How To Prove It:
A Structured Approach, Second Edition

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Solutions to: *1.3 Variables and Sets*

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Contents

Exercise 1

Analyze the logical forms of the following statements:

(a) 3 is a common divisor of 6, 9, and 15. (Note: You did this in exercise 2 of Section 1.1, but you should be able to give better answer now.)

(b) x is divisible by both 2 and 3 but not 4.

(c) x and y are natural numbers, and exactly one of them is prime.

Solution:

(a) $D(6) \land D(9) \land D(15)$, where $D(x)$ means "x is divisible by 3".

(b) $D(x, 2) \land D(x, 3) \land \neg D(x, 4)$, where $D(x, y)$ means "x is divisible by y".

(c) $N(x) \land N(y) \land [(P(x) \land \neg P(y)) \lor (P(y) \land \neg P(x))]$, where $N(x)$ means "x is a natural number" and $P(x)$ means "x is prime".

Exercise 2

Analyze the logical forms of the following statements:

(a) x and y are men, and either x is taller than y or y is taller than x.

(b) Either x or y has brown eyes, and either x or y has red hair.

(c) Either x or y has both brown eyes and red hair.

Solution:

(a) $M(x) \wedge M(y) \wedge [(T(x,y) \vee (T(y,x)]$, where $M(x)$ means "x is man" and $T(x,y)$ means "x is taller than y".

(b) $[B(x) \vee B(y)] \wedge [R(x) \vee R(y)]$, where $B(x)$ means "x has brown eyes" and $R(x)$ means "x has red eyes".

(c) $[B(x) \wedge R(x)] \vee [B(y) \vee R(y)]$, where $B(x)$ means "x has brown eyes" and $R(x)$ means "x has red eyes".

Exercise 3

Write definitions using elementhood tests for the following sets:

(a) {Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune, Pluto}.

(b) {Brown, Columbia, Cornell, Darthmouth, Harvard, Princeton, University of Pennsylvania, Yale}.

(c) {Alabama, Alaska, Arizona, ..., Wisconsin, Wyoming}.

(d) {Alberta, British Columbia, Manitoba, New Brunswick, Newfoundland and Labrador, Northwest Territories, Nova Scotia, Nunavut, Ontario, Prince Edwards Island, Quebec, Saskatchewan, Yukon}.

Solution:

(a) $\{x \mid x$ is a planet in the solar system$\}$

(b) $\{x \mid x$ is an Ivy League University$\}$

(c) $\{x \mid x$ is a state in modern United States of America$\}$

(d) $\{x \mid x$ is a province or territory in Canada$\}$

Exercise 4

Write definitions using elementhood tests for the following sets:

(a) $\{1, 4, 9, 16, 25, 36, 49, ...\}$

(b) $\{1, 2, 4, 8, 16, 32, 64, ...\}$

(c) $\{10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$

Solution:

(a) $\{1, 4, 9, 16, 25, 36, 49, ...\} \equiv \{x \mid x = n^2, n \in \mathbb{N}^+\}$
The set of squares of positive natural numbers.

(b) $\{1, 2, 4, 8, 16, 32, 64, ...\} \equiv \{x \mid x = 2^n, n \in \mathbb{N}\}$
This set contains the powers of 2.

(c) $\{10, 11, 12, 13, 14, 15, 16, 17, 18, 19\} \equiv \{x \mid x \in \mathbb{Z}^+,\ 9 < x < 20\}$
This set contains positive integers greater than 9, but less than 20.

Exercise 5

Simplify the following statements. Which variables are free and which are bound? If the statement has no free variables, say whether it is true or false.

(a) $-3 \in \{x \in \mathbb{R} \mid 13 - 2x > 1\}$

(b) $4 \in \{x \in \mathbb{R}^- \mid 13 - 2x > 1\}$

(c) $5 \notin \{x \in \mathbb{R} \mid 13 - 2x > c\}$

Solution: Recall from the discussion after **Example 1.3.2.** that "if we want to know if 5 is an element of [$\{x \mid x^2 < 9\}$] set, we simply apply the elementhood test in the definition of the set - in other words, we check whether or not $5^2 < 9$. Since $5^2 = 25 > 9$, it fails the test, so $5 \notin \{x \mid x^2 < 9\}$". We can use this discussion to assist us in understanding elementhood statements.

(a) $-3 \in \{x \in \mathbb{R} \mid 13 - 2x > 1\} \to \mathbf{(-3 \in \mathbb{R}) \wedge (13 - 2(-3) > 1)}$.

The only bound variable is x, there are no free variables and the statement is true.

(b) $4 \in \{x \in \mathbb{R}^- \mid 13 - 2x > 1\} \to \mathbf{(4 \in \mathbb{R}) \wedge (4 < 0) \wedge (13 - 2(4) > 1)}$.

The only bound variable is x, there are no free variables and the statement is false since $4 \notin \mathbb{R}^-$.

(c) $5 \in \{x \in \mathbb{R} \mid 13 - 2x > c\} \to (5 \in \mathbb{R}) \wedge (13 - 2(5) > c)$.

The above translates to 5 is an element of that set, so to express that 5 is not an element of that set, then we write

$5 \notin \{x \in \mathbb{R} \mid 13 - 2x > c\} \to \neg[\mathbf{(5 \in \mathbb{R}) \wedge (13 - 2(5) > \boldsymbol{c})}]$.

The bound variable is x and the free variable is c.

Exercise 6

Simplify the following statements. Which variables are free and which are bound? If the statement has no free variables, say whether it is true or false.

(a) $w \in \{x \in \mathbb{R} \mid 13 - 2x > c\}$

(b) $4 \in \{x \in \mathbb{R} \mid 13 - 2x \in \{y \mid y \text{ is a prime number}\}\}$

(c) $4 \in \{x \in \{y \mid y \text{ is a prime number}\} \mid 13 - 2x > 1\}$

Solution:

(a) $w \in \{x \in \mathbb{R} \mid 13 - 2x > c\} \rightarrow \boldsymbol{(w \in \mathbb{R}) \wedge (13 - 2w > c)}$.

The bound variable is x, and the free variables are w and c.

(b) $4 \in \{x \in \mathbb{R} \mid 13 - 2x \in \{y \mid y \text{ is a prime number}\}\}$

$\rightarrow \boldsymbol{(4 \in \mathbb{R}) \wedge (13 - 2(4) \in \{y \mid y} \text{ is a prime number}\boldsymbol{\})}$

$\rightarrow \boldsymbol{(4 \in \mathbb{R}) \wedge (5} \text{ is a prime number}\boldsymbol{)}$.

The bound variables are x and y, there are no free variables, and the statement is true.

(c) $4 \in \{x \in \{y \mid y \text{ is a prime number}\} \mid 13 - 2x > 1\}$

$\rightarrow \boldsymbol{(4 \in \{y \mid y} \text{ is a prime number}\boldsymbol{\}) \wedge (13 - 2(4) > 1)}$

$\rightarrow \boldsymbol{(4} \text{ is a prime number}\boldsymbol{) \wedge (5 > 1)}$.

The bound variables are x and y, there are no free variables, and the statement is false since 4 is not a prime number.

Exercise 7

What are the truth sets of the following statements? List a few elements of the truth set if you can.

(a) Elizabeth Taylor was once married to x.

(b) x is a logical connective studied in Section 1.1.

(c) x is the author of this book.

Solution:

(a) Elizabeth Taylor was famously married eight times to seven different men, so the truth set of this statement includes all of her former husbands.

$$\begin{aligned}\{x \mid \text{Elizabeth Taylor was once married to } x\} \equiv \{ & \\ & \text{Conrad Hilton Jr.,} \\ & \text{Michael Wilding,} \\ & \text{Mike Todd,} \\ & \text{Eddie Fisher,} \\ & \text{Richard Burton,} \\ & \text{John Warner,} \\ & \text{Larry Fortensky}\}\end{aligned}$$

(b) $\{x \mid x \text{ is a logical connective studied in Section 1.1}\} \equiv \{\wedge, \vee, \neg\}$

(c) $\{x \mid x \text{ is the author of this book}\} \equiv \{\text{Daniel J. Velleman}\}$

Exercise 8

What are the truth sets of the following statements? List a few elements of the truth set if you can.

(a) x is a real number and $x^2 - 4x + 3 = 0$.

(b) x is a real number and $x^2 - 2x + 3 = 0$.

(c) x is a real number and $5 \in \{y \in \mathbb{R} \mid x^2 + y^2 < 50\}$.

Solution:

(a) To find the truth set, we need to solve the equation $x^2 - 4x + 3 = 0$. Using the quadratic formula, we get:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{4 \pm \sqrt{(-4)^2 - 4(1)(3)}}{2(1)} \\ &= \frac{4 \pm \sqrt{4}}{2} \\ &= \frac{4 \pm 2}{2} \\ &= 1 \text{ or } 3 \end{aligned}$$

Therefore, the truth set is:

$$\begin{aligned} &\{x \mid x \text{ is a real number and } x^2 - 4x + 3 = 0\} \\ &\quad \equiv \{1, 3\} \end{aligned}$$

(b) Similarly, solving the equation $x^2 - 2x + 3 = 0$ using the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{2 \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)} \\ &= \frac{2 \pm \sqrt{-8}}{2} \end{aligned}$$

Since $\sqrt{-8}$ is not a real number, there are no real solutions to this equation. Therefore, the truth set is:

$$\begin{aligned} &\{x \mid x \text{ is a real number and } x^2 - 2x + 3 = 0\} \\ &\quad \equiv \{\} \end{aligned}$$

(c) The statement "$5 \in \{y \in \mathbb{R} \mid x^2 + y^2 < 50\}$" is equivalent to "$x^2 + 5^2 < 50$". Solving this inequality:

$$\begin{aligned} x^2 + 5^2 &< 50 \\ x^2 + 25 &< 50 \\ x^2 &< 25 \\ -5 < x &< 5 \end{aligned}$$

Therefore, the truth set is:

$$\begin{aligned} &\{x \mid x \text{ is a real number and } 5 \in \{y \in \mathbb{R} \mid x^2 + y^2 < 50\}\} \\ &\quad \equiv \{x \mid x \in \mathbb{R} \text{ and } -5 < x < 5\} \end{aligned}$$

Some elements of this truth set include -4.9, 0, 3.14, etc.