

Code documentation

César Garro-Marín*

July 3, 2023

1 GMM standard errors

I minimize the quadratic form:

$$Q = \left(\frac{1}{N} \sum_{i=1}^N \psi(w_i, c) \right)' A \left(\frac{1}{N} \sum_{i=1}^N \psi(w_i, c) \right) \quad (1)$$

The GMM estimates have a distribution of:

$$\sqrt{N}(\hat{\mu} - \mu) \rightarrow N(0, \tilde{V})$$

where $\tilde{V} = (D'AD)^{-1}D'AVAD(D'AD)^{-1}$. Here:

- D is the model gradient.
- V is defined as:

$$V = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \mathbb{E}(\psi(w_i, \mu)\psi(w_i, \mu)')$$

- A is the weighting matrix $(Z'Z)^{-1}$

1.1 Estimating the variance matrix

I will now describe how do I estimate each component of the variance covariance matrix A :

- **Estimating V :** our estimate is:

$$\hat{V} = \frac{1}{N} \sum_{i=1}^N \mathbb{E}(\psi(w_i, \hat{\mu})\psi(w_i, \hat{\mu})')$$

we compute this estimate this component in the function `estimate_v`

- **Estimating W :** our weighting matrix is simple $(Z'Z)^{-1}$.
- **Estimating D :** this is the gradient of the model's errors. We compute this matrix in the function `estimate_d`.

*Boston University, email: cesarlgm@bu.edu

A Notation

Our GMM model is based on the moment equations:

$$\mathbb{E}(\psi(w_i, \mu)) = 0$$

where:

- w_i is the vector of data for observation i .
- ψ is a $P \times 1$ vector of functions.
- μ is an $R \times 1$ vector of parameters.

Our estimate is the solution to the problem:

$$\hat{\mu} = \arg \min_c \left(\frac{1}{N} \sum_{i=1}^N \psi(w_i, c) \right)' A \left(\frac{1}{N} \sum_{i=1}^N \psi(w_i, c) \right)$$

which we can write as:

$$\hat{\mu} = \arg \min_c \left(\frac{1}{N} \varepsilon(c)' Z \right) A \left(\frac{1}{N} Z' \varepsilon(c) \right)$$

where $\varepsilon(c)$ is the $N \times 1$ vector of errors of the model, and Z is the $N \times P$ matrix of instruments. Here $N = N_1 + N_2 + N_3$ is the total number of observations and N_i denotes the number of observations that belong to equation i .

A.1 The gradient:

Our estimate of the gradient D is:

$$\hat{D} = \frac{1}{N} \sum_{i=1}^N \frac{\partial \psi(w_i, \hat{\mu})}{\partial c'}$$

now, let us describe the form of this gradient in more detail. Each element of $\psi(w_l, c)$ is the function $z_{lp} \varepsilon_l(c)$, where $\varepsilon_l(c)$ is the error term for the l -th observation.

To avoid making the notation a mess, define the functions $J(l)$, $E(l)$, $T(l)$, and $I(l)$ which return the job, education level, year, and skill that correspond to observation l . In addition, for the third equation observations I define as $EET(l)$ as the function that returns the education pair-year cell of the observation.

$$\varepsilon_l(\mu) = \begin{cases} \frac{\Delta \ln \overline{S_{ijet+1}} - \sum_k \theta_{ke} \overline{S_{kej t}} \pi_{kjt} + \pi_{ijt}}{1 - \sum_k \theta_{ke} \overline{S_{kej t}}} & l \leq N_1 \\ \Delta \left[\ln \frac{q_{ejt+1}}{q_{e'jt+1}} \right] - \beta_j \left[\sum_k (\theta_{ke} \overline{S_{kjet}} - \theta_{ke'} \overline{S_{kje't}}) \pi_{kjt} \right] - \gamma_{e,e't} & N_1 < l \leq N_1 + N_2 \\ \Delta \left[\ln \frac{q_{ejt+1}}{q_{e'jt+1}} \right] - \beta_j \left[\sum_k (\theta_{ke} \overline{S_{kjet}} - \theta_{ke'} \overline{S_{kje't}}) \pi_{kjt} \right] - \gamma_{e,e't} & N_1 + N_2 < l \leq N \end{cases}$$

Note that the parameter vector μ has four types of parameters: θ_{ke} , π_{kjt} , and β_j . Let us go case by case:

$$\begin{aligned}
\frac{\partial \varepsilon_l(\mu)}{\partial \beta_j} &= \begin{cases} -\beta_j [\sum_k (\theta_{ke} \overline{S_{kjet}} - \theta_{ke'} \overline{S_{kje't}}) \pi_{kjt}] & N_1 + N_2 < l \leq N, j = J(l) \\ 0 & \text{otherwise} \end{cases} \\
\frac{\partial \varepsilon_l(\mu)}{\partial \gamma_{ee't}} &= \begin{cases} -1 & N_1 + N_2 < l \leq N, (e, e', t) = EET(l) \\ 0 & \text{otherwise} \end{cases} \\
\frac{\partial \varepsilon_l(\mu)}{\partial \pi_{ij t}} &= \begin{cases} -\theta_{ie} \overline{S_{iej t}} & l \leq N_1, i \neq I(l), j = J(l), t = T(l) \\ -\theta_{ie} \overline{S_{iej t}} + 1 & l \leq N_1, i = I(l), j = J(l), t = T(l) \\ -\beta_j (\theta_{ie} \overline{S_{ijet}} - \theta_{ie'} \overline{S_{ije't}}) & N_1 + N_2 < l \leq N \\ 0 & \text{otherwise} \end{cases} \\
\frac{\partial \varepsilon_l(\mu)}{\partial \theta_{ie}} &= \begin{cases} -\overline{S_{kejt}} \pi_{ij t} & l \leq N_1, e = E(l) \\ -\overline{S_{iej t}} & N_1 < l \leq N_1 + N_2, e = E(l) \\ -\beta_j \theta_{ie} \overline{S_{ijet}} \pi_{ij t} & N_1 + N_2 < l \leq N, (e, \cdot, \cdot) = EET(l) \\ \beta_j \theta_{ie} \overline{S_{ijet}} \pi_{ij t} & N_1 + N_2 < l \leq N, (\cdot, e, \cdot) = EET(l) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Let Ξ be the $N \times R$ matrix with general term $\Xi_{ir} = \frac{\partial \varepsilon(w_i, c)}{\partial c_r}$. Then our estimate of the gradient is $P \times R$ matrix:

$$\hat{D} = \frac{1}{N} Z' \Xi$$