How To Prove It: A Structured Approach, Second Edition

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Solutions to: *1.2 Truth Tables*

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February 2024

Contents

Exercise 1

Make truth tables for the following formulas:

(a) $\neg P \vee Q$.

(b) $(S \vee G) \wedge (\neg S \vee \neg G)$

Solution:

(a)

P	Q	$\neg P$	$\neg P \vee Q$
False	False	True	True
False	True	True	True
True	False	False	False
True	True	False	True

(b)

S	G	$\neg S$	$\neg G$	$S \vee G$	$\neg S \vee \neg G$	$(S \vee G) \wedge (\neg S \vee \neg G)$
False	False	True	True	False	True	False
False	True	True	False	True	True	True
True	False	False	True	True	True	True
True	True	False	False	True	False	False

Exercise 2

Make truth tables for the following formulas:

(a) $\neg[P \wedge (Q \vee \neg P)]$.

(b) $(P \vee Q) \wedge (\neg P \vee R)$.

Solution:

(a)

P	Q	$\neg P$	$Q \vee \neg P$	$P \wedge (Q \vee \neg P)$	$\neg[P \wedge (Q \vee \neg P)]$
False	False	True	True	False	True
False	True	True	True	False	True
True	False	False	False	False	True
True	True	False	True	True	False

(b)

P	Q	R	$\neg P$	$P \vee Q$	$\neg P \vee R$	$(P \vee Q) \wedge (\neg P \vee R)$
False	False	False	True	False	True	False
False	False	True	True	False	True	False
False	True	False	True	True	True	True
True	False	False	False	True	False	False
False	True	True	True	True	True	True
True	False	True	False	True	True	True
True	True	False	False	True	False	False
True	True	True	False	True	True	True

Exercise 3

In this exercise we will use the symbol $+$ to mean *exclusive or*. In other words, $P+Q$ means "P or Q, but not both".

(a) Make a truth table for $P+Q$.

(b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P+Q$. Justify your answer with a truth table.

Solution:

(a)

P	Q	$P+Q$
False	False	False
False	True	True
True	False	True
True	True	False

(b) The formula $(P \wedge \neg Q) \vee (\neg P \wedge Q)$ is equivalent to $P+Q$ since for every possible combination of truth values for the propositions, the truth values of both formulas are the same.

P	Q	$\neg P$	$\neg Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$(P \wedge \neg Q) \vee (\neg P \wedge Q)$	$P+Q$
False	False	True	True	False	False	False	False
False	True	True	False	False	True	True	True
True	False	False	True	True	False	True	True
True	True	False	False	False	False	False	False

Exercise 4

Find a formula using only the connectives $\wedge$ and $\neg$ that is equivalent to $P \vee Q$. Justify your answer with a truth table.

Solution: Recall the truth table for $P \vee Q$, shown below:

P	Q	$P \vee Q$
False	False	False
False	True	True
True	False	True
True	True	True

The formula $\neg(\neg P \wedge \neg Q)$ is equivalent to $P \vee Q$ since for every possible combination of inputs, the truth values of both formulas are the same.

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$	$P \vee Q$
False	False	True	True	True	False	False
False	True	True	False	False	True	True
True	False	False	True	False	True	True
True	True	False	False	False	True	True

Exercise 5

Some mathematicians use the symbol $\downarrow$ to mean *nor*. In other words, $P \downarrow Q$ means "neither P nor Q".

(a) Make a truth table for $P \downarrow Q$.

(b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P \downarrow Q$.

(c) Find formulas using only the connective $\downarrow$ that are equivalent to $\neg P$, $P \vee Q$, and $P \wedge Q$.

Solution:

(a)

P	Q	$P \downarrow Q$
False	False	True
False	True	False
True	False	False
True	True	False

(b) The formula $\neg P \wedge \neg Q$ is equivalent to $P \downarrow Q$ since for every possible combination of truth values for the propositions, the truth values of both formulas are the same.

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$P \downarrow Q$
False	False	True	True	True	True
False	True	True	False	False	False
True	False	False	True	False	False
True	True	False	False	False	False

(c)

- The formula $P \downarrow P$ is equivalent to $\neg P$.

P	P	$P \downarrow P$	$\neg P$
False	False	True	True
True	True	False	False

- The formula $(P \downarrow Q) \downarrow (P \downarrow Q)$ is equivalent to $\neg(P \downarrow Q)$ (using the result from above, i.e., $\neg P \equiv P \downarrow P$), hence equivalent to $P \vee Q$. We justify the answer with our truth table.

P	Q	$P \downarrow Q$	$\neg(P \downarrow Q)$	$P \vee Q$
False	False	True	False	False
False	True	False	True	True
True	False	False	True	True
True	True	False	True	True

- The formula $(P \downarrow P) \downarrow (Q \downarrow Q)$ is equivalent to $P \wedge Q$.

$$
\begin{aligned}
P \wedge Q &\equiv \neg\neg P \wedge \neg\neg Q && \text{(Double Negation Law)} \\
&\equiv \neg(\neg P \vee \neg Q) && \text{(DeMorgan's Law)} \\
&\equiv \neg\neg(\neg P \downarrow \neg Q) && \text{(Results Above: } \neg(P \downarrow Q) \equiv P \vee Q) \\
&\equiv \neg P \downarrow \neg Q && \text{(Double Negation Law)} \\
&\equiv (P \downarrow P) \downarrow (Q \downarrow Q) && \text{(Results Above: } \neg P \equiv P \downarrow P)
\end{aligned}
$$

We also justify our answer with a truth table.

P	Q	$P \downarrow P$	$Q \downarrow Q$	$(P \downarrow P) \downarrow (Q \downarrow Q)$	$P \wedge Q$
True	True	False	False	True	True
True	False	False	True	False	False
False	True	True	False	False	False
False	False	True	True	False	False

Exercise 6

Some mathematicians write $P|Q$ to mean "P and Q are not both true". This connective is called *nand*, and is used in the study of circuits in computer science.

(a) Make a truth table for $P|Q$.

(b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P|Q$.

(c) Find formulas using only the connective $|$ that are equivalent to $\neg P$, $P \vee Q$, and $P \wedge Q$.

Solution:

(a)

P	Q	$P\|Q$
False	False	True
False	True	True
True	False	True
True	True	False

(b) The formula $\neg(P \wedge Q)$ is equivalent to $P|Q$ since for every possible combination of truth values for the propositions, the truth values of both formulas are the same.

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$P\|Q$
False	False	False	True	True
False	True	False	True	True
True	False	False	True	True
True	True	True	False	False

(c) • The formula $P|P$ is equivalent to $\neg P$.

P	P	$P\|P$	$\neg P$
False	False	True	True
True	True	False	False

• The formula $(P|Q)|(P|Q)$ is equivalent to $\neg(P|Q)$ (using the result from above, i.e., $\neg P \equiv P|P$), hence equivalent to $P \wedge Q$. We justify the answer with our truth table.

P	Q	$P\|Q$	$\neg(P\|Q)$	$P \wedge Q$
False	False	True	False	False
False	True	True	False	False
True	False	True	False	False
True	True	False	True	True

• The formula $(P|P)|(Q|Q)$ is equivalent to $P \vee Q$.

$$\begin{aligned}
P \vee Q &\equiv \neg\neg P \vee \neg\neg Q && \text{(Double Negation Law)} \\
&\equiv \neg(\neg P \wedge \neg Q) && \text{(DeMorgan's Law)} \\
&\equiv \neg\neg(\neg P|\neg Q) && \text{(Results Above: } \neg(P|Q) \equiv P \wedge Q\text{)} \\
&\equiv \neg P|\neg Q && \text{(Double Negation Law)} \\
&\equiv (P|P)|(Q|Q) && \text{(Results Above: } \neg P \equiv P|P\text{)}
\end{aligned}$$

We also justify our answer with a truth table.

P	Q	$P\|P$	$Q\|Q$	$(P\|P)\|(Q\|Q)$	$P \vee Q$
True	True	False	False	True	True
True	False	False	True	True	True
False	True	True	False	True	True
False	False	True	True	False	False

Exercise 7

Use truth tables to determine whether or not the arguments in exercise 7 of Section 1.1 are valid.

(a) Jane and Pete won't both win the math prize. Pete will win either the math prize or the chemistry prize. Jane will win the math prize. Therefore, Pete will win the chemistry prize.

(b) The main course will be either beef or fish. The vegetable will be either peas or corn. We will not have both fish as a main course and corn as a vegetable. Therefore, we will not have both beef as a main course and peas as a vegetable.

(c) Either John or Bill is telling the truth. Either Sam or Bill is lying. Therefore, either John is telling the truth or Sam is lying.

(d) Either sales will go up and the boss will be happy, or expenses will go up and the boss won't be happy. Therefore, sales and expenses will not both go up.

Solution: The argument is valid if, in every case where all premises are true, the conclusion is also true. This means there should be no row in the truth table where all the premises are true and the conclusion is false. If such a row exists, the argument is invalid because it shows a situation where the premises could all be true without the conclusion being true.

(a) Let J_m stand for Jane can win the math prize, P_c stand for Pete can with the chemistry prize, and P_m stand for Pete can with the math prize. Consider the following truth table with the premises and conclusion of the argument.

J_m	P_m	P_c	$\neg(J_m \wedge P_m)$	$P_m \vee P_c$	J_m	P_c
False	False	False	True	False	False	False
False	False	True	True	True	False	True
False	True	False	True	True	False	False
False	True	True	True	True	False	True
True	False	False	True	False	True	False
True	False	True	True	True	True	True
True	True	False	False	True	True	False
True	True	True	False	True	True	True

Observe that row 6 is the only row where all the premises are true; furthermore, the conclusion is also true. Hence, the argument is valid.

(b) Let B stand for the statement "the main course will be beef", F stand for "the main course will be fish", P stand for "the vegetables will be peas", and C stand for "the vegetables will be corn".

B	F	P	C	$B \vee F$	$P \vee C$	$\neg(F \wedge C)$	$\neg(B \wedge P)$
T	T	T	T	T	T	F	F
T	T	T	F	T	T	T	F
T	T	F	T	T	T	F	T
T	T	F	F	T	F	T	T
T	F	T	T	T	T	T	F
T	F	T	F	T	T	T	F
T	F	F	T	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	T	T	T	F	T
F	T	T	F	T	T	T	T
F	T	F	T	T	T	F	T
F	T	F	F	T	F	T	T
F	F	T	T	F	T	T	T
F	F	T	F	F	T	T	T
F	F	F	T	F	T	T	T
F	F	F	F	F	F	T	T

Observe that row 2 is a row where all the premises are true; furthermore, the conclusion is also false. Hence, the argument is invalid. We could have stopped creating the table at row 2, but we didn't.

(c) Let J stand for the statement "John is truth telling", B for "Bill is truth telling", and S for "Sam is truth telling". Consider the following truth table with the premises and conclusion of the argument.

J	B	S	$J \vee B$	$\neg S \vee \neg B$	$J \vee \neg S$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	T	F
F	F	F	F	T	T

Observe that row 2,3,4 and 6 are rows where all the premises are true; furthermore, the conclusion is also true in all said cases. Hence, the argument is valid.

(d) Let S stand for statement "sales will go up", E stand for "expenses will go up", and H stand for "the boss will be happy". Consider the following truth table with the premises and conclusion of the argument.

S	E	H	$(S \wedge H) \vee (E \wedge \neg H)$	$\neg(S \wedge E)$
T	T	T	T	F
T	T	F	T	F
T	F	T	T	T
T	F	F	F	T
F	T	T	F	T
F	T	F	T	T
F	F	T	F	T
F	F	F	F	T

Observe that the first row shows that the premise can hold true, but the conclusion remains false. Therefore, the argument is invalid.

Exercise 8

Use truth tables to determine which of the following formulas are equivalent to each other:

(a) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.

(b) $\neg P \vee Q$.

(c) $(P \vee \neg Q) \wedge (Q \vee \neg P)$.

(d) $\neg(P \vee Q)$.

(e) $(Q \wedge P) \vee \neg P$.

Solution: Consider the following substitutions.

- A: $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.
- B: $\neg P \vee Q$.
- C: $(P \vee \neg Q) \wedge (Q \vee \neg P)$.
- D: $\neg(P \vee Q)$.
- E: $(Q \wedge P) \vee \neg P$.

The combined truth table for the statements above is.

P	Q	A	B	C	D	E
T	T	T	T	T	F	T
T	F	F	F	F	F	F
F	T	F	T	F	F	T
F	F	T	T	T	T	T

We observe that for the exact same combination of truth values for the inputs, formula $A \equiv E$, $B \equiv C$ and D is not equivalent to any other formula.

Exercise 9

Use truth tables to determine which of these statements are tautologies, which are contradictions, and which are neither:

(a) $(P \vee Q) \wedge (\neg P \vee \neg Q)$.

(b) $(P \vee Q) \wedge (\neg P \wedge \neg Q)$.

(c) $(P \vee Q) \vee (\neg P \vee \neg Q)$.

(d) $[P \wedge (Q \vee \neg R)] \vee (\neg P \vee R)$.

Solution: Recall that a contradiction is always false for all combination of truth values for its inputs. Similarly, a tautology is always true for all combination of truth values for its inputs. Consider the following substitutions.

- A: $(P \vee Q) \wedge (\neg P \vee \neg Q)$.
- B: $(P \vee Q) \wedge (\neg P \wedge \neg Q)$.
- C: $(P \vee Q) \vee (\neg P \vee \neg Q)$.
- D: $[P \wedge (Q \vee \neg R)] \vee (\neg P \vee R)$.

P	Q	A	B	C
T	T	F	F	T
T	F	T	F	T
F	T	T	F	T
F	F	F	F	T

We have already determined that A is neither a tautology or contradiction; B is a contradiction, C is a tautology.

Finally, we take on D exclusively and the truth table shows it is a tautology.

P	Q	R	$Q \vee \neg R$	$P \wedge (Q \vee \neg R)$	$\neg P \vee R$	$[P \wedge (Q \vee \neg R)] \vee (\neg P \vee R)$
F	F	F	T	F	T	T
F	F	T	F	F	T	T
F	T	F	T	F	T	T
F	T	T	T	F	T	T
T	F	F	T	T	F	T
T	F	T	F	F	T	T
T	T	F	T	T	F	T
T	T	T	T	T	T	T

Exercise 10

Use truth tables to check these laws:

(a) The second DeMorgan's law.

(b) The distributive laws.

Solution:

(a) Observe that the truth values are identical for both formulas.

P	Q	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
F	F	T	T
F	T	F	F
F	T	F	F
T	T	F	F

(b) Observe that the truth values are identical for each pair of formulas.

P	Q	R	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$
F	F	F	F	F
F	F	T	F	F
F	T	F	F	F
F	T	T	F	F
T	F	F	F	F
T	F	T	T	T
T	T	F	T	T
T	T	T	T	T

P	Q	R	$P \vee (Q \wedge R)$	$(P \vee Q) \wedge (P \vee R)$
F	F	F	F	F
F	F	T	F	F
F	T	F	F	F
F	T	T	T	T
T	F	F	T	T
T	F	T	T	T
T	T	F	T	T
T	T	T	T	T

Exercise 11

Use the laws stated in the text to find simpler formulas equivalent to these formulas.

(a) $\neg(\neg P \wedge \neg Q)$.

(b) $(P \wedge Q) \vee (P \wedge \neg Q)$.

(c) $\neg(P \wedge \neg Q) \vee (\neg P \wedge Q)$.

Solution:

(a) $\boldsymbol{\neg(\neg P \wedge \neg Q) \equiv P \vee Q}$**.**

$$\begin{aligned} \neg(\neg P \wedge \neg Q) &\equiv \neg\neg P \vee \neg\neg Q && \text{(DeMorgan's Law)} \\ &\equiv P \vee Q && \text{(Double Negation Law)} \end{aligned}$$

(b) $\boldsymbol{(P \wedge Q) \vee (P \wedge \neg Q) \equiv P}$**.**

$$\begin{aligned} (P \wedge Q) \vee (P \wedge \neg Q) &\equiv ((P \wedge Q) \vee P) \wedge ((P \wedge Q) \vee \neg Q) && \text{(Distributive Law)} \\ &\equiv P \wedge ((P \wedge Q) \vee \neg Q) && \text{(Absorption Law)} \\ &\equiv P \wedge [(P \vee \neg Q) \wedge (Q \vee \neg Q)] && \text{(Distributive Law)} \\ &\equiv P \wedge [(P \vee \neg Q) \wedge \text{Tautology}] && \text{(Definition of Tautology)} \\ &\equiv P \wedge (P \vee \neg Q]) && \text{(Tautology Law)} \\ &\equiv P && \text{(Absorption Law)} \end{aligned}$$

(c) $\boldsymbol{\neg(P \wedge \neg Q) \vee (\neg P \wedge Q) \equiv \neg P \vee Q}$**.**

$$\begin{aligned} \neg(P \wedge \neg Q) \vee (\neg P \wedge Q) &\equiv (\neg P \vee \neg\neg Q) \vee (\neg P \wedge Q) && \text{(DeMorgan's Law)} \\ &\equiv (\neg P \vee Q) \vee (\neg P \wedge Q) && \text{(Double Negation Law)} \\ &\equiv [(\neg P \vee Q) \vee \neg P] \wedge [(\neg P \vee Q) \vee Q)] && \text{(Distributive Law)} \\ &\equiv [\neg P \vee Q \vee \neg P] \wedge [\neg P \vee Q \vee Q] && \text{(Associative Law)} \\ &\equiv [\neg P \vee \neg P \vee Q] \wedge [\neg P \vee Q \vee Q] && \text{(Commutative Law)} \\ &\equiv [(\neg P \vee \neg P) \vee Q] \wedge [\neg P \vee (Q \vee Q)] && \text{(Associative Law)} \\ &\equiv (\neg P \vee Q) \wedge (\neg P \vee Q) && \text{(Idempotent Law)} \\ &\equiv \neg P \vee Q && \text{(Idempotent Law)} \end{aligned}$$

Exercise 12

Use the laws stated in the text to find simpler formulas equivalent to these formulas.

(a) $\neg(\neg P \vee Q) \vee (P \wedge \neg R)$.

(b) $\neg(\neg P \wedge Q) \vee (P \wedge \neg R)$.

(c) $(P \wedge R) \vee [\neg R \wedge (P \vee Q)]$.

Solution:

(a) $\boldsymbol{\neg(\neg P \vee Q) \vee (P \wedge \neg R) \equiv P \wedge (\neg Q \vee \neg R)}$**.**

$$\begin{aligned}
\neg(\neg P \vee Q) \vee (P \wedge \neg R) &\equiv (P \wedge \neg Q) \vee (P \wedge \neg R) && \text{(DeMorgan's Law)}\\
&\equiv [(P \wedge \neg Q) \vee P] \wedge [(P \wedge \neg Q) \vee \neg R] && \text{(Distributive Law)}\\
&\equiv P \wedge [(P \wedge \neg Q) \vee \neg R] && \text{(Absorption Law)}\\
&\equiv P \wedge [(P \vee \neg R) \wedge (\neg Q \vee \neg R)] && \text{(Distributive Law)}\\
&\equiv P \wedge (P \vee \neg R) \wedge (\neg Q \vee \neg R) && \text{(Associative Law)}\\
&\equiv [P \wedge (P \vee \neg R)] \wedge (\neg Q \vee \neg R) && \text{(Associative Law)}\\
&\equiv P \wedge (\neg Q \vee \neg R) && \text{(Absorption Law)}
\end{aligned}$$

(b) $\boldsymbol{\neg(\neg P \wedge Q) \vee (P \wedge \neg R) \equiv P \vee \neg Q}$**.**

$$\begin{aligned}
\neg(\neg P \wedge Q) \vee (P \wedge \neg R) &\equiv (P \vee \neg Q) \vee (\neg P \wedge Q) && \text{(DeMorgan's Law)}\\
&\equiv [(P \vee \neg Q) \vee P] \wedge [(P \vee \neg Q) \vee \neg R] && \text{(Distributive Law)}\\
&\equiv [P \vee \neg Q \vee P] \wedge [(P \vee \neg Q) \vee \neg R] && \text{(Associative Law)}\\
&\equiv [P \vee P \vee \neg Q] \wedge [(P \vee \neg Q) \vee \neg R] && \text{(Commutative Law)}\\
&\equiv [(P \vee P) \vee \neg Q] \wedge [(P \vee \neg Q) \vee \neg R] && \text{(Associative Law)}\\
&\equiv (P \vee \neg Q) \wedge [(P \vee \neg Q) \vee \neg R] && \text{(Idempotent Law)}\\
&\equiv P \vee \neg Q && \text{(Absorption Law)}
\end{aligned}$$

(c) $\boldsymbol{(P \wedge R) \vee [\neg R \wedge (P \vee Q)] \equiv P \vee (\neg R \wedge Q).}$

$$
\begin{aligned}
(P \wedge R) \vee [\neg R \wedge (P \vee Q)] &\equiv [(P \wedge R) \vee \neg R] \wedge [(P \wedge R) \vee (P \vee Q)] && \text{(Distributive Law)} \\
&\equiv [(P \vee \neg R) \wedge (R \vee \neg R)] \wedge [(P \wedge R) \vee (P \vee Q)] && \text{(Distributive Law)} \\
&\equiv [(P \vee \neg R) \wedge \mathrm{T.}] \wedge [(P \wedge R) \vee (P \vee Q)] && \text{(T. Definition)} \\
&\equiv (P \vee \neg R) \wedge [(P \wedge R) \vee (P \vee Q)] && \text{(Tautology Law)} \\
&\equiv [(P \vee \neg R) \wedge (P \wedge R)] \vee [(P \vee \neg R) \wedge (P \vee Q)] && \text{(Distributive Law)} \\
&\equiv [(P \vee \neg R) \wedge (P \wedge R)] \vee [P \vee (\neg R \wedge Q)] && \text{(Distributive Law)} \\
&\equiv [(P \vee \neg R) \wedge P \wedge R] \vee [P \vee (\neg R \wedge Q)] && \text{(Associative Law)} \\
&\equiv [((P \vee \neg R) \wedge P) \wedge R] \vee [P \vee (\neg R \wedge Q)] && \text{(Associative Law)} \\
&\equiv (P \wedge R) \vee [P \vee (\neg R \wedge Q)] && \text{(Absorption Law)} \\
&\equiv (P \wedge R) \vee P \vee (\neg R \wedge Q) && \text{(Associative Law)} \\
&\equiv [(P \wedge R) \vee P] \vee (\neg R \wedge Q) && \text{(Associative Law)} \\
&\equiv P \vee (\neg R \wedge Q) && \text{(Absorption Law)}
\end{aligned}
$$

, where T. stands for Tautology.

Exercise 13

Use the first DeMorgan's Law and the double negation law to derive the second DeMorgan's Law.

Solution: The first DeMorgan's Law states $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$. Now, consider the following:

$$\begin{aligned} P \vee Q &\equiv \neg(\neg P \wedge \neg Q) && \text{(First DeMorgan's Law)} \\ \neg(P \vee Q) &\equiv \neg\neg(\neg P \wedge \neg Q) && \text{(Negate Both Sides)} \\ \neg(P \vee Q) &\equiv \neg P \wedge \neg Q && \text{(Double Negation Law)} \end{aligned}$$

Thus, we have derived the second DeMorgan's Law using only the first DeMorgan's Law and the double negation law.

Exercise 14

Note that the associative laws say only that the parentheses are unnecessary when combining *three* statements with $\wedge$ or $\vee$. In fact, these laws can be used to justify leaving parentheses out when more than three statements are combined. Use associative laws to show that $[P \wedge (Q \wedge R)] \wedge S$ is equivalent $(P \wedge Q) \wedge (R \wedge S)$.

Solution:

$$\begin{aligned}
[P \wedge (Q \wedge R)] \wedge S &\equiv [P \wedge Q \wedge R] \wedge S && \text{(Associative Law)} \\
&\equiv [(P \wedge Q) \wedge R] \wedge S && \text{(Associative Law)} \\
&\equiv (A \wedge R) \wedge S && \text{(Let } A \equiv P \wedge Q\text{)} \\
&\equiv A \wedge R \wedge S && \text{(Associative Law)} \\
&\equiv A \wedge (R \wedge S) && \text{(Associative Law)} \\
&\equiv (P \wedge Q) \wedge (R \wedge S) && \text{(Since } A \equiv P \wedge Q\text{)}
\end{aligned}$$

Exercise 15

How many lines will there be in the truth table for a statement containing n variables?

Solution: Consider a single proposition, for example, P, which can be either true or false. Thus, each variable has two possible values. Since each variable is independent of the others, the total number of combinations increases exponentially with the number of variables. Specifically, one variable has $2^1 = 2$ possibilities; two variables have $2 \times 2 = 2^2 = 4$ possibilities; and three variables have $2 \times 2 \times 2 = 2^3 = 8$ possibilities. Consequently, for n variables, there are 2^n possible combinations, or equivalently, lines in the truth table.

Exercise 16

Find a formula involving the connectives $\wedge$, $\vee$, and $\neg$ that has the following truth table:

P	Q	???
F	F	T
F	T	F
T	F	T
T	T	T

Solution: Consider $P \vee \neg Q$

Exercise 17

Find a formula involving the connectives $\wedge$, $\vee$, and $\neg$ that has the following truth table:

P	Q	???
F	F	F
F	T	T
T	F	T
T	T	F

Solution: Consider $(P \wedge \neg Q) \vee (\neg P \wedge Q)$ as this was the equivalent formula to $P + Q$ in **Exercise 3**.

Exercise 18

Suppose the conclusion of an argument is a tautology. What can you conclude about the validity of the argument? What if the conclusion is a contradiction? What if one of the premises is either a tautology or a contradiction?

Solution:

Conclusion is a Tautology: If the conclusion of a deductive argument is a tautology, the argument is valid. This is because a tautology is a statement that is true under all interpretations, making it impossible for the premises to be true and the conclusion to be false—since the conclusion cannot be false. However, while the argument is valid, this does not necessarily mean that the argument is a good one in terms of providing informative insight into the relationship between the premises and conclusion, since a tautology will be true regardless of the content of the premises.

Conclusion is a Contradiction: If the conclusion is a contradiction, then the argument is invalid. A contradiction is a statement that is false under all interpretations. According to the definition of validity, if it is impossible for the premises to be true and the conclusion false, the argument is valid. However, in the case of a contradiction, the conclusion is always false, so there exists at least one interpretation where the premises are true, and the conclusion is still false, thus making the argument invalid.

One of the Premises is a Tautology: If one of the premises is a tautology, it does not by itself affect the validity of the argument. Since a tautology is always true, it does not interfere with the truth of the conclusion; it simply does not contribute to making the argument invalid. The validity of such an argument depends on the form of the argument and the relationship between the remaining premises and the conclusion.

One of the Premises is a Contradiction: In deductive logic, the validity of an argument depends on a clear relationship: if the premises are true, then the conclusion cannot be false. If a premise is inherently false, it can't simultaneously be true with other premises, so the 'if' in validity's 'if-then' is never triggered. This gives rise to an argument that is vacuously valid. The impossibility of all the premises being true at once ensures there's no chance for the conclusion to be false when the premises are true.

The validity of such an argument doesn't imply that it is sound. Soundness is a stronger condition that requires not only the argument's validity but also that all its premises are actually true. With a contradictory premise, this requirement cannot be met, hence the argument is unsound. This distinction is vital: validity is about the argument's logical structure, ensuring theoretical truth preservation, whereas soundness concerns the actual truth of the premises.