Theory of Computation

Tutorial - The Pumping Lemma (for Regular Languages)

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Plan for today

1. Non-regular languages
2. The Pumping Lemma
3. Using the PL
4. Using the closure properties (again)

Non-regular languages

What is a “non-regular” language?

The Pumping Lemma

The Pumping Lemma

Lemma. Let L be an infinite regular language. Then there exists some positive integer m such that any $w \in L$ with $|w| \geq m$ can be decomposed a $w = xyz$ with $|xy| \leq m$, and $|y| \geq 1$ such that $w_i = xy^iz$ is also in L for all $i = 0, 1, 2, \ldots$.

Breaking Down the PL

Using the PL

Using the PL

Idea. If L regular then L satisfies the conditions of the PL $\equiv$ If L does not satisfy the conditions of the PL then L is *not* regular.

What is the negation of the conditions of the PL ?

Using the PL

Imagine a game where *you* play against some *opponent*. The point of the game is to win, i.e. to fool your opponent. The game follows the following 4 moves.

Move 1. **Opponent's move.** Chooses an $m > 0$. You have no knowledge of what this m could be.

Move 2. **Your move.** Pick a *particular* string w with $|w| \geq m$.

Move 3. **Opponent's move.** They decompose your string into xyz where $|xy| \leq m$, $|y| > 0$. There are many possible decompositions, you don't know which one your opponent picked.

Move 4. **Your move.** You pick a *particular* $i \geq 0$ and show that $xy^iz \notin L$. You have defeated your opponent!

Exercise

Exercise. Prove that $L = \{0^n 1^j, n > j \geq 0\}$ is not regular.

Exercise

Exercise. Prove that $L = \{a^n : n = k^3\}$ is not regular.

Exercise

Exercise. Is the language $L = \{a^n b^l : n/l \in Z\}$ regular? Prove your claim.

Exercise

Exercise. Is the language $L = \{w \in \{a, b\}^* : n_a(w) + n_b(w) = 4n_b(w)\}$ regular? Prove your claim.

Exercise

Exercise. Is the language $L = \{a^{2n} : n \geq 0\} \cup \{a^{2^n} : n \geq 0\}$ regular? Prove your claim.

Exercise

Exercise. Let $L = \{a^n b^k c^{n+k} d^p : n, k, p \geq 0\}$ be a language we are trying to show is not regular using the pumping lemma. Suppose your opponent chooses an integer $m > 0$. Which of the following strings for w would be a suitable choice to show that L is not regular? Should you pump up or pump down?

a. $w = a^m b^m c^m$
b. $w = a^m b^m$
c. $w = a^m c^m$
d. $w = a^m b^{2m} c^m d^m$

Exercise

Exercise. Show that you cannot use the pumping lemma to prove that $L = \{a^i b^j c^k : i = 0 \implies j = k\}$ is not regular.

Using the closure properties (again)

Using the closure properties

Idea. If you are trying to show that L is not regular, assume (for contradiction) that it is regular. Now "manipulate" L using the closure properties to conclude that another language L' is regular *when you know that this is not the case*.

Example. Consider the language $L = \{w \in \{a, b\}^* : n_a(w) = n_b(w)\}$. We claim this language is not regular. Suppose (for contradiction) that it is regular. Then, $L' = L \cap L(a^* b^*)$ must be regular by the closure properties. But this would mean that $L' = \{a^n b^n : n \geq 0\}$ is regular, a contradiction.

Exercise

Exercise. Prove that $L = \{w \in \Sigma^* : w \text{ has more } 0's \text{ than } 1's\}$ is not regular.

Exercise

Exercise. Prove that $L = \{w \in \Sigma^* : w \text{ has more } 0's \text{ than } 1's\}$ is not regular.

Exercise

Exercise. Use the closure properties of regular languages to prove that $L = \{a^n b^n : n \geq 335\}$ is not regular.

Exercise

Exercise. Is the language $L = \{w_1 c w_2 : w_1, w_2 \in \{a, b\}^*, w_1 \neq w_2\}$ regular? Prove your answer.