# Code documentation

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# 1 New approach (v2):

## 1.1 Counting parameters

$$\Delta\overline{\ln S^{e}_{ijet}} = \frac{\sigma_j}{\sigma_j - 1}\left(\sum_k \theta_{ke}\overline{S_{kejt}}\Delta \ln A_{kjt} - \Delta \ln A_{ijt}\right) + \varepsilon_{iej}$$

We have $12JT$ observations, and we need to identify $3JT$ $\pi$ parameters, and $12\theta$.

## 1.2 Model equations

At the job level it holds that:

$$\begin{aligned}
\Delta\overline{\ln S^{e}_{ijet}} &= \frac{\sigma_j}{\sigma_j - 1}\left(\sum_k \theta_{ke}\overline{S_{kejt}}\Delta \ln A_{kjt} - \Delta \ln A_{ijt}\right) \\
1 &= \sum_k \theta^{e}_{k}\overline{S_{keJt}} \\
\Delta\left[\ln \frac{q_{eJt}}{q_{e'Jt}}\right] &= \sum_k \left(\theta^{e}_{k}\overline{S_{kJet}} - \theta^{e'}_{k}\overline{S_{kJe't}}\right)\Delta \ln A_{kjt} + const_{e,e't}
\end{aligned}$$

## 1.3 Building GMM

In our data we have 3 education levels, 4 skills, $J$ jobs and $T$ periods. We normalize $\Delta \ln A_{4jt} = 0, \forall j, t$

- **Equation (1)**: instruments for this equation are skills by job matrices. More specifically, let $x_{ejt}$ be the row vector $\begin{pmatrix} S_{1ejt} & S_{2ejt} & S_{3ejt} \end{pmatrix}$. Define the matrix $X_{ejt}$ as the 4×3

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matrix containing the skill indexes by education. This matrix excludes the reference skill. For simplicity, I have ignored the bars in the notation.

$$X_{ejt} = \begin{pmatrix} S_{1ejt} & S_{2ejt} & S_{3ejt} \\ S_{1ejt} & S_{2ejt} & S_{3ejt} \\ S_{1ejt} & S_{2ejt} & S_{3ejt} \\ S_{1ejt} & S_{2ejt} & S_{3ejt} \end{pmatrix} = \begin{pmatrix} x_{ejt} \\ x_{ejt} \\ x_{ejt} \\ x_{ejt} \end{pmatrix}$$

Let $X_e$ be the block-diagonal matrix with $X_{ejt}$ in the diagonal.

$$X_e = \begin{pmatrix} X_{e11} & \mathbf{0} & \dots & 0 \\ \mathbf{0} & X_{e12} & \dots & 0 \\ \vdots & & & \\ \mathbf{0} & \mathbf{0} & \dots & X_{eJT} \end{pmatrix}$$

The set of instruments for equation 1 is then:

$$Z_1 = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \tag{1}$$

- **Equation** (1): instruments for the sum to restriction are the skill indexes by education level. Let $\tilde{x}_{ejt}$ be the row vector $\begin{pmatrix} S_{1ejt} & S_{2ejt} & S_{3ejt} & S_{4ejt} \end{pmatrix}$ and let $\tilde{X}_e$ be the $N_e \times 4$ matrix stacking all the $\tilde{x}_{ejt}$ for education level $e$. The instruments for equation 2 are the block diagonal matrix with elements $\tilde{X}_e$:

$$Z_2 = \begin{pmatrix} \tilde{X}_1 & 0 & 0 \\ 0 & \tilde{X}_2 & 0 \\ 0 & 0 & \tilde{X}_3 \end{pmatrix}$$

- **Putting it all together:** let $u_l$ be the stacked vector of equation $l$ errors. Define:

$$Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

the moment equations form the model are given by $\mathbb{E}\left(Z'u\right) = 0$.

I choose parameters $\theta$ to minimize the quadratic form:

$$\min_{\theta} \frac{1}{N} \left(u'Z(Z'Z)^{-1}Z'u\right) \tag{2}$$

### 1.4 Calculating the gradient

**Reminder:**

$$\frac{\partial Q}{\partial x} = x'(A + A')$$

I can write the quadratic from as:

$$Q(\gamma) = \frac{1}{N}\left(u(\gamma)'Z(Z'Z)^{-1}Z'u(\gamma)\right)$$

then,

$$\frac{\partial Q(\gamma)}{\partial \gamma} = \frac{2}{N}\left(\frac{\partial u(\gamma)}{\partial \gamma}\right)' Z(Z'Z)^{-1}Zu(\gamma)$$

now, I start element by element:

$$\begin{aligned}
\frac{\partial \Delta \ln S_{iejt}}{\partial \ln A_{kjt}} &= \theta_{ke}S_{kejt} \\
\frac{\partial \Delta \ln S_{iejt}}{\partial \ln A_{ijt}} &= \theta_{ie}S_{iejt} - 1 \\
\frac{\partial \Delta \ln S_{iejt}}{\partial \ln \theta_{ke}} &= S_{kjt}\Delta \ln A_{kjt}
\end{aligned}$$

for the sum to 1 restrictions we have:

$$\frac{\partial g_{ejt}(\gamma)}{\partial \theta_{ke}} = -S_{kejt}$$

next thing to do: write the gradient