Where do ultracategories come from?

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joint work with Joshua Wrigley

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Compact Hausdorff spaces and the ultrafilter monad

Compact Hausdorff spaces enjoy two remarkable properties:

1. they are algebras for the *ultrafilter monad* $\langle \beta, \eta, \mu \rangle$ on Set;
2. the functor $\beta \colon \mathsf{Set} \to \mathsf{Set}$ is a right Kan extension.

$$\begin{tikzcd} \mathsf{FinSet} \arrow[r, hook] \arrow[d, hook] & \mathsf{Set} \\ \mathsf{Set} \arrow[ur, bend right, "\beta"'] \arrow[ur, phantom, "\Leftarrow \rho"] & \end{tikzcd}$$

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In *no-iteration form*, the monad $\langle \beta, \eta, \mu \rangle$ can be equivalently described by

- a function $\eta_X \colon X \to \beta X$ for each set X,
- and a function $(-)^* \colon \mathsf{Set}(Y, \beta X) \to \mathsf{Set}(\beta Y, \beta X)$ for each pair of sets X, Y,

satisfying equations.

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satisfying equations. In the same spirit, a β-algebra K is equivalently described by a function

$$(-)^K \colon \mathsf{Set}(Y, K) \longrightarrow \mathsf{Set}(\beta Y, K)$$

for each set Y, satisfying equations. Intuitively, $h^K(\nu) \in K$ is the *topological limit* of a family $h \colon Y \to K$ of points of K *with respect to* the ultrafilter $\nu \in \beta Y$.

Ultracategories

Ultracategories were introduced by Makkai as categories endowed with structure meant to abstract the notion of ultraproducts from model theory. Different definitions exist, but the core is that of a category C with a functor

$$(-)^C\colon [Y, C] \longrightarrow [\beta Y, C]$$

for each set Y, which Makkai calls a *pre-ultracategory*. Intuitively, $h^C(\nu)$ is the 'ultraproduct' of a family $h\colon Y \to C$ of objects of C with respect to the ultrafilter $\nu \in \beta Y$.

Makkai, *Stone duality for first-order logic*, 1987

Lurie, *Ultracategories*, 2018

Main example

$\mathrm{Mod}(\mathbb{T})$, for a coherent theory $\mathbb{T}$, is an ultracategory: $(-)^{\mathrm{Mod}(\mathbb{T})}$ maps a tuple $(M_y)_{y\in Y}$ of models and an ultrafilter $\nu \in \beta Y$ to the actual ultraproduct $\prod_\nu M_y$.

Our contribution

Question

Ultracategories categorify compact Hausdorff spaces. Can we make an axiomatisation of the notion of ultracategory emerge just as naturally, i.e. as

1. algebras for a pseudomonad on CAT,
2. whose underlying pseudofunctor is universally-induced by $\beta \colon \mathrm{Set} \to \mathrm{Set}$?

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"[Both Makkai's and Lurie's definitions of an ultracategory are] very heavy, and come together with axioms whose choice seems quite arbitrary."

Di Liberti, *The geometry of coherent topoi and ultrastructures*, 2022

"Ultraproducts are categorically inevitable."

Leinster, *Codensity and the ultrafilter monad, 2018*

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Earlier work in this direction tries to tackle the problem directly, by defining suitable pseudomonads on CAT.

Marmolejo, *Ultraproducts and continuous families of models*, 1995

Rosolini, *Ultracompletions*, talk at CT2024

Hamad, *Ultracategories as colax algebras for a pseudo-monad on* CAT, 2025

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Our answer

We will make an axiomatisation emerge by putting ultracategories in the context of *relative monad theory*. The starting point is that Lurie's definition is *almost* that of a colax algebra for a relative 2-monad over CAT.

Altenkirch, Chapman, Uustalu, *Monads need not be endofunctors*, 2015

Fiore, Gambino, Hyland, Winskel, *Relative pseudomonads, Kleisli bicategories and substitution monoidal structures*, 2018

Relative 2-monads

Definition

Let $J\colon \mathcal{B} \to \mathrm{CAT}$ be a 2-functor.

A *J-relative 2-monad* on CAT is given by

1. a category Tb, for each $b \in \mathcal{B}$,
2. a functor $\eta_b\colon Jb \to Tb$ for each $b \in \mathcal{B}$,
3. and a functor $(-)^*\colon [Jb, Tb'] \to [Tb, Tb']$ for each pair $b, b' \in \mathcal{B}$,

satisfying the conditions:

a. $\eta_b^* = \mathrm{id}_{Tb}$,

b. $f^* \circ \eta_b = f$,

c. $(f^* \circ g)^* = f^* \circ g^*$.

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The relative ultrafilter 2-monad

If J is 2-fully-faithful, every 2-monad $\langle T, \eta, (-)^* \rangle$ on $\mathcal{B}$ yields a J-relative 2-monad $\langle JT, J\eta, J(-)^* J^{-1} \rangle$ on CAT. In particular, considering the inclusion $\mathrm{Set} \hookrightarrow \mathrm{CAT}$, the ultrafilter monad $\beta\colon \mathrm{Set} \to \mathrm{Set}$ yields the *relative ultrafilter 2-monad* $\beta\colon \mathrm{Set} \to \mathrm{CAT}$.

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$$(-)^C \colon [X, C] \longrightarrow [\beta X, C]$$

for each set X;

$$\begin{tikzcd} \beta X \arrow[dr, "h^C"] & \\ X \arrow[r, "h"'] & C \end{tikzcd}$$

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$$\Gamma_h \colon h^C \eta_X \implies h$$
for each $h \colon X \to C$;

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$$\begin{tikzcd}
Y \arrow[r, "k"'] & \beta X \arrow[dr, "h^C"] & \\
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$$\begin{tikzcd}
\beta Y \arrow[dr, "k^*"] & & \\
Y \arrow[u, "\eta_Y"] \arrow[r, "k"'] & \beta X \arrow[dr, "h^C"] & \\
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for each $h \colon X \to C$;
3. a natural transformation
$$\Delta_{h,k} \colon h^C k^* \implies (h^C k)^C$$
for each $h \colon X \to C$, $k \colon Y \to \beta X$,

$$\begin{tikzcd}
\beta Y \arrow[dr, "k^*"] \arrow[ddrr, bend left, "(h^C k)^C"] & & \\
Y \arrow[u, "\eta_Y"] \arrow[r, "k"'] & \beta X \arrow[dr, "h^C"] \arrow[ur, phantom, "\Delta_{h,k} \Nearrow"] & \\
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satisfying some coherence conditions.

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\beta Y \arrow[dr, "k^*"] \arrow[ddrr, bend left, "(h^C k)^C"] & & \\
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& X \arrow[u, "\eta_X"] \arrow[r, "h"'] \arrow[ur, phantom, "\Gamma_h"] & C
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for each $h \colon X \to C, k \colon Y \to \beta X$,

satisfying some coherence conditions.

Remark

Lurie's definition is the same, but he also requires each counitor Γ_h and some coassociators $\Delta_{h,k}$ to be invertible.

However:

- ultrafunctors coincide with *pseudomorphisms*;
- left ultrafunctors coincide with *colax morphisms*.

Left oplax Kan extensions

For a weak ultracategory C, the functors $(-)^C \colon [X, C] \to [\beta X, C]$ define a *lax transformation*

$$(-)^C \colon [J-, C] \implies [J\beta-, C]$$

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where, for 2-functors $F, G \colon \mathcal{B} \to \mathcal{A}$, a lax transformation $\sigma \colon F \Rightarrow G$ is given by

- a 1-cell $\sigma_b \colon Fb \to Gb$ for each $b \in \mathcal{B}$,
- and a 2-cell

$$\begin{tikzcd} Fb \arrow[r, "\sigma_b"] \arrow[d, "Ff"'] & Gb \arrow[d, "Gf"] \arrow[dl, Rightarrow, "\sigma_f"'] \\ Fb' \arrow[r, "\sigma_{b'}"'] & Gb' \end{tikzcd}$$

for each 1-cell $f \colon b \to b'$ in $\mathcal{B}$,

satisfying some coherence and naturality conditions. Reversing the 2-cells we obtain an *oplax* transformation.

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To exhibit C as an algebra for a monad $\widetilde{\beta}$ on CAT, such a transformation should correspond to a functor

$$\widetilde{\beta} C \longrightarrow C$$

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For 2-functors $J, T\colon \mathcal{B} \to \mathcal{A}$, a *left oplax Kan extension* of T along J is a 2-functor $\widetilde{T}\colon \mathcal{A} \to \mathcal{A}$ such that for all $a, a' \in \mathcal{A}$ there are natural isomorphisms:

$$\mathbf{Lax}\,[\mathcal{B}^{\mathrm{op}}, \mathrm{CAT}]\left(\mathcal{A}(J-, a), \mathcal{A}(T-, a')\right) \cong \mathcal{A}(\widetilde{T}a, a')$$

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$$\begin{tikzcd}
\mathcal{B} \arrow[r, "T"] \arrow[d, "J"'] \arrow[dr, phantom, "\Downarrow\forall\sigma"] & \mathcal{A} \\
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$$\begin{tikzcd}
\mathcal{B} \arrow[r, "T"] \arrow[d, "J"'] \arrow[dr, phantom, "\Downarrow\zeta"] & \mathcal{A} \\
\mathcal{A} \arrow[ur, bend left, "\widetilde{T}"] \arrow[ur, bend right=60, "S"'] \arrow[ur, phantom, "\Downarrow\exists!\underline{\sigma}"] &
\end{tikzcd}$$

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$$\mathbf{Oplax}\,[\mathcal{B}, \mathcal{A}]\,(\widetilde{T}, S) \cong \mathbf{Str}\,[\mathcal{A}, \mathcal{A}]\,(T, S \circ J)$$

$$\begin{tikzcd}
\mathcal{B} \arrow[rr, "T"] \arrow[dd, "J"'] \arrow[dr, phantom, "\Downarrow\zeta"] & & \mathcal{A} \\
& {} \arrow[dr, phantom, "\Downarrow\exists!\underline{\sigma}"] & \\
\mathcal{A} \arrow[uurr, bend left=20, "\widetilde{T}" description] \arrow[uurr, bend right=40, "S"'] & &
\end{tikzcd}$$

The main result

Theorem (T. & Wrigley)

Let $\langle T, \eta, (-)^* \rangle$ *be a J-relative 2-monad on* CAT*. Suppose that:*

1. *J is 2-fully-faithful,*
2. *$\mathcal{B}$ has a terminal object 1 that is preserved by J,*
3. *$\mathcal{B}$ has oplax colimits of shape $(Jb)^{\mathrm{op}}$ for $b \in \mathcal{B}$, which J preserves.*

The left oplax Kan extension $\widetilde{T}$ of T along J carries the structure of a pseudomonad on CAT *such that the 2-categories* $\mathrm{ColaxAlg}_J(T)$ *and* $\mathrm{ColaxAlg}(\widetilde{T})$ *are isomorphic.*

T. and Wrigley, *Ultracategories via Kan extensions of relative monads*, 2025

Left oplax Kan extensions in CAT

For $\mathcal{A} = \mathrm{CAT}$, left oplax Kan extensions exist and we can describe them explicitly. For concreteness, we describe here the extension of $\beta\colon \mathrm{Set} \to \mathrm{CAT}$ along $\mathrm{Set} \hookrightarrow \mathrm{CAT}$.

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For a category C:

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If C is a weak ultracategory, recall that each functor $h\colon X \to C$ extends to a functor $h^C\colon \beta X \to C$. The corresponding colax $\widetilde{\beta}$-algebra functor $\widetilde{\beta} C \to C$ then maps

$$(X, h\colon X \to C, \nu \in \beta X) \mapsto h^C(\nu).$$

A pseudomonad structure on $\widetilde{\beta}$

In the case of $\beta\colon \mathrm{Set} \to \mathrm{CAT}$, the inclusion $\mathrm{Set} \hookrightarrow \mathrm{CAT}$:

1. is fully-faithful,
2. preserves the terminal object 1;
3. preserves small coproducts.

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In the case of $\beta\colon \mathsf{Set} \to \mathsf{CAT}$, the inclusion $\mathsf{Set} \hookrightarrow \mathsf{CAT}$ satisfies our assumptions. Therefore, $\widetilde{\beta}$ carries the structure of a pseudomonad $\langle \widetilde{\beta}, \eta^\sharp, \mu^\sharp \rangle$ on CAT, which we now describe.

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For an element $x \in X$, write $h(x) = (Y_x, k_x\colon Y_x \to C, \theta_x \in \beta Y_x) \in \widetilde{\beta}C$.

A pseudomonad structure on $\widetilde{\beta}$

In the case of $\beta\colon \mathsf{Set} \to \mathsf{CAT}$, the inclusion $\mathsf{Set} \hookrightarrow \mathsf{CAT}$ satisfies our assumptions. Therefore, $\widetilde{\beta}$ carries the structure of a pseudomonad $\langle \widetilde{\beta}, \eta^\sharp, \mu^\sharp \rangle$ on CAT, which we now describe.

- On objects, the unit $\eta^\sharp_C\colon C \to \widetilde{\beta} C$ maps $c \in C$ to $(1, c\colon 1_{\mathsf{CAT}} \to C, * \in \beta 1)$.
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2. Let $k\colon Y \to C$ be the unique functor determined by $\{\, k_x\colon Y_x \to C \,\}_{x\in X}$, i.e. $k(y \in Y_x) := k_x(y)$.
3. The functor $q\colon X \to \beta Y$ defined by $x \mapsto \beta i_x(\theta_x)$ extends to a functor $q^*\colon \beta X \to \beta Y$, so that we can consider $q^*(\nu) \in \beta Y$. Concretely, for $S \subseteq Y$,

$$S \in q^*(\nu) \iff \{\, x \in X \mid S \cap Y_x \in \theta_x \,\} \in \nu$$

Weak ultracategories II

Applying our result to the relative ultrafilter 2-monad, we conclude that weak ultracategories are pseudomonadic over CAT.

Corollary

Weak ultracategories are the colax algebras for the pseudomonad $\widetilde{\beta}$ *on* CAT *where:*

$$\begin{tikzcd}
\mathrm{FinSet} \arrow[r, hook] \arrow[d, hook] & \mathrm{Set} \arrow[r, hook] & \mathrm{CAT} \\
\mathrm{Set} \arrow[d, hook] \arrow[ur, bend right, "\beta"'] \arrow[u, phantom, shift left=6ex, "\nwarrow\rho"] \arrow[r, phantom, shift right=2ex, "\searrow\zeta"] & & \\
\mathrm{CAT} \arrow[uurr, bend right, "\widetilde{\beta}"'] & &
\end{tikzcd}$$

- $\beta\colon \mathrm{Set} \to \mathrm{Set}$ *is the right Kan extension of* $\mathrm{FinSet} \hookrightarrow \mathrm{Set}$ *along itself;*
- $\widetilde{\beta}\colon \mathrm{CAT} \to \mathrm{CAT}$ *is the left oplax Kan extension of* $\beta\colon \mathrm{Set} \to \mathrm{CAT}$ *along* $\mathrm{Set} \hookrightarrow \mathrm{CAT}$.

Future directions

- We can apply our result to other monads of interest: in particular, the *upper prime filter monad* $B\colon \mathsf{Pos} \to \mathsf{Pos}$ whose algebras are compact ordered spaces. *Prime categories*, i.e. the colax algebras for the *relative upper prime filter 2-monad* $B\colon \mathsf{Pos} \to \mathsf{CAT}$, are then the colax algebras for $\widetilde{B}\colon \mathsf{CAT} \to \mathsf{CAT}$.
  - $\rightarrow$ Connections with *positive model theory*
  - $\rightarrow$ Towards a Priestley-like duality for first-order logic

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  - $\rightarrow$ Connections with *positive model theory*
  - $\rightarrow$ Towards a Priestley-like duality for first-order logic
- Assuming they exist, left oplax Kan extensions along $J \colon \mathcal{B} \to \mathcal{A}$ determine a 2-adjunction

$$\begin{tikzcd} \mathbf{Oplax}\,[\mathcal{B}, \mathcal{A}] \arrow[rr, bend left, "\widetilde{(-)}"] \arrow[rr, phantom, "\perp"] & & \mathbf{Str}\,[\mathcal{A}, \mathcal{A}] \arrow[ll, bend left, "J^*"] \end{tikzcd}$$

  Can we find sufficient hypotheses on $\mathcal{A}$ to obtain an abstract 'unrelativisation' procedure for J-relative pseudomonads?
  - $\rightarrow$ Connections with *skew-monoidal 2-categories* and monoidal 2-functors

Thank you!

Ultracategories via Kan extensions of relative monads,
Umberto Tarantino and Joshua Wrigley, 2025, arXiv:2506.09788

Weak ultracategories are ultracategories

Lurie's ultracategories are a (proper) subclass of weak ultracategories.
However, Lurie's ultrafunctors and left ultrafunctors coincide with pseudomorphisms and colax morphisms, so that we have 2-fully-faithful embeddings:

$$\mathsf{Ult} \hookrightarrow \mathsf{WeakUlt}^{\mathsf{pseudo}} \qquad \mathsf{Ult}^{\mathsf{L}} \hookrightarrow \mathsf{WeakUlt}^{\mathsf{colax}}$$

This ensures that weak ultracategories are a good axiomatisation of ultracategories.

Theorem (Lurie)

For a small pretopos P, the evaluation functor $\mathrm{ev}\colon P \to [\mathrm{Mod}(P), \mathrm{Set}]$ *induces equivalences of categories:*

1. $P \simeq \mathsf{WeakUlt}^{\mathsf{pseudo}}(\mathrm{Mod}(P), \mathrm{Set})$;
2. $\mathrm{Sh}(P) \simeq \mathsf{WeakUlt}^{\mathsf{colax}}(\mathrm{Mod}(P), \mathrm{Set})$.

In particular, $\mathrm{Mod}\colon \mathsf{Pretop}^{\mathsf{op}} \hookrightarrow \mathsf{WeakUlt}^{\mathsf{pseudo}}$ is 2-fully-faithful.

Left oplax Kan extensions in CAT

For $\mathcal{A} = \mathrm{CAT}$, left oplax Kan extensions exist and we can describe them explicitly. For concreteness, consider $\beta\colon \mathrm{Set} \to \mathrm{CAT}$. For a category C:

- objects of $\widetilde{T}C$ are triples $(b \in \mathcal{B}, h\colon Jb \to C, \nu \in Tb)$;
- morphisms $(b, h, \nu) \to (b', h', \nu')$ in $\widetilde{T}C$ are triples of
  1. a 1-cell $f\colon b' \to b$ in $\mathcal{B}$,
  2. a natural transformation $\alpha\colon h \circ Jf \Rightarrow h'$,
  3. and an arrow $\varphi\colon \nu \to Tf(\nu')$ in Tb,

  modulo the equivalence relation generated by $(f, \alpha_f, \varphi_f) \sim (g, \alpha_g, \varphi_g)$ if there exists a 2-cell $\sigma\colon f \Rightarrow g$ in $\mathcal{B}$ such that

$$\begin{tikzcd}
Jb' \arrow[rr, bend left=40, "Jf"] \arrow[rr, bend right=30, "Jg"{description}] \arrow[rr, phantom, "\Downarrow J\sigma"] \arrow[ddr, "h'"'] & & Jb, \arrow[ddl, "h"] \\
& \Downarrow \alpha_g & \\
& C &
\end{tikzcd}
\overset{(1)}{=}
\begin{tikzcd}
Jb' \arrow[rr, bend left=40, "Jf"] \arrow[ddr, "h'"'] & & Jb \arrow[ddl, "h"] \\
& \Downarrow \alpha_f & \\
& C &
\end{tikzcd}
\qquad
\begin{tikzcd}
& Tf(\nu') \arrow[dd, "(T\sigma)_{\nu'}"] \\
\nu \arrow[ur, "\varphi_f"] \arrow[dr, "\varphi_g"'] \arrow[r, phantom, "(2)"] & \\
& Tg(\nu')
\end{tikzcd}$$

A pseudomonad structure on $\widetilde{T}$

Under our assumptions, $\widetilde{T}$ carries the structure of a pseudomonad $\langle \widetilde{T}, \eta^\sharp, \mu^\sharp \rangle$ on CAT.

- On objects, the unit $\eta^\sharp_C \colon C \to \widetilde{T}C$ maps $c \in C$ to $(1_{\mathcal{B}}, c \colon 1_{\mathrm{CAT}} \to C, \eta_{1_{\mathcal{B}}}(*))$.
- On objects, the multiplication $\mu^\sharp_C \colon \widetilde{T}^2 C \to \widetilde{T}C$ acts by

$$\mu^\sharp_C(b, h \colon Jb \to \widetilde{T}C, \nu) = (\ell, a \colon J\ell \to C, Q^*\nu).$$

For an object $x \in Jb$, write $h(x) = (Rx, a_x \colon JRx \to C, \nu_x)$.

For an arrow $g \colon x \to y \in Jb$, write $h(g) = (Rg, \gamma_g \colon a_x \circ JRg \Rightarrow a_y, \psi_g)$.

1. Let $\ell \in \mathcal{B}$ be the oplax colimit of $R \colon (Jb)^{\mathrm{op}} \to \mathcal{B}$, with universal cocone:

$$\begin{tikzcd} Rx \arrow[dr, bend right, "c_x"'] & & Ry \arrow[ll, "Rg"'] \arrow[dl, bend left, "c_y"] \arrow[dl, phantom, shift right=6, "\Leftarrow \lambda_g"] \\ & \ell & \end{tikzcd}$$

2. As J preserves oplax colimits and C is an oplax cocone of $JR \colon (Jb)^{\mathrm{op}} \to \mathrm{CAT}$

$$\begin{tikzcd} JRx \arrow[dr, bend right, "a_x"'] & & JRy \arrow[ll, "JRg"'] \arrow[dl, bend left, "a_y"] \arrow[dl, phantom, shift right=6, "\Leftarrow \gamma_g"] \\ & C, & \end{tikzcd}$$

there is a universal functor $a \colon J\ell \to C$.

3. The map $x \mapsto Tc_x(\nu_x)$ lifts to a functor $Q \colon Jb \to T\ell$, which extends to a functor $Q^* \colon Tb \to T\ell$ via the monad structure of T, so that we can consider $Q^*\nu \in T\ell$.