Exotic Manifolds

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Topology Expository Research Paper
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June 29, 2023

Contents

Abstract

This paper will include various propositions, definitions, and proofs in the field of topology. I will also provide a detailed explanation of Milnor's construction of exotic 7-spheres. The possible manifolds will be constructed as total spaces of S^3 bundles over S^4, denoted as $M_{h,l}$. The subset of these candidates satisfying the condition $h + l = \pm 1$ will be shown to be topological spheres by Morse Theory. A subset of these that do not satisfy $(h - l)^2 \equiv 1(\text{mod}7)$ will be shown to not be differential spheres, by the Hirzebruch Signature Theorem.

1 Basic Definitions

1.1 Metric Space

Definition 1. *(X, d) is a metric space if and only if $d : X \times X \to \mathbb{R}_{\geq 0}$ a metric such that:*

1. *$d(x, y) = d(y, x) \geq 0$*
2. *$d(x, y) = 0$ iff $x = y$*
3. *$d(x, y) + d(y, z) \leq d(x, z)$ (triangle inequality).*

Example 1.1. *The Euclidean metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$ is defined by*

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}$$

where

$$x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n)$$

For $n = 1$, this metric reduces to the absolute value metric on $\mathbb{R}$, and for $n = 2$, it is the previous example. We will mostly consider the case $n = 2$ for simplicity.

Example 1.2. *Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by*

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|$$

$$x = (x_1, x_2), y = (y_1, y_2)$$

Then d is a metric on $\mathbb{R}^2$ called the ℓ^1 metric. It is also referred to informally as the "taxicab" metric since it is the distance one would travel by taxi on a rectangular grid of streets.

Example 1.3. *Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by*

$$d(x, y) = max(|x_1 - y_1|, |x_2 - y_2|)$$

$$x = (x_1, x_2), y = (y_1, y_2)$$

Then d is a metric on $\mathbb{R}^2$, called the ℓ^∞, or maximum metric.

1.2 Topological Space

Before we define topological spaces, we need to define an open and closed ball.

Definition 2. *The set $B(x, r) = \{y \in X : d(x, y) < r\}$ is called the open ball $B(x, r)$ with center x and radius r. The set $B(x, r) = \{y \in X : d(x, y) \leq r\}$ is called the closed ball $B(x, r)$ with center x and radius r. In contrast to an open ball, a closed ball contains the points of the boundary where $d(x, y) = r$. Sometimes the radius is labeled instead of r and then the ball is also called epsilon ball. Note that in $\mathbb{R}$, an open ball is simply an open interval $(x-r, x+r)$,i.e.the set $\{y \in R : x - r < y < x + r\}$,and a closed ball is simply a closed interval $(x - r, x + r)$,i.e.the set $\{y \in R : x - r \leq y \leq x + r\}$.*

Definition 3 (**Topological Space**)**.** *A topological space is a set X together with a collection $\mathcal{T}$ of subsets of X, called open sets, such that:*

1. *X and $\emptyset$ are in $\mathcal{T}$*
2. *The union of any collection of sets in $\mathcal{T}$ is still in $\mathcal{T}$*
3. *The intersection of a finite collection of sets in $\mathcal{T}$ is still in $\mathcal{T}$.*

Proposition 1. *A metric space is a topological space.*

Proof. We will show that a metric space is a topological space by showing that the three conditions of $\mathcal{T}$, a topological space, are satisfied:

1. $\emptyset$ is vacuously open since there are no points to violate the openness. Now consider $B(x, r)$ for any point $x \in X$ are $r \in \mathbb{R}$ such that $r > 0$. $B(x, r) \in X$ and $\therefore$ satisfies the definition of an open set.
2. Let $\{U_i\}$ be a collection of open sets in X. $\forall x \in U_i$, $\exists$ some i such that $x \in U_i$. Since U_i is open, $\exists B(x, r) \in U_i$. $\therefore B(x, r) \in \cup U_i$, making it open.
3. Consider $\bigcap\limits_{i=1}^{n} U_i \in X$. $\forall x \in \cap\, U_i, \therefore x \in U_i$. Since each U_i is open, $\exists B(x, r)$ $\forall i \in U_i$. Now let r be $\min(\mathbb{R})$ such that $r = r_1, r_2, ..., r_n$. $B(x, r) \in U_i$, $\therefore B(x, r) \in \cap\, U_i$, making it open.

   We have shown that the metric space (X, d) satisfies the conditions to be a topological space, as desired.

□

If we let $\mathcal{T}$ be the collection containing only $\emptyset$ and X then this is called the *trivial topology* on X. If the collection of all subsets on X defines a topology on X, then this is called the *discrete topology*.

1.3 Continuity

Definition 4. *A function $f : X \to Y$ between topological spaces X and Y is said to be continuous if for every $\varepsilon > 0$, there exists $\delta > 0$ such that dist$(f(x), f(y)) < \varepsilon$ whenever dist$(x, y) < \delta$.*

1.4 Closed Space

Definition 5. *X is a metric space $A \subseteq X$ is closed if $\{x_n\}_{n=1}^{\infty} \in A$. $x_n \to x$ then $x \in A$.*

- Note: if a function is converging to something, it must include what it is converging to in order to be closed.

Definition 6. *$A \subseteq X$ is closed if $X \setminus AB$ is closed.*

1.5 Open Space

Definition 7. *We say $x_n \to x$ if $\forall\ \varepsilon > 0$, $\exists\ N$ s.t. $\forall\ n \geq N$ $d(x_n, x) < \varepsilon$.*

Definition 8. *$U \subset X$ is open if $\forall\ n \in U$, $\exists\ \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq U$.*

Proposition 2. *$\bigcap_{i=1}^{n} U_i$ and $\bigcup_{i \in I}^{\infty} U_i$ are open if and only if $\bigcup_{i=1}^{n} U_i$ and $\bigcap_{i \in I}^{\infty} U_i$ are closed.*

Proof. We will begin this proof in the forward direction.

Let C be a collection of closed sets satisfying the topological space conditions:

1. $\emptyset$ and X are open.
2. $\forall \{U_i\} \subseteq C$, $\cup\ U_i$ is open.
3. $\forall \{U_1, U_2, ..., U_n\} \subseteq C$, $\cup\ U_i$ is open

Therefore, the complement of C satisfies the conditions for a topological space using open sets. We will now show the backward direction. Let O be a collection of open sets satisfying the topological space conditions:

1. $\emptyset$ and X are closed
2. $\forall \{U_i\} \subseteq O$, $\cap\ U_i$ is closed.
3. $\forall \{U_1, U_2, ..., U_n\} \subseteq O \cup U_i$ is closed.

Hence, the complement of O satisfies the condition for a topological space using closed sets. Therefore, we have established the equivalence between defining a topological space using a collection of closed and open sets. □

1.6 Bounded

Definition 9. *$A \subset X$ is bounded. Fix $a \in A$. $\exists\ R > 0$ s.t. $A \subseteq B_R(a)$.*

1.7 Compactness

Definition 10 (**Compact Space**). *A compact space is a space that is closed and bounded.*

Definition 11. *A topological space X is said to be* compact *if every open cover of X has a finite subcover. That is, for any collection $\{U_\alpha\}_{\alpha \in I}$ of open sets in X such that $X \subseteq \bigcup_{\alpha \in I} U_\alpha$, there exists a finite subset $J \subseteq I$ such that $X \subseteq \bigcup_{\alpha \in J} U_\alpha$.*

Theorem 1. *A topological space X is compact if and only if every infinite subset of X has an accumulation point in X.*

Proof. Assume X is compact and suppose, for contradiction, that there exists an infinite subset $A \subseteq X$ with no accumulation point in X. Then, for every $x \in X$, there exists an open neighborhood U_x of x such that $U_x \cap A$ is finite (since x cannot be an accumulation point of A). The collection $\{U_x\}_{x \in X}$ forms an open cover of X, but it cannot have a finite subcover since each $U_x \cap A$ is finite, contradicting the assumption that X is compact. Therefore, every infinite subset of X must have an accumulation point.

Assume that every infinite subset of X has an accumulation point in X. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X. Suppose, for contradiction, that there does not exist a finite subcover of X. Then, for each $n \in \mathbb{N}$, we can choose $x_n \in X$ such that $x_n \notin \bigcup_{\alpha \in J_n} U_\alpha$, where $J_n = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a finite subset of I. The set $\{x_n\}_{n \in \mathbb{N}}$ is an infinite subset of X, and by assumption, it must have an accumulation point $x \in X$. Since each U_α is open and covers X, there exists $\alpha \in I$ such that $x \in U_\alpha$. Since U_α is open, there exists an open neighborhood V of x such that $V \subseteq U_\alpha$. However, x is an accumulation point of $\{x_n\}_{n \in \mathbb{N}}$, so V contains infinitely many points of $\{x_n\}_{n \in \mathbb{N}}$. This contradicts the construction of x_n for all $n \in \mathbb{N}$. Therefore, there must exist a finite subcover of X, and hence X is compact. $\square$

Proposition 3. *If A is compact, then A is closed.*

Proof. Assume A is a compact set. We want to show that A is closed.

Suppose A is not closed. i.e., A^c is not open. $\exists x \in A^c$ such that $\forall r > 0$, $B(x, r) \cap A \neq \emptyset$.

Consider the collection $\{B(x, r) : x \in A, B(x, r) \cap A^c \neq \emptyset\}$. By compactness of A, $\exists$ finitely many open balls $\{B(x_i, r_i\}$ that cover A.

Now consider $\bigcap(B(x_i, r_i))$. By construction, $\bigcap(B(x_i, r_i))$ is non-empty and contained in A. However, $\bigcap(B(x_i, r_i))$ also intersects A^c, contradicting the assumption that A^c is disjoint from these open balls. $\therefore$ if A is compact, then A is closed. $\square$

1.8 Connectedness

Theorem 2. *If $f : X \to Y$ onto and continuous and X is connected then Y is connected.*

Proof. If not, then $Y = A \cup B$ is a separation. Then $f^{-1}(A) \cup f^{-1}(B) = X$ is a separation of X. Contradiction. $\square$

Theorem 3. $[0, 1]$ *is connected.*

Proof. If not, then $[0, 1] = A \cup B$ is a separation with $0 \in A$. $B = A^c$ implies that B is closed, and A must be closed by the same argument.
Consider $t = \inf\{x \in B\}$. As B is closed, $t \in B$. B is open, so $t > 0$ and $\exists \epsilon > 0$ such that $t - \epsilon \in B$. Contradiction. $\square$

Definition 12 (**Path Connected**). *A topological space X is path connected if $\forall a, b \in X$, $\exists$ continuous map $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$.*

Theorem 4. *If $f : X \to Y$ is continuous and onto and X is path-connected, then Y is path-connected.*

Definition 13 (**Disconnected**). *A topological space $(X, \mathcal{T})$ is said to be* disconnected *if there exist disjoint nonempty subsets $A, B \subseteq X$ such that $X = A \sqcup B$, and $\bar{A} \cap B = A \cap \bar{B} = \emptyset$. If $(X, \mathcal{T})$ is not disconnected, it is said to be* connected.

Theorem 5. *Path connected implies connected*

- $\mathbb{R}^2 \setminus \{0\}$ *is path connected. In fact, it is path connected for* $\mathbb{R}^n \setminus \{0\}, n \geq 2$.
- S^n *for $n \geq 1$ is path connected.*

Proposition 4. *X any topological space then path-connected implies connectedness.*

Proof. Suppose not. $\exists$ U_1, U_2 such that $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = X$. $x_1 \in U_1, x_2 \in U_2$. $F : I \to X$ such that $F(0) = x_1$ and $F(1) = x_2$. $F(I) = (F(I) \cap U_1) \sqcup (F(I) \cap U_2)$. So $F(I)$ is not connected if x_1 is not connected. Contradiction. $\square$

Proposition 5. *If X is a connected manifold, then it is also path-connected.*

Proof. Assume X is a connected manifold. To show that X is path-connected, suppose there are two points a and b in X with no continuous path connecting them. Define sets $A = \{x \in X : \text{there exists a continuous path from } a \text{ to } x\}$ and $B = \{x \in X : \text{there exists a continuous path from } x \text{ to } b\}$. Since a and b are distinct, a is in A and b is in B. By considering neighborhoods of points in A and B, it can be shown that A and B are open sets. Since X is connected, it cannot be written as the disjoint union of two non-empty open sets.

However, A and B are disjoint open sets whose union is X. This contradicts the assumption of X being connected. Therefore, there must exist a continuous path from a to b, establishing X as path-connected. Thus, X being a connected manifold implies its path-connectedness. $\square$

1.9 Inverse and Implicit Function Theorems

Theorem 6 (**Inverse Function Theorem**). *Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ be a continuously differentiable function. Suppose $a \in U$ such that the derivative $Df(a)$ is invertible. Then there exists open sets V and W such that $a \in V \subseteq U$, $f(a) \in W$, and $f : V \to W$ is a diffeomorphism, where $f : V \to W$ is a bijective continuously differentiable map with a continuously differentiable inverse.*

- Note: A condition for a function to be invertible in a neighborhood of a point in its domain- its derivative is continuous and non-zero at the point.

Theorem 7 (**Implicit Function Theorem**). *Let $U \subseteq \mathbb{R}^{n+m}$ be an open set and $F : U \to \mathbb{R}^m$ be a continuously differentiable function. Suppose $(a, b) \in U$ such that $F(a, b) = 0$ and the partial derivative matrix $\frac{\partial F}{\partial b}(a, b)$ is invertible. Then there exists open sets $V \subseteq \mathbb{R}^n$ containing a and $W \subseteq \mathbb{R}^m$ containing b, as well as a continuously differentiable map $g : V \to W$ such that $F(x, g(x)) = 0$ for all $x \in V$.*

- This is a tool that allows relations to be converted to functions of several real variables by representing the relation as the graph of F.

2 Manifolds

2.1 Manifolds

Definition 14 (**Manifold**). *A* manifold *is a topological space M that is locally Euclidean. More precisely, for every point $p \in M$, there exists an open neighborhood U of p and a homeomorphism $\varphi : U \to V$, where V is an open subset of Euclidean space $\mathbb{R}^n$, for some positive integer n. The pair (U, φ) is called a* chart*, and the collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ covering M is called an* atlas.

Definition 15 (**Hausdorff Space**). *A topological space X is called a* Hausdorff space *(or T_2* space*) if for any two distinct points $x, y \in X$, there exist open neighborhoods U of x and V of y such that $U \cap V = \varnothing$.*

- For any two points, there exists two disjoint, open neighborhoods such that the intersection is empty and it is vacuously true.
- An example of a space which is *not* Hausdorff is a line with two origins. A *line with two origins* is a mathematical structure consisting of a set L along with two distinct points O_1 and O_2 called *origins*, and a notion of line segment between any two points in L. The line segment between any two points A and B is denoted as $\overline{AB}$.

  The line with two origins satisfies the following properties:

  1. For any two distinct points A and B in L, there exists a unique line segment $\overline{AB}$ between them.

2. If A, B, and C are three points in L such that B lies on $\overline{AC}$, then $\overline{AB} \cup \overline{BC} = \overline{AC}$.
3. For any point A in L, there exists a unique point A' in L such that $\overline{AA'} = \overline{O_1O_2}$.

Definition 16 (**Stereographic Projection**). *A planar perspective projection, viewed from the point on the globe opposite the point of tangency. It projects points on a spheroid directly to the plane and it is the only azimuthal conformal projection. The projection is most commonly used in polar aspects for topographic maps of polar regions.*

Definition 17 (**Stereographic Projection (alternative definition)**). *Perspective projection of the sphere through a specific point on the sphere onto a plane, perpendicular to the diameter through the point.*

- *It is a smooth, bijective function from the entire sphere except the center of projection to the entire plane.*
- *It preserves angles at which curves meet and thus locally approximately preserves shapes. It is neither isometric (distance preserving) nor area-preserving.*

Definition 18 (**Differentiability on Manifolds**). *Let M and N be smooth manifolds. A map $f : M \to N$ is said to be* differentiable *(or* smooth*) at a point $p \in M$ if, for every chart (U, φ) around p and (V, ψ) around $f(p)$, the composite map $\psi \circ f \circ \varphi^{-1} : \varphi(U) \subseteq \mathbb{R}^m \to \psi(V) \subseteq \mathbb{R}^n$ is differentiable at $\varphi(p)$ in the usual sense of differentiability in Euclidean space.*

If f is differentiable at every point $p \in M$, we say that f is a differentiable map *(or* smooth map*) from M to N.*

For more clarity, I will provide a quick definition of homeomorphism and diffeomorphism.

Definition 19 (**Homeomorphism**). *A function $f : X \to Y$ between two topological spaces is a* homeomorphism *if it satisfies the following properties:*

1. *f is a bijection*
2. *f is continuous*
3. *f^{-1} is continuous*

Definition 20 (**Diffeomorphism**). *A function $f : X \to Y$ between two topological spaces is a* diffeomorphism *if it satisfies the following properties:*

1. *f is a bijection*
2. *f is smooth*
3. *f^{-1} is smooth*

where smooth means infinitely differentiable.

I would now like to introduce a more precise definition of a manifold in order to prepare for the following sections.

Definition 21. *We say that M is a topological manifold of dimension n (or a topological n-manifold) if it satisfies the following properties:*

- *M is a Hausdorff space: for every pair of distinct points $p, q \in M$ there are disjoint, open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.*
- *M is second-countable: there exists a countable basis for the topology of M.*
- *M is locally Euclidean of dimension n: each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.*

2.2 Fiber Bundles

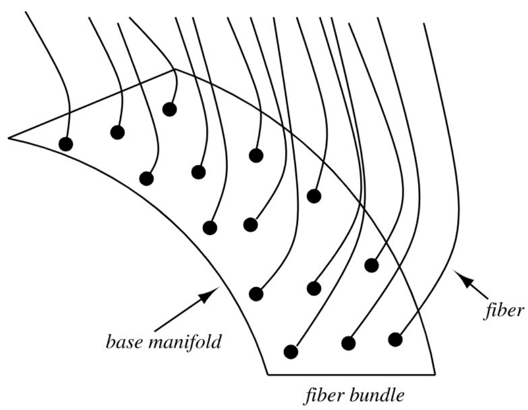


Figure 1: Diagram of a Fiber Bundle

Definition 22 (**Fiber Bundle**)**.** *Let M and B be topological spaces. A* fiber bundle *over B with typical fiber F is a quadruple (E, π, B, F) consisting of:*

1. *A topological space E called the* total space.
2. *A continuous surjective map $\pi : E \to B$ called the* projection map.
3. *Topological spaces B and F, called the* base space *and* fiber *respectively.*

4. *For each $p \in B$, there exists an open neighborhood U of p in B and a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes:*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ \downarrow & & \downarrow \\ U & \xrightarrow{proj_1} & U \end{array}$$

Here, $proj_1$ denotes the projection onto the first factor.

Definition 23 (**Trivial Vector Bundle**). *For X a topological space, then a topological vector bundle $E \to X$ over a topological field K is called trivial if its total space is the product topological spaces:*

$$E \to X \times K^n \xrightarrow{pr_1} X$$

with the topological vector space K^n for some $n \in N$. For $n = 1$, one also speaks of the trivial line bundle. An isomorphism of vector bundles over X of the form

$$E \to X \times \mathbb{R}^n$$

is called a trivialization of E. If E admits such an isomorphism, then it is called a trivializable vector bundle.

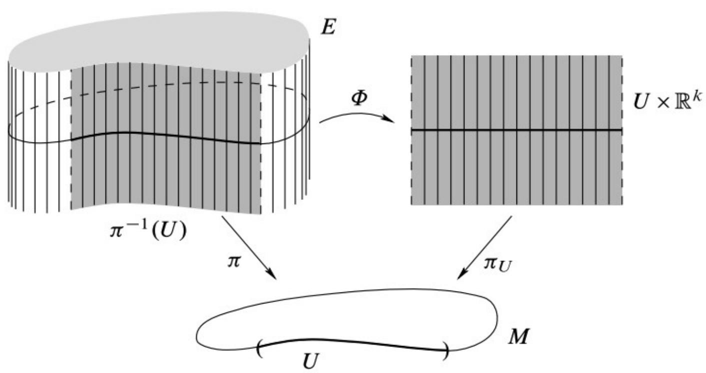


Figure 2: A local trivialization of a vector bundle (Lee).

Proposition 6. *A vector bundle is trivial if there exists a section which is nonzero.*

Proof. Assume the vector bundle is trivial: $E = M \times V$. We will now define a section $\sigma : M \to E$ such that $\sigma(p) = V_p \; \forall \; p \in M$, where V_p is a nonzero vector in V. Since the vector bundle is trivial, σ is a nonzero section. $\square$

Definition 24 (**Tangent Bundle**). *Let M be a topological space. A real vector bundle of rank k over M is a topological space E together with a surjective, continuous map $\pi : E \to M$ satisfying:*

1. *$\forall$ $p \in M$, the set $E_p = \pi^{-1}(p) < E$ (fiber of E over p) is endowed with the structure of a k-dimension real vector space*
2. *$\forall$ $p \in M$, $\exists$ nbhd $U(p) \in M$ and a homeomorphism $\Phi : \pi^{-1}(u) \to U \times \mathbb{R}^k$ such that:*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^k \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

Proposition 7. *There exists K-many linearly independent sections if any only if $E \to M$ is trivial. i.e. $E \to M$ is a vector bundle.*

Proof. If E is isomorphic to $M \times \mathbb{R}^k$ then we can define $\delta_i : M \to E$ by $\delta_i(m) = (m, p_1)$ for a fixed basis $p_1 ... p_k$ of $\mathbb{R}^k$. Then $\delta_i ... \delta_k$ are linearly independent at each point. Now suppose that the sections $\delta_1 ... \delta_k$ exist. We can define a bundle map $f : E \to M \times \mathbb{R}^k$ in the following manner: for a point $(m, v) \in E$, we have

$$V = \sum_{i=1}^{n} a_i \delta_i(m)$$

Since $\delta_1(m), ..., \delta_k(b)$ are linearly independent the expression for V above is always possible. We will now map (m, v) to $(m, (a_1, ..., a_k))$. This map is smooth since the δ_i are smooth. Therefore, $E \to M \times \mathbb{R}^k$ is a bundle isomorphism. □

Proposition 8. *The tangent bundle of a circle is trivial. (See Figure 3).*

Proof. Consider the circle as a smooth manifold embedded in the Euclidean plane. Let M be the circle and TM be its tangent bundle. To show that TM is trivial, we need to find a global, smooth frame for TM.

Define the vector field $V(\theta) = (\frac{\partial}{\partial \theta})$, which represents a tangent vector pointing in the direction of increasing θ at each point on the circle. Since the circle is 1-dim, $V(\theta)$ spans the tangent space at every point.

Now we shall construct a diffeomorphism $\varphi(p, V) = (p, \lambda)$, where p is a point on the circle, V is a tangent vector at p, and λ represents the component of V along the Euclidean line.

This map, φ, is well-defined, smooth, bijective, and its inverse is also smooth. Therefore TM is isomorphic to $M \times \mathbb{R}$, indicating its triviality. □

Proposition 9. *Möbius bundle is not the same as the trivial bundle.*

- *The proof is far too complex to cover.*

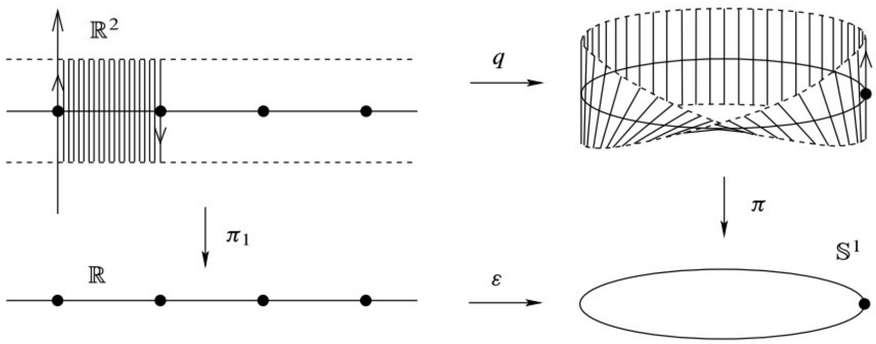


Figure 3: Non-trivial bundle of the Möbius Band (Lee).

2.3 Smooth Manifolds

Definition 25 (**Smooth Manifold**)**.** *A* smooth manifold *is a topological manifold so that if $\varphi : U \to V$ and $\varphi' : U' \to V'$ are local homeomorphic then $\varphi' \circ \varphi^{-1} : \varphi(U' \cap U) \to \varphi(U' \cap U)$ is smooth.*

Definition 26 (**Smooth Manifold (alternative definition)**)**.** *A* smooth manifold *is a second-countable Hausdorff topological space M equipped with an atlas $\mathcal{A}$ consisting of charts $(U_\alpha, \varphi_\alpha)$, where U_α is an open subset of M and $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism for each α.*

The atlas $\mathcal{A}$ must satisfy the following properties:

1. Covering property*: The union of the domains of the charts covers the entire manifold, i.e., $\bigcup_\alpha U_\alpha = M$.*

2. Smoothness condition*: For any two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) in the atlas $\mathcal{A}$ with non-empty intersection $U_\alpha \cap U_\beta$, the map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map in the sense of differential calculus on Euclidean spaces.*

The charts in the atlas are often referred to as smooth coordinate charts *(See Figure 4).*

3 Exotic Manifolds

3.1 Introduction

Definition 27. *An* exotic manifold *is a differentiable manifold that is homeomorphic but not diffeomorphic to a standard manifold of the same dimension.*

I will now provide a brief history of exotic manifolds, which gives us helpful insight into how we can find and construct them. Tibor Rado (1925) proved that

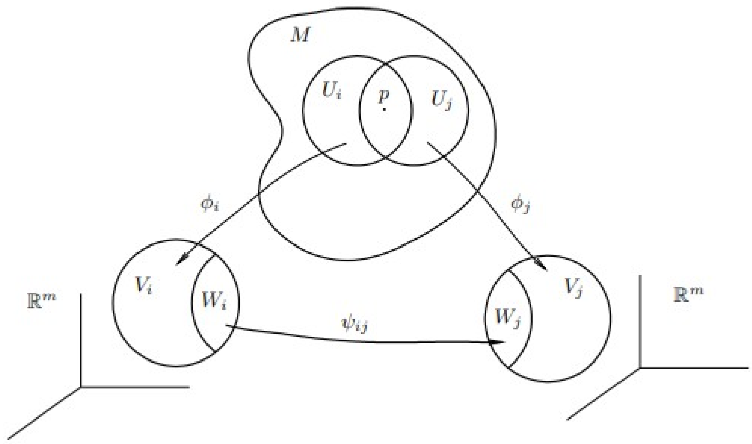


Figure 4: Any two overlapping charts of an atlas must satisfy a compatibility condition in the overlapping region (UC Berkeley).

in dimension 2 there are no exotic differentiable structures or the uniqueness of the standard structure. Moise (1952) proved that in dimension 3 there are no exotic differentiable structures, or to put in another way, 3-dimensional differentiable manifolds which are homeomorphic are diffeomorphic. In this way, S^3 inherits a unique differentiable structure, no matter which $\mathbb{R}^4$ it is considered to be embedded in. Milnor (1956) showed that a smooth manifold is homeomorphic to the S^7 but not diffeomorphic. Freedman (1982) classified simply connected 4-manifolds up to homeomorphism, and Donaldson (1983) showed that there are exotic smooth structures on $\mathbb{R}^4$.

To summarize this information:

- There are no exotic smooth structures in dimensions ≤ 3
- There exists a unique smooth structure on the Euclidean space $\mathbb{R}^n$ for $n \neq 4$.
- There exists uncountably many exotic smooth structures on the Euclidean space $\mathbb{R}^4$ of dimension 4.

3.2 Hopf Fibration

In this section, we will discuss Hopf fibrations, which are fiber bundles in which the base space, the fibers, and the total space are all spheres.

The *complex Hopf fibration* is a nontrivial circle principal bundle over the 2-sphere whose total space is the 3-sphere. This is the fiber bundle with fibers homeomorphic to S^1 and base space S^2. The total space of this fiber bundle is a 3-sphere. So, we represent this fiber bundle with the following diagram:

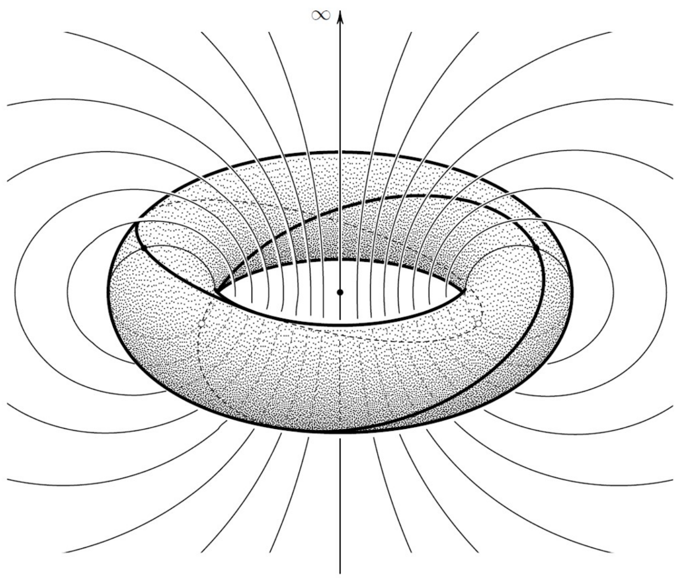


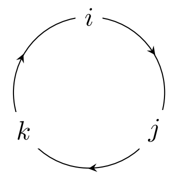
Figure 5: The Hopf fibration

$$S^1 \hookrightarrow S^3 \to S^2$$

We can explicitly write down the projection map in the complex Hopf fibration as $\pi : (z_1, z_2) \mapsto [z_1, z_2]$. Note that every two fibers are linked with linking number one in the total space. This is a qualitative difference from the relationship between the fibers in the fiber bundle whose fibers and base space are copies of S^1 and whose total space is a torus. In that case, the linking number of any two fibers is zero.

We will now introduce *Quaternionic Hopf fibration.* There are only four Hopf fibrations, which correspond to the four division algebras. The four division algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. The quaternions are denoted $\mathbb{H}$, and they have non-commutative multiplication. More formally,

Definition 28. $\mathbb{H}$ *is the set of numbers of the form* $a + bi + cj + dk$*, where* $ij = k = -ji$ *and addition is component-wise. Note the following diagram regarding the properties of addition:*



The quaternionic Hopf fibration is formed exactly analogously to the complex Hopf vibration. So it has the following diagram:

$$S^3 \hookrightarrow S^7 \to S^4$$

where S^3 is the set of unit quaternions and S^7 is the set of pairs of quaternions with the unit norm. The maps are also analogous to the complex case: $\pi : (z_1, z_2) \mapsto [z_1, z_2]$. The preimage of the point $[1; 0]$ under the projection map is π is $\{z_1 \in \mathbb{H} |\ \|z_1\|^2 = 1\}$, which is a 3-sphere embedded in the total space. The only difference is in the definition of projective space. Projective space is defined by putting an equivalence relation on any two points that are on the same line through the origin. That is:

$$(z_1, z_2) \sim (\lambda z_1, \lambda z_2)$$

Since multiplication in the quaternions is not commutative, the definition of quaternionic projective spaces includes a choice of default multiplication. We choose left-multiplication. This is a degree of freedom that did not occur in the complex Hopf fibration and is the reason why the simplest known example of exotic smooth structure occurs in seven dimensions, instead of three dimensions.

The Quaternionic Hopf Fibration plays a pivotal role in the construction of exotic manifolds, particularly in relation to the emergence of the S^7 from the set of unit norm.

3.3 Morse Theory

Morse theory is the method of studying the topology of a smooth manifold M by the study of Morse functions $M \to \mathbb{R}$ and their associated gradient flows. In other words, it is a way of translating the homotopy type of a manifold into statements about critical points of particular functions.

Example 3.1. *Let $X \to Y$ be a surjective submersion of compact smooth manifolds, and assume Y is connected. By suitable implicit function theorems, the preimage of any parametrized non-stationary curve $\gamma : (0, 1) \to Y$ is a submanifold of X, and furthermore, the parameter is a Morse function on this submanifold, having no critical points. (It is not coercive). By a little more analysis, the Morse gradient flow is therefore a smooth family of homotopy equivalences. A trivial adjustment of the Riemann structure further allows the Morse flow to send fibers to fibers diffeomorphically, so that in fact the fibers over neighboring points of Y are diffeomorphic. But since Y is connected, this implies that all* *the fibers are diffeomorphic, so that $X \to Y$ is a smooth fiber bundle over Y. (McEnroe).*

We will now discuss Reeb's Theorem and the proof of such.

Theorem 8. *If M is a compact smooth manifold of dimension n, and f is a differentiable function on M with only two nondegenerate critical points, then M is homeomorphic to a standard sphere.*

- This theorem states that if a differentiable manifold has a differentiable function with exactly two critical points, then it is homeomorphic to the standard Euclidean d-sphere

Proof. Let M and f be as in the theorem statement. Then one of the critical points, p, must be a maximum and the other, q, must be a minimum. By definition, in any small neighborhood around p, the vector field must be directed outward in every direction. So the point p corresponds to an n-cell. By definition, a minimum has the vector field directed inward everywhere. So q corresponds to a 0-cell. M is the union of an n-cell and a 0-cell and therefore is homeomorphic to an n-sphere. $\square$

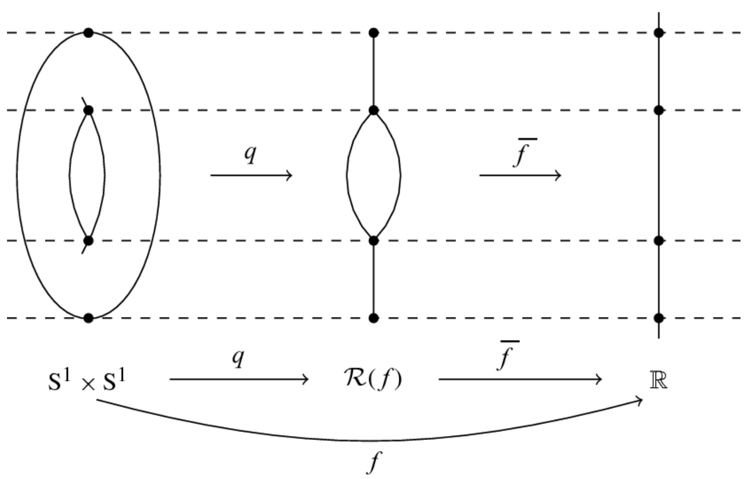


Figure 6: Example of the Reeb graph of a height function (Research Gate).

3.4 S^3 bundles over S^4

We first need to begin with the definition of homotopy as well as the definition of a loop:

Definition 29 (**Homotopy**). *A continuous transformation from one function to another. A* homotopy *between two functions f and g from a space X to a space Y is a continuous map G from $X \times [0,1] \mapsto Y$ such that $G(x,0) = f(x)$ and $G(x,1) = g(x)$, where $\times$ denotes set pairing.*

Note that a *homotopy* between two functions f and g from space X to space Y can be designed as a continuous map G where $X[0,1] \mapsto Y$ such that $G(x,0) = f(x)$ and $G(x,1) = g(x)$.

Definition 30 (**Loop**). *$F : I \to X$ such that $F(0) = F(1) = x_0$.*

Now fix $x_0 \in X$ (from the definition of a loop). If any two loops based at x_0 are homotopic, then X is simply connected. (See the definition of simply connected below).

Definition 31 (**Simply Connected**). *A topological space X is* simply connected *if it is* path connected *and any loop X defined by $f : S^1 \to X$ can be contracted to a point: there exists a continuous map $F : D^2 \to X$ such that F restricted to S^1 is f. Note that X is simply connected if and only if it is path-connected.*

Definition 32 (**Homotopic Group**). *$f, g : S^n \to X$ are same in $\pi_n(x)$ if they are homotopic.*

We will also need the formal definition of characteristic classes. Characteristic classes will be used later on to prove the exoticness of structures.

Definition 33 (**Characteristic Classes**). *Characteristic classes are global invariants that measure the deviation of a local product structure from a global product structure. (They are one of the unifying geometric concepts in algebraic topology, differential geometry, and algebraic geometry.)*

Characteristic classes provide a means to extend the concept of the linking number, as previously introduced in the section on Hopf fibrations. Our objective is to characterize the interconnections among the fibers over a manifold, akin to the linking number in the context of the complex Hopf fibration. This generalization is known as the Euler class. Furthermore, we seek additional characteristic classes that enable us to differentiate our exotic spheres. These classes offer a more nuanced understanding of the twisting and intertwining of the fibers, going beyond a generalized notion of linking.

The exotic 7-spheres constructed by Milnor in 1956 are all examples of fiber bundles over the S^4 with S^3 fibers, with the structure group the special orthogonal group $SO(4)$. We will now discuss this. Note that we will need complicated algebraic topology in order to do this, most of which will not be covered due to the complexity.

Consider S^4, which can be covered by an atlas with just two charts, as is the case for any topological sphere. Each chart corresponds to an $\mathbb{R}^4$ bundle, which is trivial because the charts are contractible. To obtain a non-trivial bundle, we introduce a notion of "twisting" in the gluing maps between the fibers. Our goal is to construct a family of such gluing maps that yield diverse total spaces.

Let us designate the first chart as U_1, encompassing all of S^4 except for the south pole, and the second chart as U_2, covering all of S^4 except for the north pole. To establish the desired maps, we introduce ϕ_1 and ϕ_2 as the mappings from each chart to $\mathbb{R}^4$, respectively.

$$\phi_1 : U_1 \to \mathbb{R}^4 = \mathbb{H}$$
$$[z; 1] \mapsto z$$
$$\phi_2 : U_2 \to \mathbb{R}^4 = \mathbb{H}$$
$$[1; w] \mapsto w$$

Now note that the transition map is:

$$\phi_2 \circ \phi_1^{-1} : \mathbb{H} - \{0\} \to \mathbb{H} - \{0\}$$
$$z \mapsto \frac{1}{z}$$

We now want to construct the local trivializations of the fiber bundle on each chart of the base space and then matched. This, however, requires algebraic topology, as mentioned previously, and will not be discussed in detail. Instead, I will outline the general process of bundling S^3 over S^4.

Once the local trivializations are constructed, we will consider the preimage of the first chart of the base space in the total space, then the preimage of the second chart in the total space of the Hopf fibration. Then we will take these preimages, and compute the transition map of the trivializations of such. This results in the following transition map:

$$\rho_2 \circ \rho_1 : \phi_1(U_1 \cap U_2) \times \mathbb{H} \to \phi_2(U_1 \cup U_2) \times \mathbb{H}$$
$$(z, y) \mapsto \left(\frac{1}{z}, \frac{yz}{||z||} \right)$$

We will perform a gluing operation that involves connecting (z, y) to $\left(\frac{1}{z}, \frac{yz}{||z||} \right)$. Notably, the coordinates on the base space remain unchanged, indicating that the primary gluing occurs within the fibers, which represent quaternion lines. This gluing process introduces a twist. Our objective is to have the fiber represent the unit quaternions, which can be visualized as 3-spheres. Consequently, it becomes necessary to normalize the coordinates on the fiber.

Through this operation, we get a total space that is a 7-sphere.

3.5 The exotic sphere is a topological 7-sphere

In order to prove that the exotic sphere, as created from bundling S^3 over S^4, we must use Morse Theory, specifically Reeb's Theorem. The details of this proof, however, are far too complex to cover, so we will instead discuss the general idea as we did in the previous section.

Proposition 10. *$M_{h,l}$ is a topological sphere if $h + l = -1$.*

Note that this notation comes from the definition of Reeb's Theorem, (Theorem 8).

Let's consider the points of the form $(z, \pm 1)$. If $||z|| \neq 0$, the

$$\nabla \left(\frac{1}{\sqrt{1 + ||z||^2}} = \frac{\pm 2||z||}{\sqrt{1 + ||z||^2}} \neq 0 \right)$$

Therefore, within the first chart, the critical points of interest are $(0, 1)$ and $(0, -1)$. Conversely, upon examining the second chart, it becomes evident that it lacks any critical points. As a result, the function g possesses solely two critical points on $M_{h,l}$ when the condition $h + l = -1$ is satisfied. Consequently, by invoking Reeb's Theorem, we can establish the homeomorphism between $M_{h,l}$ and S^7 under this particular condition.

We have shown that a homeomorphism exists between $M_{h,l}$ and S^7, but in order to show that they are an exotic pair, we have to show that they fail diffeomorphism. We will show this by constructing a smooth manifold from $M_{h,l}$ and then using the Hirzebruch Signature Theorem, a major result from characteristic classes, to find a contradiction.

Theorem 9 (**Hirzebruch Signature Theorem**)**.** *Let M be a closed orientable smooth manifold of dimension 8, with signature $\tau(M)$. Then,*

$$\tau(M) = \frac{1}{45}(7p_2(M) - p_1^2(M))$$

Now we can apply the Hirzebruch Signature Theorem to our manifold, $M_{h,l}$. We have:

$$\pm 1 = \frac{1}{45}(7p_2(M_{h,l}) - (\pm 2(h - l))^2)$$

We will mod out by 7 in order to eliminate $p_2(M_{h,l})$:

$$3 = \pm 4(h - l)^2 (\text{mod} 7)$$

$$(h - l)^2 = 1 (\text{mod} 7)$$

It is a necessary condition for $M_{h,l}$ to have a differentiable structure and there was a faulty assumption that our manifold could be glued to S^7 smoothly. Therefore, $M_{h,l}$ is not diffeomorphic to S^7 if $(h - l)^2 \neq 1(\text{mod} 7)$. So, $M_{0,1}$ and $M_{1,0}$ which are the base spaces of the standard quaternionic Hopf fibrations are both diffeomorphic to S^7, as they should be, but $M_{3,-2}$, for instance, is not.

To summarize, we have constructed a family of manifolds that are homeomorphic but not diffeomorphic to the 7-sphere. These are the simplest examples of exotic smooth structures known. The tool used to construct this manifold is the quaternionic Hopf fibration. The non-commutativity of the quaternions causes the set of possible S^3 bundles over S^4 to be classified by $Z \bigoplus Z$. This gives enough ‘room’ in the set of candidate manifolds for exotic structures to exist. The homeomorphism between the base spaces $M_{h,l}$ and S^7 was shown for $h + l \pm 1$ by Morse Theory, and, in particular, Reeb's Theorem. The non-diffeomorphism for $(h - l)^2 \neq 1(\text{mod}7)$ is shown by constructing a smooth manifold from $M_{h,l}$ and then using the Hirzebruch Signature Theorem to find a contradiction.

4 References

References

[1] Lee, John M., “Introduction to Smooth Manifolds,” June 2013.

[2] McEnroe, Rachel, “Milnor's Construction of Exotic 7-Spheres,” Reading, 2015.

[3] Milnor, John, “On Manifolds Homeomorphic to the 7-Sphere,” June 1956.

[4] Oklahoma State University, *Chapter 7: Metric Spaces*, Reading, 2011.

[5] UC Berkeley, *Phy250 Notes: Manifolds, Tangent Vectors, and Covectors*, Reading, 2015.