PARTIAL ORDERS

2.1. UNIVERSAL ALGEBRA

In this chapter we will study partial orders. Partial orders are a kind of structures. In this first section we will introduce the general notion of a structure and study general properties of structures. The mathematical theory that studies structures at this level of generality is called Universal Algebra. A very thorough overview of Universal Algebra can be found in Grätzer (1978a).

First some introductory notions.

In chapter one (Section 1.4.1.) I introduced the notions of relation and function, their domain and range and the notions of injection, surjection and bijection. We will start by developing the theory of relations and functions a bit more here.

A *binary relation* from A into B is a subset of $A \times B$.

A *binary relation* on A is a subset of $A \times A$.

Let R be a relation from A into B.

The *converse* of R,

$$R^{-1} = \{\langle b, a\rangle : \langle a, b\rangle \in R\}$$

Let $R \subseteq A \times B$ and $S \subseteq B \times C$. The *composition* of R and S, (S *over* R),

$$S \circ R = \{\langle a, c\rangle \in A \times C : \exists b \in B : \langle a, b\rangle \in R \text{ and } \langle b, c\rangle \in S\}$$

Given set A. The *identity relation* on A,

$$I_A = \{\langle a, a\rangle : a \in A\}$$

A *function* from A into B is a relation f from A into B such that:

1. $\text{dom} f = A$
2. $\forall a \in A\, \exists! b \in B : \langle a, b\rangle \in f$

Let $B \subseteq A$. The *identity function* on B, $\text{id}_B : B \to A$ is that function such that :

$$\forall b : \text{id}_B(b) = b$$

Let $B \subseteq A$. The *characteristic function* of B, $\text{ch}_B : A \to \{0, 1\}$ is that function such that $\forall a$:

$$\text{ch}_B(a) = 1 \text{ iff } a \in B$$

$$\text{ch}_B(a) = 0 \text{ iff } a \notin B$$

Let $b \in B$. The *constant function* from A on b is that function c_b: $A \to B$ such that

$$\forall a : \text{c}_b(a) = b$$

If $f : A \to A$ then f is called a one-place *operation* on A. If f is a bijection, then it is called a *permutation* of A. Similarly, if $f : A \times A \to A$, then f is called a two-place operation on A.

If f is an operation on A then A is called *closed under* f.

Let $B \subseteq A$ and $R \subseteq A \times A$ and $f : A \to C$. The *restriction* of R to B,

$$R \restriction B = \{\langle b, b'\rangle \in B \times B : \langle b, b'\rangle \in R\}$$

The *restriction* of f to B,

$$f \restriction B = \{\langle b, c\rangle : b \in B \text{ and } \langle b, c\rangle \in f\}$$

Let f be an operation on A and $B \subseteq A$. B is *closed under* f iff

$$f \restriction B : B \to B \quad (\text{i.e. } \forall b \in B : f \restriction B(b) \in B).$$

Similarly, if f is a two place operation on A,

$$f \restriction B = \{\langle\langle b, b'\rangle, c\rangle : b, b' \in B \text{ and}$$
$$\langle\langle b, b'\rangle, c\rangle \in f\}$$

and B is closed under f iff $\forall b, b' \in B : f(b,b') \in B$.

So, if A is a set, f an operation on A, and B a subset of A, then B is closed under f if the restriction of f to B is an operation on B.

Let $f : A \to B$ and $g : B \to C$. The *composition* of f and g (g *over* f),

$$g \circ f = \lambda x.g(f(x))$$

(*Note*: I use λ-abstraction in the meta-language here, for readability.)

If $f : A \to B$ is an injection, then the converse of f, f^{-1}, is a function from ran f into A. f^{-1} is called the *inverse function* of f.

You can check that $f^{-1} \circ f = \text{id}_A$ and that $f \circ f^{-1}$ is the identity function on ran f.

All these notions extend in the obvious way to n-place relations and functions.

For instance, let $h : B^2 \to C$, and $f, g : A \to B$, then:

$$h \circ (f, g) = \lambda x.h(f(x), g(x))$$

If we have a Cartesian product $A_1 \times \cdots \times A_n$ (also called A^n) then:

$$\text{pr}_i : A_1 \times \cdots \times A_n \to A_i (i \leq n),$$

the ith *projection* is that function such that $\text{pr}_i(\langle a_1, \ldots, a_i, \ldots, a_n \rangle) = a_i$.

We call a relation R on A *finitary* iff for some finite n,

$$R \subseteq A^n.$$

Similarly, an operation f on A is finitary iff for some finite n,

$$f : A^n \to A.$$

Let us now come to business. We want to study structures here. Let us define as abstractly as possible what a structure is. (Well, we could be more abstract by allowing more domains and relations and operations on and between them, but we have to stop somewhere in abstraction.)

A *structure* is a tuple $\mathbf{A} = \langle A, \mathbf{R}, \mathbf{F}, \mathbf{S} \rangle$ where:

1. A is a non-empty set
2. $\mathbf{R}$ is a set of *special*, *designated* finitary relations on A
3. $\mathbf{F}$ is a set of *special*, *designated* finitary operations on A
4. $\mathbf{S}$ is a set of *special*, *designated* elements of A.

A *relational structure* is a pair $\langle A, \mathbf{R} \rangle$ with A and $\mathbf{R}$ as above. Rather than defining everything for structures as general as this (too many subscripts and superscripts), I will be concerned with relational structures that have one two-place relation (sometimes two), and write them as $\langle A, R \rangle$, and I will assume that, if needed, you can generalize the notions to be defined to other relational structures.

An *algebra* is a pair $\langle A, \mathbf{F} \rangle$, with A and $\mathbf{F}$ as above. Again, we will only be concerned with special algebras, mostly with algebras that have one one-place operation and two two-place operations. Structures of the form $\langle A, \mathbf{R}, \mathbf{F} \rangle$ we call *relational algebras*. Further, also with respect to special elements, we will in general only be concerned with structures that have at most two of those.

So, to give some examples: $\langle A, \leq \rangle$ is a relational structure with relation $\leq$. $\langle A, -, \wedge, \vee \rangle$ is an algebra with one place operation $-$ and two place operations $\wedge$ and $\vee$. $\langle A, \leq, -, \wedge, \vee \rangle$ is a relational algebra. $\langle A, \leq, -, \wedge, \vee, 0, 1 \rangle$ is a structure (relational algebra) with special elements 0 and 1.

I will here introduce some fundamental concepts of universal algebra. I will define them for 'token'-structures of the form $\langle A, R, *, a \rangle$, where R is assumed to be a two-place relation, $*$ a two-place operation (I will write $a * b$ for $*(a, b)$) and a a special element of A.

We will assume that all the relations, operations and special elements of a structure come in a fixed order, this order is called the type of the algebra.

Most of the time we will only be concerned with structures that have the same type, that is, structures where the order puts the relations, functions and special elements in one–one correspondence, such that every relation, function and special element in the one structure has a (unique) corresponding relation, function or special element in the other structure. When we write $\langle A, R, *, a \rangle$ and $\langle B, R, *, b \rangle$, the use of the same symbols in A and in B indicates that R and $*$ in B are the corresponding relation and function of R and $*$ in A. So this does not mean that R on A and R on B are *the same* relation, only that they are the corresponding relations. Sometimes, I will want to remind you of this, and I will write $\langle A, R_A, *_A, a \rangle$ and $\langle B, R_B, *_B, b \rangle$ to indicate the corresponding relations, operations and special elements.

So, I will be concerned with structures of the form $\langle A, R, *, a \rangle$ and assume that this can be generalized (if necessary). Similarly, you get the corresponding notions for relational structures or algebras, by leaving out the irrelevant parts.

Let $\mathbf{A} = \langle A, R_A, *_A, a \rangle$ and $\mathbf{B} = \langle B, R_B, *_B, b \rangle$ be two structures.

A *is a substructure of* **B**, $\mathbf{A} \subseteq \mathbf{B}$ iff

1. $A \subseteq B$
2. $R_A = R_B \upharpoonright A$
3. $*_A = *_B \upharpoonright A$
4. $a = b$

Note that, since $\subseteq$ is defined for structures **A** and **B** here, it automatically follows that **A** can only be a substructure of **B** if **A** is closed under $*_B \upharpoonright A$.

It is very important to take this very literally. Suppose we have a structure $\langle A, R, *, a\rangle$ and a subset $B \subseteq A$. It is very well possible that on B we can define a structure $\langle B, R', *', b\rangle$ of the same type. However, that is not automatically a substructure of $\langle A, R, *, a\rangle$: only if $R' = R \upharpoonright B$ and $*' = * \upharpoonright B$. Else it is just a structure of the same type on a subset of A.

Let $\mathbf{A} = \langle A, R_A, *_A, a\rangle$ be a structure and let $B \subseteq A$.
The *restriction* of $\mathbf{A}$ to B, $\mathbf{A} \upharpoonright B$ is the unique tuple

$$\mathbf{A} \upharpoonright B = \langle B, R_A \upharpoonright B, *_A \upharpoonright B, a\rangle$$

$\mathbf{A} \upharpoonright B$ is a structure iff B is closed under $* \upharpoonright A$ and $a \in B$. Note that $\mathbf{B} \subseteq \mathbf{A}$ if $\mathbf{A} \upharpoonright B$ is a structure and $B = A \upharpoonright B$. In fact, we will from now on use the term 'the restriction of $\mathbf{A}$ to B' for the unique substructure $\mathbf{A} \upharpoonright B$, presupposing that $\mathbf{A} \upharpoonright B$ is a structure (and of course, whenever we use it, we have to check that this presupposition is fulfilled).

Exercise 1. Let $\mathbf{N}$ be the set of natural numbers, $\mathbf{E}$ the even numbers, $\mathbf{O}$ the odd numbers, $\mathbf{Z}$ the integers, $\mathbf{Q}$ the rational numbers and $\mathbf{R}$ the real numbers. For the purpose of this exercise, $A \upharpoonright B$ stands for the restriction of A to B as we first defined it. Given $\langle \mathbf{N}, <\rangle$, the natural numbers with their natural order, and $\langle \mathbf{N}, <, +\rangle$, the same structure with addition.
Define:

$$\begin{aligned}
\langle \mathbf{E}, <\rangle &:= \langle \mathbf{N}, <\rangle \upharpoonright \mathbf{E}\\
\langle \mathbf{O}, <\rangle &:= \langle \mathbf{N}, <\rangle \upharpoonright \mathbf{O}\\
\langle \mathbf{E}, <, +\rangle &:= \langle \mathbf{N}, <, +\rangle \upharpoonright \mathbf{E}\\
\langle \mathbf{O}, <, +\rangle &:= \langle \mathbf{N}, <, +\rangle \upharpoonright \mathbf{O}
\end{aligned}$$

(a) Which of the four defined structures are substructures of $\langle \mathbf{N}, <\rangle$ and which of them are substructures of $\langle \mathbf{N}, <, +\rangle$? Explain your answer.
(b) How would you define a relation: '$\mathbf{A}$ is a generalized substructure of B' that is like the notion substructure, but is a relation between structures of (possibly) different types; which of the above structures are generalized substructures of $\langle \mathbf{N}, <, +\rangle$?

Let $\mathbf{A} = \langle A, R_A, *_A, a\rangle$ and $\mathbf{B} = \langle B, R_B, *_B, b\rangle$ be two structures (of the same type).
$h: A \to B$ is a *homomorphism* from $\mathbf{A}$ into $\mathbf{B}$ iff

1. h is a function from A into B
2. h preserves the structure:
   (a) $\forall a, a' \in A$: if $R_A(a, a')$ then $R_B(h(a), h(a'))$
   (b) $\forall a, a' \in A$: $h(a *_A a') = h(a) *_B h(a')$
   (c) $h(a) = b$

Exercise 2. Let $f: A \to B$ and $g: B \to C$ be two homomorphisms. Show that $g \circ f$ is a homomorphism from A into C.

This result forms in Montague Grammar the basis for the formalization of the compositionality constraint and for the fact that the level of translation into intensional logic IL can be eliminated if wanted:

- The disambiguated language, the level of syntactic derivation trees, can be given the form of an algebra, with derivation trees as elements and syntactic rules as operations.
- IL can be given the form of an algebra of the same type, with IL-expressions as elements and operations defined in terms of the formation rules of IL as operations.
- The level of meanings can be given the structure of an algebra of the same type with meanings (in PTQ intensions) as elements and operations defined in terms of the primitive operations on meanings.

The *translation* of derivation trees into IL-formulas is a homomorphism, the *interpretation* of IL-formulas in the meaning algebra is a homomorphism, so there is a homomorphism (the composition of these two homomorphisms), the *direct interpretation* of the derivation trees into the meaning algebra. I am simplifying a bit here. The structures we need in Montague Grammar are not algebras in our sense (one domain with operations on it) but so-called 'many-sorted algebras', where we have different domains (types) and sorted operations.

Montague's paper 'Universal Grammar' was meant to give the general form that a compositional theory of interpretation for natural language should take (so 'universal' means universal in the sense of universal algebra). You see then that part of the work that had to be done in that paper was to adapt notions that are well known from universal algebra for algebras (like homomorphism) to the structures (many sorted algebras) that are needed in such a universal interpretation theory. PTQ is an example of a semantic theory cast in 'universal grammar' (where you take meanings to be intensions), it is not itself such a general theory. For details, see Montague (1970), Janssen (1983), Dowty, Wall and Peters (1981).

Let $f: A \to B$ be a homomorphism.

f is an *injective homomorphism* from A into B iff f is an injection

f is a *surjective homomorphism* or an *epimorphism* from A onto B iff f is a surjection,

f is a *bijective homomorphism* from A into B iff f is a bijection.

f is a *strong homomorphism* iff f *anti-preserves* the relations, that is:

$$\forall a, a' \in A\text{: if } R_B(f(a), f(a')) \text{ then } R_A(a, a')$$

Thus for a strong homomorphism we have:

$$\forall a, a' \in A\text{: } R_B(f(a), f(a')) \text{ iff } R_A(a, a').$$

f is an *embedding* iff f is a strong injective homomorphism.
f is an *isomorphism* iff f is a strong bijective homomorphism.

A and **B** are *isomorphic* iff there is an isomorphism between **A** and **B**.

If **A** and **B** are algebras (without special relations), then the notions of strong homomorphism and homomorphism collapse, so there we have:

An *embedding* is an injective homomorphism
An *epimorphism* is a surjective homomorphism
An *isomorphism* is a bijective homomorphism

Let h be a homomorphism from **A** into **B**.

The *homomorphic image* of **A** in **B** under h, $h(\mathbf{A})$, is the structure given as follows:

Let

$$h(A) = \{h(a)\colon a \in A\}$$
$$h(R_A) = \{\langle h(a), h(a')\rangle\colon a, a' \in A \text{ and } R_A(a, a')\}$$
$$h(*_A) = \{\langle\langle h(a), h(a')\rangle, h(a'')\rangle\colon a, a', a'' \in A \text{ and } *_A(a, a') = a''\}$$
$$h(\mathbf{A}) = \langle h(A), h(R_A), h(*_A), h(a)\rangle$$

It is easy to see with the definition of homomorphism that $h(\mathbf{A})$ is a substructure of **B**, in fact, the following holds:

$$h(\mathbf{A}) = \mathbf{B} \restriction h(\mathbf{A})$$

If h is an epimorphism then **B** is the homomorphic image of **A** under h. If h is an embedding, then $h(\mathbf{A})$ is called the isomorphic image of **A** in **B** under h. In this case, we call h an isomorphic embedding of **A** into **B**. If h is an isomorphism, **B** is the isomorphic image of **A** under h.

Exercise 3. 1. Let h be an embedding of **A** into **B**. Give a justification for calling $h(\mathbf{A})$ the isomorphic image of **A** in **B** under h.

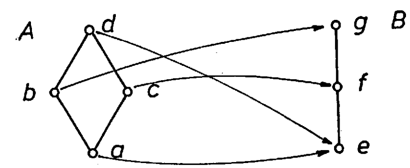
2. Give an example of two relational structures and a bijective homomorphism that is not an isomorphism.

If f is an isomorphism between **A** and **A**, i.e. $f: A \to A$ is an isomorphism, we call f an *automorphism*.

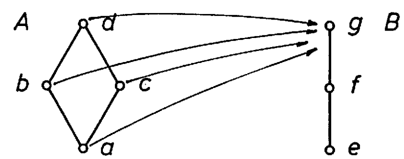
Of course, id_A always gives you an automorphism, but there may be other automorphisms.

Exercise 4. (I) The following diagrams all represent structures of the form $\langle A, \leqslant\rangle$, where $a \leqslant b$ is represented as 'from a we can follow a line up to b' ($a \leqslant a$ indicates an empty line from a up to a). The arrows indicate a function from A into B. Indicate in all cases whether the function is: (1) a homomorphism; (2) an embedding; (3) an epimorphism (4) an isomorphism (5) an automorphism.

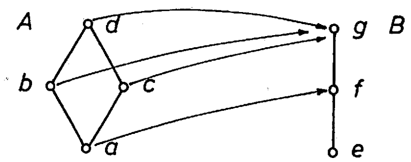
(a)



(b)



(c)



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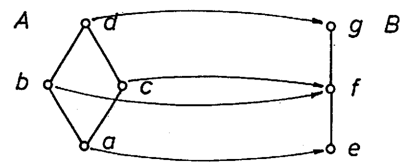

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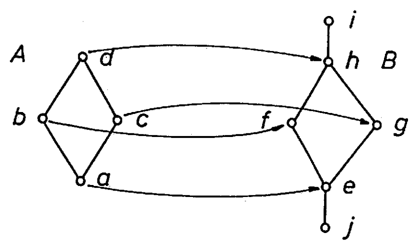
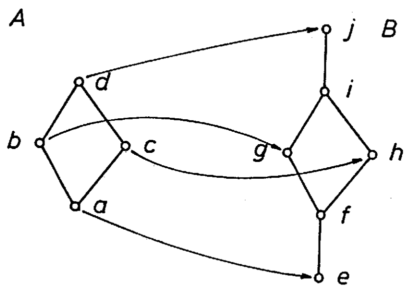
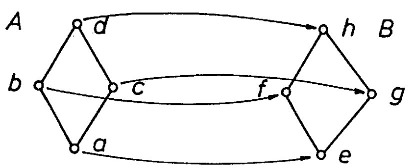


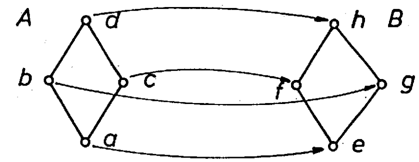
(f)



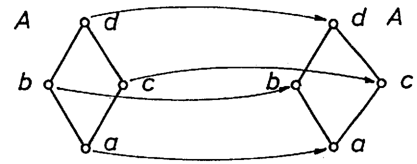
(g)



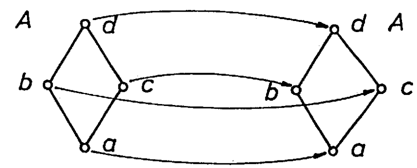
(h)



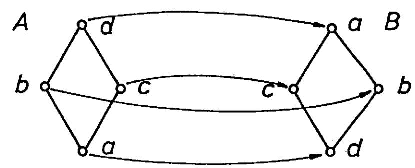
(i)



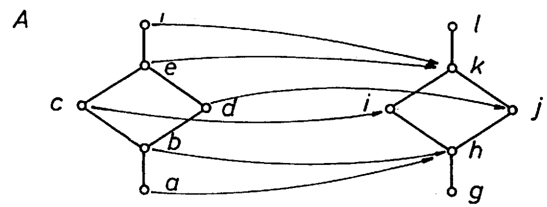
(j)



(k)



(l)



(II) We interpret the same diagrams now as algebras $\langle A, \wedge, \vee\rangle$, where $a \vee b$ is represented as the smallest element in which two lines going up from a and from b meet. (Again, from a and a, two empty lines go up and meet in a and if a line goes up from a to b, then that line and the empty line going up from b to b meet in b.) Similarly,

$a \wedge b$ means: $a \wedge b$ is the biggest element where two lines going down from a and b meet.

State for every of the given diagrams (a)–(l) whether the function is a homomorphism or not.

Let us make more precise the sense in which homomorphisms are structure preserving, by answering the question: which properties of **A** are preserved onto the homomorphic image, $h(\mathbf{A})$, by homomorphism $h{:}A \to B$. Equivalently, we will answer: which properties are preserved under epimorphisms. We will make this question precise in the following way.

We have structures of the form $\langle A, R, *, a\rangle$. Let us look at the language $L = \{R, *, a\}$, that is, we have a first order language with mathematical constants R, $*$ and a. Every structure of the form $\langle A, R_A, *_A, a_A\rangle$ is a model for L, with $i(R) = R_A$, $i(*) = *_A$, $i(a) = a_A$.

The formulas that are true in structure **A** express first order properties of **A**. In other words, by looking at our structures as models for L, we can answer which properties are preserved, by answering which formulas are preserved. Let us give the following definition:

POS, the set of positive formulas of L is the smallest set such that:

1. $ATFORM \subseteq POS$
2. If $\varphi, \psi \in POS$ then $(\varphi \wedge \psi)$, $(\varphi \vee \psi) \in POS$
3. If $\varphi \in POS$, $x \in VAR$ then $\forall x\varphi$, $\exists x\varphi \in POS$

(So positive formulas are constructed with $\wedge$, $\vee$, $\forall$, $\exists$ from atomic formulas without negation.)

Let us first give a definition and prove a lemma.

Let h be a homomorphism from **A** into **B**, and $g{:}\,VAR \to A$ an assignment function, then:

$$h(g) \text{ is defined by: } \forall x \in VAR{:}\ h(g)(x) = h(g(x))$$

LEMMA. *Let $h{:}A \to B$ be a homomorphism. Then for every $t \in TERM$, and every $g{:}\ VAR \to A$:*

$$h([\![t]\!]_{A,g}) = [\![t]\!]_{B,h(g)}$$

Proof. With induction.

1. $h([\![x]\!]_{A,g}) = h(g(x)) = h(g)(x) = [\![x]\!]_{B,h(g)}$
2. $h([\![a]\!]_{A,g}) = h(i_A(a)) = h(a_A) = a_B$

(because h is a homomorphism)
$= i_B(a) = [\![a]\!]_{B,h(g)}$

3. Induction: suppose $h([\![t_1]\!]_{A,g}) = [\![t_1]\!]_{B,h(g)}$ and $h([\![t_2]\!]_{A,g}) = [\![t_2]\!]_{B,h(g)}$

Then:

$$\begin{aligned}
&h([\![t_1 * t_2]\!]_{A,g}) = \\
&h([\![t_1]\!]_{A,g} *_A [\![t_2]\!]_{A,g}) = \text{(homomorphism)}\\
&h([\![t_1]\!]_{A,g}) *_B h([\![t_2]\!]_{A,g}) = \text{(induction)}\\
&[\![t_1]\!]_{B,h(g)} *_B [\![t_2]\!]_{B,h(g)} = \\
&[\![t_1 * t_2]\!]_{B,h(g)}
\end{aligned}$$

With this lemma a straightforward induction proves:

THEOREM. *Let $h{:}\ A \to B$ be an epimorphism. Then for every $\varphi \in POS$:*

$$\forall g\text{: } \textit{if } \mathbf{A} \models \varphi\,[g] \textit{ then } \mathbf{B} \models \varphi\,[h(g)].$$

I.e. positive formulas are preserved under epimorphisms (in other words, positive formulas are preserved onto homomorphic images).

Proof. 1. Atomic formulas.

$$\text{If } \mathbf{A} \models t_1 = t_2\ [g] \text{ then } [\![t_1]\!]_{A,g} = [\![t_2]\!]_{A,g},$$

then (h is a function) $h([\![t_1]\!]_{A,g}) = h([\![t_2]\!]_{A,g})$, then (Lemma) $[\![t_1]\!]_{B,h(g)} = [\![t_2]\!]_{B,h(g)}$, then $\mathbf{B} \models t_1 = t_2[h(g)]$

$$\text{If } \mathbf{A} \models R(t_1, t_2)\ [g] \text{ then } \langle [\![t_1]\!]_{A,g}, [\![t_2]\!]_{A,g}\rangle \in R_A,$$

then (homomorphisms preserve R_A): $\langle h([\![t_1]\!]_{A,g}), h([\![t_2]\!]_{A,g})\rangle \in R_B$, then (lemma): $\langle [\![t_1]\!]_{B,h(g)}, [\![t_2]\!]_{B,h(g)}\rangle \in R_B$ then $\mathbf{B} \models R(t_1, t_2)\ [h(g)]$

2. Conjunction, disjunction: with obvious induction, I skip them here.

3. Assume (induction clause):

$$\forall g\text{: if } \mathbf{A} \models \varphi\,[g] \text{ then } \mathbf{B} \models \varphi\,[h(g)]$$

and assume: $\mathbf{A} \models \exists x\varphi\,[g]$.

Then for some $d \in A$: $\mathbf{A} \models \varphi[g_x^d]$. Then (by induction) for some $d \in A$: $\mathbf{B} \models \varphi[h(g_x^d)]$. Since $h(g_x^d)(d) = h(d)$, it follows that:

$$h(g_x^d) = h(g)_x^{h(d)}.$$

Hence, for some $d \in A$: $\mathbf{B} \models \varphi[h(g)_x^{h(d)}]$, thus, for some $b \in B(b = h(d))$: $\mathbf{B} \models \varphi[h(g)_x^b]$, hence $\mathbf{B} \models \exists x\varphi\ [h(g)]$.

4. With the same induction clause, assume $\mathbf{A} \models \forall x\varphi\ [g]$. Then for all $d \in A$: $\mathbf{A} \models \varphi[g_x^d]$, then (induction) for all $d \in A$: $\mathbf{B} \models \varphi[h(g_x^d)]$, then for all $d \in A$: $\mathbf{B} \models \varphi[h(g)_x^{h(d)}]$. Now h is an epimorphism (this is the only place where we use this assumption), thus every $b \in B$ is the value of some $d \in A$, hence for all $b \in B$: $\mathbf{B} \models \varphi[h(g)_x^b]$, thus $\mathbf{B} \models \forall x\varphi\ [h(g)]$.

We can draw some conclusions from inspecting this proof.

Let h be a *homomorphism* (rather than an epimorphism). Then h will respect all formulas built from atomic formulas with $\wedge$, $\vee$, $\exists$ (we used the surjection assumption only for $\forall$).

Let h be an *injective homomorphism*. Then h will respect all formulas built from atomic formulas and formulas of the form $\neg(t_1 = t_2)$, with $\wedge$, $\vee$, $\exists$. (Reason: if $\mathbf{A} \models \neg(t_1 = t_2)\ [g]$ then $[\![t_1]\!]_{A,g} \neq [\![t_2]\!]_{A,g}$, then (because g is an injection) $h([\![t_1]\!]_{A,g}) \neq h([\![t_2]\!]_{A,g})$, hence $\mathbf{B} \models \neg(t_1 = t_2)[h(g)]$.

Let h be a *bijective homomorphism*. Then h will respect all formulas built from atomic formulas and formulas of the form $\neg(t_1 = t_2)$, with $\wedge$, $\vee$, $\exists$, $\forall$.

Let h be a *strong homomorphism*. Then h will respect all formulas built from atomic formulas and formulas of the form $\neg R(t_1, t_2)$, with $\wedge$, $\vee$, $\exists$.

Let h be a *strong epimorphism*. Then h will respect all formulas built from atomic formulas and formulas of the form $\neg R(t_1, t_2)$, with $\wedge$, $\vee$, $\exists$, $\forall$.

Let h be an *embedding*, then h will respect all formulas built from atomic formulas and negations of atomic formulas with $\wedge$, $\vee$, $\exists$.

Finally, for isomorphisms, we proved a stronger theorem in the previous chapter:

THEOREM. *If $h{:}A \to B$ is an* **isomorphism** *then for every first order formula φ:*

$$\forall g\text{: } \mathbf{A} \models \varphi[g] \textit{ iff } \mathbf{B} \models \varphi[h(g)]$$

The crux of the proof is that now we can prove with induction: if (induction hypothesis): $\forall g$: $\mathbf{A} \models \varphi[g]$ iff $\mathbf{B} \models \varphi[h(g)]$, then $\forall g$: $\mathbf{A} \models \neg\varphi[g]$ iff $\mathbf{B} \models \neg\varphi[h(g)]$. (Proof: $\mathbf{A} \models \neg\varphi[g]$ iff $\mathbf{A} \not\models \varphi[g]$ iff (induction) $\mathbf{B} \not\models \varphi[h(g)]$ iff $\mathbf{B} \models \neg\varphi[h(g)]$).

We know that we have the iff for atomic statements: identity statements are preserved and antipreserved, because h is an injective homomorphism; relational statements are preserved and antipreserved, because h is a strong homomorphism. For other formulas, the one side:

$$\forall g\text{: if } \mathbf{A} \models \varphi[g] \text{ then } \mathbf{B} \models \varphi[h(g)]$$

we get for $\wedge$, $\vee$, $\exists$ because h is a homomorphism; for $\forall$, because h is an epimorphism.

Finally, the other side:

$$\forall g\text{: if } \mathbf{A} \models \varphi[g] \text{ then } \mathbf{B} \models \varphi[h(g)]$$

directly follows from the induction proof for negation, because it is equivalent to:

$$\forall g\text{: if } \mathbf{A} \models \neg\varphi[g] \text{ then } \mathbf{B} \models \neg\varphi[h(g)]$$

If two structures are isomorphic, then for most mathematical purposes we do not distinguish them, they are the same structure. This ends, for this chapter, our discussion of universal algebra; more notions will be introduced in the chapter on lattices.

2.2. PARTIAL ORDERS AND EQUIVALENCE RELATIONS

We will now develop the theory of relations further. We will consider relational structures $\langle A, R\rangle$.

1. R is *reflexive* iff $\forall a \in A$: $R(a, a)$
2. R is *irreflexive* iff $\forall a \in A$: $\neg R(a, a)$
3. R is *transitive* iff $\forall a, b, c \in A$: $R(a,b) \wedge R(b,c) \to R(a,c)$
4. R is *intransitive* iff $\forall a, b, c \in A$: $R(a, b) \wedge R(b, c) \to \neg R(a,c)$
5. R is *symmetric* iff $\forall a, b \in A$: $R(a, b) \to R(b, a)$
6. R is *asymmetric* iff $\forall a, b \in A$: $R(a, b) \to \neg R(b,a)$
7. R is *antisymmetric* iff $\forall a, b \in A$: $R(a, b) \wedge R(b, a) \to a = b$
8. R is *connected* iff $\forall a, b \in A$: $R(a, b) \vee R(b, a) \vee a = b$

We can picture a relation on A with points and arrows, where $a \to b$ represents $R(a, b)$. Such a picture we call *a graph of* $\langle A, R\rangle$, or – mixing terminology – a model of $\langle A, R\rangle$.

Exercise 5. Consider conditions (1) through (7)

(a) Draw a model with three elements where:

(1) (1), (3) and (5) hold
(2) (2), (3) and (6) hold
(3) (1), (3) and (7) hold
(4) (1) holds, but none of the others
(5) the same for (2)
(6) the same for (3)
(7) the same for (5)
(8) none of them hold

(b) Show that (6) implies (2).
(c) Show that (2) and (3) imply (6).
(d) Give a model where (1) and (3) hold, but not (5), (6), (7).

R is a *preorder* iff R is reflexive and transitive.
R is a *partial order* iff R is reflexive, transitive and antisymmetric.
R is a *strict partial order* iff R is irreflexive and transitive.
R is a (strict) *total order* or *linear order* iff R is a (strict) partial order and R is connected.
R is an *equivalence relation* iff R is reflexive, transitive and symmetric.

So we understand our structures $\langle A, R\rangle$ now as ordered by R. We call R the order of $\langle A, R\rangle$, and we call $\langle A, R\rangle$ itself an order as well. If $\langle A, R\rangle$ is a set with a partial order R on it, we call both $\langle A, R\rangle$ and R and even A a partial order and we call both $\langle A, R\rangle$ and A a partially ordered set. 'Partially ordered set' is abbreviated as *poset*.

We won't be very often concerned with preorders. An example of a preorder is the relation: 'be at least as heavy as'. An example from syntax is c-command: (in a tree) node A c-commands node B iff the first branching node dominating A dominates B.

Note that there are three ways of defining a strict partial order: as an irreflexive, transitive order; as a transitive, asymmetric order; and as an irreflexive, transitive, asymmetric order. All of these are equivalent (check the last exercise). The last one is redundant, but is nonetheless the one you will find most often.

If R is a partial order, we normally write $\leqslant$ or $\sqsubseteq$ (or $\geqslant$) or other typographic variations of it; similarly we use $<$ and its variations for strict partial orders. It works the other way round as well: we usually do not write $\leqslant$ if the order is not a partial order. So if we use $\leqslant$, we have to check whether it is a partial order, or note explicitly that we use $\leqslant$, although the relation is not a partial order.

Practically everything we will discuss in this book has something to do with partial orders. It is not an exaggeration to say that, as far as ontology is concerned, semantics is the study of partial orders underlying the 'metaphysics of natural language'. You can say that what a partial order does with a set is to give a qualitative notion of comparison for it, an ordering in terms of structure (rather than in terms of numerical values).

Typical notions of 'earlier than', 'smaller than', 'part of', 'contain' are partial orders that tell us that certain elements are comparable in terms of the order, and others are not. Semantic relations between expressions in natural language like entailment, incompatibility and equivalence are typically comparison relations of the same sort and there are intimate relations between them.

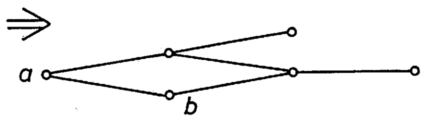
Let me make a remark here about the graphs of partial orders. In general, a relation is represented as a set of points and arrows. In the case of partial orders and strict partial orders, we can simplify these diagrams by some conventions.

In the first place, we can take transitivity to be understood. That is, if an arrow goes from a to b and from b to c, we don't write the arrow going from a to c. Secondly, if we are dealing with a partial order, we also take the reflexivity arrow to be understood; this means that the diagram won't tell us that we are dealing with a reflexive or with an irreflexive order. That is, in fact, all the better, because we want to be able to interpret the same diagram as either a partial order or a strict partial order, and we can, because partial orders and strict partial orders are just two sides of the same coin:

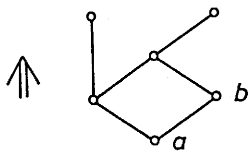
Every partial order determines a strict partial order and every strict partial order determines a partial order,

Given $\langle A, \leqslant\rangle$, we can define $a < b := a \leqslant b \wedge a \neq b$. It is easy to check that this is a strict partial order. Similarly, given $\langle A, <\rangle$, we can define $a \leqslant b := a < b \vee a = b$, and again, it is not hard to see that this is a partial order. So, in the diagram, we take the reflexivity, irreflexivity to be understood.

Finally, because of antisymmetry (asymmetry) we can take the direction of the graph to be understood: either by assuming that a line going to the right from a to b means $a < b$ or that a line going up from a to b means $a < b$ (or any other way). So:

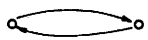


a is earlier than b

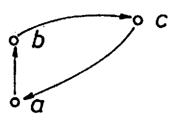


a is smaller than b

The reason is that all arrows can be understood to go in the same direction (and hence we can write lines, rather than arrows), because of antisymmetry (asymmetry) we don't have arrows going in the other direction:

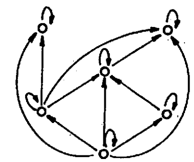


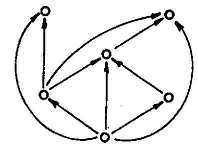
and by antisymmetry (asymmetry) and transitivity we don't have loops either:

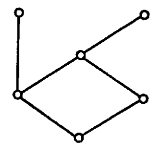


So partial orders are directed, and we simplify the graph by stipulating the direction. These simplified diagrams are called Hasse diagrams.

The advantage of these diagrams should be clear, for instance, if you see that the three diagrams below essentially represent the same order:







Exercise 6. Let $\langle A, \leqslant\rangle$ be a poset. We define $a \geqslant b := b \leqslant a$. This means that $\geqslant\, = \,\leqslant^{-1}$. Look at $\langle A, \geqslant\rangle$.

Show that $\langle A, \geqslant\rangle$ is a poset.

$\langle A, \geqslant\rangle$ is called the *dual* of $\langle A, \leqslant\rangle$.

Intuitively, the dual of a poset is the structure that you get if you turn the graph upside down.

If φ is a statement that holds for poset $\langle A, \leqslant\rangle$, then the *dual* of φ, the statement that you get by changing $\geqslant$ for $\leqslant$ everywhere in φ, will be a statement that holds for the dual of $\langle A, \leqslant\rangle$.

It is also obvious that (because the class of posets is closed under duals) if a statement is true on all posets, its dual is true on all posets as well (this is called duality).

A poset is called *partial* because it is not necessarily the case that all elements are comparable by means of the order.

a and b are called *incomparable* or *unconnected* iff neither $a \leqslant b$ nor $b \leqslant a$. A linear, or total order is an order in which all elements are comparable.

We can make the discussion of duality a little bit more general. Let

$\langle A, R\rangle$ be an order of a certain type T (this means, satisfying certain axioms, i.e. a partial order, or a total order).

If $\langle A, R^{-1}\rangle$ is also an order of type T, $\langle A, R^{-1}\rangle$ is called the *dual* of $\langle A, R\rangle$. Let φ be a statement about orders of that type. If, everywhere in φ, we replace R by R^{-1}, we get the dual of φ. Obviously the following will hold:

DUALITY PRINCIPLE: *if statement φ is true in all orders of type T then its dual is also true in all orders of type T.*

Coming back to posets. Since the class of posets is closed under duals, the duality principle applies to posets. This means that we can chose to prove something for a poset or for its dual.

Let us now turn to equivalence relations. Equivalence relations, often written as $\approx$ (or something notationally like it) are reflexive, transitive and symmetric relations.

What is the importance of equivalence relations? One of the things that you find over and over again in semantics, is that certain entities are taken as primitive, often with an order on them, and other entities are constructed out of those. The crucial things is that, for semantic reasons, we not only want a domain of constructed entities, but we want our constructed entities to be ordered as well, by an order that we can understand in terms of the order of the primitives. Equivalence relations give us a very important technique of doing precisely that. We will see several applications of this in the course of this book.

Just like directed graphs form the natural diagrams for partial orders, equivalence relations have natural and simple pictural representations. These are the pictures of partitions. Partitions are defined as follows.

Let A be a set. *A partition* of A is a set P such that:

1. $P \subseteq \text{pow } A$ and $\emptyset \notin P$

P is a set of nonempty subsets of A

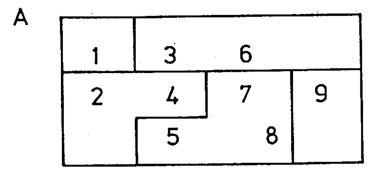
2. $A = \cup\{B: B \in P\}$.

The elements of P are called the blocks of P. So A is the union of all blocks in P (we also call this: *P minimally covers A*)

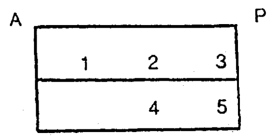
3. $\forall X, Y \in P: X \cap Y = \emptyset$ or $X = Y$

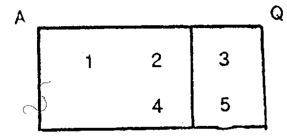
The blocks do not overlap

So, a partition of a set is a way of cutting that set into pieces (like a cake is cut into pieces):



Of course, there is more than one way of cutting a set A into pieces. Take the following two bipartitions:



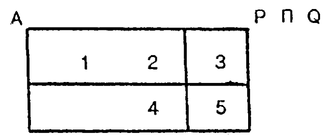


Let us define the intersection of two partitions as the set of nonempty intersections of their blocks:

The *intersection* of partitions P and Q on A

$$P \sqcap Q = \{X \cap Y: X \in P \text{ and } Y \in Q \text{ and } X \cap Y \neq \emptyset\}.$$

It is easy to see that the intersection of two partitions (on A) is again a partition:



It is much harder to define the union of two partitions. Just taking the unions of the blocks does not give you a partition (the sets you get will overlap). We can partially order the set of all partitions on A, by the following relation of refinement:

Let P and Q be partitions on A.
P is a refinement of Q, $P \sqsubseteq Q$ iff $\forall X \in P \exists Y \in Q: X \subseteq Y$.

Intuitively, P is a refinement of Q if P draws the same lines in the graph as Q does, and maybe some more (so P can refine or split some of the blocks in Q).

You can check that $P \sqcap Q$ gives you the minimal way of adding lines to P and to Q such that the result is a refinement of both. Coming back to the union of two partitions. $P \sqcup Q$ is the minimal way of *removing* lines from P and Q such that both P and Q are refinements of the result.

Now let X be a nonempty set of blocks in P. Set Z removes the lines between the blocks in X in P if $Z = \cup X$.

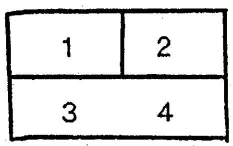
If X is a nonempty set of blocks in P and Y is a nonempty set of blocks in Q, then Z is the *unification* of X and Y iff $Z = \cup X = \cup Y$. So Z is the unification of a set of blocks X in P and a set of blocks Y in Q if you end up with the same set if you remove all the lines in X and Y.

Of course, not all such sets of blocks have unifications. But we *can* look at the set of all unifications of the sets of blocks that do: this the unification set of P and Q,

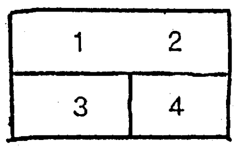
$$\mathrm{Un}(P, Q) = \{Z\colon \exists X \subseteq P \exists Y \subseteq Q[X \neq \emptyset \text{ and } Y \neq \emptyset \text{ and } Z \text{ is the unification of } X \text{ and } Y\}$$

Let me give an example: take the following two partitions of A:

P



Q



To find the unifications, we can look at the set of all unions of sets of blocks in P and the set of all unions of sets of blocks in Q. The ones that we find in both are the unifications:

For P: $\{\{1\}, \{2\}, \underline{\{3, 4\}}, \underline{\{1, 2\}}, \{1, 3, 4\}, \{2, 3, 4\}, \underline{\{1, 2, 3, 4\}}\}$
For Q: $\{\underline{\{1, 2\}}, \{3\}, \{4\}, \underline{\{3, 4\}}, \{1, 2, 3\}, \{1, 2, 4\}, \underline{\{1, 2, 3, 4\}}\}$
So $\mathrm{Un}(P, Q) = \{\{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$

$\mathrm{Un}(P, Q)$ is again partially ordered by $\subseteq$. This order tells us which unifications remove more lines: if $Z' \subseteq Z$ then Z removes more (or as many) lines as Z'. So: We call a unification Z in $\mathrm{Un}(P, Q)$ *minimal* if there is no Z' in $\mathrm{Un}(P, Q)$ such that $Z' \subseteq Z$ and $Z' \neq Z$.

Let $\mathrm{MinUn}(P, Q) = \{Z \in \mathrm{Un}(P, Q)\colon Z \text{ is minimal}\}$. Now we can define the union of two partitions: the union was the minimal way of removing lines from P and Q such that both P and Q are refinements of the resulting partition.

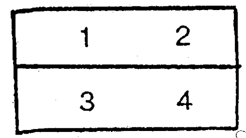
Let P and Q be two partitions on A. The *union* of P and Q, $P \sqcup Q = \mathrm{MinUn}(P, Q)$.

In our example, the set of unifications of P and Q was

$$\{\{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$$

Hence the union of P and Q, the set of minimal unifications is $\{\{1, 2\}, \{3, 4\}\}$ because $\{1, 2, 3, 4\}$ is not minimal. It is clear in this example that this is a partition:

$P \sqcup Q$



We can prove in general that $P \sqcup Q$ is indeed the smallest partition of which both P and Q are refinements.

That $P \sqsubseteq P \sqcup Q$ is easy to see. Every block X in P is subset of some block in $P \sqcup Q$. This is because the union is the set of all minimal unifications, and there is always some minimal unification that unifies blocks in P *among which* X with blocks in Q. That $P \sqcup Q$ is the smallest such partition follows from the minimality. That $P \sqcup Q$ is a partition can be seen as follows.

That $P \sqcup Q$ does not have empty elements is trivial, it has only unifications as elements. That $P \sqcup Q$ covers A is also simple: since all elements of $P \sqcup Q$ are unifications of sets of elements of A, certainly $\cup(P \sqcup Q) \subseteq A$; since $P \sqsubseteq Q$, and $A = \cup P$, $A \subseteq \cup(P \sqcup Q)$, so indeed $P \sqcup Q$ covers A.

Finally, here is a fact about the union of sets of sets (which we won't prove):

Suppose: *for all* $x, y \in A \cup B$: *if* $x \neq y$ *then* $x \cap y = \emptyset$.
Then: $\cup(A \cap B) = \cup A \cap \cup B$.

Now suppose $Z, Z' \in P \sqcup Q$. This means that for some sets of blocks X in P and Y in Q: $Z = \cup X = \cup Y$ and for some sets of blocks X' in P and Y' in Q: $Z' = \cup X' = \cup Y'$.

Look at $Z \cap Z'$. This is $\cup X \cap \cup X'$. Since X and X' are sets of blocks in P, the condition stated in the above fact holds for $X \cup X'$.

Hence $\cup X \cap \cup X' = \cup(X \cap X')$. Since the same holds for Y and Y', we see that $\cup(X \cap X') = \cup(Y \cap Y')$.

Now, suppose $Z \neq Z'$. Suppose $Z \cap Z'$ is not empty. That means that $\cup(X \cap X')$ is nonempty and smaller than either Z or Z'. But this means that $X \cap X'$ cannot be empty (else the union would be empty), and is smaller than either X or X' (else its union would be either Z or Z'). The same holds for Y and Y'. But this means that there is an element of $P \sqcup Q$, namely $Z \cap Z'$, such that there is a set of blocks in P, $X \cap X'$, and a set of blocks in Q, $Y \cap Y'$, such that $Z \cap Z' = \cup(X \cap X') = \cup(Y \cap Y')$, a unification of blocks in P and blocks in Q. But then Z and Z' are not minimal unifications, because $Z \cap Z'$ is smaller. This is a contradiction, so indeed $Z \cap Z'$ is empty.

So indeed $P \sqcup Q$ is a partition.

Exercise 7. Draw the graph of the partial order of the set of all partitions on $\{1, 2, 3, 4\}$ ordered by refinement. (In other words, draw a picture for every partition on this set and draw lines between these pictures in such a way that a line going up means $P \sqsubseteq Q$.)

Let's come back to the graphs for equivalence relations. Equivalence relations have nice graphs because equivalence relations and partitions are in one–one correspondence.

Let A be a set and $\approx$ be an equivalence relation on A and let $a \in A$. The equivalence class of a under $\approx$,

$$[a]_{\approx} = \{b \in A\colon b \approx a\}$$

a is called the representing element of $[a]_{\approx}$.

Some properties of equivalence classes.

1. Equivalence classes are independent of their representing element. That is:

$$\text{if } b \in [a]_{\approx} \text{ then } [a]_{\approx} = [b]_{\approx}$$

2. No equivalence class is empty.
3. If $[a]_{\approx} \neq [b]_{\approx}$ then $[a]_{\approx} \cap [b]_{\approx} = \emptyset$
4. Let $[A]_{\approx} = \{[a]_{\approx}\colon a \in A\}$

$$\cup[A]_{\approx} = A$$

Exercise 8. Prove (1)–(4).

Given all these properties, we have proved that if $\approx$ is an equivalence relation on A then $[A]_{\approx}$ is a partition. So every equivalence relation determines a partition with the equivalence classes as blocks.

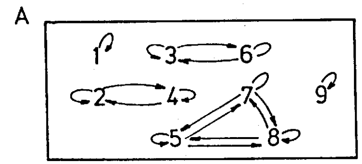
Now, suppose P is a partition on A. Define the following relation:

$$\approx\ := \lambda a \lambda b.\exists X \in P[a \in X \wedge b \in X]$$

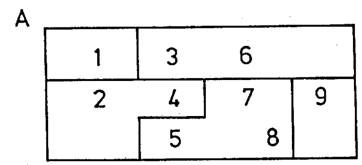
You can check that $\approx$ is indeed an equivalence relation. So every partition determines an equivalence relation. That the above defined $\approx$ is not arbitrary, may become clear from the following fact:

Let R be an equivalence relation on A. Let $P = [A]_R$ and let $\approx$ be the equivalence relation defined as above. Then $R = \approx$.

So indeed equivalence relations and partitions are two sides of the same coin. It is this that allows us to simplify the graph of the following equivalence relation:



to:



The interesting thing about equivalence relations that makes them very useful for our purposes is that if we have a relational structure, $\langle A, R\rangle$, and we have an equivalence relation $\approx$ on A, we can build a new structure out of $\langle A, R\rangle$ by taking equivalence classes and define a new order on the structure as follows:

The equivalence structure of $\langle A, R\rangle$ under $\approx$ is:

$$\langle A, R\rangle_{\approx} := \langle [A]_{\approx}, R'\rangle$$

where

$$XR'Y := \exists a \in X \exists b \in Y \colon aRb$$

Now look at the following function h:

$$\forall a \in A \colon h(a) = [a]_{\approx}$$

h maps every element of A onto the equivalence class of a under $\approx$. h is called the *natural homomorphism*.

This invites the following:

THEOREM. *h, the natural homomorphism, is a homomorphism (in fact, an epimorphism. The fact that h is a surjection follows simply from the fact that $[A]_{\approx}$ by definition of h is the image of A under h).*

What this means, then, is that:

$$\text{if } aRa' \text{ then } h(a)R'h(a')$$

Check, if you want, that this is indeed the case.

So we see that by taking equivalence classes we get a new structure that preserves the relation. Taking equivalence classes then indeed gives us a way of forming new relational structures out of old ones.

It depends on what properties R has and what the equivalence relation is what properties R' will have.

As we have seen, epimorphisms preserve certain properties of R, but don't preserve others. For instance, if $\langle A, \leqslant \rangle$ is a partial order, then it is not hard to check that $\leqslant'$ will be reflexive. We can give a direct proof of this, but we can also simply observe that reflexivity is expressed by the first order sentence $\forall x[R(x,x)]$. This is a positive sentence, hence it is preserved under epimorphisms.

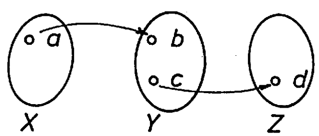
But other properties need not be preserved. The equivalence structure need not be transitive. Transitivity is expressed by:

$$\forall x \forall y \forall z[R(x,y) \wedge R(y,z) \rightarrow R(x,z)],$$

and this is not a positive sentence (i.e. it is equivalent to

$$\forall x \forall y \forall z[\neg R(x,y) \vee \neg R(y,z) \vee R(x,z)]).$$

The following picture represents a partial order and a partition on it:



Here $X \leqslant' Y$, because $a \leqslant b$. $Y \leqslant' Z$ because $c \leqslant d$, but $X \nleqslant' Z$. The relation $\leqslant$ is preserved: $a \leqslant b$, hence $X \leqslant' Y$; $c \leqslant d$, hence $Y \leqslant' Z$; but the property of transitivity that $\leqslant$ has is not preserved. Other equivalence relations do preserve transitivity (for instance, if $X = \{a, b\}$, $Y = \{c\}$ and $Z = \{d\}$, then the resulting structure is a partial order).

Equivalence relations, then, are means of creating new interesting structures if they are well chosen. I will give a few examples shortly, but let us first generalize the construction to algebras.

If a structure not only has relations, but also operations, $\langle A, \leqslant, * \rangle$, then it is not in general guaranteed that equivalence classes will preserve the operations. Here we want a rather stronger notion.

Let $\langle A, \leqslant, * \rangle$ be a relational algebra and let $\approx$ be an equivalence relation on A.

$\approx$ is a *congruence relation* iff if $a \approx a'$ and $b \approx b'$ then

$$(a * b) \approx (a' * b')$$

(So a congruence relation preserves the operations.)

The *congruence structure* of $\langle A, \leqslant, * \rangle$ is:

$$\langle A, \leqslant, * \rangle_{\approx} := \langle [A]_{\approx}, \leqslant', *' \rangle$$

where $\leqslant'$ is defined as before and:

$$[a]_{\approx} *' [b]_{\approx} = [a * b]_{\approx}$$

Again, the natural homomorphism is indeed a homomorphism if $\approx$ is a congruence relation.

Example 1. Let *FORM* be the set of formulas of propositional logic and $\neg$, $\wedge$ and $\vee$ be the syntactic formation rules (so $\neg(\varphi)$ is the result of concatenating $\neg$ and φ). Look at the syntactic algebra $\langle FORM, \neg, \wedge, \vee \rangle$. Further, let $\leftrightarrow$ be the relation of logical equivalence, as defined by the semantics.

It is easy to see that $\leftrightarrow$ is an equivalence relation, in fact a congruence relation (if $\varphi \leftrightarrow \psi$ then $\neg\varphi \leftrightarrow \neg\psi$, etc.).

So form $\langle FORM, \neg, \wedge, \vee \rangle_{\leftrightarrow} = \langle \{[\varphi]_{\leftrightarrow} \colon \varphi \in FORM\}, \neg', \wedge', \vee' \rangle$.

This structure is called the *Lindenbaum algebra* of propositional logic. We know that $h(\varphi) = [\varphi]_{\leftrightarrow}$ is a homomorphism. We can think of these equivalence classes as propositions. We can prove that the Lindenbaum algebra forms a Boolean algebra.

Example 2. This example will come back in a more refined form later. Let us assume that we have a structure of events and we think of these as intensional entities, so we don't identify them with their temporal substrata, periods of time. Let us further assume that we have a temporal relation on them, R, where we interpret R as: eRe' iff e is temporally included in e'. Intuitively, R should be a *preorder*, a reflexive and transitive relation, but not a partial order, because we do not want to say that if e is temporally included in e' and e' in e then $e = e'$. That is precisely what makes events different from periods: different events can go on on the same period of time.

So we have the preorder $\langle E, R\rangle$.

It would be nice if we could define the notion of a period, the period of time at which an event goes on. And for periods we do want a temporal order that is a partial order: if two periods completely overlap, they are the same period. We can do this, because:

THEOREM. *If $\langle A, R\rangle$ is a preorder, we can construct a partial order out of it.*

How do we do this?

Look at the relation $\approx := \lambda a\lambda b.R(a,b) \wedge R(b,a)$. It is easy to check that this is an equivalence relation. So take equivalence classes and form $\langle A, R\rangle_{\approx} = \langle [A]_{\approx} \leqslant\rangle$. I took the liberty of writing $\leqslant$ for R' already; of course we have to check that $\leqslant$ is indeed a partial order.

We know that $\leqslant$ is reflexive, we have argued that before. This time $\leqslant$ is also transitive, because of the way we have chosen $\approx$. Suppose $X \leqslant Y$ and $Y \leqslant Z$. Then $\exists a \in X\ \exists b \in Y$: $R(a,b)$ and $\exists c \in Y \exists d \in Z$: $R(c,d)$. Since both b and c are in Y: $R(b,c)$. But then, by transitivity of R, $R(a,c)$. But then, again by transitivity, $R(a,d)$. Hence $\exists a \in X \exists d \in Z$: $R(a,d)$, so $X \leqslant Z$.
$\leqslant$ is also antisymmetric: if $X \leqslant Y$ and $Y \leqslant X$ then $X = Y$.

Exercise 9. Prove this

With this theorem we know that we can construct a period structure out of our event structure, by taking the equivalence relation on events of "e and e' are cotemporal" or "e and e' go on at the same period". Identifying the period at which an event goes on with the equivalence class under this relation, i.e. the period at which an event goes on as the set of all events that are cotemporal with it, we get a period structure, that is indeed partially ordered, as we wanted.

So we can use equivalence classes to construct periods out of events. Of course, we may want more constraints on events and more constraints on periods. One of the things we will look into later is what constraints on events give us what kinds of period structures (under different ways of constructing periods).

Summarizing: the point about the proof is that in a preorder there may still be x and y such that xRy and yRx, but not $x = y$. The equivalence classes identify all such elements: at the level of equivalence classes they are replaced by one element and the relation becomes a partial order.

Example 3. This example is, semantically, a bit advanced and the discussion here can only be concise. The reader may want to consult Groenendijk and Stokhof (1985) for further details.

In Montague Grammar, the extension of a *that*-complement is a proposition, a function from possible worlds to truth values (type $\langle s, t\rangle$). Just like predicates of type $\langle e, t\rangle$ are properties of individuals (type e), expressions in type $\langle s, t\rangle$ are properties of possible worlds (type s) (that is: sets of possible worlds). The *intension* of a *that*-complement, then, is a function from worlds to extensions of *that*-complements (in fact, a constant function), that is, a function of type $\langle s, \langle s, t\rangle\rangle$. Just like functions in $\langle e, \langle e, t\rangle\rangle$ correspond to relations between individuals, functions in $\langle s, \langle s, t\rangle\rangle$ correspond to relations between possible worlds.

In Groenendijk and Stokhof's (1982) theory of questions, the above analysis is extended to questions. They argue that the extension of a question is a proposition, just like the extension of a *that*-complement, and that the intension of a question, similarly, is a relation between possible worlds.

Of course, we then have to say *which* proposition and relation. With abstraction over possible worlds in the meta language we can express what the extension and intension of a *that*-complement are as follows:

$$\text{EXT } (\textit{that } \varphi) = \lambda w.[\![\varphi]\!]_{M,w,g}$$

the function that assigns to every world w the extension of φ at w.

$$\text{INT } (\textit{that } \varphi) = \lambda v\lambda w.[\![\varphi]\!]_{M,w,g}$$

This is indeed a constant function: the function that assigns to every world v: $\lambda w.[\![\varphi]\!]_{M,w,g}$.

Groenendijk and Stokhof argue that a question like: *does John come*, or *whether John comes*, should have as its *extension*, the proposition that is true in a world w iff *John comes* has in that world w the same truth value as *John comes* has in the actual world (call this world s). This means:

$$\text{EXT}\ (\textit{whether}\ \varphi) = \lambda w.[[\![\varphi]\!]_{M,w,g} = [\![\varphi]\!]_{M,s,g}]$$

Assuming that verbs like *know* and *tell* operate on the *extension* of *whether* φ, this has the nice consequence that:

$$\frac{\begin{array}{l}\textit{John knows whether } \varphi\\ \varphi\end{array}}{\textit{John knows that } \varphi} \qquad \frac{\begin{array}{l}\textit{John knows whether } \varphi\\ \neg\varphi\end{array}}{\textit{John knows that } \neg\varphi}$$

become valid inference patterns.

On the other hand, verbs like *wonder* operate on the intension of *whether* φ:

$$\text{INT}\ (\textit{whether}\ \varphi) = \lambda s\lambda w.[[\![\varphi]\!]_{M,w,g} = [\![\varphi]\!]_{M,s,g}]$$

The analysis extends naturally to wh-questions.

The extension of *who come* is the proposition that is true in a world iff the ones that come in that world are the same as the ones that come in the real world:

$$\text{EXT}\ (\textit{who } P) = \lambda w.[[\![\lambda x.P(x)]\!]_{M,w,g} = [\![\lambda x.P(x)]\!]_{M,s,g}]$$
$$\text{INT}\ (\textit{who } P) = \lambda s\lambda w.[\![\lambda x.P(x)]\!]_{M,w,g} = [\![\lambda x.P(x)]\!]_{M,s,g}]$$

The intensions of questions, thus, are relations between possible worlds.

But they have certain properties, like the following:

$$\forall v: \lambda s\lambda w.[[\![\varphi]\!]_{M,w,g} = [\![\varphi]\!]_{M,s,g}]\ (v, v)$$

(because for all v: $[\![\varphi]\!]_{M,v,g} = [\![\varphi]\!]_{M,v,g}$).

So, this relation is reflexive. It can easily be seen that it is transitive and symmetric as well, thus:

The intension of a question forms an equivalence relation between possible worlds.

This means, thus, that the intension of a question is a *partition* of the set of all possible worlds.

Take the question: *does John come* or *whether John comes*. Its intension is:

$$\lambda s\lambda w.[[\![\textit{John comes}]\!]_w = [\![\textit{John comes}]\!]_s]$$

Since *John comes* is either true or false in a world, this equivalence relation determines a bipartition of the set of all possible worlds into the set of worlds where John comes, and the set of worlds where John doesn't come:

W

John comes
John doesn't come

The blocks of the partition are the semantically possible answers to the question. Answering the question comes down to determining in which of the blocks the real world is located.

Similarly, the question *who come* partitions *W* into the possible answers:

- the worlds where nobody comes
- the worlds where John is the only one who comes
- the worlds where Mary is the only one who comes
- the worlds where John and Mary are the only ones who come up to:
- the worlds where everybody comes:

W

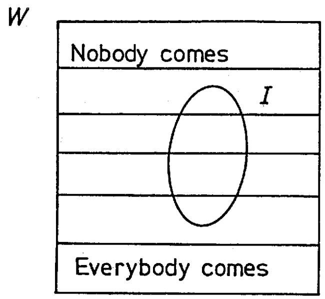
Nobody comes
John comes
John and Mary come
⋮
Here Comes Everybody

Now we can make two standard assumptions.

First we represent our partial information by what Stalnaker (1978) calls a *context set*: the set of worlds compatible with the information.

Secondly, we assume that accepting a statement φ as true means to intersect its intension with the context set (i.e. eliminating all worlds where φ doesn't hold from our information).

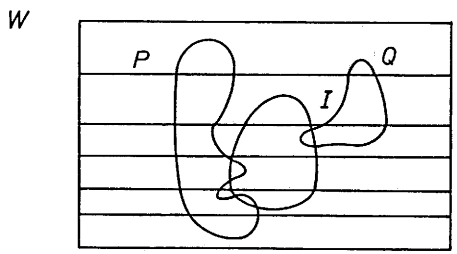
Suppose that the following diagram represents our information when the question *who comes* is asked:



We see several things: in the first place, the question also determines a partition on our information I, and since there is more than one block in I, this means that also given our information the question is still open.

Secondly, we already have the information that some semantically possible answers do not hold: according to our information it is not the case that nobody comes, neither that everybody comes.

We can then define notions of semantic and pragmatic partial and total answer: a pragmatically partial answer is an answer that, if added to I, eliminates some of the blocks on I; it is total if it eliminates all but one of the blocks on I. In the following diagram P is a partial answer, while Q is a total answer:



Note that this example also shows that a proposition need not be semantically an answer to a question at all (like P), or need not be semantically total (like Q), to answer the question partially (P) or totally (Q), relative to our information.

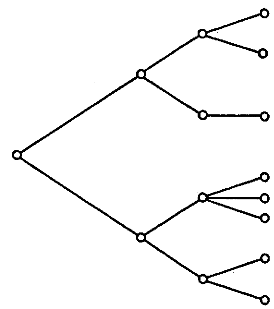
A statement like: *Mary is the only person with red hair*, will effectively answer the question *who comes*, if we already know that who comes has red hair.

For more discussion and details on the semantics and pragmatics of questions and answers and the fruitfulness of partitions, see Groenendijk and Stokhof (1985).

Example 4. The last example concerns trees. We will go into trees in more detail later. Here is a definition:

A tree is a structure $\langle A, \leqslant, \bigcirc \rangle$, where $\leqslant$ is a partial order and $\bigcirc$ is a minimal element or origin (smaller than all others) such that for every $a \in A$: $\{b\colon b \leqslant a\}$ is a finite linear order.

In the prototypical tree incomparable elements abound:



Now what if we want to compare these elements after all?

One case where we might want that occurs in branching time, where the branching represent states in which different possible futures are still open. Now we may have reason to talk about 'the same moment of time in all possible futures'. An example concerns the analysis of temporal counterfactuals with temporal indexicals, like *If she hadn't left me a week ago, I wouldn't be so miserable now*. Suppose we interpret this as: if we go back from now to the moment where it was still open for her to leave me or not, then in all the possible futures (from that moment) where she didn't leave me, I wouldn't be so miserable now. (This is much too simple, never mind about that. For an excellent discussion of branching time, see Thomason (1984).) One thing can't be the case: the indexical *now* cannot be interpreted as just the actual present moment, because there I am miserable. So those different possible future have to have their own now. But those nows have to be comparable, because if I utter this at t I do intend my statement to mean: I wouldn't be so miserable at t. In this sense, *now* is indeed an indexical, rigid: it refers to the present moment on different branches. Of course, there is no such thing because if two moments are at different branches, they are not the same moment. So the question arises: can we make sense of the idea that those *nows* are in some sense the same present moment?

And indeed we can (in a tree, that is):

THEOREM. *If $\langle A, \leqslant, \bigcirc\rangle$ is a tree we can construct a linear order out of it that preserves the structure.*

Here is a way of doing it. If $a \in A$ then $\{b: b \leqslant a\}$ is a finite linear order going back to the origin. Let us define the *distance* of a:

$$d(a) = |\{b: b \leqslant a\}| - 1$$

(the -1 is just to give the origin distance 0). d measures how many nodes a is away from the origin.

Now we define an equivalence relation:

$$\approx\ := \lambda a \lambda b. d(a) = d(b)$$

Now look at $\langle A, \leqslant, \bigcirc\rangle_{\approx} = \langle [A]_{\approx}, \leqslant', [\bigcirc]_{\approx}\rangle$

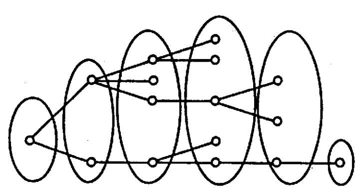
THEOREM. $\leqslant'$ is a linear order.

Reflexivity we get for free. For the other conditions, we need the following lemma:

LEMMA. *$d(a) \leqslant d(b)$ iff $\exists c$: $d(a) = d(c)$ and $c \leqslant b$.*

Exercise 10. Prove the lemma and with the lemma prove the theorem.

We can show in a picture what is going on. In the following tree the equivalence classes under equidistance are indicated:



0 1 2 3 4 5

Clearly this is a partition of the tree. The equivalence classes, so to say, collapse all equidistant points to one point in the new tree. So, even though points may not be comparable in the tree in terms of the partial order, we can always compare them in terms of their distance to the origin.

Another way of formulating this result is that there is a homomorphism from the tree into a begin segment of the natural numbers.

Exercise 11. Let us define a *semitree* as a structure $\langle A, \leqslant\rangle$ where $\leqslant$ is a partial order such that:

1. for all $a \in A$: $\{b: b \leqslant a\}$ is a linear order
2. for all $a, a' \in A$: if $a \leqslant a'$ then $\{b: a \leqslant b \leqslant a'\}$ is finite
3. for all $a, a' \in A$: $\{b: b \leqslant a\} \cap \{b: b \leqslant a'\} \neq \emptyset$.

A semitree differs from a tree in that it doesn't have to have an origin (time doesn't have to have a beginning). You cannot use the above notion of distance to the origin now, because there needn't be one. Define an equivalence relation $\approx$ of equidistance (hint: think about equidistant to what) such that for semitree $\langle A, \leqslant\rangle$, $\langle A, \leqslant\rangle_{\approx}$ is a linear order.

If we drop the finiteness requirement in the definition of tree or semitree we can no longer use our notion of distance.

A *tree-like structure* is a structure $\langle A, \leqslant\rangle$ where $\leqslant$ is a partial order and for all $a \in A$: $\{b: b \leqslant a\}$ is a linear order and for all $a, a' \in A$: $\{b: b \leqslant a\} \cap \{b: b \leqslant a'\} \neq \emptyset$.

Exercise 12. Show that in a tree-like structure (and hence in a tree or in a semi-tree) once the structure branches (to the future) the branches never meet again, that is: if a and a' are incomparable, then $\{b: a \leqslant b\} \cap \{b: a' \leqslant b\} = \emptyset$.

If we still want to measure time (say, in days) we have to play other tricks. One thing we could do is the following. Let $\langle A, \leqslant'\rangle$ be a tree-like structure.

$B \subseteq A$ is *cofinal* in A iff $\forall a \in A\ \exists b \in B$: $a \leqslant b$
B is *co-initial* in A iff $\forall a \in A\ \exists b \in B$: $b \leqslant a$
B is *extended* in A iff B is confinal and co-initial.

Example: The set of even numbers, **E**, is extended in **N**; **N** is cofinal in **Z** (but not co-initial); **Z** is extended in **Q** and **Q** is extended in **R** (all those are of course linear tree-like structures, only **N** and **E** are trees and only **N**, **E** and **Z** are semitrees).

Now we can define: a *distance structure* in $\langle A, \leqslant\rangle$ is a substructure of $\langle A, \leqslant\rangle$ that is extended in A and that is a semi-tree.

Let $\langle B, \leqslant\rangle$ be a distance structure in $\langle A, \leqslant\rangle$ and let $b \in B$. We can define the *period of A covered by b*:

$$p(b) = \{a \in A\colon a \leqslant b \text{ and } \neg\exists b' \in B\colon b' \neq b \text{ and } a \leqslant b\}$$

Intuitively, the period of b covers all the moments in A, later than b's predecessor up to b.

The *period structure* in A relative to distance structure B in A is $\langle\{p(b)\colon b \in B\}, \leqslant'\rangle$ where $\leqslant'$ is defined by:

$$p(b) \leqslant' p(b') \text{ iff } b \leqslant b'$$

It is not hard to see that the period structure is isomorphic with the distance structure.

Given that B is an extended substructure of A and a semi-tree, the period structure is a partition of A. For every $a \in A$ there is a unique $b \in B$ such that $a \in \mathrm{p}(b)$ (the smallest b such that $a \leqslant b$). For $a \in A$, let us use the same notation $\mathrm{p}(a)$ to stand for the period it is in (i.e. the period of the smallest b such that $a \leqslant b$). The equivalence relation corresponding to this partition then is: $\lambda a \lambda a'.\mathrm{p}(a) = \mathrm{p}(a')$.

We know that we can define a notion of equidistance for the distance structure (because it is a semi-tree), so exactly the same notion of equidistance applies to the period structure (because they are isomorphic). Then we can define for a and a' in A: a and a' are equidistant *with respect to distance structure* $\langle B, \leqslant\rangle$ iff $\mathrm{p}(a)$ and $\mathrm{p}(a')$ are equidistant. This is, of course, again an equivalence relation resulting in a linear order.

Summarizing: Given tree-like structure $\langle A, \leqslant\rangle$ and distance structure $\langle B, \leqslant\rangle$ in A, we construct a period structure partitioning $\langle A, \leqslant\rangle$ isomorphic to $\langle B, \leqslant\rangle$, and the latter we can homomorphically embed into a linear order (a segment of **Z**, in fact).

2.3. CHAINS AND LINEAR ORDERS

We will now define some structural notions for partial orders. We will take time as our model, but not assume any more structure at the moment than a partial order.

First some notions concerning minimal elements. Let $\langle T, <\rangle$ be a partial order.

$\langle T, <\rangle$ has a *minimal element* iff $\exists a \forall b\ \neg(b < a)$
$\langle T, <\rangle$ has a *maximal element* iff $\exists a \forall b\ \neg(a < b)$
$\langle T, <\rangle$ is *continuing to the future* iff $\langle T, <\rangle$ has no maximal element
$\langle T, <\rangle$ is *continuing to the past* iff $\langle T, <\rangle$ has no minimal element
$\langle T, <\rangle$ is *continuing* iff $\langle T, <\rangle$ is continuing to the past and to the future
$\langle T, <\rangle$ has an *origin* or *minimum* or *a zero* iff $\exists a \forall b\ a \leqslant b$

If $\langle T, <\rangle$ has a zero it has a unique one (check), we write 0.

$\langle T, <\rangle$ has a *maximum* or *one* iff $\exists a \forall b\ b \leqslant a$

Again, if $\langle T, <\rangle$ has a one it has a unique one, we write 1. Partial orders can have more than one minimal (maximal) element.

Exercise 13. (a) Give an example of a poset with more than one minimal element and more than one maximal element.
(b) Show that if a linear order has a minimal (maximal) element it has a unique one.

If a partial order is finite it has a minimal and a maximal element (not necessarily a unique one, of course). You can show this with induction to the cardinality of finite partial orders. It follows from this that if a partial order is continuing to either the future or the past it has infinitely many elements.

We can define tree-like structures with the following notions: Let $\langle T, <\rangle$ be a partial order.

$\langle T, <\rangle$ is *not branching to the past* iff
$\forall a \forall b \forall b'\ [(b < a \wedge b' < a) \rightarrow (b < b' \vee b' < b \vee b = b')]$
$\langle T, <\rangle$ is *not branching to the future* iff
$\forall a \forall b \forall b'\ [(a < b \wedge a < b') \rightarrow (b < b' \vee b' < b \vee b = b')]$

$\langle T, <\rangle$ is *not branching* iff $\langle T, <\rangle$ is not branching to the past and not branching to the future.

Exercise 14. (a) Show that there are partial orders $\langle T, <\rangle$ that are not branching, but where $<$ still is not a linear order.
(b) Find a first order sentence φ such that:

$\langle T, <\rangle$ is a linear order iff $\langle T, <\rangle$ is not branching and φ holds on $\langle T, <\rangle$.

(c) Give an example of a poset where φ holds, but which is both branching to the future and to the past.
(d) Give a first order definition of a tree-like structure.

So it is not problematic to define tree-like structures in a first order way. As we will see, this is not possible for semi-trees and trees. We can, of course, express in a first order way the difference between a tree and a semi-tree (a tree has an origin, but a semi-tree does not have to have one).

Let us restrict ourselves for the moment to linear orders $\langle T, <\rangle$. We said earlier that $\langle \mathbf{N}, <\rangle$ and $\langle \mathbf{Z}, <\rangle$ are semi-trees but $\langle \mathbf{Q}, <\rangle$ is not. The distinction can be brought out with the following notions:

$\langle T, <\rangle$ is *dense* iff $\forall a \forall a'\ [a < a' \rightarrow \exists b[a < b < a']]$

So density says that between every two points there is a third point. As we have seen before (in the appendix of Chapter One), if there are two points a and a' such that $a < a'$, then the structure is infinite, because there are infinitely many points between a and a'. How would we express that a structure is not only not dense, but is nowhere dense?

$\langle T, <\rangle$ is *discreet* iff
$\forall a[\exists b[a < b] \rightarrow \exists b[a < b \wedge \neg\exists c[a < c \wedge c < b]]]$ and
$\forall a[\exists b[b < a] \rightarrow \exists b[b < a \wedge \neg\exists c[b < c \wedge c < a]]]$

A structure is discreet if every element that has a successor has an *immediate successor* and every element that has a predecessor has an *immediate predecessor*.

Clearly $\langle \mathbf{Q}, <\rangle$ is dense and $\langle \mathbf{N}, <\rangle$ and $\langle \mathbf{Z}, <\rangle$ are discreet.

Exercise 15. For trees and semi-trees, too, we have the intuition that the order is discreet. Given that they are linearly ordered to the past it is not problematic to impose that every element that has a predecessor has an immediate predecessor. But for the (possibly branching) future it is not sufficient to impose that every element that has a successor has an immediate successor.

(a) Why is this insufficient?

(b) Define the notion of discreetness for semi-trees. That is, find a first order sentence φ such that semi-tree $\langle T, <\rangle$ is discreet (in the sense that it is nowhere dense) iff φ hold on $\langle T, <\rangle$.

Our first order theory of trees and semi-trees now has the following ingredients: a semi-tree is a partial order $\langle T, <\rangle$ such that: T is not branching to the past, every two elements in T have a common predecessor (for trees, this is replaced by: T has an origin) and T is discreet.

This is the closest we can come to trees and semi-trees in a first-order way. This is not enough to carve them out, however. To see this, let's go back to $\langle \mathbf{N}, <\rangle$.

$\mathbf{N}$ is a linear tree with origin 0 that is continuing to the future. In fact, it is not hard to see that every linear tree (under the definition of tree we have given before) that is continuing to the future is isomorphic to $\langle \mathbf{N}, <\rangle$.

We know that $\langle \mathbf{N}, <\rangle$ satisfies the following first order axioms:

1. $\mathbf{N}$ is a linear order
2. $\mathbf{N}$ has an origin
3. $\mathbf{N}$ is continuing to the future
4. $\mathbf{N}$ is discreet

Now suppose that $\langle T, <\rangle$ satisfies all these postulates. Can we prove that $\langle T, <\rangle$ is isomorphic with $\langle \mathbf{N}, <\rangle$? The answer is no. Here is an example. Let $\mathbf{N} * \mathbf{Z}$ stand for the structure that consists of $\mathbf{N}$ with a copy of $\mathbf{Z}$ put after it. Compare those two structures:

$$0 \quad 1 \quad 2 \quad 3 \quad \ldots\ldots\ldots$$
$$0 \quad 1 \quad 2 \quad 3 \quad \ldots\ldots\ldots -3' \quad -2' \quad -1' \quad 0' \quad 1' \quad 2' \quad 3' \quad \ldots\ldots$$

Clearly, the two structures are not isomorphic. $\mathbf{N} * \mathbf{Z}$ has a *gap* in the middle that is missing in $\mathbf{N}$. In $\mathbf{N}$ it holds that between every two points there are finitely many elements; this doesn't hold in $\mathbf{N} * \mathbf{Z}$: between 3 and $-3'$ there are infinitely many points. Still $\mathbf{N} * \mathbf{Z}$ satisfies the same first order postulates as does $\mathbf{N}$. It is a linear order with an origin, continuing to the future, and it is discreet: every element has an immediate predecessor and an immediate successor. **N** is a tree, but $\mathbf{N} * \mathbf{Z}$ is not. We could even take infinitely many copies of **Z** behind **N**, and it will still satisfy the same first order conditions (then it has infinitely many gaps). Similarly, though **Z** is a semi-tree, $\mathbf{Z} * \mathbf{Z}$ is not, but it satisfies the same first order conditions as **Z** does. Even the following structure does: Let $\mathbf{Q} @ \mathbf{Z}$ stand for the result of replacing every element of **Q** by a copy of **Z**. So the copies of **Z** are densely ordered with respect to each other. Still the whole structure is linear, continuing and discreet, so it has the same first order theory as **Z**. We cannot express the fact that a structure has a gap in a first order way. We can (of course) express this in a second order way.

Here is an important property: Let $\langle T, <\rangle$ be a partial order.

> $\langle T, <\rangle$ is *well founded* iff every nonempty linearly ordered subset of $\langle T, <\rangle$ has a minimal element.

Suppose $\langle T, <\rangle$ is a discreet linear order with an origin, continuing to the future and well founded. Then $\langle T, <\rangle$ is isomorphic with $\langle \mathbf{N}, <\rangle$.

It is easy to see that **N** is well founded and that $\mathbf{N} * \mathbf{Z}$ is not. The copy of **Z** is a linearly ordered subset of $\mathbf{N} * \mathbf{Z}$, and this does not have a minimal element.

Similarly, we can now define:

> $\langle T, <\rangle$ is a *tree* iff T has an origin, T is not branching to the past, T is discreet and T is well founded.

Again, we cannot define semi-trees by adding well-foundedness to the first order theory of semi-trees.

Exercise 16. Well-foundedness overlaps with discreetness. Let $\langle T, <\rangle$ be a linear order. Let us split discreetness into:

> T is *discreet to the future* iff every element that has a successor has an immediate successor.
> T is *discreet to the past* iff every element that has a predecessor has an immediate predecessor.

I. Let $\langle T, <\rangle$ be a wellfounded linear order.
(a) Show that $\langle T, <\rangle$ has an origin.
(b) Show that $\langle T, <\rangle$ is discreet to the future.

Given this, it follows that if $\langle T, <\rangle$ is a well founded linear order that is continuing to the future and discreet to the past, then $\langle T, <\rangle$ is isomorphic with $\langle \mathbf{N}, <\rangle$.

II. Let $\langle T, <\rangle$ be a well founded linear order that is continuing to the future. Clearly, $\langle \mathbf{N}, <\rangle$ satisfies this, but is every structure that satisfies this isomorphic to $\langle \mathbf{N}, <\rangle$? Give a counterexample.

Let us now introduce some more important second order concepts for partial orders.

> A *chain* in T is a linearly ordered subset of T.

So, a chain in T is a substructure $\langle T', <\rangle$ of $\langle T, <\rangle$, such that $<$ is a linear order on T'.

> chain c in T is a *maximal chain* in T iff for every chain c' in T: if $c \subseteq c'$ then $c = c'$.

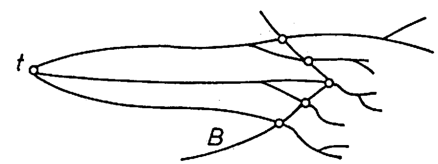
So, a maximal chain cannot be properly extended in T to a chain (it can quite well be extended to another substructure, like T itself, but not to a linear order).

> chain c is a *branch in T* iff c is a maximal chain.

A branch is also called a path.

> Branch (path) b is a *branch (path) through t* iff $t \in b$.
> A *bar for t* (also called barrier for t) is a set B intersecting every branch (path) through t.

(There are many paths you can take from Amsterdam to Rome, but all of them at some point or other cross the Italian border.) Note that $\{t\}$ is a bar for t. Usually in partial orders $\langle T, <\rangle$ we think of bars for t as lying 'in the future' of t:



But we don't put that as a formal requirement on the notion of bar (we don't have to, because we can express the fact that B is a bar for

t in the future of t by saying that $B \neq \{t\}$ and B is a bar for t in $\langle\{t' \in T: t \leqslant t'\}, <\rangle$.

> *A convex set* in $\langle T, \leqslant\rangle$ is a set $X \subseteq T$ such that
> $\forall x, y \in X \forall t \in T$[if $x \leqslant t \leqslant y$ then $t \in X$]

So a convex set is a set that is closed under 'intermediate elements'. In other words, a convex set is a substructure of T that is *uninterrupted*.

Let b be a branch in $\langle T, <\rangle$.

> $X \subseteq b$ is *convex in* b iff $\forall x, y \in X \forall t \in b$ [if $x \leqslant t \leqslant y$ then $t \in X$]
> An *interval of* b is a subset of b that is convex in b.

i is an *interval in* $\langle T, <\rangle$ iff for some branch b in $\langle T, <\rangle$: i is an interval of b.

Exercise 17. Why couldn't we define: an interval in $\langle T, <\rangle$ is a convex chain in $\langle T, <\rangle$?

Let $\langle T, <\rangle$ be a linear order once more. (Since branches are linear orders, the notions to be developed now apply to them.)

> A *cut* in $\langle T, <\rangle$ is a pair $\langle T_1, T_2\rangle$ such that:
>
> 1. $\{T_1, T_2\}$ is a *bipartition* of T
> 2. if $t \in T_1$ and $t' \in T_2$ then $t < t'$

Exercise 18. Show that if $\langle T_1, T_2\rangle$ is a cut in T, then T_1 and T_2 are intervals in T.

Let $\langle T_1, T_2\rangle$ be a cut in $\langle T, <\rangle$. We write just $\langle T_1, <\rangle$ for the restriction of $\langle T, <\rangle$ to T_1.

> $\langle T_1, T_2\rangle$ determines a *jump* iff $\langle T_1, <\rangle$ has a maximal element and $\langle T_2, <\rangle$ has a minimal element.
> $\langle T_1, T_2\rangle$ determines a *gap* iff neither $\langle T_1, <\rangle$ has a maximal element, nor $\langle T_2, <\rangle$ has a minimal element.
> $\langle T_1, T_2\rangle$ determines a *transition* iff either $\langle T_1, <\rangle$ has a maximal element or $\langle T_2, <\rangle$ has a minimal element (but not both).

With these notions we can finish our discussion of trees and semi-trees.

First linear orders. Let $\langle T, <\rangle$ be a linear order.

THEOREM. *$\langle T, <\rangle$ is dense iff no cut in $\langle T, <\rangle$ determines a jump.*

Proof. If $\langle T, <\rangle$ is not dense, then there are t_1, t_2 such that $t_1 < t_2$ but no point lies in the middle. Take $\langle\{t': t' \leqslant t_1\}, \{t'': t_2 \leqslant t''\}\rangle$. It is easy to see that this is a cut, and it determines a jump from t_1 to t_2. Vice versa, suppose that some cut $\langle T_1, T_2\rangle$ determines a jump. Let t_1 be the maximum of T_1 and t_2 the minimum of T_2. No point lies in between t_1 and t_2, because our cut is a bipartition of T, so T is not dense.

THEOREM. *$\langle T, <\rangle$ is discreet iff no cut in $\langle T, <\rangle$ determines a transition.*

Exercise 19. Prove this.

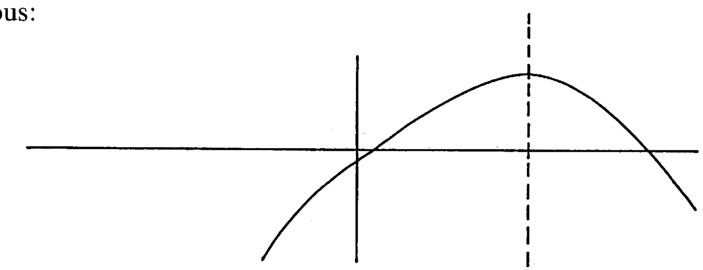
So we know that in **Q** and **R** (because they are dense) we do find cuts that determine transitions. For instance,

$$\langle\{t: t < 0\}, \{t: 0 \leqslant t\}\rangle$$

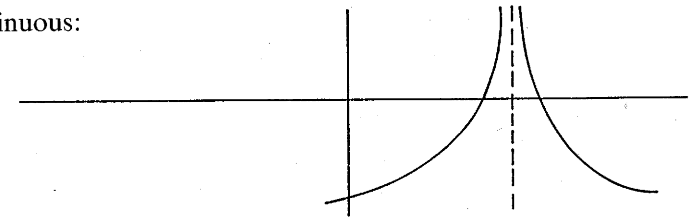
is a cut that determines a transition. Let us talk about gaps now. We have seen that $\mathbf{N} * \mathbf{Z}$ has a gap in the middle. We know how to say that precisely now: $\langle \mathbf{N}, \mathbf{Z}'\rangle$ determines a gap. So a gap is a cut where two sets come infinitely close to each other, but never touch.

Maybe you remember some things about functions from your school mathematics. We call a function continuous, intuitively, if you can keep your pen on the paper while drawing the graph.

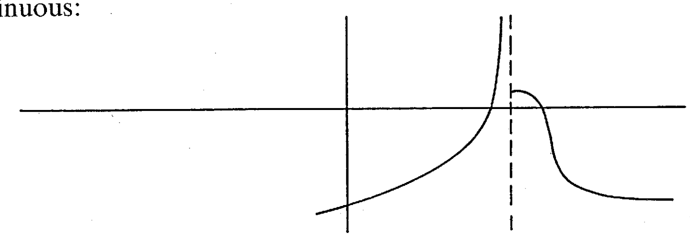
Continuous:



Non-continuous:

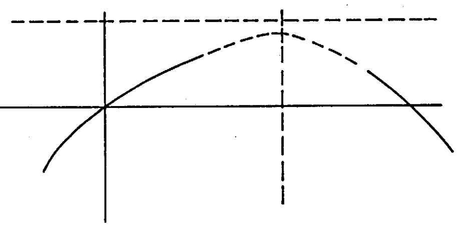


Non-continuous:

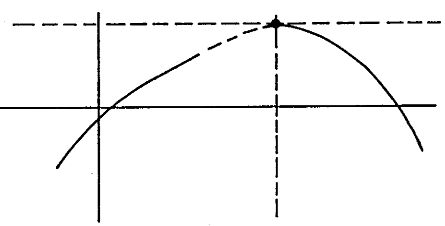


In comparing these three situations, we see that the difference between them is that in the first case both sides of the graph reach a common maximum, while the non-continuous ones don't. This is crucial for the ability to draw the graph in one line. Now if we think of both sides of the graph as approximating a common maximum, we observe a difference between the case where those approximations determine a gap and where they determine a transition. In both cases they may come infinitely close to each other:

Gap



Transition



But in the case of a transition, we have one side approximating a real point, the maximum of the graph, while in the case of a gap, both may come infinitely close to the horizontally dotted line, but there need not be a unique horizontally dotted line that they come infinitely close to. Similarly, you know that they will stay on both sides of the vertical dotted line, but there need not be a unique one. In other words, those dotted lines have a precise meaning in the second graph, we know what they are, but they are an illusion and misleading in the first graph, because there is not a point (the unique intersection of a unique horizontal and a unique vertical line), a common maximum, that both are approximating. For this reason, if the approximations determine a gap, we cannot really draw the graph in one line, because that assumes that the graph has a maximum. So, if there is a gap, the structure is not continuous. We define:

$\langle T, <\rangle$ is *continuous* iff no cut in $\langle T, <\rangle$ determines a gap.

Some facts:

THEOREM. *If* $\langle T, <\rangle$ *is a linear order with an origin, continuing to the future, discreet and continuous then* $\langle T, <\rangle$ *is isomorphic to* $\langle \mathbf{N}, <\rangle$.

THEOREM. *If* $\langle T, <\rangle$ *is a linear order, continuing, discreet and continuous then* $\langle T, <\rangle$ *is isomorphic to* $\langle \mathbf{Z}, <\rangle$.

THEOREM. *If* $\langle T, <\rangle$ *is a linear order, continuing, dense and countable then* $\langle T, <\rangle$ *is isomorphic to* $\langle \mathbf{Q}, <\rangle$.

We proved the latter theorem in the appendix of Chapter One. Note that there is no continuity requirement on $\mathbf{Q}$. For good reason, because $\mathbf{Q}$ is not continuous. This can easily seen by looking at $\mathbf{Q} * \mathbf{Q}'$, $\mathbf{Q}$ with a copy of $\mathbf{Q}$ behind it. By the above theorem, this structure is isomorphic to $\mathbf{Q}$. But $\langle \mathbf{Q}, \mathbf{Q}'\rangle$ determines a gap. Hence, $\mathbf{Q}$ is not continuous.

THEOREM. *If* $\langle T, <\rangle$ *is a linear order, continuing, continuous and: there is a countable proper subset* T' *of* T *such that*:

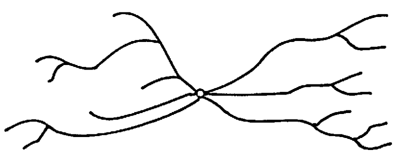
$$\forall t, t' \in T[t < t' \rightarrow \exists t'' \in T'[t < t'' < t']]$$

(*so* T' *is isomorphic to* $\mathbf{Q}$ *and extended in* T) *then* $\langle T, <\rangle$ *is isomorphic with* $\langle \mathbf{R}, <\rangle$.

Given all these results for linear orders, we can use them to define trees and semi-trees.

A *tree* is a partial order $\langle A, <\rangle$ which has an origin, is not branching to the past, such that every branch in A is discreet and continuous.
A *semi-tree* is a partial order $\langle A, <\rangle$ which is not branching to the past, where every two points have a common predecessor, and every branch in A is discreet and continuous.

Exercise 20. You are one of those persons who live totally in the present. You look back at the past and you don't know what happened. You look to the future and you don't know what will happen. Only the present is certain. We can model this relativistic view of time in a diabolo:



Where a tree has as special element an origin, a diabolo has a special element a 'middle'. Define the notions diabolo and diabolo-like structure. (Hint: you may use the notions of tree and tree-like structure.)

Let us come back to chains and branches in partial orders. Let $\langle T, \leqslant\rangle$ be a partial order and let T' be a subset of T.

T' has an *upper bound* in T iff $\exists t \in T\, \forall t' \in T'$: $t' \leqslant t$
T' has a *lower bound* in T iff $\exists t \in T\, \forall t' \in T'$: $t \leqslant t'$
T' has a *lowest upper bound* or *supremum* in T iff $\exists t \in T[t$ is an upper bound for T' and $\forall t' \in T$[if t' is an upper bound for T' then $t \leqslant t']]$
T' has a *greatest lower bound* or *infimum* in T iff $\exists t \in T[t$ is a lower bound for T' and $\forall t' \in T$[if t' is a lower bound for T' then $t' \leqslant t]]$
T' is *bounded* in T iff T' has both an infimum and a supremum.

(*Remark:* the term interval is normally used for bounded convex sets. Note that we use a different terminology here.)

T' is *lower closed* iff T' has an infimum t and $t \in T'$
T' is *upper closed* iff T' has a supremum t and $t \in T'$
T' is *closed* iff T' is lower and upper closed, else *open*.

It is easy to see that if T' has a supremum (infimum) it has a unique one. Note that it is not part of the definition of supremum (infimum) that the supremum (infimum) is an element of T'.
Look at OR:

$$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \ldots\ldots \omega \quad \omega+1 \quad \omega+2 \quad \ldots$$

N is a subset of this. It has a supremum in OR, namely ω, but $\omega \notin \mathbf{N}$. It also has an infimum, namely 0. This time $0 \in \mathbf{N}$. We see that if T' has a unique minimal element, this minimal element is the infimum of T' in T; similarly if T' has a unique maximal element this is the supremum of T' in T. And also if T' has an infimum (supremum) t and $t \in T'$ then t is the minimal (maximal) element of T'.
These notions are defined for arbitrary subsets of T and hence they apply to chains, branches, intervals, convex sets.
A principal that is often used in proofs about partial orders is *Zorn's Lemma*. The name is misleading, because it is not a lemma, not something that is proved. In fact, you can prove that Zorn's Lemma is one of the disguises of the *axiom of choice*: Zorn's Lemma is equivalent to the axiom of choice (so it is a theorem in ZF + AC, but so is AC in ZF + Zorn's lemma). If you prove something with Zorn's Lemma, you have to mention that you used it.
Let $\langle T, \leqslant\rangle$ be a partial order.

ZORN'S LEMMA. *If every chain in T has a supremum, then T has a maximal element.*

We have already proved that if T is finite Zorn's Lemma holds without problems, because every finite partial order has a maximal element anyway.
It is also easy to see that Zorn's Lemma already holds for linear orders. If every chain in linear order $\langle T, <\rangle$ has a supremum in T, then T itself has a supremum in T (because T is chain in T), hence this supremum is the maximum of T.

Zorn's Lemma is a very useful, but also very non-constructive principle. The reason is that it postulates the existence of a maximal element without giving you any way of finding it. Its non-constructive nature becomes more clear in a principle that is equivalent to it, the maximal chain principle:

MAXIMAL CHAIN PRINCIPLE. *Every chain in* $\langle T, \leq \rangle$ *can be extended to a maximal chain.*

The principle tells you that you can extend a chain, it doesn't tell you how.

Let us prove that Zorn's Lemma and the Maximal Chain Principle are equivalent.

Zorn's Lemma implies the Maximal Chain Principle. Take any chain c in $\langle T, \leq \rangle$ and let C be the set of all chains *extending* c, $C = \{c' \colon c \subseteq c'\}$. Look at $\langle C, \subseteq \rangle$. This structure is of course a partial order. Now take any chain *in this set* (such a chain is a set of increasing chains in T extending c), say C_0 and look at $\cup C_0$. $\cup C_0 \in C$, i.e. $\cup C_0$ is a chain in T. Namely, all the elements of $\cup C_0$ are obviously elements of T. Suppose $t, t' \in \cup C_0$. By definition of $\cup C_0$ we know that t and t' are in $\cup C_0$ if for some $X \in C_0$ and some $Y \in C_0$: $t \in X$ and $t' \in Y$. Because C_0 is increasing we know that $X \subseteq Y$ or $Y \subseteq X$, say, $X \subseteq Y$. Then we know that $t, t' \in Y$. But Y is a chain in T, so $t \leq t'$ or $t' \leq t$. So indeed $\cup C_0$ is a chain in T.

So, we have proved up to now that for any chain $C_0 \in C$: $\cup C_0 \in C$. But $\cup C_0$ is the supremum of C_0 in C (for all $c \in C_0$: $c \subseteq \cup C_0$ and for any $X \in C$ if for all $c \in C_0$: $c \in X$ then $\cup C_0 \subseteq X$). Now Zorn's lemma tells us that C has a maximal element (again, it postulates that, it doesn't tell us how to construct it). In other words, the partial order of all chains extending our original c has a maximal element. So there is a $c_m \in C$ such that $\forall c' \in C$: if $c_m \subseteq c'$ then $c_m = c'$. This of course means that there is a c_m such that $c \subseteq c_m$ and $\forall c'$: if $c_m \subseteq c'$ then $c_m = c'$. So indeed every chain can be extended to a maximal chain. (c was chosen arbitrarily, and we showed that there is a c_m extending it.)

The Maximal Chain Principle implies Zorn's Lemma. Assume that every chain in $\langle T, \leq \rangle$ can be extended to a maximal chain and assume that every chain in $\langle T, \leq \rangle$ has a supremum. Take any chain in $\langle T, \leq \rangle$. It can be extended to a maximal chain, say c_m. c_m is a chain, so it has a supremum, call it t. Since c_m is a maximal chain, it follows that $t \in c_m$. But that means that t is a maximal element in $\langle T, \leq \rangle$. So indeed the Maximal Chain Principle is equivalent to Zorn's Lemma and hence to the Axiom of Choice.

Note the parallel between the Maximal Chain Principle and Lindenbaum's Lemma for propositional logic: every consistent set of sentences can be extended to a maximal consistent set of sentences. We can prove Lindenbaum's Lemma directly with the Maximal Chain Principle, but we prefer the original proof because the construction we used tells us for every enumeration of formulas of propositional logic and consistent set Δ what the maximally consistent extension of Δ is.

Still, the maximal chain principle plays an important role, especially in theories of partial information, vagueness and unspecificity, where we want to define something as partial, vague, unspecific if there are still different ways of making it total, precise, specific. Total, precise, specific entities are then regarded as the maximal elements of a partial order of information growth, precisification, specification. Technically, it plays an important role in the theory of distributive lattices and Boolean Algebras. We'll see that later.

The last topic to be covered in this section is representation theory. There is a natural way of constructing posets in set theory. Take a set A. Form the power set of A and order this by subset. The result is a partial order. Let us put this in a definition:

Given set A.

> The *set theoretic poset based on* A is $\langle \text{pow } A, \subseteq \rangle$.
> $\langle X, \subseteq \rangle$ is a *set theoretic poset* iff for some set A, $\langle X, \subseteq \rangle$ is the set theoretic poset based on A.

Set theoretic posets are interesting, because it is easy to construct them and to determine what their order is. Also, they have a very regular order. Here, for instance, is one fact:

FACT. *If X and Y have the same cardinality then the set theoretic poset based on X and the set theoretic poset based on Y are isomorphic.*

Proof. Let f be a bijection between X and Y. We define a bijection f' between pow X and pow Y as follows: Let $X' \in \text{pow } X$.

$$f'(X') = \{f(x) \colon x \in X'\}$$

Clearly f' is a bijection.

It is easy to check that:

$$X' \subseteq X'' \text{ iff } \{f(x)\colon x \in X'\} \subseteq \{f(x)\colon x \in X''\}$$

So indeed f is an isomorphism.

So if two set theoretic posets have the same cardinality, they have the same structure. In other words, up to isomorphism, there is exactly one set theoretic poset for every cardinality of the base sets.

Exercise 21. Given set $A = \{a, b\}$.

(a) Draw the graph of the set theoretic poset based on A.
(b) Draw the graph of the set theoretic poset based on pow A.
(c) An application to Generalized Quantifier Theory. Let

$[\![Man]\!] = \{a\}$; $[\![woman]\!] = \{b\}$;
$[\![every\ X]\!] = \{Z \subseteq A\colon X \subseteq Z\}$, and
$[\![no\ X]\!] = \{Z \subseteq A\colon X \cap Z = \emptyset\}$,
$[\![some\ X]\!] = \{Z \subseteq A\colon X \cap Z \neq \emptyset\}$,
$[\![not\ every\ X]\!] = \{Z \subseteq A\colon X \not\subseteq Z\}$.

Indicate in the graph of (b) which sets correspond to:

Everyone, no-one, someone, not every one,
Every man, no man, some man, not every man,
Every woman, no woman, some woman, not every woman

(d) This time, let $A = \{a, b, c, d\}$,
Draw the set theoretic poset based on A.
(e) Let $[\![Woman]\!] = \{c\}$. What is $[\![no\ woman]\!]$?
(f) Let $[\![walk\ fast]\!] = \{b\}$; $[\![skate]\!] = \{a\}$; $[\![walk]\!] = \{b, d\}$ and $[\![move]\!] = \{a, b, d\}$. Check in (d) that all the following sentences are true:

No woman moves; *No woman walks*; *No woman skates*; *No woman walks fast.*

In fact, we have strong intuitions about which of these sentences imply the others. Formulate the pattern as a condition on the quantifier *no woman*. A quantifier showing this pattern is called *downward entailing*. Explain the name with help of the graphs and give another example of a downward entailing quantifier (of the form Determiner Noun).

The questions that representation theory deals with are: first, are there for structures of the type in question such natural examples that are constructed in such a regular way? Secondly, what is the relation between structures of this type in general and these natural examples. Is it, for instance, the case that all structures of this type can be (up to isomorphism) constructed in this way (that would give you an enormous amount of information about those structures: construct the natural example of the right cardinality and you know what any structure of this type of that cardinality looks like). Is it the case that every structures of this type can be embedded in a natural example? Then you would know that the natural examples are the most general structures of this type. Such questions and their answers are interesting for relational structures; as we will see later, they are extremely important for algebraic structures.

So let us ask what the relation is between posets in general and set theoretic posets. Some definitions:

> $\langle A, \leqslant\rangle$ *can be represented as* a set theoretic poset iff $\langle A, \leqslant\rangle$ is isomorphic with a set theoretic poset.
> $\langle A, \leqslant\rangle$ *can be represented in* a set theoretic poset iff $\langle A, \leqslant\rangle$ can be embedded in a set theoretic poset (i.e. there is an isomorphism between $\langle A, \leqslant\rangle$ and a sub-structure of some set theoretic poset.

It is easy to see that not every poset can be represented as a set theoretic poset. We only have to make a simple cardinality observation for that. Since set theoretic posets are power sets, their cardinality is a power of 2 (remember, if $|X| = \alpha$ then $|\text{pow}\,X| = 2^{\alpha}$). Take a poset whose cardinality is not a power of 2 (say, a poset with three elements), there is no set theoretic poset that is isomorphic to it. But we do have:

REPRESENTATION THEOREM FOR PARTIAL ORDERS. *Every poset can be represented in a set theoretic poset.*

Proof. We can in fact prove something stronger: every poset $\langle A, \leqslant\rangle$ can be represented in $\langle \text{pow}A, \leqslant\rangle$. Because the construction we use will come back over and over, I will give it the name that it only properly deserves when we look at lattices: Let $a \in A$.

The *ideal generated by a*, $(a] = \{b\colon b \leqslant a\}$.

Now look at the function $f\colon A \to \text{pow}\,A$ such that $\forall a\colon f(a) = (a]$.

1. f is an injection. Suppose $a \neq b$. Then either $a \nleqslant b$ or $b \nleqslant a$, say,

$a \nleq b$. Then $a \notin (b]$. Since $a \in (a]$ it follows that $(a] \neq (b]$. So indeed f is an injection.

2. If $a \leq b$ then $(a] \subseteq (b]$. This is obvious. If $(a] \subseteq (b]$ then $\{c: c \leq a\} \subseteq \{c: c \leq b\}$, hence $a \in (b]$, thus $a \leq b$.

So f is a homomorphism, in fact, an isomorphism between $\langle A, \leq\rangle$ and $\langle f(A), \subseteq\rangle$. Hence, indeed f is an embedding of $\langle a, \leq\rangle$ in $\langle \text{pow } A, \subseteq\rangle$.

Exercise 22. Draw the graph of the set theoretic poset with eight elements. Show that every poset with three elements can be embedded into it.

So set theoretic posets are the most general posets. There are no posets that have such a complicated structure that you can't find a set theoretic poset in which you cannot carve it out. Another way of stating the result is: you can get any poset, by starting with a set theoretic poset and leaving out lines and/or element with their connecting lines.

SEMANTICS WITH PARTIAL ORDERS

3.1. INSTANT TENSE LOGIC

I will start with Priorian tense logic. (For more thorough discussion: see Prior, 1967; van Benthem, 1983; and Burgess, 1984.) Priorian tense logic is based on three assumptions. The first is that tenses (like the past tense) are sentential operators. The second is that tenses are (implicitly) quantifiers over times. The third is that times are instants.

All these assumptions have been challenged. For instance, Bach (1979) treats tenses as VP operators, Enç (1981) as V operators. Partee (1973) and Partee (1984) discuss the deictic and anaphoric rather than quantificational uses of the past tense. Many analyses of tense adopt some version of Reichenbach's (1947) notion of reference time, which is lacking in Priorian tense logic. Finally, there is interval semantics (and event semantics), which we will discuss in a later chapter.

A discussion of these various alternatives falls outside the scope of this book.

Let me say here that, if I think that it is still worthwhile talking about Priorian tense logic, it is not because I think that the criticism that led to the alternatives is not well founded. On the contrary, I think that in particular Prior's identification of the past tense with an existential quantifier over times is misguided, and may even have had some negative influence on the development of semantic thinking about tense. I will be interested in something else here, though.

It will be part of any semantics for temporal expressions to determine what temporal property a given temporal expression expresses. I.e., whatever your framework, if you give a semantics for, say, *until*, you will have to specify what exactly the temporal relation is that is expressed by *until*.

Priorian tense logic, as a simple example of a temporal language, is very well suited to a study of the questions of how to express such temporal properties in a temporal language and what temporal properties can be expressed. So we will use Priorian tense logic as a simple example to get a grip on these problems.