

Code documentation

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1 New approach (v2):

1.1 Counting parameters

$$\Delta \overline{\ln S_{ijet}^e} = \frac{\sigma_j}{\sigma_j - 1} \left(\sum_k \theta_{ke} \overline{S_{kejt}} \Delta \ln A_{kjt} - \Delta \ln A_{ijet} \right) + \varepsilon_{iej}$$

We have $12JT$ observations, and we need to identify $3JT$ π parameters, and 12θ .

1.2 Model equations

At the job level it holds that:

$$\Delta \overline{\ln S_{ijet}^e} = \frac{\sigma_j}{\sigma_j - 1} \left(\sum_k \theta_{ke} \overline{S_{kejt}} \Delta \ln A_{kjt} - \Delta \ln A_{ijet} \right) \quad (1)$$

$$1 = \sum_k \theta_k^e \overline{S_{keJt}} \quad (2)$$

$$(3)$$

1.3 Building GMM

In our data we have 3 education levels, 4 skills, J jobs and T periods. We normalize $\Delta \ln A_{4jt} = 0, \forall j, t$

- **Equation (1):** instruments for this equation are skills by job matrices. More specifically, let x_{ejt} be the row vector $(S_{1ejt} \ S_{2ejt} \ S_{3ejt})$. Define the matrix X_{ejt} as the 4×3

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matrix containing the skill indexes by education. This matrix excludes the reference skill. For simplicity, I have ignored the bars in the notation.

$$X_{ejt} = \begin{pmatrix} S_{1ejt} & S_{2ejt} & S_{3ejt} \\ S_{1ejt} & S_{2ejt} & S_{3ejt} \\ S_{1ejt} & S_{2ejt} & S_{3ejt} \\ S_{1ejt} & S_{2ejt} & S_{3ejt} \end{pmatrix} = \begin{pmatrix} x_{ejt} \\ x_{ejt} \\ x_{ejt} \\ x_{ejt} \end{pmatrix}$$

Let X_e be the block-diagonal matrix with X_{ejt} in the diagonal.

$$X_e = \begin{pmatrix} X_{e11} & \mathbf{0} & \dots & 0 \\ \mathbf{0} & X_{e12} & \dots & 0 \\ \vdots & & & \\ \mathbf{0} & \mathbf{0} & \dots & X_{eJT} \end{pmatrix}$$

The set of instruments for equation 1 is then:

$$Z_1 = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \quad (4)$$

- **Equation (2):** instruments for the sum to restriction are the skill indexes by education level. Let \tilde{x}_{ejt} be the row vector $(S_{1ejt} \ S_{2ejt} \ S_{3ejt} \ S_{4ejt})$ and let \tilde{X}_e be the $N_e \times 4$ matrix stacking all the \tilde{x}_{ejt} for education level e . The instruments for equation 2 are the block diagonal matrix with elements \tilde{X}_e :

$$Z_2 = \begin{pmatrix} \tilde{X}_1 & 0 & 0 \\ 0 & \tilde{X}_2 & 0 \\ 0 & 0 & \tilde{X}_3 \end{pmatrix}$$

- **Putting it all together:** let u_l be the stacked vector of equation l errors. Define:

$$Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}$$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

the moment equations for the model are given by $\mathbb{E}(Z'u) = 0$.

I choose parameters θ to minimize the quadratic form:

$$\min_{\theta} \frac{1}{N} (u'Z(Z'Z)^{-1}Z'u) \quad (5)$$

1.4 Calculating the gradient

Reminder:

$$\frac{\partial Q}{\partial x} = x'(A + A')$$

I can write the quadratic form as:

$$Q(\gamma) = \frac{1}{N} (u(\gamma)' Z (Z' Z)^{-1} Z' u(\gamma))$$

then,

$$\frac{\partial Q(\gamma)}{\partial \gamma} = \frac{2}{N} \left(\frac{\partial u(\gamma)}{\partial \gamma} \right)' Z (Z' Z)^{-1} Z' u(\gamma)$$

now, I start element by element:

$$\begin{aligned} \frac{\partial \Delta \ln S_{iejt}}{\partial \ln A_{kjt}} &= \theta_{ke} S_{kejt} \\ \frac{\partial \Delta \ln S_{iejt}}{\partial \ln A_{ijt}} &= \theta_{ie} S_{iejt} - 1 \\ \frac{\partial \Delta \ln S_{iejt}}{\partial \ln \theta_{ke}} &= S_{kjt} \Delta \ln A_{kjt} \end{aligned}$$

for the sum to 1 restrictions we have:

$$\frac{\partial g_{ejt}(\gamma)}{\partial \theta_{ke}} = -S_{kejt}$$

next thing to do: write the gradient