Code documentation

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1 Notation

Our GMM model is based on the moment equations:

$$\mathbb{E}\left(\psi(w_i,\mu)\right)=0$$

where:

- w_i is the vector of data for observation i.
- ψ is a $P\times 1$ vector of functions.
- μ is an $R\times 1$ vector of parameters.

We estimate the model's parameters μ by solving he problem:

$$\hat{\mu}=\arg\min_c\left(\frac{1}{N}\sum_{i=1}^{N}\psi(w_i,c)\right)'A\left(\frac{1}{N}\sum_{i=1}^{N}\psi(w_i,c)\right)$$

which we can write as:

$$\hat{\mu}=\arg\min_c\left(\frac{1}{N}\varepsilon(c)'Z\right)A\left(\frac{1}{N}Z'\varepsilon(c)\right)$$

where $\varepsilon(c)$ is the $N\times 1$ vector of errors of the model, and Z is the $N\times P$ matrix of instruments. Here $N=N_1+N_2+N_3$ is the total number of observations and N_i denotes the number of observations that belong to equation i.

$\varepsilon(c)$ is defined as follows. Define the functions $J(l)$, $E(l)$, $T(l)$, and $I(l)$ which return the job, education level, year, and skill that correspond to observation l. In addition, for observations belonging to the employment equation, define as $EET(l)$ as the function that returns the education pair-year cell of the observation. Then the error for observation l is:

$$\varepsilon_l(\mu)=\begin{cases}\Delta\overline{\ln S_{ijet+1}}-\sum_k\theta_{ke}\overline{S_{kejt}}\pi_{kjt}+\pi_{ijt} & l\leq N_1\\ 1-\sum_k\theta_{ke}\overline{S_{kejt}} & N_1<l\leq N_1+N_2\\ \Delta\left[\ln\frac{q_{ejt+1}}{q_{e'jt+1}}\right]-\beta_j\left[\sum_k\left(\theta_{ke}\overline{S_{kjet}}-\theta_{ke'}\overline{S_{kje't}}\right)\pi_{kjt}\right]-\gamma_{e,e't} & N_1+N_2<l\leq N\end{cases}$$

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2 GMM standard errors

The GMM estimates are distributed as:

$$\sqrt{N}(\hat{\mu} - \mu) \to N(0, \tilde{V})$$

where $\bar{V} = (D'AD)^{-1}D'AVAD(D'AD)^{-1}$. Here:

- D is the model gradient.
- V is defined as:

$$V = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}\left(\psi(w_i, \mu)\psi(w_i, \mu)'\right)$$

- A is the weighting matrix $(Z'Z)^{-1}$

We estimate $\bar{V}$ as:

$$\hat{\bar{V}} = (\hat{D}'A\hat{D})^{-1}\hat{D}'A\hat{V}A\hat{D}(\hat{D}'A\hat{D})^{-1}$$

where:

- $\hat{V} = \frac{1}{N}\sum_{i=1}^{N} \psi(w_i, \hat{\mu})\psi(w_i, \hat{\mu})'$.
- $\hat{D} = \frac{1}{N}\sum_{i=1}^{N} \frac{\partial\psi(w_i,\hat{\mu})}{\partial c'}$

2.1 Computing the gradient:

Let Ξ be the $N \times R$ matrix with general term $\Xi_{lr} = \frac{\partial\varepsilon(w_l,c)}{\partial c_r}$. Then our estimate of the gradient is $P \times R$ matrix:

$$\hat{D} = \frac{1}{N} Z'\Xi$$

Thus, we jut have to compute Ξ_{lr}. We need to compute derivatives with respect to four types of parameters: θ_{ke}, π_{kjt}, and β_j:

$$
\begin{aligned}
\frac{\partial \varepsilon_l(\mu)}{\partial \theta_{ie}} &= \begin{cases} -\overline{S_{kejt}}\pi_{ijt} & l \leq N_1, e = E(l), i \neq 1 \\ -\overline{S_{iejt}} & N_1 < l \leq N_1 + N_2, e = E(l), i \neq 1 \\ -\beta_j \overline{S_{ijet}}\pi_{ijt} & N_1 + N_2 < l \leq N, (e, \cdot, \cdot) = EET(l), i \neq 1 \\ \beta_j \overline{S_{ijet}}\pi_{ijt} & N_1 + N_2 < l \leq N, (\cdot, e, \cdot) = EET(l), i \neq 1 \\ 0 & \text{otherwise} \end{cases} \\
\frac{\partial \varepsilon_l(\mu)}{\partial \theta_1} &= \begin{cases} -\overline{S_{kejt}}\pi_{ijt} & l \leq N_1 \\ -\overline{S_{iejt}} & N_1 < l \leq N_1 + N_2 \\ \beta_j \overline{S_{ijet}}\pi_{ijt} - \beta_j \overline{S_{ije't}}\pi_{ijt} & N_1 + N_2 < l \leq N \\ 0 & \text{otherwise} \end{cases} \\
\frac{\partial \varepsilon_l(\mu)}{\partial \pi_{ijt}} &= \begin{cases} -\theta_{ie}\overline{S_{iejt}} & l \leq N_1, i \neq I(l), j = J(l), t = T(l) \\ -\theta_{ie}\overline{S_{iejt}} + 1 & l \leq N_1, i = I(l), j = J(l), t = T(l) \\ -\beta_j \left(\theta_{ie}\overline{S_{ijet}} - \theta_{ie'}\overline{S_{ije't}}\right) & N_1 + N_2 < l \leq N \\ 0 & \text{otherwise} \end{cases} \\
\frac{\partial \varepsilon_l(\mu)}{\partial \beta_j} &= \begin{cases} -\left[\sum_k \left(\theta_{ke}\overline{S_{kjet}} - \theta_{ke'}\overline{S_{kje't}}\right)\pi_{kjt}\right] & N_1 + N_2 < l \leq N, j = J(l) \\ 0 & \text{otherwise} \end{cases} \\
\frac{\partial \varepsilon_l(\mu)}{\partial \gamma_{ee't}} &= \begin{cases} -1 & N_1 + N_2 < l \leq N, (e, e', t) = EET(l) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

We compute Ξ with the function `get_xi_matrix`, and compute $\bar{V}$ with `estimate_v`. Finally, the function `get_standard_errors` computes the standard errors.

2.2 Standard errors for σ_j and $d\ln A_{ijt}$

The above derivation gives standard errors for $\beta_j = \frac{\sigma_j - 1}{\sigma_j}$. Using the delta method, it is straightforward to compute standard errors for $\sigma_j = \frac{1}{1-\beta_j}$.

$$\sqrt{N}(\hat{\sigma_j} - \sigma_j) \to N\left(0, \text{var}(\sigma_j)\left[\frac{\partial \sigma_j}{\partial \beta_j}\right]^2\right)$$

where:

$$\left[\frac{\partial \sigma_j}{\partial \beta_j}\right]^2 = \frac{1}{(1-\beta_j)^2} = \sigma_j^2$$

By an analogous argument:

$$\sqrt{N}(d\hat{\ln} A_{ijt} - d\ln A_{ijt}) \to N\left(0, G(\gamma)'\tilde{V}G(\gamma)\right)$$

where $\tilde{V}$ is the variance matrix of $\psi = (\pi_{ijt}, \beta_j)'$ and $G(\psi) = (\beta_j, \pi_{ijt})'$