphism involved by (consitently) isomorphic.

For example, because the notion of limit in a category can be formulated without using equality between objects we automatically deduce that equivalence of categories preserves limits, or that if two diagram are isomorphic then a limit for one is also a limit for the other.

To be precise, this refers to first-order formula, we we can have quantifier over all objects of the category, or over all morphisms in a given hom-set "hom(X, Y)", and we can use equality between two terms of type hom(X, Y), but not betweem two terms of type "objects", or two terms that are in different hom-set.

For example, the property of an object X to be a terminal object, which can be written as

$$\texttt{isTerminal}(X) \coloneqq \forall y \in \mathrm{Ob}, (\exists v \in \mathrm{Hom}(y, X) \text{ and } \forall u, w \in \mathrm{Hom}(y, X), u = w)$$

is such a formula, but the following two formulas which says respectively that the category is skeletal or has no non-trivial isomorphism:

$$\forall X, Y \in \mathrm{Ob}, \forall f \in \mathrm{Hom}(X,Y), \forall g \in \mathrm{Hom}(Y,X), (f \circ g = \mathsf{id}_Y \text{ and } g \circ f = \mathsf{id}_X \Rightarrow X = Y)$$

$$\forall X, Y \in \mathrm{Ob}, \forall f \in \mathrm{Hom}(X,Y), \forall g \in \mathrm{Hom}(Y,X), (f \circ g = \mathsf{id}_Y \text{ and } g \circ f = \mathsf{id}_X \Rightarrow f = \mathsf{id}_X)$$

are not of this form, and they are indeed not invariant under equivalence of categories.

Note that in order for this to make sense, it is key to use a notion of "dependent type". Indeed, we need to be able to formulate the idea that a morphism f is in $\mathrm{Hom}(X,Y)$, without being able to say that $s(f) = X$ and $t(f) = Y$ as this would involve using equality between objects. So, given two objects X and Y, we need to be able to consider the type of arrows from X to Y as a primitive notion.

Now, it is natural to expect that similar results can be generalized to higher category. For example, we expect (and it can be shown) that a property of 2-categories or bicategories that does not use equality between object or between 1-arrow will also be invariant under biequivalence. One can also expect it can be generalized to other sort of higher structures, for example a result about multicategories not using equality between object should also have similar good invariance properties.

The main goal of this paper is, informally, to establish a version of this result for essentially any kind of higher structures, independently of the type of strucutre or the "categoricity level". The only requirement is that the sort of higher structure we are considering must be organised into the fibrant objects of a model category (or semi-model category, or weak model categories).

That is, we will attach to every (semi/weak) model category a "first-order language", whose formulas are statement about objects of the cateory (possibly with parameters) such that

- Replacing the value of the parameters by homotopically equivalent parameters does not change the validity of a formula.
- Two weakly equivalent fibrant objects satisfies the same formula.

We call these two results respectively the first and second invariance theorem, and their precise statement is given as **??**.

We will now go into a little more detail of how this language is defined, and explain the role of the different of the paper.

As mentioned above, we need to use dependents type. So our starting point is a "Generalised algebraic theory" T in the sense of Cartmell ([Car78]) as our basis - if we compare to traditional model theory, T plays a role similar to a signature.

We then build on top of T the associated first order language in section 2.1. The idea is that for each formula, the (free) variables are taken from a context of the theory T, and there can be no equality at all. We will wee though example how some cases of equality can sometimes be recovered indirectly using the fact that the theory T can include equality axioms (it can be any Generalized algebraic theory). Because we want to be able to do infinitary logic, we use everywhere an infinitary generalization of the notion of Generalized algebraic theory that is introduced in appendix A, however a reader familiar with Genralized algebraic theory can probably guess how it works.

In section 2.2 we review quickly some important properties of the category of models of a Genralized algebraic theory, or equivalently of the category of models of a "clan" (in the sense of Joyal), most notably their canonical weak factorization system. In section 2.3 we explain how the language defined in section 2.1 can be given an alternative more category theoretic definition that can be applied to any clan. Note that every clan can be shown to be the syntactic category of a generalized algebraic theory (and we prove more generally that in our infinitary setting any "κ-clan" is the syntactic category of a generalized κ-algebraic theory in appendix B.4, and the category theoretic definition of the language of the clan is equivalent to the syntactic definition of the language of any such Generalized algebraic theory.

This re-interpretation is the key to associate a language to any model category: Given a (weak) model category $\mathcal{M}$, We take the category $\mathcal{M}^{\text{Cof}}$ of cofibrant objects and cofibration between them, this constitute a co-clan (the opposite of a clan) and we can take the language assocaited to it. This is what we call the language of the model category $\mathcal{M}$. We review briefly the general theory of weak model category in section 3.1 and in section 3.2 we explain in details how this language of $\mathcal{M}$ actually talk about the objects of $\mathcal{M}$ and prove the first two invariances theorem mentioned above.

To give a general picture of how this language works, if $\mathcal{M}$ is our model category, each formula in the language has a "context" C, which informally can be thought of as the list of free variable that can appear in the formula as well as their types. This "context" C is concretely just a cofibrant object of $\mathcal{M}$. An interpretation of the context C into an object $X \in \mathcal{M}$ is just a map $v : C \to X$. And given ϕ a formula in context C and $v : C \to X$ a map, $\phi(v)$ can be either true or false. We write

$$M \vdash \phi(v)$$

is it is true. The first invariance theorem assert that if X is fibrant and $v : C \to X$ is homotopic to $v' : C \to X$ then $M \vdash \phi(v) \Leftrightarrow M \vdash \phi(v')$. The second invariance theorem state that if $F : X \to Y$ is a weak equivalence between fibrant object then $X \vdash \phi(v) \Leftrightarrow Y \vdash \phi(f(v))$.

For example, if $\mathcal{M}$ is the canonical of folk model structure on category, the language we obtain is essentially the language of categories we mentioned at the begining. The formula

$$\forall Z \in \mathrm{Ob}, \forall g, h \in \mathrm{Hom}(Y, Z), g \circ f = h \circ f \Rightarrow g = h$$

is a formula in context $X, Y \in \mathrm{Ob}, f \in \mathrm{Hom}(X, Y)$ which corresponds to the (cofirbant) object X which has two object (say X and Y) and a unique non-identity arrow $f : X \to Y$. A map from C to another category $\mathcal{D}$ is the choice of an arrow f in $\mathcal{D}$ and $\phi(f)$ is true if and only if f is an epimorphism. The second invariance theorem says (in this special case) that equivalence of categories preserves epimorphism, and the first invariance theorem that if f is isomorphic to another arrow then one is an epimorphism if and only if the other is.

In **??** we show how these notions pecialize to many classical model structure, and we also discuss briefly some general tool to construct this language explicitely for any model structure.

Finally, section 4 is devoted to statement and proof of the third invariance theorem, which essentially says that two Quillen equivalent model categories have the same homotopy language.

2 The first order language of a generalized algebraic theory

2.1 Syntactic approach: The first-order language of a generalized algebraic theory

In this section we give a very classical syntactical approach to the language we consider in this paper. We start from a generalized algebraic theory and we build its first-order language on top of it.

Because we want to be able to do infinitary logic, we use an infinitary version of Cartmell's notion of generalized algebraic theory, called generalized κ-algebraic theory for κ a regular cardinal, which we develop in details in Appendix A. However, this generalization is very straightforward and a reader familiar with Cartmell formalisms should be able to guess how it works and read this section directly without relying on Appendix A.

We fix two regular cardinals κ and λ, and T a generalized κ-algebraic theory, and we will define the first-order language of T denoted $\mathcal{L}^T_\lambda$ or $\mathcal{L}^T_{\lambda,\kappa}$.

More precisely, for each context Γ of T, we will have define a set $\mathcal{L}^T_\lambda(\Gamma)$ of "T-formulas in context Γ", which are essentially first-order formula with λ-small conjunction and disjunction whose free variables are the variables of the context Γ, in particular, they have less than κ-variables.

Definition 2.1.1. The sets $\mathcal{L}^T_\lambda(\Gamma)$ of T-formulas in context Γ are defined inductively using the following rules:

1. For each context Γ, the true formula $\top$ and false formula $\bot$ are in $\mathcal{L}^T_\lambda(\Gamma)$.
2. If $\Phi \in \mathcal{L}^T_\lambda(\Gamma)$ then $\neg\Phi \in \mathcal{L}^T_\lambda(\Gamma)$.
3. For each collection of formulas $\Phi_i \in \mathcal{L}^T_\lambda(\Gamma)$, indexed by a λ-small set I, the conjunction and disjunction
$$\bigvee_{i\in I} \Phi_i \qquad \bigwedge_{i\in I} \Phi_i$$
are in $\mathcal{L}^T_\lambda(\Gamma)$.
4. Given two ordinals $\gamma < \alpha < \kappa$, if $\Gamma' = \{x_\beta : \Gamma_\beta\}_{\beta<\alpha}$ is a context of length α, and $\Gamma = \{x_\beta : \Gamma_\beta\}_{\beta<\gamma}$ is the subcontext of length γ, then for any formula $\Phi \in \mathcal{L}^T_\lambda(\Gamma')$ we have formulas
$$\exists\{x_\beta : \Gamma_\beta\}_{\gamma\leqslant\beta<\alpha}\Phi \qquad \forall\{x_\beta : \Gamma_\beta\}_{\gamma\leqslant\beta<\alpha}\Phi$$
in $\mathcal{L}^T_\lambda(\Gamma)$.

Remark 2.1.2. The key point in Definition 2.1.1 is that we are not including atomic formulas other than $\top$ and $\bot$. In particular, the language *does not include any equality.* At this point the reader might be confused as to how we actually get any non-trivial formulae in this language as it might seems that applying quantifiers, conjunction or disjunction to formulas that are either $\bot$ or $\top$ will never produce any formulas that are not immediately interpreted as $\bot$ or $\top$. Or even, on how we might obtain formulas with free variables. The central idea is that free variables appear thanks to the fact we quantify over dependent types, that is types in which free variables can appears. The following examples will demonstrate this phenomena.

Example 2.1.3. Let Cat be the generalized ω-algebraic theory of categories as introduced in Example A.1.7. Then in the context $(x : \mathsf{Ob})$ we can form the formula

$$\phi(x) \coloneqq (\forall y : \mathsf{Ob},\ \exists f : \mathsf{Hom}(x, y), \top)$$

which expresses that, for any object y there is an arrow from x to y, that is, x is a weakly initial object. Indeed, $\top$ is a formula in context $(x : \mathsf{Ob}, y : \mathsf{Ob}, f : \mathsf{Hom}(x, y))$, so that $\exists f : \mathsf{Hom}(x, y), \top$ is a formula in context $(x : \mathsf{Ob}, y : \mathsf{Ob})$, and $\forall y : \mathsf{Ob}, \exists f : \mathsf{Hom}(x, y), \top$ is a formula in context $(x : \mathsf{Ob})$.

Now, the logic is still not strong enough to express many of the interesting category theoretic notions. For example, without any kind of equality predicate on morphisms there is no way to write down formulas for an initial object, or a limit. In the next example, we show how modifying the theory Cat allows to recovers equality on morphisms:

Example 2.1.4. We consider the theory $Cat_=$ obtained by adding to the theory Cat the following:

$$x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y) \vdash \mathsf{Eq}(f, g)\mathsf{Type}$$

$$x, y : \mathsf{Ob}, f : \mathsf{Hom}(x, y) \vdash r_f : \mathsf{Eq}(f, f)$$

$$x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y), a : \mathsf{Eq}(f, g) \vdash f \equiv g$$

$$x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y), a : \mathsf{Eq}(f, g) \vdash a \equiv r_f$$

One easily see that a model of $Cat_=$ is just a category, with the type $\mathsf{Eq}(f, g)$ being empty if $f \neq g$ and $\{r_f\}$ if $f = g$.

In this new theory, we can now form a formula "$f = g$" in context $(x, y : \mathsf{Ob}, f, g : \mathsf{Hom}(x, y))$ which is defined as

$$(f = g) \coloneqq (\exists v : \mathsf{Eq}(f, g), \top).$$

So in $\mathcal{L}_\omega^{Cat_=}$ we can form formula involving equality between parallel morphisms and hence we recover the "language of categories" as studied in [Bla78] and [Fre76]. For example, we can form the formula "x is initial" in context $(x : \mathsf{Ob})$ as

$$\mathsf{isInitial}(x) \coloneqq \forall y : \mathsf{Ob}, (\exists f : \mathsf{Hom}(x, y)) \wedge (\forall f, g : \mathsf{Hom}(x, y), f = g)$$

Construction 2.1.5. If $f : \Delta \to \Gamma$ is a context morphism, and if $\phi \in \mathcal{L}_\lambda^T(\Gamma)$ then we can define its pullback $f^*\phi$, which is obtained by substituting the free variables of ϕ by the components of f. Formally, this is defined inductively as:

1. $f^*\top = \top$ and $f^*\bot = \bot$.

2. $f^*(\neg\Phi) = \neg f^*\Phi$

3. $f^*\left(\bigvee_{i\in I}\Phi_i\right) = \bigvee_{i\in I} f^*\Phi_i$ and $f^*\left(\bigwedge_{i\in I}\Phi_i\right) = \bigwedge_{i\in I} f^*\Phi_i$.

4. Finally, if $\Gamma' = (\Gamma, x_1 \in X_1, \ldots, x_\alpha \in X_\alpha)$ then

   $$f^*\left(\exists(x_1 \in X_1, \ldots, x_\alpha \in X_\alpha)\Phi\right) = \exists(x_1 \in f^*X_1, \ldots, x_\alpha \in f^*X_\alpha)f^*\Phi$$

   $$f^*\left(\forall(x_1 \in X_1, \ldots, x_\alpha \in X_\alpha)\Phi\right) = \forall(x_1 \in f^*X_1, \ldots, x_\alpha \in f^*X_\alpha)f^*\Phi$$

   Where f^*X_i denotes the pullback of types, that is the context $(\Delta, f^*X_1, \ldots, f^*X_\alpha)$ is the canonical pullback of the generalised display map:

   $$\begin{tikzcd}
(\Delta, f^*X_1, \ldots, f^*X_\alpha) \arrow[r] \arrow[d, two heads] & (\Gamma, X_1, \ldots, X_\alpha) \arrow[d, two heads] \\
\Delta \arrow[r] & \Gamma.
\end{tikzcd}$$

Definition 2.1.6. We define the relation $\vdash_\Gamma$ on $\mathcal{L}^T_\lambda(\Gamma)$ as the smallest familly of relations such that:

1. For each context Γ, $\vdash_\Gamma$ is a transitive and reflexive relation on $\mathcal{L}^T_\lambda(\Gamma)$.

2. $\forall\Phi \in \mathcal{L}^T_\lambda(\Gamma)$, $\Phi \vdash_\Gamma \top$ and $\bot \vdash_\Gamma \Phi$.

3. $\forall\Phi \in \mathcal{L}^T_\lambda(\Gamma), \Phi \wedge \neg\Phi \vdash \bot$ and $\top \vdash \Phi \vee \neg\Phi$.

4. For any λ-small familly $(\Phi_i)_{i\in I} \in \mathcal{L}^T_\lambda(\Gamma)$ we have

   $$\bigvee_{i\in I}\Phi_i \vdash_\Gamma \Psi \Leftrightarrow \forall i, (\Phi_i \vdash_\Gamma \Psi)$$

   $$\Psi \vdash \bigwedge_{i\in I}\Phi_i \Leftrightarrow \forall i, (\Psi \vdash_\Gamma \Phi_i)$$

5. For $\Gamma' = \left(\Gamma, \left\{x_\beta : \Gamma'_\beta\right\}_{\gamma\leqslant\beta<\alpha}\right)$ a context extension, with $p : \Gamma' \twoheadrightarrow \Gamma$ the corresponding generalized display map, $\Psi \in \mathcal{L}^T_\lambda(\Gamma')$ and $\Phi \in \mathcal{L}^T_\lambda(\Gamma)$ we have

   $$\exists\{x_\beta : \Gamma_\beta\}_{\gamma\leqslant\beta<\alpha}\Psi \vdash_\Gamma \Phi \Leftrightarrow \Psi \vdash_{\Gamma'} p^*\Phi$$

   $$\Phi \vdash_\Gamma \forall\{x_\beta : \Gamma_\beta\}_{\gamma\leqslant\beta<\alpha}\Psi \Leftrightarrow p^*\Phi \vdash_{\Gamma'} \Psi$$

While we have not included the following in the definition we can show that:

Proposition 2.1.7. If $f : \Delta \to \Gamma$ is a context morphism in T, and $\Phi \vdash_\Gamma \Psi$ then $f^*\Phi \vdash_\Delta f^*\Psi$.

Proof. We can show that if we define the relations $\Phi \vdash'_\Gamma \Delta$ to be "For all $f : \Delta \to \Gamma$, we have $f^*\Phi \vdash_\Delta f^*\Psi$, then it satisfies all the conditions from definition 2.1.6. Wich shows that $\vdash \Rightarrow \vdash'$ and hence conclude the proof. $\square$

Construction 2.1.8. Finally, given a model X of our theory T, Γ a context, $x \in X(\Gamma)$ and $\Phi \in \mathcal{L}^T_\lambda(\Gamma)$, we can interpret $\Phi(x)$ as a proposition (I.e. true or false) in the ovious way by substituing the components of x into ϕ and interpreting all the logic symbols in the usual way. Formally we have:

1. If $\Phi = \top$, then $\Phi(x)$ is true and if $\Phi = \bot$ then $\Phi(x)$ is false.
2. If $\Phi = \neg\Psi$, then $\Phi(x)$ is true if and only if $\Psi(x)$ is false.
3. If $\Phi = \bigvee \Phi_i$, then $\Phi(x)$ is true if and only if $\Phi_i(x)$ is true for some x.
4. If $\Phi = \bigwedge \Phi_i$, then $\Phi(x)$ is true if and only $\Phi_i(x)$ is true for all i.
5. $\Phi = \exists\{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha}\Psi$ for $\Gamma' = \left(\Gamma, \left\{x_\beta : \Gamma'_\beta\right\}_{\gamma \leqslant \beta < \alpha}\right)$ a context extension, with $p : \Gamma' \twoheadrightarrow \Gamma$ the corresponding generalized display map, then $\Phi(x)$ is true if there exists a $y \in X(\Gamma')$ such that $p(y) = x$ and $\Psi(y)$.
6. If $\Phi = \forall\{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha}\Psi$ in the same situation as above, then $\Phi(x)$ is true if for any $y \in X(\Gamma')$ such that $p(y) = x$ we have $\Psi(y)$.

The following lemma is immediate by induction, the proof is left to the reader.

Lemma 2.1.9. Let X be a model of a κ-generalized algebraic theory T.

1. For $\Phi, \Psi \in \mathcal{L}^T_\lambda(\Gamma)$ and $x \in X(\Gamma)$, then if $\Psi \vdash_\Gamma \Phi$ and $\Psi(x)$ then $\Phi(x)$.
2. If $f : \Gamma \to \Delta$ is any context morphism and $\Phi = f^*\Psi$ and $x \in X(\Gamma)$ then $\Phi(x) \Leftrightarrow \Psi(f(x))$.

Definition 2.1.10. We write $\Psi \dashv\vdash_\Gamma \Phi$ to mean both $\Psi \vdash_\Gamma \Phi$ and $\Phi \vdash_\Gamma \Psi$. We denote

$$\mathbb{L}^T_\lambda(\Gamma) \coloneqq \mathcal{L}^T_\lambda(\Gamma)/(\dashv\vdash_\Gamma)$$

the quotient.

Note that $(\dashv\vdash_\Gamma)$ is indeed an equivalence relation as $\vdash_\Gamma$ is transitive and reflexive.

Remark 2.1.11. Note that it follows from proposition 2.1.7 that for a context morphism $f : \Delta \to \Gamma$ the f^* operation from $\mathcal{L}^T_\lambda(\Gamma) \to \mathcal{L}^T_\lambda(\Delta)$ is compatible to the relation $\dashv\vdash$, and hence it descent to an operation

$$f^* : \mathbb{L}^T_\lambda(\Gamma) \to \mathbb{L}^T_\lambda(\Delta)$$

It is also easy to see from definition 2.1.6 that the relation $\vdash$ is compatible to all the logical operations on $\mathcal{L}^T_\lambda$, that is $\neg, \bigvee, \bigwedge, \exists, \forall$ in the sense that for example if $\Phi_i \vdash \Psi_i$ for all $i \in I$ then $\bigvee_{i\in I} \Phi_i \vdash \bigvee_{i\in I} \Psi_i$ and hence they all descent into operation on L^T_λ.

Construction 2.1.12. At the begining of the section, we have briefly called than language $\mathcal{L}^T_{\lambda,\kappa}$ before dropping the κ from the notation as it can be read of from the fact that T is a generalized κ-algebraic theory. However, we can consider $\mathcal{L}^T_{\lambda,\kappa'}$ for any $\kappa' \geqslant \kappa$. Indeed, given T a generalized κ-algebraic theory we can define a generalized κ'-algebraic theory $T_{\kappa'}$ by taking a set of axioms for T and seeing them as axioms for a generalized κ'-algebraic theory. A model of $T_{\kappa'}$ is the same as a model of T. We then define

$$\mathcal{L}^T_{\lambda,\kappa'} \coloneqq \mathcal{L}^{T_{\kappa'}}_{\lambda,\kappa'} = \mathcal{L}^{T_{\kappa'}}_{\lambda},$$

as well as its quotient

$$\mathbb{L}^T_{\lambda,\kappa'} \coloneqq \mathbb{L}^{T_{\kappa'}}_{\lambda,\kappa'} = \mathbb{L}^{T_{\kappa'}}_{\lambda},$$

Example 2.1.13. Let Σ be a signature in the sense of traditional model theory. Then we can considere the generalized algebraic theory $T_{\Sigma,=}$, which has one sort in empty context of each type symbole X, each of these type has an equality predicate as the one constructed in 2.1.4, a term for each function symbol, and for each relation symbole $R \subset X_1, \ldots, X_n$ an additional type axiom

$$x_1 : X_1, \ldots, x_n : X_n \vdash R(x_1, \ldots, x_n)\mathsf{Type}$$

with the additional axiom

$$x_1 : X_1, \ldots, x_n : X_n, t_1, t_2 : R(x_1, \ldots, x_n) \vdash t_1 = t_2$$

Models of this theory are exactly Σ structure, and elements of $\mathbb{L}^{T_{\Sigma,=}}_{\omega,\omega}$ are essentially the same as usual first order formula in this signature. Elements of $\mathbb{L}^{T_{\Sigma,=}}_{\lambda,\kappa}$ corresponds to infinitary first-order formulas using λ-small conjunction and disjunction and where $\exists$ and $\forall$ quantifier can quantify over κ-small set of variables.

2.2 Models of Clans and their weak factorization system

We recall that:

Definition 2.2.1. A *clan*, or ω*-clan*, is a category $\mathcal{C}$ endowed with a class of maps called *fibrations* such that

- $\mathcal{C}$ has an initial object 1, and for every $X \in \mathcal{C}$ the unique map $X \twoheadrightarrow 1$ is a fibration.
- Isomorphisms are fibrations, the composite of two fibrations is a fibrations.
- Pullback of fibrations exists and are fibrations.

For κ a regular cardinal, A κ-clan is a clan which further satisfies:

- For any ordinal $\lambda < \kappa$, if $A_\bullet : \lambda^{\mathtt{op}} \to \mathcal{C}$ is a diagram in which all the transition maps $A_\beta \twoheadrightarrow A_\alpha$ for $\alpha < \beta$ are fibrations, then the limits

  $$\mathtt{Lim}_{\alpha<\lambda} A_\alpha$$

  exists, and all the projection maps $\pi_\beta : \mathtt{Lim}_{\alpha<\lambda} A_\alpha \twoheadrightarrow A_\beta$ are fibrations. We refer to these as *limits of κ-small chains of fibrations*.

A *morphism of clans* is a functor that send fibrations to fibrations, preserve the initial object and pullback of fibrations. A *morphism of κ-clans* is in addition required to preserves the limits of κ-small chains of fibrations.

Fibrations will be denoted with a double-headed arrow $\twoheadrightarrow$.

Remark 2.2.2. We define *coclans* and *κ-coclans* dually, as the category $\mathcal{C}$ endowed with a class of *cofibrations* whose opposite category are clans and κ-clans.

Definition 2.2.3. If $\mathcal{C}$ is a κ-clan, a models X of $\mathcal{C}$ is a functor $X : \mathcal{C} \to \mathbf{Set}$ that preserves the terminal object, pullback of fibrations and limits of κ-small chains of fibrations. The category $\mathrm{Mod}(\mathcal{C})$ of models of $\mathcal{C}$ is defined as a full subcategory of the category $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ of all functors.

Remark 2.2.4. A key observation is of course that if T is a generalized κ-algebraic theory and $\mathcal{C}_T$ is its contextual category, then $\mathcal{C}_T$ can be seen as a κ-clan where fibration are the map that are isomorphic to generalized display maps. Moreover, the models of T are exactly the models of this clans $\mathrm{Mod}(T) = \mathrm{Mod}(\mathcal{C}_T)$, so that models of generalized algebraic theories are a special cases of models of clans.

Also note that:

- By Corollary B.4.8 every κ-clan $\mathcal{C}$ is equivalent to a κ-contextual category.
- By Theorem B.3.35 every κ-contextual category is isomorphic to the contextual category $\mathcal{C}_T$ of a generalized κ-algebraic theory.

So combining the two, every κ-clan is equivalent to one of the form $\mathcal{C}_T$ for T a generalized κ-algebraic theory. Hence there is no fundamental difference between the models of a clans and the models of a generalized κ-algebraic theory.

Construction 2.2.5. Let $\mathcal{C}$ be a κ-clan and $よ_\bullet : \mathcal{C}^{\mathsf{op}} \to Fun(\mathcal{C}, \mathbf{Set})$ be the contravariant Yoneda embedding. Note that for every $A \in \mathcal{C}^{\mathsf{op}}$ the functor $よ_A : \mathcal{C} \to \mathbf{Set}$ preserves all limits, so in particular it is a model. Hence we have a Yoneda embedding $よ_\bullet : \mathcal{C}^{\mathsf{op}} \to \mathrm{Mod}(\mathcal{C})$. Note that by the Yoneda lemma, we have a natural isomorphism

$$\mathrm{Hom}(よ_A, X) \simeq X(A)$$

for $X \in \mathrm{Mod}(\mathcal{C})$ and $A \in \mathcal{C}$.

Remark 2.2.6. The category of models of a κ-clan $\mathcal{C}$ is caracterized by preservation of certain κ-small limits. This implies, by general category theoretic results that, for a small κ-clan $\mathcal{C}$:

- The category $\mathrm{Mod}(\mathcal{C})$ is locally κ-presentable.
- The representable models $よ_A$ for $A \in \mathcal{C}$ are κ-presentable objetcs.

Indeed, the category $\mathrm{Mod}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ is closed under κ-filtered colimits because κ-filtered colimits commutes to κ-small limits, which, because of the isomorphism $\mathrm{Hom}(よ_A, X) \simeq X(A)$ implies that the object $よ_A$ are κ-presentable in $\mathrm{Mod}(\mathcal{C})$. Moreover, as every $X \in \mathrm{Mod}(\mathcal{C})$ can be written as $X = \mathtt{Colim}_{よ_A \to X} よ_A$ this imply that the category $\mathrm{Mod}(\mathcal{C})$ is locally κ-accessible, and hence locally κ-presentable as it is also closed under small limits.

Remark 2.2.7. More generally, any κ-presentable category $\mathcal{C}$ is equivalent to the cateogry of functors $\mathcal{C}_\kappa^{\mathsf{op}} \to \mathbf{Set}$ that preserves κ-small limits, where $\mathcal{C}$ is the (essentially small) category of κ-presentable objects of $\mathcal{C}$. In particular, every κ-presentable category is the category of models of a κ-clan: One can take the category $\mathcal{C}_\kappa^{\mathsf{op}}$, with all maps being fibrations. However, the category $\mathrm{Mod}(\mathcal{C})$ of models of a κ-clans comes with an additional structure that is more specific:

Definition 2.2.8. Given a κ-clan $\mathcal{C}$, we consider the weak factorization on the category $\mathrm{Mod}(\mathcal{C})$ which is cofibrantly generated by the maps

$$よ_A \hookrightarrow よ_B$$

where $B \twoheadrightarrow A$ is a fibration in $\mathcal{C}$. The element of the left class will be called *cofibrations* and the element of right class *trivial fibrations*.

Remark 2.2.9. In the special case $\kappa = \omega$, this weak factorization was defined in [Hen16, Definition 2.4.2] and extensively studied in [Fre23]. In particular Jonas Frey gave in [Fre23] a complete characterization of which pairs of a category and a weak factorization can be obtained this way from an ω-clan. The methods used by Frey can be extended to the κ-case to obtain a similar characterization. Frey also shows that (in the $\kappa = \omega$ case, the ω-presentable cofibrant object in $\mathrm{Mod}(\mathcal{C})$ are exactly the retracts of representable models, and the same proof generalizes to the κ-case to show that when $\mathcal{C}$ is a κ-clan the κ-presentable cofibrant objects are exactly the retracts of representable. While we will not directly use these results, but we might occasionaly make some side comments that assumes these results hold for arbitrary κ.

Lemma 2.2.10. Given $\mathcal{C}$ a clan, a morphisms $f : M \to N$ of $\mathcal{C}$-models is a trivial fibration if and only if for every fibration $p : X \twoheadrightarrow Y$ in $\mathcal{C}$, the naturality square:

$$\begin{tikzcd}
M(X) \arrow[r] \arrow[d] & M(Y) \arrow[d] \\
N(X) \arrow[r] & N(Y)
\end{tikzcd}$$

is a *weak pullback square*, that is if the induced map $M(X) \to N(X) \times_{N(Y)} M(Y)$ is a surjection.

Proof. By the Yoneda lemma, there is a one-to-one correspondence between elements of $M(X)$ and morphisms of models $よ_X \to M$, such that the map $M(X) \to M(Y)$ is obtained as the composite $よ_Y \to よ_X \to M$ and the map $M(X) \to N(X)$ as the composite $よ_X \to M \to N$. An element of $N(X) \times_{N(Y)} M(Y)$ is hence the data of maps $よ_X \to N$ and $よ_Y \to M$ such that the composite $よ_Y \to M \to N$ and $よ_Y \to よ_X \to N$ coincide, that is, it is exactly a commutative square:

$$\begin{tikzcd} よ_Y \arrow[r] \arrow[d, "よ_p"'] & M \arrow[d, "f"] \\ よ_X \arrow[r] & N \end{tikzcd}$$

An element of $M(X)$ whose image in $N(X) \times_{N(Y)} M(Y)$ is the square above is then exactly a dotted diagonal filling:

$$\begin{tikzcd} よ_Y \arrow[r] \arrow[d, "よ_p"'] & M \arrow[d, "f"] \\ よ_X \arrow[r] \arrow[ur, dotted] & N \end{tikzcd}$$

Hence the surjectivity of this map is equivalent to the fact that f has the right lifting property against $よ_Y \to よ_X$ for all fibrations $X \twoheadrightarrow Y$, which concludes the proof. □

2.3 The Category theoretic approach: The first order language of a κ-clans

In this section we present another equivalent approach to the definition of the language, which is more categorical in spirit, and strongly inspired from Lawvere's theory of Hyperdoctrines ([Law69], [Law70]). This approach, while much more abstract has several advantages over the syntactic approach, but mainly it allows to works directly with the more general notion of a clan (see appendix B.4), instead of a generalized κ-algebraic theory. This will be usefull latter to define the language of a model category without having to build explicitly a syntax for it.

As before, we fix λ a regular cardinal. A λ-boolean algebra is a boolean algebra which admits join (and hence intersection) of λ-small families. We denote by $\mathbf{Bool}_\lambda$ the category whose objects are λ-boolean algebras and whose morphisms are boolean algebra morphisms preserving λ small join (and hence intersection).

We introduce the notion of λ-boolean algebra over a clan $\mathcal{C}$ which can be thought of as an axiomatization of the structure that the $\mathbb{L}^T_\lambda$ from section 2.1 have over the contextual category of T.

Definition 2.3.1. Given $\mathcal{C}$ a clan and λ a regular cardinal, a λ-boolean algebra over $\mathcal{C}$ is a functor

$$\mathcal{B} : \mathcal{C}^{op} \to \mathbf{Bool}_\lambda$$

such that

1. For each fibration $\pi : Z \twoheadrightarrow X$ in $\mathcal{C}$, $\pi^* : \mathcal{B}(X) \to \mathcal{B}(Z)$ has a left adjoint:
$$\exists_\pi : \mathcal{B}(Z) \leftrightarrows \mathcal{B}(X) : \pi^*.$$

2. The Beck-Chevalley condition holds for each pullback square along a fibration, that is, given any pullback square:
$$\begin{tikzcd} Z' \arrow[r, "f'"] \arrow[d, two heads, "\pi'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, two heads, "\pi"] \\ X' \arrow[r, "f"'] & X \end{tikzcd}$$
we have $f^* \exists_\pi = \exists_{\pi'} f'^*$.

Morphisms of λ-boolean algebras over $\mathcal{C}$ are natural transformation that commutes with the $\exists_\pi$. We call weak morphisms the general natural transformations.

Remark 2.3.2. If $\mathcal{B}$ a λ-boolean algebra over $\mathcal{C}$, then for each $X \in \mathcal{C}$, the negation $\neg : \mathcal{B}(X) \to \mathcal{B}(X)^{op}$ is a contravariant equivalence, so if $\pi : Z \to X$ is a fibration, then the map $\pi^* : \mathcal{B}(X) \to \mathcal{B}(Z)$ also has a right adjoint defined by:

$$\forall_\pi(\phi) := \neg(\exists_\pi \neg \phi)$$

We immediately have from this definition the other Beck-Chevalley condition $f^*(\forall_\pi) = \forall_\pi f^*$ and the fact that morphism of boolean algebra over $\mathcal{C}$ are also compatible to $\forall_\pi$, simply because both the f^* and morphism are compatible to both $\exists_\pi$ and the negation.

Remark 2.3.3. Definition 2.3.1 will in practice be applied to $\mathcal{C}$ a κ-clan (and not just a clan), the only reason it is stated like that is because the definition actually does not explicitely involves κ. This is related to the fact that the dependencies in κ of the language defined in the previous subsection is only through the choice of which context can our variables (including bound variables) be taken from: taking a larger κ mean we can quantify over more variable at the same time. Similarly, the dependency on κ is hidden in the dependency on $\mathcal{C}$, as $\mathcal{C}$ is playing the role of the category of κ-contexts.

Let start with our main example of such boolean algebra over a clan, which is the motivating example for the notion:

Theorem 2.3.4. If T is a generalized κ-algebraic theory, and $\mathcal{C}_T$ is the corresponding κ-contextual category, seen as a clan. Then the construction $X \mapsto \mathbb{L}^T_\lambda(X)$ from definition 2.1.10 (see also definition 2.1.1 and 2.1.6) is a λ-boolean algebra over $\mathcal{C}_T$. In fact, it is an initial object in the category of λ-boolean algebra over $\mathcal{C}_T$.

Proof. We first check that $\mathcal{L}^T_\lambda$ is a λ-boolean algebra over $\mathcal{C}_T$. We have mentioned in remark 2.1.11 that all the logical operation $\vee, \wedge, \neg, \exists$ and so one are compatible to the equivalence relation $\dashv\vdash$, so they all induce operation on the quotient $\mathbb{L}^T_\lambda$. The first four points of definition 2.1.6 immediately shows that each $\mathbb{L}^T_\lambda(X)$ is a boolean algebra, whose order relation is given by $\vdash$, and with λ-small union. By, construction 2.1.5 the map $f^* : \mathcal{L}^T_\lambda(X) \to \mathcal{L}^T_\lambda(Y)$ is compatible to all the logical operation, so it gives a morphism of boolean algebra $\mathbb{L}^T_\lambda(X) \to \mathbb{L}^T_\lambda(Y)$. It is also immediate to check by induction that $(g \circ f)^*(\phi) = f^* g^*(\phi)$ and $id^*(\phi) = \phi$, so that this form a functor $\mathcal{C}_T \to$ $\mathbf{Bool}_\lambda$. Next, the last two conditions of definition 2.1.6 shows that $\exists$ and $\forall$ defines left and right adjoint to π^*. Finally, the Beck-Chevalley condition follows from how f^* is defined on formulas starting with a $\exists$ quantifier:

$$f^* \left(\exists \{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} \Phi\right) = \exists \{x_\beta : f^* \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} f^* \Phi$$

which (after passing to the quotient $\mathcal{L} \to \mathbb{L}$) exactly says that $f^* \exists_\pi = \exists \pi f^*$ where π is the generalized display map corresponding to forgetting the variables $\{x_\beta\}_{\gamma \leqslant \beta < \alpha} \in X_\alpha$).

We now check that it is an initial object in the category of λ-boolean algebra over $\mathcal{C}_T$. Let $\mathcal{B}$ be any λ-boolean algebra over $\mathcal{C}$. Any morphism $v : \mathbb{L}^T_\lambda \to \mathcal{B}$ has to satisfies:

1. $v(\bot) = \bot_\mathcal{B}$ and $v(\top) = \top_\mathcal{B}$.
2. $v(\neg \Phi) = \neg v(\Phi)$.
3. $v(\bigvee_{i \in I} \Phi_i) = \bigvee_{i \in I} v(\Phi_i)$ and $v(\bigwedge_{i \in I} \Phi_i) = \bigwedge_{i \in I} v(\Phi_i)$.
4. $$v(\exists \{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} \Phi) = \exists \{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} v(\Phi)$$
   and
   $$v(\forall \{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} \Phi) = \forall \{x_\beta : \Gamma_\beta\}_{\gamma \leqslant \beta < \alpha} v(\Phi).$$

These form an inductive definition for a function $\mathcal{L}^T_\lambda \to \mathcal{B}$. So there is a unique such function $v : \mathcal{L}^T_\lambda \to \mathcal{B}$. To conclude we only need to check that this function v descent to a function $\mathbb{L}^T_\lambda \to \mathcal{B}$ and is a morphism of λ-boolean algebra over $\mathcal{C}$. But this is rather immediate: We first observe, by induction over definition 2.1.6, that if $\Phi \vdash \Psi$ then $v(\Phi) \leqslant v(\Psi)$. this implies that if $\Phi \dashv\vdash \Psi$ then $v(\Phi) = v(\Psi)$, so v does defines a function $\mathbb{L}^T_\Lambda \to \mathcal{B}$. The naturality equation

$$v(f^*(\Phi)) = f^*(v(\Phi))$$

can be proved by induction on the forumla Φ, and the compatibility of v with all the boolean algebra operations and the quantifier follows immediately from the definition of v. $\square$

Proposition 2.3.5. Given any (small) clan $\mathcal{C}$ and λ a regular cardinal, there is an initial λ-boolean algebra over $\mathcal{C}$, which we denote by $\mathbb{L}^{\mathcal{C}}_\lambda$.

Note that by theorem 2.3.4, if T is a generalized κ-algebraic theory, with $\mathcal{C}_T$ its κ-contextual category them

$$\mathbb{L}^{\mathcal{C}_T}_\lambda = \mathbb{L}^T_\lambda$$

This provides a way to define (or at least to characterize) the first order language of any clan, without having to explicitely give a syntactic description of the clan.

Proof. We can either remark that the λ-boolean algebras over $\mathcal{C}$ are (by their definition) the models of a multisorted λ-algebraic theory (with one sort for each object $c \in \mathcal{C}$) and hence there is an initial objects by usual results on algebraic theories. Alternatively we can use (see appendix B.4) that every clan is equivalent to the contextual category of a generalized algebraic theory and use theorem 2.3.4 to conclude. $\square$

Next, we mention a few more examples:

Example 2.3.6.

1. Let **Set** be the category of sets, considered as a clan where every arrow is a fibration. The contravariant power-set functor $\mathcal{P} : \mathbf{Set}^{op} \to \mathbf{Bool}_\lambda$ is a λ-Boolean algebra over **Set**. The Beck-Chevalley condition follow from lemma 2.3.7 below.

2. Given $F : \mathcal{C} \to \mathcal{D}$ a morphism of clans, if $\mathcal{B}$ is a λ-boolean algebra over $\mathcal{D}$, then $F^*\mathcal{B}$ defined by $F^*\mathcal{B}(\Gamma) = \mathcal{B}(F(\Gamma))$ is a λ-boolean algebra over $\mathcal{C}$.

3. Combining the two observations above, given any model M of a clan $\mathcal{C}$, that is a morphism of clans $M : \mathcal{C} \to \mathbf{Set}$, one has a boolean algebra $\mathcal{P}(M)$ over $\mathcal{C}$ given by pulling back example 1. along the morphism $M : \mathcal{C} \to \mathbf{Set}$. More explicitly:

$$\begin{array}{rccc} \mathcal{P}(M): & \mathcal{C}^{op} & \to & \mathbf{Set} \\ & \Gamma & \mapsto & \mathcal{P}(M(\Gamma)) \end{array}$$

Lemma 2.3.7. Given a square of sets:

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "h"] \\ Y \arrow[r, "k"] & Z \end{tikzcd}$$

Then the power set functor satisfies the Beck-Chevalley condition on this square, i.e. $k^*\exists_h = \exists_g f^*$ as maps $\mathcal{P}(X) \to \mathcal{P}(Y)$ if and only if the square is a weak pullback square, i.e. if and only if the cartesian gap map $W \to Y \times_Z X$ is surjective.

Proof. Given a subset $P \subset X$ one has:

$$k^* h_! P = \{y \in Y | k(y) = h(p) \text{ for some } p \in P\}$$

$$g_! f^* P = \{g(w) | f(w) \in P\}$$

Surjectivity of the map $W \to Y \times_Z X$ gives a canonical way to make any element of $k^* h_! P$ into an element of $g_! f^* P$, and conversely, applying the equality to $P = \{p\}$ produces the surjectivity of $W \to Y \times_Z X$. $\square$

So in this new setting with just a clan $\mathcal{C}$, one can still define the set of formulas $\mathbb{L}^{\mathcal{C}}_{\Lambda}$ as the initial λ-boolean algebra over $\mathcal{C}$. We now explain what it means for formulas defined this way to be "true" or "false", given a model and an interpretation of its variables in the model.

Construction 2.3.8. Given a clan $\mathcal{C}$ and a model of $M : \mathcal{C} \to \mathbf{Set}$, we have as explained in example 2.3.6 a λ-boolean algebra over $\mathcal{C}$ defined by $c \mapsto \mathcal{P}(M(c))$. So By initiality of the κ-boolean algebra $\mathbb{L}^{\mathcal{C}}_{\lambda}$ there exists a unique morphism of λ-boolean algebras over $\mathcal{C}$:

$$| - |_M : \mathbb{L}^{\mathcal{C}}_{\lambda} \to \mathcal{P}(M).$$

This morphism associates to each formula ϕ in context Γ, a subset $|\phi|_M \subseteq M(\Gamma)$. An element $x \in M(\Gamma)$ is said to satisfy ϕ if $x \in |\phi|_M$, with some abuse of notation we say tthat "$\phi(x)$ is true" in this case. We also write

$$M \vdash \phi(x)$$

when we want to insist on which model we are talking about.

When Γ is the terminal object of $\mathcal{C}$, terminal object i.e. ϕ is a closed formula, then $M(\Gamma) = \{*\}$. Therefore, $\mathcal{P}(M(\Gamma)) = \{\bot, \top\}$ so that $|\phi|_M$ is simply a proposition. One then says that M satisfies ϕ, and we write $M \vdash \phi$.

Lemma 2.3.9. When $\mathcal{C} = \mathcal{C}_T$ is the κ-contextual category of a κ-generalized algebraic theory, them through the identification $\mathbb{L}^T_\lambda = \mathbb{L}^{\mathcal{C}}_\lambda$, the two definition of validity of a formula on elements of a model given by construction 2.1.8 and construction 2.3.8 are equivalent.

Proof. Defining the validity of formulas as in construction 2.3.8 it is immediate to see that the all the explicit conditions of the inductive definition given in construction 2.1.8, simply because the map $\mathbb{L}^{\mathcal{C}}_\lambda \to \mathcal{P}(M)$ is a morphism of λ-boolean algebra. Hence it imediately follows by induction on formulas that the two definitions are equivalents. $\square$

Construction 2.3.10. Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of clans. And let $\mathbb{L}^{\mathcal{C}}_\lambda$ and $\mathbb{L}^{\mathcal{D}}_\lambda$ their respective initial λ-boolean algebras. From the fact that $\mathbb{L}^{\mathcal{C}}_\lambda$ is initial there is a morphisms of λ-boolean algebras

$$\alpha^F : \mathbb{L}^{\mathcal{C}}_\lambda \to F^* \left(\mathbb{L}^{\mathcal{D}}_\lambda\right)$$

For any $\Gamma \in \mathcal{C}$ and any formula $\Phi \in \mathbb{L}^{\mathcal{C}}_\lambda(\Gamma)$ we denote $F(\Phi) \coloneqq \alpha^F_\Gamma(\Phi)$ which is a formula in context $F(\Gamma)$ i.e an element of $\mathbb{L}^{\mathcal{D}}_\lambda(F(\Gamma))$. The following is immediate from the definition above:

Proposition 2.3.11. Let $M : \mathcal{D} \to \mathbf{Set}$, $\Phi \in \mathbb{L}^{\mathcal{C}}_\lambda(\Gamma)$ a formula in context Γ and $x \in M(F(\Gamma))$. Then, $M \vdash \alpha_F(\Phi)(x)$ if and only if $F^*M \vdash \Phi(x)$.

Finally, we finish this section by showing the key property of invariance of formula along trivial fibrations. Stronger invariance property will be established in the next section assuming we are working with a model category, but this first invariance property is purely algebraic. This is also the key observation in Makkai FOLDS [Mak95] and it is directly inspired from it.

We start with the following observation: let $\mathcal{C}$ be a clan and $f : M \to N$ a morphisms of two $\mathcal{C}$-models, then we have an obvious map $f^* : P(N) \to P(M)$ which sends a subset $A \subset N(c)$ for $c \in \mathcal{C}$ to

$$f_c^{-1}(A) \subset M(c)$$

this map is easily seen to be a *weak* morphism of boolean algebra over $\mathcal{C}$. That is it is compatible to the boolean algebra operations and the ordinary contravariant functoriality. But it does not have to be compatible with the covariant functoriality $\exists_\pi$ along fibrations. However, one has:

Lemma 2.3.12. Let $\mathcal{C}$ be a clan and let $f : M \to N$ be a morphism between two $\mathcal{C}$-models. Then f is a trivial fibration if and only if $f^* : P(N) \to P(M)$ is a morphism of λ-boolean algebras.

Proof. We only need to show that for every fibration $p : X \to Y$ the following square

$$\begin{tikzcd} P(N(X)) \arrow[r, "f^*_X"] \arrow[d, "\exists"] & P(M(X)) \arrow[d, "\exists"] \\ P(N(Y)) \arrow[r, "f^*_Y"] & P(M(Y)). \end{tikzcd}$$

commutes. From Lemma 2.3.7 this is equivalent to say that the dotted map in

$$\begin{tikzcd} M(X) \arrow[rrd, bend left, "f_X"] \arrow[rd, dashed] \arrow[rdd, bend right, "\pi_*"'] & & \\ & P \arrow[r] \arrow[d] & N(X) \arrow[d, "\pi_*"] \\ & M(Y) \arrow[r, "f_Y"] & N(Y) \end{tikzcd}$$

is surjective. But this is exaclty the characterization of trivial fibrations given in lemma 2.2.10. $\square$

This allows to deduce the key result of invariance of formula along trivial fibrations of models: basically the validity of formula is preserved by trivial fibrations of models:

Corollary 2.3.13. Let $\mathcal{C}$ be a clans and let $f : M \twoheadrightarrow N$ be a trivial fibration between two $\mathcal{C}$-models. For $c \in \mathcal{C}$, let $x \in M(c)$ be an element of M and $\phi \in \mathbb{L}^{\mathcal{C}}_{\lambda}$ any formula. Then

$$M \vdash \phi(x) \Leftrightarrow N \vdash \phi(f(x))$$

Proof. As $f : M \to N$ is a trivial fibration, it follows from that the map $f^* : \mathcal{P}(N) \to \mathcal{P}(M)$ is a morphisnm of boolean algebra over $\mathcal{C}$. Hence, by initiality of $\mathbb{L}^{\mathcal{C}}$, the unique morphism $|\_|_M : \mathbb{L}^{\mathcal{C}} \to \mathcal{P}(M)$ is obtained as a composite $\mathbb{L}^{\mathcal{C}} \xrightarrow{|\vdash|_N} \mathcal{P}(M) \xrightarrow{f^*} \mathcal{P}(N)$. By definition, $M \vdash \phi(x)$ means that $x \in |\phi|_M$ while $N \vdash \phi(f(x))$ means that $x \in f^*|\phi|_N$, hence the result immediately follows. $\square$

3 The language of a model category

3.1 Weak model categories

The most general setting in which we will show good homotopy theoretic properties of the language introduced in section 2 is for the weak model categories introduced in [Hen20], which we will briefly recall here. In practice this extra-generality compared to Quillen model structure is not extremely useful - all the examples we will consider in **??** are Quillen model structures, so it would not be unreasonable to skip the present subsection. There are two reasons we need weak model categories:

- A key construction toward the proof of the third invariance theorem in section 4 is in general only a weak model structure, and we need to use its language as an intermediate tool.
- Future applications to left and right semi-model structure: actual weak model structure that are not left or right semi-model structures are fairly uncommon, but the weak model categories include both left and right semi-model structure at the same time, which are considerably more common.

Definition 3.1.1. A *weak model category* is a category $\mathcal{M}$ with three classes of maps, *cofibrations*, *fibrations* and *weak equivalences* satisfying the following conditions:

1. $\mathcal{M}$ has an initial object 0 and a terminal object 1, the identity of 0 is a cofibration, the identity of 1 is a fibration.
2. Given a solid diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"] & B \arrow[d, tail, dotted, "j"] \\ C \arrow[r, dotted] & D \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

   Where i is a cofibration and A and B are cofibrant, then the pushout j exists and is a cofibration.
3. The dual of condition 2 holds for fibrations between fibrant objects.

4. Every arrow isomorphic to a fibration, cofibration, or weak equivalence is also one.

5. Every arrow from a cofibrant to a fibrant object can be factored as a cofibration followed by a trivial fibration.

6. Every arrow from a cofibrant to a fibrant object can be factored as a trivial cofibration followed by a fibration.

7. Given a solid square:

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"] & X \arrow[d, two heads, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

   Where A and B are cofibrant, i is a cofibration, X and Y are fibrant, p is a fibration and either p or i is a weak equivalence, then there exists a dotted map that makes the diagram commutes.

8. Given a commutative tringle:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Z \\ & Y \arrow[ur] & \end{tikzcd}$$

   where each of the three objects is either fibrant or cofibrant, if two of the maps are weak equivalences then the thrid also is.

9. Every isomorphism between objects that are either fibrant or cofibrant is a weak equivalence.

Remark 3.1.2. In definition 3.1.1 we use the usual conventions: a *cofibrant object* is an object such that the unique map $0 \to X$ is a cofibration and a fibrant object is an object such that the unique map $X \to 1$ is a fibration. A trivial (co)fibration is a map which is both an equivalence and a (co)fibration. We will also use the term *core cofibrations* to mean "cofibration between cofibrant objects" and *core fibrations* to mean "fibration between fibrant objects".

Remark 3.1.3. It is crucial to observe that definition 3.1.1 only involve the core cofibrations, core fibrations and weak equivalences between objects that are either fibrant or cofibrants. By that we mean that if given $\mathcal{M}$ a category with these three class of maps, then ($\mathcal{M}$,cofibrations, fibrations, weak equivalences) is a weak model structure if and only if ($\mathcal{M}$, core cofibrations, core fibrations, weak equivalences between objects that are either fibrant or cofirbant) is a weak model structure.

For this reason, we generally consider that only core cofibrations, core fibrations and weak equivalence between objects that are either fibrant or cofirbant are to be treated as relevant notion. Nothing we will do here depends on the three class of maps outside of these restriction. In [Hen20] it was even considered that the words cofibrations, fibrations and weak equivalences means "core cofibrations", "core fibrations" and "weak equivalences between fibrant or cofibrant objects".

Remark 3.1.4. The definition of weak model structure in [Hen20] is different from definition 3.1.1, but it is equivalent. It is stated without reference to the class of weak equivalence and using notion of (weak relative) path object and cylinder object. It is easy to show that a weak model structure in the sense of definition 3.1.1 is a weak model structure in the sense of [Hen20] by constructing the cylinder and path objects as factorization of the codiagonal and diagonal maps (see 3.1.5 below). Conversely it is shown in [Hen20] that given a weak model structure, it admit a (unique[1]) class of weak equivalences such that all conditions of definition 3.1.1 are satisfied.

It is shown in [Hen20] that most of the basic theory of Quillen model categories carries over to weak model categories, with only some aditional care taken - mostly replacing objects by fibrant and cofibrant replacement of objects before applying the usual construction. The main significant difference is that the homotopy category (defined in terms of homotopy class of maps between bifibrant objects as we will recall below) is no longer equivalent to $\mathcal{M}[W^{-1}]$ - the localization of $\mathcal{M}$ at weak equivalence, but only to $\mathcal{M}^{\text{cof}\vee\text{fib}}[W^{-1}]$ the localization the full subcategory of objects that are either fibrant or cofibrant at the weak equivalences. The problem, is that the axioms of a weak model category allows to take a fibrant replacement of a cofibrant object C as a (trivial cofibration/fibration) factorization of $C \to 1$, and similarly we can take a cofibrant replacement of a fibrant objects, but there is no way to do similar replacement with an object which is neither fibrant nor cofibrant.

We now quickly go over some aspect of the construction of the homotopy category of a weak model category, the result mentioned below are all proved in section 2.1 and 2.2 of [Hen20].

[1] Keeping in mind remark 3.1.3. Only the class of weak equivalence between fibrant or cofibrant objects is uniquely defiend, outside of this, there no restriction whatsoever on weak equivalence from definition 3.1.1.

Construction 3.1.5. If X is a bifibrant object (i.e. fibrant and cofibrant), we can form a *cylinder objects* IX for X as a (cofibration,trivial fibration) factorization:

$$X \coprod X \hookrightarrow IX \overset{\sim}{\twoheadrightarrow} X$$

and a path objects for X as a (trivial cofibration, fibration) factorization

$$X \overset{\sim}{\hookrightarrow} PX \twoheadrightarrow X \times X$$

And given a pair of maps $f, g : X \rightrightarrows Y$ between bifibrant objects, we say they are homotopic if there is a dotted map h making the diagram below comutes:

$$\begin{tikzcd} X \arrow[d] \arrow[dr, "f"] & \\ IX \arrow[r, dotted, "h"] & Y \\ X \arrow[u] \arrow[ur, "g"'] & \end{tikzcd}$$

or equivalently a map h

$$\begin{tikzcd} & Y \\ X \arrow[ur, "g"] \arrow[r, dotted, "h"] \arrow[dr, "f"'] & PY \arrow[u] \arrow[d] \\ & Y \end{tikzcd}$$

This is an equivalence relation, and the homotopy category $\mathrm{Ho}(\mathcal{M})$ of $\mathcal{M}$ can be defined as the category of bifibrant objects with homotopy class of maps between them. Moreover this category is equivalent to the formal localization $\mathcal{M}^{\mathrm{cof} \vee \mathrm{fib}}[W^{-1}]$.

Construction 3.1.6. Note that is an object $C \in \mathcal{M}$ is only cofibrant and not fibrant we cannot define a cylinder object in the same was as above, as the factorization axiom does not allows to factor the maps $X \coprod X \to X$ if X is not fibrant. In place of this, we can consider a fibrant replacement $X \overset{\sim}{\hookrightarrow} X^{\mathrm{FIB}} \twoheadrightarrow 1$, and then form a factorization:

$$\begin{tikzcd} X \coprod X \arrow[r, hook] \arrow[d, "\nabla"] & IX \arrow[d, two heads, "\sim"] \\ X \arrow[r, hook, "\sim"] & X^{\mathrm{FIB}} \end{tikzcd}$$

This object IX, and more generally any object fitting into a diagram:

$$\begin{tikzcd} X \coprod X \arrow[r, hook] \arrow[d, "\nabla"] & IX \arrow[d, "\sim"] \\ X \arrow[r, hook, "\sim"] & DX \end{tikzcd}$$

is called a weak cylinder object. Dually, if Y is fibrant we define a weak path object of Y as any object PY that fits into a diagram:

$$\begin{tikzcd} TX \arrow[r, "\sim"] \arrow[d, two heads, "\sim"] & PX \arrow[d, two heads] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

We can then show that for a pair of maps $X \rightrightarrows Y$ from a cofibrant object X to a fibrant object Y the following are equivalent:

- f is homotopic to g in terms of a weak cylinder object for X.
- f is homotopic to g in terms of a weak path objects for Y.
- f and g are equal in the localization $\mathcal{M}^{\mathrm{cof}\vee\mathrm{fib}}[W^{-1}]$.

Moreover any arrow $X \to Y$ in the localization $\mathcal{M}^{\mathrm{cof}\vee\mathrm{fib}}[W^{-1}]$ comes from an arrow $X \to Y$ in $\mathcal{M}$.

3.2 The language of a weak model category and the first two invariance theorems

Construction 3.2.1. Given $\mathcal{M}$ a weak model category, the category $\mathcal{M}^{\mathrm{Cof}}$ of cofibrant objects with cofibration between them forms a coclan. We define the language of $\mathcal{M}$ to be the language of the coclan $\mathcal{M}^{\mathrm{Cof}}$, that for any regular cardinal λ, we denote by $\mathbb{L}^{\mathcal{M}}_{\lambda}$ the κ-boolean algebra $\mathbb{L}^{\mathcal{M}^{\mathrm{Cof}}}_{\lambda}$ over $\mathcal{M}^{\mathrm{Cof}}$.

Note that we have for each *cofibrant* object $X \in \mathcal{M}$ a set (or possibly a class if $\mathcal{M}$ is large) of formulas $\mathbb{L}^{\mathcal{M}}_{\lambda}(X)$.

Remark 3.2.2. There is a size issue to be mentioned here. In most practical exemple, $\mathcal{M}^{\mathrm{Cof}}$ is a large category while the construction of $\mathbb{L}^{\mathcal{M}^{\mathrm{Cof}}}_{\lambda}$ developed in section 2.3 assume it is a small category. This is dealt with by invoking a larger Grothendieck universe, but this has a practical consequence: The set of formula $\mathbb{L}^{\mathcal{M}}_{\lambda}(X)$ might not be a small set. Indeed it lives in the same Grothendieck universe as the one in which $\mathcal{M}^{\mathrm{Cof}}$ is small.

Construction 3.2.3. If $X \in \mathcal{M}$ then we can define a model of the coclan $\mathcal{M}^{\text{Cof}}$ using the restricted Yoneda embedding:

$$よ_X : \begin{array}{ccc} (\mathcal{M}^{\text{Cof}})^{\text{op}} & \to & \textbf{Set} \\ c & \mapsto & \text{Hom}(c, X) \end{array}$$

Which defines a functor $よ : \mathcal{M} \to \text{Mod}(\mathcal{M}^{\text{Cof}})$.

Definition 3.2.4. Let $\mathcal{M}$ be a weak model category. For $c \in \mathcal{M}$ a cofibrant, and $X \in \mathcal{M}$ any object, $v : c \to X$ and $\phi \in \mathbb{L}_\lambda^{\mathcal{M}}(c)$ we define

$$X \vdash \phi(v)$$

to mean

$$よ_X \vdash \phi(v)$$

where v is seen as an element of $よ_X(c) = \text{Hom}(c, X)$.

Remark 3.2.5. In the special case where $\mathcal{M} = \text{Mod}(T)$ is the category of models of a generalized κ-algebraic theory (or more generally of a κ-coclan), then $\mathbb{L}_\lambda^{\mathcal{M}}$ is the initial λ-boolean algebra over the coclan of all cofibrant objects of $\mathcal{M}$, while the syntactic category of T is equivalent to a full sub-κ-coclan of that. In particular there is a morphism of λ-boolean algebra over the syntactic category $\mathcal{C}_T$

$$\mathbb{L}_\lambda^T(X) \to \mathbb{L}_\lambda^{\mathcal{M}}(X) \qquad (\text{For } X \in \mathcal{C}_T)$$

If we denote this map by i then for X any model of T we can easily check that

$$X \vdash \phi(v) \Leftrightarrow X \vdash i(\phi)(v)$$

for any $c \in \mathcal{C}_T$ and $\phi \in \mathbb{L}_\lambda^T(c)$, where the left hand side is interpreted in the sense of definition 2.1.1 while the right hand side is in terms of definition 3.2.4.

Note that we do expect these to be the same. Informally, $\mathbb{L}_\lambda^T$ corresponds to an $\mathcal{L}_{\kappa,\lambda}$ logic, in the sense that quatifier can only be applied to formulas in κ-small context (so, to less that κ-many variables at the same time), while $\mathbb{L}_\lambda^{\mathcal{M}}$ corresponds to an $\mathcal{L}_{\infty,\lambda}$ logic, where quantifier can be applied to arbitrarily many formula at the same time.

Theorem 3.2.6. Let $\mathcal{M}$ be a weak model category, $c \in \mathcal{M}$ a cofibrant object,X, Y two fibrant objects, $v : c \to X$ a maps, and $\phi \in \mathbb{L}_\lambda^{\mathcal{M}}(c)$ then:

- **First invariance theorem:** Let $v_1, v_2 : c \to X$ be two homotopically equivalent maps with X fibrant. Then

$$X \vdash \phi(v_1) \quad \Leftrightarrow \quad X \vdash \phi(v_2)$$

- **Second invariance theorem:** Let $f : X \to Y$ be a weak equivalence between two fibrant objects and $v : c \to X$ any maps then

$$X \vdash \phi(v) \quad \Leftrightarrow \quad Y \vdash \phi(fv)$$

Proof. We start by first observe that the second invariance theorem in the special case where f is a trivial fibration immediately follows from corollary 2.3.13 as a trivial fibration in f has the right lifting property against all core cofibrations and hence is sent to a trivial fibration in $\mod(\mathcal{M}^{\mathrm{Cof}})$ by the functor from construction 3.2.3.

We use this to prove the first invariance theorem: If $v_1, v_2 : c \to X$ are homotopic then there exists a map h:

$$\begin{tikzcd} & X \\ c \arrow[ur, "v_2"] \arrow[r, dotted, "h"] \arrow[dr, "v_1"'] & PX \arrow[u, "p_2"'] \arrow[d, "p_1"] \\ & X \end{tikzcd}$$

The two maps $p_1, p_2 : PX \to X$ are trivial fibrations (They are both fibrations and weak equivalences), and $v_1 = p_1 \circ h$ and $v_2 = p_2 \circ h$ we have by the observation above:

$$\begin{array}{cccc} & X & \vdash & \phi(v_1) \\ \Leftrightarrow & X & \vdash & \phi(p_1 h) \\ \Leftrightarrow & PX & \vdash & \phi(h) \\ \Leftrightarrow & X & \vdash & \phi(p_2 h) \\ \Leftrightarrow & X & \vdash & \phi(v_2) \end{array}$$

This concludes the proof of the first isomorphism theorem.

Next, we observe it is enough to prove the second invariance theorem when X and Y are both bifibrant. Indeed starting from $f : X \to Y$ a weak equivalence between fibrant object, $v : c \to X$ and $\phi \in \mathbb{L}_\lambda^{\mathcal{M}}(c)$ as in the theorem. We can replace both X and Y by bifibrant objects

$$\begin{tikzcd} X^{\mathrm{Cof}} \arrow[r, "\sim", "f'"'] \arrow[d, two heads, "\sim"] & Y^{\mathrm{Cof}} \arrow[d, two heads, "\sim"] \\ X \arrow[r, "\sim", "f"'] & Y \end{tikzcd}$$

by first replacing X by a cofibrant object X^{Cof} and then factoring the map $X^{\mathrm{Cof}} \to Y$, which is a weak equivalence, as a trivial cofibration followed by a trivial fibration. The map $v : C \to X$, can be lifted to map $v' : c \to X^{\mathrm{Cof}}$. As we can already apply the second isomorphisms theorem to trivial fibrations, we have that

$$X \vdash \phi(v) \Leftrightarrow X^{\mathrm{Cof}} \vdash \phi(v')$$

$$Y \vdash \phi(fv) \Leftrightarrow Y^{\mathrm{Cof}} \vdash \phi(f'v')$$

so that it is enough to show the second isomorphism for bifibrant objects.

This last step is achieved essentially using a "Brown factorization": any weak equivalence between bifibrant objects can be factored as a section of a trivial fibration followed by a trivial fibration. Indeed, if $f : X \to Y$ is a map between bifibrant object we can form the pullbacks:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "e'"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "e"] \\ X \times_Y PY \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & PY \arrow[d, two heads] \\ X \times Y \arrow[r] \arrow[d, two heads, "\pi_1"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times Y \arrow[d, two heads, "\pi_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Note that because the fibrations $PY \to Y$ are trivial fibration, the map $X \times_Y PY \to X$ in the diagran above is also a trivial fibration. The total vertical maps are both the identity. Which gives us a diagram:

$$\begin{tikzcd}
 & X \arrow[dl, "id_X"'] \arrow[d, "e'"] \arrow[dr, "f"] & \\
X & X \times_Y PY \arrow[l, two heads, "\sim", "q"'] \arrow[r, two heads, "p"'] & Y
\end{tikzcd}$$

Where p is the map $X \times_Y PY \twoheadrightarrow X \times Y \overset{\pi_2}{\twoheadrightarrow} Y$. Note that all maps in this diagram are weak equivalence due to the 2-out-of-3 condition. We can now prove the theorem, we have

$$X \vdash \phi(v) \Leftrightarrow X \times_Y PY \vdash \phi(e'v)$$

because $v = qe'v$ and q is a trivial fibration, and

$$X \times_Y PY \vdash \phi(e'v) \Leftrightarrow Y \vdash \phi(fv)$$

because p is a trivial fibration and $fv = pe'v$. Hence, combining the two

$$X \vdash \phi(v) \Leftrightarrow Y \vdash \phi(fv)$$

□

Finally, we quickly mention how Quillen adjunction acts on formulas. A Quillen adjunction between two weak model categories is an adjunction

$$L : \mathcal{C} \leftrightarrows \mathcal{D} : R$$

where the left adjunction L sends cofibrations to cofibrations and the right adjoint R sends fibrations to fibrations.

Remark 3.2.7. There is also a more general notion called “weak Quillen functors” introduced in [Hen20] which is sometime more convenient where L is only defined on cofibrant objects and R on fibrant objects, and they are only required to preserve core (co)cofibrations - all results in this section, as well as the third invariance theorem from section 4 apply to weak Quillen adjunction. We restrict ourselve to Quillen adjunction in the paper for simplicity and because this already cover most of the applications.

Such a Quillen adjunction (or weak Quillen adjunction) induce in particular a coclan morphism $L : \mathcal{C}^{\mathrm{Cof}} \to \mathcal{D}^{\mathrm{Cof}}$, which following construction 2.3.10 we have a (unique) comparison map

$$\alpha_L : \mathbb{L}^{\mathcal{C}}_{\lambda} \to L^* \mathbb{L}^{\mathcal{D}}_{\lambda}$$

As before, if $\phi \in \mathbb{L}_\lambda^{\mathcal{C}}(C)$ we often write $L(\phi)$ instead of $\alpha_L(\Phi)$, note that $L(\phi) \in \mathbb{L}_\lambda^{\mathcal{D}}(L(C))$.

Finally, exactly as in construction 2.3.10, for any (fibrant) object $X \in \mathcal{D}$, and cofibrant object $C \in \mathcal{C}$, any map $v : C \to R(X)$ corresponding to $\tilde{v} : LC \to X$, and $\phi \in \mathbb{L}_\lambda^{\mathcal{C}}$ we have

$$R(X) \vdash \phi(v) \Leftrightarrow X \vdash L(\phi)(\tilde{v})$$

The thrid invariance theorem that we will establish in section 4 show that for Quillen equivalence this construction gives an equivalence between the language of $\mathcal{C}$ and of $\mathcal{D}$ in an appropriate sense.

4 Languange invance under Quillen equivalences

The main goal of this section is to show that the language associated to a weak model category from Section 3 is invariant under Quillen equivalence between weak model categories. First, in Section 4.1 we show that the result holds for a particular class of functors, called *Barton trivial fibrations*. We use the constructions in Section 4.3 to prove in Theorem 4.3.19 that any left Quillen functor, part of a Quillen equivalence between weak model categories can be decomposed as a span of Barton trivial fibrations. This argument can be seen as analogous to a well-known result, Brown's factorization lemma, which we recall in Section 4.2.

4.1 Invariance along Barton trivial fibrations

Recall from Construction 3.2.1 that given $\mathcal{M}$ a weak model category, the language of $\mathcal{M}$ is the language of the clan $(\mathcal{M}^{\text{Cof}})^{\mathsf{op}}$ i.e. for any regular cardinal λ, it is the κ-boolean algebra $\mathbb{L}_\lambda^{(\mathcal{M}^{\text{Cof}})^{\mathsf{op}}}$ over $(\mathcal{M}^{\text{Cof}})^{\mathsf{op}}$. Which is simply denoted by $\mathbb{L}_\lambda^{\mathcal{M}}$.

Remark 4.1.1. The fibrations in the clan $(\mathcal{M}^{\text{Cof}})^{\mathsf{op}}$ are precisely the core cofibrations of $\mathcal{M}$. Therefore, the first condition of Definition 2.3.1 for the language can be readed as: For any core cofibration $f : \Gamma \hookrightarrow \Gamma'$ in $\mathcal{M}$, the induced map $f^* : \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma) \to \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma')$ has a left adjoint $\exists_\pi : \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma') \to \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$. Similarly for the Beck-Chevalley condition.

Construction 4.1.2. Recall from Construction 2.3.10 that for a functor $F : \mathcal{C} \to \mathcal{D}$ between weak model categories. And let $\mathbb{L}_\lambda^{\mathcal{C}}$ and $\mathbb{L}_\lambda^{\mathcal{D}}$ their respective initial λ-boolean algebras. From the fact that $\mathbb{L}_\lambda^{\mathcal{C}}$ is initial there is a morphisms of λ-boolean algebras

$$\mathbb{L}F : \mathbb{L}_\lambda^{\mathcal{C}} \to F^* \left(\mathbb{L}_\lambda^{\mathcal{D}}\right)$$

For any $\Gamma \in \mathcal{C}$ and any formula $\phi \in \mathbb{L}_\lambda^{\mathcal{C}}(\Gamma)$ we denote $F(\phi) \coloneqq \mathbb{L}F_\Gamma(\phi)$ which is a formula in context $F(\Gamma)$ i.e an element of $\mathbb{L}_\lambda^{\mathcal{D}}(F(\Gamma))$.

Observation 4.1.3. We can reinterpret Proposition 2.3.11 in the presence of $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$ a Quillen adjunction between weak model categories. Let $\Gamma \in \mathcal{M}$ be a cofibrant object and $X \in \mathcal{N}$ be a fibrant object. Then for any $v : F(\Gamma) \to X$ and $v' : \Gamma \to G(X)$ its adjoint transpose we have

$$G(X) \vdash \phi(v') \Leftrightarrow X \vdash F(\phi)(v).$$

Now we turn our attention to a more semantical relation on formulas. This relation will allows to construct boolean algebras over the homotopy category of $\mathcal{M}$.

Definition 4.1.4. Let Γ be a cofibrant object of $\mathcal{M}$. Two formulas $\phi, \psi \in \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ are said to be **semantically equivalent** if for all fibrant objects $X \in \mathcal{M}$ we have $|\phi|_X = |\psi|_X$. In this situation we write $\phi \approx \psi$.

Observation 4.1.5. For any cofibrant object $\Gamma \in \mathcal{M}$, the relation $\approx$ on $\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ is an equivalence relation. Furthermore, $\approx$ is compatible with the κ-boolean algebra structure of $\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$. In conclusion, for any Γ cofibrant we get a boolean algebra $h\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ whose elements are equivalence classes of formula in $\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ under the relation $\approx$. For each formula $\phi \in \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ we write $[\phi]$ for its equivalence, these are exactly the elements of $h\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$.

It is obvious, but worth mentioning, that if $\phi, \psi \in \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ are such that $\phi \approx \psi$, then

$$X \vdash \phi(v) \Leftrightarrow X \vdash \psi(v)$$

for any fibrant object $X \in \mathcal{M}$ and map $v : \Gamma \to X$.

Proposition 4.1.6. Let $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$ be a Quillen adjunction between weak model categories. The construction above restricts to a mophism of κ-boolean algebras:

$$h\mathbb{L}F : h\mathbb{L}_\lambda^{\mathcal{C}} \to F^* \left(h\mathbb{L}_\lambda^{\mathcal{D}}\right).$$

Moreover, for any cofibrant object $\Gamma \in \mathcal{M}$, fibrant object $X \in \mathcal{N}$, morphism $v : F(\Gamma) \to X$ with adjoint transpose $v' : \Gamma \to G(X)$ and $[\phi] \in h\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$, we have

$$G(X) \vdash [\phi](v') \Leftrightarrow X \vdash F([\phi])(v),$$

where $h\mathbb{L}F_\Gamma([\phi]) \coloneqq [F(\phi)]$, and we write $F[\phi]$ for this.

Proof. Note that the result boils down to show that the induced map is well-defined i.e. given formulas $\phi, \psi \in \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ such that $\phi \approx \psi$ then $F(\phi) \approx F(\psi)$, but this is easy; take $\Gamma \in \mathcal{M}$ cofibrant, $X \in \mathcal{N}$ fibrant, $v : F(\Gamma) \to X$ and $v' : \Gamma \to G(X)$ its adjoint transpose, then

$$X \vdash F(\phi)(v) \Leftrightarrow G(X) \vdash \phi(v') \Leftrightarrow G(X) \vdash \psi(v') \Leftrightarrow X \vdash F(\psi)(v)$$

The second part follows immediately from Observation 4.1.3 and the first part since $F([\phi])$ now makes sense. ☐

Lemma 4.1.7. Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ a Quillen equivalence. Then for any cofibrant object $\Gamma \in \mathcal{M}$, the induced map $h\mathbb{L}F_\Gamma : h\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma) \to h\mathbb{L}_\lambda^{\mathcal{N}}(F\Gamma)$ is injective.

Proof. Let ϕ and ψ be formulas in $\mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ such that $F\phi \approx F\psi$ i.e. $F\phi$ and $F\psi$ are equal in $h\mathbb{L}_\lambda^{\mathcal{N}}(F\Gamma)$. We must show that $\psi \approx \phi$. Let X be a fibrant object of $\mathcal{M}$. The Quillen equivalence induces an equivalence between homotopy categories $Ho(G) : Ho(\mathcal{N}^{\text{FIB}}) \to Ho(\mathcal{M}^{\text{FIB}})$. Hence, there is a fibrant object $Y \in \mathcal{N}$ such that GY is isomorphic to X in $Ho(\mathcal{M}^{\text{FIB}})$. Given any $x : \Gamma \to X$, denote by $y : \Gamma \to GY$ any map such that the following triangle

$$\begin{tikzcd} A \arrow[r, "x"] \arrow[dr, "y"'] & X \arrow[d, "\cong"] \\ & GY \end{tikzcd}$$

commutes in $Ho(\mathcal{M}^{\text{FIB}})$. Lastly, let $y' : F\Gamma \to Y$ the transpose of y via the Quillen adjuction. It follows from the first invariance theorem Theorem 3.2.6 that $X \vdash \phi(x)$ if and only if $GY \vdash \phi(y)$. From Proposition 4.1.6 this is equivalent to $Y \vdash F(\psi)(y')$. By assumption $F(\phi) \approx F(\psi)$, so $Y \vdash F(\psi)(y')$. Again, this is $GY \vdash \psi(y)$ and $X \vdash \psi(x)$. This establishes the equality $|\phi|_X = |\psi|_X$ for all $X \in \mathcal{M}$ fibrant, which demonstrates the statement. ☐

Definition 4.1.8. Let $F : \mathcal{C} \to \mathcal{D}$ a morphism between κ-coclans. We say that F is **extensive** if for every object in $X \in \mathcal{C} \to \mathcal{D}$ and for any cofibration $g : FX \hookrightarrow Y \in \mathcal{D}$ there exists $f : X \to Z$ and an isomorphism $\theta : F(Z) \cong Y$ making the obvious triangle commutative.

Dually, $F : \mathcal{C} \to \mathcal{D}$ a morphism between κ-clans is **extensive** if the induce map of κ-coclans $F^{\mathsf{op}} : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}^{\mathsf{op}}$.

In our setting, a functor $F : \mathcal{M} \to \mathcal{N}$ between weak model categories will be called extensive if the morphism of coclans $F : \mathcal{M}^{\text{COF}} \to \mathcal{N}^{\text{COF}}$ is extensive.

Lemma 4.1.9. Let $F : \mathcal{M} \to \mathcal{N}$ be an extensive morphism between κ-clans and $\Gamma \in \mathcal{M}$. Then, any formula $\Phi \in \mathbb{L}_\lambda^{\mathcal{N}}(F\Gamma)$ is the image by F of a formula $\Phi_0 \in \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$.

Proof. Note that since every κ-clan is of the form $\mathbb{C}_T$ for some T κ-generalized algebraic theory it is enough to show the result is valid for the syntactic definition of language as in Definition 2.1.1. We prove by induction on formulas $\Phi \in \mathbb{L}_\lambda^{\mathcal{N}}(\Delta)$ that given any context Γ such that $f : \Delta \cong F(\Gamma)$ there is a formula $\Phi_0 \in \mathbb{L}_\lambda^{\mathcal{M}}(\Gamma)$ such that $f^*(F\Phi_0) = \Phi$.

1. When $\Phi = \top$ or $\Phi = \bot$, then this can clearly be lifted to $\top$ and $\bot$.

2. If $\Phi = \neg\Psi$ or $\Phi = \bigvee_{i\in I} \Psi_i$ or $\Phi = \bigwedge_{i\in I} \Psi_i$ then it is also clear that Φ can be lifted. Indeed, we can simply use the inductive hypotesis to lift each Ψ_i and then use the boolean algebra structure to conclude.

3. Suppose that Φ is of the form $\exists_\pi \Psi$ or $\forall_\pi \Psi$ for some fibration $\pi : \Gamma' \twoheadrightarrow F(\Gamma)$. Here the formula $\Psi \in \mathbb{L}_\lambda^{\mathcal{N}}(\Gamma')$, so $\Phi \in \mathbb{L}_\lambda^{\mathcal{N}}(F\Gamma)$. Furthermore, we assume that Ψ can be lifted. Since F is a trivial fibration, there is a lift $\bar{\pi} : \bar{\Gamma}' \to \Gamma \in \mathcal{M}$ of $\pi : \Gamma' \twoheadrightarrow F(\Gamma)$, which comes with an isomorphism $g : \Gamma' \cong F(\bar{\Gamma}')$ such that the following triangle commutes

$$\begin{tikzcd} \Gamma' \arrow[r, two heads, "\pi"] \arrow[d, "g", "\cong"'] & F(\Gamma) \\ F(\bar{\Gamma}'). \arrow[ur, two heads, "F(\bar{\pi})"'] & \end{tikzcd}$$

Therefore, we get a commutative square as in the left, and at the level of laguages as on the right

$$\begin{tikzcd} \Gamma' \arrow[r, two heads, "\pi'"] \arrow[d, "g", "\cong"'] & \Delta \arrow[d, "f"', "\cong"] \\ F(\bar{\Gamma}') \arrow[r, two heads, "F(\bar{\pi})"'] & F(\Gamma) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{L}_\lambda^{\mathcal{N}}(F(\bar{\Gamma}')) \arrow[r, "\exists_{\pi'}"] \arrow[d, "g^*"'] & \mathbb{L}_\lambda^{\mathcal{N}}(F(\Gamma)) \arrow[d, "f^*"] \\ \mathbb{L}_\lambda^{\mathcal{N}}(\Gamma') \arrow[r, "\exists_{F(\bar{\pi})}"'] & \mathbb{L}_\lambda^{\mathcal{N}}(\Delta). \end{tikzcd}$$

By assumption $\psi \in \mathbb{L}_\lambda^{\mathcal{N}}(\Gamma')$ can be lifted. Hence, there is a formula $\Psi_0 \in \mathbb{L}_\lambda^{\mathcal{M}}(\bar{\Gamma}')$ such that $g^*(F\Psi_0) = \Psi$. Using the right hand square above, one can see that $\exists_{\bar{\pi}}\Psi_0$ is a lift for Φ.

∎

There is an immediate consequence of Lemma 4.1.9, namely when the map $F : \mathcal{M} \to \mathcal{N}$ is a trivial fibration between weak model categories where we can use the natural clan structure of each category.

Definition 4.1.10. Let $F : \mathcal{M} \to \mathcal{N}$ a functor between weak model categories. We say that F is a **Barton trivial fibration** if it is extensive and a left Quillen equivalence.

Remark 4.1.11. The terminology of extensive from Definition 4.1.8 is adapted from Reid Barton's PhD thesis [Bar19], were it is used in the context of premodel categories. Furthermore, the trivial fibrations in the previous definition are exactly the trivial fibrations in *Ibidem* for the model 2-category of model categories in Barton's work. As the reader may anticipate, in the cited documet the notion of fibration is exists as well. The same notion of fibration was also considered by [**?**].

Theorem 4.1.12. Let $F : \mathcal{M} \to \mathcal{N}$ be a Barton trivial fibration between weak model categories. Then for any cofibrant $\Gamma \in \mathcal{M}$ the induced map $h\mathbb{L}F_A : h\mathbb{L}^{\mathcal{M}}_{\lambda}(\Gamma) \to h\mathbb{L}^{\mathcal{N}}_{\lambda}(F\Gamma)$ is an isomorphism.

Proof. By the previous Lemma 4.1.7 we know that $h\mathbb{L}F_\Gamma : h\mathbb{L}^{\mathcal{M}}_{\lambda}(\Gamma) \to h\mathbb{L}^{\mathcal{N}}_{\lambda}(\Gamma)$ is injective. Next we can use Lemma 4.1.9 by observing that this sujectivity also descends at the level of $h\mathbb{L}F_\Gamma : h\mathbb{L}^{\mathcal{M}}_{\lambda}(\Gamma) \to h\mathbb{L}^{\mathcal{N}}_{\lambda}(\Gamma)$. □

This theorem is the reason for the constructions to come. With this result at hand, we simply need to reduce our problem to this case, as explained below.

4.2 Brown's Lemma for weak model categories

In this section we prove a version of Brown's lemma in the (large) category of weak model categories. As of now, there is no fibrational structure on this category, we can not simply verify the conditions for the result to be true. Hence, we must prove this result by hand. An important ingridient is the existence of specific factorizations as in Lemma 4.2.1 below. To provide such, Brown's factorization lemma make use of path objects. We will start by recalling the factorization lemma, since details are relevant to the subsequent we provide a modern account of its proof found in [vdBM16].

Lemma 4.2.1. Let $f : X \to Y$ a map in $\mathcal{C}$ a category with fibrations. There exists a factorization

$$f = p_f w_f$$

where p_f is a fibration and w_f is a section of an acyclic fibration. In particular, w_f is a weak equivalence.

Proof. First, one notes that for any product $X_1 \times X_2$ in $\mathcal{C}$, the projections $X_1 \times X_2 \to X_1$ and $X_1 \times X_2 \to X_2$ are fibrations. Also, for any object Y in $\mathcal{C}$, there is a factorisation

$$Y \xrightarrow{c} PY \xrightarrow{e} Y \times Y$$

of the diagonal map were c is a weak equivalence and e is a fibration. The two maps $PX \to X$, induced by the projections, are trivial fibrations.

We can then construct the following pullback square

$$\begin{tikzcd} Pf \arrow[r, "p_2"] \arrow[d, "p_1"'] & PY \arrow[d, "\pi_1 e"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where p_1 is a trivial fibration. There is a map $w_f := (1, cf) : Y \to Pf$ given by the universal property of the pullback Pf. By setting $p_f := \pi_2 e p_2$, one verifies that $p_f w_f = f$ and w_f is a section of p_1. It remains to show that p_f is a fibration. This true upon observing that the following square

$$\begin{tikzcd} Pf \arrow[r, "p_2"] \arrow[d, "(p_1, p_f)"'] & PY \arrow[d, "e"] \\ X \times Y \arrow[r, "f \times 1"'] & Y \times Y \end{tikzcd}$$

is a pullback. $\square$

In order to obtain factorizations as in Lemma 4.2.1 it is necessary to have path objects in the category of weak model categories. We devote the next section to construct a path object for any weak model category.

4.3 Path objects for weak model categories

4.3.1 Reedy weak model categories

Before doing all the constructions, we need to set up the formalism needed for such. In this section we study Reedy weak model categories. These are, as the name suggests, the counterpart of Reedy model categories. Most of the proofs are straightforward adaptation of the classical ones, so they are ommited.

Definition 4.3.1. A **Reedy category** is a category R together with two wide subcategories R_+ and R_- and a functor $deg : R \to \alpha$, where α is an ordinal, such that:

1. For every $a \to b \in R_+$ a non-identity arrow, $deg(a) < deg(b)$.

2. For every $a \to b \in R_-$ a non-identity arrow, $deg(b) < deg(a)$.

3. Every arrow in R factors uniquely as an arrow in R_- followed by an arrow in R_+.

When the subcategory R_- consists of identity arrows only, then R is called **direct category**. Similarly, when the subcategory R_+ consists of identity arrows only, then R is called **inverse category**

Let R be a Reedy category and $\mathcal{M}$ be a weak model category. Consider $\mathcal{M}^R$ the functor category associated to them. Let $X : R \to \mathcal{M}$ and $r \in R$ objects in the respective categories. The **latching object** at r, whenever it exists, is defined as

$$L_r X \coloneqq \mathtt{Colim}_{s \in (R_+/r) - \{Id_r\}} X_s.$$

Furthermore, we ask that this object is cofibrant in $\mathcal{M}$. Dually, the **matching object** at r, whenever it exists, is defined as

$$M_r X \coloneqq \mathtt{Lim}_{s \in (r/R_-) - \{Id_r\}} X_s.$$

Furthermore, we ask that this object is fibrant in $\mathcal{M}$.

Definition 4.3.2. A map $f : X \to Y$ in $\mathcal{M}^R$ is said to be an **(acyclic) Reedy cofibration** if for all $r \in R$ the induced map below

$$\begin{tikzcd}
L_r X \arrow[r] \arrow[d] & X_r \arrow[d] \arrow[ddr, bend left] & \\
L_r Y \arrow[r] \arrow[drr, bend right] & L_r Y \sqcup_{L_r X} X_r \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\
& & Y_r
\end{tikzcd}$$

is an (acyclic) cofibration in $\mathcal{M}$.

Dually, $f : X \to Y$ in $\mathcal{M}^R$ is said to be an **(acyclic) Reedy fibration** if for all $r \in R$ the induced map below

$$\begin{tikzcd}
X_r \arrow[dr, dashed] \arrow[ddr, bend right] \arrow[drr, bend left] & & \\
& M_r X \times_{M_r Y} Y_r \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_r \arrow[d] \\
& M_r X \arrow[r] & M_r Y
\end{tikzcd}$$

is an (acyclic) fibration in $\mathcal{M}$.

It is a classical result that for any Quillen model category $\mathcal{M}$ and a Reedy category R that the category of fuctors $\mathcal{M}^R$ carries a model structure in which the weak equivalences are the level-wise weak equivalences, the (acyclic) (co)fibrations are precisely the (acyclic) (co)fibrations. The same result can be obtained if we simply assume that the base category carries a weak model structure.

Theorem 4.3.3. Assume that $\mathcal{M}$ is a weak model category and that R is a Reedy category. Then there is a weak model structure on $\mathcal{M}^R$ such that a map $f : X \to Y$ it is:

1. A *weak equivalence* if and only if $f_r : X_r \to Y_r$ is a weak equivalence for all $r \in R$.

2. An *(acyclic) cofibration* if it is an (acyclic) Reedy cofibration.

3. An *(acyclic) fibration* if it is an (acyclic) Reedy fibration.

We quickly comment each requirement from Definition 3.1.1, but all the classical constructions generalize to our setting.

Proof. 1. This is clear. Simply use the the constant diagrams at the initial and terminal objects of $\mathcal{M}$.

2. The pushouts are computed point-wise. One can the show that cofibrations are preserved along maps between cofibrant objects.

3. Similar to the previous condition.

4. This is immediate because point-wise any arrow in $\mathcal{M}$ isomorphic to a fibration, cofibration or weak equivalence is also one in $\mathcal{M}$. For example, if $f' : X' \to Y'$ is isomorphic to a fibration $f : X \to Y$ then this isomorphism will induce an isomorphism between $X'_r \to M_r X' \times_{M_r Y'} Y'_r$ and $X_r \to M_r X \times_{M_r Y} Y_r$, from which we conclude.

5. The classical argument generalizes to when $\mathcal{M}$ is a weak model category.

6. The classical argument generalizes to when $\mathcal{M}$ is a weak model category.

7. One can adapt the classic inductive argument to obtain solutions to lifting problems of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & X \arrow[d, two heads] \\ B \arrow[r] & Y \end{tikzcd}$$

8. The 2-out-of-3 property follows directly since these are level-wise in $\mathcal{M}^J$ and the property is satisfied in $\mathcal{M}$.

9. It is immediate that isomorphisms are also weak equivalences.

$\square$

Remark 4.3.4. When the Reedy category is directed this model structure coincides with the projective weak model structure. It is straightforward to define this last weak model category. In thi model, the weak equivalences and the cofibrations are the leve-wise weak equivalences and fibrations respectively. Similarly, when the Reedy category is an inverse category, then the Reedy weak model structure is Quillen equivalent to the injective model structure. In this other case, weak equivalences and cofibrations are given level-wise.

4.3.2 Construction of path objects

We start by fixing a weak model category $\mathcal{M}$ and let J be the category

$$\begin{tikzcd} a \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & b \arrow[r, "k"] & c \end{tikzcd}$$

such that $ki = kj$.

Construction 4.3.5. Define the degree function making J into a direct category, $deg(a) = 0$, $deg(b) = 1$, $deg(c) = 2$. Consider the category of diagrams $\mathcal{M}^J$ with the Reedy weak model structure from Theorem 4.3.3. Additionally, we say that a map of diagrams $f : X \to Y$ is a **cofibration** if is a Reedy cofibration and if the induced maps

$$Y_a \sqcup_{X_a} X_c \overset{\sim}{\hookrightarrow} Y_c \text{ and } Y_b \sqcup_{X_b} X_c \overset{\sim}{\hookrightarrow} Y_c$$

are trivial cofibrations in $\mathcal{M}$. The class of **fibrations** are Reedy fibrations, which are level-wise fibrations since the category J is directed.

Observation 4.3.6. One can verify that in this new model structure the core fibrations and core trivial cofibrations coincide with the ones in the Reedy weak model structure.

The reader might suspect that this is not a fortuitous coincidence, this suspicions is well justified. One can develop a theory of right Bousfield localization of a weak model structure, the property we highlighted is a manifestation of this theory. Furthermore, one can also develop a general theory of localization of weak model categories. Unfortunately, this would divert us from the pourpose of this paper. For us, it will be enough to deal with the afore particular.

We examine the class of cofibrations. For a diagram $X \in \mathcal{M}^J$ the latching objects are $L_aX = \emptyset$, $L_bX = X_a \sqcup X_a$ and $L_cX = X_b \sqcup_{X_a} X_b$. These are cofibrant in $\mathcal{M}$. Then a map $f : X \to Y$ being a cofibration means that $X_a \hookrightarrow Y_a$,

$$X_b \sqcup_{X_a \sqcup X_a} (Y_a \sqcup Y_a) \hookrightarrow Y_b \text{ and } X_c \sqcup_{(X_b \sqcup_{X_a} X_b)} (Y_b \sqcup_{Y_a} Y_b) \hookrightarrow Y_c$$

are cofibrations in $\mathcal{M}$, additionally $Y_a \sqcup_{X_a} X_c \overset{\sim}{\hookrightarrow} Y_c$ and $Y_b \sqcup_{X_b} X_c \overset{\sim}{\hookrightarrow} Y_c$ are trivial cofibrations in $\mathcal{M}$.

Therefore, a diagram $Y \in \mathcal{M}^J$ is **cofibrant** if Y_a is a cofibrant object in $\mathcal{M}$,

$$Y_a \sqcup Y_a \hookrightarrow Y_b \text{ and } Y_b \sqcup_{Y_a} Y_b \hookrightarrow Y_c$$

are cofibrations, additionally $Y_a \overset{\sim}{\hookrightarrow} Y_c$ and $Y_b \overset{\sim}{\hookrightarrow} Y_c$ are trivial cofibrations. Spelling out the second Reedy condition give us the following commutative diagram:

$$\begin{tikzcd}
\emptyset \arrow[r, hook] \arrow[d, hook] & Y_a \arrow[d, hook] \arrow[ddr, bend left] & \\
Y_a \arrow[r, hook] \arrow[drr, bend right] & Y_a \sqcup Y_a \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, hook] & \\
& & Y_b
\end{tikzcd}$$

This says that both maps $Y_a \overset{Yi}{\underset{Yj}{\rightrightarrows}} Y_b$ are cofibrations. We can use this on the following diagram

$$\begin{tikzcd}
Y_a \arrow[r, hook] \arrow[d, hook] & Y_b \arrow[d, hook] \arrow[ddr, bend left] & \\
Y_b \arrow[r, hook] \arrow[drr, bend right] & Y_b \sqcup_{Y_a} Y_b \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, hook] & \\
& & Y_c
\end{tikzcd}$$

to conclude that $Y_b \hookrightarrow Y_c$ is a cofibration. Of course this is in principle not necessary since we also have $Y_b \overset{\sim}{\hookrightarrow} Y_c$ is a trivial cofibration, the novel aspect is that this follows only from Reedy cofibrancy. We also have a trivial cofibration $Y_a \overset{\sim}{\hookrightarrow} Y_c$, by the two-out-of-three property the maps $Y_a \overset{Yi}{\underset{Yj}{\rightrightarrows}} Y_b$ are trivial cofibrations. We collect the above in the follwing:

Remark 4.3.7. If Y is cofibrant then we obtain the following diagram:

$$\begin{tikzcd}
Y_a \sqcup Y_a \arrow[r, "\triangledown"] \arrow[d, hook] & Y_a \arrow[d, hook, "\sim"] \\
Y_b \arrow[r, hook, "\sim"] & Y_c.
\end{tikzcd}$$

This is just to say that cofibrant diagrams of $\mathcal{M}^J$ encode objects of $\mathcal{M}$ for which a weak cylinder exists in the sense of Construction 3.1.6.

Our goal is to show that the category of diagrams $\mathcal{M}^J$ has a weak model structure on it. We begin by showing:

Lemma 4.3.8. Any map between diagrams $f : X \to Y$, where X is a cofibrant diagram X and Y is a fibrant diagram in $\mathcal{M}^J$, can be factored as a trivial cofibration followed by a fibration.

Proof. Factor f as $X \overset{\sim}{\hookrightarrow} Z \twoheadrightarrow Y$ in the Reedy model structure. By Observation 4.3.6, it will be enough to show that Z is cofibrant in $\mathcal{M}^J$. Since $X \overset{\sim}{\hookrightarrow} Z$ is a Reedy trivial cofibration then $X_a \overset{\sim}{\hookrightarrow} Z_a$, $X_b \sqcup_{X_a \sqcup X_a} (Z_a \sqcup Z_a) \overset{\sim}{\hookrightarrow} Z_b$ and $X_c \sqcup_{(X_b \sqcup_{X_a} X_b)} (Z_b \sqcup_{Z_a} Z_b) \overset{\sim}{\hookrightarrow} Z_c$ are trivial cofibrations. We then obtain the following diagram:

$$\begin{tikzcd}
X_a \sqcup X_a \arrow[r] \arrow[d, hook, "\sim"] & X_b \arrow[d, hook, "\sim"] \arrow[ddr, bend left] & \\
Z_a \sqcup Z_a \arrow[r] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, hook, "\sim"] & \\
& & Z_b
\end{tikzcd}$$

This shows that $X_b \overset{\sim}{\hookrightarrow} Z_b$ is a trivial cofibration. Since X is cofibrant then all the maps in the diagram

$$\begin{tikzcd}
X_a \arrow[r, shift left] \arrow[r, shift right] & X_b \arrow[r] & X_c
\end{tikzcd}$$

are trivial cofibrations. Consider the commutative diagram where the back and front faces are pushouts

$$\begin{tikzcd}
X_a \arrow[rr, hook, "\sim"] \arrow[dr, hook, "\sim"] \arrow[dd, hook, "\sim"] & & X_b \arrow[dr, hook, "\sim"] \arrow[dd, hook, "\sim"] & \\
& Z_a \arrow[rr, hook, "\sim"] \arrow[dd, hook, "\sim"] & & Z_b \arrow[dd, hook, "\sim"] \\
X_b \arrow[rr, hook, "\sim"] \arrow[dr, hook, "\sim"] & & X_b \sqcup_{X_a} X_b \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr] & \\
& Z_b \arrow[rr, hook, "\sim"] & & Z_b \sqcup_{Z_a} Z_b, \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

which, by the two-out-of-three, shows that $X_b \sqcup_{X_a} X_b \overset{\sim}{\hookrightarrow} Z_b \sqcup_{Z_a} Z_b$ is a trivial cofibration. Remains to prove that $Z_b \overset{\sim}{\hookrightarrow} Z_c$ is a trivial cofibration. The pushout

$$\begin{tikzcd}
X_b \sqcup_{X_a} X_b \arrow[r] \arrow[d, hook, "\sim"] & X_c \arrow[d, hook, "\sim"] \arrow[ddr, bend left] & \\
Z_b \sqcup_{Z_a} Z_b \arrow[r] & \bullet \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, hook, "\sim"] & \\
& & Z_c
\end{tikzcd}$$

shows that $X_c \overset{\sim}{\hookrightarrow} Z_c$ is a trivial cofibration. Note that Z is Reedy cofibrant, hence $Z_b \hookrightarrow Z_c$ is a cofibration. By the 2-out-of-3 property we can conclude that $Z_b \overset{\sim}{\hookrightarrow} Z_c$ is indeed an acyclic cofibration. The above says that Z is cofibrant. $\square$

For the factorization of a diagram map $f : X \to Y$ in $\mathcal{M}^J$, with X cofibrant and Y fibrant, into a cofibration followed by a trivial fibration we will need an auxiliary class of diagrams. Denote by K the category J with the opposite Reedy structure given above (the degree function reversed). We endow $\mathcal{M}^K$ with the Reedy model structure. Then a diagram $Y \in \mathcal{M}^K$ is fibrant if $Y_c \twoheadrightarrow 1$, $Y_b \twoheadrightarrow Y_c$ and $Y_a \twoheadrightarrow Y_b \times_{Y_c} Y_b$ are fibrations in $\mathcal{M}$. Then Y_b is also fibrant. The limit of such diagram is simply the equalizer $Eq(Y_i, Y_j)$. Note that the following pullback also computes the limit of Y:

$$\begin{tikzcd}
P \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & Y_a \arrow[d, two heads] \\
Y_b \arrow[r] & Y_b \times_{Y_c} Y_b.
\end{tikzcd}$$

From this we conclude that $\mathtt{Lim}\, Y$ is a fibrant object of $\mathcal{M}$, and letting Z to denote the constant diagram at $\mathtt{Lim}\, Y$ then this comes with a diagram map $Z \to Y$ of the following form

$$\begin{tikzcd}
\mathtt{Lim}\, Y \arrow[r, shift left] \arrow[r, shift right] \arrow[d, two heads] & \mathtt{Lim}\, Y \arrow[r] \arrow[d, two heads] & \mathtt{Lim}\, Y \arrow[d, two heads] \\
Y_a \arrow[r, shift left] \arrow[r, shift right] & Y_b \arrow[r] & Y_c
\end{tikzcd}$$

where all top arrows are identities. Finally, note that Y being fibrant in $\mathcal{M}^K$ implies that both maps $Y_a \rightrightarrows Y_b$ are fibrations. This can be deduced from the following diagram:

$$\begin{tikzcd}
Y_a \arrow[dr, two heads] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Y_b \times_{Y_c} Y_b \arrow[r, two heads] \arrow[d, two heads] & Y_b \arrow[d, two heads] \\
& Y_b \arrow[r, two heads] & Y_c
\end{tikzcd}$$

Observation 4.3.9. Recall that the fibrations in $\mathcal{M}^J$ are the level-wise fibrations. Since $Z \in \mathcal{M}^K$ is point-wise fibrant then it is Reedy fibrant in

$\mathcal{M}^J$. Similarly, Y is Reedy fibrant in $\mathcal{M}^K$, in particular, implies that is object-wise fibrant, so it is fibrant in $\mathcal{M}^J$.

Lemma 4.3.10. The map $Z \to Y$ from above has the right lifting property with respect to any cofibration $A \hookrightarrow B$ between cofibrant objects in $\mathcal{M}^J$.

This is just to say that the map $Z \to Y$ is a trivial fibration in $\mathcal{M}^J$.

Proof. First, assume that B is a cofibrant object in $\mathcal{M}^J$ and Y a fibrant diagram in $\mathcal{M}^K$, we consider the lifting problem in $\mathcal{M}^J$:

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, hook] & Z \arrow[d] \\
B \arrow[r] & Y
\end{tikzcd}$$

From the discussion above we obtain the following commutative diagram:

$$\begin{tikzcd}
B_a \arrow[r, hook, shift left, "\sim"] \arrow[r, hook, shift right, "\sim"'] \arrow[d] & B_b \arrow[r, hook, "\sim"] \arrow[d] & B_c \arrow[d] \\
Y_a \arrow[r, two heads, shift left] \arrow[r, two heads, shift right] & Y_b \arrow[r, two heads] & Y_c
\end{tikzcd}$$

Thus, we obtain the following lifts:

$$\begin{tikzcd}
B_a \arrow[r] \arrow[d, hook, "\sim", "Bi"'] & Y_a \arrow[d, two heads, "Yi"] \\
B_b \arrow[r] \arrow[ur, dashed, "l_i"'] & Y_b
\end{tikzcd}
\quad
\begin{tikzcd}
B_a \arrow[r] \arrow[d, hook, "\sim", "Bj"'] & Y_a \arrow[d, two heads, "Yj"] \\
B_b \arrow[r] \arrow[ur, dashed, "l_j"'] & Y_b
\end{tikzcd}
\quad
\begin{tikzcd}
B_b \arrow[r] \arrow[d, hook, "\sim", "Bk"'] & Y_b \arrow[d, two heads, "Yk"] \\
B_c \arrow[r] \arrow[ur, dashed, "l_k"'] & Y_c
\end{tikzcd}$$

Using this we can construct the following commutative diagram:

$$\begin{tikzcd}
B_a \arrow[r, hook, "\sim"] \arrow[d, hook, "\sim"] & B_b \arrow[d, hook, "\sim"] \arrow[dr, "l_j"] & & \\
B_b \arrow[r, hook, "\sim"] \arrow[dr, "B_k"'] & B_b \sqcup_{B_a} B_b \arrow[r] \arrow[d, hook, "\sim"] \arrow[ul, phantom, "\ulcorner"] & Y_a \arrow[d, two heads] \arrow[dr, "Y_j"] & \\
 & B_c \arrow[r] \arrow[dr, "l_k"'] & Y_b \sqcup_{Y_c} Y_b \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & Y_b \arrow[d, two heads] \\
 & & Y_b \arrow[r, two heads] & Y_c
\end{tikzcd}$$

where the middle trivial cofibration and fibration come from B being cofibrant in $\mathcal{M}^J$ and Y being fibrant in $\mathcal{M}^K$ respectively. Then there exist

a map $B_c \xrightarrow{r} Y_a$ that fits in the diagram. Furthermore, we readily see from the diagram that $Y_j r = l_k = Y_i r$. Therefore, there is a unique arrow $B_c \xrightarrow{t} Eq(Y_i, Y_j) = \mathtt{Lim}\, Y$ making the obvious triangle commutative. By taking the appropriate compositions with the map t we can construct a diagram map $B \to Z$ such that is a solution to the lifting problem.
For the general case

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & Z \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

one can play the same game, the only change is that the diagram is a bit more involved. □

Lemma 4.3.11. If $Y \in \mathcal{M}^K$ is fibrant then there exists a trivial fibration $W \twoheadrightarrow Y \in \mathcal{M}^J$.

Proof. Since Y is fibrant $\mathcal{M}^K$, then it is fibrant in $\mathcal{M}^J$ as these are pointwise fibrant. Similarly, Z is fibrant in $\mathcal{M}^J$. We can take a Reedy cofibrant replacement in $\mathcal{M}^J$, $Z' \overset{\sim}{\twoheadrightarrow} Z$. From Lemma 4.3.8 we can factor this as $Z' \overset{\sim}{\hookrightarrow} W \twoheadrightarrow Y$. Let be $A \hookrightarrow B$ a cofibration between cofibrant objects in $\mathcal{M}^J$ and consider the lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & W \arrow[d, two heads] \\ B \arrow[r] & Y. \end{tikzcd}$$

In the diagrams

$$\begin{tikzcd} 0 \arrow[rr, hook] \arrow[d, hook] & & Z' \arrow[d, hook, "\sim"] \\ & & W \arrow[d, two heads] \\ A \arrow[r, hook] \arrow[rruu, dashed] & B \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[dd, hook] & Z' \arrow[d, hook, "\sim"] \\ & W \arrow[d, two heads] \\ B \arrow[r] \arrow[ruu, dashed] & Y \end{tikzcd}$$

we obtain the dashed arrow on the left because A is cofibrant and the vertical on the right is the fibration $Z' \overset{\sim}{\twoheadrightarrow} Z \overset{\sim}{\twoheadrightarrow} Y$, where the last arrow is a trivial fibration by Lemma 4.3.10. Similarly, for the problem on the right, the top horizontal arrow is now the diagonal on the left. We can assemble this arrows to give a solution to the original problem through Z'

$$\begin{tikzcd} A \arrow[rr] \arrow[d, hook] \arrow[dr, dashed] & & W \arrow[d, two heads] \\ & Z' \arrow[ur, hook, "\sim"] \arrow[dr] & \\ B \arrow[rr] \arrow[ur, dashed] & & Y. \end{tikzcd}$$

$\square$

Before giving the factorization, we need a technical result whose proof we ommit for the momment.

Lemma 4.3.12. For all $i : A \hookrightarrow B$ and $i' : A' \hookrightarrow B'$ cofibrations between cofibrant objects, for all $p : X \twoheadrightarrow Y$ fibration between fibrant objects, if there is a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\sim"] \arrow[d, hook, "i"'] & A' \arrow[d, hook, "i'"] \\ B \arrow[r, "\sim"] & B' \end{tikzcd}$$

then $i \pitchfork p$ if and only if $i' \pitchfork p$. The dual statement also holds: For all $i : A \hookrightarrow B$ cofibrations between cofibrant objects, for all $p : X \twoheadrightarrow Y$ and $p' : X' \twoheadrightarrow Y'$ fibrations between fibrant objects, if there is a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[d, two heads, "p"'] & X' \arrow[d, two heads, "p'"] \\ Y \arrow[r, "\sim"] & Y' \end{tikzcd}$$

then $i \pitchfork p$ if and only if $i \pitchfork p'$.

Lemma 4.3.13. Let $X \to Y$ be a map in $\mathcal{M}^J$ with X cofibrant and Y fibrant. Then such a map can be factored as a cofibration followed by a trivial fibration.

Proof. Observe first that Y can be assumed to be Reedy cofibrant in $\mathcal{M}^J$. Indeed, we can simply take a Reedy cofibrant replacement $Y' \overset{\sim}{\twoheadrightarrow} Y$, and instead use the dashed arrow

$$\begin{tikzcd} 0 \arrow[r, hook] \arrow[d, hook] & Y' \arrow[d, two heads, "\sim"] \\ X \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

Under this assumtion, Y is point-wise cofibrant, whence Reedy cofibrant in $\mathcal{M}^K$. Therefore, we can take a fibrant replacement in $\mathcal{M}^K$, $Y \overset{\sim}{\hookrightarrow} Y'$. Using [Hen20, Corollary 2.4.4] equivalences are preserved under pullbacks along fibrations, so we get the pullback square

$$\begin{tikzcd} LY \arrow[r, "\sim"] \arrow[d, two heads] & W \arrow[d, two heads] \\ Y \arrow[r, hook, "\sim"] & Y'. \end{tikzcd}$$

Furthermore, we know from Lemma 4.3.11 that $W \twoheadrightarrow Y'$ is a trivial fibration in $\mathcal{M}^J$. Therefore, it has the right lifting property against any cofibration between cofibrant objects in $\mathcal{M}^J$. We can use Lemma 4.3.12 to conclude that $LY \twoheadrightarrow Y$ satisfies the same property i.e. it is a trivial fibration in $\mathcal{M}^J$. Since X is cofibrant, we obtain a lift

$$\begin{tikzcd} 0 \arrow[r, hook] \arrow[d, hook] & LY \arrow[d, two heads, "\sim"] \\ X \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

The map $X \to LY$ can be factored in the Reedy model structure $\mathcal{M}^J$ as $X \hookrightarrow X' \overset{\sim}{\twoheadrightarrow} LY$. The diagram X' is cofibrant in $\mathcal{M}^J$ since is equivalent the cofibrant diagram LY, and X is cofibrant by assumption. Therefore, it follows from Observation 4.3.6 that the Reedy cofibration $X \hookrightarrow X'$ is a cofibration in the model $\mathcal{M}^J$. This give us the desired factorization in $\mathcal{M}^J$, $X \hookrightarrow X' \overset{\sim}{\twoheadrightarrow} Y$. □

All the previous work can be summarized in the following theorem. This exactly constructs the model structure on the category of diagrams $\mathcal{M}^J$, which as explained above, encodes objects with a weak cylinder object. Again we verify all the conditions of Definition 3.1.1

Theorem 4.3.14. The category of diagrams $\mathcal{M}^J$ has a weak model structure were

1. A map between diagrams $X \to Y$ is a cofibration if
   (a) It is a Reedy cofibration,
   (b) $Y_a \sqcup_{X_a} X_c \overset{\sim}{\hookrightarrow} Y_c$ and $Y_b \sqcup_{X_b} X_c \overset{\sim}{\hookrightarrow} Y_c$ are trivial cofibrations in $\mathcal{M}$.
2. Fibrations are level-wise fibrations.

Proof.

1. The existence of initial and terminal diagrams is clear.
2. The fact that it is a Reedy cofibration was established in Theorem 4.3.3. One can use the fact that in the weak model category $\mathcal{M}$ the acyclic cofibrations are pushout stable to conclude that the map obtained by the pushout in $\mathcal{M}^J$ is a cofibration in the new sense.
3. This is dual to the previous condition.
4. In addition to the property be true in the Reedy weak model structure on $\mathcal{M}^J$ we need to use repeatedly that maps isomorphic to weak equivalences are also weak equivalences, this is simply because the new condition we added involves the requirement that certain maps are weak equivalences.
5. The factorization of a map $f : X \to Y$, where X is cofibrant and Y is fibrant, into a cofibration followed by a trivial fibration is the content of Lemma 4.3.13.
6. The factorization of a map $f : X \to Y$, where X is cofibrant and Y is fibrant, into a tirvial cofibration followed by a fibration is the content of Lemma 4.3.8.
7. Solutions to lifting problems come from the Reedy weak model structure.
8. The 2-out-of-3 property is immediate.
9. This is also clear.

$\square$

4.3.3 Weak model on cylinders

Next, we consider a particular case of the diagram J

$$a \underset{j}{\overset{i}{\rightrightarrows}} b \xrightarrow{k} c$$

where the map k is an identity. By convenience, we denote this new diagram as $0 \to 2 \leftarrow 1$. In the resulting weak model structure $\mathcal{N}^J$, a map of diagrams $X \to Y \in \mathcal{N}^J$ is a cofibration if it is a Reedy cofibration and the maps

$$X_2 \sqcup_{X_1} Y_1 \overset{\sim}{\hookrightarrow} Y_2 \text{ and } X_2 \sqcup_{X_0} Y_0 \overset{\sim}{\hookrightarrow} Y_2$$

are trivial cofibration in N. Unwinding the definitions, a diagram $X \in \mathcal{N}^J$ is cofibrant if both maps $X_0 \overset{\sim}{\hookrightarrow} X_2$ and $X_1 \overset{\sim}{\hookrightarrow} X_2$ are trivial cofibrations.

Lemma 4.3.15. The functor $\mathcal{N}^J \to \mathcal{N}$ such that $A \to B \leftarrow C \in \mathcal{N}^J \mapsto A \in \mathcal{N}$, is a trivial fibration. Also the functor $\mathcal{N}^J \to \mathcal{N}$ such that $A \to B \leftarrow C \in \mathcal{N}^J \mapsto C \in \mathcal{N}$, is a trivial fibration.

Proof. Let $A \coloneqq a \overset{\sim}{\hookrightarrow} b \overset{\sim}{\hookleftarrow} c \in \mathcal{N}^J$ be a cofibrant diagram and $x \in \textsc{Cof}(\mathcal{N})$ a cofibrant object. We take the fibrant replacement of x and consider the pushout as indicated below, and we obtain a solution to the lifting problem on the right:

$$\begin{tikzcd}
 & a \arrow[r, hook] \arrow[d, hook, "\sim"'] \arrow[dr, phantom, "\lrcorner", very near end] & x \arrow[r, hook, "\sim"] \arrow[d, hook, "\sim"] & x^{fib} \\
c \arrow[r, hook, "\sim"] & b \arrow[r] & b \sqcup_a x \arrow[ur, dashed] &
\end{tikzcd}$$

The resulting map $c \to x^{fib}$ can be factored as $c \hookrightarrow z \overset{\sim}{\twoheadrightarrow} x^{fib}$. We can take further pushouts

$$\begin{tikzcd}
 & a \arrow[r, hook] \arrow[d, hook, "\sim"'] \arrow[dr, phantom, "\lrcorner", very near end] & x \arrow[r, hook, "\sim"] \arrow[d, hook, "\sim"] & x^{fib} \\
c \arrow[r, hook, "\sim"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near end] & b \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near end] & b \sqcup_a x \arrow[ur] \arrow[d, hook] & \\
z \arrow[r, hook, "\sim"'] \arrow[uurrr, two heads, bend right=60, "\sim"'] & z \sqcup_c b \arrow[r, hook] & P. &
\end{tikzcd}$$

There is a map $P \to x^{fib}$ which we can factor as $P \hookrightarrow y \overset{\sim}{\twoheadrightarrow} x^{fib}$, and the resulting diagram we get

$$\begin{tikzcd}
 & a \arrow[r, hook] \arrow[d, hook, "\sim"'] \arrow[dr, phantom, "\lrcorner", very near end] & x \arrow[r, hook, "\sim"] \arrow[d, hook, "\sim"] & x^{fib} \\
c \arrow[r, hook, "\sim"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near end] & b \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near end] & b \sqcup_a x \arrow[ur] \arrow[d, hook] & y \arrow[u, two heads, "\sim"'] \\
z \arrow[r, hook, "\sim"'] & z \sqcup_c b \arrow[r, hook] & P \arrow[ur, hook] &
\end{tikzcd}$$

Furthermore, there is a map $b \sqcup_a x \to y$ which is a cofibration as it is the composite of the two cofibrations. Using the 2-out-of-3 property repeatedly one concludes that the map $z \sqcup_c b \to y$ is a trivial cofibration. Thus, we have constructed the cofibrant object $X \coloneqq z \xhookrightarrow{\sim} y \xhookleftarrow{\sim} x \in \mathcal{N}^J$. The induced map $A \to X$ is a levelwise cofibration. The maps $b \sqcup_a x \to y$ and $b \sqcup_a z \to y$ are trivial cofibrations.

Remains to show that $A \to X$ is a Reedy cofibration. We already have that $a \to x$ and $c \to z$ are cofibrations. We now need to show that the induced map

$$\begin{tikzcd}
a \sqcup c \arrow[r] \arrow[d, hook] & b \arrow[d, hook] \arrow[ddr, bend left] & \\
x \sqcup z \arrow[r] \arrow[drr, bend right] & (x \sqcup z) \sqcup_{a \sqcup c} b \arrow[dr, dashed] & \\
& & y
\end{tikzcd}$$

is a cofibration. By diagram chasing one can show that the diagram

$$\begin{tikzcd}
a \sqcup c \arrow[r] \arrow[d] & b \arrow[d] \\
x \sqcup z \arrow[r] & (z \sqcup_c b) \sqcup_b (b \sqcup_a x)
\end{tikzcd}$$

commutes. One shows that the bottom right corner computes the pushout of the span. Using that the map $P \hookrightarrow y$ is a cofibration one concludes that $(x\sqcup) \sqcup_{a \sqcup c} b \to y$ is also a cofibration. This concludes the proof that $A \to X$ is a Reedy cofibration between cofibrant objects in $\mathcal{N}^J$. Therefore, it must a cofibration. We summarize our construction with the following diagram:

$$\begin{tikzcd}
c \arrow[r, hook] \arrow[d, hook, "\sim"] & z \arrow[d, hook, "\sim"] \\
b \arrow[r, hook] & y \\
a \arrow[r, hook] \arrow[u, hook, "\sim"] & x \arrow[u, hook, "\sim"] \\
a \arrow[r, hook] & x
\end{tikzcd}$$

This cofibration is a (strict) lift of $a \hookrightarrow x$, showing that the functor $\mathcal{N}^J \to N$ is a trivial fibration. Of course, the second part of the lemma is analogous. $\square$

We now want to see that any left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ part of a Quillen equivalence between weak model categories factors a section of a trivial fibration followed by a trivial fibration. To this end, consider the following category of diagrams

$$\mathcal{N}_F^J := \{Fa \to b \leftarrow c | a \in \mathcal{M}^{\mathrm{Cof}},\, b, c \in \mathcal{N}\}.$$

The weak model structure on this category is similar to that of $\mathcal{N}^J$, the only difference is that $X \to Y$ is a cofibration if $X_b \sqcup_{FX_a} FY_a \to Y_b$ is a trivial fibration. When F is the identity functor we recover $\mathcal{N}^J$. A cofibrant object in $\mathcal{N}_F^J$ is a diagram if the form

$$Fa \xhookrightarrow{\sim} b \xhookleftarrow{\sim} c.$$

Observation 4.3.16. With the set up above, it follows immediately from **??** that the projection $\pi_1 : \mathcal{N}_F^J \to \mathcal{M}$, sending each diagram $Fa \to b \leftarrow c$ to a, is a trivial fibration.

To show that the projection from $\pi_2 : \mathcal{N}_F^J \to \mathcal{N}$ sending each diagram $Fa \to b \leftarrow c$ to $c \in \mathcal{N}$ is a trivial fibration we make use of the following:

Lemma 4.3.17. Let $F : \mathcal{M} \to \mathcal{N}$ be a left Quillen functor part of a Quillen equivalence between weak model categories. For any objects $x \in \mathcal{M}^{\mathrm{Cof}}$, $y \in \mathcal{N}^{\mathrm{Fib}}$ and a map $f : Fx \to y$ there exists an object $z \in \mathcal{M}^{\mathrm{Cof}}$ such that f factors as

$$\begin{tikzcd} Fx \arrow[rr, "f"] \arrow[dr, hook] & & y \\ & Fz. \arrow[ur, "\sim"'] & \end{tikzcd}$$

Proof. We know that there is an isomorphism

$$\varphi : \mathrm{Hom}_{\mathcal{N}}(Fx, y) \simeq \mathrm{Hom}_{\mathcal{M}}(x, Gy) : \varphi^{-1}$$

given by the Quillen adjunction, natural in $x \in \mathcal{M}^{\mathrm{Cof}}$ and $y \in \mathcal{N}^{\mathrm{Fib}}$. Recall from [Hen20, 2.4.3 Proposition] that $F : \mathcal{M}^{\mathrm{Cof}} \to \mathcal{N}^{\mathrm{Cof}}$ and $G : \mathcal{N}^{\mathrm{Fib}} \to \mathcal{M}^{\mathrm{Fib}}$ preserve equivalences. Take φf the adjoint transpose of f. We can take a factorization

$$\begin{tikzcd} & z \arrow[dr, two heads, "s", "\sim"'] & \\ x \arrow[ur, hook, "r"] \arrow[rr, "\varphi f"'] & & Gy \end{tikzcd}$$

By naturality one checks that $f = \varphi^{-1} sFr$, where Fr is a cofibration. Since the Quillen pair is an equivalence we deduce from [Hen20, 2.4.5 Proposition (i)] that $\varphi^{-1}s$ is an equivalence. □

Corollary 4.3.18. Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a Quillen equivalence. Then the projection $\pi_2 : \mathcal{N}_F^J \to \mathcal{N}$sending each diagram $Fa \to b \leftarrow c$ to $c \in \mathcal{N}$ is a trivial fibration.

Proof. We show that in a situation as in

$$\begin{tikzcd}
Fa \arrow[d, hook, "\sim"] & \\
b & \\
c \arrow[u, hook, "\sim"'] \arrow[d, mapsto] & \\
c \arrow[r, hook] & z
\end{tikzcd}$$

there is a cofibrant object over z that projects onto $c \hookrightarrow z$. By taking a fibrant replacement, we can assume that the diagram is point-wise fibrant. From [Hen20, 2.2.3 Proposition] there exists a homotopy inverse of $c \overset{\sim}{\hookrightarrow} b$, this give us a map $Fa \to c$. Using Lemma 4.3.17 this last map can be factored as $Fa \hookrightarrow Fx \overset{\sim}{\to} c$. The rest of the proof continues as in Lemma 4.3.15. ☐

Theorem 4.3.19. Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a Quillen equivalence between weak model categories. Then, $F : \mathcal{M} \to \mathcal{N}$ admits a factorization $F = \pi_2 W_F$ where π_2 is a trivial fibration and W_F is the section of a trivial fibration.

Proof. The functor $F : \mathcal{M} \to \mathcal{N}$ can be factored as $\mathcal{M} \overset{W_F}{\to} \mathcal{N}_F^J \overset{\pi_2}{\to} \mathcal{N}$ where the functor $W_F : \mathcal{M} \to \mathcal{N}_F^J$ sends each $x \in \mathcal{M}$ to the constant diagram at Fx, this is a section of the projection π_1. We have shown in Lemma 4.3.15 and Corollary 4.3.18 that both projections are trivial fibrations. ☐

4.4 Proof of main theorem

Theorem 4.4.1. Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ a Quillen equivalence. Then for any cofibrant object $A \in \mathcal{M}$. The induced map $h\mathbb{L}F_A : h\mathbb{L}_\lambda^{\mathcal{M}}(A) \to h\mathbb{L}_\lambda^{\mathcal{N}}(FA)$ is an isomorphism.

Proof. Using Theorem 4.3.19 we obtain a factorization of $F = P_F W_f$ where W_F is a section of a trivial fibration. There is a span of trivial fibrations

$$\begin{tikzcd}
& \mathcal{J} \arrow[dl, two heads, "\sim"'] \arrow[dr, two heads, "\sim"] & \\
\mathcal{M} \arrow[rr, "F"'] & & \mathcal{N}
\end{tikzcd}$$

We can apply Theorem 4.1.12 to conclude that $h\mathbb{L}F_A : h\mathbb{L}_\lambda^{\mathcal{M}}(A) \to h\mathbb{L}_\lambda^{\mathcal{N}}(FA)$ is an isomorphism. This is becase the factorization of F also induces a factorization of $h\mathbb{L}F_A$ as in the diagram

$$\begin{tikzcd} & h\mathbb{L}_\lambda^{\mathcal{M}}(A) \arrow[dr, "h\mathbb{L}P_F"] & \\ h\mathbb{L}_\lambda^{\mathcal{M}}(A) \arrow[ur, "h\mathbb{L}W_F"] \arrow[rr, "h\mathbb{L}F_A"'] & & h\mathbb{L}_\lambda^{\mathcal{M}}(A) \end{tikzcd}$$

□

It is an immediate from Theorem 4.4.1 that

Corollary 4.4.2. For any Quillen equivalence $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$. The functors $Ho(F) \circ h\mathbb{L}_\lambda^{\mathcal{M}}$ and $h\mathbb{L}_\lambda^{\mathcal{N}} : Ho(\mathcal{N}) \to \mathbf{Bool}_\lambda$ are naturally isomorphic via $\mathbb{L}F$.

A Infinitary Cartmell theories

We introduce a generalization of *Cartmell theories*, also known as *generalized algebraic theories*, Cartmell [Car78]. This is straightforward and most of the proofs will be omitted since they are similar to those in [Car78], in very few cases we will need to provide new proofs. We claim no originality other than the generalization itself. We begin by recalling some definitions given in *Ibidem*. We assume to have a set of variables V whose size is $\aleph_0$ and an alphabet A. Informally, a **Cartmell generalized algebraic theory** consists of:

i) A set S, called the set of *sort symbols*,

ii) A set O, called the set of *operation symbols*,

iii) An introductory rule for each sort symbol,

iv) An introductory rule for each operation symbol,

v) A set of axioms.

To understand our generalization let us examine the previous definition in more detail, for this we need some preliminary notions. An **expression** is a finite sequence of $A \cup V \cup \{(\} \cup \{)\} \cup \{,\}$, inductively:

i) Elements of V and A are expressions,

ii) If $f \in A$ and $e_1, e_2, ..., e_n$ are expressions then $f(e_1, e_2, ..., e_n)$ is an expression.

The set of expression is denoted by E. This is simply to say that an expression is a finite string taken from the set $A \cup V \cup \{(\} \cup \{)\} \cup \{,\}$. A **premise** is a finite (possibly empty) sequence of $V \times E$. A **conclusion** will be an n-tuple of expressions i.e. any element of E^n for some $n \in \mathbb{N}$. Finally, a **rule** is given by a premise P and a conclusion C. Rules are written as: $P \vdash C$. This intends to convey the idea that under the premise P the conclusion C is a valid expression. Whenever P is a premise we will write $x_1 : \Delta_1,\ x_2 : \Delta_2, ..., x_n : \Delta_n$. For a conclusion this is slightly more involved since we differentiate depending on the size of the tuple. For example if we have a 1-tuple Δ then we write: $\Delta\,\mathsf{Type}$. We favour the notation ":" from type theory instead of the set theoretic one "ϵ" used by Cartmell. Furthermore, we will take advantage of conventions and notation from type theory.

The most important definition we will need to change is that of a *context*. In a Cartmell theory, a **context** is the premise such that a rule

$$x_1 : \Delta_1,\ x_2 : \Delta_2(x_1), ..., x_n : \Delta_n(x_1, x_2, \cdots, x_{n-1}) \vdash \Delta(x_1, x_2, \cdots, x_n)\ \mathsf{Type}$$

is a *derived rule*.

The only difference between Cartmell theories and infinitary Cartmell theories is that in the contexts we allow infinitely many variables. Just as any Cartmell theory gives rise to a contextual category, the same is true for the infinitary case with the appropriate generalized version of a contextual category.

A.1 Generalized algebraic theories

In this section we give the formal definition of an infinitary Cartmell theory. We follow Cartmell [Car78] to develop the theory, there will be some instances were a change has to be made. We could say that by changing in the definition every instance of "finite" by "size strictly less than κ" we get the correct notion, this is indeed the case. We carve out the definition with a fair amount of details since the applications we have in mind benefit from having an explicit syntax. The technicalities and motivations for introducing a generalized algebraic in the following way are presented in Cartmell [Car78].

From now on we fix a regular cardinal κ, unless otherwise stated, all other ordinals mentioned will be strictly smaller than κ.
Let V be a set such that $|V| = \kappa$, this set will be called the set of **variables**. We make an additional assumption on this set: Its elements have **canonical names**, this is $V = \{x_\alpha\}_{\alpha<\kappa}$, This also known as an **enumeration**. This is a minor assumption that allows to change variables. Otherwise we would need to proof a result similar [Car78, Corollary, pp 1.32][2]. Let A be any set which as before is called **alphabet**. Following [Car78] we define inductively the collection of **expressions** A^* over the alphabet A. An expression any λ-sequence of $A \cup V \cup \{(\} \cup \{)\} \cup \{,\}$ subject to:

i) If $x_\alpha \in V$ then $x_\alpha \in A^*$,

ii) If $F \in A$ then $F \in A^*$,

iii) If $F \in A$ and $\{e_\alpha\}_{\alpha<\lambda} \subseteq A^*$ then $F(e_\alpha)_{\alpha<\lambda} \in A^*$.

A **premise** is any λ-sequence of $V \times A^*$. We will usually write premises as $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$, where x_α are variables and Δ_α are expressions for $\alpha < \lambda$. Suppose we have a premise Γ, or later a *context*, and we need an extra premise (or *context*), according to our variable numeration we formally must to write $\Gamma, \{x_\alpha : \Delta_\alpha\}_{\lambda\leq\alpha<\mu}$ where λ represent the number of variables in Γ. This is clearly a problem when the expression complexity increases. In order to avoid overloading the notation we choose to reset the variable counting to only essential variables in use. Under this convention we will write $\Gamma, \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ instead. We will freely assume that Γ is a premise unless otherwise specified.

Definition A.1.1. A **judgment** is an expression over the alphabet A that has one of the following forms:

1. Type judgment: $\Gamma \vdash \Delta\ \mathsf{Type}$.

2. Element judgment: $\Gamma \vdash t : \Delta$.

3. Type equality judgment: $\Gamma \vdash \Delta \equiv \Delta'$.

4. Term equality judgment: $\Gamma \vdash t \equiv_\Delta t'$.

where Γ is a premise.

[2] This result states that under the substitution property the derived rules are stable under substitution of variables by another variables

Given a premise Γ, $\{e_\alpha\}_{\alpha<\lambda}$ expression and $\{x_\alpha\}_{\alpha<\lambda}$ variables then the new expression

$$\Gamma[e_\alpha|x_\alpha]_{\alpha<\lambda}$$

it is obtained by simultaneously changing the variables in Γ by the expressions. This process, unsurprisingly, is called **substitution** of variables. Along with the infinitary substitutions we will also allow operations to have possibly infinite arity. This is made explicit:

Definition A.1.2. A **κ-pretheory** T consist of the following data:

i) A set S, called the set of *sort symbols*,

ii) A set O, called the set of *operation symbols*,

iii) For each sort symbol B, a judgment of the form:

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash B(x_\alpha)_{\alpha<\lambda}\,\mathsf{Type}$$

where λ is some ordinal strictly smaller than κ,

iv) For each operator symbol F, a judgment:

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash F(x_\alpha)_{\alpha<\lambda} : \Delta$$

where λ is some ordinal strictly smaller than κ,

v) A set of judgements, each of which is either a type equality judgment or term equality judgment listed in Definition A.1.1. This is the set of **axioms** of the κ-pretheory.

The following definitions are of inductive nature:

Definition A.1.3.

1. A premise $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a **context** if the judgment

$$\{x_\beta : \Delta_\beta\}_{\beta<\alpha} \vdash \Delta_\alpha\,\mathsf{Type}$$

is a *derived judgment* of T for every $\alpha < \lambda$. Whenever we want to specify that a premise Γ is a context we will write $\vdash \Gamma\,\mathsf{Ctxt}$.

2. The judgment

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta\,\mathsf{Type}$$

is a **well-formed judgment** of T if and only if $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a context.

3. The judgment

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta$$

is **well-formed** if and only if

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \ \mathsf{Type}$$

is a *derived judgment* of T.

Definition A.1.4. Let T be a κ-pretheory. The set of **derived judgments** of T are the ones that can be derived from the following list:

1. $$\frac{\Gamma \vdash A \ \mathsf{Type}}{\Gamma \vdash A \equiv A}$$

2. $$\frac{\Gamma \vdash t : A}{\Gamma \vdash t \equiv_A t}$$

3. $$\frac{\Gamma \vdash A_1 \equiv A_2}{\Gamma \vdash A_2 \equiv A_1}$$

4. $$\frac{\Gamma \vdash t_1 \equiv_A t_2}{\Gamma \vdash t_2 \equiv_A t_1}$$

5. $$\frac{\Gamma \vdash A_1 \equiv A_2 \qquad \Gamma \vdash A_2 \equiv A_3}{\Gamma \vdash A_1 \equiv A_3}$$

6. $$\frac{\Gamma \vdash t_1 \equiv_A t_2 \qquad \Gamma \vdash t_2 \equiv_A t_3}{\Gamma \vdash t_1 \equiv_A t_3}$$

7. $$\frac{\Gamma \vdash A_1 \equiv A_2 \qquad \Gamma \vdash t_1 \equiv_{A_1} t_2}{\Gamma \vdash t_2 \equiv_{A_2} t_1}$$

8.

$$\frac{\Gamma \vdash A_1 \equiv A_2 \qquad \Gamma \vdash t : A_1}{\Gamma \vdash t : A_2}$$

9.

$$\frac{\Gamma,\ \{x_\delta : A_\delta\}_{\delta<\beta<\lambda} \vdash A_\beta \ \mathsf{Type}}{\Gamma,\ \{x_\alpha : A_\alpha\}_{\alpha<\lambda} \vdash x_\alpha : A_\alpha}$$

10.

$$\frac{\{x_\alpha : A_\alpha\}_{\alpha<\lambda} \vdash B(x_\lambda)\ \mathsf{Type}, \quad \vdash \Gamma\ \mathsf{Ctxt}, \quad \Gamma \vdash t_\alpha : B[t_\alpha|x_\alpha]}{\Gamma \vdash B(t_\lambda)\ \mathsf{Type}}$$

This is true for any B sort symbol with a well-formed introduction type judgment.

11.

$$\frac{\Gamma,\ \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash F(x_\lambda) : \Delta, \quad \Gamma \vdash t_\alpha : \Delta_\alpha[t_\alpha|x_\alpha]}{\Gamma,\ \{t_\alpha : \Delta_\alpha[t_\alpha|x_\alpha]\}_{\alpha<\lambda} \vdash F(t_\lambda) : \Delta[t_\lambda \mid x_\lambda]}$$

This is true for any F opertator symbol with a well-formed introduction type element judgment.

12.

$$\frac{\begin{array}{c}\vdash \Gamma\ \mathsf{Ctxt} \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \equiv \Delta' \\ \Gamma,\ t_\alpha : \Delta_\alpha[t_\beta \mid x_\beta]_{\beta<\alpha},\ t'_\alpha : \Delta'_\alpha[t'_\beta \mid x_\beta]_{\beta<\alpha} \vdash t_\alpha \equiv_{\Delta_\alpha[t_\beta|x_\beta]_{\beta<\alpha}} t'_\alpha\end{array}}{\begin{array}{c}\Gamma,\ \{t_\alpha : \Delta_\alpha[t_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda},\ \{t'_\alpha : \Delta'_\alpha[t'_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda} \\ \vdash \Delta[t_\alpha|x_\alpha]_{\alpha<\lambda} \equiv \Delta'[t'_\alpha|x_\alpha]_{\alpha<\lambda}\end{array}}$$

13.

$$\frac{\begin{array}{c}\vdash \Gamma\ \mathsf{Ctxt} \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t' \\ \Gamma,\ s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha},\ s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha} \vdash s_\alpha \equiv_{\Delta_\alpha[s'_\beta|x_\beta]_{\beta<\alpha}} s'_\alpha\end{array}}{\begin{array}{c}\Gamma,\ \{s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda},\ \{s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda} \\ \vdash t[s_\alpha \mid x_\alpha]_{\alpha<\lambda} \equiv_{\Delta[s_\alpha|x_\alpha]_{\alpha<\lambda}} t'[s'_\alpha \mid x_\alpha]_{\alpha<\lambda}\end{array}}$$

14. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \equiv \Delta'$ is an axiom then

$$\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta\ \mathsf{Type} \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta'\ \mathsf{Type},}{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \equiv \Delta'}$$

15. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t'$ is an axiom then

$$\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t' : \Delta}{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t'}$$

We are now ready for the following:

Definition A.1.5. A κ-pretheory T is **well-formed** if all its rules are well-formed. A κ-**generalized algebraic theory** or κ-**Cartmell theory** is a well-formed κ-pretheory.

Remark A.1.6. Observe that a generalized algebraic theory as defined by Cartmell [Car78] is the same as an ω-generalized algebraic theory in our sense.

We introduce an important example of κ-algebraic theories.

Example A.1.7. Let Cat denote the ω-algebraic theory defined in the following way:

1. Type of objects: $\vdash \mathsf{Ob}\,\mathsf{Type}$.
2. Type of morphisms: $x : \mathsf{Ob},\, y : \mathsf{Ob} \vdash \mathsf{Hom}(x, y)\,\mathsf{Type}$.
3. Composition operation: $x : \mathsf{Ob},\, y : \mathsf{Ob},\, z : \mathsf{Ob},\, f : \mathsf{Hom}(x, y),\, g : \mathsf{Hom}(y, z) \vdash g \circ f : \mathsf{Hom}(x, z)$.
4. Identity operator: $x : \mathsf{Ob} \vdash \mathsf{id}_x : \mathsf{Hom}(x, x)$.

Subject to the following axioms:

$$\frac{x : \mathsf{Ob},\, y : \mathsf{Ob},\, f : \mathsf{Hom}(x, y)}{\mathsf{id}_y \circ f \equiv f} \qquad \frac{x : \mathsf{Ob},\, y : \mathsf{Ob}, f : \mathsf{Hom}(x, y)}{f \circ \mathsf{id}_x \equiv f}$$

$$\frac{x : \mathsf{Ob},\, y :: \mathsf{Ob},\, z : \mathsf{Ob},\, w : \mathsf{Ob},\, f : \mathsf{Hom}(x, y), g : \mathsf{Hom}(y, z), h : \mathsf{Hom}(z, w)}{(h \circ g) \circ f \equiv h \circ (g \circ f)}$$

A.2 Substitution property

Let T be a κ-Cartmell theory. Recall that given Δ, $\{t_\alpha\}_{\alpha<\lambda}$ expressions and $\{x_\alpha\}_{\alpha<\lambda}$ variables then the new expression $\Delta[e_\alpha|x_\alpha]_{\alpha<\lambda}$ denotes the substitution of variables by the expressions.

Definition A.2.1. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta$ be a derived judgment of T. We say that this judgment has the **substitution property** if for every $\vdash \Gamma\,\mathsf{Ctxt}$ and expressions $\{t_\alpha\}_{\alpha<\lambda}$, such that for all $\alpha < \lambda$

$$\Gamma,\, \{t_\beta : \Delta_\beta[t_\gamma|x_\gamma]_{\gamma<\beta}\}_{\beta<\alpha} \vdash t_\alpha : \Delta_\alpha[t_\beta|x_\beta]_{\beta<\alpha}$$

are derived rules then

$$\Gamma \vdash \Delta[t_\alpha|x_\alpha]_{\alpha<\lambda}$$

is a derived rule of T.

In [Car78] it is proven that all derived judgment of a generalized algebraic theory satisfy the substitution property. This is done through a series of results that can be generalized to our setting. The proofs are omitted since they are the same as in the original reference.

Lemma A.2.2. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta$ is a derived judgment of T then the variables that appear in Δ is a subset of $\{x_\alpha\}_{\alpha<\lambda}$

Proof. See [Car78, Lemma 1, Section 1.7]. □

Lemma A.2.3.
1. The premise of a derived judgment is a context.
2. If $\vdash \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}\mathsf{Ctxt}$ then for $\alpha < \lambda$, we have

$$\{x_\beta : \Delta_\beta\}_{\beta<\alpha} \vdash \Delta_\alpha\ \mathsf{Type}$$

Proof. See [Car78, Lemma 2, Section 1.7]. □

Theorem A.2.4. Every derived judgment of a κ-Cartmell theory has the substitution property.

Proof. The same as proof as in [Car78, 1.7] applies. This goes by proving that each judgment has the substitution property. For the last two judgments in Definition A.1.1 this is part of Definition A.1.4. While for the first two it is done by induction on the derivations. It is shown that each derivation rule of Definition A.1.4 preserve the substitution property. □

This result has similar consequences to those in [Car78]. The proofs are analogous or the same. For us is only relevant to know that our κ-Cartmell theories are well defined. Meaning:

Proposition A.2.5. The derived judgments of a κ-Cartmell theory are well-formed.

Proof. Again, by induction on the derivations [Car78, pp. 1.33]. □

Both the statement and proof of the next lemma are the same as The Derivation Lemma [Car78, pp. 1.34]. The proof does not rely on the context size.

Lemma A.2.6. 1. Every derived type judgment of T is of the form

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash A(t_\alpha)_{\alpha<\lambda}$$

for some type symbol A with introductory rule

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash A(x_\alpha)_{\alpha<\lambda} \ \mathsf{Type}$$

and $\{t_\alpha\}_{\alpha<\lambda}$ are expressions such that for all $\alpha < \lambda$ the rule

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\alpha : \Delta_\alpha[t_\delta \mid x_\delta]_{\delta<\alpha}.$$

2. Every type element judgment of T is of the form

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash x_\beta : \Omega$$

for some x_β and such that $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega_\beta \equiv \Omega$, or is of the form

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash f(t_\alpha)_{\alpha<\lambda} : \Omega$$

for some operator symbol f of T with introductory judgment of the form

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash f(x_\alpha)_{\alpha<\lambda} : \Delta$$

such that for each $\alpha < \lambda$ the rules

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\alpha : \Delta_\alpha[t_\delta \mid x_\delta]_{\delta<\alpha}$$

and

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Delta[t_\alpha \mid x_\alpha]_{\alpha<\lambda} \equiv \Omega$$

are derived rules of T.

Proof. This is follows from Definition A.1.4 (10) and (11). □

A.3 Equivalence relation on judgments

Trough out this section we work in an κ-Cartmell theory. We first introduce a relation that allows us to identify context which express the same meaning, but differ on the variables that are used in it [Car78, 1.13].

There is a relation defined on the judgments of the κ-Cartmell theory T.

Definition A.3.1. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta_\lambda\ \mathsf{Type}$, $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega_\mu\ \mathsf{Type}$ be two type judgments of T. We say that

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta_\lambda\ \mathsf{Type} \approx \{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega_\mu\ \mathsf{Type}$$

if either:

1. Both ordinals are successors such that $\lambda = \mu = \nu + 1$ and for all $\alpha \leq \nu$ we have
$$\{x_\delta : \Delta_\delta\}_{\delta<\alpha} \vdash \Delta_\alpha \equiv \Omega_\alpha$$
is a derived rule of T.

2. Both ordinals are limits with $\lambda = \mu$ and for any successor ordinal $\nu + 1 < \lambda$ we have
$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\nu} \vdash \Delta_\nu\ \mathsf{Type} \approx \{x_\beta : \Omega_\beta\}_{\beta<\nu} \vdash \Omega_\nu\ \mathsf{Type}.$$

Lemma A.3.2. The relation $\approx$ is an equivalence relation on type judgments of the theory T.

Proof. x This is an immediate result since we have assumed canonical names for variables. Otherwise we could repeat the argument as in [Car78, 1.13]. □

Definition A.3.3. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ and $\{x_\beta : \Omega_\beta\}_{\beta<\mu}$ be two contexts. We say that
$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x_\beta : \Omega_\beta\}_{\beta<\mu}$$
if and only if $\lambda = \mu$ and for all $\alpha < \lambda$

$$\{x_\delta : \Delta_\delta\}_{\delta<\alpha} \vdash \Delta_\alpha\ \mathsf{Type} \approx \{x_\gamma : \Omega_\gamma\}_{\gamma<\alpha} \vdash \Omega_\alpha\ \mathsf{Type}$$

It follows that this induces an equivalence relation on contexts.

Definition A.3.4. We say that

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta \approx \{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash s : \Omega$$

if and only if $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta\ \mathsf{Type} \approx \{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega\ \mathsf{Type}$ and $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv s$.

Remark A.3.5. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ and $\{x_\beta : \Omega_\beta\}_{\beta<\mu}$ be two contexts. Assume further that

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x_\beta : \Omega_\beta\}_{\beta<\mu}.$$

Then for all derived rules

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega$$

the rule

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Omega$$

is also a derived rule.

Regardless of its simplicity this remark is useful in the next:

Corollary A.3.6. The relation $\approx$ is an equivalence relation on judgments of the form $\{x_\beta : \Delta_\beta\}_{\beta<\mu} \vdash t : \Delta$.

Proof. Reflexivity is a consequence of 2 from Definition A.1.4. Assume that $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta \approx \{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda} \vdash s : \Omega$. Hence the contexts satisfy $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda}$. Applying the symmetry of the relation $\approx$ on contexts and using Remark A.3.5 we see that that $\{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda} \vdash t \equiv s$. Then we must have $\{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda} \vdash s : \Delta$ and $\{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda} \vdash \Omega \equiv \Delta$. We can apply 4 from Definition A.1.4 to conclude that $\{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda} \vdash s \equiv t$, thus proving symmetry. Transitivity is a straightforward application of Remark A.3.5. $\square$

Definition A.3.7. A **morphism** between contexts

$$\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$$

is μ-sequence of terms $\{t_\beta\}_{\beta<\mu}$ such that for all $\beta < \mu$ we have

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t_\beta : \Omega_\beta[t_\gamma | x_\gamma]_{\gamma<\beta}.$$

Just as in the finite case, with the substitution as composition and the obvious identity, it can be shown that contexts form a category with morphism as defined above. This is called the **category of realizations** of the theory T. The composition of

$$\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$$

and

$$\langle s_\delta \rangle_{\delta<\nu} : \{x_\beta : \Omega_\beta\}_{\beta<\mu} \to \{x_\delta : \Omega'_\delta\}_{\delta<\nu}$$

is the map

$$\langle s_\delta\rangle_{\delta<\nu}\circ\langle t_\beta\rangle_{\beta<\mu}:\{x_\alpha:\Delta_\alpha\}_{\alpha<\lambda}\to\{x_\delta:\Omega'_\delta\}_{\delta<\nu}$$

defined as the sequence $\langle s_\delta[\langle t_\beta|x_\beta\rangle_{\beta<\mu}]\rangle_{\delta<\nu}$.

Using the previous relation $\approx$ on contexts and rules we induce one on morphisms between contexts. If we have morphisms

$$\langle t_\beta\rangle_{\beta<\mu}:\{x_\alpha:\Delta_\alpha\}_{\alpha<\lambda}\to\{x_\beta:\Omega_\beta\}_{\beta<\mu}\text{ and }\langle t'_\beta\rangle_{\beta<\mu}:\{x_\alpha:\Delta'_\alpha\}_{\alpha<\lambda}\to\{x_\beta:\Omega'_\beta\}_{\beta<\mu}$$

Then

$$\langle t_\beta\rangle_{\beta<\mu}\approx\langle t'_\beta\rangle_{\beta<\mu}$$

if and only if

$$\{x_\beta:\Omega_\beta\}_{\beta<\mu}\approx\{x'_\beta:\Omega'_\beta\}_{\beta<\mu}$$

and for all $\gamma<\mu$

$$\{x_\beta:\Delta_\beta\}_{\beta<\mu}\vdash t_\gamma:\Omega_\gamma[t_{\gamma'}|x_{\gamma'}]_{\gamma'<\gamma}\approx\{x_\beta:\Delta'_\beta\}_{\beta<\mu}\vdash t'_\gamma:\Omega'_\gamma[t'_{\gamma'}|x_{\gamma'}]_{\gamma'<\gamma}.$$

Unfolding the definition this means that

$$\{x_\beta:\Delta_\beta\}_{\beta<\mu}\vdash\Omega_\gamma[t_{\gamma'}|x_{\gamma'}]_{\gamma'<\gamma}\,\mathsf{Type}\approx\{x_\beta:\Delta'_\beta\}_{\beta<\mu}\vdash\Omega'_\gamma[t'_{\gamma'}|x_{\gamma'}]_{\gamma'<\gamma}\,\mathsf{Type}$$

and that $\{x_\beta:\Delta_\beta\}_{\beta<\mu}\vdash t_\gamma\equiv t'_\gamma$ for all $\gamma<\mu$.

The following remarks are results from [Car78] whose proofs are completely similar. However, it is important to make them explicit since they imply that we can define a composition operation of equivalence classes of morphisms between contexts.

Remark A.3.8. Let $\langle t_\beta\rangle_{\beta<\mu}:\{x_\alpha:\Delta_\alpha\}_{\alpha<\lambda}\to\{x_\beta:\Omega_\beta\}_{\beta<\mu}$ and $\langle t'_\beta\rangle_{\beta<\mu}:\{x_\alpha:\Delta_\alpha\}_{\alpha<\lambda}\to\{x_\beta:\Omega'_\beta\}_{\beta<\mu}$ two morphisms between contexts with $\langle t_\beta\rangle_{\beta<\mu}\approx\langle t'_\beta\rangle_{\beta<\mu}$.

1. If $\{x_\beta:\Omega_\beta\}_{\beta<\mu}\vdash\Omega\,\mathsf{Type}$ and $\{x_\beta:\Omega'_\beta\}_{\beta<\mu}\vdash\Omega'\,\mathsf{Type}$ are derived judgment of the theory such that

$$\{x_\beta:\Omega_\beta,\,x_\mu:\Omega\}_{\beta<\mu}\approx\{x_\beta:\Omega'_\beta,\,x_\mu:\Omega'\}_{\beta<\mu}$$

then

$$\{x_\alpha:\Delta_\alpha,\,x_\mu:\Omega[t_\beta|x_\beta]_{\beta<\mu}\}_{\alpha<\lambda}\approx\{x_\alpha:\Delta'_\alpha,\,x_\mu:\Omega'[t'_\beta|x'_\beta]_{\beta<\mu}\}_{\alpha<\lambda}$$

This follows by unwinding the relation $\approx$ and applying the principle 12 from Definition A.1.4. This simply means that we can extend contexts by a fresh variable. Moreover, there is a more general result:

For all $\varepsilon > 0$, if $\{x_\beta : \Omega_\beta\}_{\beta<\mu+\varepsilon}$ and $\{x_\beta : \Omega'_\beta\}_{\beta<\mu+\varepsilon}$ are contexts then

$$\{x_\alpha : \Delta_\alpha,\, x_\beta : \Omega_\beta[t_\gamma|x_\gamma]_{\gamma<\beta}\}_{\substack{\alpha<\lambda,\\ \mu\leq\beta<\mu+\varepsilon}} \approx \{x_\alpha : \Delta'_\alpha,\, x_\beta : \Omega'_\beta[t'_\gamma|x_\gamma]_{\gamma<\beta}\}_{\substack{\alpha<\lambda,\\ \mu\leq\beta<\mu+\varepsilon}}$$

2. If $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash s : \Omega$ and $\{x_\beta : \Omega'_\beta\}_{\beta<\mu} \vdash s' : \Omega'$ are derived judgment such that

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash s \equiv_\Omega s'.$$

Then

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash s[t_\beta|x_\beta]_{\beta<\mu} \equiv_{\Omega[t_\beta|x_\beta]_{\beta<\mu}} s'[t'_\beta|x_\beta]_{\beta<\mu}.$$

Observe that the principle 13 from Definition A.1.4 implies this result.

Remark A.3.9. 1. Let $\langle t_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ be a morphism between two contexts. If

$$\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x'_\alpha : \Delta'_\alpha\}_{\alpha<\lambda} \text{ and } \{x_\beta : \Omega_\beta\}_{\beta<\mu} \approx \{x'_\beta : \Omega'_\beta\}_{\beta<\mu}$$

then $\langle t_\beta\rangle_{\beta<\mu} : \{x'_\alpha : \Delta'_\alpha\}_{\alpha<\lambda} \to \{x'_\beta : \Omega'_\beta\}_{\beta<\mu}$ is also a morphisms between these contexts.

2. If we have a context $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+1}$ and $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x'_\alpha : \Delta'_\alpha\}_{\alpha<\lambda}$ then we can extend the context $\{x'_\alpha : \Delta'_\alpha\}_{\alpha<\lambda}$ to $\{x'_\alpha : \Delta'_\alpha\}_{\alpha<\lambda+1}$ such that $x'_\alpha : \Delta'_\alpha$ is $x_\lambda : \Delta_\lambda$.

Remark A.3.10. Let $\langle t_\beta\rangle_{\beta<\mu+1} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu+1}$ and $\langle s_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ be morphisms between contexts. Then we have a morphism

$$\langle s_\beta\rangle_{\beta<\mu+1} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu+1}$$

where $s_\mu \equiv t_\mu$, and such that $\{s_\beta\}_{\beta<\mu+1} \approx \{t_\beta\}_{\beta<\mu+1}$.

A.4 The category of κ-Cartmell theories

We construct a category where the objects are κ-Cartmell theories with maps *interpretations*. This is analogous the category that Cartmell constructs in [Car78, 1.11], all the results can be copied from there to our setting. Since we work with different theories the alphabets, expressions and rules are marked accordingly. If T is a theory then these sets are denoted $Alp(T)$, $Exp(T)$, $Rul(T)$ respectively.

Let T and T' two κ-Cartmell theories. Let any function $I : Alp(T) \to Exp(T')$. Using this function we can define a **preinterpretation** $\widetilde{I} : Exp(T) \to Exp(T')$ by induction on the construction of expressions:

1. If $x \in V$
$$\widetilde{I}(x) := x,$$
2. If $F \in Alp(T)$
$$\widetilde{I}(F) := I(F),$$
3. If $L \in Alp(T)$ alphabet symbol and $\{t_\alpha\}_{\alpha<\lambda}$ are expressions
$$\widetilde{I}(L(t_\alpha)_{\alpha<\lambda}) := I(L)(\widetilde{I}(t_\alpha))_{\alpha<\lambda}.$$

Definition A.4.1. Given a preinterpretation $\widetilde{I}$ we define a new function $\widehat{I} : Rul(T) \to Rul(T')$.

1. $\widehat{I}(\Gamma \vdash \Delta \ \mathsf{Type}) := \widetilde{I}(\Gamma) \vdash \widetilde{I}(\Delta) \ \mathsf{Type}$
2. $\widehat{I}(\Delta \vdash t : \Delta) := \widetilde{I}(\Delta) \vdash \widetilde{I}(t) : \widetilde{I}(\Delta)$
3. $\widehat{I}(\Delta,\, \Delta' \vdash \Delta \equiv \Delta') := \widetilde{I}(\Delta),\, \widetilde{I}(\Delta') \vdash \widetilde{I}(\Delta) \equiv \widetilde{I}(\Delta')$.
4. $\widehat{I}(\Delta,\, t, t' : \Delta \vdash t \equiv_\Delta t') := \widetilde{I}(\Delta),\, \widetilde{I}(t), \widetilde{I}(t') : \widetilde{I}(\Delta) \vdash \widetilde{I}(t) \equiv_{\widetilde{I}(\Delta)} \widetilde{I}(t')$.

This function is an **interpretation** from T into T' if all introductory judgment and axioms of T are sent to introductory judgment and axioms of T', we will simply denote this as $I : T \to T'$.

Just as in [Car78] it is possible to prove that:

Lemma A.4.2. If I is an interpretation from T to T' then it preserves derived judgment of the theory T.

Proof. From Lemma 2 [Car78, pp 1.52]. To illustrate how this is done we show that the derived judgment Definition A.1.4 (13) it is preserved by I. Consider the derived judgment

$$\frac{\begin{array}{c}\vdash \Gamma \ \mathsf{Ctxt} \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t' \\ \Gamma,\, s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha},\, s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha} \vdash s_\alpha \equiv_{\Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha}} s'_\alpha\end{array}}{\begin{array}{c}\Gamma,\, \{s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda},\, \{s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda} \\ \vdash t[s_\alpha \mid x_\alpha]_{\alpha<\lambda} \equiv_{\Delta[s_\alpha \mid x_\alpha]_{\alpha<\lambda}} t'[s'_\alpha \mid x_\alpha]_{\alpha<\lambda}\end{array}}$$

in the theory T. We may assume that the context Γ is of the form $\{x_\beta : \Omega_\beta\}_{\beta<\mu}$, so we get

$$\frac{\begin{array}{c}\vdash \{x_\beta : \Omega_\beta\}_{\beta<\mu} \ \mathsf{Ctxt} \qquad \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t' \\ \{x_\beta : \Omega_\beta\}_{\beta<\mu},\, s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha},\, s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha} \vdash s_\alpha \equiv_{\Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha}} s'_\alpha\end{array}}{\begin{array}{c}\{x_\beta : \Omega_\beta\}_{\beta<\mu},\, \{s_\alpha : \Delta_\alpha[s_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda},\, \{s'_\alpha : \Delta_\alpha[s'_\beta \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda} \\ \vdash t[s_\alpha \mid x_\alpha]_{\alpha<\lambda} \equiv_{\Delta[s_\alpha \mid x_\alpha]_{\alpha<\lambda}} t'[s'_\alpha \mid x_\alpha]_{\alpha<\lambda}\end{array}}$$

Applying the I to the hypothesis and by Lemma A.4.3 we obtain the following derivations in T'.

- $\vdash \{x_\beta : \widetilde{I}(\Omega_\beta)\}_{\beta<\mu}\,\mathsf{Ctxt}$,
- $\{x_\alpha : \widetilde{I}(\Delta_\alpha)\}_{\alpha<\lambda} \vdash \widetilde{I}(t) \equiv_\Delta \widetilde{I}(t')$,
- $\{x_\beta : \widetilde{I}(\Omega_\beta)\}_{\beta<\mu},\ s_\alpha : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s_\beta) \mid x_\beta]_{\beta<\alpha},\ \widetilde{I}(s'_\alpha) : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s'_\beta) \mid x_\beta]_{\beta<\alpha} \vdash \widetilde{I}(s_\alpha) \equiv_{\widetilde{I}(\Delta_\alpha)[\widetilde{I}(s'_\beta)|x_\beta]_{\beta<\alpha}} \widetilde{I}(s'_\alpha)$.

We have all the requirements to use Definition A.1.4 (13) for the theory T'. Thus

$$\frac{\begin{array}{c}\vdash \{x_\beta : \widetilde{I}(\Omega_\beta)\}_{\beta<\mu}\,\mathsf{Ctxt} \qquad \{x_\alpha : \widetilde{I}(\Delta_\alpha)\}_{\alpha<\lambda} \vdash \widetilde{I}(t) \equiv_\Delta \widetilde{I}(t') \\ \{x_\beta : \widetilde{I}(\Omega_\beta)\}_{\beta<\mu},\ s_\alpha : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s_\beta) \mid x_\beta]_{\beta<\alpha},\ \widetilde{I}(s'_\alpha) : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s'_\beta) \mid x_\beta]_{\beta<\alpha} \\ \vdash \widetilde{I}(s_\alpha) \equiv_{\widetilde{I}(\Delta_\alpha)[\widetilde{I}(s'_\beta)|x_\beta]_{\beta<\alpha}} \widetilde{I}(s'_\alpha)\end{array}}{\begin{array}{c}\{x_\beta : \widetilde{I}(\Omega_\beta)\}_{\beta<\mu}, \{\widetilde{I}(s_\alpha) : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s_\beta) \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda}, \{\widetilde{I}(s'_\alpha) : \widetilde{I}(\Delta_\alpha)[\widetilde{I}(s'_\beta) \mid x_\beta]_{\beta<\alpha}\}_{\alpha<\lambda} \\ \vdash \widetilde{I}(t)[\widetilde{I}(s_\alpha) \mid x_\alpha]_{\alpha<\lambda} \equiv_{\widetilde{I}(\Delta)[\widetilde{I}(s_\alpha)|x_\alpha]_{\alpha<\lambda}} \widetilde{I}(t')[\widetilde{I}(s'_\alpha) \mid x_\alpha]_{\alpha<\lambda}\end{array}}$$

is a derived rule of T'. Therefore, the rule is preserved by the interpretation I.

□

The following lemma fills the gap:

Lemma A.4.3. If I is an interpretation of T into T' and we have expressions f and $\{t_\alpha\}_{\alpha<\lambda}$ on the alphabet A_T then

$$\widetilde{I}(f[t_\alpha \mid x_\alpha]_{\alpha<\lambda}) = \widetilde{I}(f)[\widetilde{I}(t_\alpha) \mid x_\alpha]_{\alpha<\lambda}.$$

Proof. This is done by induction on the length of f in [Car78, Lemma 1, pp. 1.52]. The interesting case is when $f = F(e_\beta)_{\beta<\mu}$ for some F in the alphabet and expressions $\{e_\beta\}_{\beta<\mu}$. We assume inductively the result true for the expressions $\{e_\beta\}_{\beta<\mu}$. Then we have:

$$\begin{aligned}\widetilde{I}(f[t_\alpha \mid x_\alpha]_{\alpha<\lambda}) &= \widetilde{I}(F(e_\beta[t_\alpha \mid x_\alpha]_{\alpha<\lambda})_{\beta<\mu}) \\ &= I(F)(\widetilde{I}(e_\beta[t_\alpha \mid x_\alpha]_{\alpha<\lambda}))_{\beta<\mu} \\ &= I(F)(\widetilde{I}(e_\beta)[\widetilde{I}(t_\alpha) \mid x_\alpha]_{\alpha<\lambda})_{\beta<\mu}, \text{ by induction hypothesis} \\ &= I(F)(\widetilde{I}(e_\beta))_{\beta<\mu}[\widetilde{I}(t_\alpha) \mid x_\alpha]_{\alpha<\lambda} \\ &= \widetilde{I}(F(e_\beta)_{\beta<\mu})[\widetilde{I}(t_\alpha) \mid x_\alpha]_{\alpha<\lambda} \\ &= \widetilde{I}(f)[\widetilde{I}(t_\alpha) \mid x_\alpha]_{\alpha<\lambda}\end{aligned}$$

□

There is also a notion of composition of interpretations: If $I : S \to T$ and $J : T \to U$ are interpretations then there is an interpretation $J \circ I : S \to U$ that is defined in the obvious way. It is also easy to infer what is the identity for this composition. A crucial result to define this compositions is:

Lemma A.4.4. If $I : S \to T$ and $J : T \to U$ are interpretations then $\widetilde{J \circ I}(e) = \widetilde{J}(\widetilde{I}(e))$

Proof. This is by induction of the expression e see [Car78, Lemma 3, pp. 1.55]. □

We can define the category of κ-GAT of κ-generalized algebraic theories. There is an equivalence relation on interpretations between two theories T and T'. If I, $J : T \to T'$ are two interpretations then $I \approx J$ if an only if for every rule $r \in R_U$ we have $I(r) \approx J(r)$ in the theory T'.

Lemma A.4.5. If I and J are interpretations from T to T' such that $I \approx J$ then for all type and element judgment $\mathcal{J}$ of U, $\widehat{I}(\mathcal{J}) \approx \widehat{J}(\mathcal{J})$ in T'.

Proof. See [Car78, Lemma 1, Section 1.14]. □

Then Lemma A.4.5 implies that the compositions as given is well-defined. Finally, in order to get the correct morphisms we need to know that the equivalence relation on interpretations is compatible with the composition. Another advantageous consequence is that this it give us a criteria to establish whether two interpretations are equivalent.

Corollary A.4.6. If I and J are interpretations from T to T' then $I \approx J$ if and only if for any type element judgment r, $\widehat{I}(r) \approx \widehat{J}(r)$.

Proof. This follows from Lemma A.4.5 and (3) of Definition A.1.3. □

Corollary A.4.7. If I and J are interpretations from T to T' and I' and J' are interpretations from T' to T'' then from $I \approx J$ and $I' \approx J'$ we conclude that $I' \circ I \approx J' \circ J$.

Proof. [Car78, pp. 1.72]. □

The category κ-GAT has morphisms equivalence classes of interpretations [Car78, pp. 1.72].

A.5 Construction and properties of the category $\mathbb{C}_T$

Let be T an κ-Cartmell theory. The category $\mathbb{C}_T$ has the following data:

- Objects: Equivalence classes of contexts under the relation $\approx$. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a context then the object in $\mathbb{C}_T$ is denoted $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$.
- Morphisms: A morphism between $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$ and $[\{x_\beta : \Omega_\mu\}_{\beta<\mu}]$ it is the equivalence class of a map

$$\langle t_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$$

  induced by the relation $\approx$. We denote this set by

$$\hom_{\mathbb{C}_T}([\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}], [\{x_\beta : \Omega_\mu\}_{\beta<\mu}]).$$

- Composition: This is induced by the composition of maps between contexts. This is again well-defined in view of 2 of Remark A.3.8.
- Identity: For a context $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ its identity is the equivalence class $[\{x_\alpha\}_{\alpha<\lambda}]$.

Remark A.5.1. The category $\mathbb{C}_T$ has a unique object $1 := [\emptyset]$ the equivalence class of the empty context. Note that this is a terminal object.

Remark A.5.2. Let $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$ an object of $\mathbb{C}_T$. Then for any $\mu < \lambda$ we get a morphism $[\langle x_\beta\rangle_{\beta<\mu}] : [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : \Delta_\beta\}_{\beta<\mu}]$. Indeed, since $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a context then for any $\beta < \lambda$ we have $\{x_\delta : \Delta_\delta\}_{\delta<\delta} \vdash \Delta_\beta$ Type. Therefore, it follows from (Definition A.1.4, 9) that $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash x_\alpha : \Delta_\alpha$ for all $\alpha < \lambda$. In particular this is true for all $\beta < \mu$, this gives the morphism above.

Following the same argument if $\nu < \mu$ then we also we a map $[\langle x_\gamma\rangle_{\gamma<\nu}] : [\{x_\beta : \Delta_\beta\}_{\beta<\mu}] \to [\{x_\gamma : \Delta_\gamma\}_{\gamma<\nu}]$. Furthermore, we get a commutative diagram:

$$\begin{tikzcd}
{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \arrow[dr, "{[\langle x_\gamma\rangle_{\gamma<\nu}]}"'] & {[\{x_\beta : \Delta_\beta\}_{\beta<\mu}]} \arrow[d, "{[\langle x_\gamma\rangle_{\gamma<\nu}]}"] \\
& {[\{x_\gamma : \Delta_\gamma\}_{\gamma<\nu}]}
\end{tikzcd}$$

Remark A.5.3. Since this morphisms are somewhat canonical we will use the notation " $\twoheadrightarrow$ ", and whenever we use this arrow for a morphism it must be assumed that such map is of this form. These morphisms are called display, which is Cartmell's terminology. In contrast, our we our 'display' maps can be of arbitrary length, which we will often refer as **generalized display** maps.

Suppose there is a context $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+\varepsilon}]$ with $\varepsilon \geq 0$. Then we can consider an ε-indexed sequence of display morphisms:

$$\cdots \quad [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+2}] \longrightarrow\!\!\!\!\rightarrow [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+1}] \longrightarrow\!\!\!\!\rightarrow [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$$

Also, there is a display map $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+\varepsilon}] \twoheadrightarrow [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$. This display morphism will be by definition the composition for the sequence. If $\varepsilon = 0$ then this maps is simply the identity. We also get a factorization of the map $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \twoheadrightarrow 1$ via display maps for any $\lambda \geq 0$.

Observation A.5.4. From the previous Remark A.5.2 we can observe that if λ is a limit ordinal then $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$ is the limit of the sequence

$$\cdots \quad [\{x_1 : \Delta_1, x_2 : \Delta_2\}] \longrightarrow\!\!\!\!\rightarrow [\{x_1 : \Delta_1\}] \longrightarrow\!\!\!\!\rightarrow 1.$$

If there is another context $[\{x_\delta : \Gamma_\delta\}_{\delta<\gamma}]$ and maps

$$[\langle t_\beta\rangle_{\beta<\alpha}] : [\{x_\delta : \Gamma_\delta\}_{\delta<\gamma}] \to [\{x_\beta : \Delta_\beta\}_{\beta<\alpha}]$$

for all $\alpha < \lambda$ then we can simply take the map

$$[\langle t_\alpha\rangle_{\alpha<\lambda}] : [\{x_\delta : \Gamma_\delta\}_{\delta<\gamma}] \to [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}].$$

This can be shown the cone map (which is unique). This verifies our claim.

Using Remark A.5.2 we can define a function:

$$\nu : Ob(\mathbb{C}_T) \longrightarrow \kappa$$

as $\nu([\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]) := \lambda$. We call this the **length function**. We can use ν to construct a filtration on the objects of $\mathbb{C}_T$: we define

$$Ob_\lambda(\mathbb{C}_T) := \nu^{-1}(\lambda)$$

then $Ob(\mathbb{C}_T) = \coprod_{\lambda<\kappa} Ob_\lambda(\mathbb{C}_T)$, and so if $\alpha \leq \beta$ then $Ob_\alpha(\mathbb{C}_T) \subseteq Ob_\beta(\mathbb{C}_T)$. Furthermore, if $p : A \twoheadrightarrow B$ is a display morphism then $\nu(B) \leq \nu(A)$.

For $\alpha < \beta$ there are functions

$$\pi_\beta : Ob_\beta(\mathbb{C}_T) \to Ob_\alpha(\mathbb{C}_T)$$

that are defined in the obvious way. Additionally, $1 \in Ob_0(\mathbb{C}_T)$ is unique.

The proof of the following lemma is the same as in [Car78].

Lemma A.5.5. The pullback of a display map along arbitrary morphisms in $\mathbb{C}_T$ exists and it is also display.

Proof. We use induction over the context length. Assume we have the following diagram in $\mathbb{C}_T$:

$$\begin{tikzcd}
 & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu+1}]} \arrow[d, two heads, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \\
{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, "{[\langle t_\beta\rangle_{\beta<\mu}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu}]}
\end{tikzcd}$$

Then the pullback is given using Remark A.3.8, the context is

$$[\{x_\alpha : \Delta_\alpha,\, x_\mu : \Omega_\mu[t_\beta \mid x_\beta]_{\beta<\mu}\}_{\alpha<\lambda}].$$

Therefore we have a commutative square

$$\begin{tikzcd}
{[\{x_\alpha : \Delta_\alpha,\, x_\mu : \Omega_\mu[t_\beta \mid x_\beta]_{\beta<\mu}\}_{\alpha<\lambda}]} \arrow[r, "{[\langle t_\beta, x_\mu\rangle_{\beta<\mu}]}"] \arrow[d, two heads, "{[\langle x_\alpha\rangle_{\alpha<\lambda}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu+1}]} \arrow[d, two heads, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \\
{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, "{[\langle t_\beta\rangle_{\beta<\mu}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu}]}
\end{tikzcd} \tag{1}$$

Note that by definition the left vertical morphism is also display. If there is another commutative square

$$\begin{tikzcd}
{[\{x_\zeta : \Gamma_\zeta\}_{\zeta<\xi}]} \arrow[r, "{[\langle g_\beta\rangle_{\beta<\mu+1}]}"] \arrow[d, "{[\langle f_\alpha\rangle_{\alpha<\lambda}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu+1}]} \arrow[d, two heads, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \\
{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, "{[\langle t_\beta\rangle_{\beta<\mu}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu}]}
\end{tikzcd}$$

The map

$$[\langle f_\alpha, g_\mu\rangle_{\alpha<\lambda}] : [\{x_\zeta : \Gamma_\zeta\}_{\zeta<\xi}] \to [\{x_\alpha : \Delta_\alpha,\, x_\mu : \Omega_\mu[t_\beta \mid x_\beta]_{\beta<\mu}\}_{\alpha<\lambda}]$$

shows that the square (1) is the pullback.
Next, assume that we have a diagram

$$\begin{tikzcd}
 & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu}]} \arrow[d, two heads, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \\
{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, "{[\langle t_\beta\rangle_{\beta<\nu}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\nu}]}
\end{tikzcd}$$

where μ is a limit ordinal strictly larger than ν. We simplify the notation as follows:

$$\begin{tikzcd} & B_\mu \arrow[d, two heads] \\ A_\lambda \arrow[r, "\langle t_\beta \rangle_{\beta<\nu}"'] & B_\nu \end{tikzcd}$$

Assume that factorization of the map $B_\mu \twoheadrightarrow B_\nu$ is of the form

$$\ldots \twoheadrightarrow B_{\nu+2} \twoheadrightarrow B_{\nu+1} \twoheadrightarrow B_\nu$$

and therefore B_μ is the limit (obtained in a similar way as in Observation A.5.4 and Remark A.5.2). Then we can take the successive pullback

$$\begin{tikzcd} f^*B_\mu \arrow[r, "q(f,B_\mu)"] \arrow[dr, phantom, "\lrcorner", very near start] & B_\mu \\ \vdots & \vdots \\ q(f,B_{\nu+1})^*B_{\nu+2} \arrow[r, "q(q(f,B_{\nu+1}),B_{\nu+2})"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\nu+2} \arrow[d, two heads] \\ f^*B_{\nu+1} \arrow[r, "q(f,B_{\nu+1})"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\nu+1} \arrow[d, two heads] \\ A_\lambda \arrow[r, "f"'] & B_\nu \end{tikzcd} \tag{2}$$

where at each succesor stage it is given as before, $f := \langle t_\beta \rangle_{\beta<\nu}$, the context

$$f^*B_\mu := [\{x_\alpha : \Delta_\alpha,\, x_\beta : \Omega_\beta[t_\delta \mid x_\delta]_{\delta<\beta}\}_{\substack{\alpha<\lambda \\ \nu<\beta<\mu}}]$$

is the limits of the sequence on the left hand side, with the obvious display maps to each object in the sequence, and

$$q(f, B_\mu) := [\langle t_\beta,\, x_\gamma \rangle_{\beta<\nu<\gamma<\mu}].$$

This makes the outer rectangle in (2) commutative. Moreover, the map $q(f, B_\mu)$ is the unique cone map induced by the family of maps

$$\{[\langle t_\beta,\, x_\gamma \rangle_{\beta<\nu<\gamma<\delta}] : f^*B_\mu \to B_\delta\}_{\nu<\delta<\mu}.$$

$\square$

Using the same notation as in the lemma above we have

Remark A.5.6. 1. If $f = Id_{B_\nu}$ then $(Id_{B_\nu})^* B_\mu = B_\mu$ and $q(Id_{B_\nu}, B_\mu) = Id_{B_\mu}$.

2. For a diagram

$$\begin{tikzcd} & & A \arrow[d, two heads, "p"] \\ D \arrow[r, "g"] & C \arrow[r, "f"] & B \end{tikzcd}$$

we have that $g^*(f^*(A)) = (fg)^*(A)$ and $q(fg, A) = q(f, A)(g, f^*A)$.

We will refer the category $\mathbb{C}_T$ as the **syntactic category** associated to the κ-Cartmell theory T.

Observation A.5.7. We note that Lemma A.5.5 give us an explicit construction of pullbacks in $\mathbb{C}_T$, as well the pullback of the maps and an explicit description of $q(f, B_\mu)$.

We finish this section by characterizing the display maps in the category $\mathbb{C}_T$. This result says that display maps are somehow generic. We start with a preparatory result.

Lemma A.5.8. Let T a κ-Cartmell theory and $\mathbb{C}_T$ its syntactic κ-contextual category. Assume that there is a $f : \Delta \to \Gamma$, then any display map $B \twoheadrightarrow \Delta$ of length 1 can be obtained as a pullback of the form

$$\begin{tikzcd} B \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma' \arrow[d, two heads] \\ \Delta \arrow[r, "f"'] & \Gamma \end{tikzcd}$$

where $\Gamma' \twoheadrightarrow \Gamma$ is of length 1.

Proof. This simply a reformulation of Lemma A.2.6. Assume that

$$f = [\langle t_\beta \rangle_{\beta<\mu}] : [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : \Gamma_\beta\}_{\beta<\mu}].$$

Therefore, when the display map is of the form

$$[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+1}] \twoheadrightarrow [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}].$$

We can construct the square

$$\begin{tikzcd} {[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+1}]} \arrow[r, "\langle t_\beta, x_\lambda \rangle_{\beta<\mu}"] \arrow[d, two heads] & {[\{x : \Gamma_\beta, x_\lambda : \Delta_\lambda\}_{\beta<\mu}]} \arrow[d, two heads] \\ {[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]} \arrow[r, "\langle t_\beta \rangle_{\beta<\mu}"'] & {[\{x : \Gamma_\beta\}_{\beta<\mu}].} \end{tikzcd}$$

Since for all $\beta < \mu$, x_β does not occur in Δ_λ we have that $\Delta_\lambda[t_\beta|x_\beta]_{\beta<\mu} \equiv \Delta_\lambda$. Hence, it follows from the construction of pullbacks in $\mathbb{C}_T$ (Lemma A.5.5) that the square above is indeed a pullback diagram. $\square$

We are ready to give the full description of display maps.

Proposition A.5.9. Every Display map $B \twoheadrightarrow \Delta$ in $\mathbb{C}_T$ is a limit of a κ-small tower $V : \lambda \to \mathbb{C}_T$ where for each limit ordinal $\beta < \lambda$

$$V(\beta) = \mathtt{Lim}_{\alpha<\beta} V(\alpha)$$

and the map $V(\alpha+1) \to V(\alpha)$ is a pullback of a length one display map of the form $(\Gamma, A) \twoheadrightarrow \Gamma$ where $\Gamma \vdash A\,\mathsf{Type}$ is a type axiom of the theory T.

Proof. Each display map in $\mathbb{C}_T$ has a length λ. Just as in Remark A.5.2 it admits a decomposition into display maps. It will be enough to prove the second claim, but this follows by an inductive argument in conjunction with the previous Lemma A.5.8. The inductive step provide us with the required map $f : V(\alpha) \to \Gamma$ in Lemma A.5.8. $\square$

B Contextual categories and Cartmell theories

This section is the most relevant part. We will show that from the syntax of a κ-Cartmell theory we can construct a category, called κ-*Contextual category*, which we now introduce.

B.1 κ-contextual categories

The discussion in Appendix A.5 on the properties of the syntactic category $\mathbb{C}_T$ can be summarized with the next definition which is the natural generalization of Cartmell's [Car78] or [KL18]. We present our definition in the same way as in the later. Recall that κ is a regular cardinal.

Definition B.1.1. A category $\mathcal{C}$ is said to be a κ-**contextual category** if:

1. The objects of $\mathcal{C}$ have grading $Ob(\mathcal{C}) = \coprod_{\lambda<\kappa} Ob_\lambda(\mathcal{C})$. This grading determines the **height** of any object $B \in \mathcal{C}$, which we write as $ht(B)$.

2. There is a terminal object $1 \in \mathcal{C}$ and it is unique up to equality with height 0.

3. There is a wide subcategory $Dis(\mathcal{C})$ with distinguished maps " $\twoheadrightarrow$ " called *display morphisms*,

4. The subcategory $Dis(\mathcal{C})$ is closed under transfinite compositions: If we have

$$\cdots \twoheadrightarrow B_3 \twoheadrightarrow B_2 \twoheadrightarrow B_1 \twoheadrightarrow B_0$$

a λ-sequence of display maps then there a unique object B in $Dis(\mathcal{C})$ with height λ and for each $\mu \leq \lambda$ a display map $B \twoheadrightarrow B_\mu$ such that for any $\alpha < \lambda$ we have a factorization

$$\begin{tikzcd} B \arrow[rr, two heads] \arrow[dr, two heads] & & B_0 \\ & B_\alpha \arrow[ur, two heads] & \end{tikzcd}$$

5. The inclusion functor preserve $i : Dis(\mathcal{C}) \hookrightarrow \mathcal{C}$ transfinite compositions.

6. If $A \twoheadrightarrow B$ is an arrow in $Dis(\mathcal{C})$ then $B \in Ob_\mu(\mathcal{C})$ and $A \in Ob_\lambda(\mathcal{C})$ for some ordinals λ, μ with $\mu \leq \lambda$.

7. For any object $A \in Ob_\lambda(\mathcal{C})$ and any $\mu \leq \lambda$ there exists a unique object $B \in Ob_\mu(\mathcal{C})$ and a unique display map $A \twoheadrightarrow B$. The **length** of this display map is the unique ordinal α such that $\lambda = \mu + \alpha$, is such situation we write $lt(p)$.

8. For any $A \in Ob_\lambda(\mathcal{C})$, a map $A \twoheadrightarrow B$ and any map $f : C \to B$ there is a pullback square

$$\begin{tikzcd} f^*A \arrow[r, "q(f,A)"] \arrow[d, two heads, "f^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "p"] \\ C \arrow[r, "f"'] & B \end{tikzcd}$$

called *canonical pullback* of A along f, and we require $lt(f^*p) = lt(p)$.

9. Canonical pullbacks are strictly functorial: for ordinals with $\mu \leq \lambda$, $A \in Ob_\lambda(\mathcal{C})$

   (a) If $f = id_B$ then $id_B^* A = A$ and $q(id_B, A) = id_A$.

   (b) For a diagram

$$\begin{tikzcd} & & A \arrow[d, two heads, "p"] \\ D \arrow[r, "g"] & C \arrow[r, "f"] & B \end{tikzcd}$$

   we have that $g^*(f^*(A)) = (fg)^*(A)$ and $q(fg, A) = q(f, A)(g, f^*A)$.

10. Given display maps $p : A \twoheadrightarrow B$ and $q : B \to C$ and any $f : X \to C$, in the diagram

$$\begin{tikzcd} q(f,B)^*A \arrow[r, "q(q(f,B),A)"] \arrow[d, two heads, "q(f,B)^*p"'] \arrow[dr, phantom, "\lrcorner" very near start] & A \arrow[d, two heads, "p"] \\ f^*B \arrow[r, "q(f,B)"'] \arrow[d, two heads, "f^*r"'] \arrow[dr, phantom, "\lrcorner" very near start] & B \arrow[d, two heads, "r"] \\ X \arrow[r, "f"'] & C \end{tikzcd}$$

We have that $f^*r \circ (q(f,B)^*p) = f^*(r \circ p)$ and $q(q(f,B),A) = q(f,A)$.

Remark B.1.2. We use the term "display map" in rather different way to Cartmell. For us, a display map can have any height and it is only bounded by the regular cardinal κ.

We have already seen one example of such category.

Corollary B.1.3. For any κ-Cartmell theory T the syntactic category $\mathbb{C}_T$ is a κ-contextual category.

Proof. This is done throughout Appendix A.5. $\square$

Remark B.1.4. It follows from Definition B.1.1 that for any object $B \in \mathcal{C}$ the map $B \twoheadrightarrow 1$ can be decomposed as a transfinite composition of display maps

$$B_\lambda \twoheadrightarrow \ldots \twoheadrightarrow B_1 \twoheadrightarrow 1.$$

The length of decomposition above is given by the degree of B. This is what [Car78] calls the tree structure of the category. Whenever we refer to objects in a κ-contextual category as above, we will emphasize its height by writing B_λ. Likewise, we will denote the display maps as $p_\alpha : B_\lambda \twoheadrightarrow B_\alpha$ for each $\alpha < \lambda$.

The following lemma is a consequence of Definition B.1.1 and Remark B.1.4.

Lemma B.1.5. Let $B \in Ob_\lambda(\mathcal{C})$ such that λ is a limit ordinal. Then B itself is a limit object in $\mathcal{C}$.

Proof. From Remark A.5.2 we obtain a sequence

$$\cdots \longrightarrow\!\!\!\!\!\rightarrow B_3 \longrightarrow\!\!\!\!\!\rightarrow B_2 \longrightarrow\!\!\!\!\!\rightarrow B_1 \longrightarrow\!\!\!\!\!\rightarrow 1$$

It follows from Axiom 4 of Definition B.1.1 that B must be limit of the sequence. Finally, we use that the inclusion $Dis(\mathcal{C}) \to \mathcal{C}$ preserve limits. $\square$

Definition B.1.6. Let $\mathcal{C}$, $\mathcal{D}$ contextual categories. A functor $F : \mathcal{C} \to \mathcal{D}$ it is called **contextual functor** if it satisfies the following conditions:

1. $F(Ob_\lambda(\mathcal{C})) \subseteq Ob_\lambda(\mathcal{D})$ for all $\lambda < \kappa$,
2. F restricts to a functor $Dis(\mathcal{C}) \to Dis(\mathcal{D})$,
3. F preserve canonical pullbacks up to equality, meaning that for any square in $\mathcal{C}$

$$\begin{tikzcd} f^*A \arrow[r, "q(f,A)"] \arrow[d, two heads, "f^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, two heads, "p"] \\ C \arrow[r, "f"'] & B \end{tikzcd}$$

   we have $F(f^*A) = (Ff)^*(FA)$ and $F(q(f,A)) = q(Ff, FA)$.

Since the degree of each object is preserved by a κ-contextual functor, it makes sense to denote $F(A_\lambda) := F(A)_\lambda$ for $A_\lambda \in \mathcal{C}$. Another piece of notation we can introduce is from the functor $F : Dis(\mathcal{C}) \to Dis(\mathcal{D})$; since any display map $p_\alpha : A_\lambda \twoheadrightarrow A_\alpha$ is sent to a display map $F(p_\alpha) : F(A)_\lambda \twoheadrightarrow F(A)_\alpha$ and the degrees are preserved, we agree to omit F on this maps.
Contextual functors are the morphisms of the category of κ-contextual categories, we will denote it as κ-CON.

B.2 Interlude: categorical facts

We collect and recall some categorical facts about general κ-contextual categories.

Proposition B.2.1 (The slice κ-contextual category)**.** Let $\mathcal{C}$ be a κ-contextual category. For any object $B \in Ob_\mu(\mathcal{C})$ there is a κ-contextual category which is a full subcategory of the slice $\mathcal{C}_{/B}$ which has objects display maps $A \twoheadrightarrow B$ where $A \in Ob_\lambda(\mathcal{C})$ with $\lambda \geq \mu$.

Since we will rarely use categories other than κ-contextual categories, we will employ the slice notation $\mathcal{C}_{/B}$ for the category from the previous proposition.

Proof. The proof is purely completely formal. The important fact to remember is that the pullback of a display map is also display. $\square$

It is a well known fact that the pasting of two pullbacks give us a pullback, in our case consider the following diagram:

$$
\begin{tikzcd}
f^*B_\mu \arrow[r, "q(f,B_\mu)"] \arrow[dr, phantom, very near start, "\lrcorner"] & B_\mu \\
\vdots & \vdots \\
q(f,B_{\nu+1})^*B_{\nu+2} \arrow[r, "q(q(f,B_{\nu+1}),B_{\nu+2})"] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & B_{\nu+2} \arrow[d, two heads] \\
f^*B_{\nu+1} \arrow[r, "q(f,B_{\nu+1})"] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & B_{\nu+1} \arrow[d, two heads] \\
A_\lambda \arrow[r, "f"'] & B_\nu
\end{tikzcd}
$$

Then if μ is a limit ordinal, the object B_μ is the limit of the sequence on the right hand side. Thus f^*B_μ is the limit of the sequence on the left hand side. Note that pairwise we have $q(f,B_{\nu+1})^*B_{\nu+2} = f^*B_{\nu+2}$ and $q(f,B_{\mu+2}) = q(q(f,B_{\mu+1}),B_{\mu+2})$.
If $f : A_\lambda \to B_\nu$ and $p_\nu : B_\mu \twoheadrightarrow B_\nu$ is a display map with $\mu = \nu+1$, using the universal property of the pullback we can construct the following diagram

$$
\begin{tikzcd}
A_\lambda \arrow[dr, dashed, "\delta_f^\nu"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "Id_{A_\lambda}"'] & & \\
& (p_\nu f)^*B_\mu \arrow[r] \arrow[d, two heads, "(p_\nu f)^*p_\nu"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B_\mu \arrow[d, two heads, "p_\nu"] \\
& A_\lambda \arrow[r, "p_\nu f"'] & B_\nu.
\end{tikzcd}
$$

The map δ_f^ν makes both triangles commutative. We will focus on the fact that $((f_\nu)^*p_\nu)\delta_f^\nu = Id_{A_\lambda}$, where $f_\nu = p_\nu f$. Assume that we have a map $p : B_\mu \twoheadrightarrow B_\nu$ with μ a limit ordinal, in particular the length of p is a limit ordinal. Then a map $f : A_\lambda \to B_\mu$ is determinate by a family of maps

$\{f_\gamma : A_\lambda \to B_\gamma\}$. Then we obtain:

$$\begin{tikzcd}
A_\lambda \arrow[dr, dashed, "\delta_f^\nu"] \arrow[rr, bend left, "f"] \arrow[dddddr, bend right, "Id"'] & & \\
& f^*B_\mu \arrow[r, "{q(f,B_\mu)}"] \arrow[dr, phantom, "\lrcorner", very near start] & B_\mu \arrow[dddd, bend left, "p"] \\
& \vdots & \vdots \\
& q(f,B_{\nu+1})^*B_{\nu+2} \arrow[r, "{q(q(f,B_{\nu+1}),B_{\nu+2})}"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\nu+2} \arrow[d, two heads] \\
& f^*B_{\nu+1} \arrow[r, "{q(f,B_{\nu+1})}"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & B_{\nu+1} \arrow[d, two heads] \\
& A_\lambda \arrow[r, "f_\nu"'] & B_\nu
\end{tikzcd}$$

where the map δ_f^ν is given a the family of maps $(\delta_f^\nu)_\gamma$ each given by an intermediate pullback square in the diagram above.

Notation B.2.2. If the situation above, for $f : A_\lambda \to B_\mu$ we denote

$$\Gamma(B_\nu^\mu) := \{h : A_\lambda \to (p_\nu f)^* B_\mu \mid ((p_\nu f)^* p_\nu)h = Id_{A_\lambda}\}.$$

We can consider a more general case, if $A_\lambda \in Ob_\lambda(\mathcal{C})$ and $B_\mu \in Ob_\mu(\mathcal{C})$ with $\lambda < \mu$, then there is a unique display map $p : B_\mu \twoheadrightarrow A_\lambda$. We set

$$\Gamma(B_\lambda^\mu) := \{s : A_\lambda \to B_\mu \mid ps = Id_{A_\lambda}\}$$

for this situation as well, since the object A_λ will be inferred from the context.

If the contextual category is $\mathbb{C}_T$ then, recalling Lemma A.5.5, we can give an explicit description of the map δ_f^ν.

Lemma B.2.3. Assume that $f := [\langle t_\beta \rangle_{\beta<\nu}] : [\{x_\alpha : A_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : B_\beta\}_{\beta<\nu}]$ and there is a display map $p : [\{x_\beta : B_\beta\}_{\beta<\mu}] \twoheadrightarrow [\{x_\beta : B_\beta\}_{\beta<\nu}]$ then $\delta_f^\nu = [\langle x_\alpha, t_\beta \rangle_{\substack{\alpha<\lambda \\ \nu<\beta<\mu}}]$.

Proof. This follows by induction on μ and the explicit construction of pullbacks from Lemma A.5.5. □

In certain situations, the property above characterizes the map δ_f^ν.

Lemma B.2.4. If $[\{x_\beta : B_\beta\}_{\beta<\mu}]$ is an object of $\mathbb{C}_T$ and $\nu < \mu$ then $f \in \Gamma(B^\mu_\nu)$ if and only if $f = [\langle x_\beta, t_\gamma\rangle_{\beta<\nu<\gamma<\mu}]$, where for all $\nu < \gamma < \mu$ the rule $\{x_\beta : B_\beta\}_{\beta<\nu}, \{t_{\gamma'} : B_{\gamma'}\}_{\gamma'<\gamma} \vdash t_\gamma : B_\gamma$ is a derived rule.

The next result follows from the previous lemmas and it is used in Observation B.3.30.

Lemma B.2.5. Let A_λ, B_μ objects of $\mathcal{C}$ and for each $\beta < \mu$ we have maps $r_{\beta+1} \in \Gamma(r^*_\beta \cdots r^*_1 p^* B_{\beta+1})$ then there exists a unique sequence of maps $\{g_\beta : A_\lambda \to B_\beta\}_{\beta<\mu}$ such that for all $\beta < \mu$ we have $p_\beta g_{\beta+1} = g_\beta$ such that $\delta_{g_\beta} = r_\beta$.

Some words about the previous lemma are in order. The expression $r^*_\beta \cdots r^*_1 p^* B_{\beta+1}$ can be illustrated by the first two steps:

$$
\begin{tikzcd}
p^*B_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_2 \arrow[d, two heads] \\
p^*B_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \arrow[d, two heads] \\
A_\lambda \arrow[r, "p"'] \arrow[u, bend left, "r_1"] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
r_1^*p^*B_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & p^*B_2 \arrow[d] \\
A_\lambda \arrow[r, "r_1"'] \arrow[u, bend left, "r_2"] & p^*B_1
\end{tikzcd}
$$

B.3 The equivalence between κ-GAT and κ-CON

B.3.1 The functor $\mathbb{C} : \kappa\text{-GAT} \to \kappa\text{-CON}$

To establish this equivalence of categories we first define a functor $\mathbb{C}$: κ-GAT $\to$ κ-CON using the construction of Appendix A.5. The proof again comes from ([Car78], section 2.4.1). We register all preliminary results needed to define this functor, however again we omit the proofs since they are similar to the original ones given by Cartmell.

On objects $\mathbb{C} : \kappa\text{-GAT} \to \kappa\text{-CON}$ is defined as $\mathbb{C}_T$ for T a κ-Cartmell theory. For a map $[I] : T \to T'$ between theories we need functor $\mathbb{C}(I)$: $\mathbb{C}_T \to \mathbb{C}_{T'}$:

1. On objects; $\mathbb{C}(I)([\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]) := [\{x_\alpha : \widetilde{I}(\Delta_\alpha)\}_{\alpha<\lambda}]$,
2. On morphisms: If $[\langle t_\beta\rangle_{\beta<\mu}] : [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : \Delta_\beta\}_{\beta<\mu}]$ then $\mathbb{C}(I)([\langle t_\beta\rangle_{\beta<\mu}]) := [\langle \widetilde{I}(\langle t_\beta\rangle_{\beta<\mu})]$.

If we there is an interpretation J in the equivalence class $[I]$ then by Lemma A.4.5 any rule r of T we get $\widehat{I}(r) \approx \widehat{J}(r)$. Therefore, the definition of $\mathbb{C}(I)$ does not depend on the representative of $[I]$.

Remains to verify that $\mathbb{C}(I)$ is indeed a contextual functor. Firstly, it is primordial to verify it is well-defined.

Lemma B.3.1. Let $[I] : T \to T'$ be a map in κ-GAT then the following hold:

1. The interpretation I preserves contexts: If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a context in the theory T then $\{x_\alpha : \overline{I}(\Delta_\alpha)\}_{\alpha<\lambda}$ is a context in the theory T'.

2. The interpretation I preserves the equivalence relation $\approx$ between contexts: If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ and $\{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda}$ are contexts in the theory U with $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \approx \{x_\alpha : \Omega_\alpha\}_{\alpha<\lambda}$ then $\{x_\alpha : \overline{I}(\Delta_\alpha)\}_{\alpha<\lambda} \approx \{x_\alpha : \overline{I}(\Omega_\alpha)\}_{\alpha<\lambda}$.

3. The interpretation I preserves morphisms between contexts: If $\langle t_\beta\rangle_{\beta<\mu}\} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ is a morphism between contexts in the theory T then $\langle \overline{I}(t_\beta)\rangle_{\beta<\mu}\} : \{x_\alpha : \overline{I}(\Delta_\alpha)\}_{\alpha<\lambda} \to \{x_\beta : \overline{I}(\Omega_\beta)\}_{\beta<\mu}$ is a morphism between contexts in the theory T'.

4. The interpretation I preserves the equivalence relation $\approx$ between morphisms of contexts: If $\langle s_\beta\rangle_{\beta<\mu}, \langle t_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ are morphisms between contexts in the theory T with $\langle s_\beta\rangle_{\beta<\mu} \approx \langle t_\beta\rangle_{\beta<\mu}$ then $\langle \overline{I}(s_\beta)\rangle_{\beta<\mu} \approx \langle \overline{I}(t_\beta)\rangle_{\beta<\mu}$.

Proof. The proof of each statement is consequence of Lemma A.4.3 or Lemma A.4.2. Our enumeration of variables give us a notation simplification of the proof given by [Car78].
For example to prove 4; we have by assumption that $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t_\gamma \equiv_{\Omega_\gamma[t_\beta|x_\beta]_{\beta<\gamma}} s_\gamma$ for all $0 < \gamma \leq \mu$. Therefore, since the interpretation preserves this rule oT we get that $\{x_\alpha : \overline{I}(\Delta_\alpha)\}_{\alpha<\lambda} \vdash \overline{I}(t_\gamma) \equiv_{\overline{I}(\Omega_\gamma)[\overline{I}(t_\beta)|x_\beta]_{\beta<\gamma}} \overline{I}(s_\gamma)$ for all $0 < \gamma \leq \mu$. This exactly establishes $\langle \overline{I}(s_\beta)\rangle_{\beta<\mu} \approx \langle \overline{I}(t_\beta)\rangle_{\beta<\mu}$. □

We have seen that the definition of $\mathbb{C}(I)$ give us the correct objects and morphisms. Now we show that it is indeed a contextual functor.

Lemma B.3.2. Let $I : T \to T'$ be a morphism in κ-GAT. Then the map $\mathbb{C}(I) : \mathbb{C}_T \to \mathbb{C}_{T'}$ is a contextual functor.

Proof. The map is functor trivially. That it preserves the grading and restricts to a functor between the display subcategories $Dis(\mathbb{C}_T) \to Dis(\mathbb{C}_{T'})$

it is also immediate. To prove it preserves canonical pullbacks consider the following pullback square in the category $\mathbb{C}_T$:

$$
\begin{tikzcd}
{[\{x_\alpha : \Delta_\alpha,\, x_\gamma : \Omega_\gamma[t_\beta \mid x_\beta]_{\beta<\mu}\}_{\substack{\alpha<\kappa, \\ \mu\leq\gamma<\mu+\varepsilon}}]} \arrow[r, "{[\langle t_\beta, x_\gamma\rangle_{\substack{\beta<\mu, \\ \mu\leq\gamma<\mu+\varepsilon}}]}"] \arrow[d, "{[\langle x_\alpha\rangle_{\alpha<\kappa}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu+\varepsilon}]} \arrow[d, two heads, "{[\langle x_\beta\rangle_{\beta<\mu}]}"] \\
{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\kappa}]} \arrow[r, "{[\langle t_\beta\rangle_{\beta<\mu}]}"'] & {[\{x_\beta : \Omega_\beta\}_{\beta<\mu}]}
\end{tikzcd}
$$

Then a straightforward computation, using the definition of $\mathbb{C}(I)$ shows that this is send to a pullback square in the category $\mathbb{C}_{T'}$. $\square$

Corollary B.3.3. There is a functor $\mathbb{C} : \kappa$-GAT $\to \kappa$-CON.

B.3.2 The functor $U : \kappa$-CON $\to \kappa$-GAT

We now turn to construct a functor that to each κ-contextual category $\mathcal{C}$ associates a κ-generalized algebraic theory $U(\mathcal{C})$, this is part of [Car78, Section 2.4]. We will use the notation introduced in Remark B.1.4. This means we identify each object by its height, say B_λ, and write display maps as $p_\alpha : B_\lambda \twoheadrightarrow B_\alpha$ if $\lambda > 0$ and $\alpha < \lambda$. If $\alpha = 0$ then $B_0 = 1$ the terminal object. A morphism $f : A_\lambda \to B_\mu$ is trivial when B_μ is trivial i.e $\mu = 0$.

Definition B.3.4. We define $U(\mathcal{C}) \in \kappa$-GAT as:

1. For each non-trivial object B_μ with $\mu = \lambda + 1$ a type symbol $\overline{B_\mu}$ with introductory rule: $\{x_\beta : \overline{B_\beta}\}_{\beta<\mu} \vdash \overline{B_\mu}(x_\beta)_{\beta<\mu}\,\mathsf{Type}$. The notation emphasizes the fact that $\overline{B_\mu}$ depends on the indicated variables.

2. If $f : A_\lambda \to B_\mu$ is morphism of $\mathcal{C}$ with $\mu = \nu + 1$ we get an operator symbol $\overline{f}$. It has introductory rule;

   - If $f : A_\lambda \to B_{\mu+1}$, denote by $\rho_\mu : B_{\mu+1} \twoheadrightarrow B_\mu$. Then the operator symbol has introductory rule

   $$\{x_\alpha : \overline{A}_\alpha\}_{\alpha<\lambda} \vdash \overline{f}(x_\alpha)_{\alpha<\lambda} : \overline{(\rho_\mu f)^* B_{\mu+1}}(x_\alpha)_{\alpha<\lambda}.$$

   This does not clash with the notation from the previous point since it always refer to an object of $\mathcal{C}$ and in this case refers to map.

Subject to the following axioms in $U(\mathcal{C})$:

1. Let A_λ, B_μ, $C_{\nu+1}$ be objects of $\mathcal{C}$ and maps $f : A_\lambda \to B_\mu$, $g : B_\mu \to C_{\nu+1}$:
$$\{x_\alpha : \overline{A}_\alpha\}_{\alpha<\lambda} \vdash \overline{gf}(x_\alpha)_{\alpha<\lambda} \equiv_{\overline{(p_\nu gf)^* C_{\nu+1}}(x_\alpha)_{\alpha<\lambda}} \overline{g}\big(\overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\big)_{\beta<\mu}.$$

2. Let B_μ be a non-trivial object of $\mathcal{C}$. For each $\delta < \mu$ we have
$$\{x_\beta : \overline{B}_\beta\}_{\beta<\mu} \vdash \overline{p_\delta}(x_\beta)_{\beta<\mu} \equiv_{\overline{B}_\delta(x_\beta)_{\beta<\delta}} x_\delta.$$

3. Let A_λ, $B_{\mu+1}$ objects of $\mathcal{C}$ and a map $f : A_\lambda \to B_\mu$ then
$$\{x_\alpha : \overline{A}_\alpha\}_{\alpha<\lambda} \vdash \overline{f^* B_{\mu+1}}(x_\alpha)_{\alpha<\lambda} \equiv \overline{B_{\mu+1}}\big(\overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\big)_{\beta<\mu}$$
and
$$\{x_\alpha : \overline{A}_\alpha,\, x_\delta : \overline{f^* B_{\mu+1}}(x_\alpha)_{\alpha<\lambda}\}_{\alpha<\lambda} \vdash \overline{q(f, B_{\mu+1})}(x_\alpha, x_\delta)_{\alpha<\lambda} \equiv_{\overline{f^* B_\mu}(x_\alpha)_{\alpha<\lambda}} x_\delta.$$

Observation B.3.5. It is immediate to observe that $U(\mathcal{C})$ as defined is a κ-pretheory. We have sort symbol and operator symbols introduced by type judgment and type element judgments respectively. Note that the list of axioms we provided are well-formed rules. This is because the premise of each axiom is by definition a context.

Remark B.3.6. If $f : A_\lambda \to B_\mu$ is a map in $\mathcal{C}$, where μ is a limit ordinal i.e B_μ is a limit object, then we get a family of maps $\{f_\nu : A_\lambda \to B_\nu\}_{\nu<\mu}$. Therefore, the associated operator $\overline{f}$ is uniquely determined by the operators $\overline{f_\nu}$ for which in this case we can assume that ν is a successor ordinal.

If $F : \mathcal{C} \to \mathcal{D}$ is a functor between κ-contextual categories then we need an interpretation $U(F) : U(\mathcal{C}) \to U(\mathcal{D})$;

1. For an object A_λ, the interpretation is defined as
$$U(F)(\overline{A_\lambda}) := \overline{FA_\lambda}(x_\alpha)_{\alpha<\lambda}.$$

2. For a morphism $f : A_\lambda \to B_{\mu+1}$, the operator $\overline{f}$ is interpreted as
$$U(F)(\overline{f}) := \overline{F(f)}(x_\alpha)_{\alpha<\lambda}.$$

The next step is to prove that this is indeed an map between the κ-Cartmell theories, this is done in [Car78, pp 2.29]. For this, it is enough to show that rules and axioms of $U(\mathcal{C})$ are send to rules of $U(\mathcal{D})$. The functoriality of $U : \kappa\text{-CON} \to \kappa\text{-GAT}$ is also immediate from its definition. This is tested on each type and operator symbol. It is then enough to take the equivalence class $[U(F)]$.

B.3.3 The natural isomorphism $U \circ \mathbb{C} \cong Id_{\kappa\text{-}GAT}$

For each $T \in \kappa$-GAT we want to define an interpretation $[\varphi_T] : T \to U(\mathbb{C}_T)$, we do this by defining a preinterpretation $\varphi_T : Exp(T) \to Exp(U(\mathbb{C}_T))$:

1. If Δ is a type symbol of T with introduction rule

$$\{x_\alpha : \Delta_\beta\}_{\beta<\mu} \vdash \Delta(x_\beta)_{\beta<\mu} \ \mathsf{Type}$$

   then

$$\varphi_T(\Delta) := \overline{[\{x_\beta : \Delta_\beta,\, x_\delta : \Delta(x_\beta)_{\beta<\mu}\}_{\beta<\mu}]}(x_\beta)_{\beta<\mu}$$

2. If f is an operator symbol with introductory rule

$$\{x_\alpha : \Delta_\beta\}_{\beta<\mu} \vdash f(x_\beta)_{\beta<\mu} : \Delta$$

   then

$$\varphi_T(f) := \overline{[\langle x_\beta, f(x_\beta)_{\beta<\mu}\rangle_{\beta<\mu}]}(x_\beta)_{\beta<\mu}$$

   where $\langle x_\beta, f(x_\beta)_{\beta<\mu}\rangle_{\beta<\mu}$ is the morphism $\{x_\alpha : \Delta_\beta\}_{\beta<\mu} \to \{x_\alpha : \Delta_\beta, x_\delta : \Delta\}_{\beta<\mu}$.

We proceed to verify that as defined $\varphi_T : T \to U(\mathbb{C}_T)$ is an interpretation. This a crucial point in the proof so we spell out some details in Corollary B.3.15. The results before it are technical steps towards it.

Lemma B.3.7. If $\mathcal{C}$ is a contextual category, objects A_λ, B_μ and $f : A_\lambda \to B_\mu$ is map with $\mu = \nu + 1$ (in particular it is non-trivial) then the rule

$$\{x_\alpha : \overline{A}_\alpha(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{f}(x_\alpha)_{\alpha<\lambda} : \overline{B_\mu}\big(\overline{p_\beta \circ f}(x_\alpha)_{\alpha<\lambda}\big)_{\beta<\mu}$$

is a derived rule of $U(\mathcal{C})$.

Proof. We have the axiom

$$\{x_\alpha : \overline{A}_\alpha\}_{\alpha<\lambda} \vdash \overline{f^* B_\mu}(x_\alpha)_{\alpha<\lambda} \equiv \overline{B_\mu}\big(\overline{p_\beta \circ f}(x_\alpha)_{\alpha<\lambda}\big)_{\beta<\mu}$$

for $U(\mathcal{C})$ and the derivation rule for κ-GAT

$$\frac{\Gamma \vdash A_1 \equiv A_2 \qquad t : A_1}{\Gamma \vdash t : A_2}.$$

These put together give us the result. $\square$

Lemma B.3.8. Let $\mathcal{C}$ a κ-contextual category, objects $\{A_\alpha\}_{\alpha<\lambda}$, $\{B_\beta\}_{\beta<\mu+1}$, $\{C_\gamma\}_{\gamma<\varepsilon}$ and a commutative diagram

$$\begin{tikzcd} C_\varepsilon \arrow[r, "l"] \arrow[d, "k"'] & B_{\mu+1} \arrow[d, two heads, "p"] \\ A_\lambda \arrow[r, "f"'] & B_\mu. \end{tikzcd}$$

If $h : C_\varepsilon \to f^* B_{\mu+1}$ is the unique map given by the pullback, then the rule

$$\{x_\gamma : \overline{C_\gamma}(x_\delta)_{\delta<\gamma}\}_{\gamma<\varepsilon} \vdash \overline{h}(x_\gamma)_{\gamma<\varepsilon} \equiv_{\overline{(fk)^* B_{\mu+1}}(x_\gamma)_{\gamma<\varepsilon}} \overline{l}(x_\gamma)_{\gamma<\varepsilon}$$

is a derived rule of $U(\mathcal{C})$.

Proof. The proof is the same as [Car78, Lemma 2 pp. 2.32] using Lemma B.3.7. $\square$

Lemma B.3.9. Let $\mathcal{C}$ a κ-contextual category, objects $\{A_\alpha\}_{\alpha<\lambda}$, $\{B_\beta\}_{\beta<\mu}$, $\{C_\gamma\}_{\gamma<\varepsilon}$ and for $0 < \nu < \mu$ a commutative diagram

$$\begin{tikzcd} C_\varepsilon \arrow[r, "l_\nu"] \arrow[d, "k_\nu"'] & B_\mu \arrow[d, two heads, "p_\nu"] \\ A_\lambda \arrow[r, "f"'] & B_\nu. \end{tikzcd}$$

If $h_\nu : C_\varepsilon \to f^* B_\mu$ is the unique map given by the pullback, then the rule

$$\{x_\gamma : \overline{C_\gamma}(x_\delta)_{\delta<\gamma}\}_{\gamma<\varepsilon} \vdash \overline{h_\nu}(x_\gamma)_{\gamma<\varepsilon} \equiv_{\overline{(fk_\nu)^* B_\mu}(x_\gamma)_{\gamma<\varepsilon}} \overline{l_\nu}(x_\gamma)_{\gamma<\varepsilon}$$

is a derived rule of $U(\mathcal{C})$.

Proof. This by induction on the height of p_ν. When it is a successor ordinal, this is the previous Lemma B.3.9. When it is a limit ordinal B_μ is a limit object, therefore the result reduces to the inductive hypothesis, which is the successor case again. $\square$

Recall from Appendix B.2 we defined the set of maps $\Gamma(B)$. It follows from the previous result that

Corollary B.3.10. If $\mathcal{C}$ is a κ-contextual category and $f : A_\lambda \to B_\mu$ is a map in $\mathcal{C}$, then for all $\nu < \mu$

$$\{x_\alpha : A_\alpha(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{\delta_f^\nu}(x_\alpha)_{\alpha<\lambda} \equiv \overline{f}(x_\alpha)_{\alpha<\lambda}.$$

is a derived rule of $U(\mathcal{C})$.

If we specialize Corollary B.3.10 to the syntactic κ-contextual category of a κ-Cartmell theory T, then

Corollary B.3.11. Assume that $\{x_\beta : B_\beta\}_{\beta<\mu}$ is a context, $\nu < \mu$ and

$$f_\nu := [\langle t_\beta \rangle_{\beta<\nu}] : [\{x_\alpha : A_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : B_\beta\}_{\beta<\nu}]$$

a map in $\mathbb{C}_T$ then

$$\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{[\langle x_\alpha, t_\varepsilon\rangle_{\substack{\alpha<\lambda \\ \nu\leq\varepsilon<\mu}}]} \equiv \overline{[\langle t_\beta, t_\varepsilon\rangle_{\beta<\nu\leq\varepsilon<\mu}]}.$$

is a derived rule of $U(\mathbb{C}_T)$.

Proof. This follows from Corollary B.3.10 and the explicit description of $\delta^\nu_{f_\nu}$ given in Lemma B.2.3. ☐

Lemma B.3.12. If A_λ, B_μ are objects and $f_\nu : A_\lambda \to B_\nu$, with $\nu < \mu$, is a map in a κ-contextual category $\mathcal{C}$, then:

1. The rule

   $$\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{f^*_\nu B_\mu}(x_\alpha)_{\alpha<\lambda} \equiv \overline{B}(\delta^\gamma_{(p_\gamma f)}(x_\alpha)_{\alpha<\lambda})_{\gamma<\nu}$$

   is a derived rule of $U(\mathcal{C})$.

2. If $g : \Gamma(B^\mu_\nu)$ then the rule

   $$\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{\delta^\nu_{gf}}(x_\alpha)_{\alpha<\lambda} \equiv \overline{\delta^\nu_g}(\overline{\delta^\gamma_{p_\gamma f}}(x_\alpha)_{\alpha<\lambda})_{\gamma<\nu}$$

   is a derived rule of $U(\mathcal{C})$.

Corollary B.3.13. If T is a κ-Cartmell theory, $\{x_\beta : B_\beta\}_{\beta<\mu}$ is a context, $\nu < \mu$ and

$$f_\nu := [\langle t_\beta \rangle_{\beta<\nu}] : [\{x_\alpha : A_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : B_\beta\}_{\beta<\nu}]$$

is a map in $\mathbb{C}_T$ then;

1.

   $$\frac{\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda}}{\overline{[\{x_\alpha, x_\gamma : B_\gamma[t_\delta|x_\delta]_{\delta<\gamma}\}_{\substack{\alpha<\lambda \\ \nu\leq\gamma<\mu}}]}(x_\alpha)_{\alpha<\lambda} \equiv \overline{[\{x_\beta : B_\beta\}_{\beta<\nu}]}(\overline{g_\beta}(x_\alpha)_{\alpha<\lambda})_{\beta<\nu}}$$

   where for each $\beta < \nu$ the map $g_\beta := [\langle x_\alpha, t_\beta\rangle_{\alpha<\lambda}]$.

2. If for all γ, with $\nu < \gamma < \mu$, the rule

$$\{x_\beta : B_\beta\}_{\beta<\nu}, \{t_{\gamma'} : B_{\gamma'}\}_{\gamma'<\gamma} \vdash t_\gamma : B_\gamma$$

is a derived rule then

$$\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{[\langle x_\alpha, t_\gamma[t_{\gamma'} \mid x_{\gamma'}]_{\gamma'<\gamma}\rangle_{\substack{\alpha<\lambda \\ \nu<\gamma<\mu}}]} \equiv \overline{h}(\overline{g_\beta}(x_\alpha)_{\alpha<\lambda})_{\beta<\nu}$$

where g_β is defined as in the previous point and $h := [\langle x_\beta, t_\gamma\rangle_{\substack{\beta<\nu \\ \nu<\gamma<\mu}}]$.

Proof. This is a direct application of Lemma B.3.12. We remark that the assumption of point (2) simply give us an element of $\Gamma(B^\mu_\nu)$ and the map on the left depend on variables that according to our convention we leave implicit. □

The following lemma is key to prove that we have an interpretation $\varphi_T : T \to U(\mathbb{C}_T)$, the results above are used to prove:

Lemma B.3.14. If T is a κ-Cartmell theory then:

1. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta\ \mathsf{Type}$ is a type judgment of T, then the rule

$$\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{A}(x_\alpha)_{\alpha<\lambda+1} \equiv \widetilde{\varphi_T}(\Delta)$$

is a derived rule of $U(\mathbb{C}_T)$ where $A := \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda+1}$ and $A_\alpha := \{x_\delta : \Delta_\delta\}_{\delta\leq\alpha}$.

2. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta$ is a type element judgment of T, then the rule

$$\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{\langle x_\alpha, t\rangle_{\alpha<\lambda}}(x_\alpha)_{\alpha<\lambda+1} \equiv_{\overline{A}(x_\alpha)_{\alpha<\lambda}} \widetilde{\varphi_T}(t)$$

is a derived rule of $U(\mathbb{C}_T)$.

Proof. The proof is by induction on the derivations, by showing that rule derivation preserves the properties above. □

The important result of this section is the following.

Corollary B.3.15. For every κ-Cartmell theory T, the map $\varphi_T : U \to U(\mathbb{C}_T)$ is an interpretation.

Proof. We see that the function $\widehat{\varphi_T} : Rul(T) \to Rul(U(\mathbb{C}_T))$ is well defined. We start with a rule $\mathcal{J}$ of T and show that $\widehat{\varphi_T}(\mathcal{J})$ is a rule of $U(\mathbb{C}_T)$

1. Type judgment: Assume that $\mathcal{J} := \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta\ \mathsf{Type}$ is a rule of T, from Definition A.4.1 it follows that

$$\widehat{\varphi_T}(\mathcal{J}) = \{x_\alpha : \widetilde{\varphi}(\Delta_\alpha)\}_{\alpha<\lambda} \vdash \widetilde{\varphi_T}(\Delta)\ \mathsf{Type}.$$

From Lemma B.3.14 we have for any $\gamma < \lambda + 1$ the rule

$$\{x_\alpha : \overline{\Delta_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{A_{\gamma+1}}(x_\alpha)_{\alpha<\gamma+1} \equiv \widetilde{\varphi_T}(\Delta_\gamma)$$

is a derived rule of $U(\mathbb{C}_T)$. Thus, so it is

$$\{x_\alpha : \widetilde{\varphi_T}(\Delta_\alpha)(x_\delta)_{\delta<\alpha}\}_{\alpha<\lambda} \vdash \overline{A_{\gamma+1}}(x_\alpha)_{\alpha<\lambda+1} \equiv \widetilde{\varphi_T}(\Delta).$$

Then it must be the case that $\{x_\alpha : \widetilde{\varphi}(\Delta_\alpha)\}_{\alpha<\lambda} \vdash \widetilde{\varphi_T}(\Delta)\ \mathsf{Type}$ is a rule of $U(\mathbb{C}_T)$.

2. Element judgment: $\Gamma \vdash t : \Delta$. This very similar the previous rule.

3. Type equality judgment: $\Gamma \vdash \Delta \equiv \Delta'$. Also follows from Lemma B.3.14.

4. Term equality judgment: $\Gamma \vdash t \equiv_\Delta t'$. The same argument works.

$\square$

Corollary B.3.16. For every κ-Cartmell theory T, the map $[\varphi_T] : U \to U(\mathbb{C}_T)$ is morphism in the category κ-GAT.

Next, we will show that $[\varphi_-] : Id_{\kappa\text{-GAT}} \Rightarrow U \circ \mathbb{C}$ is a natural transformation.

Lemma B.3.17. Let T, T' two κ-Cartmell theories and $I : T \to T'$ an interpretation between them. Then, we have a commutative diagram

$$\begin{tikzcd} T \arrow[r, "{[\varphi_T]}"] \arrow[d, "{[I]}"'] & U(\mathbb{C}_T) \arrow[d, "U(\mathbb{C}(I))"] \\ T' \arrow[r, "{[\varphi_{T'}]}"'] & U(\mathbb{C}_{T'}). \end{tikzcd}$$

Proof. We use Corollary A.4.6. Therefore, it will be enough to test the commutativity of the diagram on type element judgments. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta_\lambda$ a type element judgment of T. For any $\alpha \leq \lambda$ we denote $A_\alpha := [\{x_\delta : \Delta_\delta\}_{\delta\leq\alpha}]$. It follows from Lemma B.3.14 that

$$\widehat{\varphi_T}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right) \approx \frac{\{x_\alpha : \overline{A_\alpha}\}_{\alpha<\lambda}}{\overline{[\langle x_\alpha, t\rangle_{\alpha<\lambda}]} : \overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}}.$$

We conclude that

$$U(\mathbb{C}(I))\left(\widehat{\varphi_T}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right)\right) \approx \frac{\{x_\alpha : \overline{\mathbb{C}(I)(A_\alpha)}\}_{\alpha<\lambda}}{\overline{\mathbb{C}(I)([\langle x_\alpha, t\rangle_{\alpha<\lambda}])} : \overline{\mathbb{C}(I)(A_\lambda)}(x_\alpha)_{\alpha<\lambda}}.$$

Looking at the other composition: we get

$$\widehat{I}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right) = \frac{\{x_\alpha : \widetilde{I}(\Delta_\alpha)\}_{\alpha<\lambda}}{\widetilde{I}(t) : \widetilde{I}(\Delta_\lambda)}.$$

A second use of Lemma B.3.14 give us that

$$\widehat{\varphi_{T'}}\left(\widehat{I}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right)\right) \approx \frac{\{x_\alpha : \overline{B_\alpha}\}_{\alpha<\lambda}}{\overline{[\langle x_\alpha, \widetilde{I}(t)\rangle_{\alpha<\lambda}]} : \overline{B_\lambda}(x_\alpha)_{\alpha<\lambda}}$$

where for $\alpha \leq \lambda$, $B_\alpha := [\{x_\delta : \widetilde{I}(\Delta_\delta)\}_{\delta\leq\alpha}]$. However, by definition we have $\mathbb{C}(I)(A_\alpha) = B_\alpha$ for $\alpha \leq \lambda$. This completes our verification. $\square$

Remains to show that $[\varphi_T]$ is an isomorphism and natural natural in T. We proceed to give an inverse $\underline{\psi_T : U(\mathbb{C}_T) \to T}$. Recall that a type symbol of $U(\mathbb{C}_T)$ is of the form $\overline{A_\lambda} = \overline{[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]}$. If $\lambda = \nu + 1$ then by choosing a representative of this equivalence class of the context we can define $\psi_T(\overline{A_\lambda}) := \Delta_\nu$.
If λ is a limit ordinal once we chose a representative $\Delta_\lambda = \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$. Then we know that $[\Delta_\lambda] = \lim_{\alpha<\lambda}[\Delta_\alpha]$ in $\mathbb{C}_T$, and this limit is unique. In this case the value of ψ_T is determined by non-limit ordinals $\alpha < \lambda$, which are $\psi_T(\overline{\Delta_\alpha}) = \Delta_\alpha$. Therefore we define $\psi_T(\overline{[\Delta_\lambda]}) := \Delta_\lambda$ for some choice of a representative of the equivalence class. However, note that the successor case determinate the limit case.
Operator symbols of $U(\mathbb{C}_T)$ come from morphisms of $\mathbb{C}_T$. Therefore, for a morphism $\overline{f} := [\langle t_\beta\rangle_{\beta<\mu}] : [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : \Omega_\beta\}_{\beta<\mu}]$ in order to define ψ_T on the associated operator, it is enough to assume that μ is a successor ordinal. First of all, we need to make choices for the contexts and morphism. However, the definition does not depend on this choices because of (1) from Remark A.3.9. This allows to define ψ_T as

$$\psi_T(\overline{f}) := t_\mu$$

where $t_\mu : \Omega_\mu[t_\beta|x_\beta]_{\beta<\mu}$.

Lemma B.3.18. The function ψ_T is an interpretation from $U(\mathbb{C}_T) \to T$.

Proof. We need to check that rules and axioms are preserved by ψ_T. It will be enough to deal with the case where $\lambda = \nu + 1$. Suppose that $\overline{A_\lambda}$ has

$$\frac{\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\nu}}{\overline{A_\nu}(x_\alpha)_{\alpha<\nu}\ \mathsf{Type}}$$

Furthermore, we assume that $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is such that $A_\lambda = [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]$. By definition

$$\widehat{\psi_T}\left(\frac{\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha<\nu}}{\overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}\ \mathsf{Type}}\right) = \frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\nu}}{\Delta_\nu\ \mathsf{Type}}.$$

This is obviously a derived rule of T. Preservation of the rule for operator symbols are is straightforward too. $\square$

Lemma B.3.19. For any κ-Cartmell theory T we have $\psi_T \circ \varphi_T \approx Id_T$.

Proof. From Corollary A.4.6 it is enough to verify the statement on type element judgments. Let $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t : \Delta_\lambda$ a type element judgment. For any $\alpha \leq \lambda$ we denote $A_\alpha := [\{x_\delta : \Delta_\delta\}_{\delta\leq\alpha}]$. It follows from Lemma B.3.14 that

$$\widehat{\varphi_T}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right) \approx \frac{\{x_\alpha : \overline{A_\alpha}\}_{\alpha<\lambda}}{\overline{[\langle x_\alpha, t\rangle_{\alpha<\lambda}]} : \overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}}.$$

Hence

$$\widehat{\psi_T}\left(\widehat{\varphi_T}\left(\frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}\right)\right) \approx \widehat{\psi_T}\left(\frac{\{x_\alpha : \overline{A_\alpha}\}_{\alpha<\lambda}}{\overline{[\langle x_\alpha, t\rangle_{\alpha<\lambda}]} : \overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}}\right) = \frac{\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}}{t : \Delta_\lambda}.$$

$\square$

Lemma B.3.20. For any κ-Cartmell theory T we have $\psi_T \circ \varphi_T \approx Id_T$.

Proof. The proof is similar to the previous lemma. All the definitions and technical results have been established, specially Lemma B.3.14. $\square$

Corollary B.3.21. There is a natural isomorphism $Id_{\kappa\text{-GAT}} \Rightarrow U \circ \mathbb{C}$.

Proof. We have constructed $[\varphi_-] : Id_{\kappa\text{-GAT}} \Rightarrow U \circ \mathbb{C}$. $\square$

B.3.4 The natural isomorphism $\mathbb{C} \circ U \cong Id_{\kappa\text{-}CON}$

In this section we aim to construct a natural isomorphism $\eta : Id_{\kappa\text{-CON}} \Rightarrow \mathbb{C} \circ U$. Let $\mathcal{C}$ be a κ-contextual category. For this, we first construct a κ-contextual functor $\eta_{\mathcal{C}} : \mathcal{C} \to \mathbb{C}_{U(\mathcal{C})}$. Recall that if A_λ is an object in $\mathcal{C}$ then for any $\alpha \leq \lambda$ we denoted $p_\alpha : A_\lambda \twoheadrightarrow A_\alpha$ to the canonical display map that exists. Then we can make the following definition:

1. For $\eta_{\mathcal{C}}(1) := 1$.
2. If A_μ is an object with $\mu = \lambda + 1$ then
$$\eta_{\mathcal{C}}(A_\mu) := [\{x_\alpha : \overline{A_\alpha}(x_\delta)_{\delta<\alpha}\}_{\alpha\leq\mu}].$$
3. For an object A_λ we define $\eta_{\mathcal{C}}(p_0) := \eta_{\mathcal{C}}(p)_0$ where $\eta_{\mathcal{C}}(p)_0 : \eta_{\mathcal{C}}(A) \twoheadrightarrow 1$.
4. If A_λ, B_μ are non-trivial objects, with μ a successor ordinal, and $f : A_\lambda \to B_\mu$ is a morphism in $\mathcal{C}$ then
$$\eta_{\mathcal{C}}(f) := [\langle \overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\rangle_{\beta\leq\mu}].$$

Again we observe that if μ is a limit ordinal then any map $f : A_\lambda \to B_\mu$ is determined by a family of maps $\{f_\nu : A_\lambda \to B_\nu\}_{\nu<\mu}$. Thus, in order to define η on such map it is enough to do it on ordinals $\nu < \mu$ which we can assume to be successor ordinals. The map $\eta(f)$ is the map induced by the family of maps $\{\eta(f_\nu) : \eta(A_\lambda) \to \eta(B_\nu)\}_{\nu<\mu}$. In conclusion, we simply need to prove properties of η for successor ordinals. The property for limit ordinals follows using the universal property of the limit object.

Lemma B.3.22. For any $\mathcal{C}$, $\eta_{\mathcal{C}} : \mathcal{C} \to \mathbb{C}_{U(\mathcal{C})}$ is a κ-contextual functor.

Proof. First we verify that it is a functor. Since for any $\alpha < \lambda$ we have $\overline{p_\alpha}(x_\alpha)_{\alpha<\lambda} = x_\alpha$, then it is immediate to see that $\eta_{\mathcal{C}}$ preserves the identities. Assume we have non-trivial morphisms $f : A_\lambda \to B_\mu$ and $g : B_\mu \to C_\nu$ then

$$\eta_{\mathcal{C}}(gf) = [\langle \overline{p_\gamma g f}(x_\alpha)_{\alpha<\lambda}\rangle_{\beta\leq\nu}]$$

From the first axiom in Definition B.3.4 $U(\mathcal{C})$ it follows that the above must be $\eta_{\mathcal{C}}(g)\eta_{\mathcal{C}}(f)$ whenever μ and ν are successor ordinals. When we have limits Now we must verify that it preserves display maps and canonical pullbacks. Both statements are direct consequences from the definitions. Furthermore,

the proof from [Car78] works without mayor changes.
For the preservation of pullbacks: We let $f : A_\lambda \to B_{\mu+1}$ then

$$\begin{aligned}\eta_{\mathcal{C}}(f^*B) &= [\langle x_\alpha : \overline{A_\delta}(x_\gamma)_{\gamma<\alpha},\, x_\epsilon : \overline{f^*B_{\mu+1}}(x_\alpha)_{\alpha<\lambda}\rangle_{\alpha<\lambda}] \\ &= [\langle x_\alpha : \overline{A_\delta}(x_\gamma)_{\gamma<\alpha},\, x_\epsilon : \overline{B_{\mu+1}}\big(\overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\big)_{\beta<\mu}\rangle_{\alpha<\lambda}] \\ &= [\langle \overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\rangle_{\beta\leq\mu}]^*[\langle x_\beta : \overline{B_\beta}(x_\gamma)_{\gamma<\beta}\rangle_{\beta\leq\mu}] \\ &= \eta_{\mathcal{C}}(f)^*\eta_{\mathcal{C}}(B).\end{aligned}$$

For a display map of $p_\nu : B_\mu \twoheadrightarrow B_\nu$ with successor ordinal as height the same argument shows that the pullback along $f_\nu : A_\lambda \to B_\nu$ is preserved. When the height is a limit ordinal we combine the previous case and the fact that in any κ-contextual category canonical pullbacks are unique. □

Lemma B.3.23. Let $\mathcal{C}$, $\mathcal{C}'$ be κ-contextual categories and a contextual functor $F : \mathcal{C} \to \mathcal{C}'$. Then the following diagram is commutative:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\eta_{\mathcal{C}}"] \arrow[d, "F"'] & \mathbb{C}_{U(\mathcal{C})} \arrow[d, "\mathbb{C}(U(F))"] \\ \mathcal{C}' \arrow[r, "\eta_{\mathcal{C}'}"'] & \mathbb{C}_{U(\mathcal{C}')}. \end{tikzcd}$$

Proof. If $f : A_\lambda \to B_\mu$ is a map in $\mathcal{C}$ then

$$\begin{aligned}\mathbb{C}(U(F))(\eta_{\mathcal{C}}(f)) &= \mathbb{C}(U(F))([\langle\overline{p_\beta f}(x_\alpha)_{\alpha<\lambda}\rangle_{\beta\leq\mu}]) \\ &= [\langle\overline{F(p_\beta f)}(x_\alpha)_{\alpha<\lambda}\rangle_{\beta\leq\mu}] \\ &= [\langle\overline{p_\beta F(f)}(x_\alpha)_{\alpha<\lambda}\rangle_{\beta\leq\mu}] \\ &= \eta_{\mathcal{C}'}(f).\end{aligned}$$

□

Corollary B.3.24. There is a natural transformation $Id_{\kappa\text{-CON}} \Rightarrow \mathbb{C}\circ U$.

Remains to show that this natural transformation is an isomorphism. For each κ-contextual category $\mathcal{C}$ we construct a κ-contextual functor

$$\xi_{\mathcal{C}} : \mathbb{C}_{U(\mathcal{C})} \to \mathcal{C}$$

which is a two-sided inverse to $\eta_{\mathcal{C}}$. From Lemma A.2.6 we see that:

1. Every derived type judgment of $U(\mathcal{C})$ is of the form

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \overline{A_\lambda}(t_\alpha)_{\alpha<\lambda} \ \mathsf{Type}$$

   for some object A_λ of $\mathcal{C}$ where for $\alpha \leq \lambda$ the rule

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\alpha : \overline{A_\alpha}[t_\delta \mid x_\delta]_{\delta<\alpha}$$

   is a derived rule of $U(\mathcal{C})$.

2. Every type element judgment of T is of the form

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash x_\beta : \Omega_\beta$$

   for some $\beta < \mu$, or is of the form

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \overline{f}(t_\alpha)_{\alpha<\lambda} : \Omega$$

   for some map $f : A_\lambda \to B_\mu$ of $\mathcal{C}$ such that for each $\alpha < \lambda$ the rules

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\alpha : \overline{A_\alpha}[t_\delta \mid x_\delta]_{\delta<\alpha}$$

   and

$$\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \overline{B_\mu}(t_\beta)_{\beta<\mu} \equiv \Omega$$

   are derived rules of $U(\mathcal{C})$.

We may assume that $\mu = \nu + 1$, the limit case will follow induction. Let $\mathcal{R}_\mathcal{C}$ be the set of type and element type judgments of $U(\mathcal{C})$. Next, we define $\mathcal{J} : \mathcal{R}_\mathcal{C} \to \mathcal{C}$ inductively. First we get maps:

1. A rule $r_{\Omega_\mu} := \{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega_\mu$ is sent an object $\mathcal{J}(r_{\Omega_\mu}) \in \mathcal{C}$.
2. For any $\alpha < \lambda$ the judgment $r_{t_\alpha} := \{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\alpha : \overline{A_\alpha}[t_\delta | x_\delta]_{\delta<\alpha}$ is sent to a map $\mathcal{J}(r_{t_\alpha})$.

The we can make the following definitions:

1. $\mathcal{J}(r_{A_\mu}) := (\mathcal{J}(t_\alpha)_{\alpha<\lambda})^* A_\mu$.
   Where $\mathcal{J}(t_\alpha)_{\alpha<\lambda}$ denotes the pullbacks as in Lemma B.2.5.
2. $\mathcal{J}(\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \overline{f}(t_\alpha)_{\alpha<\lambda} : \Omega) := (\mathcal{J}(t_\alpha)_{\alpha<\lambda})^* \delta^\nu_f$.
3. $\mathcal{J}(\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash x_\beta : \Omega) := \delta^\beta_{p_\beta}$ where $p_\beta : \mathcal{J}(r_{\Omega_\mu}) \to \mathcal{J}(r_{\Omega_\beta})$.

The burden of the proof falls into showing that the function $\mathcal{J}$ is well-defined. The proof is by induction on the derived rules of $U(\mathcal{C})$. We will focus on writing down the inductive hypothesis H as in [Car78] for this induction.

- For rules r_{Ω_μ} of the form $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega_\mu \,\mathsf{Type}$ then $H(r_{\Omega_\mu})$ is either:

  1. If the premise of r_{Ω_μ} is a non-empty context then $H(r_{\Omega_\beta})$ for all $\beta < \mu$.
  2. If r_{Ω_μ} is the rule $\vdash \Delta \,\mathsf{Type}$ then $ht(\mathcal{J}(r_{\Omega_\mu})) = 1$. Otherwise for all $\beta < \mu$ we have $ht(\mathcal{J}(r_{\Omega_\beta})) < ht(\mathcal{J}(r_{\Omega_\mu}))$.
  3. For a map $\langle t_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$. If for each $\beta+1 < \mu$ we have $\mathcal{J}(r_{t_{\beta+1}}) \in \Gamma(\mathcal{J}(r_{\Omega_{\beta+1}[t_\gamma|x_\gamma]_{\gamma\leq\beta}}))$ where $r_{\Omega_{\beta+1}[t_\gamma|x_\gamma]_{\gamma\leq\beta}}$ is the rule $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Omega_{\beta+1}[t_\gamma|x_\gamma]_{\gamma\leq\beta} \,\mathsf{Type}$ then
  $$\mathcal{J}(r_{\Omega_\mu[t_\beta|x_\beta]_{\beta<\mu}}) = (\mathcal{J}(t_\beta)_{\beta<\mu})^*\mathcal{J}(r_{\Omega_\mu})$$

- For rules r_{t_μ} of the form $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash t_\mu : \Omega_\mu$ then $H(r_{t_\mu})$ is either:

  1. $H(r_{\Omega_\mu})$.
  2. $\mathcal{J}(r_{t_\mu}) \in \Gamma(\mathcal{J}(r_{\Omega_\mu}))$.
  3. For a map $\langle t_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$. If for each $\beta+1 < \mu$ we have $\mathcal{J}(r_{t_{\beta+1}}) \in \Gamma(\mathcal{J}(r_{\Omega_{\beta+1}[t_\gamma|x_\gamma]_{\gamma\leq\beta}}))$ then
  $$\mathcal{J}(r_{t_\mu[t_\beta|x_\beta]_{\beta<\mu}}) = (\mathcal{J}(t_\beta)_{\beta<\mu})^*\mathcal{J}(r_{t_\mu})$$
  where $r_{t_\mu[t_\beta|x_\beta]_{\beta<\mu}}$ is the rule $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t_\mu[t_\beta|x_\beta]_{\beta<\mu} : \Omega_\mu[t_\beta|x_\beta]_{\beta<\mu}$.

- For rules $r_\equiv$ or the form $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta \equiv \Delta'$ the hypothesis $H(r_\equiv)$ is either:

  1. $H(r_{\Delta'})$ and $\mathcal{J}(r_\Delta) = \mathcal{J}(r_{\Delta'})$.
  2. $H(r_\Delta)$ and $\mathcal{J}(r_\Delta) = \mathcal{J}(r_{\Delta'})$.

- For rules r_ϵ or the form $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t \equiv_\Delta t'$ the hypothesis $H(r_\epsilon)$ is either:

  1. $H(r_t)$ and $\mathcal{J}(r_t) = \mathcal{J}(r_{t'})$.
  2. $H(r_{t'})$ and $\mathcal{J}(r_t) = \mathcal{J}(r_{t'})$.

Lemma B.3.25. Let $\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega$ a rule such that H is satisfied. If $\langle t_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ is a map such that $H(r_{t_\beta})$ for all $\beta < \mu$ then $H(\{x_\beta : \Omega_\beta\}_{\beta<\mu} \vdash \Omega[t_\beta|x_\beta]_{\beta<\mu})$

Proof. By induction on μ and treating all different cases for H. The proof in [Car78, Lemma 11 pp.2.56] works here too. □

Lemma B.3.26.

1. For any object $A_\lambda \in \mathcal{C}$, we have:
   (a) $A\lambda = \mathcal{J}(\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}\ \mathsf{Type})$.
   (b) For all $\alpha < \lambda$, $\delta_{p^\lambda_\alpha} = \mathcal{J}(\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha})$ where $p^\lambda_\alpha : A_\lambda \twoheadrightarrow A_\alpha$.
2. For any non-trivial object A_λ and $f : A_\lambda \to B_{\mu+1}$, $\delta_f = \mathcal{J}(\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{f}(x_\alpha)_{\alpha<\lambda}\overline{(p_\mu f)^* B}(x_\alpha)_{\alpha<\lambda})$ where $p_\mu : B_{\mu+1} \twoheadrightarrow B_\mu$.

Proof. This is [Car78, Lemma 12 pp.263]. □

Lemma B.3.27. Every derived rule of $U(\mathcal{C})$ satisfies the hypothesis H

Proof. This is by induction on derived rules of $U(\mathcal{C})$. Indeed, [Car78, Lemma pp.2.65] shows that every derivation from Definition A.1.4 preserves H. □

Corollary B.3.28.

1. For any type symbol $\overline{A_\lambda}$ of the theory $U(\mathcal{C})$ we have $H(\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{A_\lambda}(x_\alpha)_{\alpha<\lambda}\ \mathsf{Type})$.
2. For every operator symbol $\overline{f}$ in $U(\mathcal{C})$ where $f : A_\lambda \to B_{\mu+1}$ we have $H(\{x_\alpha : \overline{A_\alpha}(x_\gamma)_{\gamma<\alpha}\}_{\alpha<\lambda} \vdash \overline{f}(x_\alpha)_{\alpha<\lambda}\overline{(p_\mu f)^* B}(x_\alpha)_{\alpha<\lambda})$.

The foremost important result which summarizes the above is:

Corollary B.3.29.

1. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}$ is a context of the theory then for any $\alpha < \delta < \lambda$ we have $ht(r_{\Delta_\alpha}) < ht(r_{\Delta_\beta})$.
2. If there is a map $\langle t_\beta\rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ then for each $\beta < \mu$ we have $\mathcal{J}(r_{t_\beta}) \in \Gamma(\mathcal{J}(r_{\Omega_\beta[t_\gamma|x_\gamma]_{\gamma<\beta}}))$ where $r_{\Omega_\beta[t_\gamma|x_\gamma]_{\gamma<\beta}}$ is the rule $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Omega_\beta[t_\gamma|x_\gamma]_{\gamma<\beta}\ \mathsf{Type}$.
3. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \equiv \{x_\alpha : \Delta'_\alpha\}_{\alpha<\lambda}$ then $\mathcal{J}(r_{\Delta_\lambda}) = \mathcal{J}(r_{\Delta'_\lambda})$.
4. If $\langle t_\alpha\rangle_{\alpha<\lambda} \equiv \langle t'_\alpha\rangle_{\alpha<\lambda}$ then for each $\beta < \mu$, $\mathcal{J}(r_{t_\beta}) = \mathcal{J}(r_{t'_\beta})$.

We are almost ready to define a contextual functor $\xi_{\mathcal{C}} : \mathcal{C}_{U(\mathcal{C})} \to \mathcal{C}$. We only need the next:

Observation B.3.30. Let a map $\langle t_\beta \rangle_{\beta<\mu} : \{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \to \{x_\beta : \Omega_\beta\}_{\beta<\mu}$ then there are maps $\{g_\beta : \mathcal{J}(r_{\Delta_\lambda}) \to \mathcal{J}(r_{\Omega_\beta})\}_{\beta<\mu}$ with $\delta_{g_\beta} = \mathcal{J}(r_{t_beta})$ and $pg_{\beta+1} = g_\beta$. This is a consequence of Corollary B.3.29 and Lemma B.2.5. Therefore, there exists a unique $g : \mathcal{J}(r_{\Delta_\lambda}) \to \mathcal{J}(r_{\Omega_\mu})$ such that for all $\beta < \mu$ we have $\delta_{pg} = \mathcal{J}(r_{t_\beta})$ where $p : \mathcal{J}(r_{\Delta_\lambda}) \to \mathcal{J}(r_{\Omega_\beta})$.

Definition B.3.31. We define a function

$$\xi_{\mathcal{C}} : \mathcal{C}_{U(\mathcal{C})} \to \mathcal{C}$$

by:

1. For an object $[\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \in \mathcal{C}_{U(\mathcal{C})}$,

$$\xi([\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}]) := \mathcal{J}(r_{\Delta_\lambda}).$$

2. For an morphism $[\langle t_\beta \rangle_{\beta<\mu}] : [\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda}] \to [\{x_\beta : \Omega_\beta\}_{\beta<\mu}]$

$$\xi([\langle t_\beta \rangle_{\beta<\mu}]) := g$$

where $g : \mathcal{J}(r_{\Delta_\lambda}) \to \mathcal{J}(r_{\Omega_\mu})$ is the unique map from Observation B.3.30.

Lemma B.3.32. 1. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash \Delta_\lambda\ \mathsf{Type}$ is a derived rule of $U(\mathcal{C})$ then for all $\alpha \leq \lambda$, $\{x_\gamma : \Delta_\gamma\}_{\gamma<\lambda} \vdash \Delta_\alpha \equiv \mathcal{J}(r_{\Delta_\alpha})(x_\gamma)_{\gamma<\alpha}$ is a derived rule of $U(\mathcal{C})$.

2. If $\{x_\alpha : \Delta_\alpha\}_{\alpha<\lambda} \vdash t_\lambda : \Delta_\lambda$ is a derived rule of $U(\mathcal{C})$ then $\{x_\gamma : \Delta_\gamma\}_{\gamma<\lambda} \vdash t \equiv \mathcal{J}(r_{t_\lambda})(x_\alpha)_{\alpha<\lambda}$ is a derived rule of $U(\mathcal{C})$.

Proof. See [Car78, Lemma 15 pp. 2.74]. □

Corollary B.3.33. As functions we have $\eta_{\mathcal{C}}\xi_{\mathcal{C}} = id_{\mathcal{C}_{U(\mathcal{C})}}$ and $\xi_{\mathcal{C}}\eta_{\mathcal{C}} = Id_{\mathcal{C}}$

The results needed for this have been introduced throughout the section. Using that we have a bijection and that $\eta_{\mathcal{C}}$ is already a functor it follows:

Corollary B.3.34. The function $\xi_{\mathcal{C}} : \mathcal{C}_{U(\mathcal{C})} \to \mathcal{C}$ is a contextual functor.

The main result that is of our interest is:

Theorem B.3.35. There is a natural isomorphism $\mathcal{C}_{\_} \circ U \cong Id_{\kappa\text{-CON}}$.

Finally,

Corollary B.3.36. The categories κ-CON of κ-contextual categories and κ-GAT of κ-algebraic theories are equivalent.

B.4 Coclans and contextual categories

In this section we use prove that every κ-contextual category can be obtained by strictification of a κ-clan. Clans were introduced in [Joy17], a related definition appears in [Hen20] under the name category with fibrations.

Definition B.4.1. We say that a category $\mathcal{C}$ is a κ-**coclan** if it has a a collection of maps $\textsc{Cof}(\mathcal{C})$ satisfying the following conditions:

1. $\mathcal{C}$ has initial object 0.

2. For any $X \in \mathcal{C}$, the map $0 \to X \in \textsc{Cof}(\mathcal{C})$.

3. Any isomorphism is an element of $\textsc{Cof}(\mathcal{C})$.

4. $\textsc{Cof}(\mathcal{C})$ is closed under compositions.

5. $\textsc{Cof}(\mathcal{C})$ is closed under pushouts: If $f : A \to C$ is a morphism in $\mathcal{C}$ and $A \to B \in \textsc{Cof}(\mathcal{C})$ then the map $C \to C \coprod_A B \in \textsc{Cof}(\mathcal{C})$.

6. $\textsc{Cof}(\mathcal{C})$ is closed under transfinite compositions: for any $\lambda < \kappa$ and any λ-diagram of maps in $\textsc{Cof}(\mathcal{C})$

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots$$

   $\texttt{Colim}_\lambda A_\alpha$ exists and the map $A_0 \to \texttt{Colim}_\lambda A_\alpha$ belongs to $\textsc{Cof}(\mathcal{C})$.

As is usual, maps in $\textsc{Cof}(\mathcal{C})$ are called *cofibrations* and they are indicated by arrows " $\rightarrowtail$ ".

Dually, a category $\mathcal{C}$ is κ-**clan** if $\mathcal{C}^{op}$ is a κ-coclan. The distinguished maps are called *fibrations* and they are denoted by $\textsc{Fib}(\mathcal{C})$. The fibrations are indicated by arrows " $\twoheadrightarrow$ ". When working with κ-clans we keep the terminology "trasfinite compositions" from κ-coclans as there is no risk of confusion.

Observation B.4.2. The κ-contextual category $\mathbb{C}_T$ associated to a κ-generalized algebraic theory T has a natural κ-clan structure. Indeed, we can take $\textsc{Fib}(\mathbb{C}_T)$ as the display maps. All the axioms are easily verified. Moreover, this is true for any κ-contextual category not only for $\mathbb{C}_T$.

Recall that a **comprehension category** consists of a category $\mathcal{C}$, a fibration $p : \mathcal{E} \to \mathcal{C}$ and a functor $F : \mathcal{E} \to \mathcal{C}^\to$ such that:

1. $\partial_0 F = p$.

2. If f is a cartesian arrow in $\mathcal{E}$ then Ff is a pullback in $\mathcal{C}$, equivalently Ff is a cartesian arrow with respect to the codomain functor $\partial_0 : \mathcal{C}^{\rightarrow} \rightarrow \mathcal{C}$.

The fibration p is **cloven** if it comes with a choice of cartesian lifts. The comprehension category is said to be **split** is p is a split fibration. We also say that is **full** if F is fully faithful, the notation $(\mathcal{C}, \mathcal{E}, p, F)$

The following example appears in [Jac93, Example 4.5], we rewrite it in our setting of κ-clans. Let us fix a κ-clan $\mathcal{C}$, then the inclusion functor $\iota : \textsc{Fib}(\mathcal{C}) \hookrightarrow \mathcal{C}^{\rightarrow}$ and $P = \partial_0 \iota$ form a full comprehension category. More precisely: $\textsc{Fib}(\mathcal{C})$ has objects fibrations in $\mathcal{C}$ and arrows between two fibrations $\alpha : f \rightarrow g$ are commutative squares of the form

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, two heads, "f"'] & B \arrow[d, two heads, "g"] \\ \Delta \arrow[r, "l"'] & \Gamma. \end{tikzcd}$$

Whence an object in $\textsc{Fib}(\mathcal{C})_\Gamma$ over $\Gamma \in \mathcal{C}$ is a fibration $A \twoheadrightarrow \Gamma$. Observe that an arrow $\alpha : f \rightarrow g$ as above is cartesian if and only if it is a pullback square in $\mathcal{C}$. In conclusion, for an arrow $l : \Delta \rightarrow \Gamma$ and $B \twoheadrightarrow \Gamma \in \textsc{Fib}(\mathcal{C})_\Gamma$, a cartesian lift in $\textsc{Fib}(\mathcal{C})$ is a pullback square

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, two heads, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, two heads, "g"] \\ \Delta \arrow[r, "l"'] & \Gamma. \end{tikzcd}$$

This comprehension category is not necessarily split, reflecting the fact that taking pullbacks is not strictly functorial. Nevertheless, we can replace it by a split one via the functor

$$(-)_! : \mathbf{CompCat}(\mathcal{C}) \rightarrow \mathbf{SplCompCat}(\mathcal{C})$$

from the category of comprehension categories over $\mathcal{C}$ to the category of split comprehension categories over $\mathcal{C}$, the description of this functor appears in [LW15, 3.1] which we now recall. This produces a split comprehension category $(\mathcal{C}_!, \textsc{Fib}(\mathcal{C})_!, p_!, F_!)$ which is equivalent to the one we started with. Unfolding the result, we take the $\mathcal{C}_!$ to be simply $\mathcal{C}$.

The category $\textsc{Fib}(\mathcal{C})_!$ has:

- Objects: for each $\Gamma \in \mathcal{C}$ is a tuple $A := (V_A, E_A, f_A)$ where $V_A \in \mathcal{C}$, $E_A \twoheadrightarrow V_A \in \textsc{Fib}(\mathcal{C})_{V_A}$ and $f_A : \Gamma \rightarrow V_A \in \mathcal{C}$. We also employ the notation $[A] \coloneqq f_A^* E_A$ given by taking the pullback of $E_A \twoheadrightarrow V_A$ along f_A, so we get a fibration $[A] \twoheadrightarrow \Gamma$. In addition, we write $(E_A)_{f_A}$ for the arrow $[A] \to E_A$. Thus, an object over Γ is a diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} & E_A \arrow[d, two heads] \\ \Gamma \arrow[r, "f_A"'] & V_A. \end{tikzcd}$$

- Morphisms: A map between $(V_B, E_B, f_B) \to (V_A, E_A, f_A)$ over $\sigma : \Delta \to \Gamma$ is a map in $\mathcal{E}$ between $[B] \twoheadrightarrow \Delta$ and $[A] \twoheadrightarrow \Gamma$ i.e. a commutative square

$$\begin{tikzcd} {[B]} \arrow[r] \arrow[d, two heads] & {[A]} \arrow[d, two heads] \\ \Delta \arrow[r, "\sigma"'] & \Gamma. \end{tikzcd}$$

- Composition is induced by the composition in $\mathcal{E}$, consequently, given by pasting commutative squares.

- The identity for (V_A, E_A, f_A) is the identity of $[A] \twoheadrightarrow \Gamma$ as an object in $\mathcal{C}^{\to}$.

We now unpack the cartesian lifts for the induced functor $p_! : \textsc{Fib}(\mathcal{C})_! \to \mathcal{C}_!$. Let $\sigma : \Delta \to \Gamma$ and $(V_A, E_A, f_A) \in \textsc{Fib}(\mathcal{C})_!$ over Γ. Set $A[\sigma] \coloneqq (V_A, E_A, f_A\sigma)$, pulling back along $f_A\sigma$ we obtain the commutative outer rectangle below

$$\begin{tikzcd} {[A[\sigma]]} \arrow[r, dashed] \arrow[d, two heads] \arrow[rr, bend left] & {[A]} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E_A \arrow[d, two heads] \\ \Delta \arrow[r, "\sigma"'] & \Gamma \arrow[r, "f_A"'] & V_A \end{tikzcd}$$

The universal property of the pullback on the right give us the unique map $A_\sigma : [A[\sigma]] \to [A]$. Therefore, a lift for σ is given by the evident map $A_\sigma : (V_A, E_A, f_A\sigma) \to (V_A, E_A, f_A)$. From the definition of A_σ the square

$$\begin{tikzcd} {[A[\sigma]]} \arrow[r, "A_\sigma"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & {[A]} \arrow[d, two heads] \\ \Delta \arrow[r, "\sigma"'] & \Gamma \end{tikzcd}$$

is a pullback, this implies that the square as a map in $\text{FIB}(\mathcal{C})_!$ is a cartesian lift of σ for $p_!$. Most importantly, this lift is uniquely determined by the composition $f_A\sigma$. Note that the transfinite composition of fibrations play no role in the construction. We summarize the discussion above in the following:

Theorem B.4.3. For any κ-clan $\mathcal{C}$ there exist a full split comprehension category $(\mathcal{C}', \mathcal{E}, p_!, \iota_!)$ equivalent to $(\mathcal{C}, \text{FIB}(\mathcal{C}), p, \iota)$.

Proof. We apply the previous construction, this give us $(\mathcal{C}_!, \text{FIB}(\mathcal{C})_!, p_!)$. Since the putative cartesian map is uniquely determined by the composition $f_A\sigma$ we can use a slight abuse of notation and write $A_\sigma \coloneqq f_A\sigma$. Thus, if $\chi : \Xi \to \Delta$ is another map then $f(\sigma\chi) = (f\sigma)\chi$. This shows that the fibration $p_! : \text{FIB}(\mathcal{C})_! \to \mathcal{C}_!$ is split. The functor $\iota_! : \text{FIB}(\mathcal{C})_! \to \mathcal{C}^\to$ is defined as $\iota_!(V_A, E_A, f_A) \coloneqq \iota([A] \twoheadrightarrow \Gamma) = [A] \twoheadrightarrow \Gamma$, similarly for arrows. The comprehension category $(\mathcal{C}_!, \text{FIB}(\mathcal{C})_!, p_!, \iota_!)$ is full since $(\mathcal{C}, \text{FIB}(\mathcal{C}), p, \iota)$ is full. $\square$

A **category with attributes** is a comprehension category $(\mathcal{C}, \mathcal{E}, p, F)$ such that p is a discrete fibration. Equivalently, a category with attributes can be defined as:

1. A category $\mathcal{C}$ with a terminal object 1,
2. A presheaf $\mathsf{Ty} : \mathcal{C}^{op} \to \mathbf{Set}$,
3. A function that assigns to each object $A \in \mathsf{Ty}(\Gamma)$, an object $\Gamma.A \in \mathcal{C}$ together with a map $\Gamma.A \to \Gamma$,
4. For each $A \in \mathsf{Ty}(\Gamma)$ and $\sigma : \Delta \to \Gamma$, a pullback square

$$\begin{tikzcd} \sigma^*\Gamma.A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d] \\ \Delta \arrow[r, "\sigma"'] & \Gamma \end{tikzcd}$$

Corollary B.4.4. For any κ-clan $\mathcal{C}$ there exist a category with attributes equivalent to $\mathcal{C}$.

Proof. Theorem B.4.3 give us a full split comprehension category $(\mathcal{C}_!, \text{FIB}(\mathcal{C})_!, p_!, \iota_!)$. We take the category to be $\mathcal{C}_! = \mathcal{C}$. The additional data is given in the obvious way. Defining $\mathsf{Ty}(\Gamma) \coloneqq (\text{FIB}(\mathcal{C})_!)_\Gamma$, for each $A \in \mathsf{Ty}(\Gamma)$, we get $[A] \twoheadrightarrow \Gamma$ as described above. The required pullbacks are given by the cartesian lifts of $p_!$. Furthermore, these pullbacks are computed strictly along compositions since $p_!$ is a split fibration. $\square$

Our next goal is from the category with attributes given by Corollary B.4.4 define a κ-contextual equivalent to $\mathcal{C}$. In particular, for each object $\Gamma \in \mathcal{C}$ we get a κ-contextual category $\mathcal{C}(\Gamma)$. We start with the following observation:

Definition B.4.5. The category structure is given by the following data:

- **Objects**: For each ordinal $\mu < \kappa$ we define the set $Ob_\mu(\mathcal{C}(\Gamma))$ inductively over μ;
  - If $\mu = \lambda + 1$ then we define $Ob_\mu(\mathcal{C}(\Gamma)) \coloneqq \mathsf{Ty}([A_\lambda])$. More explicitly, an object $A_\mu \in Ob_\mu(\mathcal{C}(\Gamma))$ can be represented as the sequence
$$A_\mu \twoheadrightarrow A_\lambda \twoheadrightarrow \cdots \twoheadrightarrow \Gamma$$
and comes with a fibration $A_\mu \twoheadrightarrow \Gamma$.
  - If μ is a limit ordinal then $Ob_\mu(\mathcal{C}(\Gamma))$ is the collection of objects of the form $A_\mu \coloneqq \mathtt{Lim}_{\lambda<\mu} A_\lambda$ obtained as the transfinite composition of a sequence
$$\cdots \twoheadrightarrow A_\lambda \twoheadrightarrow \cdots \twoheadrightarrow \Gamma.$$
Each object comes with a fibration $A_\mu \twoheadrightarrow \Gamma$. This is given by the transfinite composition axiom of $\mathcal{C}$.
- **Morphisms**: For ordinals $\mu \leq \lambda < \kappa$ and objects $B_\lambda \in Ob_\lambda(\mathcal{C}(\Gamma))$, $A_\mu \in Ob_\mu(\mathcal{C}(\Gamma))$ we set
$$Hom_{\mathcal{C}(\Gamma)}(B_\lambda, A_\mu) \coloneqq Hom_{\mathcal{C}/\Gamma}(B_\lambda, A_\mu).$$
- The rest of the structure of $\mathcal{C}(\Gamma)$ is induced by $\mathcal{C}/\Gamma$, in particular the transfinite composition is that of $\mathcal{C}/\Gamma$.

Before proving that this gives us a κ-contextual category, let us explain the objects of this category. Recall that for $A \in \mathsf{Ty}(\Gamma)$ means we have a diagram of the form
$$\begin{tikzcd} & E_A \arrow[d, two heads] \\ \Gamma \arrow[r, "f_A"'] & V_A. \end{tikzcd}$$

When identify this object with $[A]$, then $\mathsf{Ty}([A])$ is the set of objects of the form

$$\begin{tikzcd} & E_B \arrow[d, two heads] \\ {[A]} \arrow[r, "(E_A)_{f_A}"'] & E_A. \end{tikzcd}$$

Each of such objects give $(V_A, f_A, E_B) \in \mathsf{Ty}(\Gamma)$ where $E_B \twoheadrightarrow V_A$ is the composition $E_B \twoheadrightarrow E_A \twoheadrightarrow V_A$. Equivalently, this is the composition $[B] \twoheadrightarrow [A] \twoheadrightarrow \Gamma$. Furthermore, if we write $\Gamma.A \coloneqq [A]$ then we can rewrite this in a more familiar fashion $\Gamma.A.B \twoheadrightarrow \Gamma.A \twoheadrightarrow \Gamma$. This illustrates the general procedure for successor ordinals. A related construction appears in [KL18, Definition 4.3].

Lemma B.4.6. For any κ-clan $\mathcal{C}$ and any $\Gamma \in \mathcal{C}$, the category $\mathcal{C}(\Gamma)$ is a κ-contextual category.

Each axiom follows more or less immediately. We start with the category with attributes we obtained in Corollary B.4.4 and the construction from Definition B.4.5.

Proof.

1. The objects of $\mathcal{C}(\Gamma)$ have grading $Ob(\mathcal{C}(\Gamma)) = \coprod_{\mu<\kappa} Ob_\mu(\mathcal{C}(\Gamma))$ as in Definition B.4.5. This grading determines the height of each object.

2. The terminal object is Γ.

3. Given ordinals $\mu \leq \lambda < \kappa$ and objects $A_\lambda, A_\mu \in \mathcal{C}(\Gamma)$, the display maps between them are the maps in $Hom_{\mathcal{C}(\Gamma)}(A_\lambda, A_\mu)$ which are also fibrations of $\mathcal{C}$. We group these maps and objects in $Dis(\mathcal{C}(\Gamma))$, which is easily seen to be a subcategory.

4. $Dis(\mathcal{C}(\Gamma))$ is closed under transfinite compositions since $\mathcal{C}$ is itself closed under such compositions.

5. The inclusion functor $i : Dis(\mathcal{C}(\Gamma)) \hookrightarrow \mathcal{C}(\Gamma)$ preserve transfinite compositions: ??.

6. If $A \twoheadrightarrow B$ is an arrow in $Dis(\mathcal{C}(\Gamma))$ then $B \in Ob_\mu(\mathcal{C}(\Gamma))$ and $A \in Ob_\lambda(\mathcal{C}(\Gamma))$ for some ordinals λ, μ with $\mu \leq \lambda$: This follows directly by definition of the objects of $\mathcal{C}(\Gamma)$

7. For any object $A \in Ob_\lambda(\mathcal{C}(\Gamma))$ and any $\mu \leq \lambda$ there exists a unique object $B \in Ob_\mu(\mathcal{C}(\Gamma))$ and a unique display map $A \twoheadrightarrow B$: We can easily obtain this by induction on λ and verify that the map has the correct length

8. Canonical pullbacks: This is given by the category with attributes structure on $\mathcal{C}$ as explained in Corollary B.4.4.

9. Canonical pullbacks are strictly functorial: This is exactly what Corollary B.4.4 achieves.

10. It follows from the description of objects given above.

□

Before we can state our main result, we first need state the appropiate notion of equivalence between κ-clans. We borrow the definitions from [Joy17] addapted to our setting. Let $\mathcal{C}$ and $\mathcal{E}$ be two κ-coclans. We say that a functor $F : \mathcal{C} \to \mathcal{E}$ is a **morphism of κ-coclans** if

1. sends initial objects to initial objects,

2. preserves cofibrations,

3. preserves pushouts of cofibrations along any map

4. preserves transfinite compositions.

Furthermore, a morphism between κ-coclans $F : \mathcal{C} \to \mathcal{E}$ is an **equivalence of κ-coclans** if there exists another morphism of κ-coclans $G : \mathcal{E} \to \mathcal{C}$ and natural isomorphisms $GF \cong Id_{\mathcal{C}}$ and $FG \cong Id_{\mathcal{E}}$.

Similarly, $F : \mathcal{C} \to \mathcal{E}$ is a **morphism of κ-clans** simply if $F^{op} : \mathcal{C}^{op} \to \mathcal{E}^{op}$ morphism of κ-coclans, and an **equivalence of κ-clans** if $F^{op} : \mathcal{C}^{op} \to \mathcal{E}^{op}$ is an equivalence κ-coclans.

Proposition B.4.7. A morphism of clans $F : \mathcal{C} \to \mathcal{E}$ is equivalence of clans if and only if F reflects fibrations and tranfinite compositions in $Dis(\mathcal{E})$, this is; if $F(\mathtt{Lim}_\lambda A_\alpha) \twoheadrightarrow F(A_0)$ is the trasfinite composition of the sequence

$$F(\mathtt{Lim}_\lambda A_\alpha) \cdots \twoheadrightarrow FA_2 \twoheadrightarrow FA_1 \twoheadrightarrow FA_0$$

then $\mathtt{Lim}_\lambda A_\alpha \twoheadrightarrow A_0$ is the transfinite composition of the sequence

$$\cdots \twoheadrightarrow A_2 \twoheadrightarrow A_1 \twoheadrightarrow A_0.$$

The equivalence of Theorem B.4.3 give us an equivalence between clans.

Corollary B.4.8. For any κ-coclan $\mathcal{C}$ there exists a κ-contextual category equivalent to it.

Proof. Let us take the κ-clan given by $\mathcal{D} := \mathcal{C}^{op}$. We can then observe that $\mathcal{D} \cong \mathcal{D}(1)$ where $\mathcal{D}(1)$ is the κ-contextual category obtained from Lemma B.4.6. We can take the opposites again to get $\mathcal{C}$. $\square$

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