How To Prove It: A Structured Approach, Second Edition

By Daniel J. Velleman

Solutions to: *1.5 The Conditional and Biconditional Connectives*

By Cesar Gonzalez

July 2024

Contents

Exercise 1

Analyze the logical forms of the following statements:

(a) If this gas either has an unpleasant smell or is not explosive, then it isn't hydrogen.

(b) Having both a fever and a headache is a sufficient condition for George to go to the doctor.

(c) Both having a fever and having a headache are sufficient conditions for George to go to the doctor.

(d) If $x \neq 2$, then a necessary condition for x to be prime is that x be odd.

Solution:

(a) Let U stand for the statement "**gas has an unpleasant smell**" and E stand for "**gas is explosive**", and H stand for "**gas is hydrogen**". Then the logical form of the statement given can be expressed as $(U \vee \neg E) \rightarrow \neg H$.

(b) Let F stand for the statement "**George has a fever**", H stand for "**George has a headache**", and D stand for "**George has to go to the doctor**". Then the logical form of the statement given can be expressed as $(F \wedge H) \rightarrow D$.

(c) Let F stand for the statement "**George has a fever**", H stand for "**George has a headache**", and D stand for "**George has to go to the doctor**". Then the logical form of the statement given can be expressed as $(F \rightarrow D) \wedge (H \rightarrow D)$.

(d) Let T stand for the statement $\boldsymbol{x \neq 2}$, let $P(x)$ stand for "x **is prime**" and let $O(x)$ stand for "x **is odd**". Then the logical form of the statement given can be expressed as $[T \wedge P(x)] \rightarrow O(x)$, or $T \rightarrow [P(x) \rightarrow O(x)]$.

Exercise 2

Analyze the logical forms of the following statements:

(a) Mary will sell her house only if she can get a good price and find a nice apartment.

(b) Having both a good credit history and an adequate down payment is a necessary condition for getting a mortgage.

(c) John will kill himself, unless someone stops him. (Hint: First try to rephrase this using the words *if* and *then* instead of *unless*.)

(d) If x is divisible by either 4 or 6, then it isn't prime.

Solution:

(a) Let S stand for the statement "**May will sell her house.**" Let P stand for "**The house will sell for a good price.**" Finally, let A stand for "**May can find a nice apartment.**" Then the logical form of the given statement can be expressed as $S \to (P \wedge A)$.

(b) Let C stand for the statement "**Have a good credit history.**" Let D stand for the statement "**Have an adequate down payment.**" Let M stand for "**Having a mortgage.**" The logical form of the given statement is $M \to (C \wedge D)$.

(c) Let K stand for the statement "**John will kill himself.**" Let S stand for "**Someone will not stop John from killing himself.**" The logical form of the given statement is $\neg S \to K$.

(d) Let $A(x)$, $B(x)$, and $C(x)$ stand for the statements: "$\boldsymbol{x}$ **is divisible by 4**", "$\boldsymbol{x}$ **is divisible by 6**", and "$\boldsymbol{x}$ **is a prime number**," correspondingly. The logical form of the given statement is $[A(x) \vee B(x)] \to \neg\, C(x)$.

Exercise 3

Analyze the logical form of the following statement:

(a) If it is raining, then it is windy and the sun is not shining.

Now analyze the following statements. Also, for each statement determine whether the statement is equivalent to either statement (a) or its converse.

(b) It is windy and not sunny only if it is raining.

(c) Rain is a sufficient condition for wind with no sunshine.

(d) Rain is a necessary condition for wind with no sunshine.

(e) It's not raining, if either the sun is shining or it's not windy.

(f) Wind is a necessary condition for it to be rainy, and so is a lack of sunshine.

(g) Either it is windy only if it is raining, or it is not sunny only if it is raining.

Solution:

(a) Let the following symbols represent the corresponding statements:

- R: It is raining
- W: It is windy
- S: The sun is shining

The logical form of the original statement can be expressed as:

$$R \to (W \land \neg S)$$

The converse of the statement can be expressed as:

$$(W \land \neg S) \to R$$

If we let $P = (W \land \neg S)$ and $Q = R$, then the converse can be stated in various equivalent ways:

- If it is windy and the sun is not shining, then it is raining. (P implies Q)
- It is raining, if it is windy and the sun is not shining. (Q, if P)
- It is windy and the sun is not shining only if it is raining. (P only if Q)
- Being windy and the sun not shining is a sufficient condition for it to be raining. (P is a sufficient condition for Q)
- That it is raining is a necessary condition for it to be windy and the sun not to shine. (Q is a necessary condition for P)
- It is a necessary condition for it to be windy and the sun not shining that it is raining (Exercise 1d).

(b) "It is windy and not sunny only if it is raining" can be expressed as:

$$(W \wedge \neg S) \rightarrow R.$$

(c) "Rain is a sufficient condition for wind with no sunshine" can be expressed as:

$$R \rightarrow (W \wedge \neg S).$$

(d) "Rain is a necessary condition for wind with no sunshine" can be expressed as:

$$(W \wedge \neg S) \rightarrow R.$$

(e) "It's not raining, if either the sun is shining or it's not windy" can be expressed as:

$$\begin{array}{rl}
(\neg W \vee S) \rightarrow \neg R & \textbf{Conditional statement} \\
\neg(\neg W \vee S) \vee \neg R & \textbf{Conditional Law} \\
(W \wedge \neg S) \vee \neg R & \textbf{DeMorgan's Law} \\
\neg R \vee (W \wedge \neg S) & \textbf{Commutative Law} \\
R \rightarrow (W \wedge \neg S) & \textbf{Conditional Law}
\end{array}$$

(f) "Wind is a necessary condition for it to be rainy, and so is a lack of sunshine" can be expressed as:

$$\begin{array}{rl}
(R \rightarrow W) \wedge (R \rightarrow \neg S) & \textbf{Compound sentence} \\
(\neg R \vee W) \wedge (\neg R \vee \neg S) & \textbf{Conditional Law} \\
\neg R \vee (W \wedge \neg S) & \textbf{Distributive law} \\
R \rightarrow (W \wedge \neg S) & \textbf{Conditional Law}
\end{array}$$

(g) "Either it is windy only if it is raining, or it is not sunny only if it is raining" can be expressed as:

$$\begin{array}{rl}
(W \rightarrow R) \vee (\neg S \rightarrow R) & \textbf{Compound sentence} \\
(\neg W \vee R) \vee (S \vee R) & \textbf{Conditional Law} \\
\neg W \vee R \vee S \vee R & \textbf{Associative law} \\
R \vee R \vee \neg W \vee S & \textbf{Commutative law} \\
(R \vee R) \vee (\neg W \vee S) & \textbf{Associative law} \\
R \vee (\neg W \vee S) & \textbf{Idempotent law} \\
\neg R \rightarrow (\neg W \vee S) & \textbf{Conditional Law} \\
(W \wedge \neg S) \rightarrow R & \textbf{Contrapositive}
\end{array}$$

We observe that the original statement **3a** is logically equivalent with statements **3c, 3e and 3f**.
We observe that the converse of statement **3a** is logically equivalent with statements **3b, 3d, and 3g**.

Exercise 4

Use truth tables to determine whether or not the following arguments are valid:

(a) Either sales or expenses will go up. If sales go up, then the boss will be happy. If expenses go up, then the boss will be unhappy. Therefore, sales and expenses will not both go up.

(b) If the tax rate and the unemployment rate both go up, then there will be a recession. If the GNP goes up, then there will not be a recession. The GNP and taxes are both going up. Therefore, the unemployment rate is not going up.

(c) The warning light will come on if and only if the pressure is too high and the relief valve is clogged. The relief valve is not clogged. Therefore, the warning light will come on if and only if the pressure is too high.

Solution:

(a) Let the following symbols represent the corresponding statements:

- S: Sales will increase
- E: Expenses will increase
- H: Boss is happy

The logical form of the argument can be expressed as:

$S \vee E$	**Sales will increase or expenses will increase**
$S \to H$	**If sales increase, the boss will be happy**
$E \to \neg H$	**If expenses increase, the boss will be unhappy**
$\therefore \neg(S \wedge E)$	**Therefore, sales and expenses will not both increase**

S	E	H	$\neg S$	$\neg E$	$\neg H$	$S \vee E$	$S \to H$	$E \to \neg H$	$\neg(S \wedge E)$
0	0	0	1	1	1	0	1	1	1
0	0	1	1	1	0	0	1	1	1
0	1	0	1	0	1	1	1	1	1
0	1	1	1	0	0	1	1	0	1
1	0	0	0	1	1	1	0	1	1
1	0	1	0	1	0	1	1	1	1
1	1	0	0	0	1	1	0	1	0
1	1	1	0	0	0	1	1	0	0

To determine if the argument is valid based on the truth table, we must identify the premises and conclusion, and then check if there is any scenario where all the premises are true and the conclusion is false. If no such scenario exists, the argument is valid. We see that rows 3 and 6 are the only rows where all the premises are true. More importantly, the conclusion in both these rows are true.
Hence, the argument is valid.

(b) Let the following symbols represent the corresponding statements:

- T: Tax rate increases
- U: Unemployment rate increases
- R: There is a recession
- G: GNP increases

The logical form of the argument can be expressed as:

$(T \wedge U) \to R$	**If both the tax rate and unemployment rate increase, then there's a recession**
$G \to \neg R$	**If the GNP increases, then there's no be a recession**
$G \wedge T$	**Both the GNP and taxe rate are increasing**
$\therefore \neg U$	**Therefore, the unemployment rate will not increase**

T	U	R	G	$\neg R$	$(T \wedge U) \to R$	$G \to \neg R$	$G \wedge T$	$\neg U$
0	0	0	0	1	1	1	0	1
0	0	0	1	1	1	1	0	1
0	0	1	0	0	1	1	0	1
0	0	1	1	0	1	0	0	1
0	1	0	0	1	1	1	0	0
0	1	0	1	1	1	1	0	0
0	1	1	0	0	1	1	0	0
0	1	1	1	0	1	0	0	0
1	0	0	0	1	1	1	0	1
1	0	0	1	1	1	1	1	1
1	0	1	0	0	1	1	0	1
1	0	1	1	0	1	0	1	1
1	1	0	0	1	0	1	0	0
1	1	0	1	1	0	1	1	0
1	1	1	0	0	1	1	0	0
1	1	1	1	0	1	0	1	0

To determine if the argument is valid based on the truth table, we must identify the premises and conclusion, and then check if there is any scenario where all the premises are true and the conclusion is false. If no such scenario exists, the argument is valid. We see that row 10 is the only row where the premises are true. More importantly, the conclusion is true. **Hence, the argument is valid.**

(c) Let the following symbols represent the corresponding statements:

- W: Warning lights will turn on
- P: Pressure is high
- V: Relief valve is clogged

The logical form of the argument can be expressed as:

$W \leftrightarrow (P \land V)$	**Warning light turns on iff the pressure is high and the valve is clogged**
$\neg V$	**The valve is not clogged**
$\therefore W \leftrightarrow P$	**Therefore, the warning light turns on iff the pressure is too high**

W	P	V	$P \land V$	$W \leftrightarrow (P \land V)$	$\neg V$	$W \leftrightarrow P$
0	0	0	0	1	1	1
0	0	1	0	1	0	1
0	1	0	0	1	1	0
0	1	1	1	0	0	0
1	0	0	0	0	1	0
1	0	1	0	0	0	0
1	1	0	0	0	1	1
1	1	1	1	1	0	1

To determine if the argument is valid based on the truth table, we must identify the premises and conclusion, and then check if there is any scenario where all the premises are true and the conclusion is false. If no such scenario exists, the argument is valid. We see that row 3 is one of two rows where the premises are true. More importantly, the conclusion is false in row 3. **Hence, the argument is invalid.**

Exercise 5

(a) Show that $P \leftrightarrow Q$ is equivalent to $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.

(b) Show that $(P \to Q) \vee (P \to R)$ is equivalent to $P \to (Q \vee R)$.

Solution:

(a) Below we prove that $P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$.

$$\begin{aligned}
(P \wedge Q) \vee (\neg P \wedge \neg Q) &\equiv [(P \wedge Q) \vee \neg P] \wedge [(P \wedge Q) \vee \neg Q] && \textbf{Distributive Law} \\
&\equiv [(P \vee \neg P) \wedge (Q \vee \neg P)] \wedge [(P \vee \neg Q) \wedge (Q \vee \neg Q)] && \textbf{Distributive Law} \\
&\equiv [\text{Tautology} \wedge (Q \vee \neg P)] \wedge [(P \vee \neg Q) \wedge \text{Tautology}] && \textbf{Def. of Tautology} \\
&\equiv (Q \vee \neg P) \wedge (P \vee \neg Q) && \textbf{Law of Tautology} \\
&\equiv (\neg P \vee Q) \wedge (\neg Q \vee P) && \textbf{Commutative Law} \\
&\equiv (P \to Q) \wedge (Q \to P) && \textbf{Conditional Law} \\
&\equiv P \leftrightarrow Q && \textbf{Biconditional Law}
\end{aligned}$$

(b) Below we prove that $(P \to Q) \vee (P \to R) \equiv P \to (Q \vee R)$.

$$\begin{aligned}
(P \to Q) \vee (P \to R) &\equiv (\neg P \vee Q) \vee (\neg P \vee R) && \textbf{Conditional Law} \\
&\equiv \neg P \vee Q \vee \neg P \vee R && \textbf{Associative Law} \\
&\equiv \neg P \vee \neg P \vee Q \vee R && \textbf{Commutative Law} \\
&\equiv (\neg P \vee \neg P) \vee (Q \vee R) && \textbf{Associative Law} \\
&\equiv \neg P \vee (Q \vee R) && \textbf{Idempotent Law} \\
&\equiv P \to (Q \vee R) && \textbf{Conditional Law}
\end{aligned}$$

Exercise 6

(a) Show that $(P \to R) \wedge (Q \to R)$ is equivalent to $(P \vee Q) \to R$.

(b) Formulate and verify a similar equivalence involving $(P \to R) \vee (Q \to R)$.

Solution:

(a) Below we prove that $(P \to R) \wedge (Q \to R) \equiv (P \vee Q) \to R$.

$$\begin{aligned}
(P \to R) \wedge (Q \to R) &\equiv (\neg P \vee R) \wedge (\neg Q \vee R) && \textbf{Conditional Law} \\
&\equiv (\neg P \wedge \neg Q) \vee R && \textbf{Distributive Law} \\
&\equiv \neg(P \vee Q) \vee R && \textbf{DeMorgan's Law} \\
&\equiv (P \vee Q) \to R && \textbf{Conditional Law}
\end{aligned}$$

(b) Below we prove that $(P \to R) \vee (Q \to R) \equiv (P \wedge Q) \to R$.

$$\begin{aligned}
(P \to R) \vee (Q \to R) &\equiv (\neg P \vee R) \vee (\neg Q \vee R) && \textbf{Conditional Law} \\
&\equiv \neg P \vee R \vee \neg Q \vee R && \textbf{Associative Law} \\
&\equiv \neg P \vee \neg Q \vee R \vee R && \textbf{Commutative Law} \\
&\equiv (\neg P \vee \neg Q) \vee (R \vee R) && \textbf{Associative Law} \\
&\equiv (\neg P \vee \neg Q) \vee R && \textbf{Idempotent Law} \\
&\equiv \neg(P \wedge Q) \vee R && \textbf{DeMorgan's Law} \\
&\equiv (P \wedge Q) \to R && \textbf{Conditional Law}
\end{aligned}$$

Exercise 7

(a) Show that $(P \to Q) \wedge (Q \to R)$ is equivalent to $(P \to R) \wedge [(P \leftrightarrow Q) \vee (R \leftrightarrow Q)]$.

(b) Show that $(P \to Q) \vee (Q \to R)$ is a tautology.

Solution:

(a) Below we prove that $(P \to Q) \wedge (Q \to R) \equiv (P \to R) \wedge [(P \leftrightarrow Q) \vee (R \leftrightarrow Q)]$.

First we prove **Lemma 7.a.1**: $(P \to Q) \wedge (Q \to R) \equiv [P \to (Q \wedge R)] \wedge (Q \to R)$.

$$
\begin{aligned}
(P \to Q) \wedge (Q \to R) &\equiv (\neg P \vee Q) \wedge (\neg Q \vee R) && \textbf{Conditional Law}\\
&\equiv [(\neg P \vee Q) \wedge \neg Q] \vee [(\neg P \vee Q) \wedge R] && \textbf{Distributive Law}\\
&\equiv [(\neg P \wedge \neg Q) \vee (Q \wedge \neg Q)] \vee [(\neg P \wedge R) \vee (Q \wedge R)] && \textbf{Distributive Law}\\
&\equiv [(\neg P \wedge \neg Q) \vee \text{Contradiction}] \vee [(\neg P \wedge R) \vee (Q \wedge R)] && \textbf{Def. of Contradiction}\\
&\equiv [\neg P \wedge \neg Q] \vee [(\neg P \wedge R) \vee (Q \wedge R)] && \textbf{Law of Contradiction}\\
&\equiv (\neg P \wedge \neg Q) \vee (\neg P \wedge R) \vee (Q \wedge R) && \textbf{Associtive Law}\\
&\equiv [(\neg P \wedge \neg Q) \vee (\neg P \wedge R)] \vee (Q \wedge R) && \textbf{Associtive Law}\\
&\equiv [\neg P \wedge (\neg Q \vee R)] \vee (Q \wedge R) && \textbf{Distributive Law}\\
&\equiv [\neg P \vee (Q \wedge R)] \wedge [(\neg Q \vee R) \vee (Q \wedge R)] && \textbf{Distributive Law}\\
&\equiv [\neg P \vee (Q \wedge R)] \wedge \{[(\neg Q \vee R) \vee Q] \wedge [(\neg Q \vee R) \vee R]\} && \textbf{Distributive Law}\\
&\equiv [\neg P \vee (Q \wedge R)] \wedge \{[(\neg Q \vee Q) \vee R] \wedge [\neg Q \vee (R \vee R)]\} && \textbf{Associative/Commutative Laws}\\
&\equiv [\neg P \vee (Q \wedge R)] \wedge \{[(\neg Q \vee Q) \vee R] \wedge (\neg Q \vee R)\} && \textbf{Idempotent Law}\\
&\equiv [\neg P \vee (Q \wedge R)] \wedge \{[\text{Tautology} \vee R] \wedge (\neg Q \vee R)\} && \textbf{Def. of Tautology}\\
&\equiv [\neg P \vee (Q \wedge R)] \wedge [\text{Tautology} \wedge (\neg Q \vee R)] && \textbf{Law of Tautology}\\
&\equiv [\neg P \vee (Q \wedge R)] \wedge (\neg Q \vee R) && \textbf{Law of Tautology}\\
&\equiv [P \to (Q \wedge R)] \wedge (Q \to R) && \textbf{Conditional Law}
\end{aligned}
$$

We also prove **Lemma 7.a.2**: $(P \to Q) \wedge (P \to R) \equiv [P \to (Q \wedge R)]$.

$$
\begin{aligned}
(P \to Q) \wedge (P \to R) &\equiv (\neg P \vee Q) \wedge (\neg P \vee R) && \textbf{Conditional law}\\
&\equiv \neg P \vee (Q \wedge R) && \textbf{Distributive law}\\
&\equiv P \to (Q \wedge R) && \textbf{Conditional law}
\end{aligned}
$$

We also prove **Lemma 7.a.3**: $Q \to Q \equiv$ Tautology.

$$
\begin{aligned}
Q \to Q &\equiv \neg Q \vee Q && \textbf{Conditional law}\\
&\equiv \text{Tautology} && \textbf{Definition of Tautology}
\end{aligned}
$$

Proof of **7.a**:

$$
\begin{array}{ll}
(P \to R) \wedge [(P \leftrightarrow Q) \vee (R \leftrightarrow Q)] & \textbf{RHS} \\
\equiv (P \to R) \wedge \{[(P \wedge Q) \vee (\neg P \wedge \neg Q)] \vee [(R \wedge Q) \vee (\neg R \wedge \neg Q)]\} & \textbf{Exercise 5.a} \\
\equiv (P \to R) \wedge \{[(P \wedge Q) \vee (R \wedge Q)] \vee [(\neg P \wedge \neg Q) \vee (\neg R \wedge \neg Q)]\} & \textbf{Associative/Commutative Laws} \\
\equiv (P \to R) \wedge \{[(P \vee R) \wedge Q] \vee [(\neg P \vee \neg R) \wedge \neg Q]\} & \textbf{Distributive Law} \\
\equiv \{(P \to R) \wedge [(P \vee R) \wedge Q]\} \vee \{(P \to R) \wedge [(\neg P \vee \neg R) \wedge \neg Q]\} & \textbf{Distributive Law} \\
\equiv [(P \to R) \wedge (P \vee R) \wedge Q] \vee [(P \to R) \wedge (\neg P \vee \neg R) \wedge \neg Q] & \textbf{Associative Law} \\
\equiv [(\neg P \vee R) \wedge (P \vee R) \wedge Q] \vee [(\neg P \vee R) \wedge (\neg P \vee \neg R) \wedge \neg Q] & \textbf{Conditional Law} \\
\equiv [\{(\neg P \vee R) \wedge (P \vee R)\} \wedge Q] \vee [\{(\neg P \vee R) \wedge (\neg P \vee \neg R)\} \wedge \neg Q] & \textbf{Associative Law} \\
\equiv [\{(\neg P \wedge P) \vee R)\} \wedge Q] \vee [\{\neg P \vee (R \wedge \neg R)\} \wedge \neg Q] & \textbf{Distributive Law} \\
\equiv [\{\text{Contradiction} \vee R)\} \wedge Q] \vee [\{\neg P \vee \text{Contradiction}\} \wedge \neg Q] & \textbf{Def. of Contradiction} \\
\equiv [R \wedge Q] \vee [\neg P \wedge \neg Q] & \textbf{Law of Contradiction} \\
\equiv [(R \wedge Q) \vee \neg P] \wedge [(R \wedge Q) \vee \neg Q] & \textbf{Distributive Law} \\
\equiv [\neg P \vee (R \wedge Q)] \wedge [\neg Q \vee (R \wedge Q)] & \textbf{Commutative Law} \\
\equiv [P \to (R \wedge Q)] \wedge [Q \to (R \wedge Q)] & \textbf{Conditional Law} \\
\equiv [(P \to R) \wedge (P \to Q)] \wedge [(Q \to R) \wedge (Q \to Q)] & \textbf{Lemma 7.a.2} \\
\equiv [(P \to R) \wedge (P \to Q)] \wedge [(Q \to R) \wedge \text{Tautology}] & \textbf{Lemma 7.a.3} \\
\equiv [(P \to R) \wedge (P \to Q)] \wedge (Q \to R) & \textbf{Law of Tautology} \\
\equiv [P \to (R \wedge Q)] \wedge (Q \to R) & \textbf{Lemma 7.a.2} \\
\equiv (P \to Q) \wedge (Q \to R) & \textbf{Lemma 7.a.1}
\end{array}
$$

(b) We prove that $(P \to Q) \vee (Q \to R) \equiv$ Tautology.

$$
\begin{array}{rll}
(P \to Q) \vee (Q \to R) & \equiv (\neg P \vee Q) \vee (\neg Q \vee R) & \textbf{Conditional Law} \\
& \equiv \neg P \vee Q \vee \neg Q \vee R & \textbf{Associative Law} \\
& \equiv \neg P \vee (Q \vee \neg Q) \vee R & \textbf{Associative Law} \\
& \equiv \neg P \vee \text{Tautology} \vee R & \textbf{Def. of Tautology} \\
& \equiv \text{Tautology} \vee \neg P \vee R & \textbf{Commutative Law} \\
& \equiv \text{Tautology} \vee (\neg P \vee R) & \textbf{Associative Law} \\
& \equiv \text{Tautology} & \textbf{Tautology Law}
\end{array}
$$

Exercise 8

Find a formula involving only the connectives $\neg$ and $\rightarrow$ that is equivalent to $P \wedge Q$.

Solution: I contend that $P \wedge Q \equiv \neg[(P \rightarrow Q) \rightarrow \neg P]$.

P	Q	$P \wedge Q$	$P \rightarrow Q$	$\neg P$	$(P \rightarrow Q) \rightarrow \neg P$	$\neg[(P \rightarrow Q) \rightarrow \neg P]$
0	0	0	1	1	1	0
0	1	0	1	1	1	0
1	0	0	0	0	1	0
1	1	1	1	0	0	1

Exercise 9

Find a formula involving only the connectives $\neg$ and $\rightarrow$ that is equivalent to $P \leftrightarrow Q$.

Solution: I contend that $P \leftrightarrow Q \equiv \neg[\neg(P \rightarrow Q) \rightarrow (Q \rightarrow P)]$.

P	Q	$P \leftrightarrow Q$	$P \rightarrow Q$	$Q \rightarrow P$	$\neg(P \rightarrow Q)$	$(Q \rightarrow P) \rightarrow \neg(P \rightarrow Q)$	$\neg[(Q \rightarrow P) \rightarrow \neg(P \rightarrow Q)]$
0	0	1	1	1	0	0	1
0	1	0	1	0	0	1	0
1	0	0	0	1	1	1	0
1	1	1	1	1	0	0	1

Exercise 10

Which of the following formulas are equivalent?

(a) $P \to (Q \to R)$.

(b) $Q \to (P \to R)$.

(c) $(P \to Q) \land (P \to R)$.

(d) $(P \land Q) \to R$.

(e) $P \to (Q \land R)$.

Solution:

(a) We show that $P \to (Q \to R) \equiv (P \land Q) \to R$.

$$\begin{aligned}
P \to (Q \to R) &\equiv \neg P \lor (Q \to R) && \textbf{Conditional Law}\\
&\equiv \neg P \lor (\neg Q \lor R) && \textbf{Conditional Law}\\
&\equiv (\neg P \lor \neg Q) \lor R && \textbf{Associative Law}\\
&\equiv \neg(P \land Q) \lor R && \textbf{DeMorgan's Law}\\
&\equiv (P \land Q) \to R && \textbf{Conditional Law}
\end{aligned}$$

(b) We show that $Q \to (P \to R) \equiv (P \land Q) \to R$.

$$\begin{aligned}
Q \to (P \to R) &\equiv \neg Q \lor (P \to R) && \textbf{Conditional Law}\\
&\equiv \neg Q \lor (\neg P \lor R) && \textbf{Conditional Law}\\
&\equiv (\neg Q \lor \neg P) \lor R && \textbf{Associative Law}\\
&\equiv \neg(Q \land P) \lor R && \textbf{DeMorgan's Law}\\
&\equiv \neg(P \land Q) \lor R && \textbf{Commutative Law}\\
&\equiv (P \land Q) \to R && \textbf{Conditional Laws}
\end{aligned}$$

(c) We show that $(P \to Q) \land (P \to R) \equiv P \to (Q \land R)$ by **Lemma 7.a.2**.

(d) $(P \land Q) \to R$.

(e) $P \to (Q \land R)$.

We conclude that by noting that exercises **10a, 10b, 10d** are all equivalent.
And exercises **10c, 10e** are equivalent.