L9: Linear Regression

Prof. Xun Jiao

Before class

- HW1 Answers
- HW2 due now
- HW3 will be posted

What is a linear equation?

Linear regression assumes a predictor of the form

$$X\theta = y$$

matrix of features (data)

unknowns (which features are relevant)

vector of outputs (labels)

(or $Ax = b$ if you prefer)

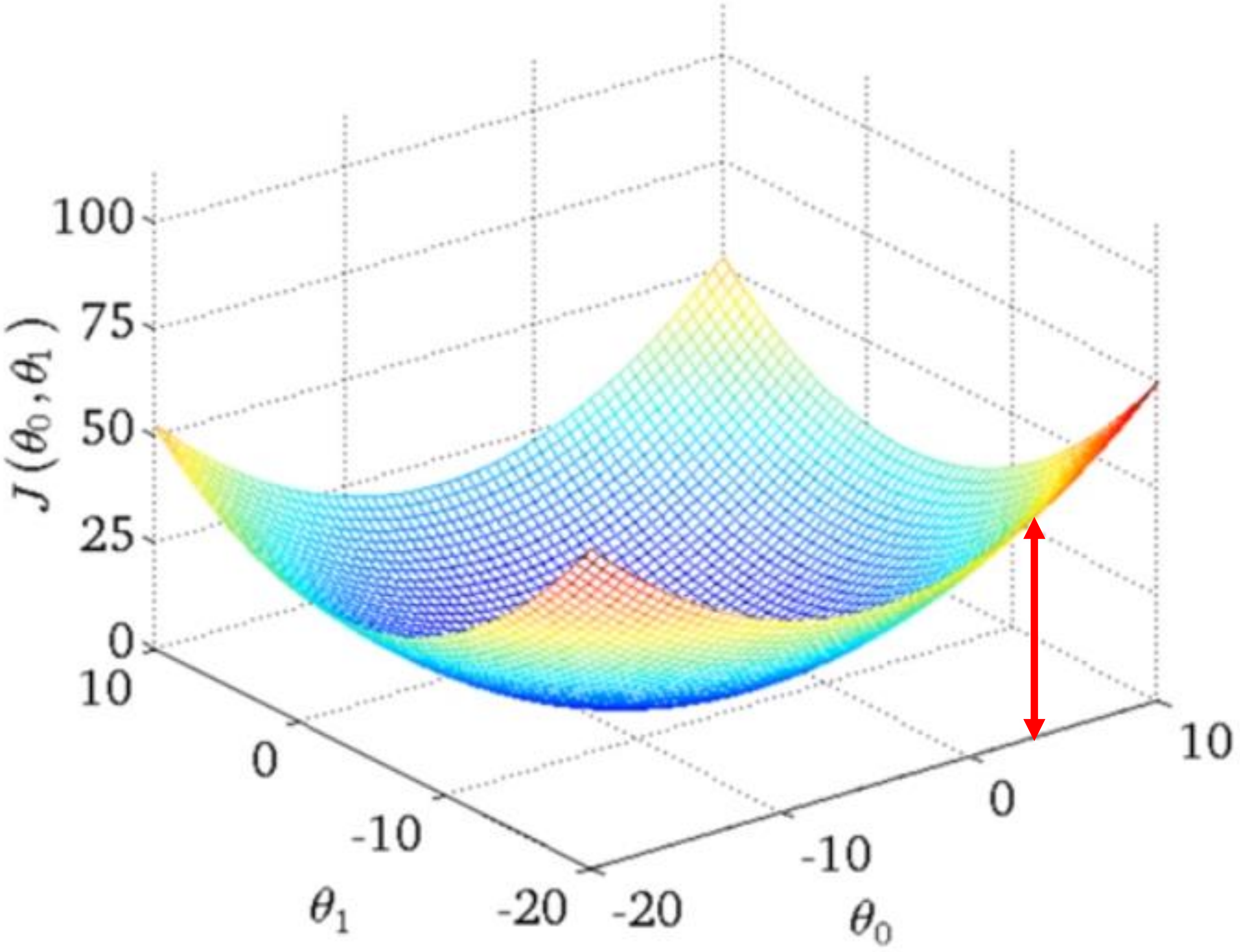
Hypothesis: $h_\theta(x) = \theta_0 + \theta_1 x$

Parameters: θ_0, θ_1

Cost Function: $J(\theta_0, \theta_1) = \frac{1}{2m} \sum_{i=1}^{m} \left(h_\theta(x^{(i)}) - y^{(i)}\right)^2$

Goal: $\underset{\theta_0, \theta_1}{\text{minimize}}\ J(\theta_0, \theta_1)$

Squared error function: most-widely used cost function.



100
75
50
25
0
$J(\theta_0, \theta_1)$
10
0
-10
-20
θ_1
-20
-10
0
10
θ_0

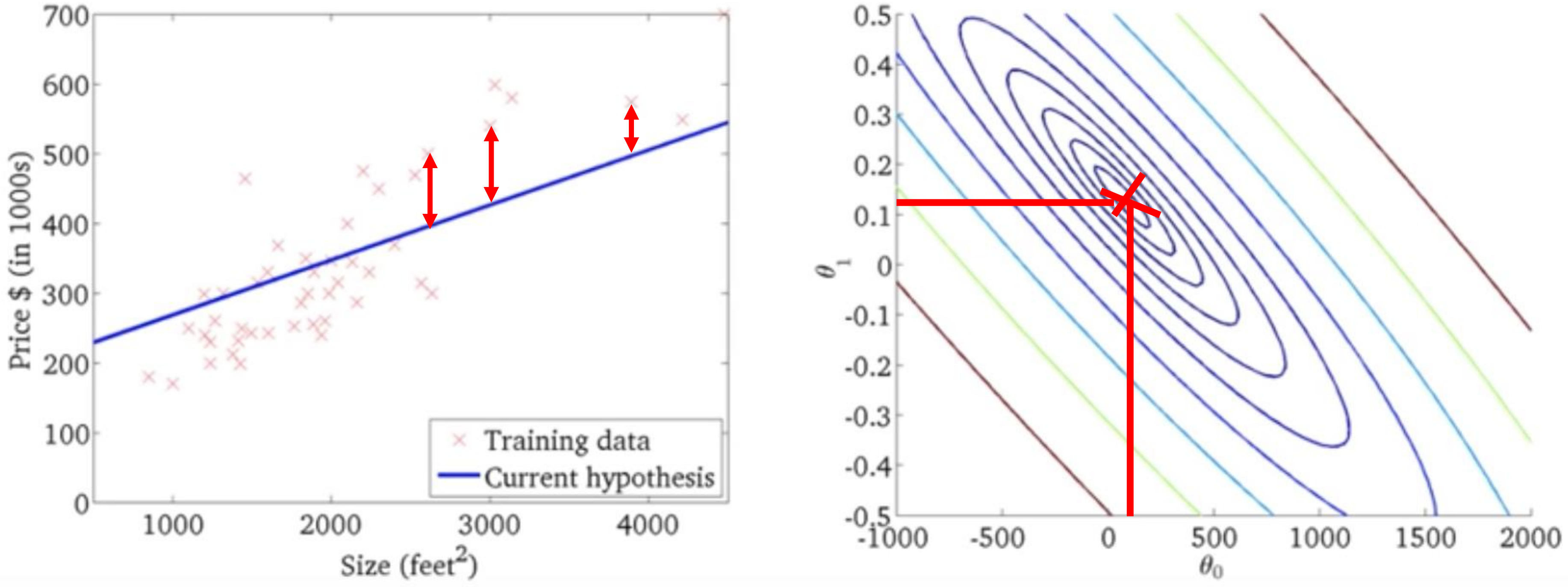
Contour plot: central point

$$h_\theta(x)$$

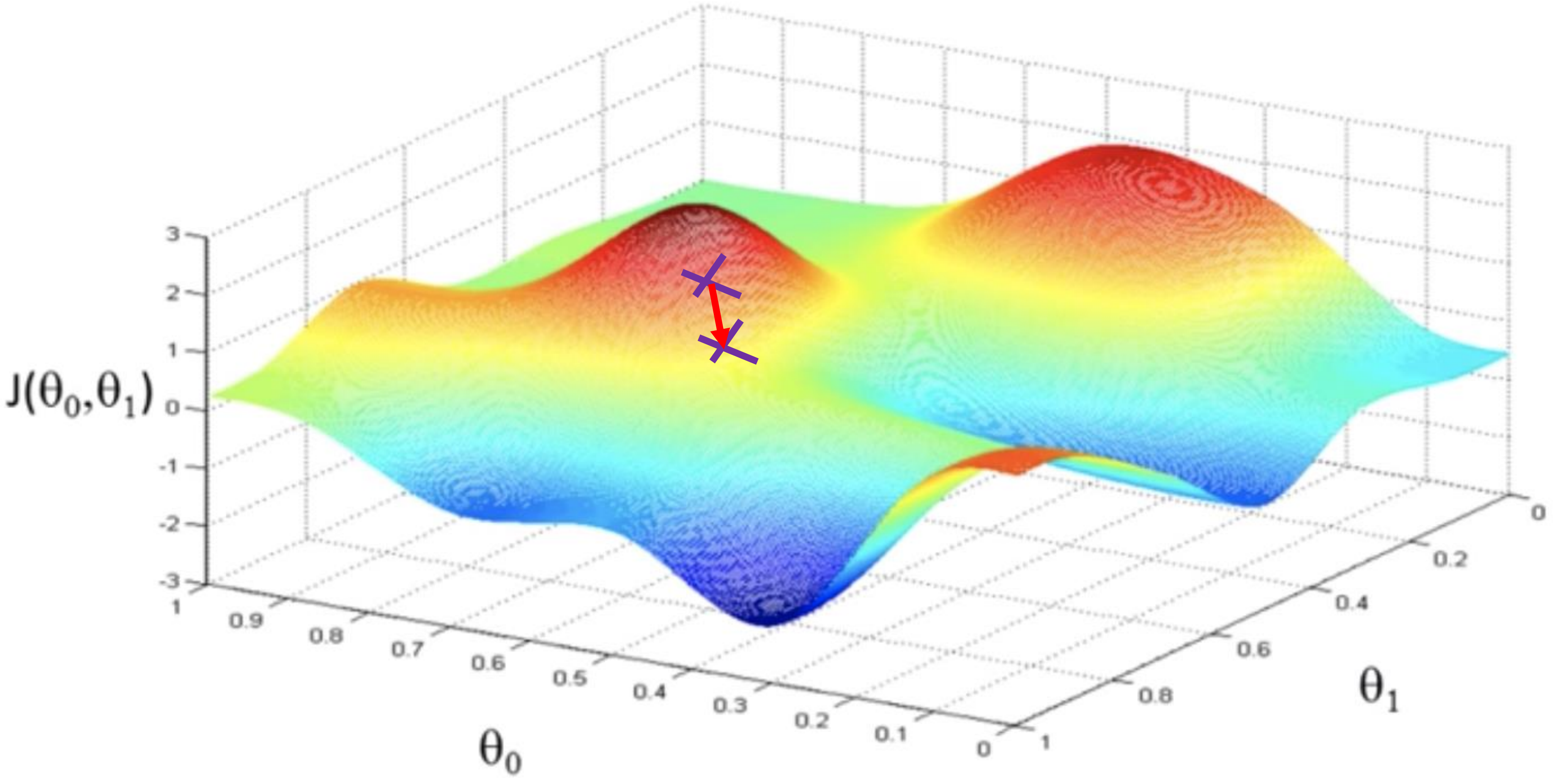
(for fixed θ_0, θ_1, this is a function of x)

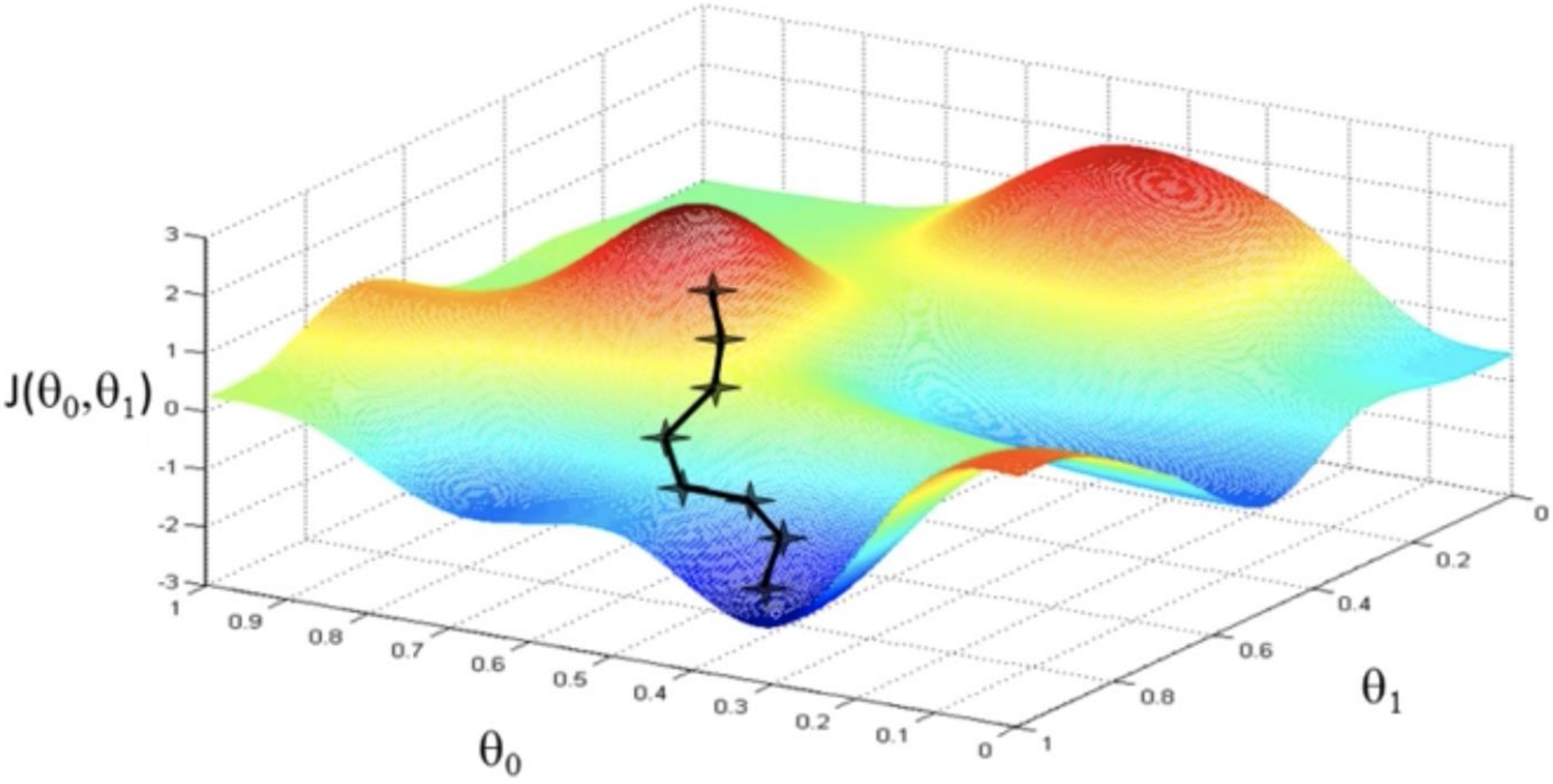
$$J(\theta_0, \theta_1)$$

(function of the parameters θ_0, θ_1)



Which direction I should move if I want to go down (as rapidly as possible) ?





$J(\theta_0,\theta_1)$
θ_0
θ_1
3
2
1
0
-1
-2
-3
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1

Gradient descent algorithm

repeat until convergence {

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1) \qquad (\text{for } j = 0 \text{ and } j = 1)$$

}

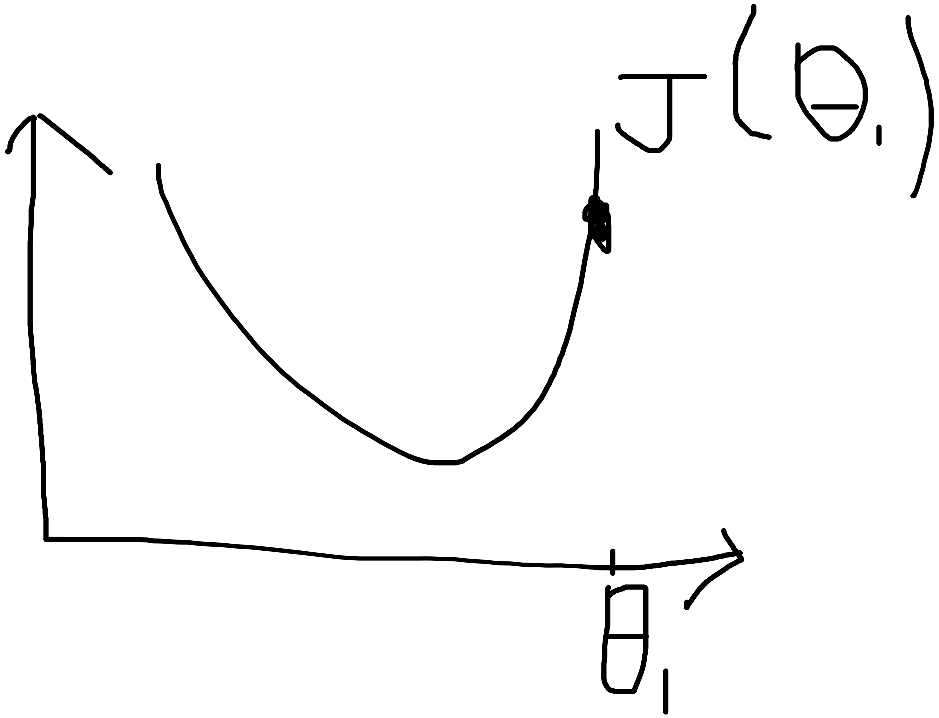
learning rate

Correct: Simultaneous update

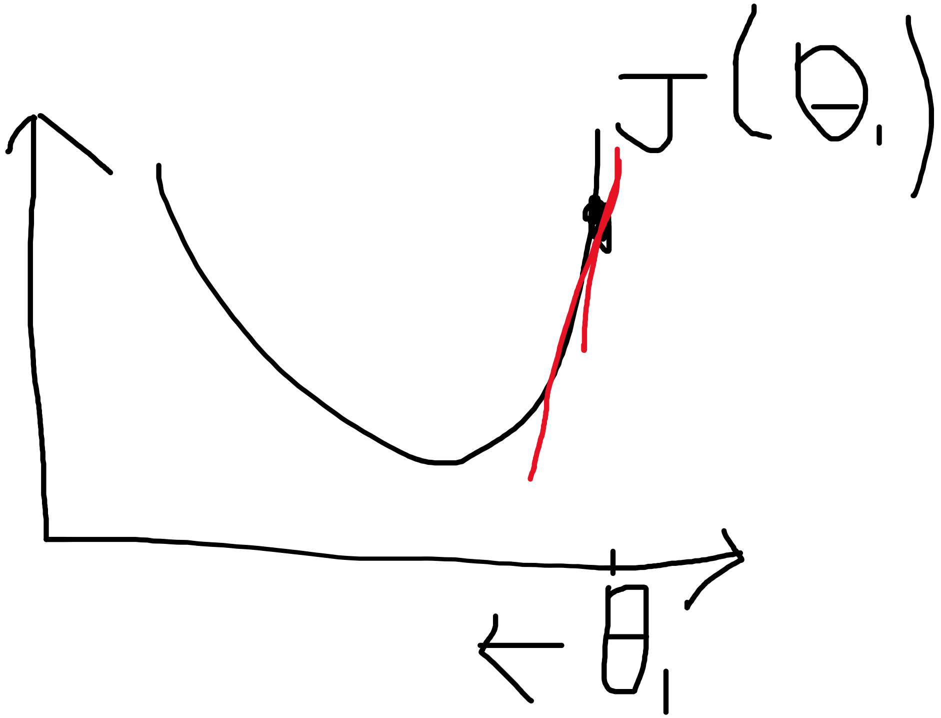
$$\begin{aligned} \text{temp0} &:= \theta_0 - \alpha \tfrac{\partial}{\partial \theta_0} J(\theta_0, \theta_1) \\ \text{temp1} &:= \theta_1 - \alpha \tfrac{\partial}{\partial \theta_1} J(\theta_0, \theta_1) \\ \theta_0 &:= \text{temp0} \\ \theta_1 &:= \text{temp1} \end{aligned}$$

Incorrect:

$$\begin{aligned} \text{temp0} &:= \theta_0 - \alpha \tfrac{\partial}{\partial \theta_0} J(\theta_0, \theta_1) \\ \theta_0 &:= \text{temp0} \\ \text{temp1} &:= \theta_1 - \alpha \tfrac{\partial}{\partial \theta_1} J(\theta_0, \theta_1) \\ \theta_1 &:= \text{temp1} \end{aligned}$$

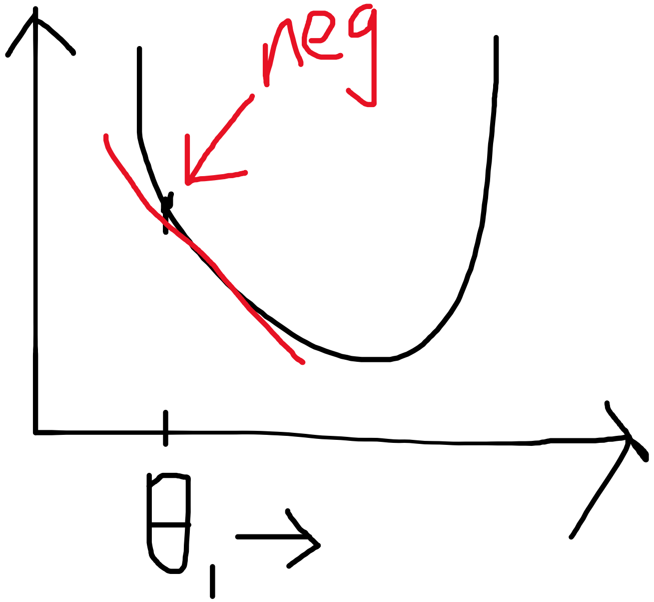


$$\theta_1 := \theta_1 - \alpha \frac{d}{d\theta_1} J(\theta_1)$$



$$\theta_1 := \theta_1 - \alpha \frac{d}{d\theta_1} J(\theta_1)$$

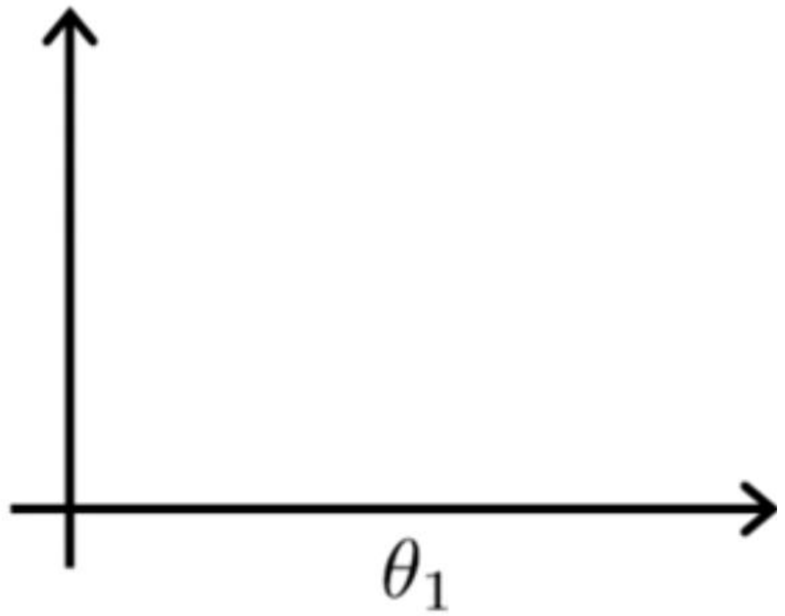
$$\theta_1 = \theta_1 - \alpha(\text{pos})$$



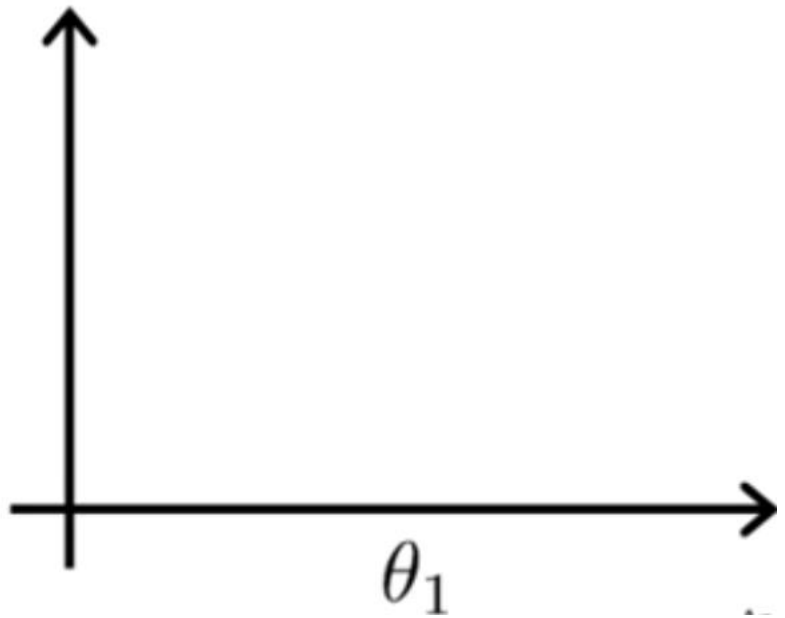
$$\theta_1 = \theta_1 - \alpha(\text{neg})$$

$$\theta_1 := \theta_1 - \alpha \frac{\partial}{\partial \theta_1} J(\theta_1)$$

If α is too small, gradient descent can be slow.



If α is too large, gradient descent can overshoot the minimum. It may fail to converge, or even diverge.



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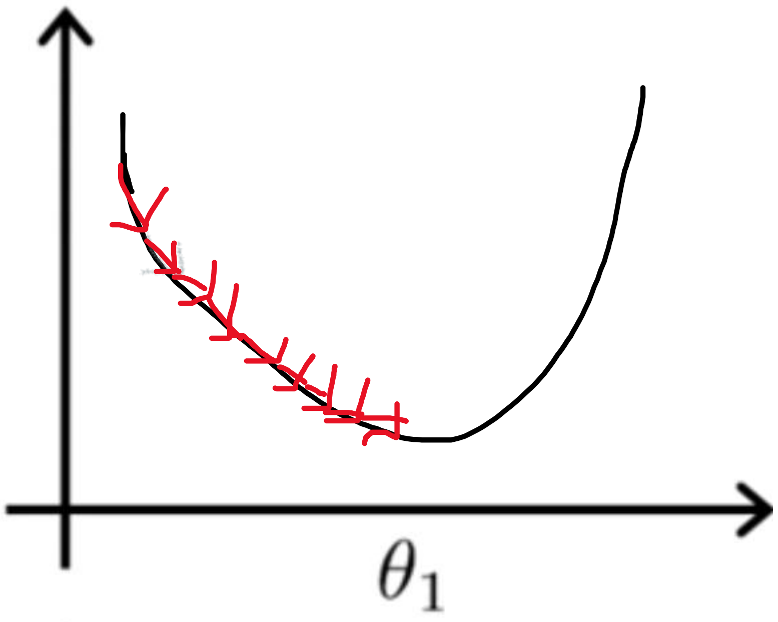

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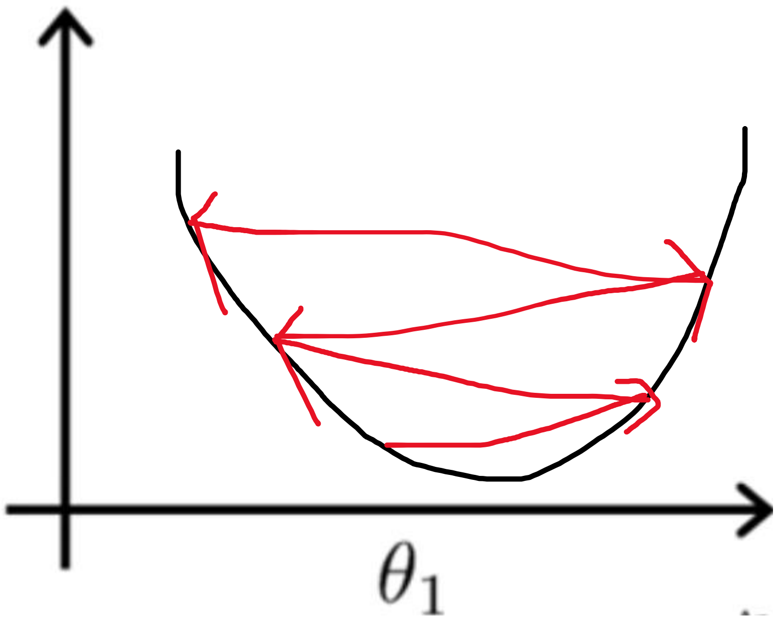


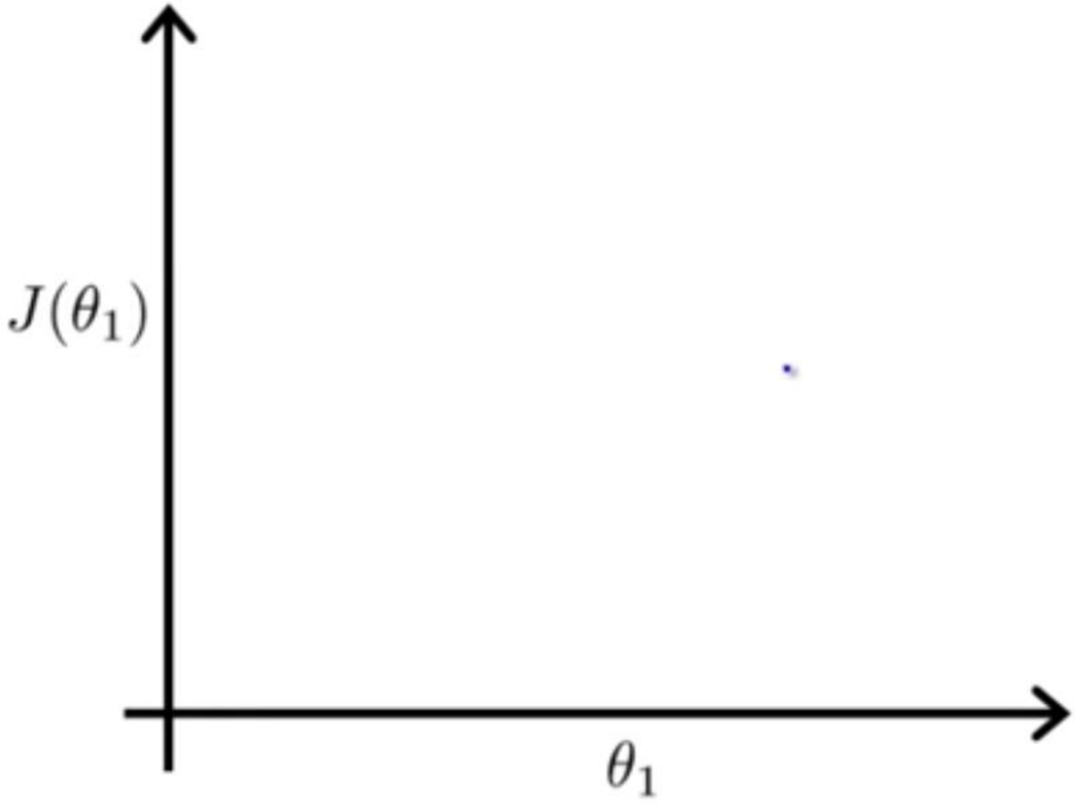
Question?

- What if already in local optimum?

Gradient descent can converge to a local minimum, even with the learning rate α fixed.

$$\theta_1 := \theta_1 - \alpha \frac{d}{d\theta_1} J(\theta_1)$$

As we approach a local minimum, gradient descent will automatically take smaller steps. So, no need to decrease α over time.



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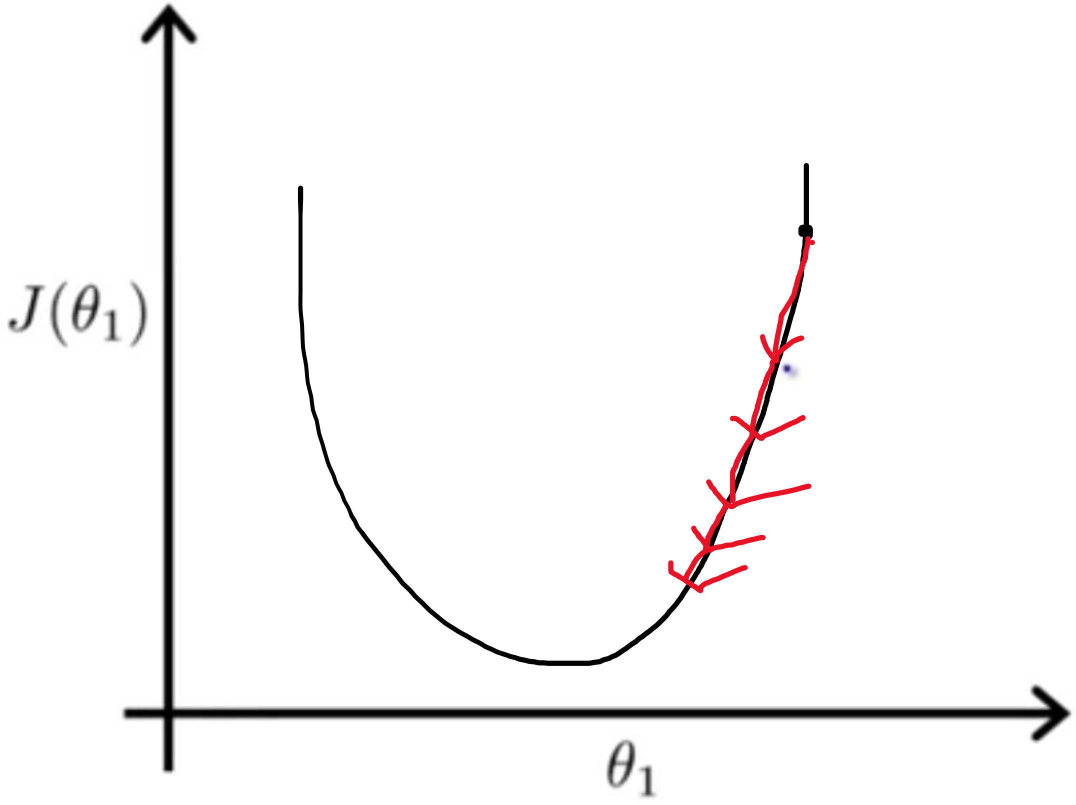

Now, we should apply GD to our problem

Gradient descent algorithm

repeat until convergence {

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1)$$

$$\text{(for } j = 1 \text{ and } j = 0)$$

}

Linear Regression Model

$$h_\theta(x) = \theta_0 + \theta_1 x$$

$$J(\theta) = \frac{1}{2m} \sum_{i=1}^{m} \left(h_\theta(x^{(i)}) - y^{(i)}\right)^2$$

Now, we should apply GD to our problem

Gradient descent algorithm

repeat until convergence {

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1)$$

(for $j = 1$ and $j = 0$)

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Gradient Descent

- Initialize $\boldsymbol{\theta}$
- Repeat until convergence

$$\theta_j \leftarrow \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\boldsymbol{\theta})$$

simultaneous update
for j = 0 ... d

For Linear Regression: $\frac{\partial}{\partial \theta_j} J(\boldsymbol{\theta}) = \frac{\partial}{\partial \theta_j} \frac{1}{2n} \sum_{i=1}^{n} \left(h_{\boldsymbol{\theta}} \left(\boldsymbol{x}^{(i)} \right) - y^{(i)} \right)^2$

Gradient Descent

- Initialize $\boldsymbol{\theta}$
- Repeat until convergence

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$$= \frac{\partial}{\partial \theta_j} \frac{1}{2n} \sum_{i=1}^{n} \left(\sum_{k=0}^{d} \theta_k x_k^{(i)} - y^{(i)} \right)^2$$

Gradient Descent

- Initialize $\boldsymbol{\theta}$
- Repeat until convergence

$$\theta_j \leftarrow \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\boldsymbol{\theta})$$

simultaneous update
for j = 0 ... d

For Linear Regression:
$$\frac{\partial}{\partial \theta_j} J(\boldsymbol{\theta}) = \frac{\partial}{\partial \theta_j} \frac{1}{2n} \sum_{i=1}^{n} \left(h_{\boldsymbol{\theta}} \left(\boldsymbol{x}^{(i)} \right) - y^{(i)} \right)^2$$
$$= \frac{\partial}{\partial \theta_j} \frac{1}{2n} \sum_{i=1}^{n} \left(\sum_{k=0}^{d} \theta_k x_k^{(i)} - y^{(i)} \right)^2$$
$$= \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{k=0}^{d} \theta_k x_k^{(i)} - y^{(i)} \right) \times \frac{\partial}{\partial \theta_j} \left(\sum_{k=0}^{d} \theta_k x_k^{(i)} - y^{(i)} \right)$$

Gradient Descent

- Initialize $\boldsymbol{\theta}$
- Repeat until convergence

$$\theta_j \leftarrow \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\boldsymbol{\theta})$$

simultaneous update
for j = 0 ... d

For Linear Regression:
$$\frac{\partial}{\partial \theta_j} J(\boldsymbol{\theta}) = \frac{\partial}{\partial \theta_j} \frac{1}{2n} \sum_{i=1}^{n} \left(h_{\boldsymbol{\theta}} \left(\boldsymbol{x}^{(i)} \right) - y^{(i)} \right)^2$$
$$= \frac{\partial}{\partial \theta_j} \frac{1}{2n} \sum_{i=1}^{n} \left(\sum_{k=0}^{d} \theta_k x_k^{(i)} - y^{(i)} \right)^2$$
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$$= \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{k=0}^{d} \theta_k x_k^{(i)} - y^{(i)} \right) x_j^{(i)}$$

Gradient Descent for Linear Regression

- Initialize $\boldsymbol{\theta}$
- Repeat until convergence

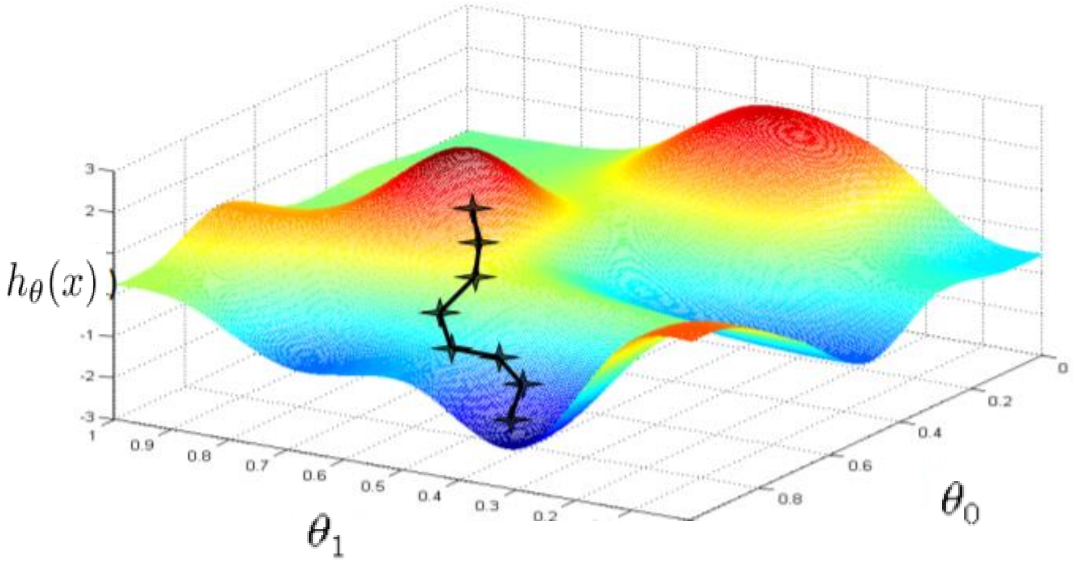
$$\theta_j \leftarrow \theta_j - \alpha \frac{1}{n} \sum_{i=1}^{n} \left(h_{\boldsymbol{\theta}} \left(\boldsymbol{x}^{(i)} \right) - y^{(i)} \right) x_j^{(i)}$$

simultaneous update for $j = 0 \ldots d$

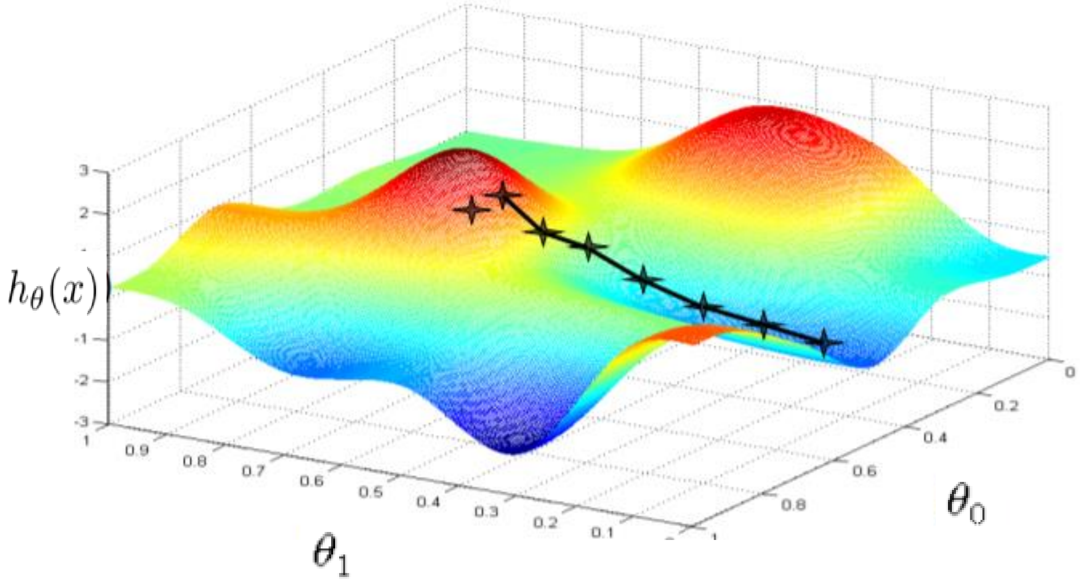
- To achieve simultaneous update
  - At the start of each GD iteration, compute $h_{\boldsymbol{\theta}} \left(\boldsymbol{x}^{(i)} \right)$
  - Use this stored value in the update step loop

- Assume convergence when $\|\boldsymbol{\theta}_{new} - \boldsymbol{\theta}_{old}\|_2 < \epsilon$

L_2 norm: $\|\boldsymbol{v}\|_2 = \sqrt{\sum_i v_i^2} = \sqrt{v_1^2 + v_2^2 + \ldots + v_{|v|}^2}$



$h_\theta(x)$
θ_1
θ_0

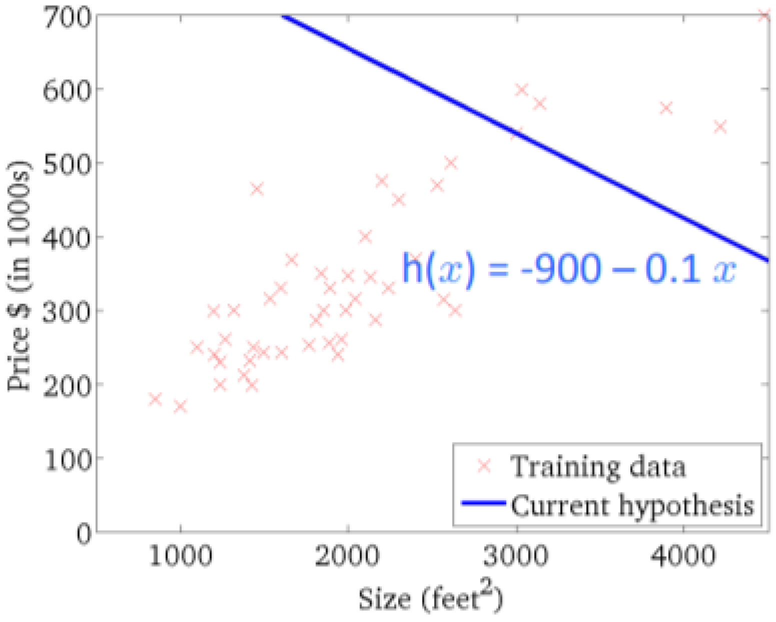


$h_\theta(x)$
θ_1
θ_0

Gradient Descent

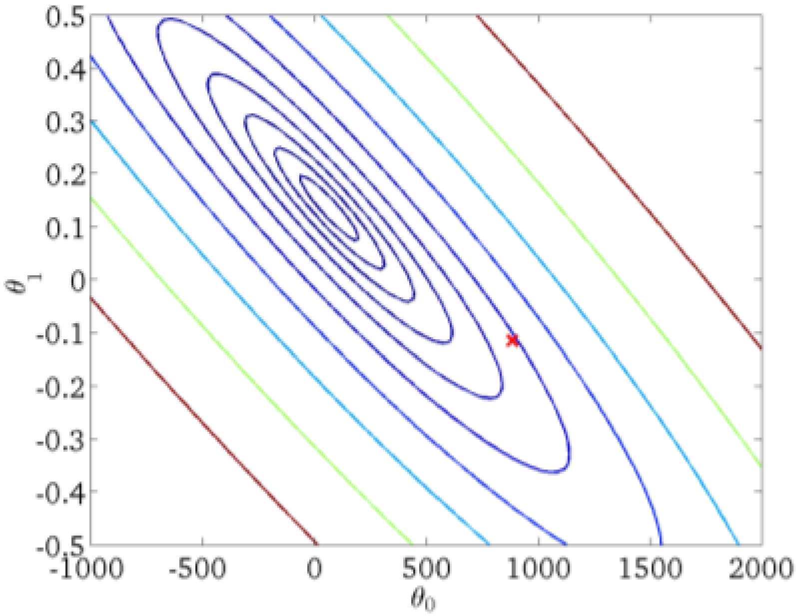
$h_\theta(x)$

(for fixed θ_0, θ_1, this is a function of x)



$J(\theta_0, \theta_1)$

(function of the parameters θ_0, θ_1)



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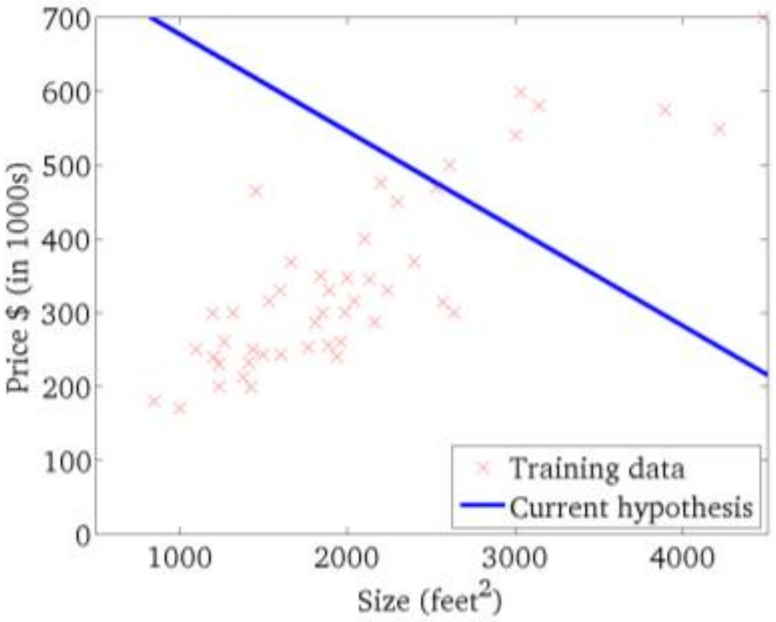

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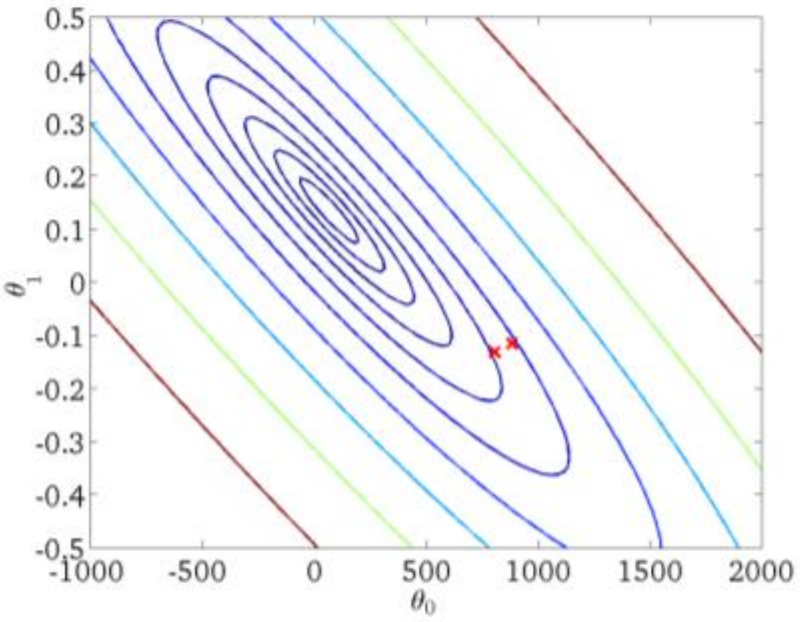

Gradient Descent

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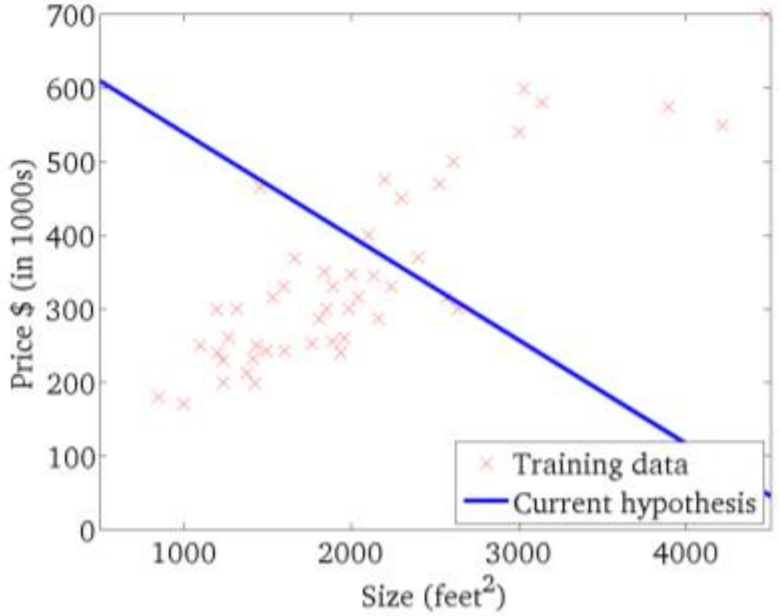

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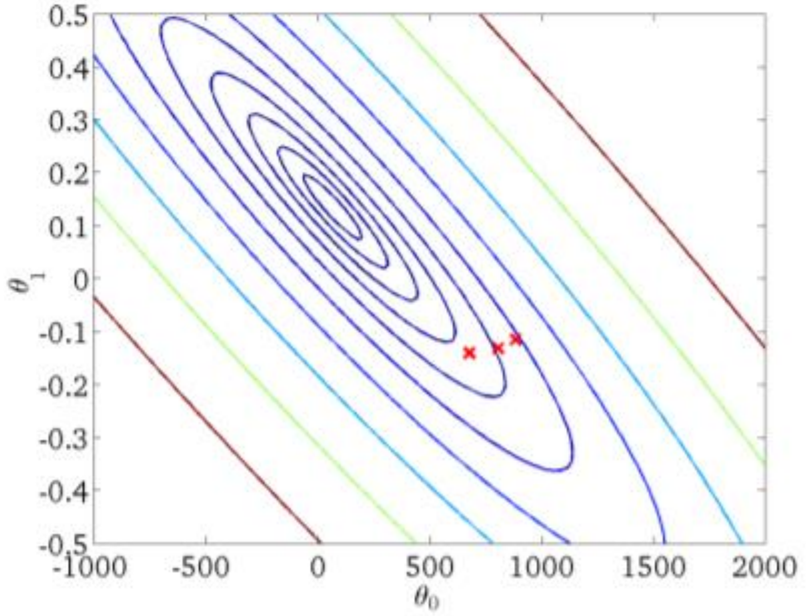

Gradient Descent

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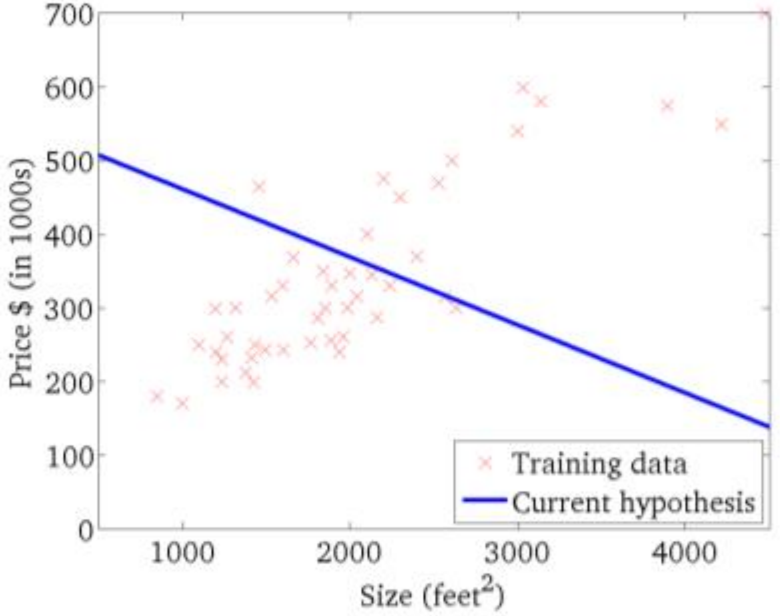

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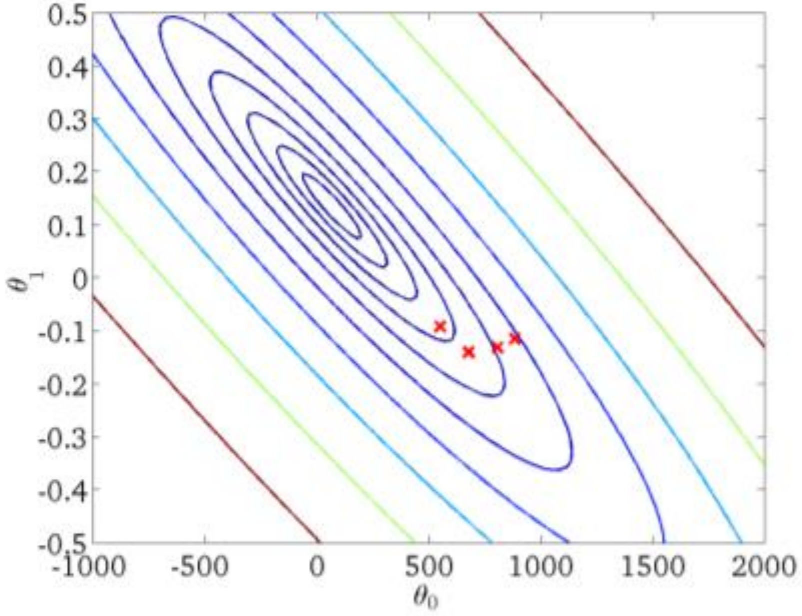

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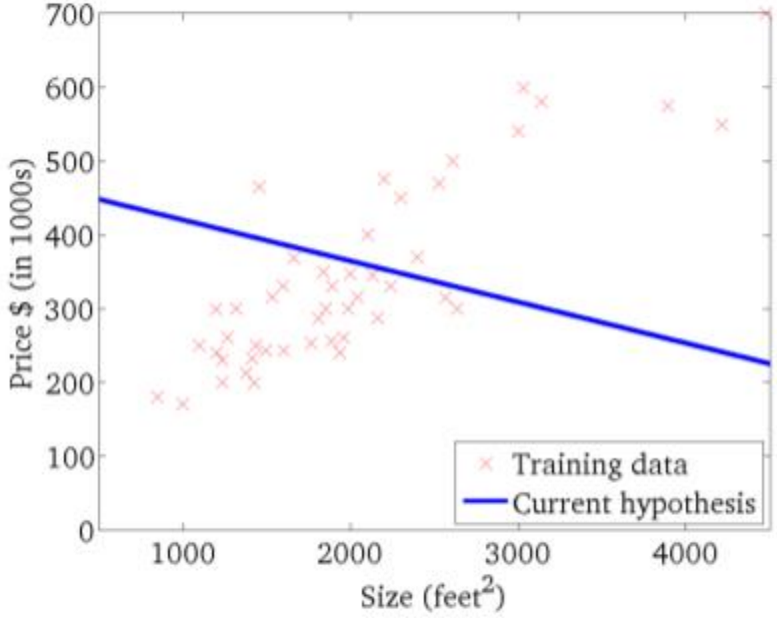

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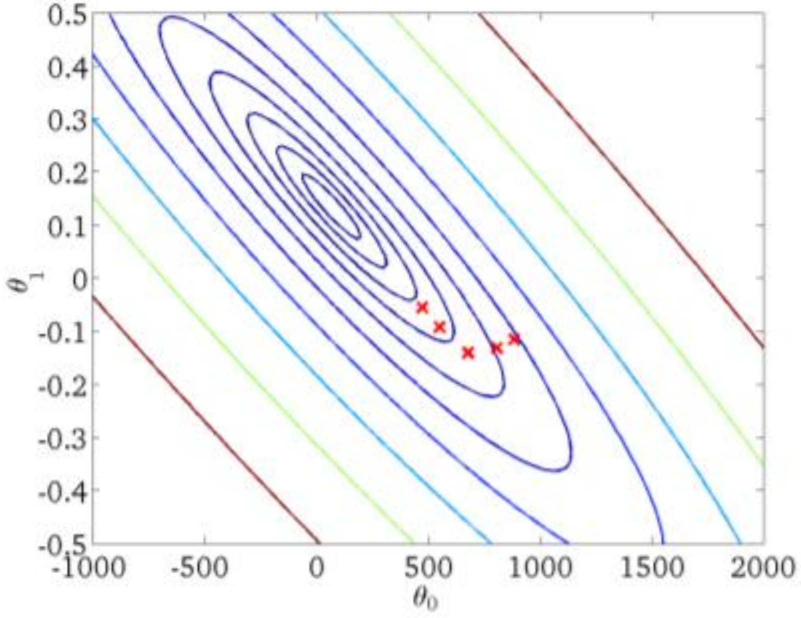

Gradient Descent

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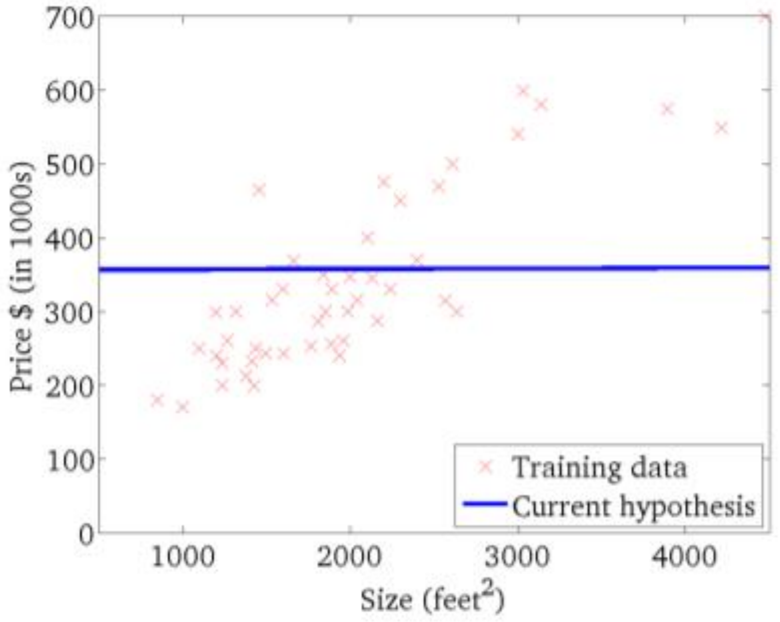

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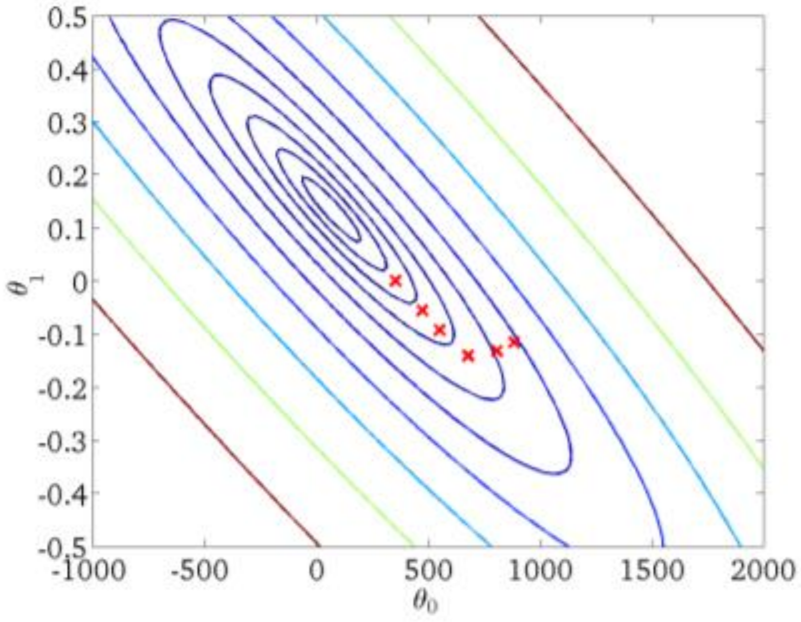

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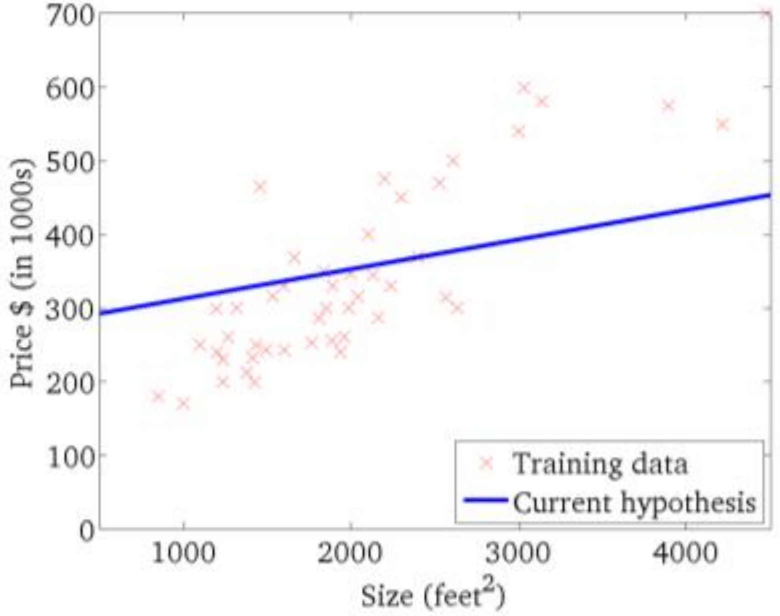

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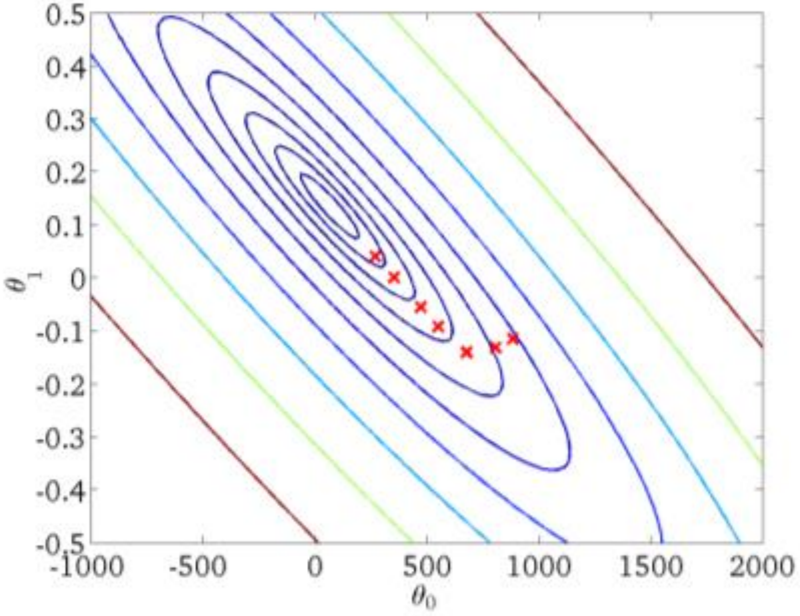

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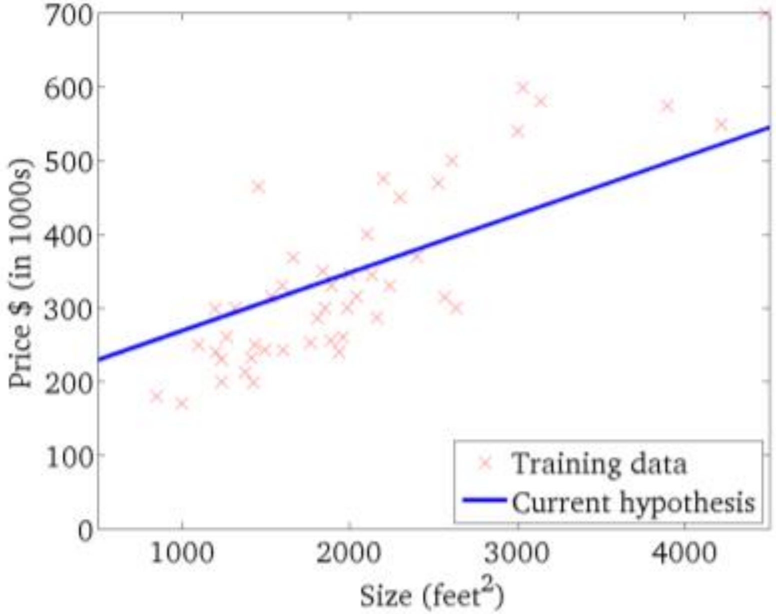

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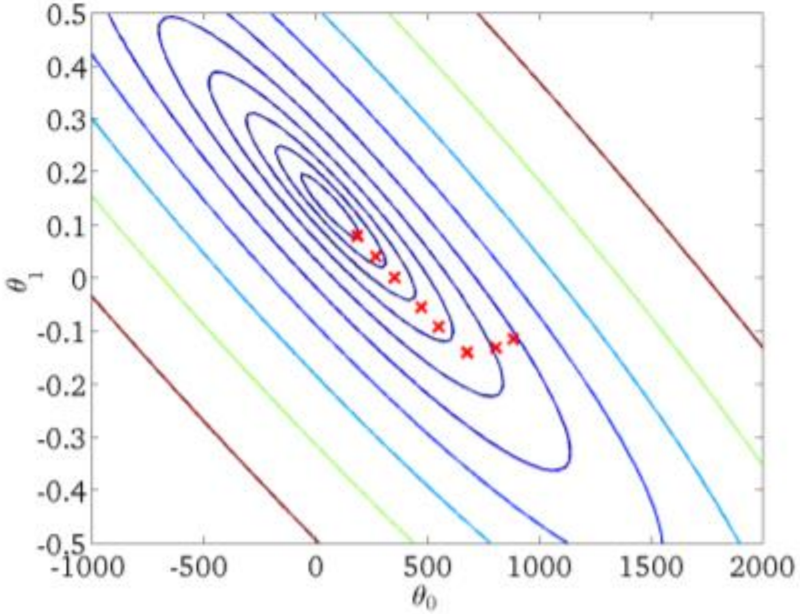

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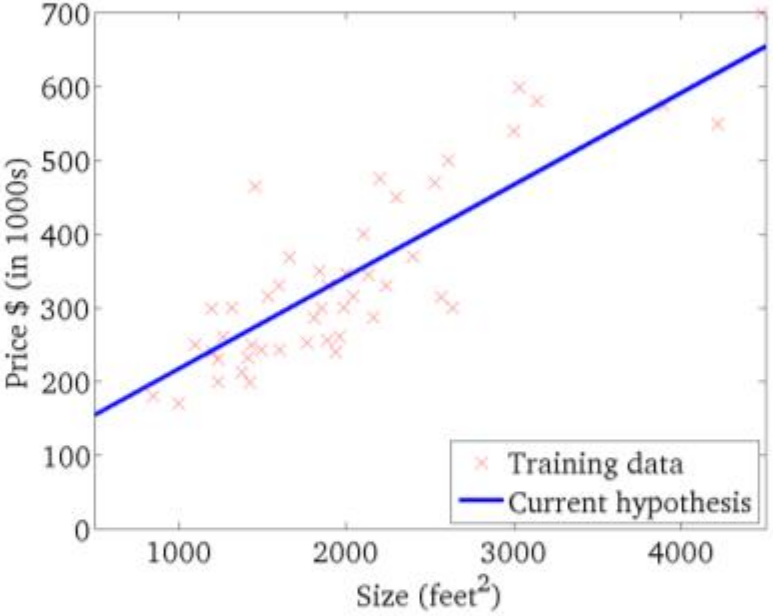

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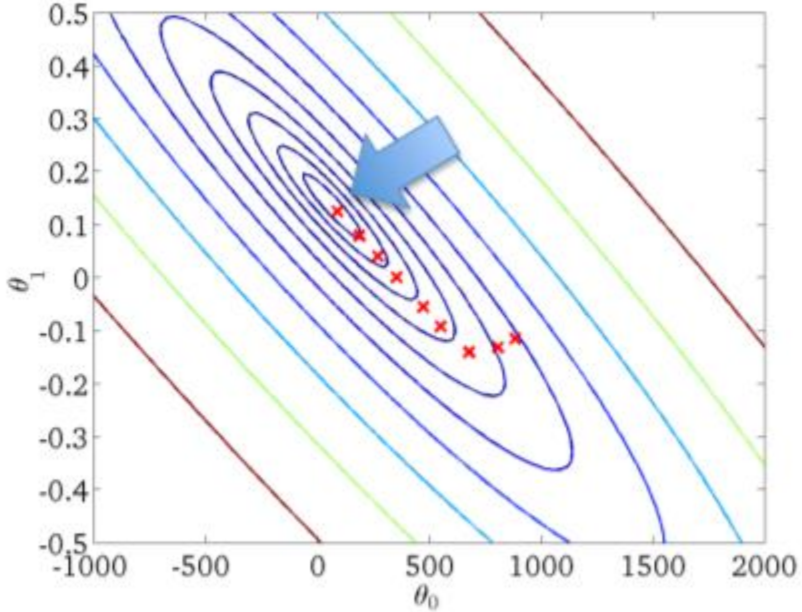


Extending Linear Regression to More Complex Models

- The inputs X for linear regression can be:
  - Original quantitative inputs
  - Transformation of quantitative inputs
    - e.g., log, exp, square root, square, etc.
  - Polynomial transformation
    - example: $y = \boldsymbol{\theta}_0 + \boldsymbol{\theta}_1 * X + \boldsymbol{\theta}_2 * X^2$

This allows use of linear regression techniques
to fit non-linear datasets.

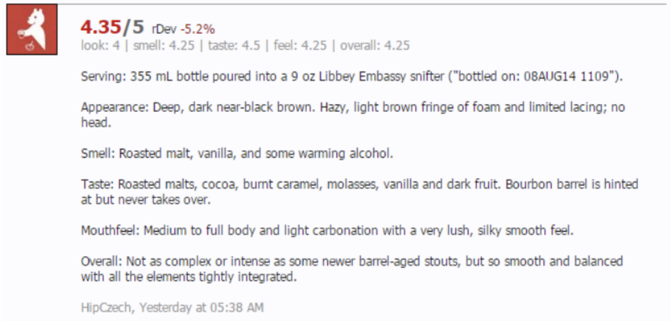
Solve real problems

- How to train a predictor for beer rating?

Beeradvocate

Beers:

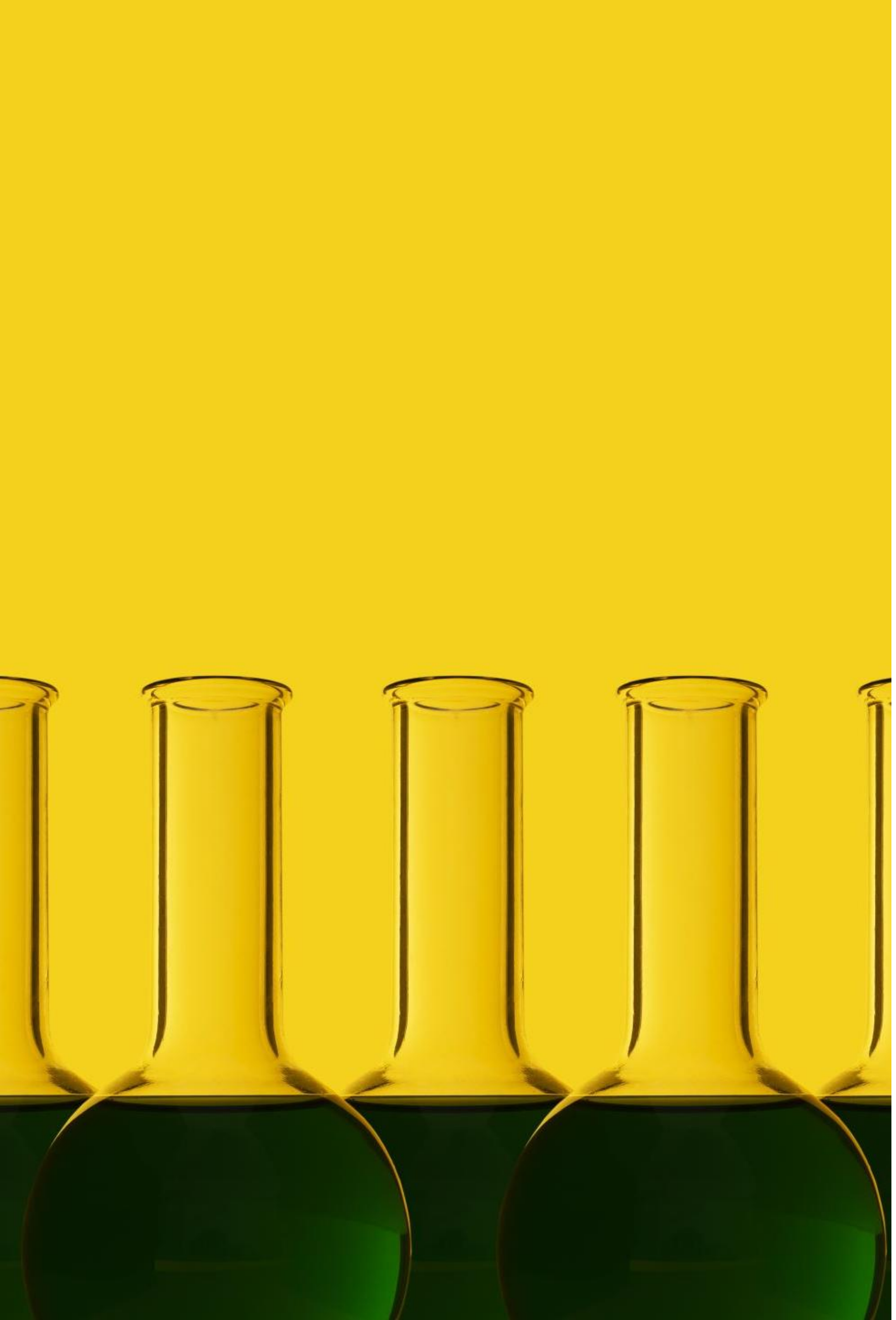
Ratings/reviews:



User profiles:

50,000 beer rating reviews are available on

- https://www.ece.villanova.edu/~xjiao/course/ECE5400/dataset/beer_50000.json



Example 1

- Using ordinary linear regression, train a predictor that uses the age (`ageInSeconds') to predict the beer rating (`review/overall'), i.e.,
  - review/overall $= \boldsymbol{\theta}_0 + \boldsymbol{\theta}_1$ * ageInSeconds
  - You may use Python libraries to do so. What are the fitted values of $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_1$?

```
import numpy
import urllib
import urllib.request
import scipy.optimize
import random

def parseData(fname):
  for l in urllib.request.urlopen(fname):
    yield eval(l)

print ("Reading data")
data = list(parseData("https://www.ece.villanova.edu/~xjiao/course/ECE5400/dataset/beer_50000.json"))
print ("done")

### Do older people rate beer more highly? ###

data2 = [d for d in data if 'user/ageInSeconds' in d]

def feature(datum):
  feat = [1]
  feat.append(datum['user/ageInSeconds'])
  return feat

X = [feature(d) for d in data2]
y = [d['review/overall'] for d in data2]
theta,residuals,rank,s = numpy.linalg.lstsq(X, y)
print(theta)

~
```