How To Prove It: A Structured Approach, Second Edition

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Solutions to: *Introduction*

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Exercise 1

(a) Factor $2^{15} - 1 = 32,767$ into a product of two smaller positive integers.

(b) Find an integer x such that $1 < x < 2^{32767} - 1$ and $2^{32767} - 1$ is divisible by x.

Solution:

(a) From the *Proof of Conjecture 2*, we are given that

$$xy = (2^b - 1)(1 + 2^b + 2^{2b} + 2^{3b} + ... + 2^{(a-1)(b)}) = 2^{ab} - 1 = 2^n - 1$$

where n is not prime, x is not prime, y is not prime, and n, x, y, a, b are all positive integers. If $n = 15$, then let $a = 3$ and $b = 5$. Hence,

$$\begin{aligned} 2^{15} - 1 &= 2^{(3)(5)} - 1 \\ &= (2^5 - 1)(1 + 2^5 + 2^{(3-1)(5)}) \\ &= (32 - 1)(1 + 32 + 2^{10}) \\ &= (31)(33 + 1024) \\ &= (31)(1057) \\ &= 32767 \end{aligned}$$

We have thus factored $32,767$ as a product of 31 and 1057.

(b) Similarly as with (a), if $n = 32767$, then let $a = 31$ and $b = 1057$. Hence,

$$\begin{aligned} 2^{32767} - 1 &= 2^{(31)(1057)} - 1 \\ &= (2^{31} - 1)(1 + 2^{31} + 2^{2(31)} + 2^{3(31)} + ... + 2^{(1057-1)(31)}) \\ &= (2^{31} - 1)(1 + 2^{31} + 2^{2(31)} + ... + 2^{(1056)(31)}) \end{aligned}$$

where $x = 2^{31} - 1$ and $y = 1 + 2^{31} + 2^{2(31)} + ... + 2^{(1056)(31)}$. We know from the *Proof of Conjecture 2* that both x, y are positive integers each smaller than n. Either can serve as the solution to the problem. For example, x is greater than one and less than $2^{32767} - 1$, and x divides $2^{32767} - 1$.

Exercise 2

Make some conjectures about the values of n for which $3^n - 1$ is prime or the values of n for which $3^n - 2^n$ is prime.

Solution:

n	Is n prime?	3^n	2^n	$3^n - 1$	Is $3^n - 1$ prime?	$3^n - 2^n$	Is $3^n - 2^n$ prime?
2	yes	9	4	8	no	5	yes
3	yes	27	8	26	no	19	yes
4	no	81	16	80	no	65	no
5	yes	243	32	242	no	211	yes
6	no	729	64	728	no	665	no
7	yes	2187	128	2186	no	2059	no
8	no	6561	256	6560	no	6305	no
9	no	19683	512	19682	no	19171	no
10	no	59049	1024	59048	no	58025	no
11	yes	177147	2048	177146	no	175099	no
12	no	531441	4096	531440	no	527345	no
13	yes	1594323	8192	1594322	no	1586131	no
14	no	4782969	16384	4782968	no	4766585	no
15	no	14348907	32768	14348906	no	14316139	no

Consult a site that lists as many prime numbers as possible to speed up the table creation process. From the pattern in the table, we make the following conjectures:

Conjecture 1: Suppose n is an integer larger than 1. Then $3^n - 1$ is not prime and is an even number.

Conjecture 2: Suppose $3^n - 2^n$ is prime, then n is prime.

Exercise 3

The proof of Theorem 3 gives a method for finding a prime number different from any in a given list of prime numbers.

(a) Use this method to find a prime different from 2,3,5, and 7.

(b) Use this method to find a prime different from 2,5, and 11.

Solution: Recall that **Theorem 3** gave the following method for finding a prime number different from those in the given list of prime numbers in its premise. If $m = p_1 p_2 \cdots p_n + 1$, where $p_1, p_2, ..., p_n$ is a list of prime numbers and m is a positive integer, then m is a prime or a product of primes.

(a) If $p_1 p_2 \cdots p_n = (2)(3)(5)(7)$, then $m = (2)(3)(5)(7) + 1 = 211$. Note that 211 is prime and not in the list.

(b) If $p_1 p_2 \cdots p_n = (2)(5)(11)$, then $m = (2)(5)(11) + 1 = 111 = (3)(37)$. Note that 3 and 37 are prime and not in the list.

Exercise 4

Find 5 consecutive integers that are not prime.

Solution: Recall that **Theorem 4** states that for every positive integer n, there is a sequence of n consecutive positive integers containing no primes. More over, if n is a positive integer, and $x = (n+1)! + 2$, then $x, x+1, x+2, ..., x+(n-1)$ are not prime. Take $n = 5$, then

$$\begin{aligned}
x &= 6! + 2 = 720 + 2 = 722 = (2)(361) \\
x + 1 &= 6! + 3 = 720 + 3 = 723 = (3)(241) \\
x + 2 &= 6! + 4 = 720 + 4 = 724 = (2)(362) \\
x + 3 &= 6! + 5 = 720 + 5 = 725 = (5)(145) \\
x + 4 &= 6! + 6 = 720 + 6 = 726 = (2)(363)
\end{aligned}$$

We have thus found 5 consecutive integers that are not prime.

Exercise 5

Use the table in Figure 1 and the discussion on p. 5 to find two more perfect numbers.

Solution: Recall the Figure 1 table, some of it shown below for convenience.

n	Is n prime?	$2^n - 1$	Is $2^n - 1$ prime?
2	yes	3	yes
3	yes	7	yes
4	no	15	no
5	yes	31	yes
6	no	63	no
7	yes	127	yes

And the relevant part of the discussion on p. 5 was that "Euclid proved that if $2^n - 1$ is prime, then $(2^n - 1) \cdot (2^{n-1})$ is a perfect". Also recall that a positive integer n is said to be perfect if n is equal to the sum of all positive integers smaller than n that divide n. And we have established already that $6 = 1 + 23 = 2^1(2^2 - 1)$ and
$28 = 1 + 2 + 4 + 7 + 14 = 2^2(2^3 - 1)$ are perfect numbers.

From the table we see that the other Mersenne primes are 5 and 7. Hence,

$$(2^5 - 1) \cdot (2^{5-1}) = (2^5 - 1)(2^4) = (32 - 1)(16) = (31)(16) = 496$$

and

$$(2^7 - 1) \cdot (2^{7-1}) = (2^7 - 1)(2^6) = (128 - 1)(64) = (127)(64) = 8128$$

are two more perfect numbers. We verify that these are perfect numbers using the definition of perfect numbers. The divisors of 496 (and smaller than 496) are 1,2,4,8,16,31,62,124, and 248; and $1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248 = 496$. The divisors of 8128 (and smaller than 8128) are 1,2,4,8,16,32,64,127,254,508,1016,2032, and 4064. Their sum is 8128.

Exercise 6

The sequence 3,5,7 is a list of three prime numbers such that each pair of adjacent numbers in the list differ by two. Are there any more such "triplet primes"?

Solution: That is a unique sequence of triplet primes. No other sequence exists since any other triplet of numbers has exactly one term divisible by 3.

Consider $(n, n+2, n+4)$ where n is a positive integer.

Case 1: If n is divisible by three, then we are done.

Case 2: If n has remainder 1 when divided by 3, then $n = 3k + 1$, so $n + 2 = 3k + 3 = 3(k + 1)$ for some positive integer k.

Case 3: If n has remainder 2 when divided by 3, then $n = 3k + 2$, so $n + 4 = 3k + 6 = 3(k + 2)$ for some positive integer k.

Hence for all values of n greater than 3, we will always get at least one number in the triplet sequence that is divisible by 3, thus not prime. The proof above is not expected now, so revisit later when the tools have been learned.