Comp 330 - Lec 24 - Nov 23rd

:) $\text{prime}^2 - 1 \mid 24$

:(

IE : Hai voluto la bicicletta ?
Adesso pedala !

Added a video about Lec 23
$P_M(L(M))$

Undecidable problems about CFLs

Recall DPs about CFLs which are decidable

ISIN (G, w) : "Given a CFG G & string $w \in \Sigma^*$, is $w \in L(G)$?" → CYK

EMPTY-CF : "Given a CFG G, is $L(G) = \emptyset$?" → GEN

Many simple DPs about CFLs are undecidable

ALL-CF (G) : "Given CFG G, $L(G) = \Sigma^*$?"

INTERSECT-CF(G_1, G_2) : "Given CFG G_1, G_2,

$L(G_1) \cap L(G_2) \neq \emptyset$?"

Show that $\overline{HP} \leq_m$ ALL-CF

1. Convert $I_{\overline{HP}} = \langle M, x \rangle$ To
   $I_{ALL\text{-}CF} = \underbrace{\langle G \rangle}_{\text{binary encoding CFG } G}$
2. Show M loops on $x \iff L(G) = \Sigma^*$

How can we create G? Requires computation history of M run on x. Key insight: If a TM M loops on x then there is no finite string which represents its computation history.

<u>Def</u> Given a TM M & an input string $x \in \Sigma^*$, a <u>valid</u> computation history of M run on x is a string of the form

$$\# \underbrace{\alpha_0}_{\text{starting IC}} \# \alpha_1 \# \alpha_2 \# \ldots \# \underbrace{\alpha_N}_{\text{halting IC} \to q_a \text{ or } q_r} \#$$

$\# \notin \Gamma$; α_1, α_2: ICs.

where

1) $\#$: separator token, $\# \notin \Gamma$

2) $\forall i \in 0 \ldots N$, $\alpha_i \in \Delta^*$, $\Delta = \Gamma \cup Q$.

α_i is an instantaneous configuration

states interpreted as letters

3) $\alpha_0 = \vdash s\, x\, \square$

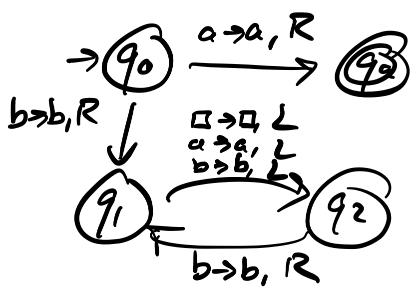
$\alpha_N = \vdash u\, q_a\, v\, \square$

OR $= \vdash u\, q_r\, v\, \square$

4) $\forall i \in 0 \ldots N-1$, α_i, α_{i+1}

$\alpha_i \xrightarrow[M]{1} \alpha_{i+1}$

Ex TM M



$x = abb$

$\# \underbrace{\vdash q_0\, abb\, \square}_{\alpha_0} \# \underbrace{\vdash a\, q_a\, bb\, \square}_{\alpha_1 = \alpha_N} \#$

$x = ba$ α_0 α_1

$\# \vdash q_0\, ba\, \square \# \vdash b\, q_1\, a\, \square \#$

$\vdash q_2\, ba\, \square \# \ldots$

α_2

Def Given a TM M & input string $x \in \Sigma^*$, $\Delta = Q \cup \Gamma \cup \{\#\}$

VALCOMPS $(M, x) = \{ \alpha : \alpha$ is a valid computation history of M running on x $\}$

$\subseteq \Delta^*$

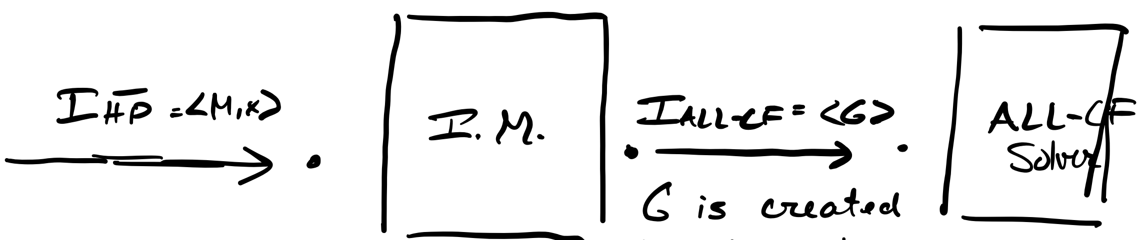
Key observation: $A = B \iff \overline{A} = \overline{B}$

M loops on $x \iff \text{VALCOMPS}(M, x) = \emptyset$

$\iff \overline{\text{VALCOMPS}(M, x)} = \overline{\emptyset}$

$\iff \overline{\text{VALCOMPS}(M, x)} = \Delta^*$

How is this relevant to $\overline{HP} \leq_m$ ALL-CF?



G is created by I.M. s.t. $L(G) = \overline{\text{VALCOMPS}}$

$\text{ANS}(I_{ALLCF}) = \text{Yes} \Rightarrow L(G) = \Delta^*$

$\Rightarrow \overline{\overline{\text{VALCOMPS}(M, x)}} = \overline{\Delta^*}$

$\Rightarrow \text{VALCOMPS}(M, x) = \emptyset$

$\Rightarrow M$ loops on x

$\Rightarrow \text{ANS}(I_{\overline{HP}}) = \text{Yes}$

$\text{ANS}(I_{ALLCF}) = \text{No} \Rightarrow L(G) \neq \Delta^*$

$$\Rightarrow \text{VALCOMPS}(M, x) \neq \emptyset$$
$$\Rightarrow M \text{ halts on } x$$
$$\Rightarrow \text{ANS}(I_{HP}) = \text{No}$$

Show that $\overline{\text{VALCOMPS}}$ is CF.

To do this, we list the conditions for $\alpha \in \Delta^*$ to be a valid computation history i.e. $\alpha \in \text{VALCOMPS}(M, x)$ $\in \Sigma^*$

c_1) $\alpha = \# \alpha_0 \# \alpha_1 \# \dots \# \alpha_N \#$

$\# \in \Delta$ $\quad \alpha_i \in (\Delta - \{\#\})^*$
$= (\Gamma \cup Q)^*$

Tape alphabet of T.M.

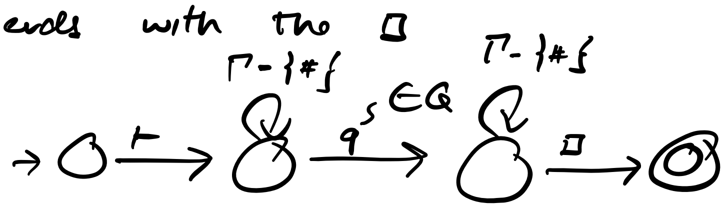
$\boxed{\Delta = Q \cup \Gamma \cup \{\#\}}$

Described with a reg exp:

$r = \# \Delta^* \# \Delta^* \dots$

$(a + b + \square + q_0 + q_1 + q_2 + q_a + q_r)^*$

c_2) Every $\alpha_i \in (\Delta - \{\#\})^*$ contains exactly one $q \in Q$, starts $\vdash$ and ends with the $\square$



c_3) $\alpha_0 \in \{\vdash s u \square : u \in (\Gamma - \{\#\})^*\}$

$\rightarrow$ Reg exp.

c_4) α_N must be a halting config

$$\alpha_N \in \{ \vdash u q_a v \sqcup : u, v \in (\Gamma - \{\#\})^* \} \cup \{ \vdash u q_r v \sqcup : u, v \in (\Gamma - \{\#\})^* \}$$

→ Regexp

c_5) $\forall i \in 0 \ldots N-1$ α_i, α_{i+1} must follow $\alpha_i \xrightarrow[M]{1} \alpha_{i+1}$

$$\# \underbrace{u\,a\,q\,b\,v\,\#}_{\alpha_i} \underbrace{u\,a\,b\,p\,v}_{\alpha_{i+1}} \#$$

$\{ww : w \in \{a,b\}^*\}$ is not CF

$$\delta(q, b) = (p, b, R)$$

$$\mathrm{VALCOMPS}(M,x) = \{\alpha \in \Delta^* : c_1 \wedge \ldots \wedge c_5\}$$
$$= \{\alpha \in \Delta^* : c_1\} \cap \ldots \cap \{\alpha \in \Delta^* : c_5\}$$
$$= L_1 \cap \ldots \cap L_5$$

$$\overline{\mathrm{VALCOMPS}(M,x)} = \underbrace{\underset{\text{REG}}{\overline{L_1}} \cup \underset{\text{REG}}{\overline{L_2}} \cup \ldots \underset{\text{REG}}{\overline{L_4}}}_{\text{REG}} \cup \underset{\text{is CF}}{\overline{L_5}}$$

Show that $\overline{L_5}$ is CF: Design a PDA which accepts $\overline{L_5}$

$\alpha \in \overline{L_5} \Rightarrow \exists\, \alpha_i, \alpha_{i+1}$ $\quad \# \alpha_i \# \alpha_{i+1} \#$

$\alpha_i \underset{M}{\overset{1}{\not\to}} \alpha_{i+1}$

$\hookrightarrow$ PDA accept

$\alpha \notin \overline{L_5} \Rightarrow$ if no such inconsistency occurs.

$\hookrightarrow$ PDA reject

Examples of inconsistencies?

<u>Ex1</u> $\quad \# \underbrace{a\,b\,b\,q_1\,a \;\#\; b}\,b\,b\,q_2\,a\,\# \rightarrow$ miscopying

$\alpha_i \qquad \alpha_{i+1}$

<u>Ex2</u> $\quad \#\, a\,b\,b\,q_1\,a \;\#\; a\,b\,q_2\,b\,b\,\#$

$\delta(q_1, a) = (q_2, a, L)$

PDA needs to look at a 3 letter window (3 l.w.) to find a pair of inconsistent α_i, α_{i+1}:

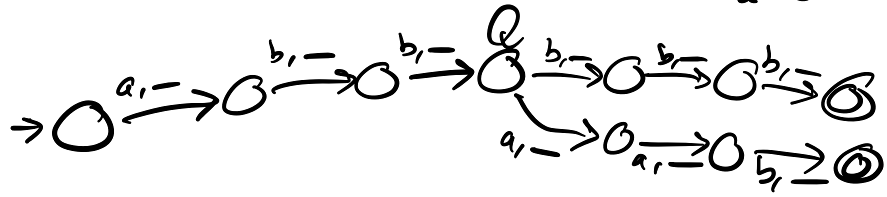
1) At most one of the 3 l.w. contained a state $q \in Q$ and a miscopy occurs (Ex 1)
2) Both of the 3 l.w. contain $q \in Q$ and δ is violated (Ex 2)

<u>Idea 1</u> $\quad \Gamma, Q \;\&\; \delta$ are all finite

$\rightarrow$ The PDA can hardcode every possible inconsistency

Inconsistency 1

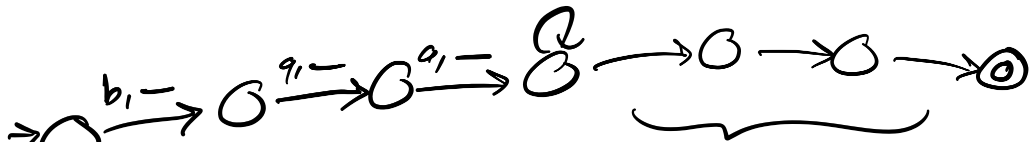
$\overbrace{\# \ldots\ abb \ldots}^{\alpha_i} \overbrace{\# \ldots\ \begin{matrix} bbb \\ aab \end{matrix} \ldots \#}^{\alpha_{i+1}}$



Inconsistency 2

$\# \ \ldots \ b q_1 a \ldots \# \ \ldots \ q_2 bb \ldots \#$

$\delta(q_1, a) = (q_2, a, L)$



All of the branches except for $q_2 ba$

Problem with idea 1: False positives

$\# \ aaa \ \ldots \quad \# \ aaab$

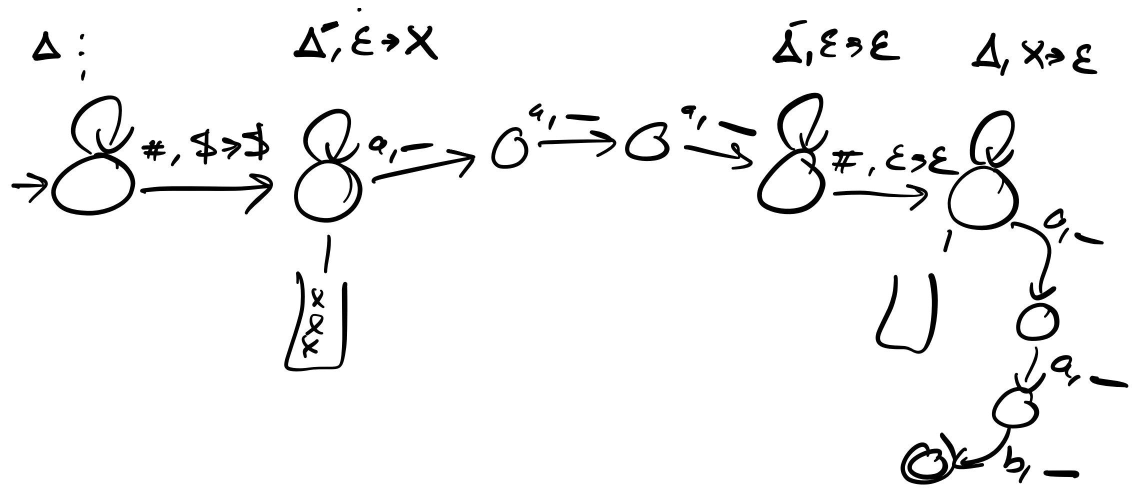
Idea 2 Use the stack to ensure that

$\# \ u_1 \ x \ u_2 \ \# \ w_1 \ y \ w_2 \ \#$

inconsistent 3.L.w.

x & y are the same distance from their respective #

$\ni \#\in \qquad \ni \# \in$



What's the point? PDA accepts

$\overline{L_5} \rightarrow$ CF

$$\overline{VALCOMPS} = \underbrace{\underbrace{\overline{L_1} \cup \overline{L_2} \dots}_{REG} \cup \underbrace{\overline{L_5}}_{CF}}_{CF}$$

Given $I_{\overline{HP}}$ create a PDA which accepts $\overline{VCOMPS}$ → create a CFG which accepts $\overline{VCOMPS}$ → $L(G) = \overline{VCOMPS} = \Delta^*$ $\Longleftrightarrow$ M loops on x.

⇒ ALL-CF is undecidable □.