Skill acquisition under certainty w/ education types.

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Abstract

OK so now we have ‘reasonable’ statics and have pinned down the velocity term. How readable is this? Are we missing any low-hanging statics?

1 Introduction

2 Model

We will start with the planner's-problem version of our model. Since the model has constant returns to scale, a single final good, and infinitely many workers producing each intermediate good[1], the planner's and competitive solutions coincide[2].

2.1 Planner's Problem

The Planner is a net-production maximizer. She has a population of workers divided into types in the finite set Θ.[3] Each type θ comes in a quantity q_θ. Endowing workers with skills is costly to the planner, but workers lacking skills produce no output.

Endowing a single worker of type θ with skills $I \in \mathbb{R}_+^N$ costs

$$C(I, \theta) = \sum_i I_i^\rho \theta_i \tag{1}$$

where $\rho > 1$.

Workers must be assigned to jobs. The set of jobs $\mathcal{J}$ is the standard simplex in $\mathbb{R}^N$, that is

$$\mathcal{J} = \{J \in \mathbb{R}^N | \forall i, J_i \geq 0; \sum_i J_i = 1\}. \tag{2}$$

A worker with skills I assigned to job J produces $J^T I$ units of the good indexed by J, or 'J-widgets'.

Widgets are aggregated into final goods using a CES aggregator. If workers produce $y : \mathcal{J} \to \mathbb{R}^+$ then global production is

$$Y = \left(\int_{\mathcal{J}} h(J)^\varepsilon y(J)^\varepsilon dJ \right)^{\frac{1}{\varepsilon}} \tag{3}$$

[1]This guarantees price-taking, so that workers will not distort their skill investment.

[2]OK, technically, the competitive market also has an equilibrium where noone invests, produces, or consumes. Do you care? Write your senator.

[3]We assume no two $\theta, \theta' \in \Theta$ are collinear. This is necessary for uniqueness of the assignment of workers to jobs (although wages net of investment costs would still be unique). Our model is therefore concerned with worker types that are qualitatively different in their aptitudes, not more or less apt in all skills in a uniform way.

where $\varepsilon < 1$ and h is a demand-shifting function; it is non-negative and integrable.

The planner's problem involves:

- Choosing how to distribute the workers of each type across the job set;
- Choosing the skill investment in each worker.

The planner's problem can be written as:

$$\max_{y:\mathcal{J}\to\mathbb{R}^+,(f_\theta:\mathcal{J}\to\mathbb{R}_+)_{\theta\in\Theta},I:\mathcal{J}\times\Theta\to\mathbb{R}^n_+} \left[\left(\int_{\mathcal{J}} h(J)^\varepsilon y(J)^\varepsilon dJ\right)^{\frac{1}{\varepsilon}} - \textstyle\sum_\theta \left[\int_{\mathcal{J}} f_\theta(J) C(I(J),\theta) dJ\right]\right] \tag{4}$$
$$\text{such that for each } J, \quad y(J) = \textstyle\sum_\theta [f_\theta(J) J^T I(J,\theta)]$$
$$\text{and for each } \theta, \quad \textstyle\int_{\mathcal{J}} f_\theta(J) dJ = q_\theta$$

where we have already restricted workers of the same type doing the same job to identical investments as investment costs are convex.

In the interest of intuition, we will tackle this problem in its dual cost-minimization form. Suppose a quantity $\bar{Y}$ is to be produced; the planner seeks to

$$\min_{y:\mathcal{J}\to\mathbb{R}^+,(f_\theta:\mathcal{J}\to\mathbb{R}_+)_{\theta\in\Theta},I:\mathcal{J}\times\Theta\to\mathbb{R}^n_+} \sum_\theta \left[\int_{\mathcal{J}} f_\theta(J) C(I(\theta,J),\theta) dJ\right] \tag{5}$$
$$\text{such that for each } J, \quad y(J) = \sum_\theta [f_\theta(J) J^T I(J,\theta)]$$
$$\text{and for each } \theta, \quad \int_{\mathcal{J}} f_\theta(J) dJ = q_\theta$$
$$\text{and finally} \quad \left(\int_{\mathcal{J}} h(J)^\varepsilon y(J)^\varepsilon dJ\right)^{\frac{1}{\varepsilon}} = \bar{Y}$$

We'll build from the ground up. Suppose $y(J)$ units of good J are to be produced by $f_\theta(J)$ workers of type θ. To find the cheapest skill investment accomplishing this we

$$\min_{I\in\mathbb{R}^n_+} \quad f_\theta(J) C(I(J)) \tag{6}$$
$$\text{subject to} \quad f_\theta(J) J^T I = y(J)$$

which is solved by the individual worker's investments given by

$$I_i^*(J, f_\theta(J), y(J), \theta) = \frac{f_\theta(J)^{-\rho} y(J)^\rho J_i^{\frac{1}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}}{\sum_n J_n^{\frac{\rho}{\rho-1}} \theta_n^{-\frac{1}{\rho-1}}}. \tag{7}$$

and therefore the total costs of optimally producing a quantity $y(J)$ of J-widgets using $f_\theta(J)$ workers of type θ are

$$f_\theta(J) C(I^*) = y(J)^\rho f_\theta(J)^{1-\rho} \left(\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}} \right)^{1-\rho}.$$

This shows that the *cost ratio* at which identical measures of workers of two types produce a given quantity of a widget *does not depend* on the quantity produced. Therefore, f_θ and $f_{\theta'}$ can have supports that overlap only at points at which this productivity ratio is constant. Thus assignment must be a partition[4]. For convenience, we'll treat all cells of the partition as closed (which does not affect the minimand).

Using such a closed-partitional assignment given by $\Gamma : \Theta \to 2^{\mathcal{J}}$, we can break down the integral in the CES production function by the component due to each type. Call X_θ a 'production target for type θ' and solve the problem of minimizing the cost of investment such that type θ's component of the production integral equals X_θ:

$$X_\theta = \int_{\Gamma(\theta)} h(J)^\varepsilon y(J)^\varepsilon dJ.$$

Notice this expression lacks the outer power $\frac{1}{\varepsilon}$; this is as production targets will be summed across types and then the power will be applied, to get final goods production. Let's minimize investment costs for which θ workers jointly produce X_θ in jobs $\Gamma(\theta)$:

$$\min_{y:\Gamma(\theta)\to\mathbb{R}_+^n, f_\theta:\Gamma(\theta)\to\mathbb{R}_+} \int_{\Gamma(\theta)} y(J)^\rho f_\theta(J)^{1-\rho} \left(\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}} \right)^{1-\rho} dJ \tag{8}$$

$$\text{subject to the target} \quad \int_{\Gamma(\theta)} h(J)^\varepsilon y(J)^\varepsilon dJ = X_\theta$$

$$\text{and the } \theta \text{ population} \quad \int_{\Gamma(\theta)} f_\theta(J) dJ = q_\theta$$

[4]Or rather, differ from a partition by a set of measure 0.

Now, we write the Lagrangian:

$$\min_{y:\Gamma(\theta)\to\mathbb{R}^n_+, f_\theta:\Gamma(\theta)\to\mathbb{R}_+} \int_{\Gamma(\theta)} y(J)^\rho f_\theta(J)^{1-\rho}\left(\sum_i J_i^{\frac{\rho}{\rho-1}}\theta_i^{-\frac{1}{\rho-1}}\right)^{1-\rho} dJ \tag{9}$$

$$-\lambda\left[\int_{\Gamma(\theta)} h(J)^\varepsilon y(J)^\varepsilon dJ - X_\theta\right] - \mu\left[\int_{\Gamma(\theta)} f_\theta(J)dJ - q_\theta\right]$$

The pointwise first-order conditions with respect to $y(J)$ and $f_\theta(J)$ are

$$\rho y(J)^{\rho-1} f(J)^{1-\rho}\left(\sum_i J_i^{\frac{\rho}{\rho-1}}\theta_i^{-\frac{1}{\rho-1}}\right)^{1-\rho} = \varepsilon\lambda h(J)^\varepsilon y(J)^{\varepsilon-1} \tag{10}$$

and

$$(\rho-1) f_\theta(J)^{-\rho} y(J)^\rho \left(\sum_i J_i^{\frac{\rho}{\rho-1}}\theta_i^{-\frac{1}{\rho-1}}\right)^{1-\rho} = \mu \tag{11}$$

which sure seem uninformative! But, if we take the *ratio* of each FOC at J to it's value at J' we get

$$\left(\frac{y(J)}{y(J')}\right)^{\rho-1}\left(\frac{f_\theta(J)}{f_\theta(J')}\right)^{1-\rho}\left(\frac{\sum_i J_i^{\frac{\rho}{\rho-1}}\theta_i^{-\frac{1}{\rho-1}}}{\sum_i J_i'^{\frac{\rho}{\rho-1}}\theta_i^{-\frac{1}{\rho-1}}}\right)^{1-\rho} = \left(\frac{h(J)}{h(J')}\right)^\varepsilon\left(\frac{y(J)}{y(J')}\right)^{\varepsilon-1} \tag{12}$$

and

$$\left(\frac{y(J)}{y(J')}\right)^{\rho}\left(\frac{\sum_i J_i^{\frac{\rho}{\rho-1}}\theta_i^{-\frac{1}{\rho-1}}}{\sum_i J_i'^{\frac{\rho}{\rho-1}}\theta_i^{-\frac{1}{\rho-1}}}\right)^{1-\rho} = \left(\frac{f_\theta(J)}{f_\theta(J')}\right)^{\rho}. \tag{13}$$

Now we can substitute out $y(J)/y(J')$ and rearrange to get

$$f_\theta(J) = f_\theta(J')\left(\frac{\sum_i J_i^{\frac{\rho}{\rho-1}}\theta_i^{-\frac{1}{\rho-1}}}{\sum_i J_i'^{\frac{\rho}{\rho-1}}\theta_i^{-\frac{1}{\rho-1}}}\right)^{\frac{\varepsilon}{1-\varepsilon}\frac{\rho-1}{\rho}}\left(\frac{h(J)}{h(J')}\right)^{\frac{\varepsilon}{1-\varepsilon}}.$$

We can now rearrange this again and integrate dJ'

$$f_\theta(J)\int_{\Gamma(\theta)}\left(\sum_i J_i'^{\frac{\rho}{\rho-1}}\theta_i^{-\frac{1}{\rho-1}}\right)^{\frac{\varepsilon}{1-\varepsilon}\frac{\rho-1}{\rho}} h(J')^{\frac{\varepsilon}{1-\varepsilon}}dJ' = \int_{\Gamma(\theta)} f_\theta(J')dJ'\left(\sum_i J_i^{\frac{\rho}{\rho-1}}\theta_i^{-\frac{1}{\rho-1}}\right)^{\frac{\varepsilon}{1-\varepsilon}\frac{\rho-1}{\rho}} h(J)^{\frac{\varepsilon}{1-\varepsilon}}$$

so that recalling that f_θ has to integrate to q_θ by the constraint, we get

$$f_\theta(J) = \frac{q_\theta \left(\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}\right)^{\frac{\varepsilon}{1-\varepsilon}\frac{\rho-1}{\rho}} h(J)^{\frac{\varepsilon}{1-\varepsilon}}}{\int_{\Gamma(\theta)} \left(\sum_i J_i'^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}\right)^{\frac{\varepsilon}{1-\varepsilon}\frac{\rho-1}{\rho}} h(J')^{\frac{\varepsilon}{1-\varepsilon}} dJ'}.$$

What joy then to plug this in to (13) and get

$$\frac{y(J)}{y(J')} \left(\frac{\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}}{\sum_i J_i'^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}}\right)^{\frac{1-\rho}{\rho}} = \frac{\left(\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}\right)^{\frac{\varepsilon}{1-\varepsilon}\frac{\rho-1}{\rho}} h(J)^{\frac{\varepsilon}{1-\varepsilon}}}{\left(\sum_i J_i'^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}\right)^{\frac{\varepsilon}{1-\varepsilon}\frac{\rho-1}{\rho}} h(J')^{\frac{\varepsilon}{1-\varepsilon}}} \tag{14}$$

or

$$y(J) = y(J') \left(\frac{\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}}{\sum_i J_i'^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}}\right)^{\frac{\rho-1}{\rho}\frac{1}{1-\varepsilon}} \left(\frac{h(J)}{h(J')}\right)^{\frac{\varepsilon}{1-\varepsilon}} \tag{15}$$

which we can make use of in the y-constraint:

$$\int_{\Gamma(\theta)} h(J)^\varepsilon \left(y(J') \left(\frac{\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}}{\sum_i J_i'^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}}\right)^{\frac{\rho-1}{\rho}\frac{1}{1-\varepsilon}} \left(\frac{h(J)}{h(J')}\right)^{\frac{\varepsilon}{1-\varepsilon}} \right)^\varepsilon dJ = X_\theta$$

$$X_\theta h(J')^{\frac{\varepsilon^2}{1-\varepsilon}} y(J')^{-\varepsilon} \left(\sum_i J_i'^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}\right)^{\frac{\rho-1}{\rho}\frac{\varepsilon}{1-\varepsilon}} = \int_{\Gamma(\theta)} h(J)^{\frac{\varepsilon}{1-\varepsilon}} \left(\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}\right)^{\frac{\rho-1}{\rho}\frac{\varepsilon}{1-\varepsilon}} dJ$$

or finally

$$y(J') = X_\theta^{\frac{1}{\varepsilon}} \frac{h(J')^{\frac{\varepsilon}{1-\varepsilon}} \left(\sum_i J_i'^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}\right)^{\frac{\rho-1}{\rho}\frac{1}{1-\varepsilon}}}{\left(\int_{\Gamma(\theta)} h(J)^{\frac{\varepsilon}{1-\varepsilon}} \left(\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}\right)^{\frac{\rho-1}{\rho}\frac{\varepsilon}{1-\varepsilon}} dJ\right)^{\frac{1}{\varepsilon}}} \tag{16}$$

Now we can sub for both y and f into the minimand to get that the cost of optimally producing the target X_θ using workers of type θ using only jobs in $\Gamma(\theta)$ is

$$\left(\int_{\Gamma(\theta)} h(J)^{\frac{\varepsilon}{1-\varepsilon}} \left(\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}\right)^{\frac{\rho-1}{\rho}\frac{\varepsilon}{1-\varepsilon}} dJ\right)^{-\rho\frac{1-\varepsilon}{\varepsilon}} q_\theta^{-(\rho-1)} X_\theta^{\frac{\rho}{\varepsilon}}$$

Now let's look at the choice of production targets. Fixing $\Gamma(\cdot)$, the set of jobs assigned to each type, the planner now seeks to

$$\min_{X_{(\cdot)}:\Theta\to\mathbb{R}_+} \sum_\theta \left(\int_{\Gamma(\theta)} h(J)^{\frac{\varepsilon}{1-\varepsilon}} \left(\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}\right)^{\frac{\rho-1}{\rho}\frac{\varepsilon}{1-\varepsilon}} dJ\right)^{-\rho\frac{1-\varepsilon}{\varepsilon}} q_\theta^{-(\rho-1)} X_\theta^{\frac{\rho}{\varepsilon}} \tag{17}$$
$$\text{subject to } \left(\textstyle\sum_\theta X_\theta\right)^{\frac{1}{\varepsilon}} = \bar{Y}$$

Let's call that integral $\Xi(\theta, \Gamma(\theta))$. We now have the optimal production targets

$$X_\theta = \frac{\Xi(\theta,\Gamma(\theta))^{\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}} q_\theta^{\frac{\varepsilon(\rho-1)}{\rho-\varepsilon}}}{\sum_{\theta'} \Xi(\theta',\Gamma(\theta'))^{\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}} q_{\theta'}^{\frac{\varepsilon(\rho-1)}{\rho-\varepsilon}}} \bar{Y}^\varepsilon$$

and thereby the value of the minimand

$$\bar{Y}^\rho \left[\sum_\theta \Xi(\theta,\Gamma(\theta))^{\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}} q_\theta^{\frac{\varepsilon(\rho-1)}{\rho-\varepsilon}}\right]^{\frac{\varepsilon-\rho}{\varepsilon}} \tag{18}$$

2.2 The Planner's Job Allocation

Given (18) we can now attack the problem of allocating worker types to jobs. We're looking to

$$\min_{\Gamma:\Theta\to 2^{\mathcal{J}}} \bar{Y}^\rho \left[\sum_\theta \Xi(\theta,\Gamma(\theta))^{\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}} q_\theta^{\frac{\varepsilon(\rho-1)}{\rho-\varepsilon}}\right]^{\frac{\varepsilon-\rho}{\varepsilon}} \tag{19}$$

where Γ is the assignment function. We return to the primal problem, which we can now phrase as

$$\max_{\bar{Y},\Gamma:\Theta\to 2^{\mathcal{J}}} \bar{Y} - \bar{Y}^\rho \left[\sum_\theta \Xi(\theta,\Gamma(\theta))^{\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}} q_\theta^{\frac{\varepsilon(\rho-1)}{\rho-\varepsilon}}\right]^{\frac{\varepsilon-\rho}{\varepsilon}} \tag{20}$$

Solving for the optimal total production $\bar{Y}$ given a Γ we reduce the Γ problem to maximizing

$$\max_{\Gamma:\Theta\to 2^{\mathcal{J}}} (\rho-1)\rho^{-\frac{\rho}{\rho-1}} \left[\sum_\theta \Xi(\theta,\Gamma(\theta))^{\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}} q_\theta^{\frac{\varepsilon(\rho-1)}{\rho-\varepsilon}}\right]^{\frac{\rho-\varepsilon}{\varepsilon(\rho-1)}} \tag{21}$$

Given an optimal Γ, if $J \in \Gamma(\theta)$ and

$$\frac{\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}}{\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i'^{-\frac{1}{\rho-1}}} < \frac{\sum_i J_i'^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}}{\sum_i J_i'^{\frac{\rho}{\rho-1}} \theta_i'^{-\frac{1}{\rho-1}}}$$

then $J \notin \Gamma(\theta')$, as otherwise the two jobs could be exchanged between types and improve the objective.

Then, given that the planner will allocate every job to at least one $\Gamma(\theta)$, we can reduce the problem to finding $\#\Theta$ constants $(k_\theta)_{\theta\in\Theta}$ and define Γ as follows:

$$\Gamma(\theta) := \left\{ J \in \mathcal{J} \middle| \forall\theta' \in \Theta, \quad \frac{\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}}{\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i'^{-\frac{1}{\rho-1}}} \geq \frac{k_\theta}{k_{\theta'}} \right\} \tag{22}$$

We can normalize one of the ks to 1 so we have one less choice variable.

Given the solution has this structure, then all we have to do is to set all $\#\Theta - 1$ derivatives wrt the ks equal to zero.

3 Tackling the Partition Problem

The Planner maximizes her objective, which is

$$\max_{\Gamma:\Theta\to 2^{\mathcal{J}}} (\rho-1)\rho^{-\frac{\rho}{\rho-1}} \left[\sum_\theta \Xi(\theta,\Gamma(\theta))^{\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}} q_\theta^{\frac{\varepsilon(\rho-1)}{\rho-\varepsilon}}\right]^{\frac{\rho-\varepsilon}{\varepsilon(\rho-1)}} \tag{23}$$

To do this the planner chooses a vector of ks, one for each type except the first (which is normalized) that define the boundaries as above. Taking a derivative of the objective with respect to k_θ and setting it to zero, we get

$$\sum_{\theta'} \frac{d\Xi(\theta',\Gamma(\theta')}{dk_\theta} \cdot \left[\frac{\Xi(\theta',\Gamma(\theta'))}{q_{\theta'}}\right]^{\frac{\varepsilon(1-\rho)}{\rho-\varepsilon}} = 0 \tag{24}$$

When we differentiate $\Xi(\theta',\Gamma(\theta'))$ with respect to k_θ the integral turns into a surface intergral over boundaries that involve types θ and θ'.

$$\int_{\Gamma(\theta)\cap(\cup_{\theta'}\Gamma(\theta'))} h(J)^{\frac{\varepsilon}{1-\varepsilon}} \left(\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}\right)^{\frac{\rho-1}{\rho}\frac{\varepsilon}{1-\varepsilon}} (-v(J,\theta,\theta'))dJ \left[\frac{\Xi(\theta,\Gamma(\theta))}{q_\theta}\right]^{\frac{\varepsilon(1-\rho)}{\rho-\varepsilon}}$$
$$+\sum_{\theta'\neq\theta} \int_{\Gamma(\theta)\cap\Gamma(\theta')} h(J)^{\frac{\varepsilon}{1-\varepsilon}} \left(\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i'^{-\frac{1}{\rho-1}}\right)^{\frac{\rho-1}{\rho}\frac{\varepsilon}{1-\varepsilon}} v(J,\theta,\theta')dJ \left[\frac{\Xi(\theta',\Gamma(\theta'))}{q_{\theta'}}\right]^{\frac{\varepsilon(1-\rho)}{\rho-\varepsilon}} = 0 \tag{25}$$

where $v(J,\theta,\theta')$ is a term that describes the velocity of expansion (contraction) of $\Gamma(\theta')$ ($\Gamma(\theta)$) at J with respect to k_θ. The summands for types not sharing a border with θ are, naturally, zero.

Now, all points on the boundary between $\Gamma(\theta)$ and $\Gamma(\theta')$ must satisfy

$$\frac{\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}}{\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i'^{-\frac{1}{\rho-1}}} = \frac{k_\theta}{k_{\theta'}} \tag{26}$$

So we can substitute for all types $\theta'\neq\theta$ with $\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i'^{-\frac{1}{\rho-1}} = \sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}} \times \frac{k_{\theta'}}{k_\theta}$

Using this we can pull out the integral and get

$$0 = \sum_{\theta'\neq\theta} \int_{\Gamma(\theta)\cap\Gamma(\theta')} h(J)^{\frac{\varepsilon}{1-\varepsilon}} \left(\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}\right)^{\frac{\rho-1}{\rho}\frac{\varepsilon}{1-\varepsilon}} v(J,\theta,\theta')dJ$$
$$\cdot\left(-\left[\frac{\Xi(\theta,\Gamma(\theta))}{q_\theta}\right]^{\frac{\varepsilon(1-\rho)}{\rho-\varepsilon}} + \left(\frac{k_{\theta'}}{k_\theta}\right)^{\frac{\rho-1}{\rho}\frac{\varepsilon}{1-\varepsilon}} \left[\frac{\Xi(\theta',\Gamma(\theta'))}{q_{\theta'}}\right]^{\frac{\varepsilon(1-\rho)}{\rho-\varepsilon}}\right)$$

One solution to the system is that for each θ,θ' we have

$$\frac{k_\theta}{k_{\theta'}} = \left(\frac{q_\theta \Xi(\theta,\Gamma(\theta))^{-1}}{q_{\theta'}\Xi(\theta',\Gamma(\theta'))^{-1}}\right)^{-\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}}. \tag{27}$$

We'll now show that this condition is in fact necessary for optimality. Suppose the planner already did a good job and attained the maximized value of the problem:

$$V = \max_{\Gamma:\Theta\to 2^{\mathcal{J}}} (\rho-1)\rho^{-\frac{\rho}{\rho-1}} \left[\sum_\theta \Xi(\theta,\Gamma(\theta))^{\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}} q_\theta^{\frac{\varepsilon(\rho-1)}{\rho-\varepsilon}}\right]^{\frac{\rho-\varepsilon}{\varepsilon(\rho-1)}} \tag{28}$$

By the envelope theorem when we differentiate V with respect to the parameter q_θ, the optimal choice of sets $\Xi(\theta, \Gamma(\theta))$ can be held constant. This gives us the value of an extra worker to the planner or equivalently the utility (wage net of skill acquisition costs) of a type θ worker in the decentralized problem:

$$u_\theta := \frac{\partial V}{\partial q_\theta} = (\rho-1)\rho^{-\frac{\rho}{\rho-1}} \left[\Xi(\theta, \Gamma(\theta))^{\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}} q_\theta^{\frac{\varepsilon(\rho-1)}{\rho-\varepsilon}-1} \right] V^{\frac{\rho-\varepsilon}{\varepsilon(\rho-1)}-1}. \tag{29}$$

We can alternatively compute the net value of an additional worker of type θ in job J to be

$$\rho^{-\frac{\rho}{\rho-1}}(\rho-1)P(J)^{\frac{\rho}{\rho-1}} \sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}} \tag{30}$$

where $P(J)$ is the value of the marginal unit of widget J to the planner.[5] If $J \in \Gamma(\theta)$, then this expression must equal $\frac{\partial V}{\partial q_\theta}$, as the marginal worker *is* assigned to J. If J is a job on the boundary of types θ and θ' partitions, that is, if $J \in \Gamma(\theta) \cap \Gamma(\theta')$, then we can use (26),(29) and (30) to get

$$\frac{u_\theta}{u_{\theta'}} = \left[\frac{\Xi(\theta, \Gamma(\theta)) q_\theta^{-1}}{\Xi(\theta', \Gamma(\theta')) q_{\theta'}^{-1}} \right]^{\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}} = \frac{\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}}{\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i'^{-\frac{1}{\rho-1}}} = \frac{k_\theta}{k'_\theta} \tag{31}$$

which implies this particular solution to the FOCs is *necessary* for optimality.

[5]In the decentralized equilibrium, $P(J)$ is something called a *price*.

4 Statics

For this section, we will restrict ourselves to three types (which - what luck! - happens to be what we're using for empirics), dubbed $\theta_0, \theta_1, \theta_2$. Much of the intuition carries over to the general case as well. We'll use shorthand replacing $\Xi(\theta_i, \Gamma(\theta_i))$ by Ξ_i, q_{θ_i} by q_i and k_{θ_i} by k_i.

First, let's notice that the definition of $\Gamma(\theta)$ in (22) is homogeneous of degree 0 in the vector k; as a consequence, we have that

$$\sum_i \frac{d\Xi_j}{d\ln k_i} = 0. \tag{32}$$

Furthermore, it is immediate from (22) that

$$\frac{d\Xi_i}{d\ln k_i} < 0 \tag{33}$$

and for $j \neq i$ we have

$$\frac{d\Xi_i}{d\ln k_j} \geq 0. \tag{34}$$

Finally, given (32), (33) and (34), we have that

$$\frac{d\Xi_i}{d\ln k_j} + \frac{d\Xi_i}{d\ln k_i} \leq 0 \tag{35}$$

Now, we log both sides of expression (27):

$$ln(\frac{k_i}{k_j}) = -\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}\left[lnq_i - ln\Xi_i - lnq_j + ln\Xi_j, \right]. \tag{36}$$

We normalize the k corresponding to θ_0 to one and totally differentiate the version of the above relation featuring θ_1 and θ_2. We get

$$\frac{dk_1}{k1} - \frac{dk2}{k_2} = -\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}\left[\left(\frac{dq_1}{q_1} - \frac{dq_2}{q_2}\right) - \left(\sum \frac{d\ln\Xi_1}{dk_i}dk_i - \sum \frac{d\ln\Xi_2}{dk_i}dk_i\right)\right]$$

which we can rearrange into

$$\left[\frac{1}{k_1} - \frac{\rho(1-\varepsilon)}{\rho-\varepsilon}\left(\frac{d\ln\Xi_1}{dk_1} - \frac{d\ln\Xi_2}{dk_1}\right)\right]dk_1$$

$$= -\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}\left(\frac{dq_1}{q_1}-\frac{dq_2}{q_2}\right)+\left[\frac{1}{k_2}+\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}\left(\frac{d\ln\Xi_1}{dk_2}-\frac{d\ln\Xi_2}{dk_2}\right)\right]dk_2 \tag{37}$$

Now, totally differentiating the relation (36) for θ_0 and θ_2, we have

$$\frac{dk_0}{k_0}-\frac{dk_2}{k_2} = -\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}\left[\left(\frac{dq_0}{q_0}-\frac{dq_2}{q_2}\right)-\left(\sum\frac{d\ln\Xi_1}{dk_i}dk_i-\sum\frac{d\ln\Xi_2}{dk_i}dk_i\right)\right]$$

Now because the k_0 is normalized to 1, we've set $dk_0=0$, and therefore

$$\left[-\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}\left(\frac{d\ln\Xi_0}{dk_1}-\frac{d\ln\Xi_2}{dk_1}\right)\right]dk_1$$

$$= -\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}\left(\frac{dq_0}{q_0}-\frac{dq_2}{q_2}\right)+\left[\frac{1}{k_2}+\frac{\rho(1-\varepsilon)}{\rho-\varepsilon}\left(\frac{d\ln\Xi_0}{dk_2}-\frac{d\ln\Xi_2}{dk_2}\right)\right]dk_2 \tag{38}$$

Let's first investigate the statics in q_0. We can simplify (37) to

$$\left[\frac{\rho-\varepsilon}{\rho(1-\varepsilon)}-\frac{d\ln\Xi_1}{d\ln k_1}+\frac{d\ln\Xi_2}{d\ln k_1}\right]\frac{d\ln k_1}{d\ln q_0}-\left[\frac{\rho-\varepsilon}{\rho(1-\varepsilon)}+\frac{d\ln\Xi_1}{d\ln k_2}-\frac{d\ln\Xi_2}{d\ln k_2}\right]\frac{d\ln k_2}{d\ln q_0}=0 \tag{39}$$

And (38) to

$$\left[\frac{d\ln\Xi_0}{d\ln k_1}-\frac{d\ln\Xi_2}{d\ln k_1}\right]\frac{d\ln k_1}{d\ln q_0}+\left[\frac{\rho-\varepsilon}{\rho(1-\varepsilon)}+\frac{d\ln\Xi_0}{d\ln k_2}-\frac{d\ln\Xi_2}{d\ln k_2}\right]\frac{d\ln k_2}{d\ln q_0}=1. \tag{40}$$

Let's do some linear algebra! Who's excited? This is our matrix equation.

$$\begin{bmatrix} \frac{\rho-\varepsilon}{\rho(1-\varepsilon)}-\frac{d\ln\Xi_1}{d\ln k_1}+\frac{d\ln\Xi_2}{d\ln k_1} & -\frac{\rho-\varepsilon}{\rho(1-\varepsilon)}-\frac{d\ln\Xi_1}{d\ln k_2}+\frac{d\ln\Xi_2}{d\ln k_2} \\ \frac{d\ln\Xi_0}{d\ln k_1}-\frac{d\ln\Xi_2}{d\ln k_1} & \frac{\rho-\varepsilon}{\rho(1-\varepsilon)}+\frac{d\ln\Xi_0}{d\ln k_2}-\frac{d\ln\Xi_2}{d\ln k_2}\end{bmatrix}\begin{bmatrix}\frac{d\ln k_1}{d\ln q_{\theta_0}}\\ \frac{d\ln k_2}{d\ln q_{\theta_0}}\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix} \tag{41}$$

Inverting the matrix, we get

$$\begin{bmatrix}\frac{d\ln k_1}{d\ln q_{\theta_0}}\\ \frac{d\ln k_2}{d\ln q_{\theta_0}}\end{bmatrix}=\frac{1}{DetA}\begin{bmatrix}\frac{\rho-\varepsilon}{\rho(1-\varepsilon)}+\frac{d\ln\Xi_1}{d\ln k_2}-\frac{d\ln\Xi_2}{d\ln k_2}\\ \frac{\rho-\varepsilon}{\rho(1-\varepsilon)}-\frac{d\ln\Xi_1}{d\ln k_1}+\frac{d\ln\Xi_2}{d\ln k_1}\end{bmatrix} \tag{42}$$

where A is the matrix in (41). Notice that from properties (33) and (34) the terms in the right-hand vector are positive. Let's sign that nasty determinant! We'll group terms by their powers of $\frac{\rho-\varepsilon}{\rho(1-\varepsilon)}$.

The coefficient on $(\frac{\rho-\varepsilon}{\rho(1-\varepsilon)})^2$ is 1, so that's positive. The coefficient on $(\frac{\rho-\varepsilon}{\rho(1-\varepsilon)})^1$ is

$$-\frac{d\ln\Xi_1}{d\ln k_1}+\cancel{\frac{d\ln\Xi_2}{d\ln k_1}}+\frac{d\ln\Xi_0}{d\ln k_2}-\frac{d\ln\Xi_2}{d\ln k_2}+\frac{d\ln\Xi_0}{d\ln k_1}-\cancel{\frac{d\ln\Xi_2}{d\ln k_1}}, \tag{43}$$

and using properties (33) and (34), every term in this expression is positive. Finally, the remaining term is

$$\left[-\frac{d\ln\Xi_1}{d\ln k_1}+\frac{d\ln\Xi_2}{d\ln k_1}\right]\left[\frac{d\ln\Xi_0}{d\ln k_2}-\frac{d\ln\Xi_2}{d\ln k_2}\right]-\left[-\frac{d\ln\Xi_1}{d\ln k_2}+\frac{d\ln\Xi_2}{d\ln k_2}\right]\left[\frac{d\ln\Xi_0}{d\ln k_1}-\frac{d\ln\Xi_2}{d\ln k_1}\right] \tag{44}$$

which expands out to

$$\begin{aligned}&-\frac{d\ln\Xi_1}{d\ln k_1}\frac{d\ln\Xi_0}{d\ln k_2}+\frac{d\ln\Xi_1}{d\ln k_1}\frac{d\ln\Xi_2}{d\ln k_2}+\frac{d\ln\Xi_2}{d\ln k_1}\frac{d\ln\Xi_0}{d\ln k_2}-\cancel{\frac{d\ln\Xi_2}{d\ln k_1}\frac{d\ln\Xi_2}{d\ln k_2}}\\&+\frac{d\ln\Xi_1}{d\ln k_2}\frac{d\ln\Xi_0}{d\ln k_1}-\frac{d\ln\Xi_1}{d\ln k_2}\frac{d\ln\Xi_2}{d\ln k_1}-\frac{d\ln\Xi_2}{d\ln k_2}\frac{d\ln\Xi_0}{d\ln k_1}+\cancel{\frac{d\ln\Xi_2}{d\ln k_2}\frac{d\ln\Xi_2}{d\ln k_1}}.\end{aligned}$$

Given properties (33) and (34), all terms are positive with the sole exception of $-\frac{d\ln\Xi_1}{d\ln k_2}\frac{d\ln\Xi_2}{d\ln k_1}$; however, given (35) we have that $\frac{d\ln\Xi_1}{d\ln k_1}\frac{d\ln\Xi_2}{d\ln k_2}-\frac{d\ln\Xi_1}{d\ln k_2}\frac{d\ln\Xi_2}{d\ln k_1}\geq 0$. Therefore, the determinant is positive.

Therefore, types 1 and 2 both increase their ks due to an increase in q_0; as these are ratios of wages to type 0, this means that their wage relative to 0's increases.

Proposition 1 *When the prevalence of a type increases, its equilibrium wage declines relatively to any other type's;* $\frac{d\left(\frac{k_i}{k_j}\right)}{dq_i}<0$.

Now, let's see what happens to the relative wages of 1 and 2 when the proportion of type 0 increases. We want to sign $\frac{d\left(\frac{k_1}{k_2}\right)}{dq_0}$. This shares a sign with $\frac{d\ln k_1}{d\ln q_0}-\frac{d\ln k_2}{d\ln q_0}$ which we can get from equation (45):

$$\frac{d\ln k_1}{d\ln q_0}-\frac{d\ln k_2}{d\ln q_0}=\frac{1}{DetA}\left[\frac{d\ln\Xi_1}{d\ln k_2}-\frac{d\ln\Xi_2}{d\ln k_2}+\frac{d\ln\Xi_1}{d\ln k_1}-\frac{d\ln\Xi_2}{d\ln k_1}\right] \tag{45}$$

Now, using property (32) we can rewrite this as

$$\frac{d\ln k_1}{d\ln q_0}-\frac{d\ln k_2}{d\ln q_0}=\frac{1}{DetA}\left[-\frac{d\ln\Xi_1}{d\ln k_0}+\frac{d\ln\Xi_2}{d\ln k_0}\right] \tag{46}$$

We discuss and interpret the k-derivatives of $\Xi_{(\cdot)}$ in the next section.

5 The velocity of boundary shifts

The statics depend heavily on terms of the form $\frac{d \ln \Xi_{\theta'}}{d \ln k_\theta}$. For $\theta \neq \theta'$, this derivative is

$$\frac{k_\theta}{\Xi_{\theta'}} \int_{\Gamma(\theta)\cap\Gamma(\theta')} h(J)^{\frac{\varepsilon}{1-\varepsilon}} \left(\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i'^{-\frac{1}{\rho-1}} \right)^{\frac{\rho-1}{\rho}\frac{\varepsilon}{1-\varepsilon}} v(J,\theta,\theta') dJ. \tag{47}$$

Most of the components of the integrand are easy to interpret; h governs the relative demand of jobs on the boundary, so that joined with the next term (which describes the ability to produce there) it describes the 'productive density' at a job J on the boundary. The velocity term $v(J,\theta,\theta')$ is different. It is essentially a 'Leibniz-rule' term that describes the rate of change of the boundary at the point J. In particular, it is the rate of expansion of $\Gamma(\theta')$ with k_θ in the direction normal to $\Gamma(\theta')$ at J.

We've got the definition of the boundary

$$f(J) := \frac{\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i^{-\frac{1}{\rho-1}}}{\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i'^{-\frac{1}{\rho-1}}} = \frac{k_\theta}{k_{\theta'}}. \tag{48}$$

What we're looking for is $v(J,\theta,\theta') = \left|\left|\frac{dJ}{dk_\theta}\right|\right|$ in order to properly compute $\frac{d\Xi_{\theta'}}{dk_\theta}$. Of course, $\left|\left|\frac{dJ}{dk_\theta}\right|\right|$ is not generally well-defined as there are many directions we can change J in. But for our purposes, we're looking for a $\frac{dJ}{dk_\theta}$ perpendicular to $\Gamma(\theta')$ at J that also satisfies $\left[\frac{dJ}{dk_\theta}\right]^T \mathbf{1} = 0$.

The orthogonal projection matrix to the plane $\{x : x^T\mathbf{1} = 0\}$ is $P = I - \frac{1}{n}\mathbf{1}\mathbf{1}^T$. The direction of the derivative in question is therefore shared by $P\nabla f(J)$, so that

$$\frac{dJ}{dk} = \alpha P \nabla f(J) \tag{49}$$

for some α. Differentiating (48) wrt k_θ, we get

$$[\nabla f(J)]^T \frac{dJ}{dk} = \frac{1}{k_{\theta'}} \tag{50}$$

so that substituting in (49) we get

$$\alpha [\nabla f(J)]^T P \nabla f(J) = \frac{1}{k_{\theta'}}. \tag{51}$$

As an orthogonal projection matrix, P is both symmetric and idempotent, so we have $P = P^T P$. Using this, we get

$$\alpha[P\nabla f(J)]^T P\nabla f(J) = \frac{1}{k_{\theta'}}. \tag{52}$$

We solve for α:

$$\alpha = \frac{1}{k_{\theta'}\,||P\nabla f(J)||^2} \tag{53}$$

and finally get

$$\frac{dJ}{dk} = \frac{P\nabla f(J)}{k_{\theta'}\,||P\nabla f(J)||^2}. \tag{54}$$

from which we have that the velocity term is

$$v(J,\theta,\theta') = \left|\left|\frac{dJ}{dk}\right|\right| = \frac{1}{k_{\theta'}\,||P\nabla f(J)||}. \tag{55}$$

Using (48) we have

$$v(J,\theta,\theta') = \frac{1}{k_{\theta'}\left|\left|P\nabla \frac{\sum_i J_i^{\frac{\rho}{\rho-1}}\theta_i^{-\frac{1}{\rho-1}}}{\sum_i J_i^{\frac{\rho}{\rho-1}}\theta_i'^{-\frac{1}{\rho-1}}}\right|\right|} \tag{56}$$

$$v(J,\theta,\theta') = \frac{\frac{\rho-1}{\rho}[\sum_i J_i^{\frac{\rho}{\rho-1}}\theta_i'^{-\frac{1}{\rho-1}}]^2}{k_{\theta'}\left|\left|P\left((\sum_i J_i^{\frac{\rho}{\rho-1}}\theta_i'^{-\frac{1}{\rho-1}})J_j^{\frac{1}{\rho-1}}\theta_j^{-\frac{1}{\rho-1}} - (\sum_i J_i^{\frac{\rho}{\rho-1}}\theta_i^{-\frac{1}{\rho-1}})J_j^{\frac{1}{\rho-1}}\theta_j'^{-\frac{1}{\rho-1}}\right)_{i\le n}^T\right|\right|} \tag{57}$$

Now, we use a substitution from (48) to get

$$v(J,\theta,\theta') = \frac{\frac{\rho-1}{\rho}[\sum_i J_i^{\frac{\rho}{\rho-1}}\theta_i'^{-\frac{1}{\rho-1}}]}{\left|\left|P\left(J_i^{\frac{1}{\rho-1}}(k'_\theta\theta_i^{-\frac{1}{\rho-1}} - k_\theta\theta_i'^{-\frac{1}{\rho-1}})\right)_{i\le n}^T\right|\right|} \tag{58}$$

We can think of $||P(\cdot)||$ as being a sort of standard deviation operator, as

$$||Px|| = \sqrt{\sum_i \left[x_i - \frac{1}{n}\sum_j x_j\right]^2} = \sqrt{n}\sqrt{Var(x)}. \tag{59}$$

Is there any explanatory value to writing our expression as:

$$v(J,\theta,\theta') = \frac{\frac{\rho-1}{\rho}[\sum_i J_i^{\frac{\rho}{\rho-1}} \theta_i'^{-\frac{1}{\rho-1}}]}{\sqrt{\sum_i \left[J_i^{\frac{1}{\rho-1}} (k_\theta' \theta_i^{-\frac{1}{\rho-1}} - k_\theta \theta_i'^{-\frac{1}{\rho-1}}) - \frac{1}{n} \sum_j J_j^{\frac{1}{\rho-1}} (k_\theta' \theta_j^{-\frac{1}{\rho-1}} - k_\theta \theta_j'^{-\frac{1}{\rho-1}}) \right]^2}} \tag{60}$$

The key to understanding the velocity term is that the denominator depends on how *different* θ and θ' are. When the types are very different, a change in the boundary leads to a large change in their relative ability to perform boundary tasks. So that, inverting this argument, a change in their relative utilities is met with a small change in the location of the boundary.**Is this at all understandable?**