# Comp 330 - Lecture 9 - September 28th

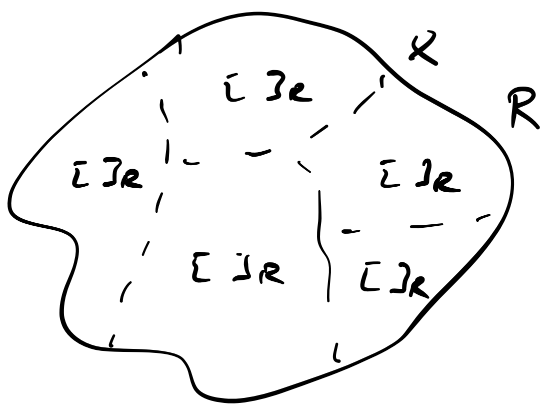
Italian expression: Mangiare la foglia

→ To eat the leaf

→ Understand quickly

## Myhill - Nerode Thm + Minimal DFA

**Recall** Given set $X$ & an eq. relation $R$ on $X$, $R$ partitions $X$ with its eq. classes.




index of $R$ is 5

**Def** Set $X$, eq. relation on $X$, $R$, then The *index* of $R$ is its number of eq. classes

**Def** $\Sigma \neq \emptyset$, eq. relation $R$ on $\Sigma^*$, $R$ is *right-invariant* if

$$\forall x,y \in \Sigma^* \quad xRy \Rightarrow \forall z \in \Sigma^* \quad xz\,R\,yz$$

**Def** ($R_M$) Fix a DFA $M = (Q, \Sigma, \delta, q_0, F)$

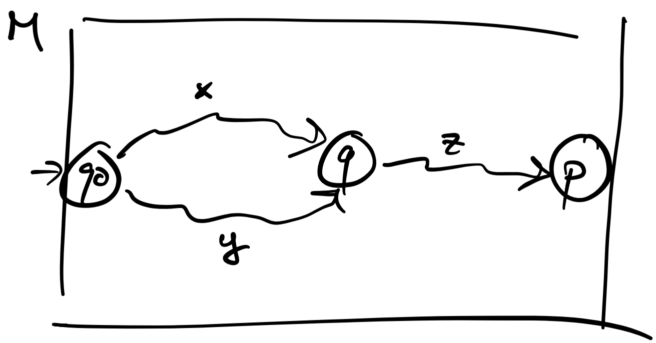
$x, y \in \Sigma^*$ $\quad x R_M y \iff \delta^*(q_0, x) = \delta^*(q_0, y)$

Note the difference with $p \approx q$ → This was a relation on states of a machine.

**Remark** 1) $R_M$ is an eq. relation

2) $R_M$ is right-invariant

$x, y \in \Sigma^*$, $x R_M y$, $z \in \Sigma^*$, $xz, yz$




$\delta^*(q_0, xz)$
$\parallel$
$\delta^*(q_0, yz)$

$xz R_M yz$

→ The Myhill-Nerode relation

**Def** ($\equiv_L$) $\Sigma \neq \emptyset$, $L \subseteq \Sigma^*$

$x, y \in \Sigma^*$ $\quad x \equiv_L y \iff [\forall z \in \Sigma^*$
$xz \in L$
$\Updownarrow$
$yz \in L]$

**Remark** 1. $\equiv_L$ is an equivalence relation

2. $\equiv_L$ is right-invariant

Pf $x, y \in \Sigma^*$, $x \equiv_L y$, pick $z \in \Sigma^*$,

W.T.S. $xz \equiv_L yz$

W.T.S. $\forall u \in \Sigma^*$ $\underset{\in L}{(xz)u} \iff \underset{\in L}{(yz)u}$

($\Rightarrow$) Assume $(xz)u \in L$

$\Rightarrow \quad x(zu) \in L$ (by associativity).

$\Rightarrow \quad y(zu) \in L$ (by def of $\equiv_L$)

$\Rightarrow \quad (yz)u \in L$

($\Leftarrow$) Same. $\blacksquare$

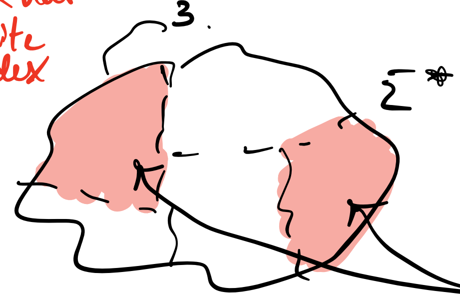
<u>Thm</u> (M-N Thm) Given $\Sigma \neq \phi$, $L \subseteq \Sigma^*$, the following 3 statements are equivalent

1. The language $L$ is accepted by some DFA $M$ $(L(M) = L)$
2. $L$ is the union of some of the eq. classes of some right-inv eq relation $\underline{R}$ of finite index.
3. $\equiv_L$ has finite index.

(1) $L \subseteq \Sigma^*$

DFA $M$ $\boxed{L(M) = L}$

(2) $R$ : 1. eq. relation
2. right-inv
3.
4. R has finite index




(3) $\equiv_L$

$[\ ]_{\equiv_L}$
$[\ ]_{\equiv_L}$
$\vdots$
$[\ ]_{\equiv_L}$

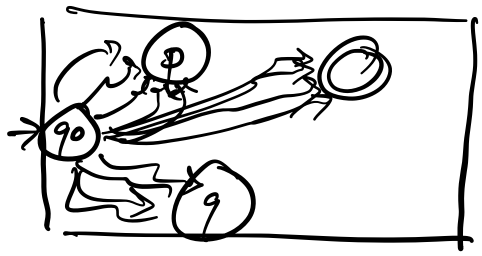
$n < \infty$

$L$ = Union of

<u>Proof</u> (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (1)

(1) $\Rightarrow$ (2) Assume (1), show (2). Find a relation $R$ which satisfies all the properties in (2). $R = R_M$.

1. Eq relation ✓
2. Right-inv ✓
3. M $\quad L(M) = L$




What are the eq classes of $R_M$?

$q \in Q$, $S_q = \{ x \in \Sigma^* : \delta^*(q_0, x) = q \}$

$L = \{ x \in \Sigma^* : \exists f \in F, \delta^*(q_0, x) = f \}$

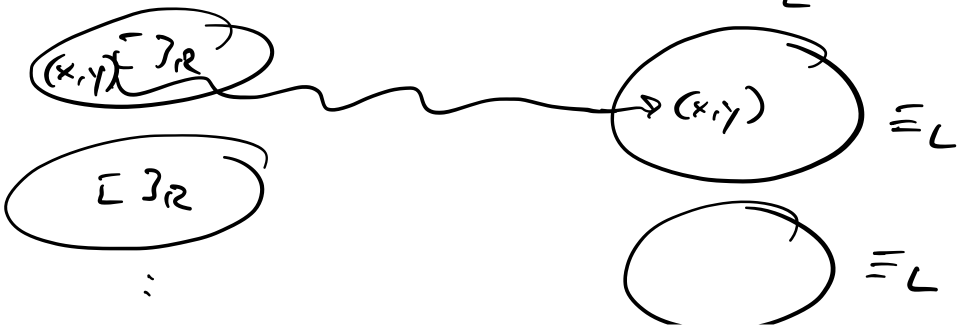
$= \bigcup_{f \in F} S_f$

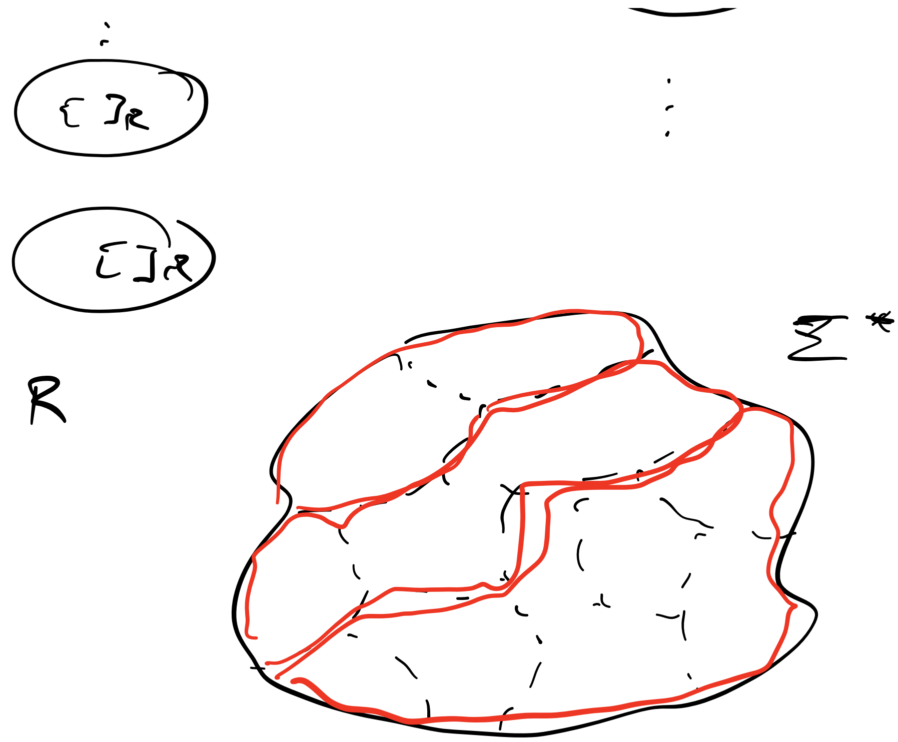
4. $R_M$ is of finite index b/c M has a finite # of states

(2) => (3) Assume (2) => show (3), $\equiv_L$ has finite index

index of $\equiv_L$ $\leq$ index of R (where R satisfies 2)

R. $\qquad\qquad\qquad \equiv_L$






$x, y \in \Sigma^*$, $\quad xRy$, $\quad$ W.T.S. $\quad x \equiv_L y$

W.T.S. $\quad \forall z \in \Sigma^* \quad xz \in L \Leftrightarrow yz \in L$

($\Rightarrow$) $\quad z \in \Sigma^*$, $\quad xz \in L$

$\Rightarrow$ $xz$ is in an eq class of $R$ that makes up $L$

$\Rightarrow$ $yz$ is in the same eq. class. because it's right-invariant

$\Rightarrow$ $yz \in L$

($\Leftarrow$) Identical


This shows that if $xRy \Rightarrow x \equiv_L y$

$$R \subseteq \equiv_L$$

(3) $\Rightarrow$ (1) Assume $\equiv_L$ has finite index

$\Rightarrow$ $\exists$ DFA s.t. $L(M) = L$

Create a DFA $M' = (Q', \Sigma, \delta', q_0', F')$ based on the eq. classes of $\equiv_L$

$Q' :=$ the eq classes of $\equiv_L \Leftrightarrow \Sigma^* / \equiv_L$

$q_0' := [\varepsilon]_{\equiv_L}$

$F' := \{ [x]_{\equiv_L} : x \in L \}$

$\delta'([\underline{x}]_{\equiv_L}, a) = [x \cdot a]_{\equiv_L}$

$\delta'([y]_{\equiv_L}, a) = [ya]_{\equiv_L}$ This is ok by right-inv of $\equiv_L$

<u>Claim 1</u> $\delta'^*([x]_{\equiv_L}, y) = [xy]_{\equiv_L}$

<u>Proof</u> By ind on $|y|$.

<u>Claim 2</u> $L(M') = L$

$x \in L(M') \Leftrightarrow \delta'^*(q_0', x) \in F'$

$\Leftrightarrow \delta'^*([\varepsilon]_{\equiv_L}, x) \in \{ [x]_{\equiv_L} : x \in L \}$

$\Leftrightarrow [\varepsilon \cdot x]_{\equiv_L} \in \{ [x]_{\equiv_L} : x \in L \}$

$\Leftrightarrow [x]_{\equiv_L} \in \{ [x]_{\equiv_L} : x \in L \}$

$\Leftrightarrow x \in L$.

$\blacksquare$

<u>Remark</u> : Remarkably, $M'$ was a minimal DFA that accepted $L$!

<u>Prop</u> The DFA $M'$ from $(3) \Rightarrow (1)$ was a minimal DFA.

<u>Pf</u> Consider DFA $M$ s.t. $L(M) = L(M')$.
Recall the eq. relation $R_M$.
In $(2) \Rightarrow (3)$, we showed that any relation $R$ satisfying (2)

$$\begin{array}{ccc} \text{index of } R & \geq & \text{index of } \equiv_L \\ \| & & \| \\ \text{index of } R_M & \geq & \underline{\#\text{ states in } M'} \\ \| & & \\ \underline{\#\text{ states of } M} & & \end{array}$$

By def $M'$ is a minimal DFA. $\square$

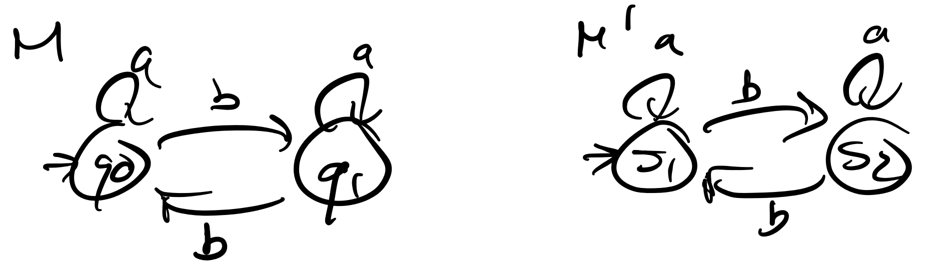
<u>Implication</u> Given a reg lang $L$, the # of states in a minimal DFA $M$ s.t. $L(M) = L$ <u>is</u> the index of $\equiv_L$.

We can use the construction of $(3) \Rightarrow (1)$ to show that minimal DFA are unique i.e. take any min DFA $M$ s.t. $L(M)=L$ show that it is the same as $M'$ (the one based on $\equiv_L$)

**Def** Two DFAs $M=(Q, \Sigma, \delta, q_0, F)$ & $M'=(Q', \Sigma, \delta', q_0', F')$ are isomorphic if $\exists$ bijection $\phi: Q \to Q'$ s.t.

1. $\phi(q_0) = q_0'$
2. $q \in F \iff \phi(q) \in F'$
3. $\phi(\delta(q,a)) = \delta'(\phi(q), a)$

[ $M$ & $M'$ are the same if $\exists$ a relabeling function from states $M$ to $M'$ ]




$\phi(q_0) = s_1 \quad \phi(q_1) = s_2$

<u>**Prop 2**</u> Given a reg language $L$ & the minimal DFA $M'$ from (3) => (1), any minimal DFA is isomorphic to $M'$.

**Pf** Omit. Post video.

A2Q5 Minimal DFA $M = (Q, \Sigma, \delta, q_0, F)$

b). $\delta^*(q_0, x) = q \iff \delta'^*([\varepsilon]_{\equiv_L}, x) = [x]_{\equiv_L}$

Remark Min Alg + Min DFA unique
=> Decision procedures for decision problems about FAs & reg. exps.

"Given 2 reek $M_1, M_2$, $L(M_1) = L(M_2)$?"

for x in $\Sigma^*$:
check if the match of x w/ $M_1$ = match of x w/ $M_2$.

Alg

$$\begin{array}{ccccccc} & & \text{NFA} & & \text{DFA} & & \text{Min} \\ M_1 & \to & M_1 & \to & M_1' & \to & M_1'' \\ & & \text{NFA} & & \text{DFA} & & \text{Min} \\ M_2 & \to & M_2 & \to & M_2' & \to & M_2'' \end{array}$$

check if $\exists \varphi$.

↳ A3Q1.