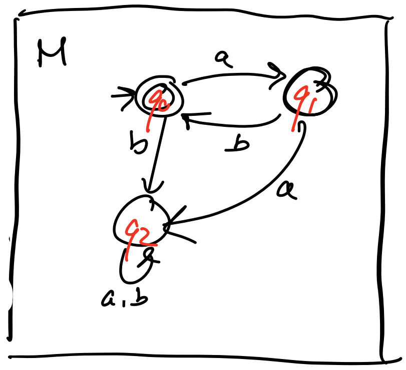
Given the following DFA M




Prove $P(M, w)$: conjunction of implications

$P(M,w)$ : " if $\underbrace{\delta^*(q_0, w) = q_0}_{p_1}$ then $\underbrace{w = (ab)^n, n \geq 0}_{q_1}$

AND if $\underbrace{\delta^*(q_0, w) = q_1}_{p_2}$ then $\underbrace{w = (ab)^n a}_{q_2}$, $n \geq 0$

AND if $\underbrace{\delta^*(q_0, w) = q_2}_{p_3}$ then $\underbrace{\forall n \geq 0 \; w \neq (ab)^n \text{ and } w \neq (ab)^n a}_{q_3}$ "

$P(M,w)$ : " $\underbrace{p_1 \to q_1 \text{ AND } p_2 \to q_2 \text{ AND } p_3 \to q_3}$ "

Proof by induction on $|w|$.

$\boxed{p \to q}$

BC $|w| = 0 \Rightarrow w = \varepsilon$

$p_2$ : " $\delta^*(q_0, w) = q_1$ "  By def of $\delta^*$

$$\delta^*(q_0, w) = \delta^*(q_0, \varepsilon) = q_0$$

$p_2$ is false so $\boxed{p_2 \to q_2}$ is true

$p_3$ : " $\delta^*(q_0, w) = q_2$ " $\to$ via similar argument

$\boxed{p_3 \Rightarrow q_3}$ is true.

$p_1$ : "$\delta^*(q_0, w) = q_0$" $\rightarrow$ This $p_1$ is true for $w = \varepsilon$

Need to show that $q_1$ is also true

$q_1$ : "$w = (ab)^n, \; n \geq 0$" $\quad w = \varepsilon = (ab)^0$ ↳ By Def

$\therefore$ $q_1$ is true $\therefore$ $\boxed{p_1 \rightarrow q_1}$ is true

**IH** Assume $P(M, w)$ $\forall w \in \Sigma^*, \; |w| = n$ $\quad (n \in \mathbb{N})$

$=$ "$p_1 \Rightarrow q_1 \wedge p_2 \Rightarrow q_2 \wedge p_3 \Rightarrow q_3$"

**IS** W.T.S. $\forall w \in \Sigma^*, \; |w| = n+1, \; P(M, w)$

Pick some arbitrary $w \in \Sigma^*, \; |w| = n+1$

$w = x\sigma, \; x \in \Sigma^*, \; \sigma \in \Sigma$

$|x| = n$

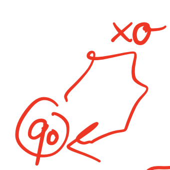
Need to show that for $w$, $\boxed{p_1 \rightarrow q_1} \wedge \underline{p_2 \Rightarrow q_2} \wedge \underline{p_3 \rightarrow q_3}$

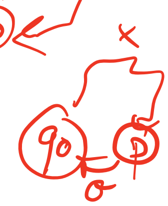
$\underline{p_1 \rightarrow q_1}$ : "if $\delta^*(q_0, w) = q_0$ then $w = (ab)^n, \; n \geq 0$"

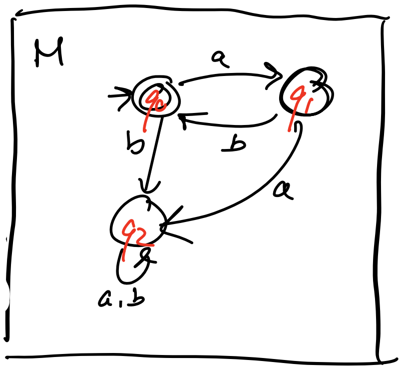
Assume $p_1$ is true $\Rightarrow \delta^*(q_0, w) = q_0$

$\Rightarrow \delta^*(q_0, x\sigma) = q_0$

Def of $\delta^*$ $\Rightarrow \delta(\underbrace{\delta^*(q_0, x)}_{\text{state}}, \underbrace{\sigma}_{\text{symbol}}) = q_0$






$\Rightarrow$ by construction of $M$

$\delta^*(q_0, x) = q_1 \wedge \sigma = b$

IH

$\Rightarrow$ Recall that in $P(M, w)$

$p_2 \to q_2$ : "if $\delta^*(q_0, w) = q_1$ then $w = (ab)^n a \quad n \geq 0$"

$(|w| = n)$

$\Rightarrow \quad \underbrace{x = (ab)^n a}_{\sigma = b}$ by IH

$\Rightarrow \quad w = x\sigma = (ab)^n a \cdot b$

$= (ab)^{n+1} \qquad m = n+1$

$= (ab)^m$

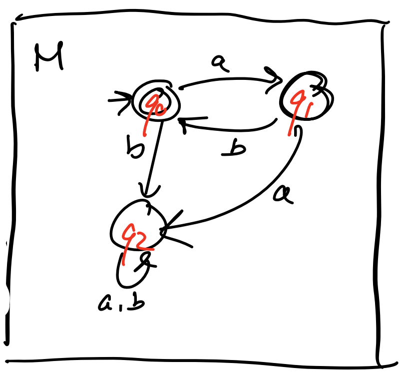
$\underline{p_2 \to q_2}$ "if $\delta^*(q_0, w) = q_1$ then $w = (ab)^n a, n \geq 0$"

Assume $p_2$ is true $\Rightarrow \delta^*(q_0, w) = q_1$

$\Rightarrow \delta^*(q_0, x\sigma) = q_1$

$\Rightarrow \delta(\delta^*(q_0, x), \sigma) = q_1$

$\Rightarrow \delta^*(q_0, x) = q_0$ AND $\sigma = a$




by analyzing $M$

$\Rightarrow \quad x = (ab)^n, n \geq 0$

by IH

$\Rightarrow \quad w = x\sigma = (ab)^n a, n \geq 0$

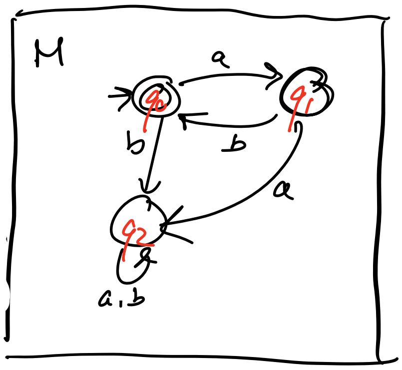
$p_3 \to q_3$ "if $\delta^*(q_0, w) = q_2$ then $\forall n \geq 0$ $w \neq (ab)^n$, AND $w \neq (ab)^n a$"

Assume $p_3$ is true $\Rightarrow \delta^*(q_0, w) = q_2$

$\Rightarrow \delta^*(q_0, x\sigma) = q_2$

$\Rightarrow \delta(\delta^*(q_0, x), \sigma) = q_2$

$\Rightarrow$ Either




(A) $\delta^*(q_0, x) = q_0$, $\sigma = b$

OR (B) $\delta^*(q_0, x) = q_1$, $\sigma = a$

OR (C) $\delta^*(q_0, x) = q_2$, $\sigma = a$ or $b$

(A) $x = (ab)^n$, $n \geq 0$
$w = x\sigma = (ab)^n b$

(B) $x = (ab)^n a$, $n \geq 0$
$w = x\sigma = (ab)^n aa$

(C) $x \neq (ab)^n$ & $x \neq (ab)^n a$ for any $n$, appending $\sigma$ to $x$ will not change this for $w$

Altenatively, could have said

if $\delta^*(q_0, w) = q_2$ then $w = (ab)^n b y$, $n \geq 0$, $y \in \Sigma^*$
OR $(ab)^n aa y$, $n \geq 0$, $y \in \Sigma^*$