How To Prove It:
A Structured Approach, Second Edition

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Solutions to: *1.4 Operations on Sets*

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Contents

Exercise 1

Let $A = \{1, 3, 12, 35\}$, $B = \{3, 7, 12, 20\}$, and $C = \{x \mid x \text{ is a prime number}\}$. List the elements of the following sets. Are any of the sets below disjoint from any of the others? Are any of the sets below subsets of any others?

(a) $A \cap B$.

(b) $(A \cup B) \setminus C$.

(c) $A \cup (B \setminus C)$.

Solution:

(a) Since $3, 12 \in A$ and $3, 12 \in B$, then $A \cap B = \mathbf{\{3, 12\}}$.

(b) Since $A \cup B = \{1, 3, 7, 12, 20, 35\}$ and $3, 7 \in C$, then $(A \cup B) \setminus C = \mathbf{\{1, 12, 20, 35\}}$.

(c) Since $3, 7 \in C$, then $B \setminus C = \{12, 20\}$. Hence $A \cup (B \setminus C) = \mathbf{\{1, 3, 12, 20, 35\}}$.

Observe that $\{3, 12\} \subset \{1, 3, 12, 20, 35\}$, so **(a)** is a subset of **(c)**.
Observe that $\{1, 12, 20, 35\} \subset \{1, 3, 12, 20, 35\}$, so **(b)** is a subset of **(c)**.
Observe that 12 is in both **(a)** and **(b)**, so **(a)** and **(b)** are not disjoint.
Since **(a)**, **(b)** and **(c)** are non-empty subsets of each other, or share elements, then none of them are disjoint in relation to each other.

Exercise 2

Let $A = \{\text{United States, Germany, China, Australia}\}$, $B = \{\text{Germany, France, India, Brazil}\}$, and $C = \{x \mid x \text{ is a country in Europe}\}$. List the elements of the following sets. Are any of the sets below disjoint from any of the others? Are any of the sets below subsets of any others?

(a) $A \cup B$.

(b) $(A \cap B) \setminus C$.

(c) $(B \cap C) \setminus A$.

Solution:

(a) $A \cup B = \boldsymbol{\{}\text{United States, Germany, China, Australia, France, India, Brazil}\boldsymbol{\}}$.

(b) Since $A \cap B = \{\text{Germany}\}$ and C is all European counties, then $(A \cap B) \setminus C = \boldsymbol{\emptyset}$.

(c) Since Germany $\in B \cap C = \{\text{Germany, France}\}$ and Germany $\in A$, then $(B \cap C) \setminus A = \boldsymbol{\{}\text{France}\boldsymbol{\}}$.

Observe that $(B \cap C) \setminus A = \{\text{France}\} \in A \cup B$, so **(c)** is a subset of **(a)**.
Observe that $\emptyset$ is the subset of every set, so **(b)** is a subset of both **(a)** and **(c)**.

The empty set is a subset of every set, including itself; however, the empty set is also disjoint from any set that has elements. The definition of disjoint sets is that they have no elements in common. Since the empty set has no elements at all, its intersection with any set that has elements is also empty. The property of being a subset does not contradict being disjoint in the case of the empty set. The empty set being a subset of a non-empty set simply means all elements of the empty set (of which there are none) are also elements of the non-empty set. This statement is vacuously true and does not require the empty set to share any actual elements with the non-empty set.

Exercise 3

Verify that the Venn diagrams for $(A \cup B) \setminus (A \cap B)$ and $(A \setminus B) \cup (B \setminus A)$ both look like Figure 5, as stated in this section.

Solution:

$$
\begin{aligned}
& x \in (A \cup B) \setminus (A \cap B) \\
& \equiv x \in (A \cup B) \land \neg(x \in A \cap B) && \textbf{(1)} \\
& \equiv (x \in A \lor x \in B) \land \neg(x \in A \land x \in B) && \textbf{(2)} \\
& \equiv (x \in A \lor x \in B) \land [\neg(x \in A) \lor \neg(x \in B)] && \textbf{(3)} \\
& \equiv [(x \in A \lor x \in B) \land \neg(x \in A)] \lor [(x \in A \lor x \in B) \land \neg(x \in B)] && \textbf{(4)} \\
& \equiv [(x \in A \land \neg(x \in A)) \lor (x \in B \land \neg(x \in A)] \lor [(x \in A \lor x \in B) \land \neg(x \in B)] && \textbf{(5)} \\
& \equiv [\text{Contradiction} \lor (x \in B \land \neg(x \in A)] \lor [(x \in A \lor x \in B) \land \neg(x \in B)] && \textbf{(6)} \\
& \equiv [x \in B \land \neg(x \in A)] \lor [(x \in A \lor x \in B) \land \neg(x \in B)] && \textbf{(7)} \\
& \equiv [x \in B \land \neg(x \in A)] \lor [(x \in A \land \neg(x \in B)) \lor (x \in B \land \neg(x \in B))] && \textbf{(8)} \\
& \equiv [x \in B \land \neg(x \in A)] \lor [(x \in A \land \neg(x \in B)) \lor \text{Contradiction}] && \textbf{(9)} \\
& \equiv [x \in B \land \neg(x \in A)] \lor [x \in A \land \neg(x \in B)] && \textbf{(10)} \\
& \equiv [x \in A \land \neg(x \in B)] \lor [x \in B \land \neg(x \in A)] && \textbf{(11)} \\
& \equiv [x \in A \setminus B] \lor [x \in B \setminus A] && \textbf{(12)} \\
& \equiv x \in (A \setminus B) \cup (B \setminus A) && \textbf{(13)}
\end{aligned}
$$

We have shown that the two original statements are equivalent, hence their Venn diagrams must be identical. Drawing the Venn diagram will result in a diagram identical to Figure 5.

(1) Definition of Difference of Sets
(2) Definition of Intersection and Union of Sets
(3) DeMorgan's Law
(4) Distributive Law: Resulting in a central $\lor$
(5) Distributive Law: Only on the LHS of the central $\lor$
(6) Definition of Contradiction
(7) Contradiction Law
(8) Distributive Law: Only on the RHS of the central $\lor$
(9) Definition of Contradiction
(10) Contradiction Law
(11) Commutative Law
(12) Definition of Difference of Sets
(13) Definition of Union of Sets

Exercise 4

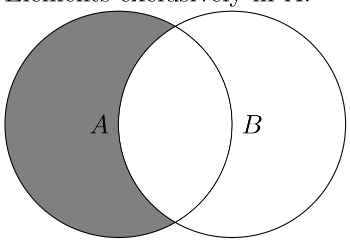
Use Venn diagrams to verify the following identities:

(a) $A \setminus (A \cap B) = A \setminus B$.

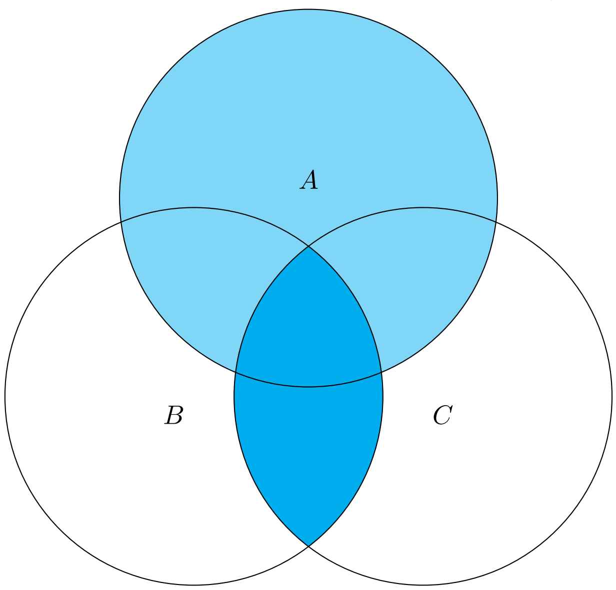
(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution:

(a) Elements exclusively in A.



(b) Elements in A or in the intersection of B and C.



Exercise 5

Verify the identities in exercise 4 by writing out (using logical symbols) what it means for an object x to be an element of each set and then using logical equivalences.

Solution:

(a) We are to prove that $A \setminus (A \cap B) = A \setminus B$.

$$
\begin{aligned}
x \in A \setminus (A \cap B) &\equiv x \in A \wedge \neg(x \in A \cap B) && \textbf{(Def. of Difference of Sets)}\\
&\equiv x \in A \wedge \neg(x \in A \wedge x \in B) && \textbf{(Def. of Intersection of Sets)}\\
&\equiv x \in A \wedge [\neg(x \in A) \vee \neg(x \in B)] && \textbf{(DeMorgan's Law)}\\
&\equiv [x \in A \wedge \neg(x \in A)] \vee [x \in A \wedge \neg(x \in B)] && \textbf{(Distributive Law)}\\
&\equiv \text{Contradiction} \vee [x \in A \wedge \neg(x \in B)] && \textbf{(Def. of Contradiction)}\\
&\equiv x \in A \wedge \neg(x \in B) && \textbf{(Law of Contradiction)}\\
&\equiv x \in A \setminus B && \textbf{(Def. of Difference of Sets)}
\end{aligned}
$$

(b) We are to prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

$$
\begin{aligned}
x \in A \cup (B \cap C) &\equiv x \in A \vee (x \in B \cap C) && \textbf{(Def. of Union of Sets)}\\
&\equiv x \in A \vee (x \in B \wedge x \in C) && \textbf{(Def. of Intersection of Sets)}\\
&\equiv (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) && \textbf{(Distributive Law)}\\
&\equiv (x \in A \cup B) \wedge (x \in A \cup C) && \textbf{(Def. of Union of Sets)}\\
&\equiv x \in (A \cup B) \cap (A \cup C) && \textbf{(Def. of Intersection of Sets)}
\end{aligned}
$$

Exercise 6

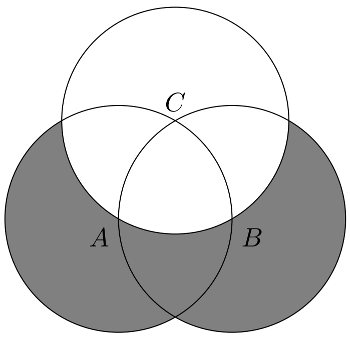
Use Venn diagrams to verify the following identities:

(a) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.

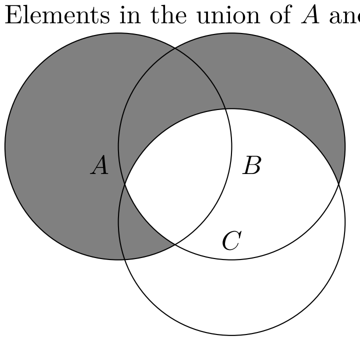
(b) $A \cup (B \setminus C) = (A \cup B) \setminus (C \setminus A)$.

Solution:

(a) Elements in the union of A and B and none in C.



(b) Elements in the union of A and unique to B.



Exercise 7

Verify the identities in exercise 6 by writing out (using logical symbols) what it means for an object x to be an element of each set and then using logical equivalences.

Solution:

(a) We are to prove that $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.

$$\begin{aligned}
x \in (A \cup B) \setminus C &\equiv (x \in A \cup B) \wedge \neg(x \in C) && \textbf{(Def. Difference of Sets)} \\
&\equiv (x \in A \vee x \in B) \wedge \neg(x \in C) && \textbf{(Def. Union of Sets)} \\
&\equiv [x \in A \wedge \neg(x \in C)] \vee [x \in B \wedge \neg(x \in C)] && \textbf{(Distributive Law)} \\
&\equiv (x \in A \setminus C) \vee (x \in B \setminus C) && \textbf{(Def. Difference of Sets)} \\
&\equiv x \in (A \setminus C) \cup (B \setminus C) && \textbf{(Def. Union of Sets)}
\end{aligned}$$

(b) We are to prove that $A \cup (B \setminus C) = (A \cup B) \setminus (C \setminus A)$.

$$\begin{aligned}
x \in A \cup (B \setminus C) &\equiv (x \in A) \vee (x \in B \setminus C) && \textbf{(Def. of Union of Sets)} \\
&\equiv (x \in A) \vee [x \in B \wedge \neg(x \in C)] && \textbf{(Def. of Difference of Sets)} \\
&\equiv [x \in A \vee x \in B] \wedge [x \in A \vee \neg(x \in C)] && \textbf{(Distributive Law)} \\
&\equiv [x \in A \vee x \in B] \wedge \neg[\neg(x \in A) \wedge x \in C] && \textbf{(DeMorgan's Law)} \\
&\equiv [x \in A \vee x \in B] \wedge \neg[x \in C \wedge \neg(x \in A)] && \textbf{(Commutative Law)} \\
&\equiv [x \in A \vee x \in B] \wedge \neg(x \in C \setminus A) && \textbf{(Def. of Difference of Sets)} \\
&\equiv (x \in A \cup B) \wedge \neg(x \in C \setminus A) && \textbf{(Def. of Union of Sets)} \\
&\equiv x \in (A \cup B) \setminus (C \setminus A) && \textbf{(Def. of Difference of Sets)}
\end{aligned}$$

Exercise 8

For each of the following sets, write out (using logical symbols) what it means for an object x to be an element of the set. Then determine which of these sets must be equal to each other by determining which statements are equivalent.

(a) $(A \setminus B) \setminus C$.

(b) $A \setminus (B \setminus C)$.

(c) $(A \setminus B) \cup (A \cap C)$.

(d) $(A \setminus B) \cap (A \setminus C)$.

(e) $A \setminus (B \cup C)$.

Solution:

(a) Below we prove that $(A \setminus B) \setminus C \equiv A \setminus (B \cup C)$, hence **(a)** and **(e)** are equivalent. We will see later that **(d)** is also equivalent.

$$\begin{aligned}
x \in (A \setminus B) \setminus C &\equiv (x \in A \setminus B) \wedge \neg(x \in C) && \textbf{(Def. of Difference of Sets)}\\
&\equiv [x \in A \wedge \neg(x \in B)] \wedge \neg(x \in C) && \textbf{(Def. of Difference of Sets)}\\
&\equiv x \in A \wedge [\neg(x \in B) \wedge \neg(x \in C)] && \textbf{(Associative Law)}\\
&\equiv x \in A \wedge \neg(x \in B \vee x \in C) && \textbf{(DeMorgan's Law)}\\
&\equiv x \in A \wedge \neg(x \in B \cup C) && \textbf{(Def. of Union of Sets)}\\
&\equiv x \in A \setminus (B \cup C) && \textbf{(Def. of Difference of Sets)}
\end{aligned}$$

(b) Below we prove that $A \setminus (B \setminus C) \equiv A \setminus B) \cup (A \cap C)$, hence **(b)** and **(c)** are equivalent.

$$\begin{aligned}
x \in A \setminus (B \setminus C) &\equiv x \in A \wedge \neg(x \in B \setminus C) && \textbf{(Def. Difference of Sets)}\\
&\equiv x \in A \wedge \neg[x \in B \wedge \neg(x \in C)] && \textbf{(Def. Difference of Sets)}\\
&\equiv x \in A \wedge [\neg(x \in B) \vee x \in C] && \textbf{(DeMorgan's Law)}\\
&\equiv [x \in A \wedge \neg(x \in B)] \vee (x \in A \wedge x \in C) && \textbf{(Distributive Law}\\
&\equiv (x \in A \setminus B) \vee (x \in A \wedge x \in C) && \textbf{(Def. Difference of Sets)}\\
&\equiv (x \in A \setminus B) \vee (x \in A \cap C) && \textbf{(Def. Intersection of Sets)}\\
&\equiv x \in (A \setminus B) \cup (A \cap C) && \textbf{(Def. Union of Sets)}
\end{aligned}$$

(c) See **(b)**.

(d) Below we prove that $(A \setminus B) \cap (A \setminus C) \equiv A \setminus (B \cup C)$. But we showed in **(a)** that $A \setminus (B \cup C) \equiv (A \setminus B) \setminus C$. Hence, **(a)**, **(d)** and **(e)** are equivalent.

$$
\begin{aligned}
x \in (A \setminus B) \cap (A \setminus C) &\equiv (x \in A \setminus B) \wedge (x \in A \setminus C) && \textbf{(Def. Intersection of Sets)} \\
&\equiv [x \in A \wedge \neg(x \in B)] \wedge (x \in A \setminus C) && \textbf{(Def. Difference of Sets)} \\
&\equiv [x \in A \wedge \neg(x \in B)] \wedge [x \in A \wedge \neg(x \in C)] && \textbf{(Def. Difference of Sets)} \\
&\equiv x \in A \wedge \neg(x \in B) \wedge x \in A \wedge \neg(x \in C) && \textbf{(Associative Law)} \\
&\equiv x \in A \wedge x \in A \wedge \neg(x \in B) \wedge \neg(x \in C) && \textbf{(Commutitative Law)} \\
&\equiv (x \in A \wedge x \in A) \wedge \neg(x \in B) \wedge \neg(x \in C) && \textbf{(Associative Law)} \\
&\equiv x \in A \wedge \neg(x \in B) \wedge \neg(x \in C) && \textbf{(Idempotent Law)} \\
&\equiv x \in A \wedge \neg(x \in B \vee x \in C) && \textbf{(DeMorgan's Law)} \\
&\equiv x \in A \wedge \neg(x \in B \cup C) && \textbf{(Def. Union of Sets)} \\
&\equiv x \in A \setminus (B \cup C) && \textbf{(Def. Difference of Sets)}
\end{aligned}
$$

(e) See **(a)**.

Exercise 9

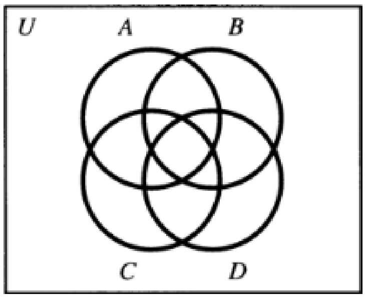
It was shown in this section that for any sets A and B, $(A \cup B) \setminus B \subseteq A$. Give an example of two sets A and B for which $(A \cup B) \setminus B \neq A$.

Solution: Suppose $A = \{1, 2, 3, 4\}$ and $B = \{4, 5, 6, 7\}$, then $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$. Hence, $(A \cup B) \setminus B = \{1, 2, 3\} \not\equiv \{1, 2, 3, 4\} = A$.

Exercise 10

It is claimed in this section that you cannot make a Venn diagram for four sets using overlapping circles.

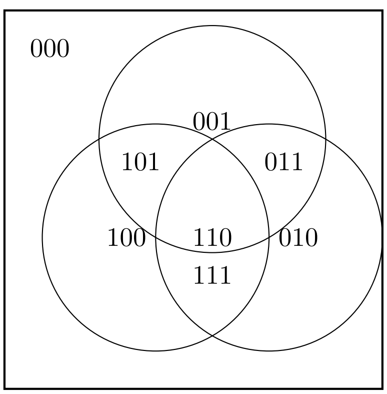
(a) What's swrong with the following diagram? (Hint: Where's the set $(A \cap D) \setminus (B \cup C)$?)



(b) Can you make a Venn diagram for four sets using shapes other than circles?

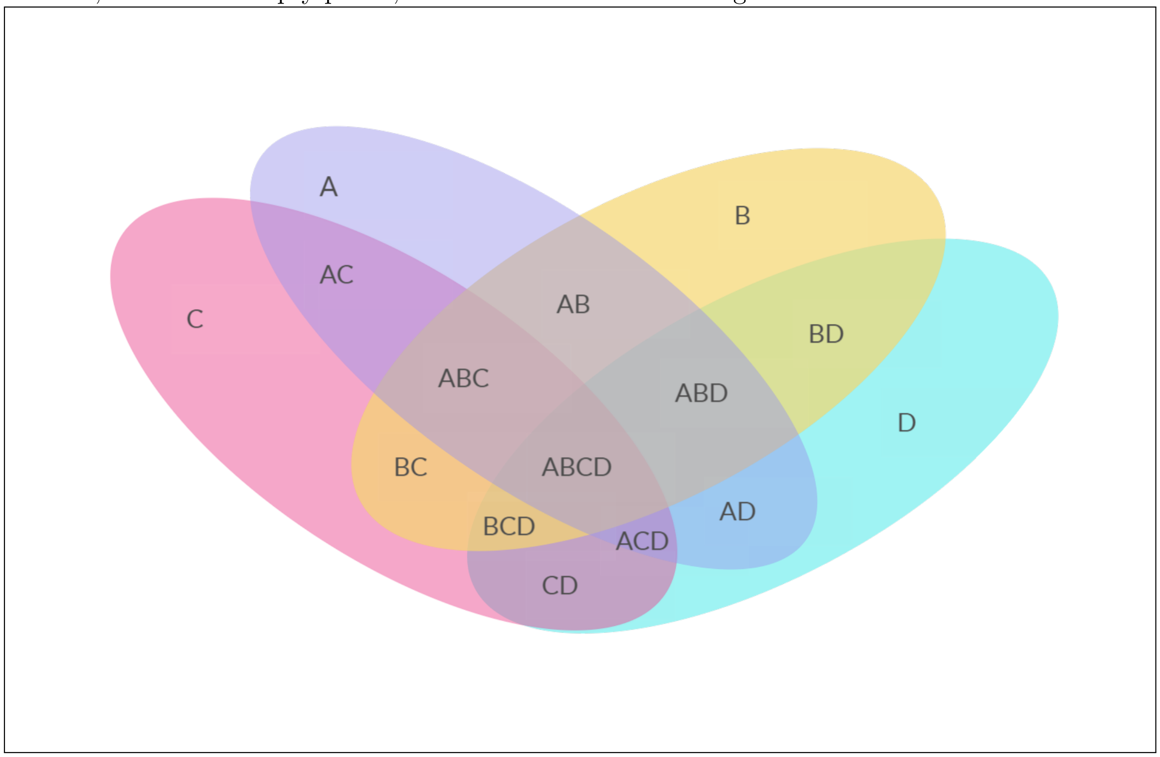
Solution:

(a) Fun fact, the number of sections in a Venn diagram of n sets corresponds to the number of rows of a truth table of n sets, that is to say, there is always 2^n. Below is the case for $n = 3$.



If you count the number of regions in $(A \cap D) \setminus (B \cup C)$, you will count 14 sections, when there should be $2^4 = 16$. Also, in the given diagram $(A \cap D) \subseteq (B \cup C)$, so $(A \cap D) \setminus (B \cup C)$ does not exist, but in the corresponding truth table there is a row where elements belong to only A and D, but not the others. Visually, that row should be present in the diagram, but it is not.

(b) It is beyond our interest as to why precisely circles are unable to illustrate a Venn diagram of 4 sets. Know that it can be done by replacing circles with ellipses, and note that we do in fact have 16 sections, the 15 non-empty pieces, and the container of the diagram.



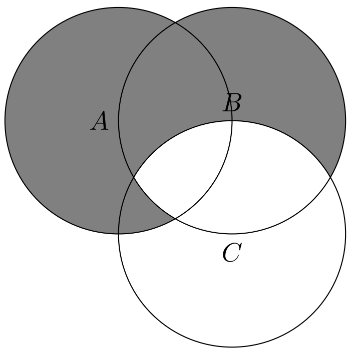
Exercise 11

(a) Make a Venn diagram for the sets $(A \cup B) \setminus C$ and $A \cup (B \setminus C)$. What can you conclude about whether one of these sets is necessarily, a subset of the other?

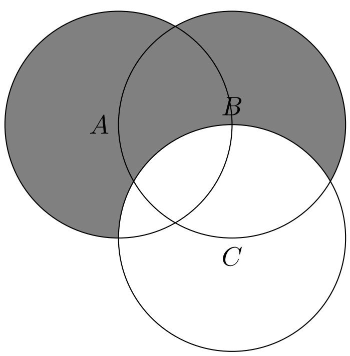
(b) Give an example of sets A, B, and C for which $(A \cup B) \setminus C \neq A \cup (B \setminus C)$.

Solution:

(a) We first draw $A \cup (B \setminus C)$ below.



Next, we draw $(A \cup B) \setminus C$ below.



Note that in $A \cup (B \setminus C)$, $A \cap C$ is nonempty. Meanwhile in $(A \cup B) \setminus C$ the set $A \cap C$ is empty. Everything else is equivalent. Hence, $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.

(b) Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$ and $C = \{1, 5\}$. Now, $(A \cup B) \setminus C$ gives us $\{1, 2, 3, 4, 5\} \setminus \{1, 5\} = \{2, 3, 4\}$, while $A \cup (B \setminus C)$ gives us $\{1, 2, 3\} \cup \{3, 4\} = \{1, 2, 3, 4\}$. These two results are clearly different: $\{2, 3, 4\} \neq \{1, 2, 3, 4\}$. This example correctly illustrates the inequality between $(A \cup B) \setminus C$ and $A \cup (B \setminus C)$.

Exercise 12

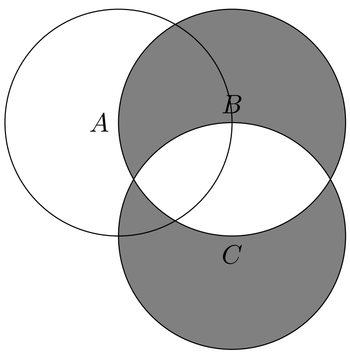
Use Venn diagrams to show that the associative law holds for symmetric difference; that is, for any sets A, B, and C, $A \triangle (B \triangle C) = (A \triangle B) \triangle C$.

Solution: We will tackle the first expression, piece by piece.

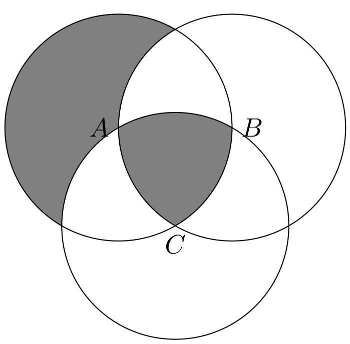
- By the definition of the symmetric difference we can write the expressions shown above as shown below:

$$\begin{aligned} A \triangle (B \triangle C) &\equiv (A \setminus (B \triangle C)) \cup ((B \triangle C) \setminus A) \\ &\equiv (A \setminus [(B \setminus C) \cup (C \setminus B)]) \cup ([(B \setminus C) \cup (C \setminus B)] \setminus A) \end{aligned}$$

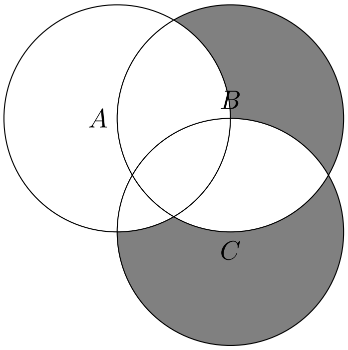
We tackle the LHS of the logical sentence first, namely $A \setminus (B \triangle C)$. For now, we ignore A, in this context we know that $B \triangle C$ is shown below:



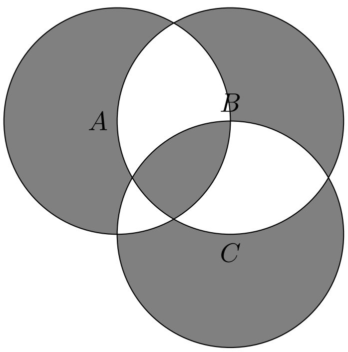
Hence, $A \setminus (B \triangle C)$ looks like:



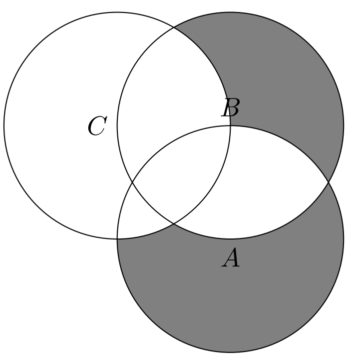
Meanwhile, $(B \triangle C) \setminus A$ looks like as shown below:



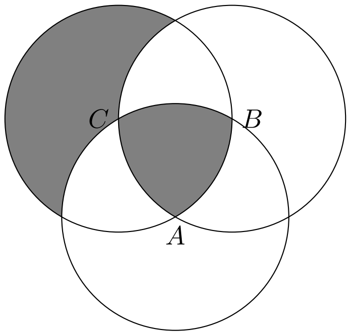
Finally, we consider the union, i.e., $A \triangle (B \triangle C) \equiv (A \setminus (B \triangle C)) \cup ((B \triangle C) \setminus A)$ to get:



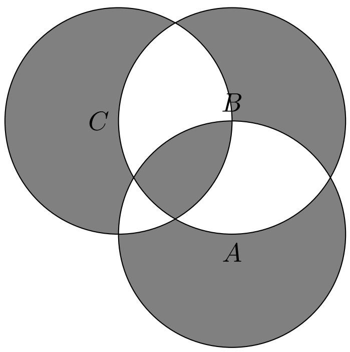
- Now, when we consider $(A \triangle B) \triangle C \equiv ((A \triangle B) \setminus C) \cup (C \setminus (A \triangle B))$, we first draw the LHS of the union below, i.e., $(A \triangle B) \setminus C$.



Meanwhile $C \setminus (A \triangle B)$ looks like:



Finally, we consider the union, i.e., $(A \triangle B) \triangle C \equiv ((A \triangle B) \setminus C) \cup (C \setminus (A \triangle B))$ to get:



We note that we get the same diagram in both cases. Thus proving that the associative law holds.

Exercise 13

Use any method you wish to verify the following identities:

(a) $(A \triangle B) \cup C = (A \cup C) \triangle (B \setminus C)$.

(b) $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$.

(c) $(A \triangle B) \setminus C = (A \setminus C) \triangle (B \setminus C)$.

Solution:

13(a) Below we prove that $(A \triangle B) \cup C = (A \cup C) \triangle (B \setminus C)$.

$x \in (A \cup C) \triangle (B \setminus C)$ **(1)**

$\equiv x \in [(A \cup C) \cup (B \setminus C)] \setminus [(A \cup C) \cap (B \setminus C)]$ **(2)**

$\equiv [x \in (A \cup C) \cup (B \setminus C)] \wedge \neg[x \in (A \cup C) \cap (B \setminus C)]$ **(3)**

$\equiv [(x \in A \vee x \in C) \vee (x \in B \wedge \neg(x \in C))] \wedge \neg[(x \in A \vee x \in C) \wedge (x \in B \wedge \neg(x \in C))]$ **(4)**

$\equiv [(x \in A \vee x \in C) \vee x \in B] \wedge [(x \in A \vee x \in C) \vee \neg(x \in C)] \wedge \neg[(x \in A \vee x \in C) \wedge (x \in B \wedge \neg(x \in C))]$ **(5)**

$\equiv [(x \in A \vee x \in B) \vee x \in C] \wedge (x \in A \vee x \in C \vee \neg(x \in C)) \wedge \neg[(x \in A \vee x \in C) \wedge (x \in B \wedge \neg(x \in C))]$ **(6)**

$\equiv [x \in Q \wedge (x \in A \vee x \in C \vee \neg(x \in C))] \wedge \neg[(x \in A \vee x \in C) \wedge (x \in B \wedge \neg(x \in C))]$ **(7)**

$\equiv [x \in Q \wedge \text{Tautology}] \wedge \neg[(x \in A \vee x \in C) \wedge (x \in B \wedge \neg(x \in C))]$ **(8)**

$\equiv [x \in Q \wedge \text{Tautology}] \wedge \neg[x \in A \vee x \in C] \vee \neg[x \in B \wedge \neg(x \in C)]$ **(9)**

$\equiv [x \in Q \wedge \text{Tautology}] \wedge [\neg(x \in A) \wedge \neg(x \in C)] \vee [\neg(x \in B) \vee x \in C]$ **(10)**

$\equiv [x \in Q \wedge \text{Tautology}] \wedge [(\neg(x \in A) \wedge \neg(x \in C)) \vee \neg(x \in B)] \vee [(\neg(x \in A) \wedge \neg(x \in C)) \vee x \in C]$ **(11)**

$\equiv [x \in Q \wedge \text{Tautology}] \wedge [(\neg(x \in A) \vee \neg(x \in B)) \wedge (\neg(x \in C) \vee \neg(x \in B))] \vee [(\neg(x \in A) \wedge \neg(x \in C)) \vee x \in C]$ **(12)**

$\equiv [x \in Q \wedge \text{Tautology}] \wedge [x \in P \wedge (\neg(x \in C) \vee \neg(x \in B))] \vee [(\neg(x \in A) \wedge \neg(x \in C)) \vee x \in C]$ **(13)**

$\equiv [x \in Q \wedge \text{Tautology}] \wedge [x \in P \wedge (\neg(x \in C) \vee \neg(x \in B))] \vee [(\neg(x \in A) \vee x \in C) \wedge (x \in C \vee \neg(x \in C))]$ **(14)**

$\equiv [x \in Q \wedge \text{Tautology}] \wedge [x \in P \wedge (\neg(x \in C) \vee \neg(x \in B))] \vee [(\neg(x \in A) \vee x \in C) \wedge \text{Tautology}]$ **(15)**

$\equiv (x \in Q) \wedge [x \in P \wedge (\neg(x \in C) \vee \neg(x \in B))] \vee [\neg(x \in A) \vee x \in C]$ **(16)**

$\equiv (x \in Q) \wedge [x \in P \wedge \neg(x \in C)] \vee [P \wedge \neg(x \in B)] \vee [\neg(x \in A) \vee x \in C]$ **(17)**

$\equiv (x \in Q) \wedge [(x \in P \wedge \neg(x \in C)) \vee \neg(x \in A) \vee [P \wedge \neg(x \in B)] \vee x \in C]$ **(18)**

$\equiv (x \in Q) \wedge [(x \in P \wedge \neg(x \in C)) \vee \neg(x \in A) \vee [(\neg(x \in A) \vee \neg(x \in B)) \wedge \neg(x \in B)] \vee x \in C]$ **(19)**

$\equiv (x \in Q) \wedge [(x \in P \wedge \neg(x \in C)) \vee \neg(x \in A) \vee \neg(x \in B) \vee x \in C]$ **(20)**

$\equiv (x \in Q) \wedge [(x \in P \wedge \neg(x \in C)) \vee x \in P \vee x \in C]$ **(21)**

$\equiv (x \in Q) \wedge [((x \in P \wedge \neg(x \in C)) \vee x \in P) \vee x \in C]$ **(22)**

$\equiv (x \in Q) \wedge [(x \in P \vee (x \in P \wedge \neg(x \in C))) \vee x \in C]$ **(23)**

$\equiv (x \in Q) \wedge [x \in P \vee x \in C]$ **(24)**

$\equiv (x \in Q) \wedge [(\neg(x \in A) \vee \neg(x \in B)) \vee x \in C]$ **(25)**

$\equiv [(x \in A \vee x \in B) \vee x \in C] \wedge [(\neg(x \in A) \vee \neg(x \in B)) \vee x \in C]$ **(26)**

$\equiv [(x \in A \vee x \in B) \wedge (\neg(x \in A) \vee \neg(x \in B))] \vee (x \in C)$ **(27)**

$\equiv [(x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)] \vee (x \in C)$ **(28)**

$\equiv [(x \in A \cup B) \wedge \neg(x \in A \cap B)] \vee (x \in C)$ **(29)**

$\equiv [x \in (A \cup B) \setminus (A \cap B)] \vee (x \in C)$ **(30)**

$\equiv (x \in A \triangle B) \vee (x \in C)$ **(31)**

$\equiv x \in (A \triangle B) \cup C$ **(32)**

$\square$

Below is the corresponding chain of justification of the **13(a)** proof.

(1) RHS
(2) Def. of Symmetric Difference
(3) Def. of Difference of Sets
(4) Def. of Union, Intersection and Difference of Sets
(5) Distributive Law
(6) Associative and Commutative Law
(7) Substitution where $x \in Q \equiv (x \in A \vee x \in B) \vee x \in C$
(8) Definition of Tautology
(9) DeMorgan's Law
(10) DeMorgan's Law
(11) Distributive Law
(12) Distributive Law
(13) Substitution where $x \in P \equiv \neg(x \in A) \vee \neg(x \in B)$
(14) Definition of Tautology
(15) Tautology Law
(16) Commutative Law and Associative Law
(17) Distributive Law
(18) Commutative Law
(19) Substitution where $x \in P \equiv \neg(x \in A) \vee \neg(x \in B)$
(20) Absorption Law $(\neg(x \in B) \equiv (\neg(x \in A) \vee \neg(x \in B)) \wedge \neg(x \in B))$
(21) Substitution where $x \in P \equiv \neg(x \in A) \vee \neg(x \in B)$
(22) Associative Law
(23) Commutative Law
(24) Absorption Law $(P \equiv P \vee (P \wedge \neg C))$
(25) Substitution where $x \in P \equiv \neg(x \in A) \vee \neg(x \in B)$
(26) Substitution where $x \in Q \equiv (x \in A \vee x \in B) \vee x \in C$
(27) Distributive Law
(28) DeMorgan's Law
(29) Def. of Union and Intersection of Sets
(30) Def. of Difference of Sets
(31) Def. of Symmetric Difference
(32) Def. of Union of Sets, LHS

13(b) Below we prove that $(A \triangle B) \cap C = (A \cap C) \triangle (B \cap C)$.

$$
\begin{aligned}
& x \in (A \cap C) \triangle (B \cap C) && \textbf{(1)} \\
& \equiv x \in [(A \cap C) \cup (B \cap C)] \setminus [(A \cap C) \cap (B \cap C)] && \textbf{(2)} \\
& \equiv [x \in (A \cap C) \cup (B \cap C)] \wedge \neg [x \in (A \cap C) \cap (B \cap C)] && \textbf{(3)} \\
& \equiv \{[x \in A \wedge x \in C] \vee [x \in B \wedge x \in C]\} \wedge \neg \{[x \in A \wedge x \in C] \wedge [x \in B \wedge x \in C]\} && \textbf{(4)} \\
& \equiv \{(x \in A \vee x \in B) \wedge x \in C\} \wedge \neg \{[x \in A \wedge x \in C] \wedge [x \in B \wedge x \in C]\} && \textbf{(5)} \\
& \equiv [(x \in A \vee x \in B) \wedge x \in C] \wedge \neg [x \in A \wedge x \in C \wedge x \in B \wedge x \in C] && \textbf{(6)} \\
& \equiv [(x \in A \vee x \in B) \wedge x \in C] \wedge \neg [x \in A \wedge x \in B \wedge (x \in C \wedge x \in C)] && \textbf{(7)} \\
& \equiv [(x \in A \vee x \in B) \wedge x \in C] \wedge \neg [x \in A \wedge x \in B \wedge x \in C] && \textbf{(8)} \\
& \equiv [(x \in A \vee x \in B) \wedge x \in C] \wedge [\neg (x \in A \wedge x \in B) \vee \neg (x \in C)] && \textbf{(9)} \\
& \equiv \{[(x \in A \vee x \in B) \wedge x \in C] \wedge \neg (x \in A \wedge x \in B)\} \vee \{[(x \in A \vee x \in B) \wedge x \in C] \wedge \neg (x \in C)\} && \textbf{(10)} \\
& \equiv \{[(x \in A \vee x \in B) \wedge x \in C] \wedge \neg (x \in A \wedge x \in B)\} \vee \{(x \in A \vee x \in B) \wedge [x \in C \wedge \neg (x \in C)]\} && \textbf{(11)} \\
& \equiv \{[(x \in A \vee x \in B) \wedge x \in C] \wedge \neg (x \in A \wedge x \in B)\} \vee \{(x \in A \vee x \in B) \wedge \text{Contradiction}\} && \textbf{(12)} \\
& \equiv \{[(x \in A \vee x \in B) \wedge x \in C] \wedge \neg (x \in A \wedge x \in B)\} \vee (\text{Contradiction}) && \textbf{(13)} \\
& \equiv \{[(x \in A \vee x \in B) \wedge x \in C] \wedge \neg (x \in A \wedge x \in B)\} && \textbf{(14)} \\
& \equiv (x \in A \vee x \in B) \wedge (x \in C) \wedge \neg (x \in A \wedge x \in B) && \textbf{(15)} \\
& \equiv (x \in A \vee x \in B) \wedge \neg (x \in A \wedge x \in B) \wedge (x \in C) && \textbf{(16)} \\
& \equiv (x \in A \cup B) \wedge \neg (x \in A \cap B) \wedge (x \in C) && \textbf{(17)} \\
& \equiv [(x \in A \cup B) \wedge \neg (x \in A \cap B)] \wedge (x \in C) && \textbf{(18)} \\
& \equiv (x \in A \triangle B) \wedge (x \in C) && \textbf{(19)} \\
& \equiv x \in (A \triangle B) \cap C && \textbf{(20)}
\end{aligned}
$$

$\square$

Below is the corresponding chain of justification of the **13(b)** proof.

(1) RHS
(2) Def. of Symmetric Difference
(3) Def. of Difference of Sets
(4) Def. of Union and Intersection of Sets
(5) Distributive Law
(6) Associative Law
(7) Associative Law and Commutative Law
(8) Idempotent Law
(9) DeMorgan's Law
(10) Distributive Law
(11) Associative Law
(12) Def. of Contradiction
(13) Contradiction Law
(14) Contradiction Law
(15) Associative Law
(16) Commutative Law
(17) Def. of Union and Intersection of Sets
(18) Associative Law
(19) Def. of Symmetric Difference
(20) Def. of Intersection of Sets, LHS

13(c) Below we prove that $(A \triangle B) \setminus C = (A \setminus C) \triangle (B \setminus C)$.

$$
\begin{aligned}
& x \in (A \setminus C) \triangle (B \setminus C) && \textbf{(1)}\\
& \equiv x \in [(A \setminus C) \cup (B \setminus C)] \setminus [(A \setminus C) \cap (B \setminus C)] && \textbf{(2)}\\
& \equiv [x \in (A \setminus C) \cup (B \setminus C)] \wedge \neg [x \in (A \setminus C) \cap (B \setminus C)] && \textbf{(3)}\\
& \equiv [(x \in A \setminus C) \vee (x \in B \setminus C)] \wedge \neg [(x \in A \setminus C) \wedge (x \in B \setminus C)] && \textbf{(4)}\\
& \equiv \{[x \in A \wedge \neg (x \in C)] \vee [x \in B \wedge \neg (x \in C)]\} \wedge \neg \{(x \in A \setminus C) \wedge (x \in B \setminus C)\} && \textbf{(5)}\\
& \equiv [(x \in A \vee x \in B) \wedge \neg (x \in C)] \wedge \neg \{(x \in A \setminus C) \wedge (x \in B \setminus C)\} && \textbf{(6)}\\
& \equiv [(x \in A \vee x \in B) \wedge \neg (x \in C)] \wedge \neg \{[x \in A \wedge \neg (x \in C)] \wedge [x \in B \wedge \neg (x \in C)]\} && \textbf{(7)}\\
& \equiv [(x \in A \vee x \in B) \wedge \neg (x \in C)] \wedge \neg [x \in A \wedge \neg (x \in C) \wedge x \in B \wedge \neg (x \in C)] && \textbf{(8)}\\
& \equiv [(x \in A \vee x \in B) \wedge \neg (x \in C)] \wedge \neg [x \in A \wedge x \in B \wedge [\neg (x \in C) \wedge \neg (x \in C)]] && \textbf{(9)}\\
& \equiv [(x \in A \vee x \in B) \wedge \neg (x \in C)] \wedge \neg [x \in A \wedge x \in B \wedge \neg (x \in C)] && \textbf{(10)}\\
& \equiv [(x \in A \vee x \in B) \wedge \neg (x \in C)] \wedge [\neg (x \in A \wedge x \in B) \vee x \in C] && \textbf{(11)}\\
& \equiv \{[(x \in A \vee x \in B) \wedge \neg (x \in C)] \wedge \neg (x \in A \wedge x \in B)\} \vee \{[(x \in A \vee x \in B) \wedge \neg (x \in C)] \wedge x \in C\} && \textbf{(12)}\\
& \equiv \{[(x \in A \vee x \in B) \wedge \neg (x \in C)] \wedge \neg (x \in A \wedge x \in B)\} \vee \{(x \in A \vee x \in B) \wedge [\neg (x \in C) \wedge x \in C]\} && \textbf{(13)}\\
& \equiv \{[(x \in A \vee x \in B) \wedge \neg (x \in C)] \wedge \neg (x \in A \wedge x \in B)\} \vee [(x \in A \vee x \in B) \wedge \text{Contradiction}] && \textbf{(14)}\\
& \equiv \{[(x \in A \vee x \in B) \wedge \neg (x \in C)] \wedge \neg (x \in A \wedge x \in B)\} \vee \text{Contradiction} && \textbf{(15)}\\
& \equiv [(x \in A \vee x \in B) \wedge \neg (x \in C)] \wedge \neg (x \in A \wedge x \in B) && \textbf{(16)}\\
& \equiv (x \in A \vee x \in B) \wedge \neg (x \in C) \wedge \neg (x \in A \wedge x \in B) && \textbf{(17)}\\
& \equiv (x \in A \vee x \in B) \wedge \neg (x \in A \wedge x \in B) \wedge \neg (x \in C) && \textbf{(18)}\\
& \equiv [(x \in A \vee x \in B) \wedge \neg (x \in A \wedge x \in B)] \wedge \neg (x \in C) && \textbf{(19)}\\
& \equiv [(x \in A \cup B) \wedge \neg (x \in A \cap B)] \wedge \neg (x \in C) && \textbf{(20)}\\
& \equiv [x \in (A \cup B) \setminus (A \cap B)] \wedge \neg (x \in C) && \textbf{(21)}\\
& \equiv x \in (A \triangle B) \wedge \neg (x \in C) && \textbf{(22)}\\
& \equiv x \in (A \triangle B) \setminus C && \textbf{(23)}
\end{aligned}
$$

$\square$

Below is the corresponding chain of justification of the **13(c)** proof.

(1)	RHS
(2)	Def. of Symmetric Difference
(3)	Def. of Difference of Sets
(4)	Def. of Union and Intersection of Sets
(5)	Def. of Difference of Sets
(6)	Distributive Law
(7)	Def. of Difference of Sets
(8)	Associative Law
(9)	Associative and Commutative Law
(10)	Idempotent Law
(11)	DeMorgan's Law
(12)	Distributive Law
(15)	Contradiction Law
(14)	Def. of Contradiction
(15)	Contradiction Law
(16)	Contradiction Law
(17)	Associative Law
(18)	Commutative Law
(19)	Associative Law
(20)	Def. of Union and Intersection of Sets
(21)	Def. of Difference of Sets
(22)	Def. of Symmetric Difference
(23)	Def. of Difference of Sets, LHS

Exercise 14

Use any method you wish to verify the following identities:

(a) $(A \cup B) \triangle C = (A \triangle C) \triangle (B \setminus A)$.

(b) $(A \cap B) \triangle C = (A \triangle C) \triangle (A \setminus B)$.

(c) $(A \setminus B) \triangle C = (A \triangle C) \triangle (A \cap B)$.

14(a) Below we prove that $(A \cup B) \triangle C = (A \triangle C) \triangle (B \setminus A)$.
But first, we prove the following lemma, which we will call **Lemma 14(a)**: $A \triangle (B \setminus A) \equiv A \cup B$.

$$
\begin{aligned}
& x \in A \triangle (B \setminus A) && \textbf{(1)} \\
& \equiv x \in [A \cup (B \setminus A)] \setminus [A \cap (B \setminus A)] && \textbf{(2)} \\
& \equiv [x \in A \cup (B \setminus A)] \wedge \neg [x \in A \cap (B \setminus A)] && \textbf{(3)} \\
& \equiv [x \in A \vee x \in (B \setminus A)] \wedge \neg [x \in A \wedge x \in (B \setminus A)] && \textbf{(4)} \\
& \equiv \{x \in A \vee [x \in B \wedge \neg (x \in A)]\} \wedge \neg \{x \in A \wedge [x \in B \wedge \neg (x \in A)]\} && \textbf{(5)} \\
& \equiv \{(x \in A \vee x \in B) \wedge (x \in A \vee \neg (x \in A))]\} \wedge \neg \{x \in A \wedge [x \in B \wedge \neg (x \in A)]\} && \textbf{(6)} \\
& \equiv \{(x \in A \vee x \in B) \wedge \text{Tautology}]\} \wedge \neg \{x \in A \wedge [x \in B \wedge \neg (x \in A)]\} && \textbf{(7)} \\
& \equiv (x \in A \vee x \in B) \wedge \neg \{x \in A \wedge [x \in B \wedge \neg (x \in A)]\} && \textbf{(8)} \\
& \equiv (x \in A \vee x \in B) \wedge \neg [x \in A \wedge x \in B \wedge \neg (x \in A)] && \textbf{(9)} \\
& \equiv (x \in A \vee x \in B) \wedge \neg [x \in A \wedge \neg (x \in A) \wedge x \in B] && \textbf{(10)} \\
& \equiv (x \in A \vee x \in B) \wedge \neg \{[x \in A \wedge \neg (x \in A)] \wedge x \in B\} && \textbf{(11)} \\
& \equiv (x \in A \vee x \in B) \wedge \neg [\text{Contradiction} \wedge x \in B] && \textbf{(12)} \\
& \equiv (x \in A \vee x \in B) \wedge \neg \text{Contradiction} && \textbf{(13)} \\
& \equiv (x \in A \vee x \in B) \wedge \text{Tautology} && \textbf{(14)} \\
& \equiv (x \in A \vee x \in B) && \textbf{(15)} \\
& \equiv x \in (A \cup B) && \textbf{(16)}
\end{aligned}
$$

$\square$

Proof for **14(a)**:

$$
\begin{aligned}
& x \in (A \triangle C) \triangle (B \setminus A) && \textbf{(1)} \\
& \equiv x \in A \triangle [C \triangle (B \setminus A)] && \textbf{(2)} \\
& \equiv x \in A \triangle [(B \setminus A) \triangle C] && \textbf{(3)} \\
& \equiv x \in [A \triangle (B \setminus A]) \triangle C && \textbf{(4)} \\
& \equiv x \in (A \cup B) \triangle C && \textbf{(5)}
\end{aligned}
$$

$\square$

Below is the corresponding chain of justification for **Lemma 14(a)**.

(1) LHS
(2) Def. of Symmetric Difference
(3) Def. of Difference of Sets
(4) Def. of Union and Intersection of Sets
(5) Def. of Difference of Sets
(6) Distributive Law
(7) Definition of Tautology
(8) Law of Tautology
(9) Associative Law
(10) Commutative Law
(11) Associative Law
(12) Definition of Contradiction
(13) Law of Contradiction
(14) Negation Law
(15) Law of Tautology
(16) Def. of Union of Sets, RHS

Below is the corresponding chain of justification for **Exercise 14(a)**.

(1) RHS
(2) **Exercise 12** Associative Law
(3) $C \triangle (B \setminus A) \equiv [C \setminus (B \setminus A)] \cup [(B \setminus A) \setminus C] \equiv [(B \setminus A) \setminus C] \cup [C \setminus (B \setminus A)] \equiv (B \setminus A) \triangle C$
(4) **Exercise 12** Associative Law
(5) By Lemma 14(a), LHS

14(b) Below we prove that $(A \cap B) \triangle C = (A \triangle C) \triangle (A \setminus B)$.
But first, we prove the following lemma, which we will call **Lemma 14(b)**: $A \triangle (A \setminus B) \equiv A \cap B$.

$$
\begin{aligned}
& x \in A \triangle (A \setminus B) && \textbf{(1)} \\
& \equiv x \in [A \cup (A \setminus B)] \setminus [A \cap (A \setminus B)] && \textbf{(2)} \\
& \equiv [x \in A \cup (A \setminus B)] \land \neg [x \in A \cap (A \setminus B)] && \textbf{(3)} \\
& \equiv [x \in A \lor x \in (A \setminus B)] \land \neg [x \in A \land x \in (A \setminus B)] && \textbf{(4)} \\
& \equiv \{x \in A \lor [x \in A \land \neg (x \in B)]\} \land \neg \{x \in A \land [x \in A \land \neg (x \in B)]\} && \textbf{(5)} \\
& \equiv x \in A \land \neg \{x \in A \land [x \in A \land \neg (x \in B)]\} && \textbf{(6)} \\
& \equiv x \in A \land \neg \{[x \in A \land x \in A] \land \neg (x \in B)\} && \textbf{(7)} \\
& \equiv x \in A \land \neg [x \in A \land \neg (x \in B)] && \textbf{(8)} \\
& \equiv x \in A \land [\neg (x \in A) \lor x \in B] && \textbf{(9)} \\
& \equiv [x \in A \land \neg (x \in A)] \lor [x \in A \land x \in B] && \textbf{(10)} \\
& \equiv \text{Contradiction} \lor [x \in A \land x \in B] && \textbf{(11)} \\
& \equiv x \in A \land x \in B && \textbf{(12)} \\
& \equiv x \in A \cap B && \textbf{(13)}
\end{aligned}
$$

$\square$

Proof for **14(b)**:

$$
\begin{aligned}
& x \in (A \triangle C) \triangle (A \setminus B) && \textbf{(1)} \\
& \equiv x \in A \triangle [C \triangle (A \setminus B)] && \textbf{(2)} \\
& \equiv x \in A \triangle [(A \setminus B) \triangle C] && \textbf{(3)} \\
& \equiv x \in [A \triangle (A \setminus B]) \triangle C && \textbf{(4)} \\
& \equiv x \in (A \cap B) \triangle C && \textbf{(5)}
\end{aligned}
$$

$\square$

Below is the corresponding chain of justification for **Lemma 14(b)**.

(1) LHS
(2) Def. of Symmetric Difference
(3) Def. of Difference of Sets
(4) Def. of Union and Intersection of Sets
(5) Def. of Difference of Sets
(6) Absorption Law
(7) Associative Law
(8) Idempotent Law
(9) DeMorgan's Law
(10) Distributive Law
(11) Definition of Contradiction
(12) Law of Contradiction
(13) Def. of Intersection of Sets, RHS

Below is the corresponding chain of justification for **Exercise 14(b)**.

(1) RHS
(2) **Exercise 12** Associative Law
(3) $C\triangle(A \setminus B) \equiv [C \setminus (A \setminus B)] \cup [(A \setminus B) \setminus C] \equiv [(A \setminus B) \setminus C] \cup [C \setminus (A \setminus B)] \equiv (A \setminus B)\triangle C$
(4) **Exercise 12** Associative Law
(5) By Lemma 14(b), LHS

14(c) Below we prove that $(A \setminus B) \triangle C = (A \triangle C) \triangle (A \cap B)$. But first, we prove the following lemma, which we will call **Lemma 14(c)**: $A \triangle (A \cap B) \equiv A \setminus B$.

$$\begin{aligned}
& x \in A \triangle (A \cap B) && \textbf{(1)} \\
& \equiv x \in [A \cup (A \cap B)] \setminus [A \cap (A \cap B)] && \textbf{(2)} \\
& \equiv x \in [A \cup (A \cap B)] \wedge \neg[x \in A \cap (A \cap B)] && \textbf{(3)} \\
& \equiv [x \in A \vee x \in A \cap B] \wedge \neg[x \in A \wedge x \in A \cap B] && \textbf{(4)} \\
& \equiv [x \in A \vee (x \in A \wedge x \in B)] \wedge \neg[x \in A \wedge (x \in A \wedge x \in B)] && \textbf{(5)} \\
& \equiv (x \in A) \wedge \neg[x \in A \wedge (x \in A \wedge x \in B)] && \textbf{(6)} \\
& \equiv (x \in A) \wedge \neg[(x \in A \wedge x \in A) \wedge x \in B] && \textbf{(7)} \\
& \equiv (x \in A) \wedge \neg[x \in A \wedge x \in B] && \textbf{(8)} \\
& \equiv (x \in A) \wedge [\neg(x \in A) \vee \neg(x \in B)] && \textbf{(9)} \\
& \equiv [x \in A \wedge \neg(x \in A)] \vee [x \in A \wedge \neg(x \in B)] && \textbf{(10)} \\
& \equiv \text{Contradiction} \vee [x \in A \wedge \neg(x \in B)] && \textbf{(11)} \\
& \equiv x \in A \wedge \neg(x \in B) && \textbf{(12)} \\
& \equiv x \in A \setminus B && \textbf{(13)} \\
& \quad \square
\end{aligned}$$

Proof for **14(c)**:

$$\begin{aligned}
& x \in (A \triangle C) \triangle (A \cap B) && \textbf{(1)} \\
& \equiv x \in A \triangle [C \triangle (A \cap B)] && \textbf{(2)} \\
& \equiv x \in A \triangle [(A \cap B) \triangle C] && \textbf{(3)} \\
& \equiv x \in [A \triangle (A \cap B]) \triangle C && \textbf{(4)} \\
& \equiv x \in (A \setminus B) \triangle C && \textbf{(5)} \\
& \quad \square
\end{aligned}$$

Below is the corresponding chain of justification for **Lemma 14(c)**.

(1) LHS
(2) Def. of Symmetric Difference
(3) Def. of Difference of Sets
(4) Def. of Union and Intersection of Sets
(5) Def. of Intersection of Sets
(6) Absorption Law
(7) Associative Law
(8) Idempotent Law
(9) DeMorgan's Law
(10) Distributive Law
(11) Definition of Contradiction
(12) Law of Contradiction
(13) Def. of Difference of Sets, RHS

Below is the corresponding chain of justification for **Exercise 14(c)**.

(1) RHS
(2) **Exercise 12** Associative Law
(3) $C\triangle(A\cap B) \equiv [C \setminus (A\cap B)] \cup [(A\cap B)\setminus C] \equiv [(A\cap B)\setminus C] \cup [C\setminus (A\cap B)] \equiv (A\cap B)\triangle C$
(4) **Exercise 12** Associative Law
(5) By Lemma 14(c), LHS

Exercise 15

Fill in the blanks to make true identities:

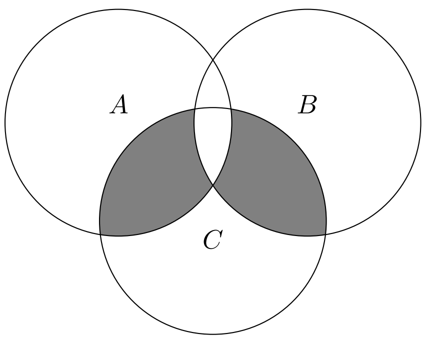
(a) $(A \triangle B) \cap C = (C \setminus A) \triangle$ ______.

(b) $C \setminus (A \triangle B) = (A \cap C) \triangle$ ______.

(c) $(B \setminus A) \triangle C = (A \triangle C) \triangle$ ______.

Solution:

15(a): Consider that $(A \triangle B) \cap C = [(A \wedge C) \vee (B \wedge C)] \wedge \neg(A \wedge B)$.
The corresponding Venn diagram is shown below.
Note the symmetry, we guess that $(A \triangle B) \cap C = (C \setminus A) \triangle (C \setminus B)$.



Below is the prove for **15(a)**.

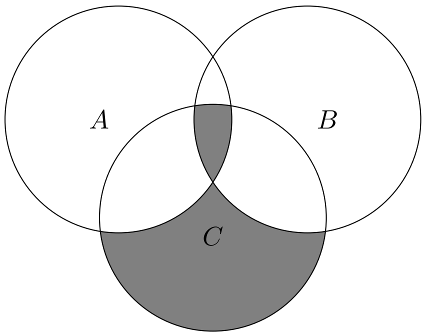
$$
\begin{aligned}
& x \in (C \setminus A) \triangle (C \setminus B) && \textbf{(1)}\\
& \equiv x \in [(C \setminus A) \cup (C \setminus B)] \setminus [(C \setminus A) \cap (C \setminus B)] && \textbf{(2)}\\
& \equiv x \in [(C \setminus A) \cup (C \setminus B)] \setminus Q && \textbf{(3)}\\
& \equiv [x \in (C \setminus A) \cup (C \setminus B)] \land \neg(x \in Q) && \textbf{(4)}\\
& \equiv [x \in C \setminus A \lor x \in C \setminus B] \land \neg(x \in Q) && \textbf{(5)}\\
& \equiv \{[x \in C \land \neg(x \in A)] \lor [x \in C \land \neg(x \in B)]\} \land \neg(x \in Q) && \textbf{(6)}\\
& \equiv \{([x \in C \land \neg(x \in A)] \lor x \in C) \land ([x \in C \land \neg(x \in A)] \lor \neg[x \in B])\} \land \neg(x \in Q) && \textbf{(7)}\\
& \equiv \{(x \in C) \land ([x \in C \land \neg(x \in A)] \lor \neg[x \in B])\} \land \neg(x \in Q) && \textbf{(8)}\\
& \equiv \{(x \in C \land [x \in C \land \neg(x \in A)]) \lor [x \in C \land \neg(x \in B)]\} \land \neg(x \in Q) && \textbf{(9)}\\
& \equiv \{([x \in C \land x \in C] \land \neg(x \in A)) \lor [x \in C \land \neg(x \in B)]\} \land \neg(x \in Q) && \textbf{(10)}\\
& \equiv \{[x \in C \land \neg(x \in A)] \lor [x \in C \land \neg(x \in B)]\} \land \neg(x \in Q) && \textbf{(11)}\\
& \equiv \{x \in C \land [\neg(x \in A) \lor \neg(x \in B)]\} \land \neg(x \in Q) && \textbf{(12)}\\
& \equiv \{x \in C \land \neg[x \in A \land x \in B]\} \land \neg(x \in Q) && \textbf{(13)}\\
& \equiv \{x \in C \land \neg[x \in A \land x \in B]\} \land \neg\{x \in [(C \setminus A) \cap (C \setminus B)]\} && \textbf{(14)}\\
& \equiv \{x \in C \land \neg[x \in A \land x \in B]\} \land \neg\{[x \in C \land \neg(x \in A)] \land [x \in C \land \neg(x \in B)]\} && \textbf{(15)}\\
& \equiv \{x \in C \land \neg[x \in A \land x \in B]\} \land \neg\{x \in C \land \neg(x \in A) \land x \in C \land \neg(x \in B)\} && \textbf{(16)}\\
& \equiv \{x \in C \land \neg[x \in A \land x \in B]\} \land \neg\{(x \in C \land x \in C) \land \neg(x \in A) \land \neg(x \in B)\} && \textbf{(17)}\\
& \equiv \{x \in C \land \neg[x \in A \land x \in B]\} \land \neg\{x \in C \land \neg(x \in A) \land \neg(x \in B)\} && \textbf{(18)}\\
& \equiv \{x \in C \land \neg[x \in A \land x \in B]\} \land \neg\{x \in C \land \neg[x \in A \lor x \in B]\} && \textbf{(19)}\\
& \equiv \{x \in C \land \neg[x \in A \land x \in B]\} \land \{\neg(x \in C) \lor [x \in A \lor x \in B]\} && \textbf{(20)}\\
& \equiv \{(x \in C \land \neg[x \in A \land x \in B]) \land \neg(x \in C)\} \lor \{(x \in C \land \neg[x \in A \land x \in B]) \land [x \in A \lor x \in B]\} && \textbf{(21)}\\
& \equiv \{[x \in C \land \neg(x \in C)] \land \neg[x \in A \land x \in B]\} \lor \{x \in C \land \neg[x \in A \land x \in B] \land [x \in A \lor x \in B]\} && \textbf{(22)}\\
& \equiv \{\text{Contradiction} \land \neg[x \in A \land x \in B]\} \lor \{x \in C \land \neg[x \in A \land x \in B] \land [x \in A \lor x \in B]\} && \textbf{(23)}\\
& \equiv \text{Contradiction} \lor \{x \in C \land \neg[x \in A \land x \in B] \land [x \in A \lor x \in B]\} && \textbf{(24)}\\
& \equiv x \in C \land \neg[x \in A \land x \in B] \land [x \in A \lor x \in B] && \textbf{(25)}\\
& \equiv [(x \in A \lor x \in B) \land \neg(x \in A \land x \in B)] \land x \in C && \textbf{(26)}\\
& \equiv [x \in A \cup B \land \neg(x \in A \cap B)] \land x \in C && \textbf{(27)}\\
& \equiv [x \in (A \cup B) \setminus (A \cap B)] \land x \in C && \textbf{(28)}\\
& \equiv [x \in A \triangle B] \land x \in C && \textbf{(29)}\\
& \equiv x \in (A \triangle B) \cap C && \textbf{(30)}
\end{aligned}
$$

$\square$

Below is the chain of justification corresponding to the proof of **15(a)**.

(1) RHS
(2) Def. of Symmetric Difference
(3) Substitution where $Q \equiv (C \setminus A) \cap (C \setminus B)$
(4) Def. of Difference of Sets
(5) Def. of Union of Sets
(6) Def. of Difference of Sets
(7) Distributive Law
(8) Absorption Law $C \equiv C \vee (C \wedge \neg A)$
(9) Distributive Law
(10) Associative Law
(11) Idempotent Law
(12) Distributive Law
(13) DeMorgan's Law
(14) Substitution where $Q \equiv (C \setminus A) \cap (C \setminus B)$
(15) Def. of Difference and Intersection of Sets
(16) Associative Law
(17) Associative Law and Commutative Law
(18) Idempotent Law
(19) DeMorgan's Law
(20) DeMorgan's Law
(21) Distributive Law
(22) Associative Law and Commutative Law
(23) Def. of Contradiction
(24) Law of Contradiction
(25) Law of Contradiction
(26) Associative Law and Commutative Law
(27) Def. of Union and Intersection of Sets
(28) Def. of Difference of Sets
(29) Def. of Symmetric Difference
(30) Def. of Intersection of Sets, LHS

15(b): Consider that $C \setminus (A \triangle B) \equiv [A \cap B \cap C] \cup [C \cap \neg(A \cup B)]$.
The corresponding Venn diagram is shown below.
Note the symmetry, we guess that $C \setminus (A \triangle B) = (A \cap C) \triangle (B \setminus C)$.



Below is the prove for **15(b)**.

$$
\begin{aligned}
& x \in (A \cap C) \triangle (B \setminus C) && \textbf{(1)} \\
& \equiv x \in [(A \cap C) \setminus (C \setminus B)] \cup [(C \setminus B) \setminus (A \cap C)] && \textbf{(2)} \\
& \equiv x \in [(A \cap C) \setminus (C \setminus B)] \cup Q && \textbf{(3)} \\
& \equiv [x \in (A \cap C) \setminus (C \setminus B)] \vee x \in Q && \textbf{(4)} \\
& \equiv [(x \in A \wedge x \in C) \wedge \neg(x \in C \wedge \neg(x \in B))] \vee x \in Q && \textbf{(5)} \\
& \equiv [(x \in A \wedge x \in C) \wedge (\neg(x \in C) \vee x \in B)] \vee x \in Q && \textbf{(6)} \\
& \equiv \{[(x \in A \wedge x \in C) \wedge \neg(x \in C)] \vee [(x \in A \wedge x \in C) \wedge x \in B]\} \vee x \in Q && \textbf{(7)} \\
& \equiv \{[x \in A \wedge (x \in C \wedge \neg(x \in C))] \vee [(x \in A \wedge x \in C) \wedge x \in B]\} \vee x \in Q && \textbf{(8)} \\
& \equiv \{[x \in A \wedge \text{Contradiction}] \vee [(x \in A \wedge x \in C) \wedge x \in B]\} \vee x \in Q && \textbf{(9)} \\
& \equiv \{\text{Contradiction} \vee [(x \in A \wedge x \in C) \wedge x \in B]\} \vee x \in Q && \textbf{(10)} \\
& \equiv [(x \in A \wedge x \in C) \wedge x \in B] \vee x \in Q && \textbf{(11)} \\
& \equiv [x \in A \wedge x \in C \wedge x \in B] \vee x \in Q && \textbf{(12)} \\
& \equiv [x \in C \wedge (x \in A \wedge x \in B)] \vee x \in Q && \textbf{(13)} \\
& \equiv x \in P \vee x \in Q && \textbf{(14)} \\
& \equiv x \in P \vee [x \in (C \setminus B) \setminus (A \cap C)] && \textbf{(15)} \\
& \equiv x \in P \vee \{(x \in C \wedge \neg(x \in B)) \wedge \neg(x \in A \wedge x \in C)\} && \textbf{(16)} \\
& \equiv x \in P \vee \{(x \in C \wedge \neg(x \in B)) \wedge (\neg(x \in A) \vee \neg(x \in C))\} && \textbf{(17)} \\
& \equiv x \in P \vee \{[(x \in C \wedge \neg(x \in B)) \wedge \neg(x \in A)] \vee [(x \in C) \wedge \neg(x \in B)) \wedge \neg(x \in C)]\} && \textbf{(18)} \\
& \equiv x \in P \vee \{[(x \in C \wedge \neg(x \in B)) \wedge \neg(x \in A)] \vee [(x \in C \wedge \neg(x \in C)) \wedge \neg(x \in B)]\} && \textbf{(19)} \\
& \equiv x \in P \vee \{[(C \wedge \neg B) \wedge \neg A] \vee [\text{Contradiction} \wedge \neg B]\} && \textbf{(20)} \\
& \equiv x \in P \vee \{[(x \in C \wedge \neg(x \in B)) \wedge \neg(x \in A)] \vee \text{Contradiction}\} && \textbf{(21)} \\
& \equiv x \in P \vee [(x \in C \wedge \neg(x \in B)) \wedge \neg(x \in A)] && \textbf{(22)} \\
& \equiv x \in P \vee [x \in C \wedge (\neg(x \in B) \wedge \neg(x \in A))] && \textbf{(23)} \\
& \equiv x \in P \vee [x \in C \wedge \neg(x \in A \vee x \in B)] && \textbf{(24)} \\
& \equiv [x \in C \wedge (x \in A \wedge x \in B)] \vee [x \in C \wedge \neg(x \in A \vee x \in B)] && \textbf{(25)} \\
& \equiv x \in C \wedge [(x \in A \wedge x \in B) \vee \neg(x \in A \vee x \in B)] && \textbf{(26)} \\
& \equiv x \in C \wedge [\neg(x \in A \vee x \in B) \vee (x \in A \wedge x \in B)] && \textbf{(27)} \\
& \equiv x \in C \wedge \neg[(x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)] && \textbf{(28)} \\
& \equiv x \in C \wedge \neg[(x \in A \cup B) \wedge \neg(x \in A \cap B)] && \textbf{(29)} \\
& \equiv x \in C \wedge \neg[x \in (A \cup B) \setminus (A \cap B)] && \textbf{(30)} \\
& \equiv x \in C \setminus [(A \cup B) \setminus (A \cap B)] && \textbf{(31)} \\
& \equiv x \in C \setminus (A \triangle B) && \textbf{(32)}
\end{aligned}
$$

$\square$

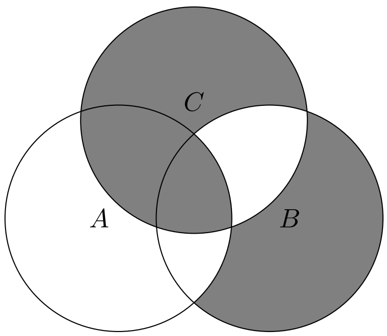
Below is the chain of justification corresponding to the proof of **15(b)**.

(1) RHS
(2) Def. of Symmetric Difference
(3) Substitution: $Q = [(C \setminus B) \setminus (A \cap C)]$
(4) Def. of Union of Sets
(5) Def. of Difference and Intersection of Sets
(6) DeMorgan's Law
(7) Distributive Law
(8) Associative Law
(9) Def. of Contradiction
(10) Law of Contradiction
(11) Law of Contradiction
(12) Associative Law
(13) Commutative and Associative Law
(14) Substitution: $P = [C \cap (A \cap B)]$
(15) Substitution: $Q = [(C \setminus B) \setminus (A \cap C)]$
(16) Def. of Difference and Intersection of Sets
(17) DeMorgan's Law
(18) Distributive Law
(19) Commutative and Associative Law
(20) Def. of Contradiction
(21) Law of Contradiction
(22) Law of Contradiction
(23) Associative Law
(24) DeMorgan's Law
(25) Substitution: $P = [C \cap (A \cap B)]$
(26) Distributive Law
(27) Commutative Law
(28) DeMorgan's Law
(29) Def. of Union and Intersection of Sets
(30) Def. of Difference of Sets
(31) Def. of Difference of Sets
(32) Def. of Symmetric Difference, LHS

15(c): Consider that $(B \setminus A) \triangle C \equiv [A \cap B \cap C] \cup [C \cap \neg(A \cup B)]$.
The corresponding Venn diagram is shown below.
The diagram is drawn by noting that (via diagrams):

$$(B \setminus A) \triangle C \equiv [B \setminus (A \cup C)] \cup [C \setminus (B \setminus A)] \equiv [B \setminus (A \cup C)] \cup (C \setminus B) \cup (C \cap A)$$

Drawing Venn diagrams, and trial and error will reveal that $(B \setminus A) \triangle C = (A \triangle C) \triangle (A \cup B)$.



Below is the prove for **15(c)**.

$$\begin{aligned}
x \in (A \triangle C) \triangle (A \cup B) &\equiv x \in A \triangle [C \triangle (A \cup B)] && \textbf{(1)}\\
&\equiv x \in A \triangle [(A \cup B) \triangle C] && \textbf{(2)}\\
&\equiv x \in [A \triangle (A \cup B)] \triangle C && \textbf{(3)}\\
&\equiv x \in [(A \cup B) \triangle A] \triangle C && \textbf{(4)}\\
&\equiv x \in [(A \triangle A) \triangle (B \setminus A)] \triangle C && \textbf{(5)}\\
&\equiv x \in [\emptyset \triangle (B \setminus A)] \triangle C && \textbf{(6)}\\
&\equiv x \in ([\emptyset \cup (B \setminus A)] \setminus [\emptyset \cap (B \setminus A)]) \triangle C && \textbf{(7)}\\
&\equiv x \in [(B \setminus A) \setminus \emptyset] \triangle C && \textbf{(8)}\\
&\equiv x \in (B \setminus A) \triangle C && \textbf{(9)}
\end{aligned}$$

Below is the corresponding chain of justification for **Exercise 15(c)**.

- **(1)** **Exercise 12** Associative Law, RHS
- **(2)** **Exercise 14(c) Proof Line 3** Commutative Law
- **(3)** **Exercise 12** Associative Law
- **(4)** **Exercise 14(c) Proof Line 3** Commutative Law
- **(5)** **Exercise 14(a)** Result of Exercise
- **(6)** $A \triangle A \equiv (A \cup A) \setminus (A \cap A) \equiv A \setminus A \equiv \emptyset$
- **(7)** $\emptyset \cup X \equiv X$ and $\emptyset \cap X \equiv \emptyset$
- **(8)** $X \setminus \emptyset \equiv X \wedge \neg\emptyset \equiv X \wedge U \equiv X$, where U is the universal set and $\neg\emptyset \equiv U$
- **(9)** LHS