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Programming Models for Facility Dispersion: The p-Dispersion and Maxisum Dispersion Problems

The p-dispersion problem is to locate p facilities on a network so that the minimum separation distance between any pair of open facilities is maximized. This problem is applicable to facilities that pose a threat to each other and to systems of retail or service franchises. In both of these applications, facilities should be as far away from the closest other facility as possible. A mixed-integer program is formulated that relies on reversing the value of the 0-1 location variables in the distance constraints so that only the distance between pairs of open facilities constrain the maximization. A related problem, the maxisum dispersion problem, which aims to maximize the average separation distance between open facilities, is also formulated and solved. Computational results for both models for locating 5 and 10 facilities on a network of 25 nodes are presented, along with a multicriteria approach combining the dispersion and maxisum problems. The p-dispersion problem has a weak duality relationship with the (p-1)-center problem in that one-half the maximin distance in the p-dispersion problem is a lower bound for the minimax distance in the center problem for (p-1) facilities. Since the p-center problem is often solved via a series of set-covering problems, the p-dispersion problem may prove useful for finding a starting distance for the series of covering problems.

1. INTRODUCTION

Over the past several years, a number of papers have appeared in the operations research literature on the p-dispersion problem, following the pioneering work of Shier (1977). The p-dispersion problem can be summarized as locating p facilities on a network so that the minimum distance between any pair of facilities is maximized. Unlike the well-known p-median and p-center problems, there are no demand nodes and no allocation of nodes to other nodes in the

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p-dispersion problem. The p-dispersion problem has been referred to as the problem of "mutually obnoxious" facilities, such as nuclear power plants, oil storage tanks, or ammunition dumps, which should be spread out to the greatest extent possible in order that an attack or accident befalling one facility will have the least likelihood of damaging any of the others. Another application is for locating franchises so that each franchise competes as little as possible with other members of the same franchise system. In this respect, the p-dispersion problem is related to central place theory, which is similarly concerned with a system of locations in which no two facilities offering the same good can be too close together.

The greatest interest in the p-dispersion problem thus far has stemmed from its dual relationship with the p-center problem. The p-center problem, developed by Hakimi (1964), is to locate p facilities on a network so that the maximum distance between any of the demand points and its closest facility is minimized. If the set of nodes for the p-center problem is defined consistently with the set of possible facility locations for the p-dispersion problem, Shier (1977) and others have shown that, on a tree network, and considering all points on the arcs of the network, the minimum distance between facilities in a dispersion problem for p facilities is exactly twice as large as the maximum distance between points and their closest facility in a center problem for p-1 facilities. Furthermore, on a general network, the p-dispersion problem defines a lower bound to the (p-1)-center problem.

To this date, the literature on the p-dispersion problem has been limited to algorithmic solutions for tree networks; to my knowledge, no mathematical programming formulations have appeared in the location theory literature. This paper presents a mixed-integer programming model of the discrete version of the p-dispersion problem (locating p facilities among n candidate nodes) on a general (not necessarily a tree) network. Another model for dispersing facilities, known as the maxisum dispersion problem, which maximizes the *average* distance between open facilities, is also presented. The formulations, interpretations, applications, and results of these two dispersal problems are compared and contrasted.

Section 2 of this paper reviews the existing literature on the p-dispersion and other facility dispersal problems, and dicusses the relationship between the dispersion and center problems. Section 3 presents a mixed-integer programming formulation of the p-dispersion problem, which may have general applicability for solving other optimization problems in which the objective is to maximize some minimum value. Section 4 presents a model of the discrete maxisum dispersion problem, which maximizes the *average* distance between facilities. In section 5, markedly different computational results for both models on the same 25-node network, for both 5 and 10 facilities, are compared. The two models have different properties, which are combined in a multicriteria approach in section 6.

2. LITERATURE REVIEW

Shier (1977) seems to have been the first to recognize the p-dispersion problem as a location problem in its own right. He treated the continuous case in which facilities can locate at any node or along any arc of a tree. Shier drew on the graph theory results of Meir and Moon (1975) in showing that, *on a tree network*, the maximin distance solution to the p-dispersion problem is exactly twice as large as the minimax distance solution to the (p-1)-center problem. He

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developed an algorithm for finding what he termed a "2-divergent set" or "3-divergent set" of a tree, which are the sets of two or three points of a tree that maximize the minimum distance between them. However, Shier recognized that his algorithm could not be extended to a number of facilities p greater than 3.

Shier's proof of the weak duality relationship between the p-dispersion and (p-1)-center problem can be extended to the discrete case where both demand points and facility sites are restricted to the same set of candidate nodes on the network. Shier's proof consists of two parts. The first part establishes a weak duality relationship between the two problems: one-half the optimal objective function value of a p-dispersion problem provides a lower bound to the objective function of the (p-1)-center problem on the same network. In other words, even if, as in a center problem, (p-1) facilities are located to minimize the furthest distance from any node to a facility, and even if, as in a dispersion problem, the p facilities are located to maximize the smallest distance between facilities, the minimum distance in the (p-1)-center problem cannot be less than one-half the maximum distance in the p-dispersion problem. Shier points out that this part of the proof holds true regardless of whether the network is a tree or a general network containing cycles, as long as distances satisfy the properties of a metric.

For there to be a strong duality relationship between the two problems, the maximin solution to the p-dispersion problem would have to be *exactly* twice as large as the minimax solution to the (p-1)-center problem, but it does so only on a tree network. Shier provided a graphical counterexample for a general network. A different counterexample shows that, even on a tree, the strong duality fails to hold when candidate sites for the (p-1)-center problem are restricted to the set of nodes. Picture a tree network composed of a line segment with a node at each endpoint and one somewhere in the middle. The solution to the 2-dispersion problem is to locate at the two nodes furthest apart, while the solution to the 1-center problem is to locate at the middle node. But unless the middle node happens to be exactly halfway between the two outside nodes, the minimax distance in the 1-center problem will be *more than half as large as* the maximin distance separating the two facilities in the 2-dispersion problem. Thus, *either* a discrete set of facility sites *or* a general network is sufficient to obviate the strong duality relationship.

The same line segment network also provides a counterexample to any notion that the search for an optimal solution to the continuous p-dispersion problem can be restricted to the set of nodes of a network. Church and Garfinkel (1978) showed that Hakimi's theorem does not apply to the p-maxian problem, which is to maximize the average distance between demand points and their respective nearest facilities. Similar logic applies to the case of the maximin problem. The solution to the 3-dispersion problem on the line segment mentioned above is one facility at either end of the line segment, and one halfway in between. Only if a node happens to lie exactly halfway in between will the discrete problem yield as good a solution as the continuous problem. However, this paper will consider only the discrete version of the p-dispersion problem.

Chandrasekaran and Daughety (1981) termed this problem the "n-center dispersion problem." They developed a polynomial algorithm for solving the p-dispersion problem based on solving a finite series of "anti-cover" or "location set-packing" problems.[1] In an anti-cover problem the objective is to locate as many facilities as possible in such a way that every pair of facilities is separated

[1] The use of the term "dispersion" problem and also that of "anti-cover" problem follows the terminology used by Moon and Chaudhry (1984) in their review article on network location problems with distance constraints.

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by a distance of at least λ. By gradually increasing the minimum separation distance λ, one eventually reaches the distance such that p facilities can no longer be packed into the network without some pair of facilities being separated by less than λ. Chandrasekaran and Daughety present an algorithm for solving the anti-cover problem and an efficient search procedure for finding the largest λ for which p facilities can be packed into the network, that is, the solution to the p-dispersion problem. Chandrasekaran and Tamir (1982) extend Shier's proof of the dual relationship with the (p-1)-center problem to the case where the set of possible facility locations for the dispersion problem is limited to a set of nodes S on the tree, while for the center problem, the set of demand nodes, but not the set of possible facility sites, is limited to the same set of nodes S. Their solution procedure is also based on solving a series of anti-cover problems, but their sequence of minimum separation distances is determined by the sorted sequence of distances between all distinct pairs of points in S.

Tansel, Francis, Lowe, and Chen (1982) developed an algorithm for solving both a nonlinear p-center problem with maximum distance constraints and the p-dispersion problem, which they term the "dual threat" problem. Their objective is to minimize the maximum loss, where the loss is defined as some nonlinear function of distance. Kulshrestha (1984) used the term *distant point* to refer to the point that maximizes the minimum weighted distance from any vertex of the graph. He developed an algorithm for finding a distant point on a tree subject to an upper bound on the sum of the weighted distances from the vertices to the point, and showed a dual relationship with a version of the p-median problem.

Moon and Chaudhry (1984) discuss various geographical applications of the p-dispersion and/or maxisum dispersion problems. One criteria in spacing oil storage tanks is to place them as far apart from one another as possible so that a fire or a terrorist attack on one will not damage the others. Similarly, strategic facilities must be protected against simultaneous enemy attacks. Moon and Chaudhry also discuss and formulate a distance constraint which requires facilities to be separated by some minimum distance, and suggest applicability to franchises such as fast food restaurants and gasoline stations. Thus, dispersal as a goal can be motivated by noncompetition between facilities or "coverage" of a region, or both. As another example of noncompetition, they cite a government regulation, no longer in effect, of minimum spacing between radio stations.

Other location problems are also directed towards the dispersal of facilities strictly amongst themselves, that is, not in the context of allocating demand points to facilities. A discrete version of the p-defense problem, which maximizes the *average* distance from any facility to its closest neighbor, was considered by Moon (1977), whose solution procedure involved construction of a dual graph and its decomposition into planar graphs. No programming formulation was developed. The anti-cover problem, which maximizes the number of facilities that can be located subject to a minimum separation distance, has been addressed by Francis et al. (1978), Chandrasekaran and Tamir (1979), Chandrasekaran and Daughety (1981), and Moon (1983). The vertex packing problem is a third model used for spreading facilities apart. A vertex packing is a set of vertices such that if i and j are in the packing set, then (i,j) is not an arc of the network; the vertex packing problem is to find the packing set of the largest cardinality. This problem has been studied widely (Balas and Samuelsson 1973 and Nemhauser and Trotter 1974). Although no two linked nodes can be in the solution set, two nonadjacent nodes can still be very close together. Houck and Vemuganti (1977) suggest that arcs can be added to the network between nonlinked nodes that are too close together based on some known minimum separation distance.

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3. MIXED INTEGER FORMULATION OF THE p-DISPERSION PROBLEM

The p-dispersion problem is closely related to the anticover problem in the following way. The p-dispersion problem aims to maximize the minimum separation distance between facilities subject to the existence of p facilities, while the anti-cover problem aims to maximize the number of facilities subject to a given minimum separation distance. The constraint of one is the objective of the other. In formulating the p-dispersion problem as a mathematical program, one maximizes the minimum separation distance subject to constraints that place some kind of relevant upper bound on that distance. Of course, the relevant upper bound is the distance between two locations, and the bound should only apply when facilities are opened at both locations. In order to formulate the constraints so that they remove the upper bound for pairs of locations that do not both have facilities, rather than setting the upper bound to zero, it is necessary to reverse the value of location variables for *open* facilities from 1 to 0, and for facilities that are *not opened* from 0 to 1.

$$\text{Maximize } D \tag{1}$$

subject to

$$\sum_{i=1}^{n} X_i = p \tag{2}$$

$$D \leqslant d_{ij}(1 + M(1-X_i) + M(1-X_j)) \qquad \text{for all } i,j \in N \,|\, i<j \tag{3}$$

$$X_i \in \{0,1\} \qquad \text{for all } i \in N \tag{4}$$

where

D = smallest separation distance between any pair of open facilities
X_i = { 1, if a facility locates at node i}
0, otherwise
n = number of potential facility sites
p = number of facilities to be located
N = set of potential facility sites
M = a very large number
d_{ij} = shortest path distance between node i and node j.

The objective function of the model is to maximize D, the minimum distance between open facilities. Constraint (3), written once for each pair of potential facility sites, effectively places an upper bound on D equal to d_{ij} *only if* facilities are open at *both* i and j, since $(1-X_i)$ and $(1-X_j)$ will both be zero. However, if there is no facility at *either* i or j, then either $(1-X_i)$ or $(1-X_j)$, or both, will have a value of one, in which case the upper bound on D in (3) will be extremely large—equal to either $d_{ij}(1+M)$ or $d_{ij}(1+2M)$—that is, effectively ∞. Thus, only the distance between pairs of nodes that both have facilities will have a limiting effect on D; the maximal value of D will be limited by the smallest d_{ij} for which both X_i and X_j equal one. Constraint (2) requires p facilities to be open. The formulation is fairly compact, with $n(n-1)/2$ constraints of type (3) and one constraint of type (2), and with 1 continuous variable and n integer variables.

This method of formulating a maximin problem (i.e., using $M(1-X_i)$ in (3) to impose an upper bound on the minimum separation distance only when $X_i = 1$) may have some generality for formulating other problems that involve creating upper bounds only when a facility is opened at a node, that is, problems with *contingent upper bounds*. For instance, Ratick and White (1987) develop a complementary anti-cover model for siting hazardous waste facilities. In a complex, multiobjective model, the "associated equity" for a town is determined additively by the number of facilities that do not pose a risk to it. A system-wide equity index is determined by the minimum of these, and a $M(1-X_i)$ term is added to each town's associated equity so that effective upper bounds are set *only by towns with facilities in them.*

Unfortunately, the relaxed linear program does not have the desired property of producing integer solutions because if all X_i have fractional values, every constraint of type (3) will bound D from above with a number larger than, and in most cases much larger than, d_{ij}. On the other hand, the problem does not require complete enumeration of all integer solutions because, when using a standard branch-and-bound algorithm, some of the branches are pruned as soon as one integer solution is found, and more as the algorithm proceeds. For as soon as one integer solution has been found, it establishes a D by which to judge other integer solutions. Other branches of the tree can be pruned before they are completely fathomed in the following way. If a partly integer branch contains two $X_i = 1$ closer together than the D for the best all-integer solution found thus far, then all solutions further down the same branch will also contain those two facilities. Computational experience with this formulation is related in a subsequent section.

The objective function of the p-dispersion problem evaluates the system in terms of the smallest distance between any pair of facilities. An alternative dispersal objective could be to maximize the average distance between facilities, which is presented in the following section as the maxisum problem. Maximizing the minimum distance ensures that no two facilities will be especially close to one another. This objective should lead to relatively equally spaced locations, assuming the distribution of potential facility sites allows it. Thus, each facility will be surrounded by a certain radius of "empty" area. If a known radius of empty area around each facility is required, the analyst could use an anti-cover model. But if the analyst wishes to maximize that minimum separation distance, the p-dispersion model may be the appropriate choice.

The p-dispersion model could be of great use in locating a system of franchises. The Boston Globe recently reported that an estimated 34 percent of all retail sales in 1986 were through franchises, and that figure is expected to climb to 50 percent by the year 2000 (McKibben 1986). Since franchises are coordinated by a single decision maker, there exists a real opportunity for applied locational analysis. Members of the same franchise system are by definition competitors, since they offer identical products. A main concern of the franchise holders will be to minimize intrachain competition. It is not suggested here that this should be the only or even the main concern governing the selection of franchise locations. Location on high traffic routes and in population centers are obviously important criteria. Perhaps some type of population-distance measure (distance in population space) could be used instead of road or Euclidean distances. Alternatively, it is suggested that the p-dispersion model be used as a secondary criteria in choosing a set of franchise locations. The primary problem could allocate demand nodes to franchise locations in any number of ways: minimizing average or maximum distance, or maximizing the number of franchises that can be supported in the market, while ensuring that each franchise surpasses a certain minimum "threshold" level of customers to which it is the closest fran-

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chise. The secondary criteria could be the maximum dispersal of franchises without degrading the primary criteria. The parallel of this problem to central place theory, with its equal spacing of facilities and its threshold requirements, is very strong. In fact, the author's research on optimization models of central place theory led to consideration of the p-dispersion problem as a means to space central places equally. Section 6 presents a multicriteria analysis using the p-dispersion problem, which shows how it can be incorporated as constraints on the minimum allowable separation distance.

In the context of protecting facilities from accidents or attacks at neighboring facilities, the maximin objective applies in cases in which the probability of the incident spreading to its neighbors is extremely sensitive to the separation distance, such as for fires. In a case, for example, where a minimum separation distance of 120 yards would significantly reduce the chance of fire spreading from any facility to any other vis à vis a separation distance of 100 yards, the maximin objective would be appropriate. But if the probability of the attack or accident spreading to other facilities declines only gradually with distance, then a more appropriate objective might want to consider the other, nonminimum distances between facilities, or might want to group facilities into several groups so as to prevent a chain reaction and at least preserve a few of them in the event of the accident.

Maximin problems are often characterized by the existence of alternate optima, and the p-dispersion problem is no exception. Specifically, in the p-dispersion problem, there will generally be one pair of facilities that determines the maximin distance; all other pairs of facilities are separated by more than that distance. As long as the other pairs are separated by distances *greater than the maximin distance*, their exact location does not effect the objective function. There is often a small bit of leeway as far as the other nodes are concerned, as will be seen in the computational examples, and it may be necessary to choose among alternate optima according to another dispersal objective. This aspect is addressed in section 6 on a multicriteria dispersion approach.

Although the weak duality theorem suggests a dual relationship with the p-center problem, the dual of the formulation in equations (1)-(4) does not suggest the p-center problem. However, the discrete p-dispersion problem could be useful for solving the discrete center problem for (p-1) facilities in the following way. The most common method (Handler and Mirchandani 1979) for solving center problems involves solving a series of set-covering problems (Toregas et al. 1971) over a range of maximum separation distances. The optimal solution to a (p-1)-center problem is found when the distance used in the covering problem requires (p-1) facilities to cover all nodes but any smaller distance will require one extra facility to cover all nodes. Shier's proof that one-half the optimal solution in a p-dispersion problem cannot be greater than the optimal solution in the corresponding (p-1)-center problem holds for the discrete version of the problem on a nontree network, as explained in section 2. The fact that the latter distance is exactly one-half the former on continuous tree networks suggests that solving a p-dispersion problem might provide a very good starting distance for a series of set-covering problems, and might substantially reduce the number of distances for which covering problems must be solved.

4. THE MAXISUM DISPERSION PROBLEM

The maxisum dispersion problem is related to the p-dispersion problem (maximin) in the same way that the p-median problem (minisum) is related to the p-center problem (minimax). The discrete maxisum problem locates p facilities

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among n discrete nodes so as to maximize the sum of distances (or average of distances) between open facilities. This maxisum objective function is an alternative criteria for spreading facilities apart. An obvious drawback to this interpretation of "spreading apart" is that some pairs of facilities could conceivably be placed very near to each other. This is not an idle concern, as the computational examples will show.

The maxisum problem is formulated below as a mixed-integer program. The formulation is straightforward but not as compact as that for the p-dispersion problem. To avoid a non-linear X_iX_j term in the objective function, variables Z_{ij} are created that can only take on a value of 1 if both X_i and X_j have values of 1.

$$\text{Maximize} \quad \sum_{i=1}^{n} \sum_{j=i+1}^{n} Z_{ij} d_{ij} \tag{5}$$

subject to

$$\sum_{i=1}^{n} X_i = p \tag{6}$$

$$Z_{ij} \leqslant X_i \qquad \text{for all } i,j \mid j > i \tag{7}$$

$$Z_{ij} \leqslant X_j \qquad \text{for all } i,j \mid j > i \tag{8}$$

$$X_i \epsilon \{0,1\} \qquad \text{for all } i \epsilon N \tag{9}$$

where

$$Z_{ij} = \{\, 1, \text{ if facilities are located at both } i \text{ and } j\} \\ \quad 0, \text{ otherwise}$$

The objective function (5) sums the distances between all pairs of open facilities. The index j is summed over the range $i+1$ to n so that d_{ij} is not double-counted. Constraints (7) and (8) allow Z_{ij} to take on a value of 1 only if both X_i and X_j are equal to 1. Although Z_{ij} is a continuous variable, it will always take on an integer value if the X_i variables are integer.

The maxisum problem is a combinatorial, NP-complete mixed-integer programming problem. Like the p-dispersion problem, it tends to yield a fractional solution when run as a continuous problem. This is because if all X_i variables have a fractional value, then every d_{ij} counts at least fractionally toward the objective function. The number of distances counting toward the objective goes up by a greater degree than their fractional contribution goes down. For instance, with $n = 25$ and $p = 5$, the continuous solution had every $X_i = p/n = 0.20$. All 300 distances in the distance matrix were multiplied by 0.2 and counted in the objective. Compare this to the integer solution in which only 10 distances were fully counted. To make matters worse, unlike the p-dispersion problem, the branch and bound tree cannot be easily pruned, since an all-integer solution must be found on each branch before its objective function value can be known. For this reason, the maxisum dispersion problem may be of interest to geographers mainly in comparison to or in combination with the p-dispersion problem.

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5. COMPUTATIONAL EXAMPLES

Six sample problems were run on a network of 25 nodes, using the Euclidean distance between nodes. These particular problems were solved on the Boston University IBM 3090, using MPSX/370 with the MIP/370 mixed-integer programming option. Both the p-dispersion and maxisum dispersion problems were solved for p=5 and p=10.

Figure 1 shows the network for the sample problem and the locations of the 5 maximally dispersed nodes using the p-dispersion model. Facilities are located at nodes 1, 5, 17, 21, and 25. The shortest distance between any two facilities is 35.23, separating nodes 1 and 5. Notice that nodes 18 and 19 are further away from all other facilities than is node 17. However, because the closest facility to node 17, at node 5, is a distance of 41.23 away, there was no incentive to locate a facility at 18 or 19 instead of 17, since it does not reduce the minimal distance. Even if the facility at node 17 were to move to node 19, the facility at node 5 could not move to node 13 or 14 because the distances from nodes 13 and 14 to node 21 is under 35.23. Solution at nodes 1, 5, 21, 25 and either 18 or 19 are thus two of several alternate optima that exist for this particular problem. MPSX identified the existence of several alternate optima, but printed just this one.

Solution time for this problem was 24 seconds of CPU time. The first integer solution was found at the 289th node on the branch and bound tree; the optimal solution was found on the 1,105th node; and optimality was proved on the 1,131st node. The p-dispersion problem is a combinatorial problem that, in this example, has 53,130 possible ways of choosing 5 out of 25 nodes. Thus, the branch and bound search required the computation or estimation of 2.1 percent of the number of possible integer solutions. The number of branches abandoned during the search was 1,045, which confirms that most of the possible integer solutions do not have to be enumerated.

Figure 2 shows an optimal solution to the 10-dispersion problem—one of twelve found by MPSX with the same objective function value. In all 12 solutions, including this one, the maximin distance was 18.25, the distance between nodes 11 and 22. CPU time was 53 seconds, but the number of possible integer solutions for the 10-node problem is 3,268,760. Solutions amounting to only 0.074 percent of the number of possible integer solutions had to be evaluated to prove

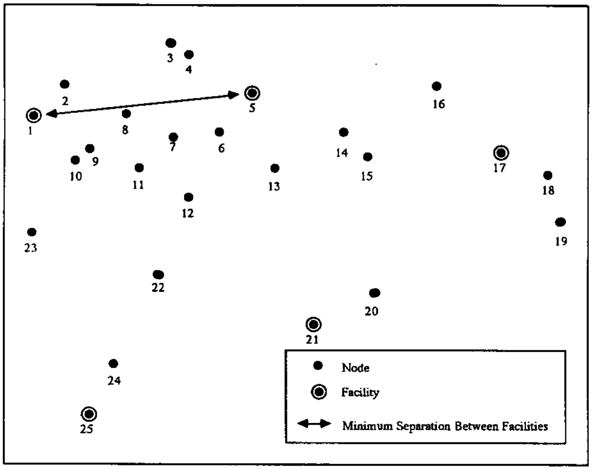

FIG. 1. Solution to the 5-Dispersion Problem

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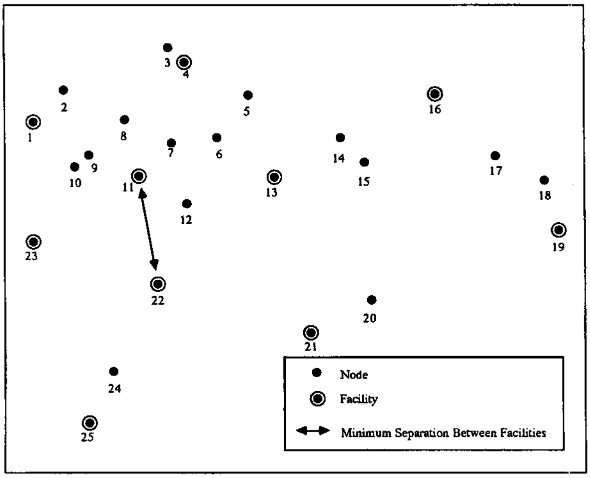

FIG. 2. Solution to the 10-Dispersion Problem

optimality. Both the 5-node and 10-node solutions to the p-dispersion problem give the appearance of fairly regular spacing, as desired.

Both the regular spacing and the relatively rapid solution times are lost when using the maxisum dispersion problem. Figures 3 and 4 show the best solutions found by MPSX for the 5-maxisum and 10-maxisum problems, respectively. Optimality was not proved in either model, despite being allotted 9 and 14 minutes of CPU time, respectively. These two "best" integer solutions were found after only 38 seconds for the 5-maxisum problem and 7 minutes, 16 seconds for the 10-maxisum problem. One indication of the enormity of NP-complete problems such as the maxisum dispersion problem is that after 15 minutes of CPU time, only 3,090 nodes of the branch and bound tree had been evaluated, which amounts to only 0.09 percent of the number of possible integer solutions—almost all of which would have to be evaluated to prove optimality.

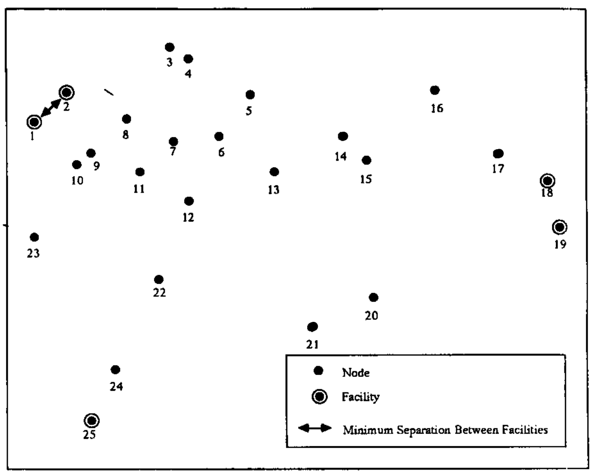

FIG. 3. Solution to the 5-Maxisum Dispersion Problem

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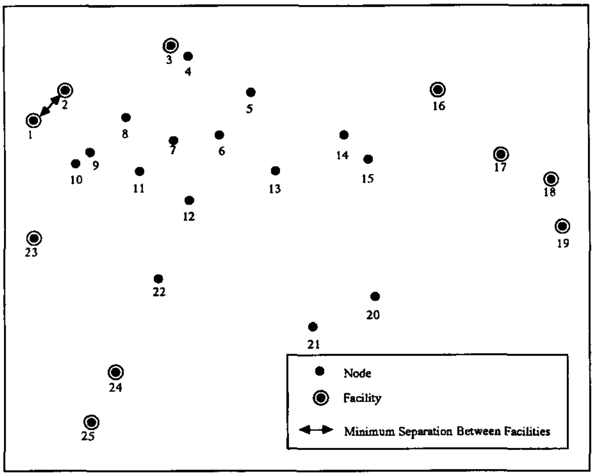

FIG. 4. Solution to the 10-Maxisum Dispersion Problem

The interesting result from the solutions to the p-defense problems is that in both the 5 and 10 node problems, facilities were located on the outer edges of the network, often in very tight groupings. If judged by the criteria of maximizing the minimum distance, the models performed extremely poorly, with the closest pair of facilities, at nodes 1 and 2, separated by only 7.07 in both models. But by the criteria of maximizing the sum of distance between all facilities, the models performed significantly better than the p-dispersion models. The average distance between open facilities in the 5-maxisum problem is 62.10, compared with 51.97 in the 5-dispersion model. Similarly, the average distance between open facilities in the 10-maxisum problem is 53.11, compared with 43.67 for the 10-dispersion problem. These comparisons and the solution times are shown in Table 1.

Comparative analysis reveals why the maxisum dispersion solutions consist of tight groupings at the extreme ends of the network. Starting with the 5-dispersion solution in Figure 1, one can see that if one moves the facility at node 21 towards node 17, the distance to node 17 gets smaller, but the distances to nodes 1 *and* 25 get larger. Two distances each increase by about the same amount as one decreases. Similarly, in the 10-maxisum solution, facility locations gravitate to the outside of the network, forming an almost circular pattern: any facility in the

TABLE 1
Summary of Computational Results

Model	Number of Facilities	Minimum Separation Distance	Average Separation Distance	CPU Solution Time
p-Dispersion	5	35.23	51.97	0.40 minutes
p-Dispersion	10	18.25	43.67	0.87 minutes
p-Maxisum Dispersion	5	7.07	62.10	9.00 minutes*
p-Maxisum Dispersion	10	7.07	53.11	15.00 minutes*
Multicriteria	5	35.23	54.97	0.67 minutes
Multicriteria	10	18.25	44.46	4.55 minutes

*Optimality not proved

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middle could do better by moving away from the center of the network toward the side with the fewest facilities.

In what situations could the maxisum dispersion problem be a useful model? Neither the obective, nor the distribution patterns make much sense for franchises. The only relevant application seems to be with facilities that pose a hazard to each other for which the probability of any accident occuring is extremely low, but the probability of it spreading to another facility is high except at the great distances. For instance, following the sample 10-dispersion problem, if the chance of a fire spreading from any of 10 oil storage tanks to another is, say, 75 percent at distances under 25.00, and the maximin dispersion distance is 18.25, it would not make much sense to spread the facilities evenly at distances of 18.25 apart. It would be much better in that case to have separate groupings so that a fire would not place all facilities at risk to a chain reaction. In fact, if a safe separation distance were known, one might want to maximize the number of groups separated from all other groups by the safe distance (which might be called the group anti-cover problem). This is not exactly what the maxisum problem does, although it seems to tend in that direction. Another relevant consideration is whether the time frame for defensive response negates any concern about a chain reaction. A location solution in which all facilities form a circle around the edge of the network would not make much sense if a chain reaction could leap from facility to facility.

6. A MULTICRITERIA APPROACH

To summarize the preceding discussion, the maxisum dispersion problem has a rather specialized applicability, as opposed to the more general applications of the p-dispersion problem. This is fortunate, given the difficulty of solving the maxisum problem. However, one advantage it has over the p-dispersion problem is that *all* the distances are considered in the maxisum problem, versus only the minimum in the p-dispersion problem. An ideal dispersal model might combine the two problems in a multicriteria approach that preserves the maximin distance from the p-dispersion problem but otherwise maximizes the average distance between open facilities.

This usage of the term "multicriteria" follows Moon and Chaudhry (1984). Whereas a true multiobjective analysis describes the *trade-off* between two objectives, a multicriteria approach incorporates one criteria into the constraints. Of course, the objective placed in the constraints does not have to be constrained at its optimal value. In this paper, it is suggested that the maxisum model is an appropriate way to choose among the many alternate optima of the p-dispersion problem.

$$\text{Maximize} \quad \sum_{i=1}^{n} \sum_{j=i+1}^{n} Z_{ij} d_{ij} \tag{10}$$

subject to

$$\sum_{i=1}^{n} X_i = p \tag{11}$$

$$Z_{ij} \leqslant X_i \qquad \text{for all } i,j \mid j > i \tag{12}$$

$$Z_{ij} \leqslant X_j \qquad \text{for all } i,j \mid j > i \tag{13}$$

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$$D \leqslant d_{ij}(1 + M(1-X_i) + M(1-X_j)) \qquad \text{for all } i,j \epsilon N \,|\, i < j \qquad (14)$$

$$D \geqslant d_{opt} \qquad (15)$$

$$X_i \epsilon \{0,1\} \qquad \text{for all } i \epsilon N \qquad (16)$$

where

d_{opt} = the optimal maximin dispersion distance from a previous run of the p-dispersion model.

Constraints (10) to (13) are unchanged from the maxisum model. Constraints (14) and (15) incorporate the p-dispersion model into the constraint set by limiting D from below by d_{opt} and from above by the d_{ij} for any pair of open facilities. These two constraints together constitute a lower bound on distances between open facilities—constraints that usually accompany the anti-cover objective. Moon and Chaudhry (1984) suggest that heuristics for solving other distance-constrained models, by Hillsman and Rushton (1975) and Khumawala (1975), could be adapted for this purpose.

The results of the multicriteria models for p=5 and p=10 are shown in Figures 5 and 6, respectively, and also summarized in Table 1. In both problems, the two facilities that defined the upper bound on the maximin distance were forced open. For p=5, the only change from the 5-dispersion problem was that the facility at node 17 moved to node 19. This single change increases the average distance between open facilities from 51.97 to 54.97. Despite the larger number of constraints in the multicriteria problems, this model solved to optimality in 40 seconds because it is more tightly constrained and more integer solutions can be eliminated. For the p=10 multicriteria problem, nodes 3 and 20 are substituted for nodes 4 and 21, and the average distance increases from 43.67 to 44.46. Four and a half minutes of CPU time were required to prove optimality.

The multicriteria dispersion models were successful in alleviating the biggest drawback of the p-dispersion: the minimum separation is still maximized, but

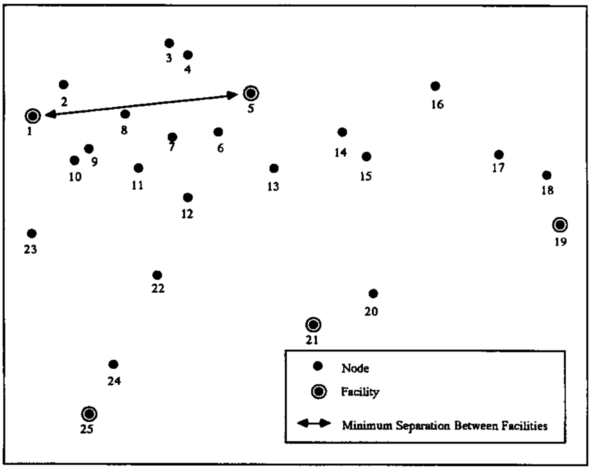

FIG. 5. Solution to the Multicriteria Problem for 5 Facilities

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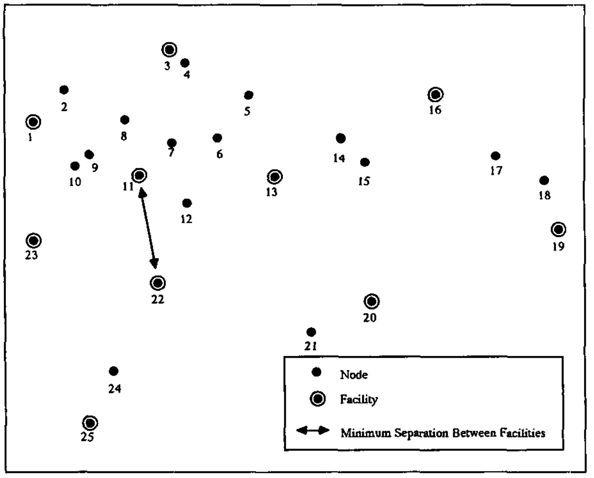

FIG. 6. Solution to the Multicriteria Problem for 10 Facilities

the other distances are also taken into account. Other dispersal-type problems that could be studied are (a) the p-facility, group anti-covering problem suggested in section 5; (b) to maximize the average distance between facilities and their closest neighboring facility (the p-defense problem); or (c) maximize the minimum distance and subsequently maximize the second shortest minima.

The multicriteria problems are also valuable because they suggest combining the dispersion obective or constraint with other location-allocation problems. In particular, other objectives relating to franchise locations, central place theory, or public services could be optimized and then incorporated into the constraint set of the p-dispersion problem, which could then maximally disperse the facilities without degrading the initial objective more than some predetermined amount.

7. CONCLUDING REMARKS

A mixed-integer programming formulation of the p-dispersion problem has been developed that relies on using $(1-X_i)$ instead of X_i in the distance constraints. This allows the minimum separation distance D to be bounded from above by d_{ij} only when X_i and X_j both equal one. Previous efforts to solve the p-dispersion problem have involved the solution of a series of related location problems using algorithmic methods. It is hoped that this paper might spur other work that capitalizes on the reversal of the definition of $0-1$ variables to solve other previously unformulated or poorly formulated location problems that involve setting upper bounds whose enforcement is contingent on one or more facilities being opened. Specialized branch and bound algorithms might significantly reduce the search for integer solutions, which could make it possible for the dispersion problem to be a time-saving and cost-saving step in the solution of center problems. The p-dispersion problem itself has applications in any situation where facilities must be spread out as far as possible from one another and be as evenly spaced as possible. The maxisum dispersion problem, in contrast, does not space facilities evenly. Applied location analysts may find it useful to combine the p-dispersion problem with other location problems in a multicriteria approach, in which one objective of the decision maker is regular spacing of facilities.

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