Lecture 5: Analyzing Growth and Endogenous Growth

Hisam Sabouni

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Overview

In our last lecture we walked through the cannonical Solow growth model. We concluded that while the model is elegant and provides meaningful insights into the accumulation of capital on GDP growth, the effects of capital accumulation on GDP growth seem to be quite small as indicated by the large Solow residual. The Solow residual, or, total factor productivity, of a country contains all of the residual growth not associated with capital and labor accumulation. The interpretation of the Solow residual is as a measure of technological progress, which the Solow model assumes to be exogenous of the model (taken as given). Endogenous growth theory explicitly models the dynamics of technological progress in an economy by allowing capital and labor to be allocated to a production sector (for consumption) or to a research and development sector (for production of new knowledge).

Before walking through a simplified model of endogenous growth, lets walk through data on a variety of countries to see how actual economic growth has evolved. A variety of data sources exist for economic growth (some going back hundreds of years), such as FRED, the World Bank, and the Penn World Tables. To keep things simple we'll work with the Penn World Tables data as the data is fairly clean and contains historical measures of economic activity from 1950 - 2014 for 182 countries (Version 9.0: https://www.rug.nl/ggdc/productivity/pwt/).

A full list of the Penn World Table variables is provided: https://cran.r-project.org/web/packages/pwt9/pwt9.pdf

A formal write up on the data is avaiable as well: http://cid.econ.ucdavis.edu/Papers/Feenstra_Inklaar_Timmer_AER.pdf

Proportions of Output

Beginning with our first lecture we described that most production functions are assumed to be Cobb-Douglas equations, following the observations that capital and labor have historically had relatively constant shares of income:

$$Y = AK^{\alpha}L^{1-\alpha}$$

A producer has the following profit maximization problem:

$$Profit = PY - wL - rK$$

$$\frac{\partial Profit}{\partial L} : P * MPL = W \rightarrow MPL = \frac{W}{P} = \text{Real wage}$$

$$\frac{\partial Profit}{\partial K} : P * MPK = R \rightarrow MPK = \frac{R}{P} = \text{Real rental price of captial}$$

The fraction of resources spent on labor is:

$$\text{Labor Share} = (\frac{W}{P})L/Y$$

and the fraction of resources spent on income is:

$$\text{Capital Share} = (\frac{R}{P})K/Y$$

Given that MPL and MPK represent the first order conditions of the production function:

$$(MPL)\frac{\partial Y}{\partial L} : (1-\alpha)A(\frac{K}{L})^{\alpha}$$

$$(MPK)\frac{\partial Y}{\partial L} : \alpha A(\frac{L}{K})^{1-\alpha}$$

Going back to our profit maximization problem:

$$(1-\alpha)A(\frac{K}{L})^{\alpha} = \frac{W}{P} = \text{Real wage}$$

$$\alpha A(\frac{L}{K})^{1-\alpha} = \frac{R}{P} = \text{Real rental price of captial}$$

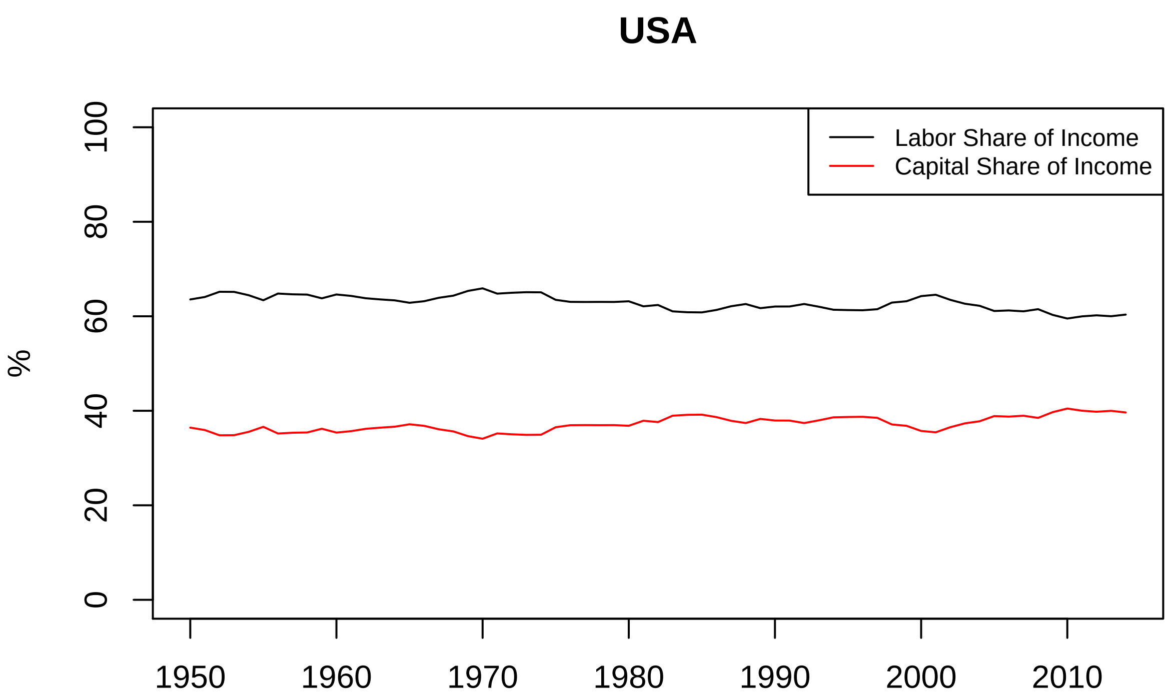
$$\text{Labor Share} = (\frac{W}{P})L/Y = (1-\alpha)A(\frac{K}{L})^{\alpha}(\frac{L}{AK^{\alpha}L^{1-\alpha}}) = (1-\alpha)$$

$$\text{Capital Share} = (\frac{R}{P})K/Y = \alpha A(\frac{L}{K})^{1-\alpha}(\frac{K}{AK^{\alpha}L^{1-\alpha}}) = \alpha$$

Fortuneately for us, the Penn World Tables include calculations for labor's share of income over time for 182 countries. If our Cobb-Douglas specification is correct, we should expect labors share of total income to be fairly constant and assume that the remaining income goes to capital.[1].

```
# install.packages('pwt9')
library(pwt9)
df <- pwt9.0
df$country <- as.character(df$country)
df$isocode <- as.character(df$isocode)
# labsh Share of labour compensation in GDP at
# current national prices.
usa_data <- df[df$isocode == "USA", ]

plot(usa_data$year, 100 * usa_data$labsh, ylim = c(0,
    100), typ = "l", ylab = "%", xlab = "", main = "USA")
lines(usa_data$year, 100 * (1 - usa_data$labsh),
    col = 2)
legend("topright", legend = c("Labor Share of Income",
    "Capital Share of Income"), lty = 1, col = 1:2,
    cex = 0.75)
```

[1] Labor's share of income is compliled as from a variety of sources on total income earned by employees/self-employeed individuals divided by the total GDP




```
mean(usa_data$labsh)   #0.628523
```

```
## [1] 0.628523
```

As we can see in the plot above, labor's share of income is fairly close to around $\frac{2}{3}$ of USA GDP, allowing us to proxy that the remaining $\frac{1}{3}$ is associated with returns to the capital stock. The United States is however just one country; how about the rest of the world? Lets calculate average labor share and average capital share over time across our sample of countries.

```
country_labor_share_mean <- tapply(df$labsh, df$isocode,
    function(x) mean(x, na.rm = T))
country_labor_share_mean
```

```
##       ABW       AGO       AIA       ALB       ARE       ARG       ARM
## 0.6466552       NaN       NaN       NaN       NaN 0.5007833 0.7413535
##       ATG       AUS       AUT       AZE       BDI       BEL       BEN
##       NaN 0.6403704 0.6225844 0.4622360 0.7354902 0.6389382 0.6374211
##       BFA       BGD       BGR       BHR       BHS       BIH       BLR
## 0.6351585       NaN 0.5613804 0.3197409 0.4126325 0.6442799 0.5347508
##       BLZ       BMU       BOL       BRA       BRB       BRN       BTN
##       NaN 0.6232925 0.5264869 0.5231254 0.7502117       NaN       NaN
##       BWA       CAF       CAN       CHE       CHL       CHN       CIV
## 0.3259567 0.2208344 0.6727201 0.6633102 0.4490850 0.6290265 0.5365637
```

```
##       CMR       COD       COG       COL       COM       CPV       CRI 
## 0.5346958       NaN       NaN 0.6868508       NaN       NaN 0.6223080 
##       CUW       CYM       CYP       CZE       DEU       DJI       DMA 
##       NaN 0.5321523 0.5057274 0.5107932 0.6589134 0.6084759       NaN 
##       DNK       DOM       DZA       ECU       EGY       ESP       EST 
## 0.6438676 0.6489443       NaN 0.5813955 0.3888875 0.6375236 0.6661466 
##       ETH       FIN       FJI       FRA       GAB       GBR       GEO 
##       NaN 0.6390449 0.6077001 0.7030071 0.3824896 0.6097692 0.4960019 
##       GHA       GIN       GMB       GNB       GNQ       GRC       GRD 
##       NaN 0.3612134       NaN       NaN       NaN 0.5262623       NaN 
##       GTM       HKG       HND       HRV       HTI       HUN       IDN 
## 0.4565490 0.4792392 0.5839580 0.6647285       NaN 0.6418051 0.4500308 
##       IND       IRL       IRN       IRQ       ISL       ISR       ITA 
## 0.6488550 0.4909140 0.3272116 0.1804591 0.6711737 0.5853110 0.5835675 
##       JAM       JOR       JPN       KAZ       KEN       KGZ       KHM 
## 0.5617884 0.4745224 0.6623223 0.4959399 0.4277333 0.6330669       NaN 
##       KNA       KOR       KWT       LAO       LBN       LBR       LCA 
##       NaN 0.6145677 0.2332705 0.5026200 0.4445315       NaN       NaN 
##       LKA       LSO       LTU       LUX       LVA       MAC       MAR 
## 0.7608327 0.6866666 0.5092135 0.6515745 0.5582728 0.3955358 0.5159656 
##       MDA       MDG       MDV       MEX       MKD       MLI       MLT 
## 0.5494455       NaN       NaN 0.4835440 0.7308632       NaN 0.5447368 
##       MMR       MNE       MNG       MOZ       MRT       MSR       MUS 
##       NaN       NaN 0.3676918 0.4315540 0.5779623       NaN 0.4934789 
##       MWI       MYS       NAM       NER       NGA       NIC       NLD 
##       NaN 0.5785532 0.5978728 0.5734452 0.2963974 0.5482810 0.6872290 
##       NOR       NPL       NZL       OMN       PAK       PAN       PER 
## 0.6127051       NaN 0.5939467 0.3021316       NaN 0.4379359 0.4826420 
##       PHL       POL       PRT       PRY       PSE       QAT       ROU 
## 0.4248751 0.6330979 0.6399392 0.5041681       NaN 0.2842099 0.5122907 
##       RUS       RWA       SAU       SDN       SEN       SGP       SLE 
## 0.7224796 0.7828406 0.3118593 0.7033067 0.3849505 0.4285984 0.5374110 
##       SLV       SRB       STP       SUR       SVK       SVN       SWE 
##       NaN 0.7240927 0.7383183 0.4651578 0.5585534 0.7056502 0.5599847 
##       SWZ       SXM       SYC       SYR       TCA       TCD       TGO 
## 0.6354203       NaN       NaN       NaN       NaN 0.5009889 0.8519685 
```

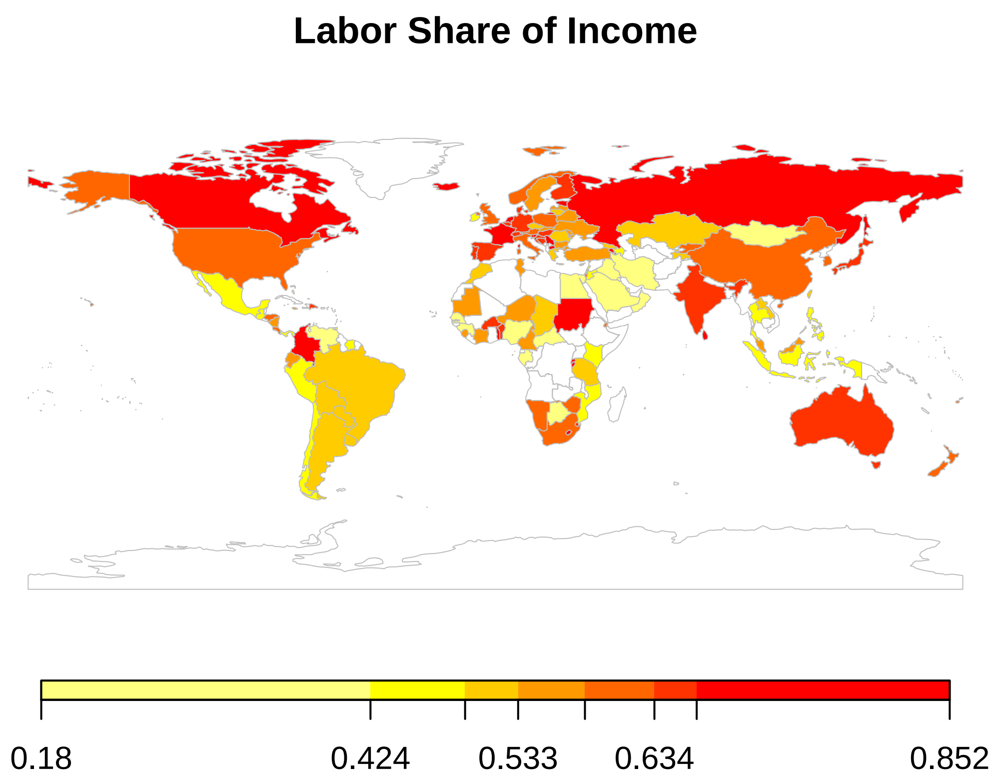
```
##       THA       TJK       TKM       TTO       TUN       TUR       TWN
## 0.4340827 0.4938746       NaN 0.4903177 0.5335571 0.5348639 0.5301800
##       TZA       UGA       UKR       URY       USA       UZB       VCT
## 0.5005295       NaN 0.5395025 0.5105828 0.6285230       NaN       NaN
##       VEN       VGB       VNM       YEM       ZAF       ZMB       ZWE
## 0.4174560 0.4003465       NaN       NaN 0.6020250       NaN 0.6241803
```

Rather than look at the numbers, lets make a map.

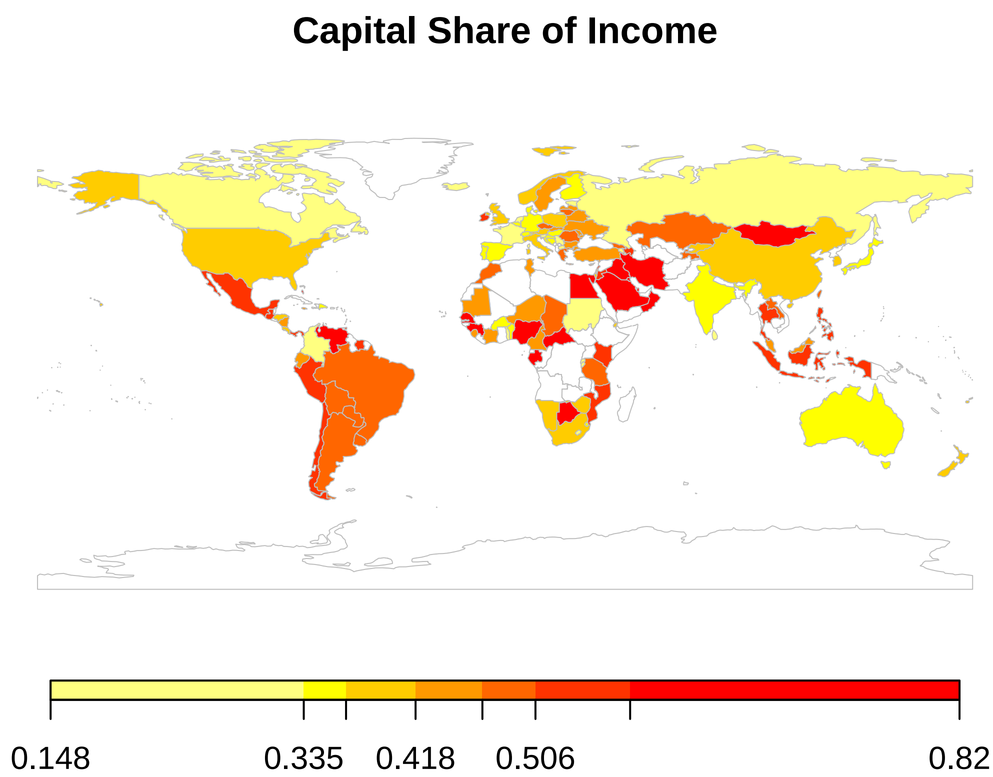
```r
# Run the line below to install on your
# machine# install.packages('rworldmap')
library(rworldmap)

labor_share_mean <- data.frame(country = names(country_labor_share_mean),
    labor_share = as.numeric(country_labor_share_mean),
    capital_share = 1 - as.numeric(country_labor_share_mean))

labor_share_mean <- na.omit(labor_share_mean)

country_map <- joinCountryData2Map(labor_share_mean,
    joinCode = "ISO3", nameJoinColumn = "country")
```

```
## 133 codes from your data successfully matched countries in the map
## 0 codes from your data failed to match with a country code in the map
## 110 codes from the map weren't represented in your data
```

```r
# ?joinCountryData2Map
map_data_labor <- mapCountryData(country_map,
    nameColumnToPlot = "labor_share", mapTitle = "Labor Share of Income",
    colourPalette = "heat", addLegend = FALSE)
# ?mapCountryData
do.call(addMapLegend, c(map_data_labor, legendLabels = "all",
    legendWidth = 0.5))
```

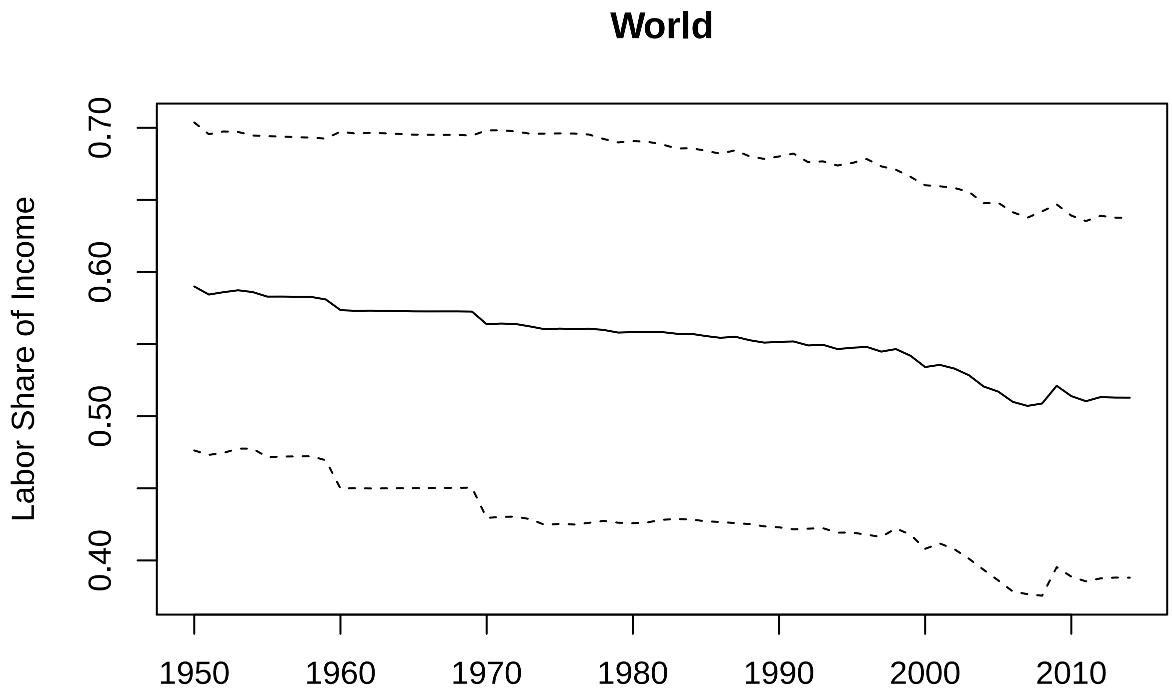



```r
map_data_capital <- mapCountryData(country_map,
    nameColumnToPlot = "capital_share", mapTitle = "Capital Share of Income",
    colourPalette = "heat", addLegend = FALSE)
do.call(addMapLegend, c(map_data_capital, legendLabels = "all",
    legendWidth = 0.5))
```



Rather than looking simply at the average, we can also take a look at how labors share of income has varied across the globe over time.

```
global_labor_share_mean <- tapply(df$labsh, df$year,
    function(x) mean(x, na.rm = T))
# global_labor_share_mean
global_labor_share_sd <- tapply(df$labsh, df$year,
    function(x) sd(x, na.rm = T))
# global_labor_share_sd
plus_one_sd <- global_labor_share_mean + global_labor_share_sd
minus_one_sd <- global_labor_share_mean - global_labor_share_sd

plot(unique(df$year), global_labor_share_mean,
    typ = "l", ylim = c(min(minus_one_sd), max(plus_one_sd)),
    xlab = "", ylab = "Labor Share of Income",
    main = "World")
lines(unique(df$year), plus_one_sd, lty = 2)
lines(unique(df$year), minus_one_sd, lty = 2)
```



Interestingly, there has been a decline in labor's share of income across the globe.

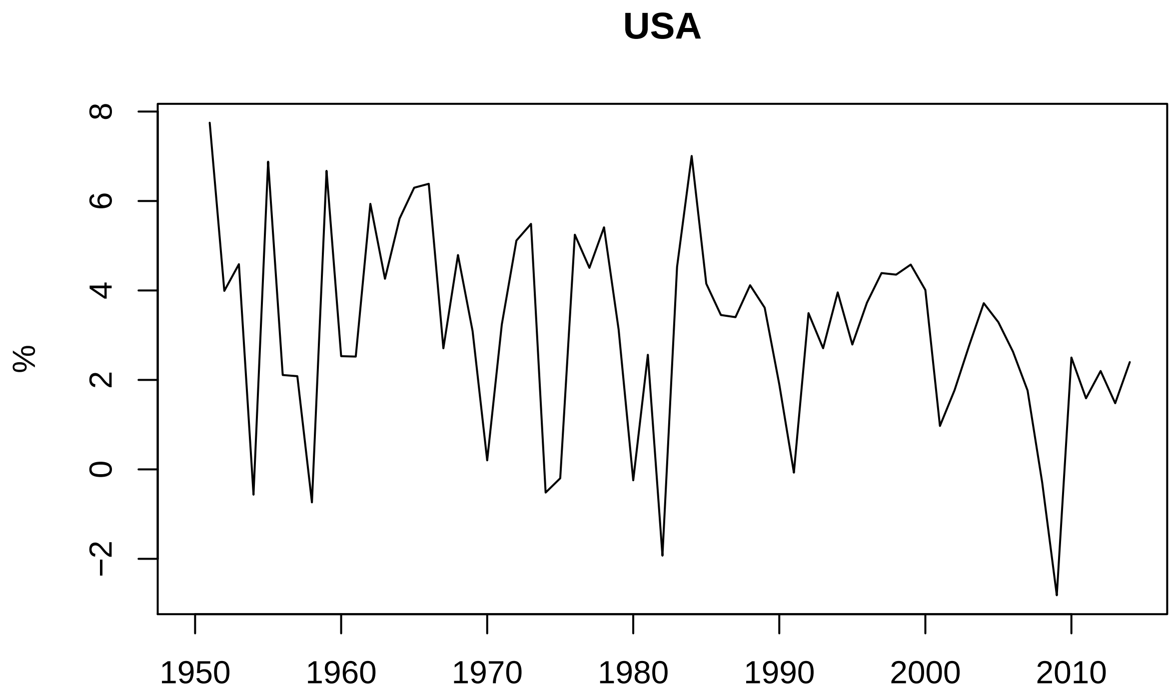
Karabarbounis and Neiman, 2014 http://www.nber.org/papers/w19136: "The stability of the labor share of income is a key foundation in macroeconomic models. We document, however, that the global labor share has significantly declined since the early 1980s, with the decline occurring within the large majority of countries and industries. We show that the decrease in the relative price of investment goods, often attributed to advances in information technology and the computer age, induced firms to shift away from labor and toward capital. The lower price of investment goods explains roughly half of the observed decline in the labor share, even when we allow for other mechanisms influencing factor shares such as increasing profits, capital-augmenting technology growth, and the changing skill composition of the labor force. We highlight the implications of this explanation for welfare and macroeconomic dynamics."

GDP Growth

Now that we have an idea of how the factors of production have evolved over time, lets take a look at the actual GDP growth.

```
# rgdpna Real GDP at constant 2011 national
# prices (in million 2011 USD).
usa_data$gdp_growth <- c(NA, diff(log(usa_data$rgdpna)))
```

```r
plot(usa_data$year, 100 * usa_data$gdp_growth,
    typ = "l", ylab = "%", xlab = "", main = "USA")
```




```r
mean(usa_data$gdp_growth, na.rm = T)  #3.1%
```

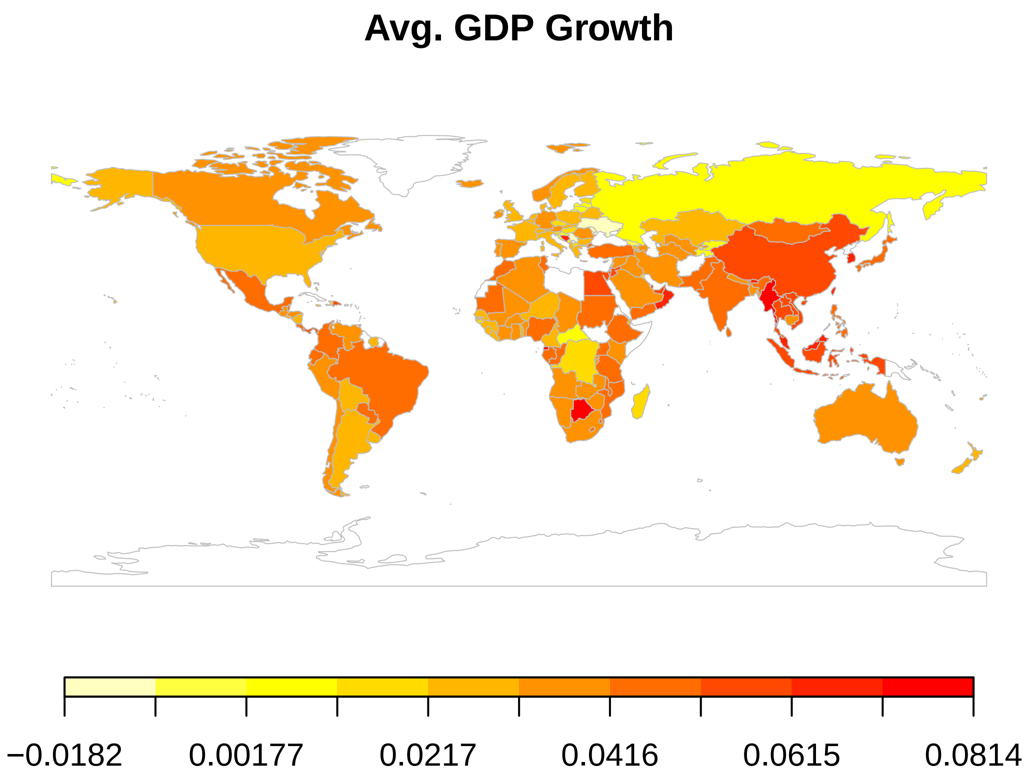
```
## [1] 0.03109286
```

As with analyzing the production factors we can see how average GDP growth varies by geography:

```r
country_gdp_mean <- tapply(df$rgdpna, df$isocode,
    function(x) mean(diff(log(x)), na.rm = T))
gdp_share_mean <- data.frame(country = names(country_gdp_mean),
    gdp_growth = as.numeric(country_gdp_mean))
gdp_share_mean <- na.omit(gdp_share_mean)

country_map <- joinCountryData2Map(gdp_share_mean,
    joinCode = "ISO3", nameJoinColumn = "country")
```

```
## 182 codes from your data successfully matched countries in the map
## 0 codes from your data failed to match with a country code in the map
## 61 codes from the map weren't represented in your data
```

```r
# ?joinCountryData2Map
map_data_gdp <- mapCountryData(country_map, nameColumnToPlot = "gdp_growth",
    mapTitle = "Avg. GDP Growth", colourPalette = "heat",
    addLegend = FALSE, numCats = 10, catMethod = "fixedWidth")
# ?mapCountryData
do.call(addMapLegend, c(map_data_gdp, legendLabels = "all",
    legendWidth = 0.5))
```



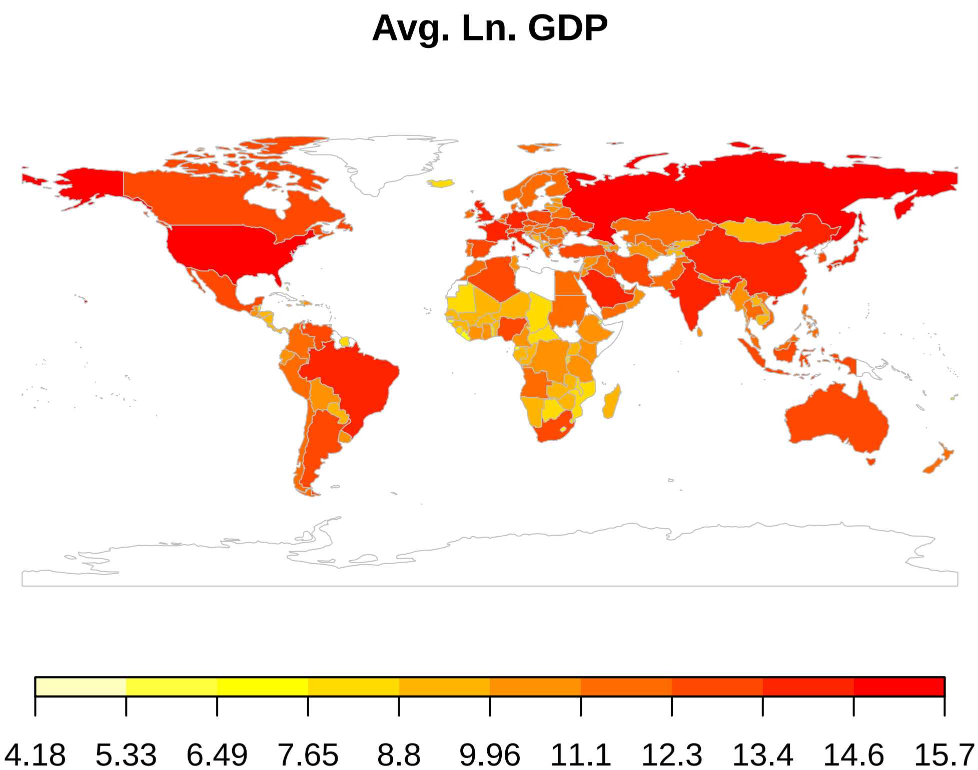
We could also generate a map for log(GDP).

```r
country_gdp_mean <- tapply(df$rgdpna, df$isocode,
    function(x) mean((log(x)), na.rm = T))
gdp_share_mean <- data.frame(country = names(country_gdp_mean),
    gdp_growth = as.numeric(country_gdp_mean))
gdp_share_mean <- na.omit(gdp_share_mean)

country_map <- joinCountryData2Map(gdp_share_mean,
    joinCode = "ISO3", nameJoinColumn = "country")
```

```
## 182 codes from your data successfully matched countries in the map
## 0 codes from your data failed to match with a country code in the map
```

```
## 61 codes from the map weren't represented in your data
```

```r
# ?joinCountryData2Map
map_data_gdp <- mapCountryData(country_map, nameColumnToPlot = "gdp_growth",
    mapTitle = "Avg. Ln. GDP", colourPalette = "heat",
    addLegend = FALSE, numCats = 10, catMethod = "fixedWidth")
# ?mapCountryData
do.call(addMapLegend, c(map_data_gdp, legendLabels = "all",
    legendWidth = 0.5))
```

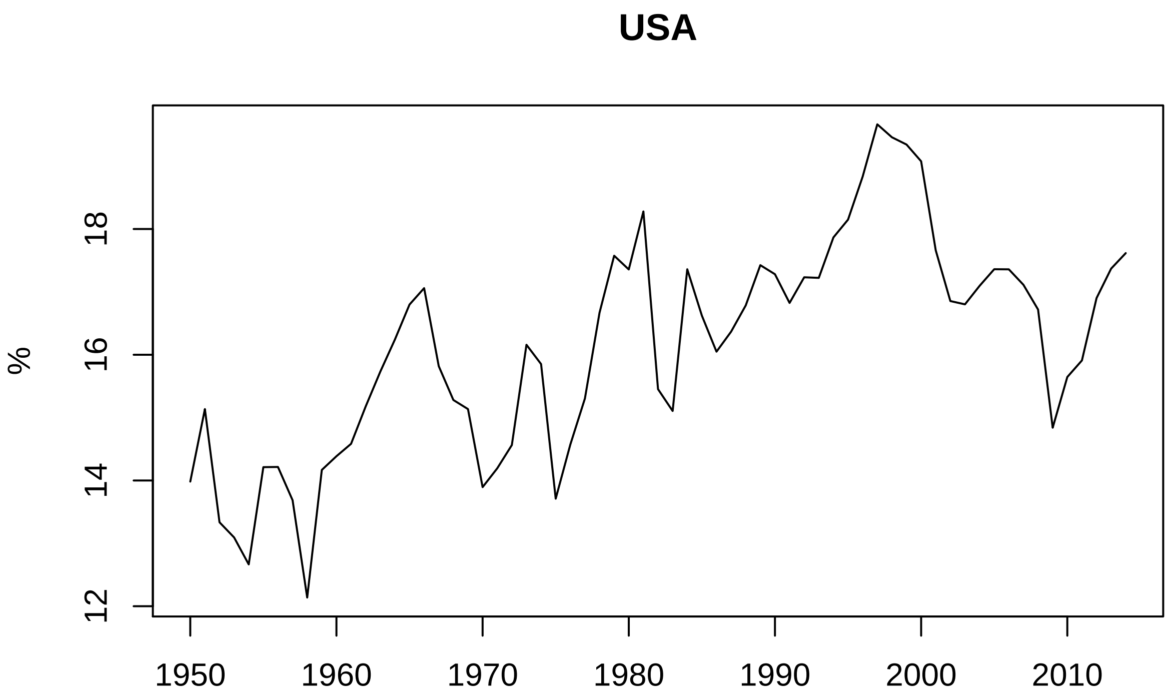


Savings Rates

Recall that savings is in essence total output less consumption. On average the savings rate in the United States has been approximately 16% since 1950 and has slowly been on the rise thanks to increases in financial innovation.

```r
# rgdpna Real GDP at constant 2011 national
# prices (in million 2011 USD). rconna Real
# consumption at constant 2011 national prices
# (in million 2011 USD). rdana Real domestic
```

```r
# absorption at constant 2011 national prices
# (in million 2011 USD).
df$Savings <- (df$rgdpna - df$rconna)/df$rgdpna
# tapply(df$Savings,df$isocode,function(x)mean(x[1:20],na.rm
# = T))
usa_data <- df[df$isocode == "USA", ]

plot(usa_data$year, 100 * usa_data$Savings, typ = "l",
    ylab = "%", xlab = "", main = "USA")
```

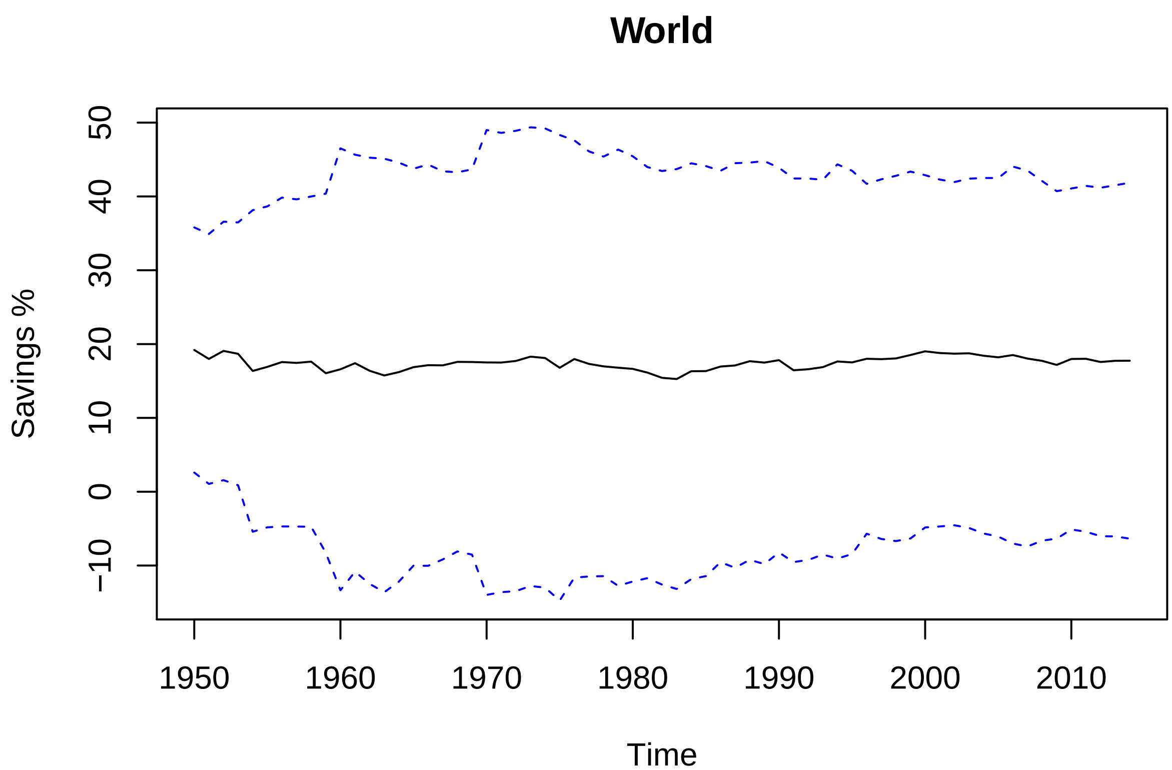



```r
mean(usa_data$Savings)
```

```
## [1] 0.1612817
```

```r
# Global average over time
global_savings_mu <- tapply(df$Savings, df$year,
    function(x) mean(x, na.rm = T))
global_savings_sd <- tapply(df$Savings, df$year,
    function(x) sd(x, na.rm = T))
minus_one_sd_savings <- global_savings_mu - global_savings_sd
plus_one_sd_savings <- global_savings_mu + global_savings_sd
```

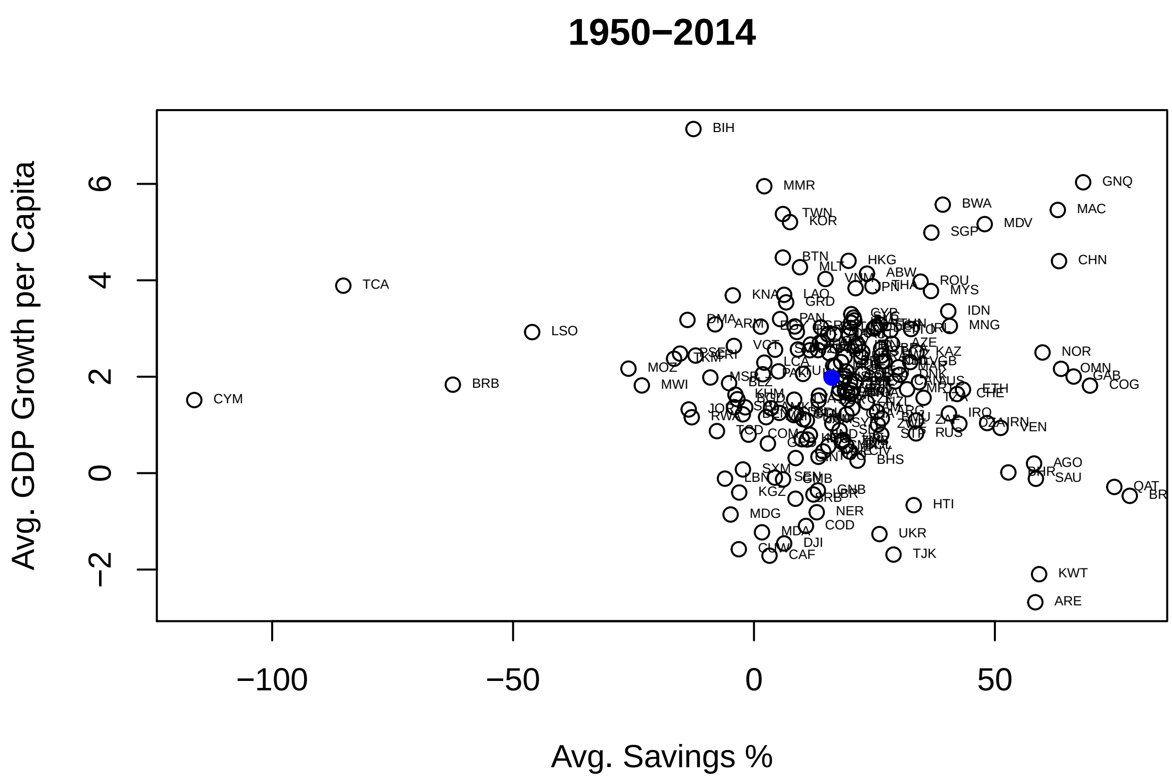
```r
plot(unique(df$year), global_savings_mu * 100,
    typ = "l", ylab = "Savings %", xlab = "Time",
    main = "World", ylim = c(100 * min(minus_one_sd_savings),
        100 * max(plus_one_sd_savings)))
lines(unique(df$year), 100 * minus_one_sd_savings,
    lty = 2, col = 4)
lines(unique(df$year), 100 * plus_one_sd_savings,
    lty = 2, col = 4)
```



Recall that savings was one of the core componenets of the Solow-Swan model. Where the higher the savings rate the higher the steady state of capital. A higher steady state of capital implied higher output (as output is a function of capital). Let's compare how savings rates and GDP growth have related to one another across our sample:

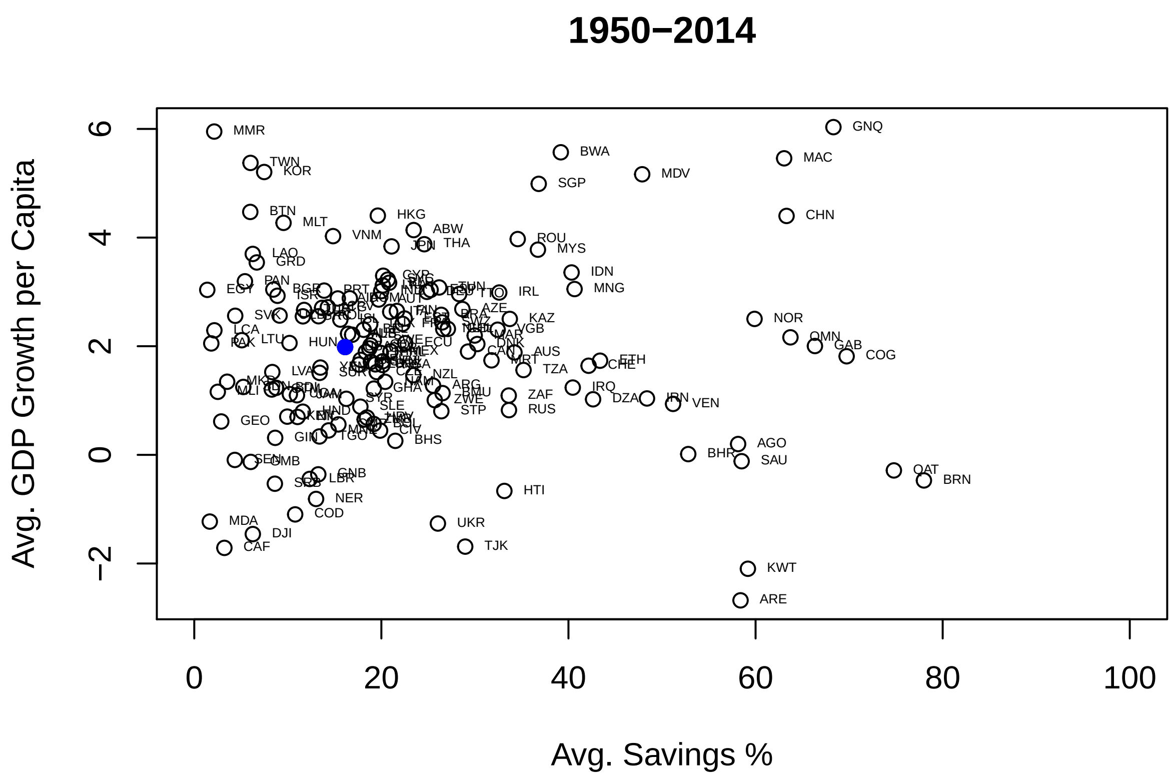
```r
global_savings_avg <- tapply(df$Savings, df$isocode,
    function(x) mean(x, na.rm = T))
global_gdp_growth_avg <- tapply(df$rgdpna/df$pop,
    df$isocode, function(x) mean(diff(log(x)),
        na.rm = T))
```

```r
plot(global_savings_avg * 100, global_gdp_growth_avg *
    100, xlab = "Avg. Savings %", ylab = "Avg. GDP Growth per Capita",
    main = "1950-2014")
text(global_savings_avg * 100, global_gdp_growth_avg *
    100, names(global_gdp_growth_avg), cex = 0.4,
    pos = 4)
points(100 * global_savings_avg[names(global_savings_avg) ==
    "USA"], 100 * global_gdp_growth_avg[names(global_savings_avg) ==
    "USA"], col = 4, pch = 19)
```




```r
# Limit to only those with positive savings..
positive_savings_countries <- names(global_savings_avg)[global_savings_avg >
    0]
global_gdp_growth_avg <- global_gdp_growth_avg[global_savings_avg >
    0]
global_savings_avg <- global_savings_avg[global_savings_avg >
    0]
```

```r
plot(global_savings_avg * 100, global_gdp_growth_avg *
    100, xlab = "Avg. Savings %", ylab = "Avg. GDP Growth per Capita",
    main = "1950-2014", xlim = c(0, 100))
text(global_savings_avg * 100, global_gdp_growth_avg *
    100, names(global_gdp_growth_avg), cex = 0.4,
    pos = 4)
points(100 * global_savings_avg[names(global_savings_avg) ==
    "USA"], 100 * global_gdp_growth_avg[names(global_savings_avg) ==
    "USA"], col = 4, pch = 19)
```




```r
mdl <- lm(global_gdp_growth_avg ~ global_savings_avg)
summary(mdl)
```

```
##
## Call:
## lm(formula = global_gdp_growth_avg ~ global_savings_avg)
##
## Residuals:
##        Min         1Q    Median         3Q        Max
## -0.045658 -0.008799  0.000658  0.009725  0.041495
```

```
##
## Coefficients:
##                      Estimate Std. Error t value Pr(>|t|)
## (Intercept)         0.0191408  0.0022713   8.427 2.68e-14 ***
## global_savings_avg -0.0004514  0.0078708  -0.057    0.954
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.01626 on 150 degrees of freedom
## Multiple R-squared:  2.193e-05,  Adjusted R-squared:  -0.006645
## F-statistic: 0.00329 on 1 and 150 DF,  p-value: 0.9543
```

```r
# What if we control for large/small economies
# prior to 1980?
summary(df$rgdpna[df$year <= 1980])
```

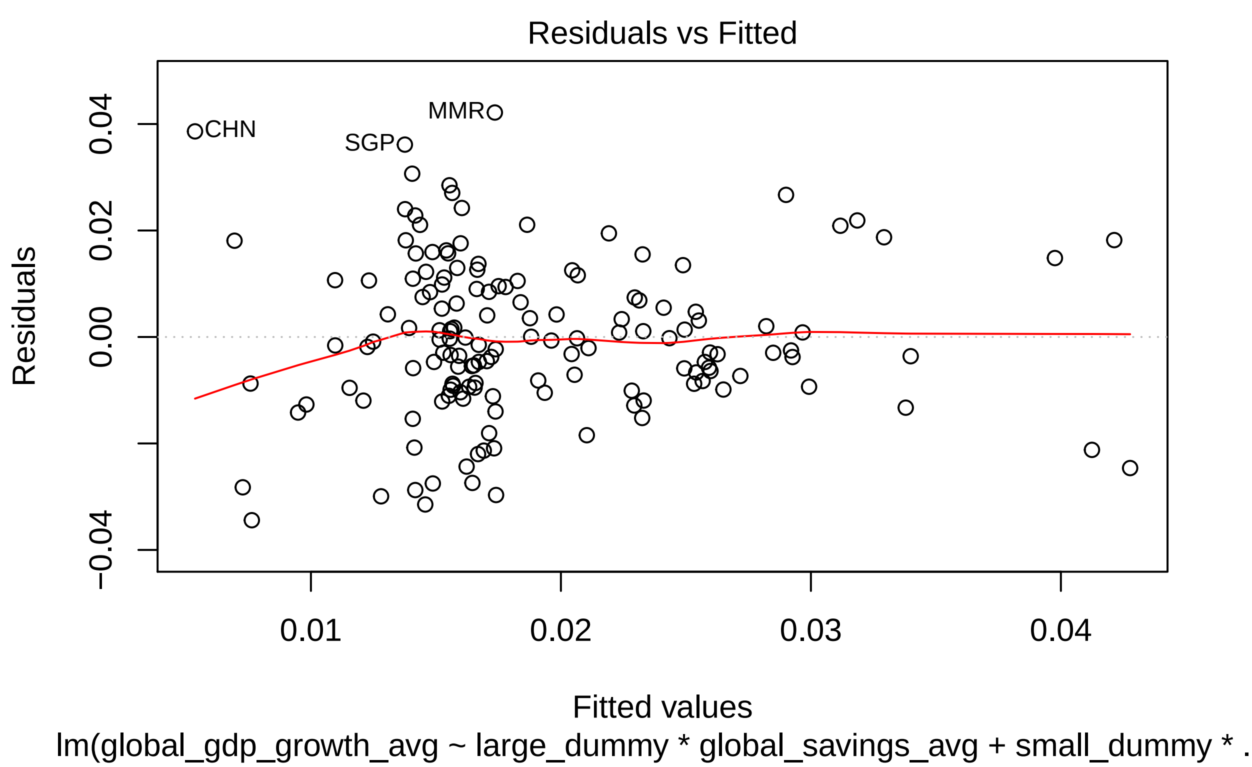
```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.    NA's
##       8    4403   19100  150400  103000 6673000    2128
```

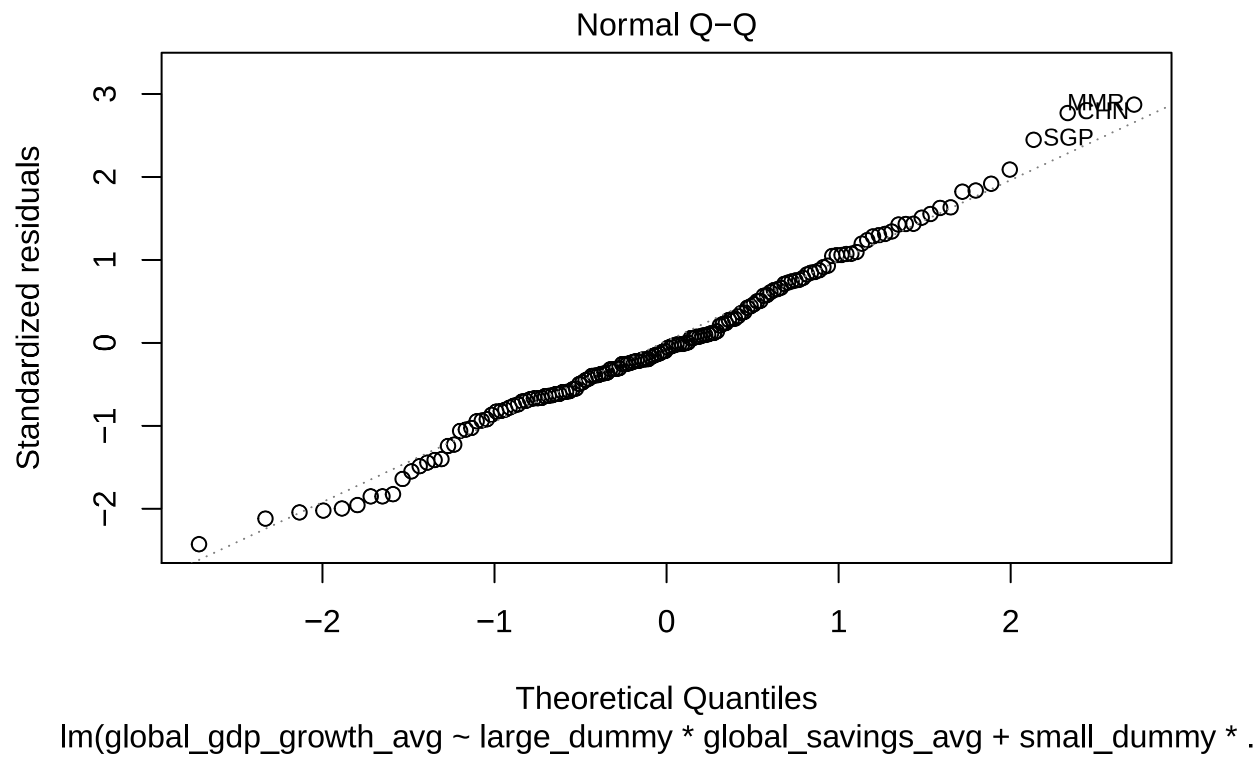
```r
large_economies <- unique(df[which(df$rgdpna >=
    103000 & df$year <= 1980), "isocode"])
small_economies <- unique(df[which(df$rgdpna <=
    4403 & df$year <= 1980), "isocode"])

large_dummy <- ifelse(names(global_gdp_growth_avg) %in%
    large_economies, 1, 0)
small_dummy <- ifelse(names(global_gdp_growth_avg) %in%
    small_economies, 1, 0)

mdl <- lm(global_gdp_growth_avg ~ large_dummy *
    global_savings_avg + small_dummy * global_savings_avg)
summary(mdl)
```

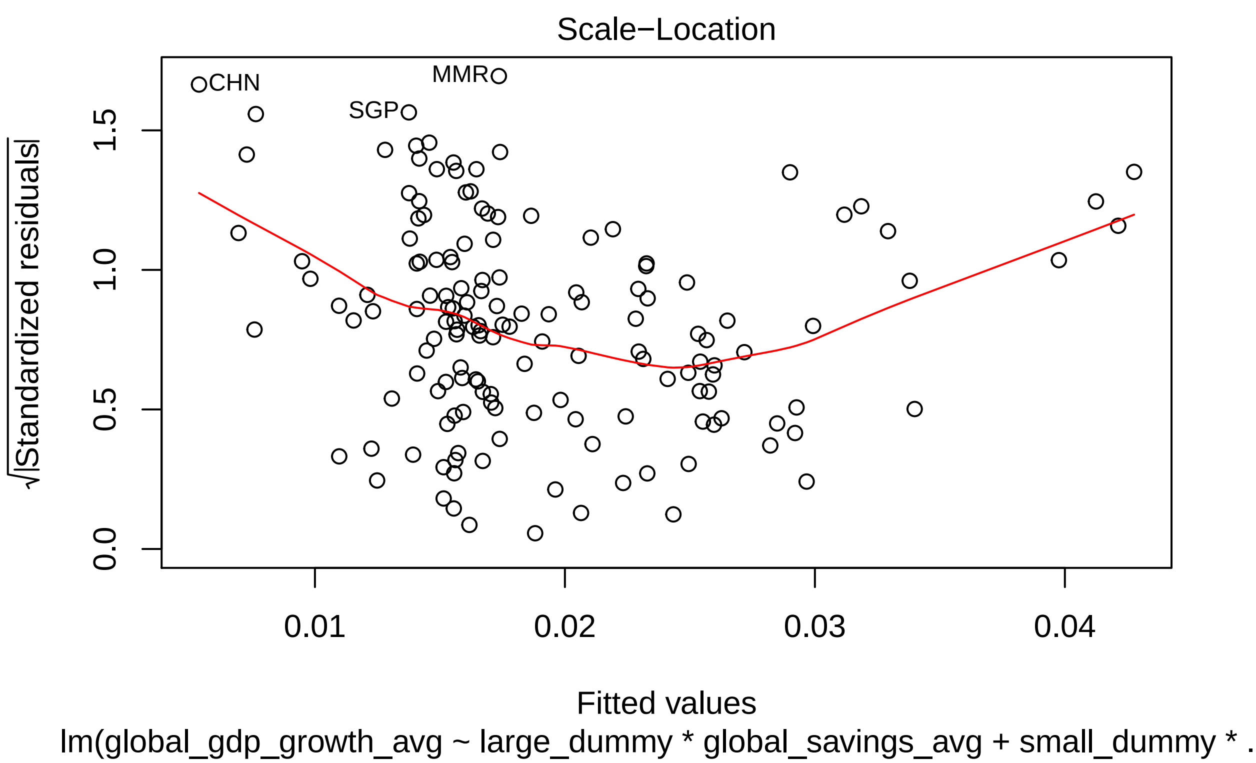
```
##
## Call:
## lm(formula = global_gdp_growth_avg ~ large_dummy * global_savings_avg +
##     small_dummy * global_savings_avg)
##
```

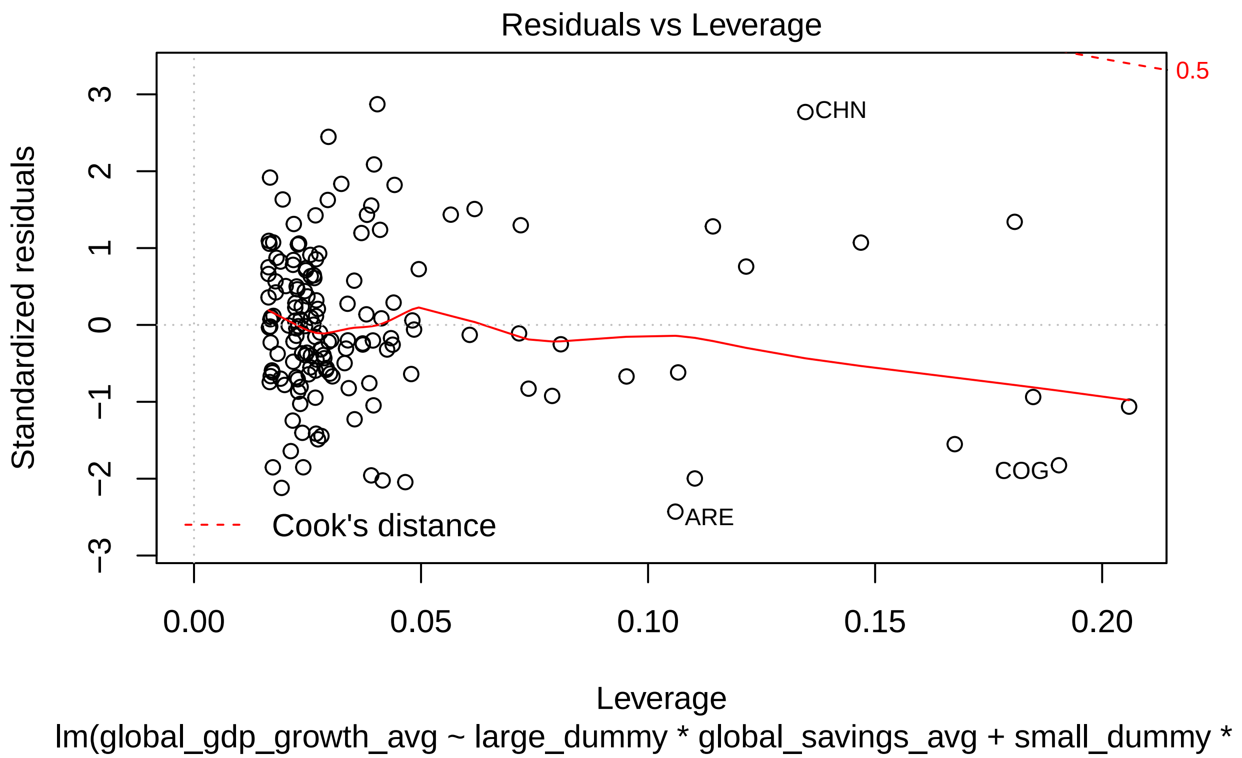
```
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.034417 -0.009344 -0.001176  0.010020  0.042155
##
## Coefficients:
##                                Estimate Std. Error t value Pr(>|t|)
## (Intercept)                    0.017578   0.003208   5.480 1.82e-07 ***
## large_dummy                    0.017056   0.005466   3.120  0.00218 **
## global_savings_avg            -0.010376   0.011661  -0.890  0.37503
## small_dummy                   -0.006224   0.004781  -1.302  0.19499
## large_dummy:global_savings_avg -0.035858   0.018200  -1.970  0.05070 .
## global_savings_avg:small_dummy  0.055426   0.017319   3.200  0.00168 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.01499 on 146 degrees of freedom
## Multiple R-squared:  0.1731, Adjusted R-squared:  0.1447
## F-statistic: 6.111 on 5 and 146 DF,  p-value: 3.603e-05
```

```r
plot(mdl)
```




Normal Q−Q
MMR
CHN
SGP
Standardized residuals
3
2
1
0
−1
−2
−2
−1
0
1
2
Theoretical Quantiles
lm(global_gdp_growth_avg ~ large_dummy * global_savings_avg + small_dummy * .


Scale−Location
CHN
MMR
SGP
√|Standardized residuals|
1.5
1.0
0.5
0.0
0.01
0.02
0.03
0.04
Fitted values
lm(global_gdp_growth_avg ~ large_dummy * global_savings_avg + small_dummy * .



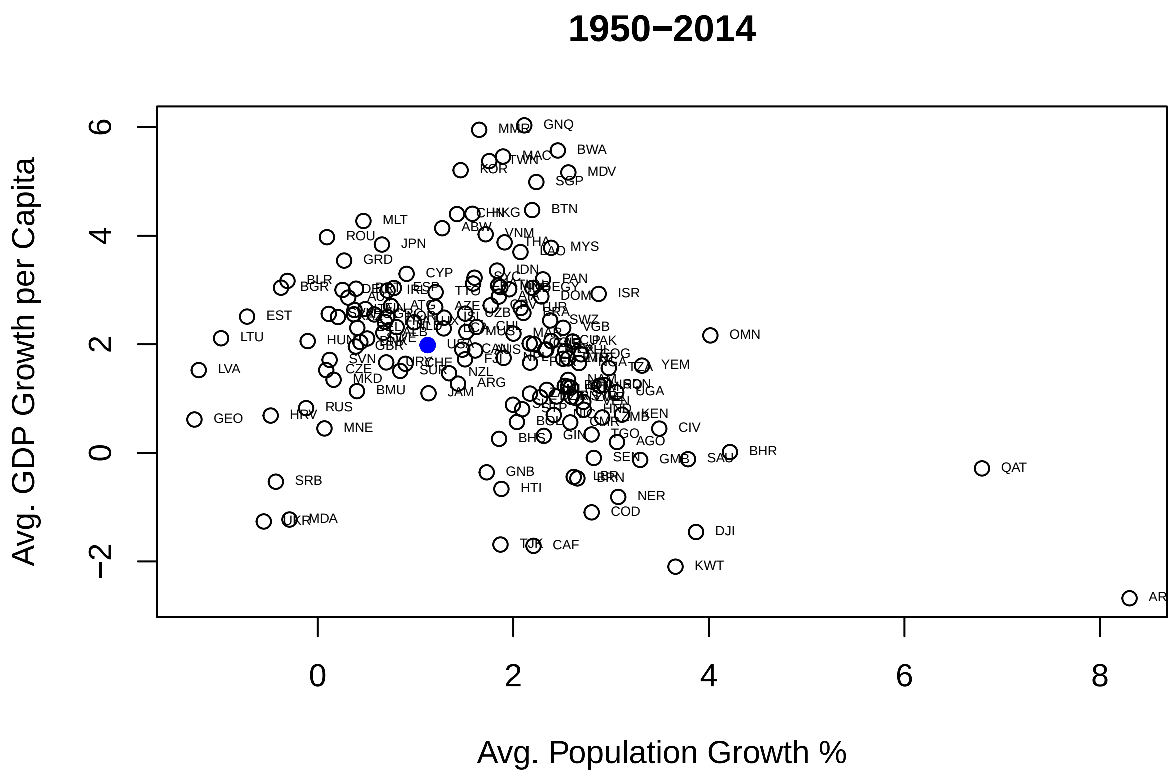
As our analysis of the Solow residual suggested, we are only explaining a small part of GDP growth through the Solow model. The above simple regressions on average GDP growth on average savings rates confirm this with the quite low adjusted-R^2.

Population Growth

Recall that in the Solow-Swan model, increases in population growth were negatively related to output per unit of effective labor. We can check this prediction by plotting average population growth against average output growth on a per capita basis to see if any patterns emerge.

```
global_pop_growth_avg <- tapply(df$pop, df$isocode,
    function(x) mean(diff(log(x)), na.rm = T))
global_pop_growth_avg <- global_pop_growth_avg[names(global_pop_growth_avg) %in%
    positive_savings_countries]  #Keep only countries with positive savings rates


plot(global_pop_growth_avg * 100, global_gdp_growth_avg *
    100, xlab = "Avg. Population Growth %", ylab = "Avg. GDP Growth per Capita",
    main = "1950-2014")
```

```r
text(global_pop_growth_avg * 100, global_gdp_growth_avg *
    100, names(global_gdp_growth_avg), cex = 0.4,
    pos = 4)
points(100 * global_pop_growth_avg[names(global_pop_growth_avg) ==
    "USA"], 100 * global_gdp_growth_avg[names(global_pop_growth_avg) ==
    "USA"], col = 4, pch = 19)
```




```r
mdl_pop <- lm(global_gdp_growth_avg ~ global_pop_growth_avg)
summary(mdl_pop)
```

```
##
## Call:
## lm(formula = global_gdp_growth_avg ~ global_pop_growth_avg)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.040013 -0.007915  0.000551  0.007788  0.042734
##
## Coefficients:
##                         Estimate Std. Error t value Pr(>|t|)
```

```
## (Intercept)             0.02535    0.00207  12.246  < 2e-16 ***
## global_pop_growth_avg -0.36701    0.09555  -3.841  0.00018 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.01551 on 150 degrees of freedom
## Multiple R-squared:  0.08955,    Adjusted R-squared:  0.08348
## F-statistic: 14.75 on 1 and 150 DF,  p-value: 0.0001802
```

```r
mdl_pop_savings <- lm(global_gdp_growth_avg ~
    global_pop_growth_avg + global_savings_avg)
summary(mdl_pop_savings)
```

```
##
## Call:
## lm(formula = global_gdp_growth_avg ~ global_pop_growth_avg +
##     global_savings_avg)
##
## Residuals:
##       Min        1Q    Median        3Q       Max
## -0.041413 -0.008176 -0.000582  0.007600  0.042549
##
## Coefficients:
##                        Estimate Std. Error t value Pr(>|t|)
## (Intercept)            0.023603   0.002420   9.752  < 2e-16 ***
## global_pop_growth_avg -0.416282   0.101734  -4.092 6.98e-05 ***
## global_savings_avg     0.011033   0.007996   1.380     0.17
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.01547 on 149 degrees of freedom
## Multiple R-squared:  0.101,  Adjusted R-squared:  0.08897
## F-statistic: 8.373 on 2 and 149 DF,  p-value: 0.0003578
```

There are plenty of more variables in the Penn World tables that I urge you to explore and create your own visualizations within (For instance, how about checking the relationship with depreciation on GDP growth?). Now let's move on to endogenous growth theory.

Endogenous Growth Theory

As mentioned before, endogenous growth theory explicitly models the dynamics of technological progress in an economy by allowing capital and labor to be allocated to a production sector (for consumption) or to a research and development sector (for production of new knowledge).

As before, we will assume a Cobb-Douglas production function with technology augmenting labor:

$$Y_t = [(1-\alpha_K)K_t]^{\alpha}[A_t(1-\alpha_L)L_t]^{1-\alpha}$$

However we now have fractions of capital α_K and fractions of labor α_L that are being allocated towards research and development, implying $1-\alpha_K$ of the capital stock and $1-\alpha_L$ of the labor stock being used for production of goods in our economy for consumption (output). This production function still exhibits constant returns to scale in capital and labor with a given technology $\rightarrow$ doubling the inputs doubles the amount that can be produced.

Previously we assumed the dynamics of A_t were exogenous to the model; following a growth rate g. Now we will explicity model the dynamics of A_t as

$$\dot{A}_t = B[\alpha_K K_t]^{\beta}[\alpha_L L_t]^{\gamma} A_t^{\theta}, \text{ where } B > 0;\ \beta \geq 0,\ \gamma \geq 0$$

B acts as a shift parameter (like g). Notice that we are not imposing constant returns to scale to capital and labor. In the case of knowledge production, exactly replicating what the existing inputs were doing would cause the same set of discoveries to be made twice, hence we could actually have diminishing returns to research and development. θ governs the effects of the existing stock of knowledge on the success of R&D. Where past discoveries may provide ideas and tools that make future discoveries easier ($\theta > 0$), or, past discoveries may have been the easy ones, making future discoveries more difficult ($\theta < 0$).

For simplicity, we will impose the following dynamics on capital and labor:

$$\dot{K}_t = sY_t$$

$$\dot{L}_t = nY_t,\ n \geq 0$$

Notice in this version of endogenous growth that when $\beta = \gamma = \alpha_K = \alpha_L = 0$ and $\theta = 1$ we are back to the Solow model (where $B = g$ and $\delta = 0$).

We start off by making further simplifications by removing capital from the model.

Endogenous Growth without Capital

A model without capital is achieved by setting $\alpha = \beta = 0$ in our production function and in our model for the dynamics of A_t. Therefore we have that

$$Y_t = A_t(1-\alpha_L)L_t$$

$$\dot{A}_t = B[\alpha_L L_t]^\gamma A_t^\theta$$

$$\dot{L}_t = nY_t,\ n \geq 0$$

Where we can get output on a per unit of labor basis by dividing by L_t:

$$\frac{Y_t}{L_t} = y_t = A_t(1-\alpha_L)$$

Which immediately implies that the growth rate of output per unit of labor relies entirely on the growth rate of A_t. Recall that the growth rate of a variable is its change over time divided by its previous value:

$$g_{A,t} = \frac{\dot{A}_t}{A_t} = B[\alpha_L L_t]^\gamma A_t^{\theta-1}$$

This tells us that the growth rate of technology initially is really a function of our model inputs. Of more interest is how the growth rate of the growth rate of A varies over time (not a typo). Taking logarithms and differentiating with respect to time:

$$\frac{\partial Ln(g_{A,t})}{\partial t} = \frac{\dot{g}_{A,t}}{g_{A,t}} = \gamma\frac{\dot{L}_t}{L_t} + (\theta-1)\frac{\dot{A}_t}{A_t}$$

$$\frac{\partial Ln(g_{A,t})}{\partial t} = \frac{\dot{g}_{A,t}}{g_{A,t}} = \gamma n + (\theta - 1)g_{A,t}$$

$$\dot{g}_{A,t} = (\gamma n + (\theta - 1)g_{A,t})g_{A,t}$$

Again, the initial values of L and A, along with the paramters of the model determine the initial values of $g_{A,t}$, now we can describe how the growth rate changes over time through $\dot{g}_{A,t}$. Of particular interest is the relationship between θ and 1.

Case 1: $\theta < 1$

When $\theta < 1$ we have the case where the contribution of additional knowledge to the production of new knowledge is not strong enough to be self-sustaining. We can solve for the growth rate at which point the growth rate stops changing ($\dot{g}_{A,t} = 0$):

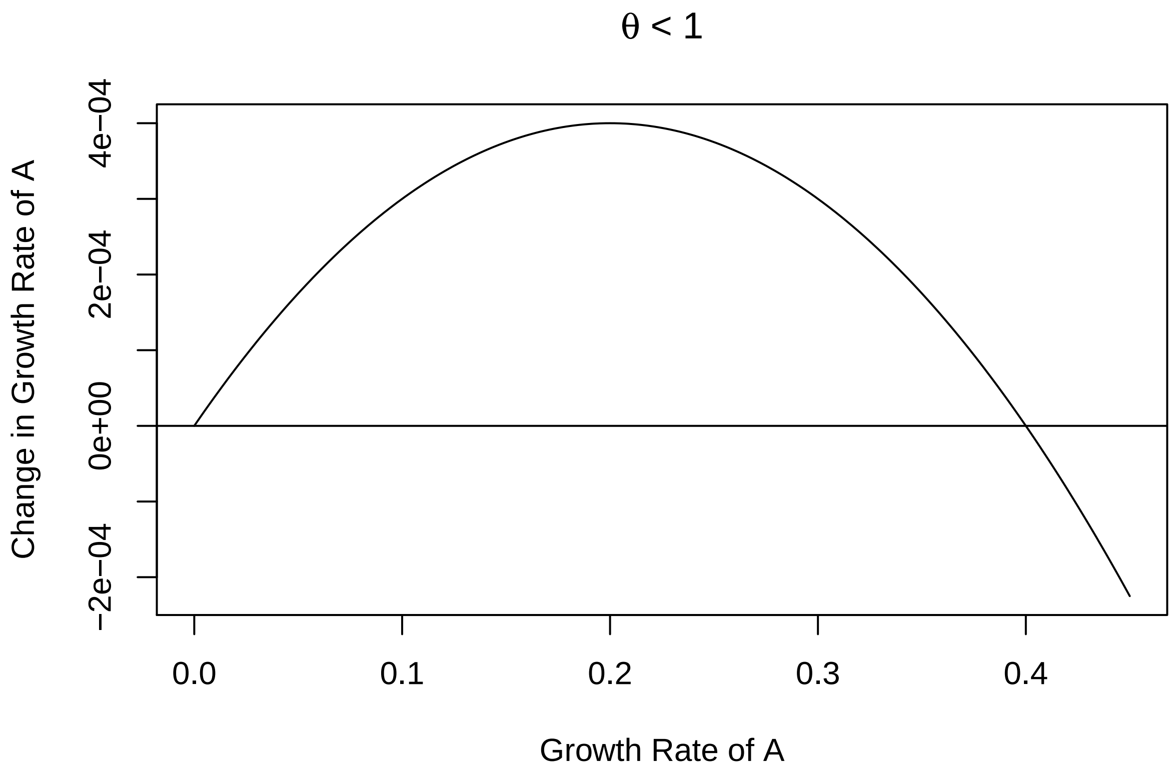
$$0 = (\gamma n + (\theta - 1)g_{A,t})g_{A,t}$$

$$g^*_{A,t} = \frac{-\gamma n}{(\theta - 1)}$$

Exactly at $g^*_{A,t}$ the growth rate is no longer changing, at which point we are on a balanced growth path.

```
A_t <- 1  #Initial technology level
L_t <- 1  #Initial population level
gamma <- 0.2  #Sensitivity to labor in R&D
alpha_l <- 0.5  #Fraction of labor working in R&D
n <- 0.02  #Population Growth Rate
theta <- 0.99  #Sensitivity to previous knowledge base
B <- 0.02  #Base R&D rate
steady_ga <- (gamma * n)/(1 - theta)  #Steady State
g_at <- seq(0, 0.45, by = 0.001)  #Vary growth rates
g_at_chng <- gamma * n * g_at + (theta - 1) *
    (g_at^2)  #See how growth rate changes
plot(g_at, g_at_chng, typ = "l", ylab = "Change in Growth Rate of A",
```

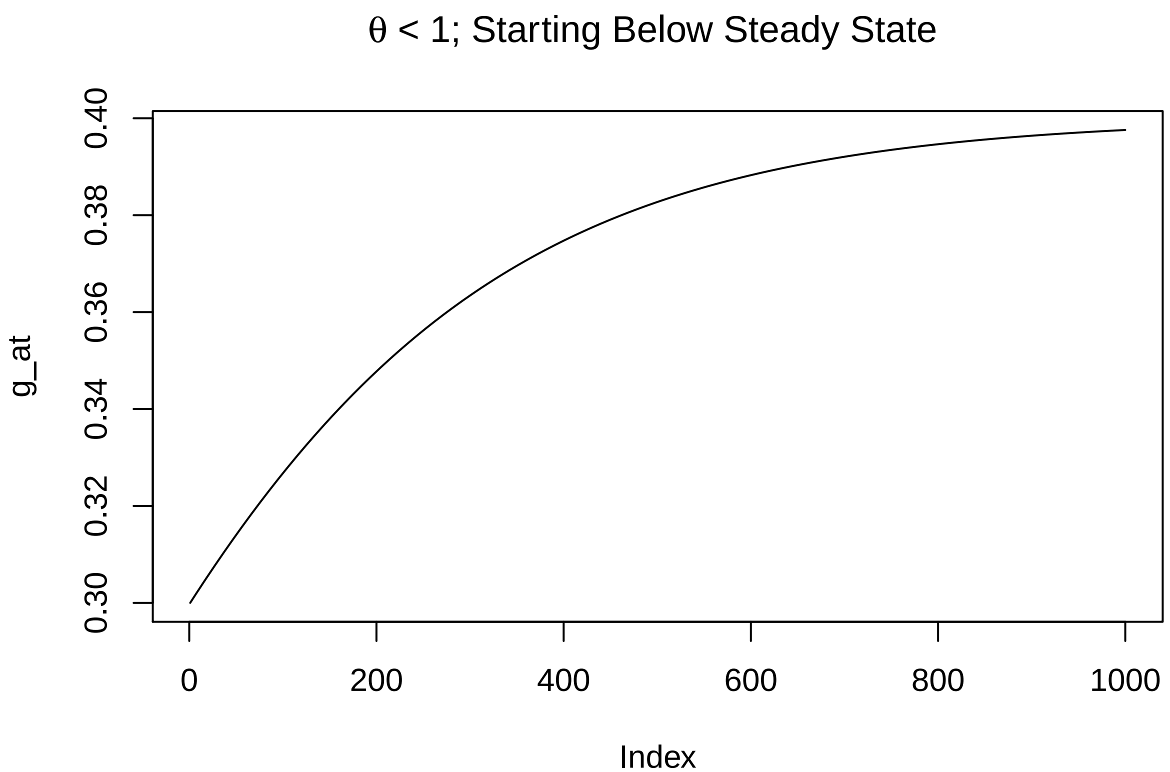
```
    xlab = "Growth Rate of A", main = expression(paste(theta,
        " < 1", sep = "")))
abline(h = 0)
```



This tells us that when we are below the steady state growth rate the growth rate will accelerate towards the steady state and once we exceed the steady state growth rate the growth rate will reverse back towards the steady state. Lets expand the code above to keep going until $g_{A,t}$ is no longer changing and see what happens when the growth rate is above or below its steady state:

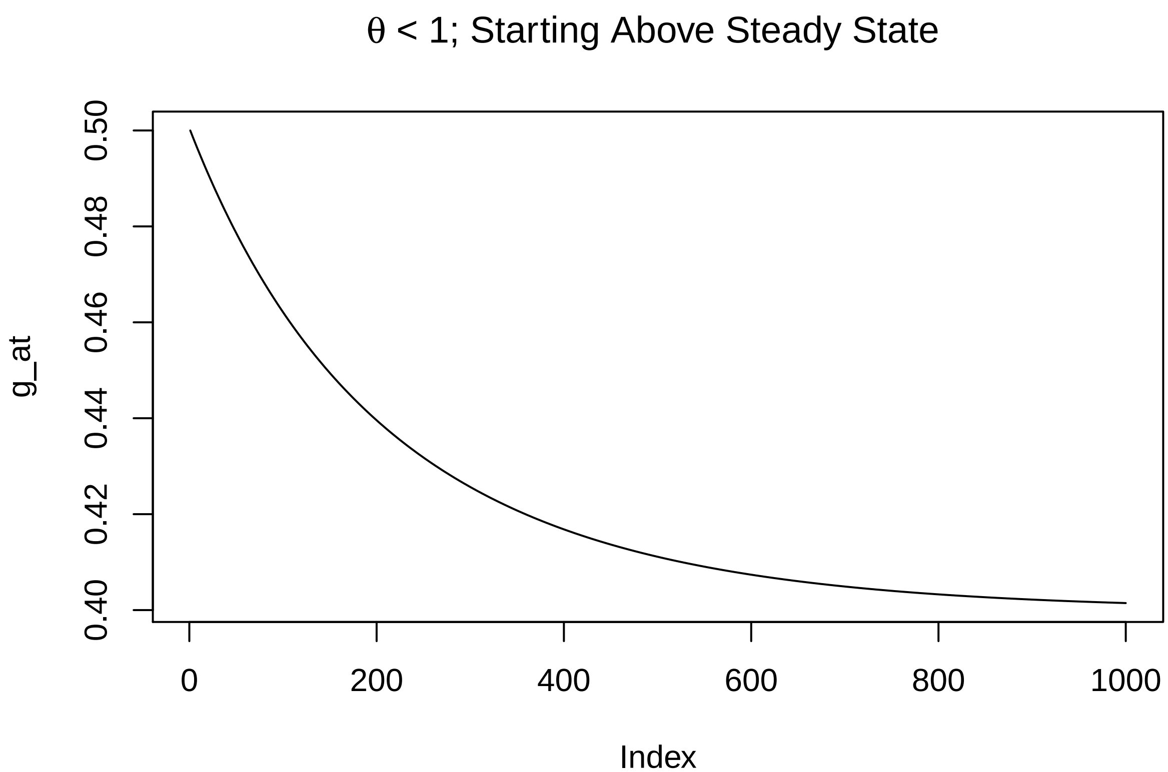
```
A_t <- 1  #Initial technology level
L_t <- 1  #Initial population level
gamma <- 0.2  #Sensitivity to labor in R&D
alpha_l <- 0.5  #Fraction of labor working in R&D
n <- 0.02  #Population Growth Rate
theta <- 0.99  #Sensitivity to previous knowledge base
B <- 0.02  #Base R&D rate
steady_ga <- (gamma * n)/(1 - theta)  #Steady State
g_at <- c(0.3)  #Initialize growth rate below steady state value
g_at_chng <- gamma * n * g_at[length(g_at)] +
    (theta - 1) * (g_at[length(g_at)]^2)  #Initialize change
```

```
itr <- 1
while (abs(g_at_chng - steady_ga) > 1e-04 & itr <
    1000) {
    g_at <- c(g_at, g_at[length(g_at)] + g_at_chng)
    g_at_chng <- gamma * n * g_at[length(g_at)] +
        (theta - 1) * (g_at[length(g_at)]^2)  #Initialize change
    itr <- itr + 1
}
plot(g_at, typ = "l", main = expression(paste(theta,
    " < 1; Starting Below Steady State", sep = " ")))
```




```
g_at <- c(0.5)  #Initialize growth rate above steady state value
g_at_chng <- gamma * n * g_at[length(g_at)] +
    (theta - 1) * (g_at[length(g_at)]^2)  #Initialize change
itr <- 1
while (abs(g_at_chng - steady_ga) > 1e-04 & itr <
    1000) {
    g_at <- c(g_at, g_at[length(g_at)] + g_at_chng)
    g_at_chng <- gamma * n * g_at[length(g_at)] +
        (theta - 1) * (g_at[length(g_at)]^2)  #Initialize change
```

```
    itr <- itr + 1
}
plot(g_at, typ = "l", main = expression(paste(theta,
    " < 1; Starting Above Steady State", sep = " ")))
```



As expected, the model converges back to its steady state value. Interestingly, this model implies that the long-run growth rate of output per worker, $g_{A,t}$, is increasing in population growth, which constrasts the Solow model and empirical data. Another interesting thing to note is that our steady state growth rate, $g^*_{A,t}$, does not rely on the fraction of the labor force working in the R&D sector, α_L! However, α_L does impact the level of A through $g_{A,t} = \frac{\dot{A}_t}{A_t} = B[\alpha_L L_t]^\gamma A_t^{\theta-1}$.

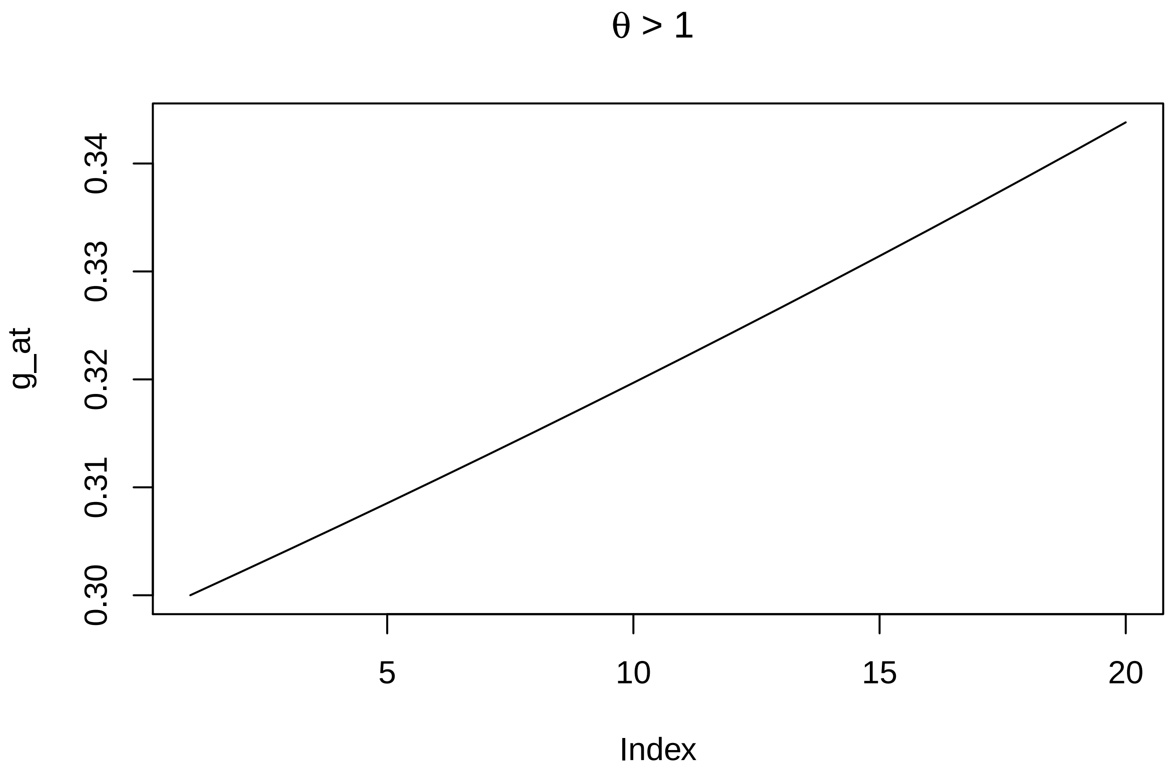
Case 2: $\theta > 1$

This case corresponds to the case where the production of new knowledge rises more than proportionally with the existing stock. From our derivation of how the growth rate evolves:

$$\dot{g}_{A,t} = (\gamma n + (\theta - 1)g_{A,t})g_{A,t}$$

The second compnent implies that $\dot{g}_{A,t}$ is positive for all possible values of $g_{A,t}$. In this case the economy exhibits ever-increasing growth rather than convergence to a balanced growth path.

```
A_t <- 1  #Initial technology level
L_t <- 1  #Initial population level
gamma <- 0.2  #Sensitivity to labor in R&D
alpha_l <- 0.5  #Fraction of labor working in R&D
n <- 0.02  #Population Growth Rate
theta <- 1.01  #Sensitivity to previous knowledge base
B <- 0.02  #Base R&D rate
g_at <- c(0.3)  #Initialize growth rate below steady state value
g_at_chng <- gamma * n * g_at[length(g_at)] +
    (theta - 1) * (g_at[length(g_at)]^2)  #Initialize change
itr <- 1
while (itr < 20) {
    g_at <- c(g_at, g_at[length(g_at)] + g_at_chng)
    g_at_chng <- gamma * n * g_at[length(g_at)] +
        (theta - 1) * (g_at[length(g_at)]^2)  #Initialize change
    itr <- itr + 1
}
plot(g_at, typ = "l", main = expression(paste(theta,
    " > 1", sep = " ")))
```



Case 3: $\theta = 1$

When $\theta = 1$ existing knowledge is just productive enough in generating new knowledge that the production of new knowledge is proportional to the stock. Our growth becomes:

$$g_{A,t} = \frac{\dot{A}_t}{A_t} = B[\alpha_L L_t]^\gamma$$

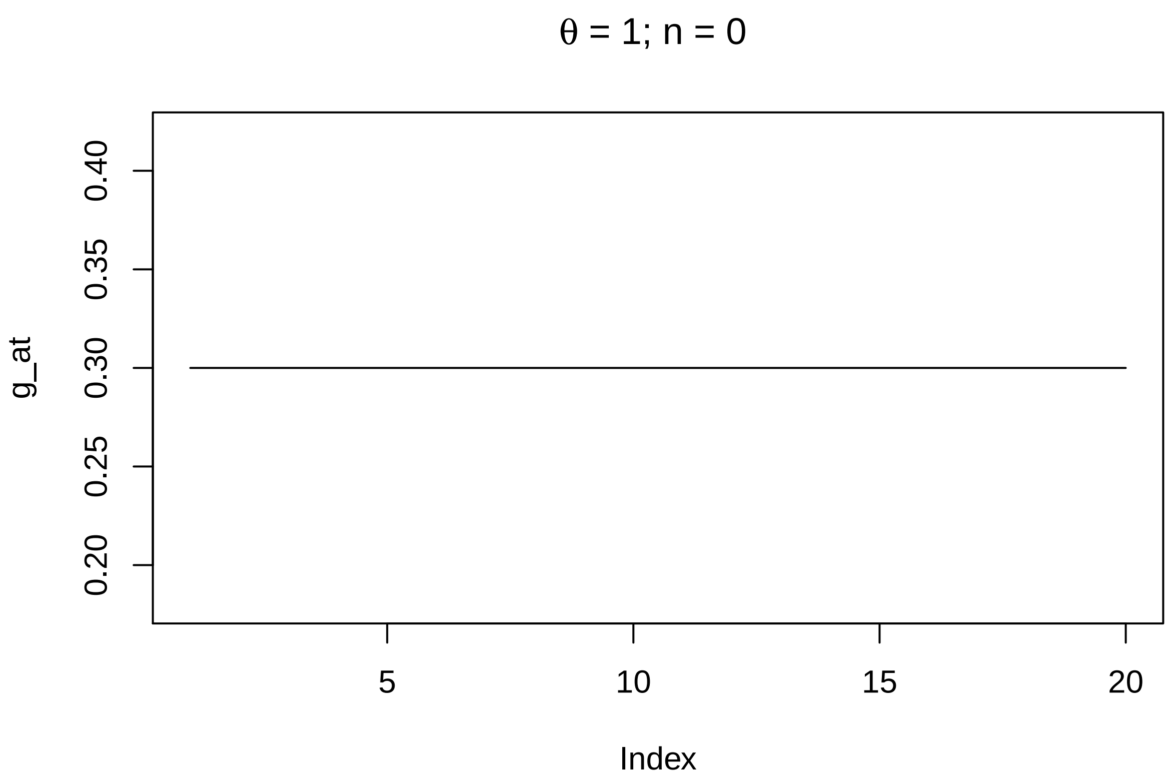
And evolves according to:

$$\dot{g}_{A,t} = \gamma n g_{A,t}$$

In this particular case, if population growth is positive, $g_{A,t}$ is perpetually growing over time (similar to the case of $\theta > 1$). If population growth is zero, we are immediately at the steady state growth rate of $B[\alpha_L L_t]^\gamma$.

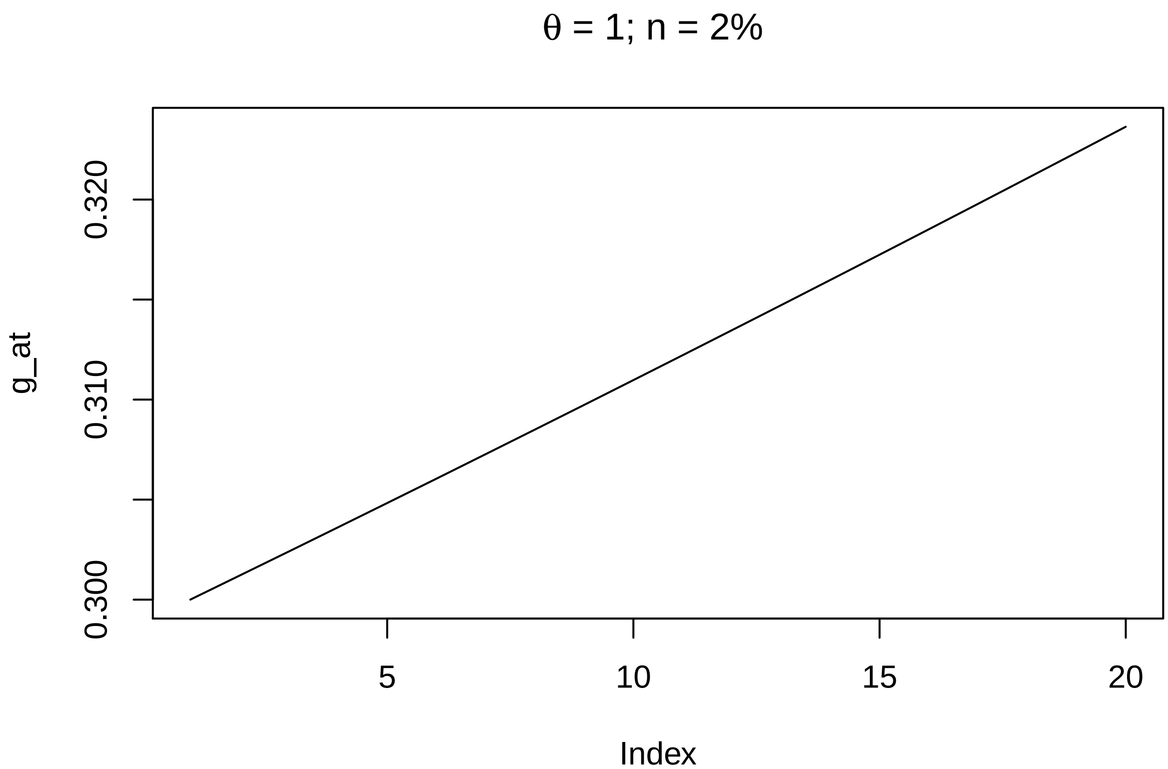
```
A_t <- 1  #Initial technology level
L_t <- 1  #Initial population level
gamma <- 0.2  #Sensitivity to labor in R&D
```

```
alpha_l <- 0.5  #Fraction of labor working in R&D
n <- 0  #Population Growth Rate
theta <- 1  #Sensitivity to previous knowledge base
B <- 0.02  #Base R&D rate
g_at <- c(0.3)  #Initialize growth rate below steady state value
g_at_chng <- gamma * n * g_at[length(g_at)] +
    (theta - 1) * (g_at[length(g_at)]^2)  #Initialize change
itr <- 1
while (itr < 20) {
    g_at <- c(g_at, g_at[length(g_at)] + g_at_chng)
    g_at_chng <- gamma * n * g_at[length(g_at)] +
        (theta - 1) * (g_at[length(g_at)]^2)  #Initialize change
    itr <- itr + 1
}
plot(g_at, typ = "l", main = expression(paste(theta,
    " = 1; n = 0", sep = " ")))
```




```
A_t <- 1  #Initial technology level
L_t <- 1  #Initial population level
gamma <- 0.2  #Sensitivity to labor in R&D
```

```r
alpha_l <- 0.5  #Fraction of labor working in R&D
n <- 0.02  #Population Growth Rate
theta <- 1  #Sensitivity to previous knowledge base
B <- 0.02  #Base R&D rate
g_at <- c(0.3)  #Initialize growth rate below steady state value
g_at_chng <- gamma * n * g_at[length(g_at)] +
    (theta - 1) * (g_at[length(g_at)]^2)  #Initialize change
itr <- 1
while (itr < 20) {
    g_at <- c(g_at, g_at[length(g_at)] + g_at_chng)
    g_at_chng <- gamma * n * g_at[length(g_at)] +
        (theta - 1) * (g_at[length(g_at)]^2)  #Initialize change
    itr <- itr + 1
}
plot(g_at, typ = "l", main = expression(paste(theta,
    " = 1; n = 2%", sep = " ")))
```



Endogenous Growth with Capital

Now lets go back and add back in capital to our model:

$$Y_t = [(1-\alpha_K)K_t]^{\alpha}[A_t(1-\alpha_L)L_t]^{1-\alpha}$$

$$\dot{A}_t = B[\alpha_K K_t]^{\beta}[\alpha_L L_t]^{\gamma}A_t^{\theta},\ \text{where } B > 0;\ \beta \geq 0,\ \gamma \geq 0$$

$$\dot{K}_t = sY_t$$

$$\dot{L}_t = nY_t,\ n \geq 0$$

Lets focus for a moment on the dynamics of capital accumulation $\dot{K}_t$:

$$\dot{K}_t = sY_t = s(1-\alpha_K)^{\alpha}(1-\alpha_L)^{1-\alpha}K_t^{\alpha}(A_tL_t)^{1-\alpha}$$

Let $c = s(1-\alpha_K)^{\alpha}(1-\alpha_L)^{1-\alpha}$

$$g_{K,t} = \frac{\dot{K}_t}{K_t} = c(\frac{A_tL_t}{K_t})^{1-\alpha}$$

Our growth rate of capital is initialized by the parameters of the model. We can now look at how the growth rate evolves over time by taking logs and differentiating with respect to time:

$$Ln(g_{K,t}) = Ln(c) + (1-\alpha)Ln(A_t) + (1-\alpha)Ln(L_t) - (1-\alpha)Ln(K_t)$$

$$\frac{\partial t Ln(g_{K,t})}{\partial t} = \frac{\dot{g}_{K,t}}{g_{K,t}} = (1-\alpha)(g_{A,t} + g_{L,t} - g_{K,t})$$

Given that $g_{K,t} > 0$ the model implies that the growth rate of capital is increasing when $g_{A,t} + n - g_{K,t} > 0$, decreasing when $g_{A,t} + n - g_{K,t} < 0$, and constant when $ g_{A,t} + n - g_{K,t}= 0$. When $g_{A,t} = 0$, we have that the growth rate of capital is growing at the rate of n. The equation implies that the growth rate in capital increases 1 for 1 with the growth rate of knowledge. When $\dot{g}_{K,t} = 0$, that is the growth rate is no longer changing we have:

$$g_{K,t} = g_{A,t} + n$$

Now lets take a look at how knowledge is evolving over time:

$$\dot{A}_t = B[\alpha_K K_t]^\beta [\alpha_L L_t]^\gamma A_t^\theta$$

$$g_{A,t} = \frac{\dot{A}_t}{A_t} = B[\alpha_K K_t]^\beta [\alpha_L L_t]^\gamma A_t^{\theta-1}$$

Our growth rate of knowledge is initialized by the parameters of the model. Let $c_A = B\alpha_K^\beta \alpha_L^\gamma$ We can now look at how the growth rate evolves over time by taking logs and differentiating with respect to time:

$$Ln(g_{A,t}) = Ln(c_A) + \beta Ln(K_t) + \gamma Ln(L_t) + (\theta - 1)Ln(A_t)$$

$$\frac{\partial Ln(g_{A,t})}{\partial t} = \frac{\dot{g}_{A,t}}{g_{A,t}} = \beta g_{K,t} + \gamma n + (\theta - 1)g_{A,t}$$

The model implies that the growth rate of knowledge is increasing when $\beta g_{K,t} + \gamma n + (\theta - 1)g_{A,t} > 0$, decreasing when $\beta g_{K,t} + \gamma n + (\theta - 1)g_{A,t} < 0$, and constant when $\beta g_{K,t} + \gamma n + (\theta - 1)g_{A,t} = 0$. When $\dot{g}_{A,t} = 0$, that is the growth rate is no longer changing we have:

$$g_{K,t} = \frac{-\gamma n}{\beta} + \frac{(1-\theta)}{\beta} g_{A,t}$$

Going back to our production function, whether there are net increasing, decreasing, or constant returns to scale to the produced factors depends on their returns to scale in the production function for knowledge. The degree of returns to scale to K and A by a factor of X, increases A by a factor of $X^{\beta+\theta}$. Therefore, in our model with capital and knowledge we need to analyze the relationship between $\beta + \theta$ and 1.

Case: $\beta + \theta < 1$

When $\beta + \theta < 1$, we have that $\frac{(1-\theta)}{\beta} > 1$ (Recall that $\beta > 0$). The point at which the growth rates of capital and knowledge are no longer changing will be our equilbrium growth rates for both variables:

$$g^*_{K,t} = g^*_{A,t} + n$$

$$\beta g^*_{K,t} = -\gamma n + (1-\theta) g^*_{A,t}$$

$$\beta (g^*_{A,t} + n) = -\gamma n + (1-\theta) g^*_{A,t}$$

$$(\beta + \gamma) n = (1 - \theta - \beta) g^*_{A,t}$$

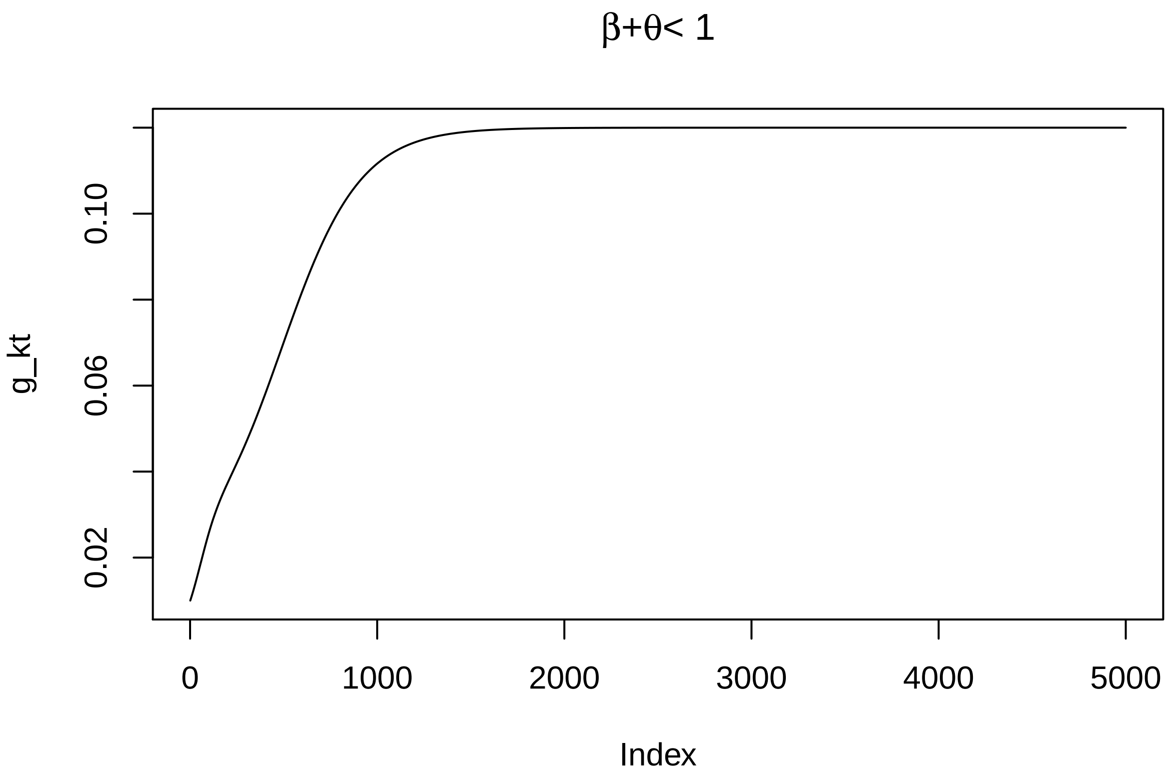
$$g^*_{A,t} = \frac{(\beta + \gamma) n}{1 - (\theta + \beta)}$$

$$g^*_{K,t} = \frac{(\beta + \gamma) n}{1 - (\theta + \beta)} + n$$

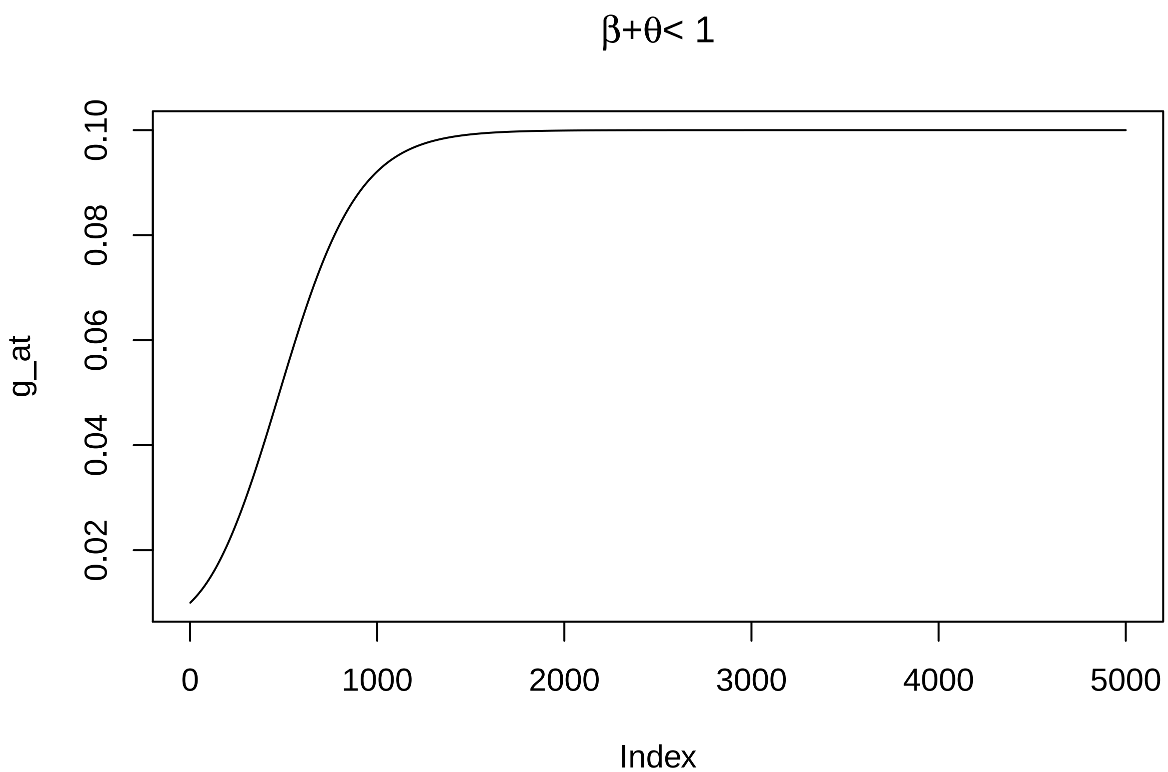
This implies that output is growing at rate $g^*_{K,t}$, when A is growing at $g^*_{A,t}$ and K is growing at $g^*_{K,t}$. Output per worker is growing at $g^*_{A,t}$. This case is similar to the case when $\theta < 1$ in the model without capital. The long-run growth rate of the economy is endogenous, and the long-run growth rate is an increasing function of population growth and is zero if population growth is zero. The fractions of the labor force and the capital stock engaged in R&D, do not affect long-run growth. The endogeneity of growth is emerging really from population growth and the parameters of knowledge production that govern the dynamics of $\dot{A}_t$.

```
K_t <- 1  #Initial capital level
A_t <- 1  #Initial technology level
L_t <- 1  #Initial population level
gamma <- 0.2  #Sensitivity to labor in R&D
alpha_l <- 0.5  #Fraction of labor deployed in R&D
```

```r
alpha_k <- 0.5  #Fraction of capital deployed in R&D
n <- 0.02  #Population Growth Rate
theta <- 0.9  #Sensitivity to previous knowledge base
beta <- 0.05  #Sensitivity to capital base
alpha <- 1/3  #Cobb-Douglas Production Function
B <- 0.02  #Base R&D rate
g_at_steady <- ((gamma + beta) * n)/(1 - (beta +
    theta))
g_kt_steady <- g_at_steady + n
g_kt <- c(0.01)
g_at <- c(0.01)

g_kt_chng <- (1 - alpha) * (g_at[length(g_at)] +
    n - g_kt[length(g_kt)]) * g_kt[length(g_kt)]
g_at_chng <- gamma * n * g_at[length(g_at)] +
    (theta - 1) * (g_at[length(g_at)]^2) + beta *
    g_kt[length(g_kt)] * g_at[length(g_at)]  #Initialize change

itr <- 1
while (itr < 5000) {
    g_kt <- c(g_kt, g_kt[length(g_kt)] + g_kt_chng)
    g_at <- c(g_at, g_at[length(g_at)] + g_at_chng)

    g_kt_chng <- (1 - alpha) * (g_at[length(g_at)] +
        n - g_kt[length(g_kt)]) * g_kt[length(g_kt)]
    g_at_chng <- gamma * n * g_at[length(g_at)] +
        (theta - 1) * (g_at[length(g_at)]^2) +
        beta * g_kt[length(g_kt)] * g_at[length(g_at)]  #Initialize change
    itr <- itr + 1
}
plot(g_kt, typ = "l", main = expression(paste(beta,
    "+", theta, "< 1", sep = "")))
```



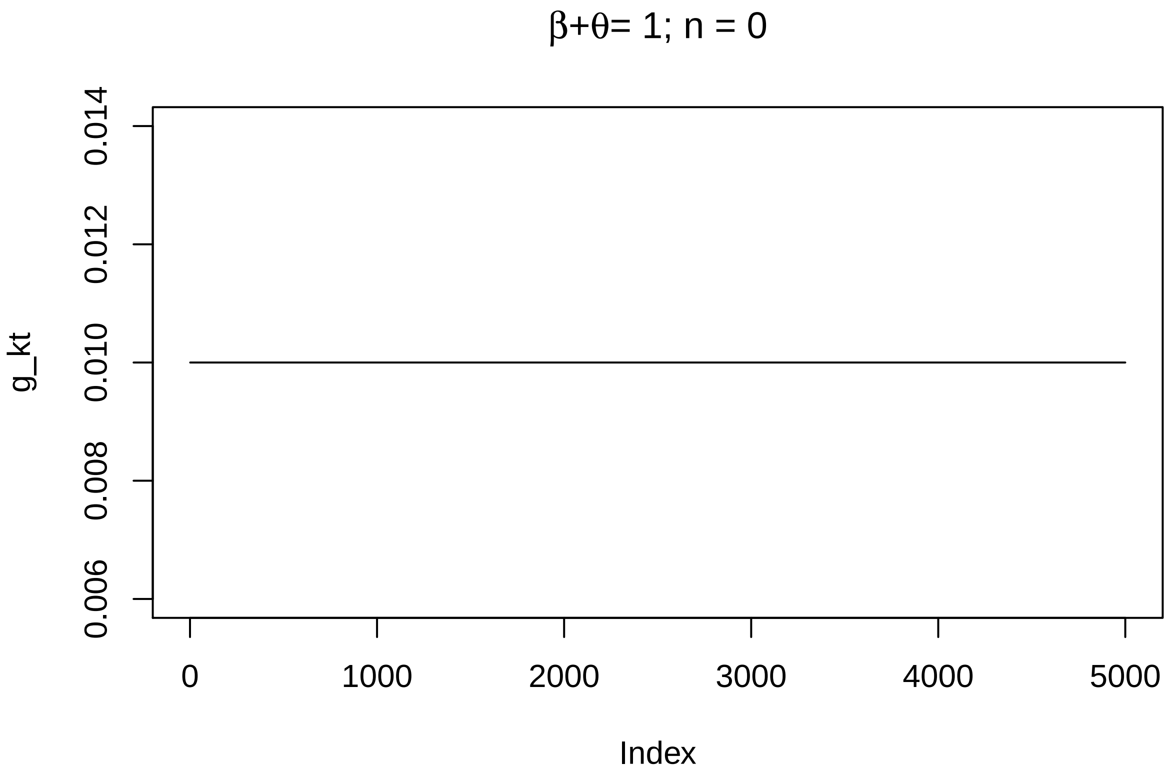

```
plot(g_at, typ = "l", main = expression(paste(beta,
    "+", theta, "< 1", sep = "")))
```



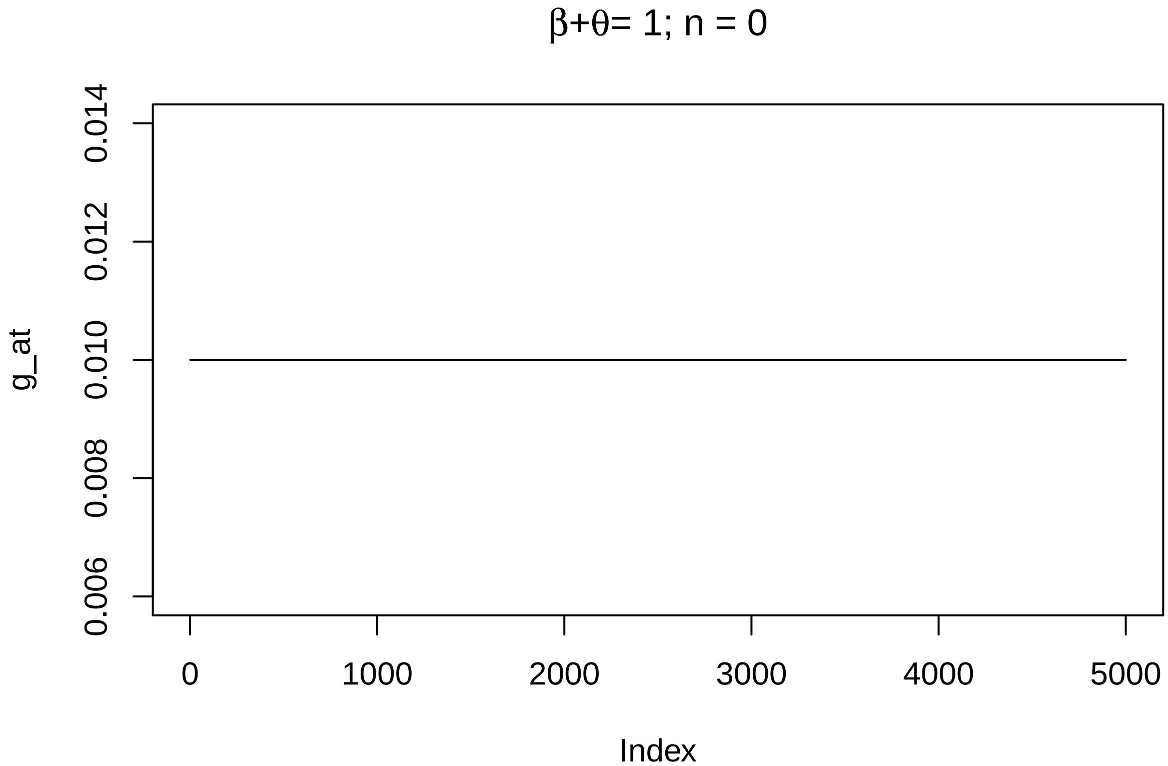
In the case where $\beta + \theta = 1$ and $n = 0$, we have that our model collapses down to simply $g_{A,t} = g_{K,t}$, which implies that the economy is immediately on a balanced growth path that is paramertized by the assumptions we make about population growth and the parameters of knowledge production.

```
K_t <- 1  #Initial capital level
A_t <- 1  #Initial technology level
L_t <- 1  #Initial population level
gamma <- 0.2  #Sensitivity to labor in R&D
alpha_l <- 0.5  #Fraction of labor deployed in R&D
alpha_k <- 0.5  #Fraction of capital deployed in R&D
n <- 0  #Population Growth Rate
theta <- 0.9  #Sensitivity to previous knowledge base
beta <- 0.1  #Sensitivity to capital base
alpha <- 1/3  #Cobb-Douglas Production Function
B <- 0.02  #Base R&D rate
g_at_steady <- ((gamma + beta) * n)/(1 - (beta +
    theta))
g_kt_steady <- g_at_steady + n
g_kt <- c(0.01)
g_at <- c(0.01)

g_kt_chng <- (1 - alpha) * (g_at[length(g_at)] +
    n - g_kt[length(g_kt)]) * g_kt[length(g_kt)]
g_at_chng <- gamma * n * g_at[length(g_at)] +
    (theta - 1) * (g_at[length(g_at)]^2) + beta *
    g_kt[length(g_kt)] * g_at[length(g_at)]  #Initialize change

itr <- 1
while (itr < 5000) {
    g_kt <- c(g_kt, g_kt[length(g_kt)] + g_kt_chng)
    g_at <- c(g_at, g_at[length(g_at)] + g_at_chng)

    g_kt_chng <- (1 - alpha) * (g_at[length(g_at)] +
        n - g_kt[length(g_kt)]) * g_kt[length(g_kt)]
    g_at_chng <- gamma * n * g_at[length(g_at)] +
        (theta - 1) * (g_at[length(g_at)]^2) +
        beta * g_kt[length(g_kt)] * g_at[length(g_at)]  #Initialize change
```

```
    itr <- itr + 1
}
plot(g_kt, typ = "l", main = expression(paste(beta,
    "+", theta, "= 1; n = 0", sep = "")))
```




```
plot(g_at, typ = "l", main = expression(paste(beta,
    "+", theta, "= 1; n = 0", sep = "")))
```


β+θ= 1; n = 0
g_at
0.006
0.008
0.010
0.012
0.014
0
1000
2000
3000
4000
5000
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