Nonsets

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"Logic is not logic" (Béziau 2010).

Abstract Our topics are set theory and the problem of the definition of a set. With the existence of non-collectivizing relations and inconsistent multiplicities, the usual definition meets some limits. But, what is a nonset? Making a difference between sets and collections could appear as a subtle concern. In fact, this does not work very well. Indeed, philosophers show a lot of examples when naive sets and collectivizing relations fail, and modern mathematicians, from Cantor and Dedekind to Aczel suspect themselves that rising objections to the uncritical use of collectivizing relations is not unreasonable. A solution of the problem may be found in a rational model (logic but not Logic, as Beziau would say) that can formalize the notion of uncomparability between objects. An example is given with the ethical relation of "absolute alterity" developed by the French philosopher Emmanuel Levinas. Then, non-transitivity, trellis and weakly associative structures can be used to formalize such a situation.

Keywords Sets · Non-collectivizing relations · Uncomparability · Ethical relations · Pseudo-ordered sets · Trellis

Mathematics Subject Classification 03E99 · 03B80 · 06D99

1 Collectivizing Relations

In usual set theories, one often think of sets as displaying the following characteristics (among others):

1. No set is a member of itself;[1]
2. Sets (unlike properties) have their extensions essentially; hence no set can exist if one of its members has not;

[1]Of course, we know that some particular theories like, for instance, Finsler's set theory (see [4]), and the revival of it in Aczel (see [1]) or in Barwise and Etchemendy (see [2]), admit circular sets. Some others even assume the existence of a "universal set" (see [16]). But it is not the case in ZF.

A. Koslow, A. Buchsbaum (eds.), *The Road to Universal Logic*, Studies in Universal Logic,
DOI 10.1007/978-3-319-10193-4_19

3. Sets form an iterated structure: at the first level, sets whose members are nonsets, at the second, sets whose members are nonsets or first level sets, etc.[2]

Cantor also inclined to think of sets as collections, i.e. things whose existence depends upon a certain sort of intellectual activity, a collecting or "thinking together". Here is Cantor's definition of sets in 1895 (see [6, p. 282]):

> By a "set" we understand any collection M into a whole of definite well-distinguished objects x of our intuition or our thought (which will be called the "elements" of M).

For doing so, Bourbaki [5, E, II, p. 3] suggests that there must exist a relation R such that the x in question are put together into the whole. This one is named a "collectivizing relation" and noted as follows: $Coll_x R$ says that "R is collectivizing in x".

2 The Problem of "Collectivizing"

One of the problems that can be raised about such a definition is that many objects or beings in the world can form a collection possibly without the result being understood as a whole. Very often, it is not the case.

Of course, a basic feature of reality is that there exist many things that we can collect. And when a multitude of given objects can be collected together, we arrive at a set.

For example, there are two tables in this room. We are ready to view them as given both separately and as a unity, and justify this by pointing to them or looking at them or thinking about them either one after the other or simultaneously.

Somehow the viewing of certain objects together suggests a loose link which seems to tie the objects together in our intuition. If the so-called objects are simple ones, there is no problem at all. If they are exactly the same, you can even add them.

But suppose these objects are very different, or very complex, or separated by some infinite gap (for instance, the gap existing between real numbers and natural numbers); or assume that the member of a set is the same that the set itself, so that the set is effectively a member of itself; or imagine again that not all the nonsets of a set can be collected together because some of them, for instance, still do not exist at the moment. What happens in all these cases? It seems obvious that you have got real nonsets, but none of them can be considered as elements of a set, whatever it can be.

3 Non-collectivizing Relations

Take the example of the Russell's set $E = \{x : x \notin x\}$. As Schoenfiels (see [27, p. 238]) writes, a closer examination of the paradox ($E \in E \iff E \notin E$) shows that it does not really contradict the intuitive notion of a set. According to this notion, a set E is formed by gathering together certain objects to form a single object, which is the set E. Thus before the set E is formed, we must have available all of the objects which are to be

[2] It is not only the case of the famous Russell's theory of types. ZF theory, as also Gödel's theory $V = L$ (see [19] or, easier to read [13]), carries on a hierarchical vision of the world.

members of E. But though these objects exist in a plain acception, in fact, there is no set such like E. Using the Bourbaki language, it means that the relation between all the objects of E is not a "collectivizing relation".

Another example of a non-collectivizing relation is what Cantor called in 1899, in a letter to Dedekind, "absolutely infinite or inconsistent multiplicities":

> A (definite) multiplicity is a system or a totality of things. But a multiplicity may be such that the assumption that all of its elements "are together" leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as a "one finished thing" [29, pp. 113–117].

One example, Cantor said, is the "totality of everything thinkable". Other ones are the system Ω of all the ordinals and the system T of all the alephs.

Nelson's non-standard analysis gave recently another example of an inconsistent multiplicity. In that theory, some objects or sets are said to be internal or external, according to the fact that they can be formalized or not within the Zermelo–Fraenkel axiomatics (ZF). A new predicate P, meaning "to be standard", is then introduced, which does not stand in ZF. Then an internal or external formula f is defined, so that the set $F = \{x : f(x)\}$ will be itself either an external or internal set. In the first case, one cannot apply the rules of ZF axiomatics. F is not a true set and f is not a collectivizing relation. For instance, the set of all non-standard integers is not a usual set of ZF. It is an external set. Of course, non-standard analysis introduces a "standardization axiom" so that, nevertheless, we may get some access to a substitute of this set. But we cannot work with the set itself.

4 Some Well-Known Theorems

We give here some examples of well-known theorems about non-collectivizing relations and their consequences.

Theorem 4.1 (Bourbaki) *Non* $\mathrm{Coll}_x(x \notin x)$.

Proof Assume $x \notin x$ is a collectivizing relation. Then there must exist a set E so that E is the set of all the x such that $x \notin x$. But now, a question: Do we have $E \in E$, or not? If $E \in E$, E is one of the x such that $x \notin x$. So $E \in E$. But if $E \in E$, E is not one of the x such that $x \notin x$. So E is not in E, and we have $E \notin E$. So we get $E \in E \iff E \notin E$, a contradiction. □

Theorem 4.2 (Cantor) $(\nexists X)(\forall x)(x \in X)$.

Proof The theorem says that there does not exist a set X whose all objects are elements. If it were the case, all relations would be collectivizing ones. But we have seen (Theorem 4.1) that $x \notin x$ is not collectivizing. □

Theorem 4.3 (Cantor) *Let* $\Omega = \omega_0, \omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_{\omega_0}, \ldots$, *the sequence of infinite ordinals. Ω is not a set, so we have*: non $\mathrm{Coll}_{\omega_\alpha}(\omega_\alpha \in \Omega)$.

Theorem 4.4 (Cantor) *Let* $T = \aleph_0, \aleph_1, \ldots, \aleph_n, \aleph_{n+1}, \ldots, \aleph_{\omega_0}, \ldots$, *the sequence of infinite cardinals. T is not a set, so we have*: non $\mathrm{Coll}_{\aleph_\alpha}(\aleph_\alpha \in T)$.

The problem of inconsistent multiplicities is that they are, in fact, contradictory entities. So we cannot use them to prove anything, as Cantor quickly realized. In fact, we cannot use them at all.

Nevertheless, a French philosopher Quentin Meillassoux [23] tried to make use of them in philosophy of science: Thinking that experience is pure contingency, he assumes that it is an inconsistent multiplicity. As a consequence, we cannot apply there the concepts of Probability Calculus because experience—or, in other words, the multiplicity of facts—*is not a set*! So we cannot assure the existence of invariants or prove the stability of scientific laws by some inductive reasoning (that probability, otherwise, could have justified).

But solving the epistemological problem of scientific laws at this cost is too much expansive. And there is no convincing evidence that experience, even if it is taken through the course of time, might be identified with a so problematic entity.

5 Collectivities, Sets, Collections

As far as I know, most of "working mathematicians" would agree with me that taking non-collectivizing relations as a basis for a theory is not a solution to set-theoretic problems.

A lot of them, in front of the question raised above, will rather make some difference between *collectivities* and *sets*. In this case, they will give to collectivities, not to sets, the best part, that is, the property of being built from true relations (see [22]).

Indeed, we can observe collectivities in the not living world (universe galaxies, solar systems, crystalline units) as in the living world (ant hills, bee swarms, nations). What properties are behind the relations who tie the collectivities? Maybe is the gravity, the symmetry or the survival instinct? In a word, it is the structural self-organization. The self-organization can be structural and functional. The paper [22] refers to the structural self-organization applied to the interconnected collectivities.

First, let us define the collectivity. Therefore, we must answer another question: What is a set? A set can be selected by a membership, or by a relation which substantiates the membership, or by bringing in the set field elements which fulfill the relation.

Because Bourbaki names "collectivizing relation" the relation defining a set, the authors name "collectivities" only the sets selected or built up by the help of the relations. Therefore, they exclude the sets selected by the membership, the most general. A collectivity means not a set made, for example, of a star, a planet, a crystal, an ant, a bee and a man. The relation that substantiates the membership of a collectivity is connected with its structure: a collectivity is made of the least structural entities, e.g. an interconnection means nodes and links, equivalent to the graph definition.

As it is said, an assemble grouping very different things such a star, a planet, a crystal, an ant, a bee and a man, etc. can still be viewed as a "set", but it is certainly not a "collectivity" because no clear relations exist between these elements. Then, to be the member of a set is not sufficient to make a collectivity. So, for such authors, it is clear that a set is not a collectivity, even if forming a set supposes the existence of a "collectivizing" relation R in the sense of Bourbaki.

So we must strongly differentiate *collectivities* and *collections*. A set is not necessarily a collectivity because the whole that their members belong to has not necessarily a well-define structure. But a set is an actual collection, that is, a collection of objects that we can enumerate, if it is finite, and to which corresponds some aleph, if it is infinite.

Now a collection cannot be necessarily a set, and the problem is that we do not know, for the moment, how to study collections that are not sets, i.e. those entities that we may call, using a negative and very imprecise term, “nonsets”.

Making a difference between sets and collections could appear as a subtle concern. But we must remember that Cantor’s (or Dedekind’s) view was that a set is in fact a “system”—in the sense of Kant, i.e. a platonistic idea:

> Par un “ensemble” ou “système”, j’entends en effet de façon générale toute multiplicité qui peut être pensée comme une unité, c’est-à-dire toute collection d’éléments déterminés qui peut être par une loi combinée en un tout : je crois définir ainsi quelque chose d’apparenté à l’“image” ou l’“idée” platonicienne.

6 Philosophy

In the past, for the German philosopher I. Kant, some problems already raised with the use of concepts like God, World or Soul (see [20]), which mean strictly nothing from a cognitive point of view. The main argument of Kant was that we have no intuitions (in particular, no sense data) of these objects (which are not, moreover, space–time entities). Besides, as knowledge was, for him, a kind of “synthesis” between concepts and intuitions, we cannot get positive ideas about such things. God and Soul are made of an invisible stuff and so, stay beyond our perception. But even the World itself, in what we are, cannot be viewed as a whole: we always see *a part of* the world, but not *the whole* world because, for seeing such a big entity, we should have to get out of it and move, for instance, in the direction of some extra-world, from which it should be possible to see this one as a whole. So our World cannot be identified with the “set of all things”. In the same way, our soul cannot be the “set of all the ideas of our mind”. As Dedekind had already shown—and Cantor says nothing different—the system of all the ideas is an inconsistent multiplicity.[3]

In modern philosophy, God himself may be seen as the “Absolute Other”, that is, not only a transcendent point that could be located at a finite distance in a geometric space and that we could reach by a combination of finite algebra operations with homogeneous coordinates, as we can see in projective geometry over finite fields (see [15, 18]), but an absolutely infinite entity which does not enter any set operations. If the relation to God becomes a model for the relations between men, as in Levinas’ philosophy, then, even two men cannot be put together into a set (see [21, p. 9]). So, the axiom of pairing does not work in affective or social relations, that always are relations between “absolute others”. And no more the axiom of arithmetic, which precisely allows us to make collections. Of course, adding certain elements to others does obey to some rule fixing the extent of that possibility: we only add entities of the same stuff. But the question is: Is there, in the world, any entity of the same stuff? Of course, there is no problem if you add balls, books, cats, etc. But can you add two human beings, especially when they have complex relationships? Everybody may understand that a couple is not a pair, even when its members have the same sex.

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[3] A consequence is that the notion of an “infinite mode” (of thought), in Spinoza’s Ethics, is certainly a contradiction.

Table 1 Profile of merit

Wars	2
Legs	1
Wives	2
Children	4
Wounds	2
Total	11

In front of these problems, some people feel inclined to get an extension of the addition law. Philip J. Davis and Reuben Hersh [11, p. 71] recall the story of a beggar in Time Square, who pined on himself the following information, proving that the addition rule is not so well known—or perhaps must we say it might be transgressed if we decided to define a profile of merit based on some new computation of troubles and offenses (see Table 1).

For sure, this apologue will make every mathematician laugh. And I have been laughing as well for a long time.

But, one day, I came across this famous letter from Cantor to Dedekind (Hahnenklee 1899, August 28th) [7, p. 447–448] where Cantor asks about the consistency of finite and infinite well-orderings. And suddenly, I stop laughing because Cantor wrote the worrying following lines, where he said that even finite multiplicities cannot be proved to be consistent. The fact of their consistency is a simple unprovable truth—“the axiom of arithmetic (in the old sense)”; the fact of the consistency of infinite multiplicities, i.e. multiplicities that have an aleph as their cardinal number, is in exactly the same way an axiom, “the axiom of the extended transfinite arithmetic”:

> One must throw up the question, where from I know that the well ordered multiplicities or sequences to which I ascribe the cardinal numbers,
>
> $$\aleph_0, \aleph_1, \ldots, \aleph_{\omega_0}, \ldots, \aleph_{\omega-1}, \ldots,$$
>
> are really “sets” in the sense of the word explained above, i.e. “consistent multiplicities”. Could not we conceive that already *those* multiplicities be “inconsistent” ones, but that the contradiction which there is to suppose a “simultaneous existence (Zusammensein) of all their elements” does not get noticed yet? My answer is that this question *can be also widened to the finite multiplicities*, and that, if we think about it exactly, we end then in the result: even for finite multiplicities, *there is no* proof of their “consistency”. In other words: the fact of the “consistency” of the finite multiplicities is a simple, unprovable truth, it is the “*axiom* of the arithmetic” (in the old sense of the word). And, also, the “consistency” of the multiplicities to which I attribute the alephs as cardinal numbers, is “the axiom of the transfinite widened arithmetic”.

Suppose now a person who is unreasonable enough[4] for refusing, at less in some cases, the very pragmatic (though completely unprovable) “axiom of arithmetic (in the old sense)”. That is exactly the position of the French philosopher Emmanuel Levinas. As Levinas says, one cannot add the other to me, there is no number to be associated with such a relation, which is a non-symmetric relation, and, in fact, an *ethical* one.

[4]If you wonder there can exist some, you may think of an imaginary animal like the tortoise of Lewis Carroll (see [8]), which never admitted a quite simple deduction relation before it has been set up into a group of axioms.

7 The Myth of "Naive Sets"

In front of these facts, we are led to a tricky question: Are collectivizing relations an error? But where do they come from? Indeed, raising such a problem is asking in fact about the origin of logic and rational thought.

For what reason, at a particular date in the history of man, some members of our species began to need gathering things and putting them into the same whole cluster, constituting what we call now "naive sets"?

The least we can say is that there is no evidence that the first grouping or classification[5] that has been constructed by men all around the World is particularly clear and undisputable. For example, Nietzsche thought it was the result of some handicap or infirmity:

> *Origin of the logical.*—How did logic come into existence in man's head? Certainly out of illogic, whose realm originally must have been immense. Innumerable beings who made inferences in a way different from ours perished: for all that, their ways might have been truer! Those, for example, who did not know how to find often enough what is "equal" as regards both nourishment and hostile animals, who subsumed things too slowly and cautiously, were favored with a lesser probability of survival than those who guessed immediately upon encountering similar instances that they must be equal. The dominant tendency, however, to treat as *equal* what is merely *similar*, an illogical tendency—for nothing is really equal—is what first created any basis for logic. In order that the concept of substance could originate—which is indispensable for logic although in the strictest sense nothing real corresponds to it—it was likewise necessary that for a long time one did not see nor perceive the changes in things; the beings that did not see so precisely had an advantage over those that saw everything "in flux" [24, § 111].

As Nietzsche says, men who did not see very well, and so, could not distinguish small details and changes in their perceptions of objects or other entities of the world, were led to identify most of them and came to consider that they belong to the same invariant class, the same *species*. Here is the birth of early classifications, but also the beginning of logic, because one of the most famous philosophical class invariant is the concept of "substance", which is also one of the basic concept of the Aristotelian ontology (to which corresponds the concept of "subject" in logic). So, with an uncomparable sense of humor, Nietzsche tells us that logic and classifications could have been based on a kind of shortsightedness, which led to identify—in modern words—*equivalence relations* (reflexive, symmetric and transitive) with similarity relations (only reflexive and symmetric, but not transitive). But it means, in fact, that the main categories of our thought could be based on a prime error, and that putting together things of our environment is not a trivial operation. In fact, there are no "naive" sets.

8 The Modern View

Let us return now to Cantor's letter to Dedekind (Hahnenklee 1899, August 28th).

In his famous book about the cantorian construction of set theory, the French philosopher Jean Cavaillès commented the quotation of Cantor concerning the lack of a true evidence for consistent and inconsistent multiplicities, and the existence of implicit (and

[5] For an overview on classifications problems, see [26].

quite unprecise) axioms, respectively named the "axiom of arithmetic" and the axiom of transfinite arithmetic, in the following way:

> To these axioms "in the old sense" whose no evidence is a guarantee, the successors of Cantor will try, for maintaining the essentials of the abstract theory, to substitute, as far as it is possible, the consistency of a modern axiomatics. As a beginning for their endeavors, were naturally introduced the systematic construction and the methods, by which the algebraic spirit of Dedekind had initiated a fragment of set theory that is the less intuitive possible [9, p. 118].

In particular, Von Neumann [30] admitted that a nonset could be an ordinal number. But then, all its segments must be true sets, since they stand for an argument in the function by which is defined a correspondence between the segment and the element which determines it. The result is that Ω, the system of all the ordinals, is the only one ordinal number which is not a set.

9 Problems of Non-collectivizing

The main problem of set theorists was to avoid paradoxes. Different strategies have been set up. Russell's theory of types, Zermelo–Fraenkel axiomatics (with the Schema of Separation), and finally, Von Neumann's theory (with the axiom of foundation). But there is a cost to pay. Russell's theory of types yields a hierarchical universe with a lot of internal problems; ZF-axiomatics leads to an artificial difference between sets and proper classes (classes that are not sets); and Von Neuman's theory assumes an axiom which is reputed completely unnecessary (the axiom of foundation). When replaced by its negation (the anti-foundation axiom), it makes possible a new theory, the theory of circular sets, whose origin, as we have seen, lays in Finsler's theory of sets of 1925 (see [4]), and its later developments in Aczel (see [1]), or in Barwise and Etchemendy (see [2]).

So raising objections to the uncritical use of collectivizing relations is not, from the viewpoint of pure theory, so unreasonable.

Having said this, we must, however, observe that we can hardly get rid of collectivizing relations and save a reasonable mathematical theory.

The main fact is that when we speak of more than one thing at a time, we implicitly consider the things we talk about as a consistent multiplicity. And if we take apart the questions of the Russell's set, of the sets of all ordinals or cardinals, and of the sets of all sets, we are generally logically right.

So, if we want to suggest that the things we put together in the same class are absolutely different from one another (so that they make a collection but not necessarily a collectivity), it will probably be better to indicate that in a definition.

In this case, defining a particular relation on such a set, may reveal some hidden structure, to which we can associate, further on, some kind of algebra, or order, or topology, so that we can continue to investigate the properties or things in the collection with the help of reason.

If not, I am afraid, we could not do any mathematics at all.

So, as a solution to the problem raised by the Levinas' counter-examples to set theory, I allow myself to introduce the following rational model.

10 A Rational Model

Let $\parallel$ be a relation of *absolute alterity*. We say that $a \parallel b$ (that we read as "a is absolutely different from b", if a and b cannot be connected in any way. This kind of relation is very closed to the uncomparability relation, where $a \parallel b$ is interpreted as "a and b are uncomparable elements" (which means that we have neither $a \leq b$ nor $b \leq a$).

Let us also introduce a new relation $\otimes$ which will be named a *hostage relation*. This relation $a \otimes b$ means that "a is the hostage of b" or that "b has taken a hostage".

10.1 Properties of ∥ and ⊗

Let us first explain the properties of $\parallel$:

- $\parallel$ is irreflexive (we do not have $a \parallel a$ because a is not absolutely different from itself, except in the case of the "simulacrum"[6]);
- $\parallel$ is symmetric ($a \parallel b$ iff $b \parallel a$, for all $a \neq b$);
- $\parallel$ is not transitive (if $a \parallel b$ and $b \parallel c$, then we do not have in general $a \parallel c$, even for all $a \neq c$). This means that $\parallel$ is not an equivalence relation (because of irreflexivity) or an order or a preorder relation (because of non-transitivity).

Assume now we have, for $\otimes$, the following properties:

- $a \otimes a$ (reflexivity, a may be a hostage of himself).
- $\neg(a \otimes b \Rightarrow b \otimes a)$ if $a \neq b$. (If $a \neq b$, the fact that a is the hostage of b does not yield that b is the hostage of a; from the viewpoint of a, it is not required. It is the problem of b.)[7]

From this follows that the relation is not symmetric (as $a \otimes b$ does not imply $b \otimes a$, *a fortiori*, $a \otimes b$ is not equivalent to $b \otimes a$).

But $\otimes$ is not transitive. The fact that $a \otimes b$ and $b \otimes c$ does not necessarily mean that $a \otimes c$.

So $\otimes$ is a reflexive, antisymmetric and non-transitive relation. As we have no transitivity, it is not an order or a preorder relation.

We shall define now a third relation $\trianglelefteq$, that we shall name an "ethical relation" and that we define as the logical sum of the previous ones. Let us set:

$$a \trianglelefteq b =_{df} a \parallel b \oplus a \otimes b.$$

[6]This concept, coming from Plato's images (eikones), is emphasized in Deleuze (see [12]). But we can already find an example of it in Diderot (see [14]) where, for example, the person of Rameau's nephew, constantly differs from itself (see our comments in [25]).

[7]Cf. [21, p. 94]. The following is a part of a dialog between Philippe Nemo (PH. N.) and Emmanuel Levinas (E.L.): "PH. N.—But do not others be, also, responsible for me? E.L.—Maybe, but this is *their* business. One of the fundamental themes (...) of *Totalité et Infini* is that the intersubjective relation is not a symmetric relation. This way, I am responsible for others without waiting for the reverse, had this to cost me the life. It is exactly because, between the others and me, the relation is not mutual that I am subjected to the others; and I am a "subject" essentially this way."

10.2 Properties of Relation $\trianglelefteq$

- (Reflexivity) $a \trianglelefteq a =_{df} a \parallel a \oplus a \otimes a = a \otimes a$. But we do not in general[8] have $a \parallel a$. So $a \trianglelefteq a$ almost always means $a \otimes a$.
- As we may have $a \otimes b$ without necessarily having $b \otimes a$, symmetry is not verified. But we can say that if x has an ethical relation with y and if y has an ethical relation with x, then x and y are "equals" in the democratic sense. So, we have a kind of antisymmetry.
- In general, transitivity is not verified.

Proof $a \trianglelefteq b =_{df} a \parallel b \oplus a \otimes b$. Then $b \trianglelefteq c =_{df} b \parallel c \oplus b \otimes c$. But even though $a \parallel b$ and $b \parallel c$ do not yield $a \parallel c$ (non-transitivity of $\parallel$), even for $a \neq c$, as well, $a \otimes b$ and $b \otimes c$ do not imply $a \otimes c$. □

- Some elements of the set may be incomparable.

11 Trellis

We have seen that the "ethical relation" is not a transitive relation. In the 1970s, Helen Skala [28] has developed some attempt to take into account such situations. The material presented in her mathematical memoir contains a foundation for the theory of non-transitive orderings. Let us recall here first some definitions.

Definition 11.1 (Pseudo-ordered sets) Any reflexive and antisymmetric binary relation $\trianglelefteq$ on a set A will be called a *pseudo-order* on A, that is, $x \trianglelefteq x$ for any element x of A, and if $x \trianglelefteq y$ and $y \trianglelefteq x$ then $x = y$.

As Skala said, a natural example of a pseudo-order on the set of real numbers is obtained by setting $x \trianglelefteq y$ iff $0 \leq y - x \leq a$, where a is a positive number.

Another example, connected with classification problems, is the following one. Let X be a set and χ, a family of subsets of X. For any subsets R and S of X, set $R \trianglelefteq S$ iff $R \subseteq S$ and the set-theoretic difference $S - R$ belongs to χ. In this way, 2^X is pseudo-ordered.

Now, the ethical relation between persons, according to Emmanuel Levinas, may be also considered as a pseudo-order.

Let us now introduce the following theorems and definitions:

Theorem 11.1 *Each pseudo-ordered set is isomorphic with a contraction set of a partially ordered set.*

Definition 11.2 (Pseudo-chain, cycle) For each subset B of a pseudo-ordered set A, a transitive and reflexive, but not necessarily antisymmetric relation $\trianglelefteq_B$ can be defined on B by setting $b \trianglelefteq_B b'$ iff there exists a finite sequence $(b_1, \ldots, b_n)$ of elements from B such that $b \trianglelefteq b_1 \trianglelefteq \cdots \trianglelefteq b_n \trianglelefteq b'$. If for each pair of elements b and b' at least one of the relations $b \trianglelefteq b'$ or $b' \trianglelefteq b$ holds, then B will be called a *pseudo-chain*. If both these relations hold for each pair of elements, B is said to be a *cycle*.

[8] Except in the case of the simulacrum.

Fig. 1 A pseudo-ordered set

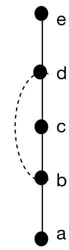

Table 2 Trellis

$\wedge$ \ $\vee$	a	b	c	d	e
a		b	c	d	e
b	a		e	e	e
c	a	b		d	e
d	a	a	a		e
e	a	b	c	d	

Theorem 11.2 *The elements of a finite pseudo-chain can be arranged* (*with possible repetitions*) *in a sequence* $(b_1, \ldots, b_n)$ *such that* $b_1 \trianglelefteq \cdots \trianglelefteq b_n$.

Definition 11.3 A pseudo-order on a set A is said to be *linear* if A itself is a pseudo-chain.

It can easily be shown that any pseudo-order can be extended to a linear one, but there is a sharper result:

Theorem 11.3 *Any pseudo-order* $\trianglelefteq$ *on a set* A *can be extended to a linear pseudo-order* $\trianglelefteq'$ *on* A *such that any cycle with respect to* $\trianglelefteq'$ *is also a cycle with respect to* $\trianglelefteq$.

We have also:

Theorem 11.4 *Any pseudo-order is the intersection of all its linear extension.*

Usually, finite pseudo-ordered sets are represented by Hasse-type diagrams with the convention that if x is below y and connected to y without $x \trianglelefteq y$ holding, then x and y will be joined by a dashed curve. For example, the pseudo-ordered set $\{a, b, c, d, e\}$ where $a \trianglelefteq b \trianglelefteq c \trianglelefteq d \trianglelefteq e$, $a \trianglelefteq x \trianglelefteq e$ for each x, would be represented as indicated in Fig. 1.

Definition 11.4 (Trellis) By a trellis, we mean a pseudo-ordered set, any two of whose elements have a least upper bound (l.u.b.) and a greatest lower bound (g.l.b.).

The pseudo-ordered set of Fig. 1, for example, is a trellis. Its l.u.b. and g.l.b. are given in Table 2. (It is understood that the operations are commutative and idempotent).

"Ethical relations" are of this kind. No doubt that they present a lot of interesting properties that have to be carefully explored.

To conclude (and also extend) our main view, we can say that the material presented above as a generalization of the concepts of partial order and lattices, offers many advantages.

1. By starting out with a reflexive and antisymmetric but not necessarily transitive order, one can define the least upper bound and greatest lower bound similarly as for partially ordered sets, thus obtaining the structure called a "trellis", in which these operations are not necessarily associative (see [28]). But with this approach, one can prove nearly all the basic theorems of lattice theory, thus revealing the superfluity of the assumption of associativity, which seems, philosophically, compatible with the possibility of ethical relations. So Ethics (at least, ethics in the sense of Levinas) seems to be a pseudo-ordered domain in which, consequently, nearly the same theorems may hold.
2. Trellis or weakly associated structures can also formalize the concept of a "tournament", i.e. a structure $(T, <)$, where T with a binary relation $<$ is such that for all $a, b \in T$ exactly one of $a = b$, $a < b$, and $b < a$ holds. This structure is nearly the one we got when we defined an $\otimes$ relation on a set, if we admit that $\otimes$ may be extended to signify "Exactly one of the following relations holds: a is the hostage of b or b is the hostage of a, or a is the same as b". It is also an algebra $(T, \vee, \wedge)$ defined by the rule: if $x < y$ then $x = x \wedge y = y \wedge x$ and $y = x \vee y = y \vee x$, and $x = x \wedge x = x \vee x$ for all x. In this algebra $(T, \vee, \wedge)$, neither $\wedge$ nor $\vee$ is associative unless $(T, <)$ is a chain, that is, $<$ is transitive. However, the two operations are idempotent, commutative; the absorption identities hold, and a weak form of the associative identities holds (see [17]).
3. On weakly associative lattices, we can also define some *tolerance relations* (that may be derived as well on lattices). Remember that a relation R on a set E is said to be a *tolerance* if it is reflexive and symmetric (see [10]). So, note that tolerance is exactly the relation $||$ when we admit it is reflexive, that is, in the case mentioned above of the "simulacrum". In France, tolerance relations are rather named "similarity relations". This kind of relations gives covers instead partitions, these being connected with equivalence relations, which suppose transitivity to be verified. A lot of interesting theorems may be derived from that.

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