Exotic Manifolds and Super String Theory

MSU Summer Topology Program

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Overview

Definition (Exotic Manifolds)

Let M be a differentiable manifold. M' is said to be an *exotic copy* of M if M' is homeomorphic to M but M' is not diffeomorphic to M.

Conditions of homeomorphism and diffeomorphism:

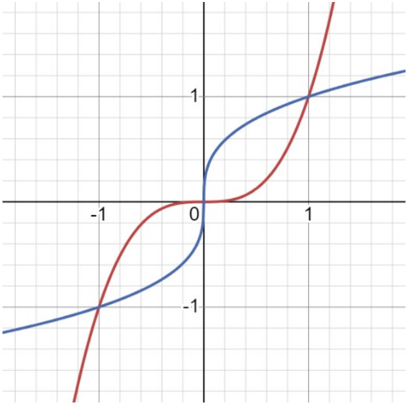
Homeomorphism	Diffeomorphism
Bijective	Bijective
f is continuous	f is smooth
f^{-1} is continuous	f^{-1} is smooth

Example of Exotic Map

$F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = x^3$

$$F'(x) = 3x^2$$
$$F''(x) = 6x$$
$$F'''(x) = 6$$
$$F''''(x) = 0 \; \ldots$$

$$x = F(x)^3$$
$$F(x)^{-1} = \sqrt[3]{x}$$
$$F'(x)^{-1} = \frac{1}{3x^{2/3}}$$



DNE at $x = 0 \;\; \therefore F(x)^{-1}$ is *not* smooth.

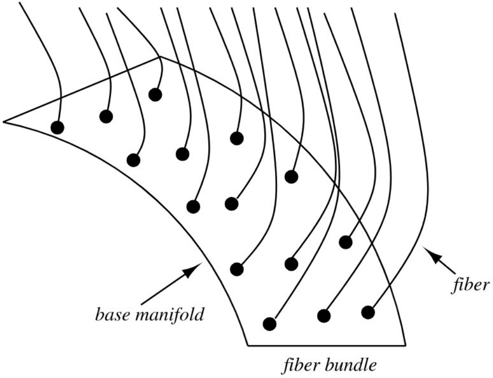
Beginning the Construction Process

We are now going to begin the process of constructing an exotic manifold.

Fiber Bundles

Definition

A fiber bundle (also called a bundle) with fiber F is a map $f : E \to B$ where E is called the total space of the fiber bundle and B is the base space of the fiber bundle.



Vector Bundles

A *vector bundle* follows the same conditions as a fiber bundle with additional characteristics.

For every point $p \in X$, there is an open neighborhood $U \subseteq X$ of p, a natural number k, and a homeomorphism:

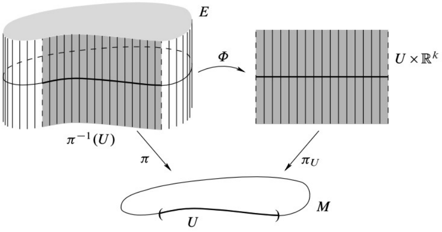
$$\varphi : U \times \mathbb{R}^k \to \pi^{-1}(U)$$

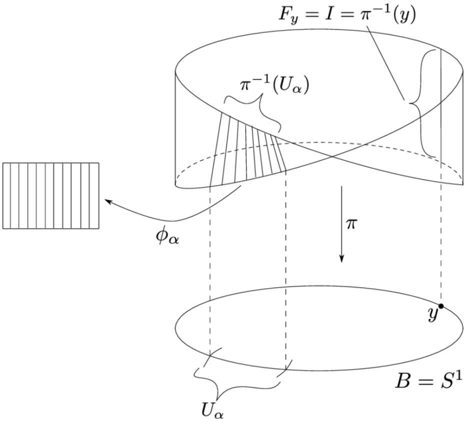
such that $\forall\ x \in U$:

1. $(\pi \circ \varphi)(x, v) = x$ for all vectors $v \in \mathbb{R}^k$
2. the map $v \to \varphi(x, v)$ is a *linear isomorphism* between the vector spaces $\mathbb{R}^k$ and $\pi^{-1}(\{x\})$

Vector Bundles

The simplest nontrivial vector bundle is a line bundle on the circle, and is analogous to the Möbius strip.





Hopf Fibrations

Definition

Hopf fibrations are fiber bundles in which the base space, the fibers, and the total space are all spheres.

The base space is S^4, and the total space is a S^7. The projection map takes points from the total space and “projects” them onto the base space.

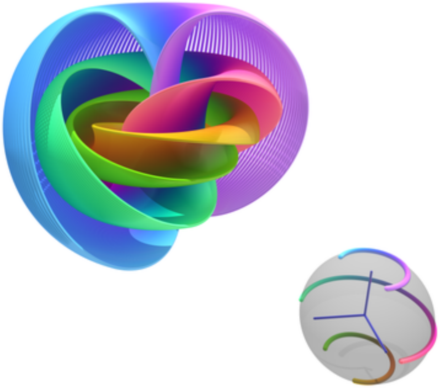
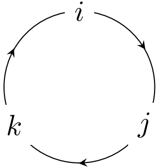


Figure: 2D projection of a Hopf Fibration

Quaternionic Hopf Fibrations

Definition

$\mathbb{H}$ is the set of numbers of the form $a + bi + cj + dk$, where $ij = k = -ji$ and addition is component-wise.



Non-commutative property of $\mathbb{H}$.

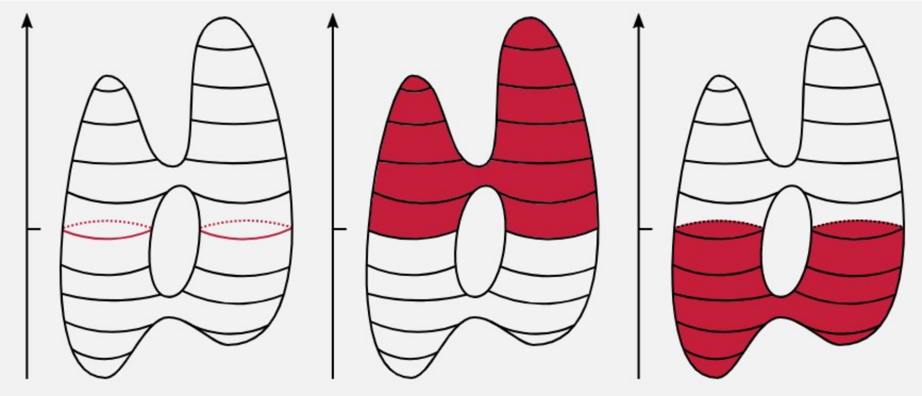
S^3 bundles over S^4

We will use Morse Theory, more specifically, Reeb's Theorem in order to show that many of the S^3 bundles over S^4 are homeomorphic to S^7.

Visualization Aid: At every point on S^4 there is an S^3.

Morse Theory

Morse theory is the method of studying the topology of a smooth manifold M by the study of Morse functions $M \to \mathbb{R}$ and their associated gradient flows.



Reeb's Theorem

Theorem

If M is a compact smooth manifold of dimension n, and f is a smooth function on M with only two nondegenerate critical points, then M is homeomorphic to a standard sphere.

Characteristic Classes

Definition

Characteristic classes are used to prove that these structures are in fact exotic.

Major results of the theory of characteristic classes:

- Euler Class
- Chern Class
- Pontryagin Class
- **Hirzebruch Signature Theorem**

Hirzebruch Signature Theorem

Theorem (Hirzebruch Signature Theorem)

Let M be a closed orientable smooth manifold of dimension 8, with signature $\tau(M)$. Then,

$$\tau(M) = \frac{1}{45}(7p_2(M) - p_1^2(M))$$

Hirzebruch's signature theorem, which gives a formula to compute the signature of a (smooth) compact oriented manifold.

Summarizing the Construction Process

We have constructed a family of manifolds that are homeomorphic but not diffeomorphic to the 7-sphere.

- Constructed the manifold with quaternionic Hopf fibrations
- The non-commutativity of the quaternions gives enough 'room' in the set of candidate manifolds for exotic structures to exist.
- The homeomorphism between the base spaces was shown by Reeb's Theorem
- The non-diffeomorphism was shown by the Hirzebruch Signature Theorem.

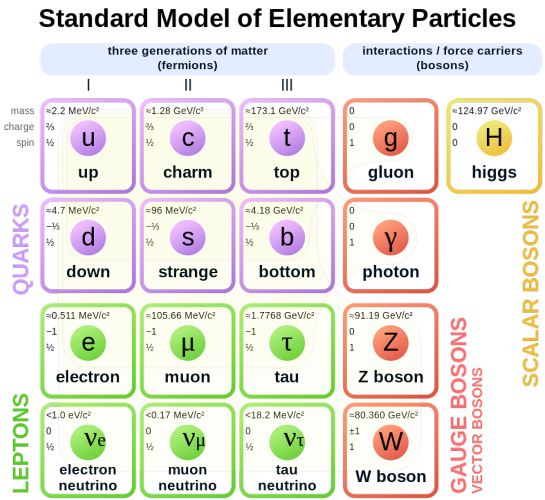
Super String Theory

Now we're going to apply exotic manifolds to theoretical physics!

Standard Model Equation (1975):

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&+ i\bar{\psi}D\psi + h.c. \\
&+ \bar{\psi}_i y_{ij}\psi_j\phi + h.c. \\
&+ |D_\mu\phi|^2 - V(\phi)
\end{aligned}
$$

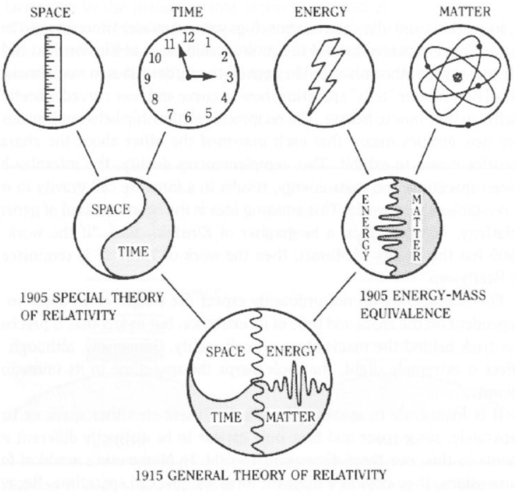
Splits the observable universe into two categories: *Fermions* and *Bosons*.



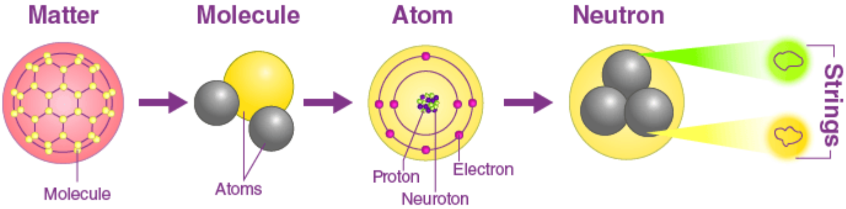
What about Gravity?

There is one problem with the Standard Model- gravity.

Gravitons, which are used to model gravity on the quantum level, *cannot* be explained on a quantum level through this model.



String Theory Model



What makes Superstring Theory, super? (*other than it's cool*).

Super String Theory Requirements

Superstring theory (SST) requires 10 spatial dimensions (11 dimensions if you include time) for its mathematical consistency.

SST has a very important symmetry, and this symmetry is broken when you try to describe a quantum string in $\mathbb{R}^4$.

But how can we exist in $\mathbb{R}^{10}$?

Kaluza-Klein Theory

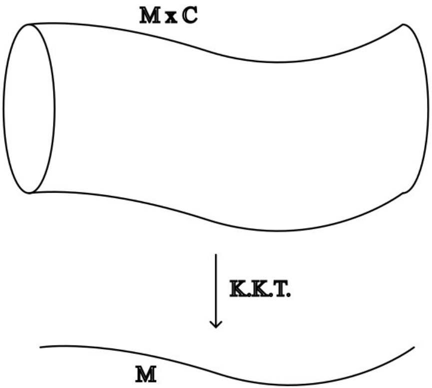
Kaluza–Klein theory (1919) is a unified field theory built around the idea of a fifth dimension beyond $\mathbb{R}^4$. It is considered an important precursor to string theory.

It was proposed that you could have 1 extra spatial dimension but if that dimension was smaller than any other scale that was around, then you wouldn't notice it.

Kaluza-Klein Mechanism

Through the idea of the Kaluza-Klein theory, some scientists believe that particles exhibit mass through their motion through compact spaces.

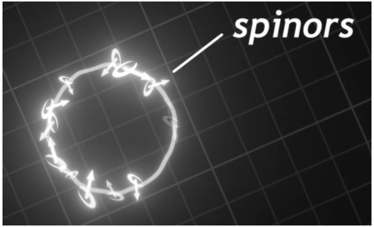
This mechanism introduced the idea of **dimensional compactification.**



Spinors on Strings

In order to physically describe the compactification of $\mathbb{R}^{10}$ into $\mathbb{R}^4$, there is a notion of spinors.

How do we describe the compacting mathematically?



Calabi-Yau Manifolds

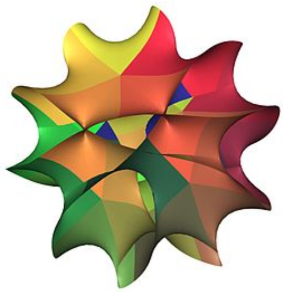


Figure: A 2D slice of a Calabi–Yau quintic manifold.

The connection between smooth structures on Calabi-Yau manifolds and exotic manifolds lies in the smooth structures that may exist on Calabi-Yau manifolds.

Conclusion

We covered:

- Vector Bundles
- Reeb's Theorem
- Hirzebruch Signature Theorem
- Constructed exotic manifolds
- Failure of the Standard Model
- Kaluza-Klein Theory
- Calabi-Yau Manifolds

String-Theory by ScienceClic:

(not a rick-roll I promise)

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Most importantly, my mentor, Amey, for helping me every step of the way.