How To Prove It: A Structured Approach, Second Edition

By Daniel J. Velleman

Solutions to: *1.1 Deductive Reasoning and Logical Connectives*

By Cesar Gonzalez

February 2024

Contents

Exercise 1

Analyze the logical forms of the following statements:

(a) We'll have either a reading assignment or homework problems, but we won't have both homework problems and a test.

(b) You won't go skiing, or you will and there won't be any snow.

(c) $\sqrt{7} \nleq 2$.

Solution:

(a) We denote by R the proposition "we will have a reading assignment", by H the proposition "we will have homework problems", and by T the proposition "we will have a test". The given statement can be translated into propositional logic as follows:

$$(R \vee H) \wedge \neg(H \wedge T).$$

This expresses that we will have either a reading assignment or homework problems, but not both homework problems and a test.

(b) Let S represent "you will go skiing" and W represent "there will be snow". The statement can be rewritten as:

$$\neg S \vee (S \wedge \neg W).$$

(c) The statement $\sqrt{7} \nleq 2$ can be understood as "the square root of 7 is not less than or equal to 2". It can be formally written as:

$$\neg \left((\sqrt{7} < 2) \vee (\sqrt{7} = 2) \right),$$

which denies the possibility that the square root of 7 is less than or equal to 2.

Exercise 2

Analyze the logical forms of the following statements:

(a) Either John and Bill are both telling the truth, or neither of them is.

(b) I'll have either fish or chicken, but I won't have both fish and mashed potatoes.

(c) 3 is a common divisor of 6,9, and 15.

Solution:

(a) Let J represent the statement ”John is telling the truth”, and B represent the statement ”Bill is telling the truth”. The given statement can be thus translated into propositional logic as follows:

$$(J \wedge B) \vee (\neg J \wedge \neg B).$$

(b) Let F represent the statement ”I will have fish”, C represent the statement ”I will have chicken” and P represent the statement ”I will have mashed potatoes”. The given statement can be thus translated into propositional logic as follows:

$$(F \vee C) \wedge \neg(F \wedge P).$$

(c) The statement ”3 is a common divisor of 6,9, and 15” can be understood as ”3 is a common divisor of 6”, and ”3 is a common divisor of 9”, and ”3 is a common divisor of 15”. It can be formally written as:

$$(3|6) \wedge (3|9) \wedge (3|15).$$

Exercise 3

Analyze the logical forms of the following statements:

(a) Alice and Bob are not both in the room.

(b) Alice and Bob are both not in the room.

(c) Either Alice or Bob is not in the room.

(d) Neither Alice nor Bob is in the room.

Solution: Let A represent the statement ”Alice is in the room” and B the statement ”Bob is in the room”. The given statements can be thus translated into propositional logic as follows:

(a) It is not the case that both Alice is in the room and Bob is in the room at the same time: $\neg(A \wedge B)$

(b) Both Alice is not in the room, and Bob is not in the room. There are no other possibilities; both individuals are absent from the room: $\neg A \wedge \neg B$

(c) Either Alice is not in the room, or Bob is not in the room, or both are not in the room: $\neg A \vee \neg B$

(d) Alice is not in the room, and Bob is also not in the room. There is no scenario where either Alice or Bob is present in the room; both are absent: $\neg A \wedge \neg B$

NOTE **”Neither Alice nor Bob is in the room”** uses a construction that directly states that both individuals are absent from the room. It’s a single statement that negates the presence of both individuals simultaneously.

”Alice and Bob are both not in the room” effectively communicates the same information, indicating that both Alice and Bob are absent from the room. This construction explicitly states the absence of each individual.

Logically, both sentences assert the absence of both Alice and Bob from the room. In terms of logical operators, both sentences can be understood as expressing a conjunction (logical AND) of two negations.

Exercise 4

Which of the following expressions are well-formed?

(a) $\neg(\neg P \vee \neg\neg R)$

(b) $\neg(P, Q, \wedge R)$

(c) $P \wedge \neg P$

(d) $(P \wedge Q)(P \vee R)$

Solution:

(a) Yes, this expression is well-formed. It uses standard logical operators correctly: negation ($\neg$), disjunction ($\vee$), and double negation. It correctly negates the disjunction of $\neg P$ and $\neg\neg R$, which is a valid logical expression.

(b) No, this expression is not well-formed. The syntax $\neg(P, Q, \wedge R)$ is incorrect because it attempts to use a comma and parentheses in a way that is not standard in logical expressions. Logical operators like conjunction ($\wedge$) should directly connect propositions without commas.

(c) Yes, this expression is well-formed, although it is a contradiction. It uses the conjunction ($\wedge$) operator to connect P and its negation $\neg P$ correctly. A contradiction is always false, but the expression itself is syntactically correct.

(d) No, this expression is not well-formed because it lacks an operator between the two grouped expressions $(P \wedge Q)$ and $(P \vee R)$. For an expression to be well-formed, there should be a logical operator (like $\wedge$, $\vee$, $\rightarrow$, etc.) connecting any two sub-expressions.

Exercise 5

Let P stand for the statement ”I will buy the pants” and S for the statement ”I will buy the shirt.” What English sentences are represented by the following expressions?

(a) $\neg(P \wedge \neg S)$.

(b) $\neg P \wedge \neg S$.

(c) $\neg P \vee \neg S$.

Solution:

(a) It is not the case that I will buy the pants and not the shirt.

(b) I will neither buy the pants nor the shirt.

(c) I will not buy either the pants or the shirt

Exercise 6

Let S stand for the statement "Steve is happy" and G for the statement "George is happy". What English sentences are represented by the following expressions?

(a) $(S \vee G) \wedge (\neg S \vee \neg G)$.

(b) $[S \vee (G \wedge \neg S)] \vee \neg G$.

(c) $S \vee [G \wedge (\neg S \vee \neg G)]$.

Solution:

(a) This expression means that at least one of Steve or George is happy, but they are not both happy. It combines the possibilities that either or both might be happy with the restriction that it cannot be the case that both are happy simultaneously. **Either Steve or George is happy, but not both.**

(b)
- $(G \wedge \neg S)$ translates to "George is happy and Steve is not happy".
- $S \vee (G \wedge \neg S)$ translates to "Either Steve is happy, or George is happy but Steve is not happy".
- $\neg G$ translates to "George is not happy".

Combining these parts with the OR operator we get: "Either it is true that Steve is happy, or it is true that George is happy but Steve is not happy, or George is not happy". In a more natural English sentence, this logical expression could be interpreted as **Either Steve is happy, or George is happy while Steve is not, or George is not happy.** Which can be further simplified to the sentence in **(a).**

(c)
- $(\neg S \vee \neg G)$ translates to "Either Steve is not happy, or George is not happy".
- $G \wedge (\neg S \vee \neg G)$ translates to "George is happy, and either Steve is not happy or George is not happy".
- $S \vee [G \wedge (\neg S \vee \neg G)]$ translates to "Either Steve is happy, or George is happy and at the same time, it is also true that either Steve is not happy or George is not happy".

However, the part "George is happy and George is not happy" is contradictory, which means that for the entire expression to be true, the only possible scenario is "George is happy and Steve is not happy." Given that the statement includes an OR operator with "Steve is happy," it simplifies to **Either Steve is happy, or George is happy, but not both.**

NOTE Remove the parentheses from all the statements: $S \vee G \wedge \neg S \vee \neg G$.

Exercise 7

Identify the premises and conclusions of the following deductive arguments and analyze their logical forms. Do you think the reasoning is valid? (Although you will have only your intuition to guide you in answering this last question, in the next section we will develop some techniques for determining the validity of arguments.)

(a) Jane and Pete won't both win the math prize. Pete will win either the math prize or the chemistry prize. Jane will win the math prize. Therefore, Pete will win the chemistry prize.

(b) The main course will be either beef or fish. The vegetable will be either peas or corn. We will not have both fish as a main course and corn as a vegetable. Therefore, we will not have both beef as a main course and peas as a vegetable.

(c) Either John or Bill is telling the truth. Either Sam or Bill is lying. Therefore, either John is telling the truth or Sam is lying.

(d) Either sales will go up and the boss will be happy, or expenses will go up and the boss won't be happy. Therefore, sales and expenses will not both go up.

Solution:
In all of the above, each sentence is a premise, except the sentence that begins with "Therefore" since that is the conclusion.

(a) **The argument is valid.** Pete will win the chemistry or math prize, but he can't win the math prize because of Jane, so he is left with winning the chemistry prize.

(b) **The argument is invalid** since we are ruling out possibilities without just cause.

(c) **The reasoning is valid**, but not for the reasons that might seem obvious. The conclusion seems to restate part of the premises rather than providing a new insight. However, assuming that telling the truth and lying are mutually exclusive and collectively exhaustive (everyone is either lying or telling the truth), and given that Bill's role is ambiguous (he could be the one telling the truth or lying), the conclusion that either John is telling the truth or Sam is lying does follow from the premises but doesn't exclude other possibilities (e.g., both could be true). The argument's structure is somewhat valid but doesn't necessarily provide meaningful insight beyond the premises.

(d) **The argument is invalid.** It incorrectly infers the mutual exclusivity of sales and expenses rising from the conditions of the boss's happiness. The singular premise do not establish a necessary link between the increase in sales and expenses, merely associating them with the boss's emotional state. This association does not logically prevent both sales and expenses from increasing together. There is no logical connection between the premise and conclusion.