Eigenvalues of Alternating Spring Systems

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I have looked at the following set of alternating spring systems where identical masses connected by the spring k_c is considered a subsystem, and subsystems are coupled by the spring k_k:

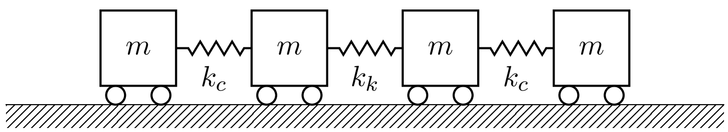

Figure 1: The spring system for $n = 2$ subsystems

Setting up the general differential equation, assuming no friction, we get

$$m\frac{d^2}{dt^2}\hat{\mathbf{x}} + \mathbf{K}\hat{\mathbf{x}} = 0$$

where $\hat{\mathbf{x}}$ is a vector containing the relative positions of each mass, $\mathbf{K}$ is the compression matrix shown below:

$$\begin{pmatrix}
k_c & -k_c & & & \dots & 0 \\
-k_c & k_c + k_k & -k_k & & \dots & \\
& -k_k & k_c + k_k & -k_c & \dots & \vdots \\
& & -k_c & k_c + k_k & \dots & \\
\vdots & & & & \ddots & \\
0 & & & \dots & -k_c & k_c
\end{pmatrix}$$

Solving for the eigenvalues of $\mathbf{K}$ symbolically using the following sympy code for the case $n = 4$ subsystems (or, 8 masses):

```
from sympy import *
init_printing(use_unicode = True)

w, k, kappa = symbols("w k k'")
firstLine = [k, -k]
evenLine = [-k, k + kappa, -kappa]
oddLine = [-kappa, k+kappa, -k]
lastLine = [-k, k]

def generateMatrix(size):
    mat = []
    mat.append([*firstLine, *[0]*2*(size-1)])
    for i in range((size - 1) * 2):
        if i % 2 is 0:
            mat.append([*[0]*i, *evenLine, *[0]*(2*(size-1)-i - 1)])
        else:
            mat.append([*[0]*i, *oddLine, *[0]*(2*(size - 1) - i - 1)])
    mat.append([*[0]*2*(size - 1), *lastLine])
    return Matrix(mat)

fourCase = generateMatrix(4)
fourEigen = fourCase.eigenvals()
pprint(simplify(fourEigen))
```

From running the above code on cases for $n = 2, 3, 4, 5, 6$, we determined that the eigenvalues follow the following order: First, there will always be two "trivial" eigenvalues that follow for all couplings of the subsystems, even $n = 1$:

$$\lambda = 0, 2k_c$$

For cases $n > 1$, there are the following additional eigenvalues:

$$\lambda = k_c + k_k \pm \sqrt{k_c^2 + k_k^2 \pm \lambda' k_c k_k}$$

Where λ' was an undetermined variable. Using the code above, we determined the following values of λ' for the following values n:

n	λ'
2	0
3	± 1
4	$0,\ \pm\sqrt{2}$
5	$\frac{\pm 1 \pm \sqrt{5}}{2}$
6	$0,\ \pm 1,\ \pm\sqrt{3}$

As it turns out, λ' is actually:

$$\pm\lambda' = 2\cos\theta$$

so we can build the following relation:

n	λ'	θ
2	0	$\frac{\pi}{2}$
3	± 1	$\frac{\pi}{3}, \frac{2\pi}{3}$
4	$0,\ \pm\sqrt{2}$	$\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$
5	$\frac{\pm 1 \pm \sqrt{5}}{2}$	$\frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$
6	$0,\ \pm 1,\ \pm\sqrt{3}$	$\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}$
$\vdots$	$\vdots$	$\vdots$
n	$2\cos(a_j)$	$a_j = \frac{j\pi}{n}$; $j \in \mathbb{N} < n$

This allows us to generalize the eigenvalues and combine the "trivial" case with the cases for $n > 1$ into:

$$\lambda = k_c + k_k \pm \sqrt{k_c^2 + k_k^2 - 2k_c k_k \cos(\theta)}$$

for the θs given above. It is thus easy to represent this visually, assuming that a is the greater of k_c and k_k and b is the lesser, with c being the hypotenuse of both these legs:

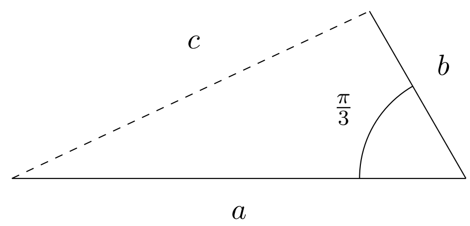

Figure 2: eigenvalue for $n = 3$ and $\theta = \frac{\pi}{3}$

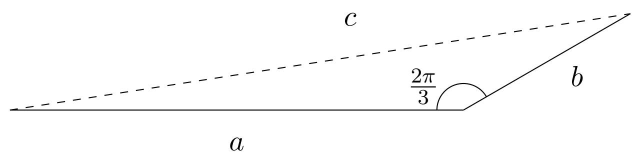

Figure 3: eigenvalue for $n = 3$ and $\theta = \frac{2\pi}{3}$

Because the eigenvalues are dependent on adding or subtracting c. The next phase of research involves figuring out how the eigenvalues are geometrically related to these specific triangles. If we take the limits as $n \to \infty$, we would find the following separate bands of eigenvalues forming: The upper band, from $2k_c$ to $2(k_c + k_k)$; and the lower band, from 0 to $k_c + k_k$. Notably if $k_c = k_k = k$, we get a single band of eigenvalues from 0 to $4k$.

We can now go back to the original differential equation:

$$m\frac{d^2}{dt^2}\hat{\mathbf{x}} = -\mathbf{K}\hat{\mathbf{x}}$$

We can now make the following assumption:

$$\hat{\mathbf{x}} = \begin{pmatrix} x_1 e^{i(\omega t + \phi)} \\ x_2 e^{i(\omega t + \phi)} \\ x_3 e^{i(\omega t + \phi)} \\ \vdots \\ x_{2n} e^{i(\omega t + \phi)} \end{pmatrix}$$

thus,

$$-m(\omega^2)\hat{\mathbf{x}} = -\mathbf{K}\hat{\mathbf{x}} = -\lambda\hat{\mathbf{x}}$$

$$\omega^2\hat{\mathbf{x}} = \frac{\lambda}{m}\hat{\mathbf{x}}$$

This is only true for all cases if $\hat{\mathbf{x}}$ is not the zero vector, and thus our normal modes can be constructed as just

$$\omega = \pm\sqrt{\frac{\lambda}{m}}$$

Because λ encompasses bands from $2k_c$ to $2(k_c+k_k)$ and 0 to k_c+k_k, we can thus define the normal modes of oscillation to have three bands: from $-\sqrt{2(\omega_c^2+\omega_k^2)}$ to $-\omega_c\sqrt{2}$, from $-\sqrt{\omega_c^2+\omega_k^2}$ to $\sqrt{\omega_c^2+\omega_k^2}$, and from $\omega_c\sqrt{2}$ to $\sqrt{2(\omega_c^2+\omega_k^2)}$ where $\omega_{(c,k)}^2 = \frac{k_{(c,k)}}{m}$

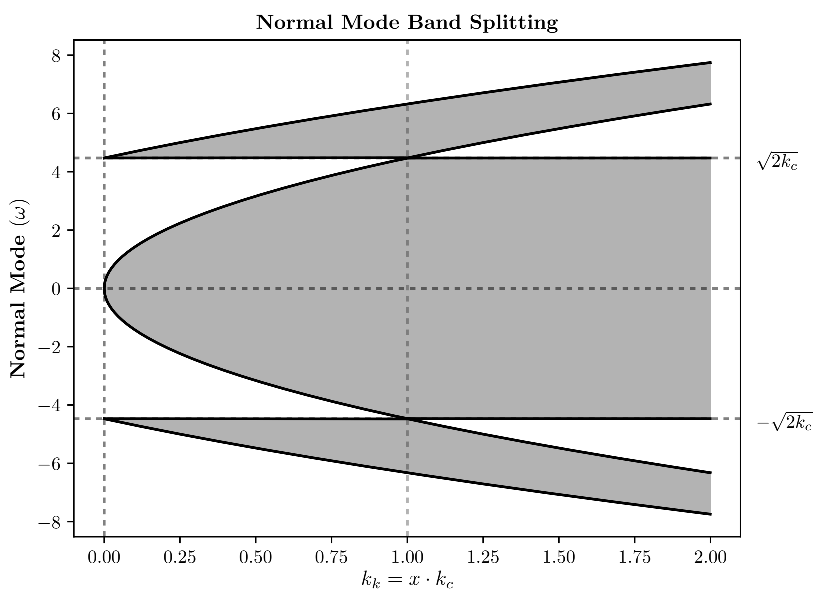

Figure 4: Graph for n=500 with $m = 1$