Eigenvalues of Alternating Spring Systems

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August 8, 2019

I have looked at the following set of alternating spring systems where identical masses are connected by alternating between the spring a with spring constant $k = a$ and spring b with spring constant $k = b$.

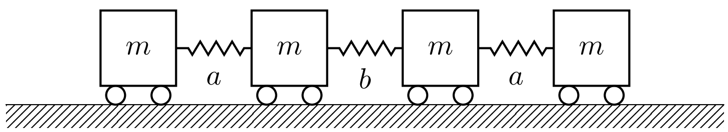

Figure 1: The spring system for $n = 4$ masses

Setting up the general differential equation, assuming no friction, we get

$$m\frac{d^2}{dt^2}\hat{\mathbf{x}} + \mathbf{K}\hat{\mathbf{x}} = 0$$

where $\hat{\mathbf{x}}$ is a vector containing the relative positions of each mass, $\mathbf{K}$ is the compression matrix shown below if n is even:

$$\begin{pmatrix} a & -a & & & \dots & 0 \\ -a & a+b & -b & & \dots & \\ & -b & a+b & -a & \dots & \vdots \\ & & -a & a+b & \dots & \\ \vdots & & & & \ddots & \\ 0 & & & \dots & -a & a \end{pmatrix}$$

and if n is odd:

$$\begin{pmatrix} a & -a & & & \dots & 0 \\ -a & a+b & -b & & \dots & \\ & -b & a+b & -a & \dots & \vdots \\ & & -a & a+b & \dots & \\ \vdots & & & & \ddots & \\ 0 & & & \dots & -b & b \end{pmatrix}$$

Solving for the eigenvalues of $\mathbf{K}$ symbolically using the following sympy code for the case $n = 8$ masses:

```
from sympy import *

a, b = symbols("a b'")

def generateMatrix(li):
    mat = []
    li.append('q')
    for i,x in enumerate(li):
        if i is 0:
            mat.append([x, - x, *[0]*(len(li)-2)])
        elif i is (len(li) - 1):
            mat.append([*[0]*(len(li)-2),- li[-1],li[-1]])
        else:
            last = li[i - 1]
            mat.append([*[0]*(i-1), - last, last + x, - x, \
            *[0]*(len(li) - 2 - i)])
    return Matrix(mat)

sevenSpring = [a,b,a,b,a,b,a]
eightMasses = generateMatrix(sevenSpring)
eightEigen = eightMasses.eigenvals()
pprint(simplify(eightEigen))
```

From running the above code on cases for $n = 4, 6, 8, 10, 12$, we determined that the eigenvalues follow the following order: First, there will always be two "trivial" eigenvalues that follow for all even n:

$$\lambda = 0, 2a$$

For cases $n > 2$, there are the following additional eigenvalues:

$$\lambda = a + b \pm \sqrt{a^2 + b^2 \pm \lambda' ab}$$

Where λ' was an undetermined variable. Using the code above, we determined the following values of λ' for the following values n:

n	λ'
4	0
6	± 1
8	$0,\ \pm\sqrt{2}$
10	$\frac{\pm 1 \pm \sqrt{5}}{2}$
12	$0,\ \pm 1,\ \pm\sqrt{3}$

It is thus my conjecture that λ' is actually:

$$\pm\lambda' = -2\cos\theta$$

which allows us to build the following relation:

n	λ'	θ
4	0	$\frac{\pi}{2}$
6	± 1	$\frac{\pi}{3}, \frac{2\pi}{3}$
8	$0,\ \pm\sqrt{2}$	$\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$
10	$\frac{\pm 1 \pm \sqrt{5}}{2}$	$\frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$
12	$0,\ \pm 1,\ \pm\sqrt{3}$	$\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}$
$\vdots$	$\vdots$	$\vdots$
$2k$	$-2\cos(a_j)$	$a_j = \frac{j\pi}{k}$; $j \in 0 < \mathbb{N} < k$

This allows us to generalize the eigenvalues into:

$$\lambda = a + b \pm \sqrt{a^2 + b^2 - 2ab\cos(\theta)}$$

for the θs given above. It is thus easy to represent this visually as vectors, assuming that $a > b$, $a\dot{b} = ab\cos(\theta)$ with c being the hypotenuse of both these legs:

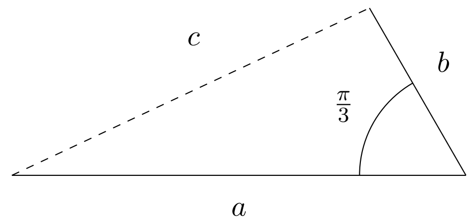

Figure 2: eigenvalue for $n = 6$ and $\theta = \frac{\pi}{3}$

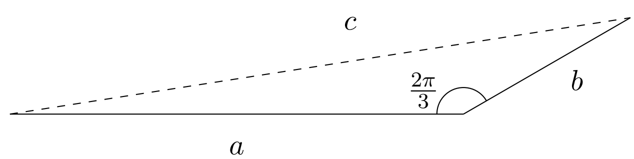

Figure 3: eigenvalue for $n = 6$ and $\theta = \frac{2\pi}{3}$

For odd n, the scheme is similar but has some peculiarities. To illustrate these peculiarities, I will compare the cases for n is odd and n is even:

n	θ_n	$2n$	θ_{2n}
3	$\frac{\pi}{3}$	6	$\frac{\pi}{3}, \frac{2\pi}{3}$
5	$\frac{\pi}{5}, \frac{3\pi}{5}$	10	$\frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$
7	$\frac{\pi}{7}, \frac{3\pi}{7}, \frac{5\pi}{7}$	14	$\frac{\pi}{7}, \frac{2\pi}{7}, \frac{3\pi}{7}, \frac{4\pi}{7}, \frac{5\pi}{7}, \frac{6\pi}{7}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$a_j = \frac{(2j-1)\pi}{k}; j \in 0 < \mathbb{N} < k$	$2k$	$a_j = \frac{j\pi}{k}$; $j \in 0 < \mathbb{N} < k$

Additionally,the "trivial" eigenvalue of $2a$ is no longer an eigenvalue of $\mathbf{K}$ when n is odd, but 0 remains an eigenvalue. This makes sense, as the 0 eigenvalue for the free ends represents uniform translation of all masses in the system. Additionally, this means that swapping the values of a and

b will have no change whatsoever on systems with odd n, while the only change on systems with even n will be trivial eigenvalue $2a$ being replaced with $2b$.