Multiple Types: Theory & Inevitable Falsification

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Abstract

Key words:

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1 A Model of Skill Acquisition and Production

What follows is a model of workers of various levels of education (types), who must each acquire skills in a way affected by their education, then choose a job at which they'll put their skills to use producing some intermediate good. We think of these goods as occupation-specific labor output, e.g. "farrier output" or "academic economist output" rather than horseshoes and *Econometrica*s. These labor outputs will be directly aggregated into a single final consumption good.

The key objects of study are the *allocation of workers (by education) to occupations* and the *skills which they acquire*. We are interested on the dependence of these objects on the production technology - both at the intermediate and final good levels - and the prevalence of different levels of education. Our ambition is to investigate these questions while assuming as little as possible about the relation of skills to output and the content - if there is any - of education . Despite this paucity of assumptions, the model does produce some empirical predictions; these are tested and refuted in Section **??**.

1.1 The Model's Primitives

There are a continuum of workers, each endowed with a type $\theta \in \Theta$, where Θ is a finite set of educational qualifications. For each type θ, the measure of workers with that type is q_θ.

Each worker must choose a skill bundle S. This choice is not unrestricted, however; a worker of type θ must choose a skill bundle from θ's *skill menu* $\mathcal{S}_\theta \subseteq \mathbb{R}^n_+$. Skill menus are assumed to be compact and convex, and to have smooth boundaries in $\mathbb{R}^n_{++}$.[1]

After workers have acquired skills, each one chooses a job $J \in \mathcal{J}$, where $\mathcal{J} \subset \mathbb{R}^k$ is the compact, convex, full-dimensional, and smooth-boundaried set of all jobs.[2]

Production is governed by a continuously differentiable production function

$$y : \mathbb{R}^n_+ \times \mathcal{J} \to \mathbb{R}_+, \tag{1}$$

where for all $J \in \mathcal{J}$, $y(\cdot, J)$ is assumed to be a standard constant-returns neoclassical production function[3]. A worker with skills S working at job J will produce $y(S, J)$ J-widgets. These widgets will then be aggregated into final goods using a CES aggregation function, for total output

$$Y(x) = \left[\int_{\mathcal{J}} h(J) x(J)^\varepsilon\right]^{\frac{1}{\varepsilon}} \tag{2}$$

where h is a technological demand shifter, and $x(J)$ denotes the total amount of J-widget input.

[1] For example, think of skill menus taking the form of standard budget sets: $\mathcal{S}_\theta := \{S \in R^n_+ | \sum_i S_i\theta_i \leq 1\}$.

[2] We do not assume that $n = k$ in general, but it is apt if one thinks of each J_i as denoting the importance of skill i at the particular J, as in Section 2.4

[3] $y(\cdot, J)$ is strictly increasing in each argument on $\mathbb{R}^n_{++}$, is twice continuously differentiable, features a bordered Hessian with non-vanishing determinant on $\mathbb{R}^n_{++}$, is strictly quasiconcave, and $y(S, J) = 0$ iff $S_i = 0$ for some i.

1.2 The Workers' Problem

Suppose a worker of type θ is assigned to job J. The worker then solves

$$\max_{S \in \mathcal{S}_\theta} y(S, J). \tag{3}$$

We denote the solution by $S^*_\theta(J)$. We also define *type-specific production functions* as follows:

$$y^*_\theta : \mathcal{J} \to \mathbb{R}_+ \quad \text{via} \quad y^*_\theta(J) = y(S^*_\theta(J), J) \tag{4}$$

Entirely by chance, our assumptions on y allow us to relate the production of a given type at nearby jobs via the Envelope Theorem:

$$\nabla y^*_\theta(J) = \nabla_J y(S, J)|_{S=S^*(\theta,J)}. \tag{5}$$

We feel this may come in handy later. In the meantime, we impose an assumption that clarifies the distinction between types. We term this [**C: We can choose to call this either...**] *non-stationary absolute advantage* [**or**] *non-vanishing local comparative advantage*:

$$\forall \theta, \theta' \in \Theta, J \in \mathcal{J}, \quad \left\| \nabla \frac{y^*_\theta(J)}{y^*_{\theta'}(J)} \right\| \neq 0. \tag{6}$$

This is a sufficient[4] condition for the planner's problem to have an essentially unique solution for every q. It implies that it is always optimal to sort workers of different types across jobs locally.

1.3 The Planner's Problem

The planner can now allocate workers to jobs in order to maximize aggregate production, taking as given the (already skill-optimized) type-specific production functions. Thus, the planner has to choose how to allocate the mass of each type of worker across $\mathcal{J}$ to maximize total production. This is done by way of choosing pdfs $\{f_\theta\}_{\theta \in \Theta}$, such that each f_θ integrates to q_θ - the mass of available type θ workers. The production of each widget J is the sum over types of the production of J-widget due to each type, $f_\theta(J) y^*_\theta(J)$.

Thus for a given allocation of workers f, final good production is

$$Y(f) = \left[\int_{\mathcal{J}} h(J) \left(\sum_{\theta \in \Theta} f_\theta(J) y^*_\theta(J) \right)^\varepsilon \right]^{\frac{1}{\varepsilon}}.$$

[4] This condition is not quite necessary, but it must not fail over any set of positive measure.

The planner's problem can therefore be written as

$$\max_{\{f_\theta\}_{\theta\in\Theta}} \left[\int_{\mathcal{J}} h(J) \left(\sum_{\theta\in\Theta} f_\theta(J) y^*_\theta(J) \right)^{\varepsilon} \right]^{\frac{1}{\varepsilon}} \tag{7}$$

$$\text{subject to } \forall\theta\in\Theta \quad \int_{\mathcal{J}} f_\theta = q_\theta$$

$$\text{and } \forall\theta\in\Theta, J\in\mathcal{J}, \quad f_\theta(J) \geq 0.$$

From here, omitting the non-negativity constraints, we can write out the associated Lagrangian

$$\left[\int_{\mathcal{J}} h(J) \left(\sum_{\theta\in\Theta} f_\theta(J) y^*_\theta(J) \right)^{\varepsilon} dJ \right]^{\frac{1}{\varepsilon}} - \sum \lambda_\theta \left[\int_{\mathcal{J}} f_\theta(J) - q_\theta \right] \tag{8}$$

Taking a first-order condition w.r.t f_θ where its non-negativity does not bind, and denoting by Y^* the maximized value of (7), we have

$$Y^{*1-\varepsilon} h(J) \left(\sum_{\theta'\in\Theta} f_{\theta'}(J) y^*_{\theta'}(J) \right)^{\varepsilon-1} y^*_\theta(J) = \lambda_\theta \tag{9}$$

so that if $f_\theta, f_{\theta'}$ both have support at J, then

$$\frac{y^*_\theta(J)}{y^*_{\theta'}(J)} = \frac{\lambda_\theta}{\lambda_{\theta'}}. \tag{10}$$

Given our assumption of **¡non-stationary absolute advantage¿** (6), equation (10) can only hold on a set of measure zero. Thus, if $\mathcal{J}^*_\theta$ and $\mathcal{J}^*_{\theta'}$ are the supports of f_θ and $f_{\theta'}$ respectively, then $\mathcal{J}^*_\theta \cap \mathcal{J}^*_{\theta'}$ has measure 0. Therefore, without loss of optimality, we can rewrite the first order condition (9) as

$$Y^{*1-\varepsilon} h(J) f_\theta(J)^{\varepsilon-1} y^*_\theta(J)^{\varepsilon} = \lambda_\theta \tag{11}$$

$$\implies f_\theta(J) = \lambda_\theta^{\frac{1}{\varepsilon-1}} Y^* y^*_\theta(J)^{\frac{\varepsilon}{1-\varepsilon}} h(J)^{\frac{1}{1-\varepsilon}} \tag{12}$$

$$\tag{13}$$

Now, we can integrate over $\mathcal{J}^*_\theta$, the support of f_θ, and use the constraint $\int_{\mathcal{J}} f_\theta = q_\theta$, to get

$$q_\theta = \lambda_\theta^{\frac{1}{\varepsilon-1}} Y^* \int_{\mathcal{J}^*_\theta} y^*_\theta(J)^{\frac{\varepsilon}{1-\varepsilon}} h(J)^{\frac{1}{1-\varepsilon}} dJ \tag{14}$$

$$\implies \lambda_\theta = \left[\frac{Y^*}{q_\theta} \int_{\mathcal{J}^*_\theta} y^*_\theta(J)^{\frac{\varepsilon}{1-\varepsilon}} h(J)^{\frac{1}{1-\varepsilon}} dJ \right]^{1-\varepsilon}. \tag{15}$$

The multiplier λ_θ is the productivity of an additional θ-type worker; in a decentralized equilibrium, this is type θ's wage. Consider now $y^*_\theta(J)/\lambda_\theta$, the unit marginal product of widget J in final production at which the planner assigns job J to type θ. If $\theta \in \mathcal{J}^*_\theta$, in the decentralized equilibrium this is the price of an additional unit of J. We can rewrite (15) in terms of it:

$$\lambda_\theta = \frac{Y^*}{q_\theta} \int_{\mathcal{J}^*_\theta} h(J)^{\frac{1}{1-\varepsilon}} \left(\frac{y^*_\theta(J)}{\lambda_\theta}\right)^{\frac{\varepsilon}{1-\varepsilon}} dJ \tag{16}$$

[The above is not currently in use. Might delete?]

Thus, from the first-order conditions and given an optimal $\{\mathcal{J}^*_\theta\}_{\theta\in\Theta}$, we found how workers are distributed within each $\mathcal{J}^*_\theta$. From here, in light of the fact that $\min_\theta(y^*_\theta(\cdot)/\lambda_\theta)$ represents widget prices, we have that $J \in \mathcal{J}^*_\theta$ iff

$$\forall\theta' \neq \theta \quad \frac{y^*_\theta(J)}{\lambda_\theta} \geq \frac{y^*_{\theta'}(J)}{\lambda_{\theta'}} \tag{17}$$

and thus, for each θ,

$$\mathcal{J}^*_\theta = \left\{ J \in \mathcal{J} \;\middle|\; \frac{y^*_\theta(J)}{\lambda_\theta} = \max_{\theta'\in\Theta} \frac{y^*_{\theta'}(J)}{\lambda_{\theta'}} \right\}. \tag{18}$$

Put together, expressions (15) and (18) - one of each for each θ - characterize the equilibrium uniquely up to a measure 0 of jobs.

2 Equilibrium Comparative Statics

We are interested in how job assignment and wages by type respond to changes in model parameters.

2.1 Changes to the Abundance of Types

We first turn to the equilibrium effects of altering the education makeup of the labor market. Each λ_θ is continuously differentiable w.r.t. each $q_{\theta'}$ almost everywhere, so we venture to calculate this derivative where it is continuous.

Differentiating (15) with respect to $q_{\theta'}$, for any $\theta' \neq \theta$, gives us

$$\frac{\partial\lambda_\theta}{\partial q_{\theta'}} = (1-\varepsilon)\left[\frac{\partial Y^*}{\partial q_{\theta'}}\frac{\lambda_\theta}{Y^*} + \frac{\partial \int_{\mathcal{J}^*_\theta} y^*_\theta(J)^{\frac{\varepsilon}{1-\varepsilon}} h(J)^{\frac{1}{1-\varepsilon}} dJ}{\partial q_{\theta'}} \frac{\lambda_\theta}{\int_{\mathcal{J}^*_\theta} y^*_\theta(J)^{\frac{\varepsilon}{1-\varepsilon}} h(J)^{\frac{1}{1-\varepsilon}} dJ}\right]. \tag{19}$$

Using (15) and the fact that $\frac{\partial Y^*}{\partial q_{\theta'}} = \lambda_{\theta'}$ (from the Lagrangian), and rearranging, we have

$$\frac{\partial\lambda_\theta}{\partial q_{\theta'}} = (1-\varepsilon)\left[\frac{\lambda_\theta\lambda_{\theta'}}{Y^*} + \frac{Y^*}{q_\theta}\lambda_\theta^{\frac{-\varepsilon}{1-\varepsilon}} \frac{\partial}{\partial q_{\theta'}} \int_{\mathcal{J}^*_\theta} y^*_\theta(J)^{\frac{\varepsilon}{1-\varepsilon}} h(J)^{\frac{1}{1-\varepsilon}} dJ\right] \tag{20}$$

Now, using the technique[5] in Appendix A and breaking up the boundary of $\mathcal{J}^*_\theta$ we can manipulate

[5] With $\ln\lambda_{\theta''} - \ln\lambda_\theta$ standing in for r and $\ln y^*_{\theta''}(J) - \ln y^*_\theta(J)$ for M.

the integral's derivative on the right:

$$\frac{\partial \lambda_\theta}{\partial q_{\theta'}} = (1-\varepsilon)\left[\frac{\lambda_\theta \lambda_{\theta'}}{Y^*} - \frac{Y^*}{q_\theta}\lambda_\theta^{\frac{-\varepsilon}{1-\varepsilon}} \sum_{\theta'' \neq \theta}\left(\frac{\partial \ln \lambda_\theta}{\partial q_{\theta'}} - \frac{\partial \ln \lambda_{\theta''}}{\partial q_{\theta'}}\right) \int_{\mathcal{J}_\theta^* \cap \mathcal{J}_{\theta''}^*} \frac{y_\theta^*(J)^{\frac{\varepsilon}{1-\varepsilon}} h(J)^{\frac{1}{1-\varepsilon}}}{\left|\left|\nabla \ln \frac{y_\theta^*(J)}{y_{\theta''}^*(J)}\right|\right|} dJ\right]. \tag{21}$$

Now, using (12) we can substitute for the integrand's numerator

$$\frac{\partial \lambda_\theta}{\partial q_{\theta'}} = (1-\varepsilon)\left[\frac{\lambda_\theta \lambda_{\theta'}}{Y^*} - \frac{Y^*}{q_\theta}\lambda_\theta^{\frac{-\varepsilon}{1-\varepsilon}} \sum_{\theta'' \neq \theta}\left(\frac{\partial \ln \lambda_\theta}{\partial q_{\theta'}} - \frac{\partial \ln \lambda_{\theta''}}{\partial q_{\theta'}}\right) \int_{\mathcal{J}_\theta^* \cap \mathcal{J}_{\theta''}^*} \frac{f_\theta(J) \lambda_\theta^{\frac{1}{1-\varepsilon}} Y^{*-1}}{\left|\left|\nabla \ln \frac{y_\theta^*(J)}{y_{\theta''}^*(J)}\right|\right|} dJ\right]. \tag{22}$$

Finally, we multiply both sides by $q_{\theta'}/\lambda_\theta$ and simplify the expression:

$$\frac{\partial \ln \lambda_\theta}{\partial \ln q_{\theta'}} = (1-\varepsilon)\left[\frac{\lambda_{\theta'} q_{\theta'}}{Y^*} - \sum_{\theta'' \neq \theta}\left(\frac{\partial \ln \lambda_\theta}{\partial \ln q_{\theta'}} - \frac{\partial \ln \lambda_{\theta''}}{\partial \ln q_{\theta'}}\right) \int_{\mathcal{J}_\theta^* \cap \mathcal{J}_{\theta''}^*} \frac{f_\theta(J)/q_\theta}{\left|\left|\nabla \ln \frac{y_\theta^*(J)}{y_{\theta''}^*(J)}\right|\right|} dJ\right]. \tag{23}$$

Surprisingly, we've arrived at something interpretable. The first term in the bracket is straightforward: it comes from the increase in total production due to the increase in type θ' workers. The second term comes from the reallocation of workers across jobs, and depends on the 'velocity' with which jobs are reallocated on boundaries, weighted by the relative mass of workers there. This velocity term is the inverse of the modulus of the gradient of (log) absolute advantage, which reflects how quickly absolute advantage shifts on each point on the boundary.[6] Also of importance is the relative number of workers swept by the motion of the boundary $f_\theta(J)/q_\theta$. The parenthetical term weights these boundary pressures by how the change in $q_{\theta'}$ relatively affects workers of types θ and θ''; if both types are equally hurt, say, $\mathcal{J}_\theta^* \cap \mathcal{J}_{\theta''}^*$ will not shift either way.

Intuitively, a worker type with sharp absolute advantage gradients against neighboring types is insulated from job reallocation, and therefore mainly experiences changes in wages due to changing total production. A less-insulated type will see the gains from total productivity increases ablated by boundary shifts, and they can turn negative.

The own-q_θ derivative is not much different:

$$\frac{\partial \ln \lambda_\theta}{\partial \ln q_\theta} = (1-\varepsilon)\left[\frac{\lambda_\theta q_\theta}{Y^*} - 1 - \sum_{\theta'' \neq \theta}\left(\frac{\partial \ln \lambda_\theta}{\partial \ln q_\theta} - \frac{\partial \ln \lambda_{\theta''}}{\partial \ln q_\theta}\right) \int_{\mathcal{J}_\theta^* \cap \mathcal{J}_{\theta''}^*} \frac{f_\theta(J)/q_\theta}{\left|\left|\nabla \ln \frac{y_\theta^*(J)}{y_{\theta''}^*(J)}\right|\right|} dJ\right]. \tag{24}$$

As for any θ, θ' we have $\frac{\partial \lambda_\theta}{\partial q_\theta} < 0$ and $\frac{\partial \lambda_\theta}{\partial q_\theta} < \frac{\partial \lambda_{\theta'}}{\partial q_\theta}$, we can think of the boundary term as cushioning the blow to the income of the most-afflicted workers by reallocating workers over jobs.

[6] That the gradient of the absolute advantage is (a.e.) perpendicular to the boundary follows from the boundary's definition. If the job set was instead a lower dimensional set embedded in $\mathbb{R}^k$ this would not be the case, and we would have to project the gradient first.

The reallocation *moderates* the effects of the shock for those worst hit.

Now, (23) and (24) fit no-one's ideas of closed-form expressions. However, they are linear in in $\left(\frac{\partial \ln \lambda_{\hat{\theta}}}{\partial \ln q_{\theta'}}\right)_{\hat{\theta}\in\Theta}$ and, combining the relevant formulae for all types, for a collection of equations equinumerous to the unknown terms; thus the statics can be solved for using matrix algebra.

How about changes to the density of workers? The best way to understand changes in the allocation of worker types to jobs is the border term in (23) and (24). However, this does not cover the changes to the allocation among those jobs that do not change hands. So, we now examine how the density of workers changes in such jobs.

Logging both sides of (12) and taking a derivative with respect to $\ln q_{\theta'}$, we obtain

$$\frac{\partial \ln f_\theta(J)}{\partial \ln q_{\theta'}} = -\frac{1}{1-\varepsilon}\left[\frac{\partial \ln \lambda_\theta}{\partial \ln q_{\theta'}}\right] + \frac{\partial \ln Y^*}{\partial \ln q_{\theta'}}. \tag{25}$$

Notice that this does not actually depend on J, so long as J is not on a boundary. Thus, we'll write $\frac{\partial \ln f_\theta}{\partial \ln q_{\theta'}}$. Now, if $\theta \neq \theta'$ we can plug in $\frac{\partial \ln \lambda_\theta}{\partial \ln q_{\theta'}}$ from equation (23), and use the fact that $\lambda_{\theta'} = \frac{\partial Y^*}{\partial q_{\theta'}}$ (from the Lagrangian), to get

$$\frac{\partial \ln f_\theta}{\partial \ln q_{\theta'}} = \sum_{\theta''\neq\theta}\left(\frac{\partial \ln \lambda_\theta}{\partial \ln q_{\theta'}} - \frac{\partial \ln \lambda_{\theta''}}{\partial \ln q_{\theta'}}\right)\int_{\mathcal{J}_\theta^*\cap\mathcal{J}_{\theta''}^*}\frac{f_\theta(J')/q_\theta}{\left|\left|\nabla \ln \frac{y_\theta^*(J')}{y_{\theta''}^*(J')}\right|\right|}dJ. \tag{26}$$

We can further substitute $\left(\frac{\partial \ln \lambda_\theta}{\partial \ln q_{\theta'}} - \frac{\partial \ln \lambda_{\theta''}}{\partial \ln q_{\theta'}}\right) = -(1-\varepsilon)\left(\frac{\partial \ln f_\theta}{\partial \ln q_{\theta'}} - \frac{\partial \ln f_{\theta''}}{\partial \ln q_{\theta'}}\right)$ to get

$$\frac{\partial \ln f_\theta}{\partial \ln q_{\theta'}} = -(1-\varepsilon)\sum_{\theta''\neq\theta}\left(\frac{\partial \ln f_\theta}{\partial \ln q_{\theta'}} - \frac{\partial \ln f_{\theta''}}{\partial \ln q_{\theta'}}\right)\int_{\mathcal{J}_\theta^*\cap\mathcal{J}_{\theta''}^*}\frac{f_\theta(J')/q_\theta}{\left|\left|\nabla \ln \frac{y_\theta^*(J')}{y_{\theta''}^*(J')}\right|\right|}dJ \tag{27}$$

so that along with the expression for $\theta = \theta'$

$$\frac{\partial \ln f_\theta}{\partial \ln q_\theta} = 1-(1-\varepsilon)\sum_{\theta''\neq\theta}\left(\frac{\partial \ln f_\theta}{\partial \ln q_\theta} - \frac{\partial \ln f_{\theta''}}{\partial \ln q_\theta}\right)\int_{\mathcal{J}_\theta^*\cap\mathcal{J}_{\theta''}^*}\frac{f_\theta(J')/q_\theta}{\left|\left|\nabla \ln \frac{y_\theta^*(J')}{y_{\theta''}^*(J')}\right|\right|}dJ \tag{28}$$

we are left with a rather straightforward system of linear equations. If a type other than θ is increased in abundance, the number of workers at jobs that stay with type θ changes only uniformly, increasing via the pressure exerted by the reassignment of jobs to types. When θ is increased in abundance, the density of workers at each job held by θ is increased, but this increase is attenuated by the outward movement of $\mathcal{J}_\theta$'s boundaries.

2.2 Changes to Skill-Augmenting Technology

We enrich the problem by including a *skill-augmenting* technological parameter vector A. This will allow us to model changes to the productivity of skills. Production at each job J given skills S is now $y(A \odot S, J)$, where $\odot$ denotes the Hadamard product. Modifying our previous formulation we have $y^*_\theta(A, J) = \max_{S \in \mathcal{S}_\theta} y(A \odot S, J)$. From there, and with work closely resembling that in Section 2.1, we arrive at[7]

$$\frac{\partial \ln \lambda_\theta}{\partial A_i} = (1-\varepsilon)\frac{\partial \ln Y^*}{\partial A_i} + \varepsilon \int_{\mathcal{J}^*_\theta} \frac{f_\theta}{q_\theta}\frac{\partial \ln y^*_\theta}{\partial A_i} dJ - (1-\varepsilon)\sum_{\theta' \neq \theta} \int_{\mathcal{J}^*_\theta \cap \mathcal{J}^*_{\theta'}} \frac{f_\theta}{q_\theta} \frac{\frac{\partial}{\partial A_i} \ln \left[\frac{y^*_{\theta'}}{y^*_\theta}\frac{\lambda_\theta}{\lambda_{\theta'}}\right]}{\left|\left|\nabla_J \ln \frac{y^*_\theta}{y^*_{\theta'}}\right|\right|} dJ. \tag{29}$$

The first term represents type θ's share of the increase in aggregate production. The second term represents θ's additional share of the increase in their own production. The last term represents the effect of shifting boundaries. The derivative term in the integrand's numerator combines the boundary-shifting effects of both the change in local absolute advantage and the change in wage ratios.

We turn again to changes to the distribution of workers within a type's allocated jobs. Take any type θ and J in the interior of $\mathcal{J}^*_\theta$. Logging (12) and differentiating with respect to A_i,

$$\frac{\partial \ln f_\theta(J)}{\partial A_i} = -\frac{1}{1-\varepsilon}\left[\frac{\partial \ln \lambda_\theta}{\partial A_i}\right] + \frac{\partial \ln Y^*}{\partial A_i} + \frac{\varepsilon}{1-\varepsilon}\frac{\partial \ln y^*_\theta(J)}{\partial A_i}. \tag{30}$$

At this point, we can substitute for $\frac{\partial \ln \lambda_\theta}{\partial A_i}$ using (29) to obtain

$$\frac{\partial \ln f_\theta(J)}{\partial A_i} = \frac{\varepsilon}{1-\varepsilon}\left[\frac{\partial \ln y^*_\theta(J)}{\partial \ln A_i} - \int_{\mathcal{J}^*_\theta} \frac{f_\theta}{q_\theta}\frac{\partial \ln y^*_\theta}{\partial A_i} dJ\right] + \sum_{\theta' \in \Theta} \int_{\mathcal{J}^*_\theta \cap \mathcal{J}^*_{\theta'}} \frac{f_\theta}{q_\theta} \frac{\frac{\partial}{\partial A_i} \ln \left[\frac{y^*_{\theta'}}{y^*_\theta}\frac{\lambda_\theta}{\lambda_{\theta'}}\right]}{\left|\left|\nabla_J \ln \frac{y^*_\theta}{y^*_{\theta'}}\right|\right|} dJ. \tag{31}$$

Unlike in the expression for $\frac{\partial \ln f_\theta(J)}{\partial q_{\theta'}}$, the job J plays an important role here. Interpreting (31) when final goods production is substitutionally elastic (inelastic), the change in the amount of θ workers allocated to J is *increasing (decreasing)* in J's change in productivity relative to the average job for θ workers. The movement of the boundary, as always, plays a role, but we can get a clearer interpretation when comparing the change at two jobs J, J':

$$\frac{\partial \ln f_\theta(J)}{\partial A_i} - \frac{\partial \ln f_\theta(J')}{\partial A_i} = \frac{\varepsilon}{1-\varepsilon}\left[\frac{\partial \ln y^*_\theta(J)}{\partial \ln A_i} - \frac{\partial \ln y^*_\theta(J')}{\partial \ln A_i}\right] \tag{32}$$

Here, it's clear that the elasticity of substitution of final goods production modulates the way worker numbers readjust in the face of a skill-augmenting technical change. Workers move towards (away from) jobs that benefit from the technical change most when final goods production is substitutionally elastic (inelastic).

[7] Arguments suppressed to fit into a row.

2.3 Changes to Final-Good Production Technology

We now wonder what happens when we change the relative importance of different intermediate goods to final good production. We first examine the effect of changes in $h(\cdot)$ on wages.[8] Logging both sides of (15) and totally differentiating, we have

$$d\ln\lambda_\theta = (1-\varepsilon)\left[d\ln Y^* - d\ln q_\theta + d\ln\left(\int_{\mathcal{J}_\theta^*} y_\theta^*(J)^{\frac{\varepsilon}{1-\varepsilon}} h(J)^{\frac{1}{1-\varepsilon}} dJ\right)\right]. \tag{33}$$

As we are only allowing h to vary exogenously, we set $d\ln q_\theta = 0$.

$$d\ln\lambda_\theta = (1-\varepsilon)\left[d\ln Y^* + \frac{d\left(\int_{\mathcal{J}_\theta^*} y_\theta^*(J)^{\frac{\varepsilon}{1-\varepsilon}} h(J)^{\frac{1}{1-\varepsilon}} dJ\right)}{\int_{\mathcal{J}_\theta^*} y_\theta^*(J)^{\frac{\varepsilon}{1-\varepsilon}} h(J)^{\frac{1}{1-\varepsilon}} dJ}\right]. \tag{34}$$

Given that y_θ^* is independent of h, we can break up the rightmost term:

$$d\ln\lambda_\theta = (1-\varepsilon)d\ln Y^* + \frac{\int_{\mathcal{J}_\theta^*} y_\theta^*(J)^{\frac{\varepsilon}{1-\varepsilon}} h(J)^{\frac{1}{1-\varepsilon}} d\ln h(J) dJ}{\int_{\mathcal{J}_\theta^*} y_\theta^*(J)^{\frac{\varepsilon}{1-\varepsilon}} h(J)^{\frac{1}{1-\varepsilon}} dJ} - (1-\varepsilon)\sum_{\theta'\neq\theta}(d\ln\lambda_\theta - d\ln\lambda_{\theta'})\int_{\mathcal{J}_\theta^*\cap\mathcal{J}_{\theta'}^*} \frac{f_\theta(J)/q_\theta}{\left\|\nabla\ln\frac{y_\theta^*(J)}{y_{\theta'}^*(J)}\right\|} dJ. \tag{35}$$

Now, from (12) we have

$$\frac{f_\theta(J')}{f_\theta(J)} = \frac{y_\theta^*(J')^{\frac{\varepsilon}{1-\varepsilon}} h(J')^{\frac{1}{1-\varepsilon}}}{y_\theta^*(J)^{\frac{\varepsilon}{1-\varepsilon}} h(J)^{\frac{1}{1-\varepsilon}}} \tag{36}$$

so that we can integrate dJ' and use $\int_{\mathcal{J}_\theta^*} f_\theta(J')dJ' = q_\theta$ to obtain

$$y_\theta^*(J)^{\frac{\varepsilon}{1-\varepsilon}} h(J)^{\frac{1}{1-\varepsilon}} = \frac{f_\theta(J)}{q_\theta}\int_{\mathcal{J}_\theta^*} y_\theta^*(J')^{\frac{\varepsilon}{1-\varepsilon}} h(J')^{\frac{1}{1-\varepsilon}} dJ. \tag{37}$$

Now, we can substitute (37) into (35) to arrive at

$$d\ln\lambda_\theta = (1-\varepsilon)d\ln Y^* + \int_{\mathcal{J}_\theta^*} \frac{f_\theta(J)}{q_\theta} d\ln h(J) dJ - (1-\varepsilon)\sum_{\theta'\neq\theta}(d\ln\lambda_\theta - d\ln\lambda_{\theta'})\int_{\mathcal{J}_\theta^*\cap\mathcal{J}_{\theta'}^*} \frac{f_\theta(J)/q_\theta}{\left\|\nabla\ln\frac{y_\theta^*(J)}{y_{\theta'}^*(J)}\right\|} dJ. \tag{38}$$

[C- Raghav, this last expression above differs from the one you had. Could you check to see which one is right?]

Now we turn to the effects of changes in h on f_θ among jobs in the interior of $\mathcal{J}_\theta$. Logging (12)

[8] We do this by manipulating infinitesimals directly. More rigorously, these results obtain by replacing $h(J)$ with $h(J) + \iota g(J)$, taking the desired derivative with respect to ι, and evaluating it at $\iota = 0$.

and taking a total derivative, we get,

$$d \ln f_\theta(J) = -\frac{1}{1-\varepsilon} d \ln \lambda_\theta + d \ln Y^* + \frac{1}{1-\varepsilon} d \ln h(J) \tag{39}$$

So that the relative effects on two jobs of a shift in h are given by

$$d \ln f_\theta(J) - d \ln f_\theta(J') = \frac{1}{1-\varepsilon} \left(d \ln h(J) - d \ln h(J')\right). \tag{40}$$

In other words, the relative change in the number of workers doing the two jobs is equal to the relative change in their importance in final good production, modulated by the elasticity of substitution of final good production.

2.4 Some Structure, Please!

For our application, we'll assume that the skill menu of each type is a standard *budget set.* That is, for each $\theta \in \Theta$, we define $\mathcal{S}_\theta = \{S \in \mathbb{R}^n_+ | \sum_i S_i \theta_i \leq 1\}$ as in the homogeneous worker model in **other paper**.

The worker's problem Lagrangian is

$$L = y(A \odot S, J) - \mu \left(\sum_i S_i \theta_i - 1 \right) \tag{41}$$

yielding, for each i, the FOC with respect to S_i,

$$y_i'(A \odot S_\theta^*(J), J) A_i = \mu \theta_i = \theta_i y_\theta^*(A \odot S_\theta^*(J), J) \tag{42}$$

with the second equality coming from constant returns to scale. We now turn to statics. From the Envelope Theorem, we have

$$\frac{\partial y_\theta^*(A, J)}{\partial A_i} = \left. \frac{\partial y(A \odot S, J)}{\partial A_i} \right|_{S = S_\theta^*(J)} = y_i'(A \odot S_\theta^*(J), J) S_{\theta,i}^*(J) \tag{43}$$

so that, combining (42) and (43) we arrive at

$$\frac{\partial \ln y_\theta^*(A \odot J)}{\partial \ln A_i} = \theta_i S_{\theta,i}^*(J). \tag{44}$$

Combining (44) and

A That velocity calculation

Take a function

$$I(k) = \int_{M(X)\leq r(k)} g(X,k)$$

where $M : \mathbb{R}^a \to \mathbb{R}$ is twice-continuously differentiable with non-vanishing gradient, $r : \mathbb{R} \to \mathbb{R}$ is increasing[9] and continuously differentiable, and $g : \mathbb{R}^a \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable. We now use first principles to try to figure out an expression for the derivative of I.

$$\frac{d}{dk} I = \lim_{\iota \to 0} \frac{I(k+\iota) - I(k)}{\iota}$$

This can be expanded as

$$\frac{1}{\iota}\left[\int_{M(X)\leq r(k+\iota)} g(X,k+\iota)dX - \int_{M(X)\leq r(k)} g(X,k)dX\right] \tag{45}$$

$$= \frac{1}{\iota}\left[\int_{M(X)\leq r(k+\iota)} \left[g(X,k) + \iota\frac{dg(X,k)}{dk}\right] dX - \int_{M(X)\leq r(k)} g(X,k)dX\right] \tag{46}$$

$$= \int_{M(X)\leq r(k)} \frac{dg(X,k)}{dk} dX + \frac{1}{\iota}\int_{r(k)\leq M(X)\leq r(k+\iota)} g(X,k)dX + \int_{r(k)\leq M(X)\leq r(k+\iota)} \frac{dg(X,k)}{dk} dX \tag{47}$$

The final term vanishes in the limit as $\iota \to 0$ (which is what we're going for as traditional Greek ιs are truly very small), so we're left with

$$\int_{M(X)\leq r(k)} \frac{dg(X,k)}{dk} dX + \frac{1}{\iota}\int_{r(k)\leq M(X)\leq r(k+\iota)} g(X,k)dX \tag{48}$$

The second term can be thought of as integrating g over the shell of the expansion of the original domain of integration on the interval $(k, k+\iota)$. In other words, we can rewrite the integral on the right as a double integral: the outer one over X such that $M(X) = r(k)$, and the inner one at each such X over those points normal to $M(X) = r(k)$ included in the original integral:

$$\int_{M(X)\leq r(k)} \frac{dg(X,k)}{dk} dX + \frac{1}{\iota}\int_{M(X)=r(k)} \int_0^{t(X)} g\left(X + \tau\frac{\nabla M(X)}{||\nabla M(X)||}, k\right) d\tau dX, \tag{49}$$

where the thickness $t(X)$ solves $M\left(X + t(X)\frac{\nabla M(X)}{||\nabla M(X)||}\right) = r(k+\iota)$. In first-order approximation, the shell has thickness equal to ι times the hypersurface velocity at each point:

$$\frac{t(X)}{\iota} = \frac{r'(k)}{||\nabla M(X)||}$$

[9] Assuming that r is increasing eases exposition significantly but is not, strictly speaking, necessary.

So that we can now write

$$\int_{M(X)\leq r(k)} \frac{dg(X,k)}{dk}dX + \frac{1}{\iota}\int_{M(X)=r(k)}\int_0^{\frac{\iota r'(k)}{||\nabla M||}} g\left(X+\tau\frac{\nabla M(X)}{||\nabla M(X)||},k\right)d\tau dX, \tag{50}$$

and therefore in the limit as $\iota \to 0$ arrive at

$$\frac{d}{dk}I = \int_{M(X)\leq r(k)} \frac{dg(X,k)}{dk}dX + r'(k)\int_{r(k)=M(X)} \frac{g(X,k)}{||\nabla M(X)||}dX. \tag{51}$$

Now, we will need to extend this technique to domains of integration where the relation of X and k is less explicit: consider domains given by some $\{X : M(X,k) \leq 0\}$. All we have to do is recalculate the hypersurface velocity from

$$M\left(X + t(X)\frac{\nabla_X M(X,k)}{||\nabla_X M(X,k)||}, k+\iota\right) = 0, \tag{52}$$

yielding

$$\frac{t(X)}{\iota} = -\frac{\frac{\partial M(X,k)}{\partial k}}{||\nabla_X M(X,k)||} \tag{53}$$

and thus in this case

$$\frac{d}{dk}\int_{M(X,k)\leq 0} g(X,k)dX = \int_{M(X,k)\leq 0}\frac{dg(X,k)}{dk}dX - \int_{M(X,k)=0}\frac{\frac{\partial M(X,k)}{\partial k}}{||\nabla_X M(X,k)||}g(X,k)dX \tag{54}$$