Comp 330 - Lecture 14 - Oct 19th

IE: Morto un papa se ne fa
un altro
Life goes on

Makeup midterm: Oct 26th 8:30AM
in MCMED 325

Midterm eval :

A3 due tomorrow night, release A4 tomorrow
↳ Due Friday at Midnight

Grammars continued

Grammars are language generators.

$$L_{RLG} \subset L_{CFG} \subset L_{CS} \subset L_{UNR.}$$

Exercise What is language generated by the following RLG G.

$G = (V = \{S, A, B\}, S, T = \{a, b, c\}, P)$

P:

$S \to abS \mid A$

$A \to bbA \mid bbB$

$B \to ccB \mid c \mid cS$

$S \to abS \to ab\,abS \to ab\,ab\,abS$

$\xrightarrow{*} (ab)^n S$

$\to (ab)^n A$

$\to (ab)^n (bb)^k A$

$\to (ab)^n (bb)^k bb\,B$

$\to (ab)^n (bb)^k bb\,(cc)^m B$

$\to (ab)^n (bb)^k bb\,(cc)^m c$

$$L(G) = L\left[\left((ab)^*(bb)^*bb\,(cc)^*c\right)^* \cdot \left((ab)^*(bb)^*bb\,(cc)^*c\right)\right]$$

<u>Proof technique</u> Formally showing that $L \in L_{CF}$

1. Design a CFG G s.t. $L(G) = L$
2. Prove the correctness of G. By strong induction
   A. $L(G) \subseteq L \Rightarrow$ Induction on the # of derivation steps.
   B. $L \subseteq L(G) \Rightarrow$ Induction on the length of $w \in L$.

<u>Ex</u> Show $L = \{ w \in \{a,b\}^* : n_a(w) = n_b(w) \}$ is CF.

1. Design a CFG G s.t. $L(G) = L$.

Base case : $S \to \varepsilon$

Recursive case :

$w = aabb \in L$

$w = bbaa\,abba \in L.$

Imagine $w \in L$ $\quad |w| = n_a + n_b = 2 \cdot n_a$

$w = \sigma_1 \sigma_2 \sigma_3 \ldots \sigma_{2k-1} \sigma_{2k} \quad k \geq 1$

$\sigma_i \in \{a, b\}$

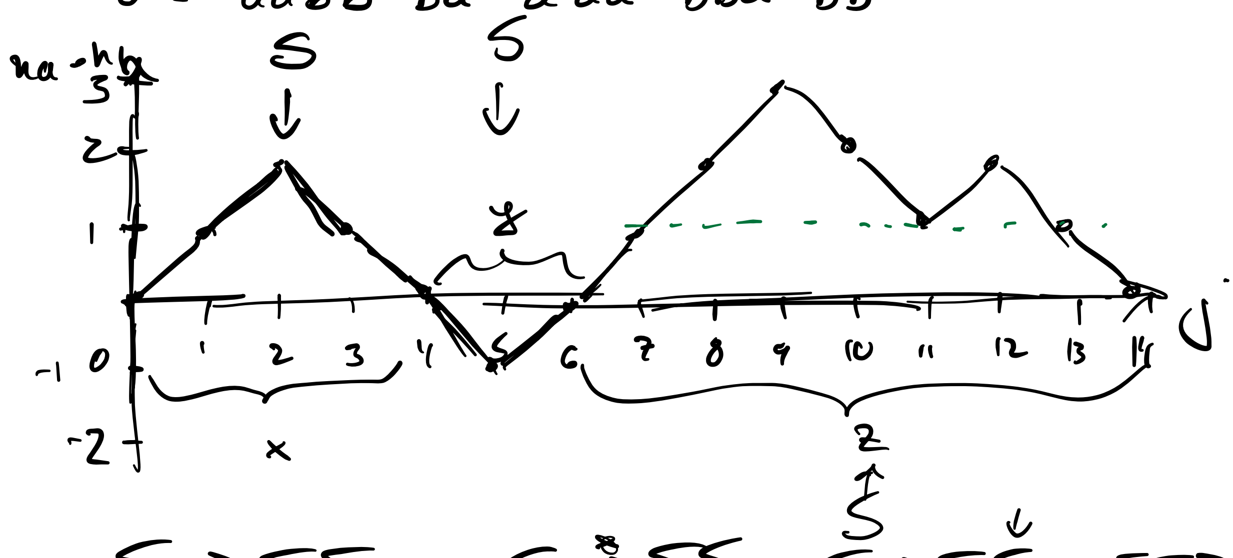
inclusive

Let's make a plot $\quad n_a(\sigma_{0:j}) - n_b(\sigma_{0:j})$

vs $\quad j$

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

$w =$ aabb ba aaa bba bb



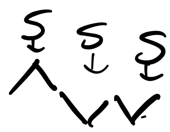
$S \to SS \qquad S \to SS \qquad S \to SS \to SSS.$

$S \to aSb \qquad S \to bSa$

Altogether: $G = (V=\{S\}, S, T=\overset{\Sigma}{\overset{\shortparallel}{\{a,b\}}}, P)$

P $\quad S \to \varepsilon \mid SS \mid aSb \mid bSa$.

$w = abbaba$



$$
\begin{aligned}
S \to SS &\to SSS \\
&\to aSbSS \\
&\to abSS \\
&\to abbSaS \\
&\to abbaS \\
&\to abbabSa \\
&\to abbaba.
\end{aligned}
$$

2. Prove the correctness of G.

$G = (V=\{S\}, S, T=\overset{\Sigma}{\overset{\shortparallel}{\{a,b\}}}, P)$

$S \to \varepsilon \mid SS \mid aSb \mid bSa$.

$L(G) \subseteq L$: $\underset{P(n)}{\overset{\shortparallel}{\{}}$ If $S \xrightarrow[G]{n} w$ & $w \in T^*$ then $w \in L.\}$

Pf By strong induction on n.

BC $n=1$ $S \xrightarrow[G]{1} w$, $w \in T^*$

then $w \in L$.

By construction $w = \varepsilon \in L$.

$n=2$ $S \xrightarrow[G]{2} w$, $w \in T^*$

$S \xrightarrow{1} SS \xrightarrow{2} \varepsilon$ x

~~$SS \to \varepsilon$~~

$S \to \varepsilon$

$\xrightarrow{1} aSb \to \boxed{ab} \in L$

$\xrightarrow{1} bSa \to \boxed{ba} \in L$

IH For some $n \in \mathbb{N}$, $n \geq 2$ &

$\forall k \in \mathbb{N}$, $1 \leq k \leq n$, if

$S \xrightarrow[G]{k} w \in T^*$ then $w \in L$.

assume this.

IS. Show for $n+1$. If $S \xrightarrow[G]{n+1} w \in T^*$

then $\underline{w \in L}$

w.t.s

$S \xrightarrow[G]{n+1} w \in T^* \Rightarrow S \xrightarrow{1} \begin{cases} \varepsilon \text{ x} \\ SS \\ aSb \\ bSa \end{cases}$

}Case 2 ✓

}Case 1

Case 1 If $S \overset{1}{\Rightarrow} aSb$
bSa

Therefore
$S \overset{n+1}{\underset{G}{\Rightarrow}} axb$
S OR $\overset{n+1}{\underset{G}{\Rightarrow}} bxa$

S has n derivations To produce a string $x \in T^*$
$S \overset{n}{\underset{G}{\Rightarrow}} x \in T^*$
by IH $x \in L$

$w = axb \Rightarrow n_a(w) = n_b(w) \Rightarrow w \in L.$

Similarly for $w = bxa$.

Case 2 If $S \overset{1}{\Rightarrow} S^1 S^2$ then from $S^1 \& S^2$ I have n derivation steps left to generate a string of Terminals. But each of $S^1 \& S^2$ must use at least one production.

n=1

Therefore $S^i \overset{t}{\underset{G}{\Rightarrow}} w_i \in T^* \quad 1 \le t \le n-1$

$\le n$

$(n+1)$
-1
-1
$= n-1$

By IH $w_1, w_2 \in L$

$S \overset{n+1}{\underset{G}{\Rightarrow}} w_1 w_2 = w$

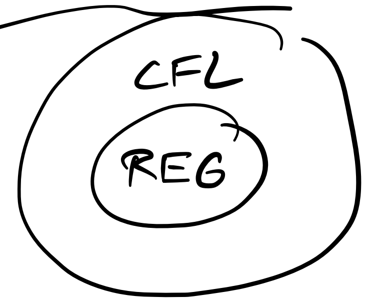
$$n_a(w) = n_a(w_1) + n_a(w_2)$$
$$= n_b(w_1) + n_b(w_2)$$
$$= n_b(w) \quad \therefore w \in L. \quad \boxtimes$$

$L \subseteq L(G)$

Claim $P(n)$ $n \in \mathbb{N}$ If $w \in L$ & $|w| = 2n$ Then $w \in L(G)$.

Proof by strong induction on n. Post notes but try as exercise.

Closure properties of CFL



Thm $\Sigma \neq \emptyset$, $L_1, L_2 \subseteq \Sigma^*$.

If L_1 & L_2 are CF then so are
1) $L_1 \cup L_2$
2) $L_1 \cdot L_2$
3) L_1^*

Spoiler: CFL are not closed under $\cap$ & $\bar{\ }$
$\{a^n b^n c^n\} \Rightarrow$ Not CF.

Pf L_1, L_2 CF $\Rightarrow \exists$ CFG $G_1 = (V_1, S_1, \Sigma, P_1)$
$G_2 = (V_2, S_2, \Sigma, P_2)$
s.t. $L(G_1) = L_1$
$L(G_2) = L_2$.

1) Consider CFG $G' = (V', S', \Sigma, P')$

$V' = V_1 \cup V_2 \cup \{S'\}$
$P' = P_1 \cup P_2 \cup \{S' \to S_1, S' \to S_2\}$

G' generates $L_1 \cup L_2$.

<u>Exercise for you</u>
(What about the infinite union?)

2) $L_1 \cdot L_2$ is CF
Consider CFG $G' = (V', S', \Sigma, P')$

$V' = V_1 \cup V_2 \cup \{S'\}$
$P' = P_1 \cup P_2 \cup \{S' \to S_1 S_2\}$

G' generates $L_1 \cdot L_2$.

3) L_1^* For you :).

<u>Ex</u> Give a CFG G which generates
$L = \{a^n b^m c^k : \underbrace{n = m+k}_{①} \text{ OR } \underbrace{k = n+m}_{②}\}$

G $S \to S_1 \mid S_2$
$w = a^{m+k} b^m c^k = a^k a^m b^m c^k$
$S_1 \to a S_1 c \mid A_1$
$A_1 \to a A_1 b \mid \varepsilon$

$w = a^n b^m c^{n+m} = a^n b^m c^{m+n} = a^n b^m c^m c^n$

$S_2 \to a S_2 c \mid A_2$

$A_2 \to b A_2 c \mid \varepsilon$.

if $n = m + K$ AND $K = n + m$ then

$m = K - n$. if $K > n$ then m is $+$ & $-$ $\boxed{\text{so } K = n \text{ \& } m = 0}$

$m = n - K$ if $K < n$ then m is $+$ & $-$

$a^n b^0 c^n$ can be generated using either S_1 or S_2.

Equivalent definitions to L(G)

CFG G

Leftmost & rightmost derivations

Ex $G: S \to SS \mid aSb \mid bSa \mid \varepsilon$

Rightmost derivation of abba

$S \to SS \to SbSa \to Sb\varepsilon a$
$\to aSbba$
$\to abba$

R rim derivation because whenever G has a choice it replaces the rim variable

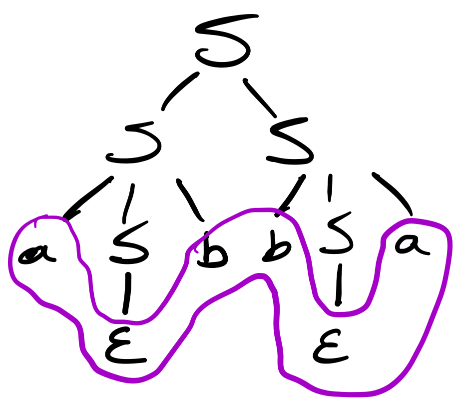
Leftmost

$S \to SS \to aSbS$
$\to a\varepsilon bS$
$\to ab\,bSa$
$\to ab\,ba$

Parse trees

$G: \quad S \to SS \mid aSb \mid bSa \mid \varepsilon$

The parse tree of G for $w = abba$



yield of the parse tree of G.