Code documentation

César Garro-Marín*

July 6, 2023

1 GMM standard errors

I minimize the quadratic form:

$$Q = \left(\frac{1}{N}\sum_{i=1}^{N}\psi(w_i, c)\right)' A \left(\frac{1}{N}\sum_{i=1}^{N}\psi(w_i, c)\right) \tag{1}$$

The GMM estimates have a distribution of:

$$\sqrt{N}(\hat{\mu} - \mu) \to N(0, \tilde{V})$$

where $\bar{V} = (D'AD)^{-1}D'AVAD(D'AD)^{-1}$. Here:

- D is the model gradient.
- V is defined as:

$$V = \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(\psi(w_i, \mu)\psi(w_i, \mu)'\right)$$

- A is the weighting matrix $(Z'Z)^{-1}$

1.1 Estimating the variance matrix

I will now describe how do I estimate each component of the variance covariance matrix A:

- **Estimating V:** our estimate is:

$$\hat{V} = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(\psi(w_i, \hat{\mu})\psi(w_i, \hat{\mu})'\right)$$

  we compute this estimate this component in the function `estimate_v`
- **Estimating W:** our weighting matrix is simple $(Z'Z)^{-1}$.
- **Estimating D:** this is the gradient of the model's errors. We compute this matrix in the function `estimate_d`.

*Boston University, email: cesarlgm@bu.edu

A Notation

Our GMM model is based on the moment equations:

$$\mathbb{E}\left(\psi(w_i,\mu)\right)=0$$

where:

- w_i is the vector of data for observation i.
- ψ is a $P\times 1$ vector of functions.
- μ is an $R\times 1$ vector of parameters.

Our estimate is the solution to the problem:

$$\hat{\mu}=\arg\min_c \left(\frac{1}{N}\sum_{i=1}^{N}\psi(w_i,c)\right)' A \left(\frac{1}{N}\sum_{i=1}^{N}\psi(w_i,c)\right)$$

which we can write as:

$$\hat{\mu}=\arg\min_c \left(\frac{1}{N}\varepsilon(c)'Z\right) A \left(\frac{1}{N}Z'\varepsilon(c)\right)$$

where $\varepsilon(c)$ is the $N\times 1$ vector of errors of the model, and Z is the $N\times P$ matrix of instruments. Here $N=N_1+N_2+N_3$ is the total number of observations and N_i denotes the number of observations that belong to equation i.

A.1 The gradient:

Our estimate of the gradient D is:

$$\hat{D}=\frac{1}{N}\sum_{i=1}^{N}\frac{\partial\psi(w_l,\hat{\mu})}{\partial c'}$$

now, let us describe the form of this gradient in more detail. Each element of $\psi(w_l,c)$ is the function $z_{lp}\varepsilon_i(c)$, where $\varepsilon_l(c)$ is the error term for the l-th observation.

To avoid making the notation a mess, define the functions $J(l)$, $E(l)$, $T(l)$, and $I(l)$ which return the job, education level, year, and skill that correspond to observation l. In addition, for the third equation observations I define as $EET(l)$ as the function that returns the education pair-year cell of the observation.

$$\varepsilon_l(\mu)=\begin{cases}\Delta\overline{\ln S_{ijet+1}}-\sum_k\theta_{ke}\overline{S_{kejt}}\pi_{kjt}+\pi_{ijt} & l\le N_1\\ 1-\sum_k\theta_{ke}\overline{S_{kejt}} & N_1<l\le N_1+N_2\\ \Delta\left[\ln\frac{q_{ejt+1}}{q_{e'jt+1}}\right]-\beta_j\left[\sum_k\left(\theta_{ke}\overline{S_{kjet}}-\theta_{ke'}\overline{S_{kje't}}\right)\pi_{kjt}\right]-\gamma_{e,e't} & N_1+N_2<l\le N\end{cases}$$

Note that the parameter vector μ has four types of parameters: θ_{ke}, π_{kjt}, and β_j. Let us go case by case:

$$
\begin{aligned}
\frac{\partial \varepsilon_l(\mu)}{\partial \beta_j} &= \begin{cases} -\left[\sum_k \left(\theta_{ke}\overline{S_{kjet}} - \theta_{ke'}\overline{S_{kje't}}\right)\pi_{kjt}\right] & N_1 + N_2 < l \leq N, j = J(l) \\ 0 & \text{otherwise} \end{cases} \\
\frac{\partial \varepsilon_l(\mu)}{\partial \gamma_{ee't}} &= \begin{cases} -1 & N_1 + N_2 < l \leq N, (e, e', t) = EET(l) \\ 0 & \text{otherwise} \end{cases} \\
\frac{\partial \varepsilon_l(\mu)}{\partial \pi_{ijt}} &= \begin{cases} -\theta_{ie}\overline{S_{iejt}} & l \leq N_1, i \neq I(l), j = J(l), t = T(l) \\ -\theta_{ie}\overline{S_{iejt}} + 1 & l \leq N_1, i = I(l), j = J(l), t = T(l) \\ -\beta_j\left(\theta_{ie}\overline{S_{ijet}} - \theta_{ie'}\overline{S_{ije't}}\right) & N_1 + N_2 < l \leq N \\ 0 & \text{otherwise} \end{cases} \\
\frac{\partial \varepsilon_l(\mu)}{\partial \theta_{ie}} &= \begin{cases} -\overline{S_{kejt}}\pi_{ijt} & l \leq N_1, e = E(l) \\ -\overline{S_{iejt}} & N_1 < l \leq N_1 + N_2, e = E(l) \\ -\beta_j\overline{S_{ijet}}\pi_{ijt} & N_1 + N_2 < l \leq N, (e, \cdot, \cdot) = EET(l) \\ \beta_j\overline{S_{ijet}}\pi_{ijt} & N_1 + N_2 < l \leq N, (\cdot, e, \cdot) = EET(l) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Let Ξ be the $N \times R$ matrix with general term $\Xi_{ir} = \frac{\partial \varepsilon(w_i, c)}{\partial c_r}$. Then our estimate of the gradient is $P \times R$ matrix:

$$\hat{D} = \frac{1}{N} Z'\Xi$$