CHAPTER SIX

LATTICES

For a very thorough and systematic textbook in lattice theory, with lots of good exercises (some of which I have used here), see Grätzer (1978b).

6.1. BASIC CONCEPTS

Let $\langle A, \leqslant \rangle$ be a partial order.

For a set $X \subseteq A$, we have defined the notions of supremum and infimum before:

a is a *lower bound* for X if $\forall x \in X$: $a \leqslant x$

Let $LB(X)$ be the set of all lower bounds for X:

a is the *infimum* of X iff $a \in LB(X)$ and

$\forall b \in LB(x)$: $b \leqslant a$

The notion of supremum is defined similarly. We write the infimum of X as $\bigwedge X$ and the supremum of X as $\bigvee X$.

Let $a, b \in A$.

the *meet* of a and b, $a \wedge b := \bigwedge\{a, b\}$
the *join* of a and b, $a \vee b := \bigvee\{a, b\}$

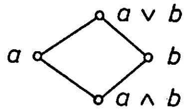
Hence, meets are infimums of two element sets and joins are supremums of two element sets: i.e.

$a \wedge b$ is the greatest element of A such that

$a \wedge b \leqslant a$ and $a \wedge b \leqslant b$

$a \vee b$ is the smallest element of A such that

$a \leqslant a \vee b$ and $b \leqslant a \vee b$



A *lattice* is a partial order $A = \langle A, \leqslant \rangle$ which is closed under meet and join: i.e.:

$$\forall a \forall b \in A: a \wedge b \in A \text{ and } a \vee b \in A$$

THEOREM. *A is a lattice iff for any non-empty finite subset X of A:*

$$\bigwedge X \in A \text{ and } \bigvee X \in A.$$

Basically what you can prove is that for any set $\{a_1, \ldots, a_n\}$,

$$(\ldots((a_1 \wedge a_2) \wedge a_3) \wedge \ldots) \wedge a_n = \bigwedge\{a_1, \ldots, a_n\},$$

and the same for $\vee$. I won't prove it here (because it will follow easily from the algebraic properties of $\wedge$ and $\vee$ that we will see in a moment).

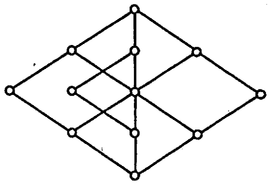
THEOREM. *Duality for lattices. If $\langle A, \leqslant \rangle$ is a lattice, then its dual $\langle A, \geqslant \rangle$ is a lattice as well.*

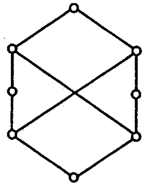
So the duality principle applies to lattices. This is nice, because it saves us proofs: if we prove a statement for $\wedge$, we automatically have proved the dual statement for $\vee$.

A lattice $\langle A, \leqslant \rangle$ *is complete* iff for any (not just finite) $X \subseteq A$:

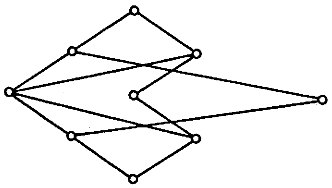
$$\bigwedge X \in A \text{ and } \bigvee X \in A$$

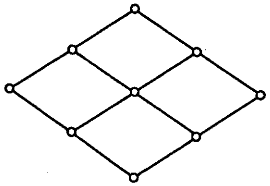
Exercise 1. (a) Are the following posets lattices:





Show that the following diagrams represent the same lattice:





Prove the following:

(b) Let $\langle A, \leqslant\rangle$ be a partial order such that for all $X \subseteq A$: $\bigwedge X \in A$. Then $\langle A, \leqslant\rangle$ is a complete lattice. (Hint: what is $\bigwedge\emptyset$? And what requirement does $\bigwedge\emptyset \in A$ put on the lattice?)

(c) Let $\langle A, \leqslant\rangle$ be a complete lattice. Then $\langle A, \leqslant\rangle$ has a maximum and a minimum.

(d) If a lattice has a maximal element, this maximal element is a maximum (1).

(e) Every finite lattice is complete.

We have given a definition of lattices as partial orders (relational structures); I will now give a definition of lattices as algebras:

A *lattice* is an algebra $\langle A, \wedge, \vee\rangle$, where $\wedge$ and $\vee$ are two place operations satisfying:

1.	*idempotency*:	$(a \wedge a) = a$
		$(a \vee a) = a$
2.	*commutativity*:	$(a \wedge b) = (b \wedge a)$
		$(a \vee b) = (b \vee a)$
3.	*associativity*:	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$
		$(a \vee b) \vee c = a \vee (b \vee c)$
4.	*absorption*:	$a \wedge (a \vee b) = a$
		$a \vee (a \wedge b) = a$

We have two notions of lattices now. We will prove that the two notions coincide completely.

THEOREM. *The two concepts of lattices coincide.*

(a) *If $\langle A, \leqslant\rangle$ is a lattice then $\langle A, \wedge, \vee\rangle$ is a lattice, where $\wedge$ and $\vee$ are meet and join.*

(b) *If $\langle A, \wedge, \vee\rangle$ is a lattice then $\langle A, \leqslant\rangle$ is a lattice, where $\leqslant$ is defined as: $a \leqslant b$ iff $a \wedge b = a$.*

(c) *Let $\langle A,\leqslant\rangle$ be a lattice, if we transform $\langle A, \leqslant\rangle$ into an algebra as under* (a) *and we transform the resulting algebra into a poset as under* (b) *we get $\langle A, \leqslant\rangle$ back. The same result for lattice $\langle A, \wedge, \vee\rangle$.*

Proof. (a) Let $\langle A, \leqslant\rangle$ be a lattice.

1. Idempotency and commutativity: these obviously hold for meets and joins: the greatest element smaller than or equal to a and a is obviously a; the greatest element smaller than or equal to a and b is obviously the greatest element smaller than or equal to b and a.

2. Associativity: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.
$a \wedge (b \wedge c)$ is the greatest element below a and $b \wedge c$. Since $b \wedge c \leqslant b$ and $b \wedge c \leqslant c$, it follows with transitivity that $a \wedge (b \wedge c) \leqslant a$ and $a \wedge (b \wedge c) \leqslant b$ and $a \wedge (b \wedge c) \leqslant c$. Now for any z: if $z \leqslant x$ and $z \leqslant y$ then $z \leqslant x \wedge y$, by definition of meet. Applying this observation twice, we get: $a \wedge (b \wedge c) \leqslant (a \wedge b)$ and hence $a \wedge (b \wedge c) \leqslant (a \wedge b) \wedge c$.

The same argument shows, of course, that $(a \wedge b) \wedge c \leqslant a \wedge (b \wedge c)$. Hence, by antisymmetry, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$. We can make a similar argument for $\vee$.

3. Absorption: $a \wedge (a \vee b) = a$.
By definition of $\wedge$, $a \wedge (a \vee b) \leqslant a$. Now, $a \leqslant a$ and $a \leqslant a \vee b$, hence, again by definition of $\wedge$, $a \leqslant a \wedge (a \vee b)$. So indeed absorption holds. The other one is done in the same way. This proves (a).

(b) Let us start by proving some helpful lemmas:

LEMMA 1. *$a \wedge b = a$ iff $a \vee b = b$.*

Proof. 1. Suppose $a \wedge b = a$.

2.	$b \vee (b \wedge a) = b$	(absorption)
3.	$(b \wedge a) \vee b = b$	(commutativity)
4.	$(a \wedge b) \vee b = b$	(commutativity, use of this will be suppressed from now on)

5. $a \vee b = b$ (substitute a for $(a \wedge b)$ in 4, using 1)

The other side goes the same, with the other absorption law.

LEMMA 2. *Given the definition of* $\leqslant$ *as* $a \wedge b = a$:

(1) $z \leqslant a \wedge b$ *iff* $z \leqslant a$ *and* $z \leqslant b$

(2) $a \vee b \leqslant z$ *iff* $a \leqslant z$ *and* $b \leqslant z$.

Proof. (2.1) Suppose $z \leqslant a$ and $z \leqslant b$. Then $z \wedge a = z$ and $z \wedge b = z$. $z = z \wedge z$, hence $z = (z \wedge a) \wedge (z \wedge b) = (z \wedge z) \wedge (a \wedge b) = z \wedge (a \wedge b)$. This means that $z \leqslant a \wedge b$.

Suppose $z \leqslant a \wedge b$. That means

$$z \vee (a \wedge b) = a \wedge b.$$

This means

$$(z \vee (a \wedge b)) \vee a = (a \wedge b) \vee a$$

i.e.

$$z \vee ((a \wedge b) \vee a) = a$$

i.e.

$$z \vee a = a$$

Hence by Lemma 1, $z \leqslant a$. That $z \leqslant b$ is proved in the same way.

(2.2) This is done in a mirror argument.

Now we have to prove that $\leqslant$ is a partial order and that $\wedge$ and $\vee$ are meet and join in $\leqslant$.

1. $\leqslant$ is reflexive: $a \leqslant a$. Of course, $a \wedge a = a$ (idempotency).
2. $\leqslant$ is transitive. Suppose $a \leqslant b$ and $b \leqslant c$. That is, $a \wedge b = a$ and $b \wedge c = b$

$$\begin{aligned}(a \wedge b) \wedge c &= a \wedge (b \wedge c)\\ a \wedge c &= a \wedge b\\ a \wedge c &= a\end{aligned}$$

Hence, $a \leqslant c$

3. $\leqslant$ is antisymmetric. Suppose $a \leqslant b$ and $b \leqslant a$: i.e. $a \wedge b = a$ and $b \wedge a = b$. Then clearly $a = b$.
4. $\wedge$ is meet.

(a) We have to prove: $a \wedge b \leqslant a$ and $a \wedge b \leqslant b$. This means $(a \wedge b) \vee a = a$ and $(a \wedge b) \vee b = b$, i.e. absorption.

(b) We have to prove: if $z \leqslant a$ and $z \leqslant b$ then $z \leqslant a \wedge b$. We have just proved this above. The proof that $\vee$ is join goes in the same way. This completes the proof of (b).

(c) Given lattice $\langle A, \leqslant \rangle$, $\wedge$ and $\vee$ are meet and join in this lattice. In this partial order we know that $a \leqslant b$ iff $a \vee b = a$ (by definition of meet). So if we form the algebra with $\wedge$ and $\vee$, and then define $a \leqslant' b$ as $a \wedge b = a$, it follows that $\leqslant = \leqslant'$. A similar argument shows the other part. This completes the proof of the theorem.

This result is very important. We know now that we can regard a lattice as an algebra.

If we check the algebraic definition of a lattice, we see that the concept of lattice is defined with identity statements, or equations. This is standard in algebra: types of algebras are defined in terms of identity statements.

Let I be a set of identity statements (in a language with functional expressions $f_1, \ldots, f_n$). The class of all algebras (with operations $f_1, \ldots, f_n$) satisisfying every identity statement in I is called an *equational class*. In other words, the equational class of I is the class of all models for I.

Algebraic structures are generally defined through identities, and hence determine equational classes.

Now an important observation is the following. We have seen in Chapter Two that homomorphisms do not in general preserve partial orders (i.e. if we have a homomorphism from a partial order onto another structure, it is not necessarily the case that the other structure will be a partial order as well). However, as we showed in Chapter Two, homomorphisms preserve positive sentences, in particular, they preserve identity statements (i.e. statements of the form $\forall x \forall y\, i = j$). This means that if we have a homomorphism from an algebra in some equational class onto some other algebra, this other algebra will be in the same equational class.

So, if we have a homomorphism from a lattice onto some other structure, that other structure will be a lattice as well, and similarly if we have a homomorphism from a particular type of lattice (say, a Boolean lattice) onto some other lattice, the other lattice will be of that same type. We see then why homomorphisms are so important in algebra: they are indeed structure preserving in a strong sense.

With the correspondence we have just proved, we now know that even if we would be more interested in lattices as partial orders, we can use the algebraic results: for instance, if we have a homomorphism from one poset with meets and joins onto another structure, we do not have to check whether the other structure will be a poset and whether it will have meets and joins, because that follows from the algebraic correspondence.

Not every lattice has a 0 and a 1, not every lattice is complete. Here are a relational and an algebraic definition of a bounded lattice:

A *bounded lattice* is a lattice $\langle A, \leqslant \rangle$ which has both a 0 and a 1.

A *bounded lattice* is a structure $\langle A, \wedge, \vee, 0, 1 \rangle$, where:

1. $\langle A, \wedge, \vee \rangle$ is a lattice.
2. *Laws of 0 and 1:* $a \wedge 0 = 0$
   $a \wedge 1 = a$

It is not hard to see that the two concepts of bounded lattice coincide (remember, $a \wedge 0 = 0$ iff $a \vee 0 = a$ iff $0 \leqslant a$).

We know that every complete lattice is bounded and also that finite lattices are both complete and bounded. So we will have to look at infinite lattices to find examples that are not complete or not bounded. Here is an example of a lattice that is not bounded:

Look at $\langle \mathbf{Z}, \wedge, \vee \rangle$, where:

$$n \wedge m := \min(n, m)$$
$$n \vee m := \max(n, m)$$

You can check that this is a lattice, but it doesn't have a 0 nor a 1. Had we taken $\langle \mathbf{N}, \wedge, \vee \rangle$ then the lattice would have had a 0 but not a 1.

We have introduced the notion of a set theoretic poset earlier. In fact, powersets are lattices, so we define:

The *set theoretic lattice* or *powerset lattice* based on A is

$$\langle \text{pow} A, \cap, \cup \rangle.$$

A *set theoretic lattice* is a lattice that is the powerset lattice of some set.

It is, of course, trivial to check that the powerset lattice is indeed a lattice. Moreover:

Every powerset lattice is complete.

Exercise 2. Prove this.

The above lattice $\langle \mathbf{Z}, \wedge, \vee \rangle$ is not complete (why?) For another incomplete lattice, look at the following.

We know that $\langle \text{pow}\, \mathbf{N}, \cap, \cup \rangle$ is complete. Now look at:

$$B = \{X \subseteq \mathbf{N}\colon X \text{ is finite}\} \text{ and } \langle B, \cap, \cup \rangle.$$

It is not hard to check that $\langle B, \cap, \cup \rangle$ is a sublattice of $\langle \text{pow}\, \mathbf{N}, \cap, \cup \rangle$ (the union/intersection of two finite sets is finite). B has a 0 ($\emptyset$), but B does not have a 1. B is not complete. For instance, take the following subset B' of B:

$$B' = \{\{0\}, \{1\}, \{2\}, \ldots\}.$$

$\bigcup B' = \mathbf{N}$, and, of course, $\mathbf{N} \notin B$.

Let $\langle A, \leqslant \rangle$ be a lattice. $a \in A$ is an *atom* in $\langle A, \leqslant \rangle$ iff $a \neq 0$ and $\neg \exists b \in A\colon 0 < b < a$.

Lattice $\langle A, \leqslant \rangle$ is *atomic* iff

$$\forall a \in A\ [a \neq 0 \rightarrow \exists b \in A\colon b \leqslant a \text{ and } b \text{ is an atom}]$$

So, an atom is an element that is minimally greater than 0 (elements minimally smaller than 1 are called *dual atoms*). A lattice L is atomic if every maximal chain in $L - \{0\}$ has a minimal element (no chains infinitely descending to 0). Some properties:

– *Every powerset lattice is atomic.*

Exercise 3. What are the atoms?

Note that lattices may have atoms without being atomic, a lattice without atoms is called *atomless*.

– It should be clear that if a is an atom then for all $b \in A$: either $a \wedge b = 0$ or $a \wedge b = a$ (not both) and that if a and b are both atoms, then $a \wedge b = 0$.

– *Every finite lattice is atomic*. (Obviously, every maximal chain that doesn't contain 0 has a minimal element, because every chain is finite.)

Summarizing, we see that finite lattices and powerset lattices are examples of complete atomic lattices.

Exercise 4. In Chapter Four I talked about the interval lattice based on a dense linear order: the set of all convex subsets with intersection and convex closure. This interval lattice is a complete atomic lattice. How could you turn this lattice into an atomless complete lattice? And into a complete lattice that is neither atomic, nor atomless?

A *join semilattice* is a poset $\langle A, \leqslant\rangle$ such that

$$\forall a, b \in A\colon a \vee b \in A$$

Dually, a *meet semilattice* is a poset $\langle A, \leqslant\rangle$ such that

$$\forall a, b \in A\colon a \wedge b \in A$$

Obviously a poset is a lattice iff it is both a join semilattice and a meet semilattice. Algebraically we have the following:

A *semilattice* is an algebra $\langle A, *\rangle$ where:

$*$ is idempotent, commutative and associative.

It should be clear that if $\langle A, \leqslant\rangle$ is a lattice, then $\langle A, \vee\rangle$ is a semilattice and $\langle A, \wedge\rangle$ is a semilattice. Similarly, that if $\langle A, \leqslant\rangle$ is a join semilattice (meet semilattice), then $\langle A, \vee\rangle(\langle A, \wedge\rangle)$ is a semilattice.

Exercise 5. (A) Let $\langle A,*\rangle$ be a semilattice. Define $a \leqslant_\wedge b := (a = a * b)$ and $a \leqslant_\vee b := (b = a * b)$. Prove that $\langle A, \leqslant_\wedge\rangle$ is a meet semilattice in which $a \wedge b = a * b$ and $\langle A, \leqslant_\vee\rangle$ is a join semilattice in which $a \vee b = a * b$.

(B) Prove that if $\langle a, \leqslant\rangle$ is join semilattice and we turn it in the above way into a semilattice $\langle A, \vee\rangle$, and this one we turn again into a join semilattice, we get $\langle A,\leqslant\rangle$ back (the same holds, of course, for meet semilattices). If we turn semilattice $\langle A, *\rangle$ into a join semilattice in the above way, and we turn that back again into a semilattice we get $\langle A, *\rangle$ back (again, the same for meet semilattices).

(C) This shows, that also for semilattices the partial order perspective and the algebraic perspective coincide. What about the following putative argument: we know that $\langle A, \leqslant\rangle$ is a lattice iff $\langle A, \leqslant\rangle$ is both a join semilattice and a meet semilattice. Now let $\langle A,\leqslant\rangle$ be a join semilattice. Then we know that its corresponding algebra is a semilattice. If this is a semilattice, then we know that its corresponding partial order is a meet semilattice. The corresponding poset is $\langle A, \leqslant\rangle$, hence we know that $\langle A, \leqslant\rangle$ is both a join semilattice and a meet semilattice. Hence, $\langle A, \leqslant\rangle$ is a lattice.

What is wrong with this argument

Let $\langle A, \leqslant\rangle$ be a join semilattice.

$\langle A, \leqslant\rangle$ is *complete* iff $\forall B \subseteq A\colon \bigvee B \in A$.

Given what we have proved earlier (we proved the dual in Exercise 1.b.), complete join semilattices are just complete lattices. However, we will be interested later in join semilattices that are closed under arbitrary join, without being lattices (in particular without having a minimal element). For this we use a slightly modified notion of completeness:

Let $\langle A, \leqslant\rangle$ be a join semilattice.

$\langle A, \leqslant\rangle$ is *complete** iff for every *non-empty* $B \subseteq A\colon \bigvee B \in A$.

A complete* join semilattice is not automatically a complete lattice, it need not have a minimal element (why?). On the other hand, a complete* join semilattice will have a maximal element (why?).
Examples of complete* join semilattices that are not lattices:

Take again the interval lattice based on a dense linear order. Now cut out the minimal element $\emptyset$. The result is a complete* join semilattice, but, of course, no longer a lattice, because it is not closed under meet any longer.

Another example: take the same structure and cut out the empty period and the singleton periods: the result is again a complete* join semilattice, this time, without minimal elements.

Yet another (but related) example: take a powerset algebra and cut out the empty set.

Exercise 6. Let $\langle A, \leqslant\rangle$ be a complete* atomic join semilattice. Prove that $\langle A, \leqslant\rangle$ is a complete atomic lattice.

This exercise shows that for join semilattices that are not lattices, we also need another notion of atomicity, if we want to characterize the difference between the two interval examples given above.

Let $\langle A, \leqslant \rangle$ be a join semilattice.

$a \in A$ is an *atom** in A iff a is a minimal element in $A - \{0\}$.

The difference between this and the definition of atom is that you can only be an atom in A if A has a 0. On the present definition, if A has a 0, then the atom*s are the atoms, but if A doesn't have a 0, the atom*s are the minimal elements.

$\langle A, \leqslant \rangle$ is *atomic** iff $\forall b \in A$: $b \neq 0 \rightarrow \exists a \in A$: $a \leqslant b$ and a is an atom*.

It is not the case that if $\langle A, \leqslant \rangle$ is a complete* atomic* join semilattice that $\langle A, \leqslant \rangle$ is a complete atomic lattice. The interval structure without the empty period is a complete* atomic* join semilattice, but not a lattice; the interval structure without the empty and singleton periods is a complete* atom*less join semilattice; the powerset without the empty set is a complete* atomic* join semilattice.

Let us turn to distributive lattices. In any lattice the following hold:

1. $a \vee (b \wedge c) \leqslant (a \vee b) \wedge (a \vee c)$

2. $(a \wedge b) \vee (a \wedge c) \leqslant a \wedge (b \vee c)$

Proof. 1. (a) $a \leqslant (a \vee b) \wedge (a \vee c)$. This follows from Lemma 2, discussed earlier.

(b) $(b \wedge c) \wedge ((a \vee b) \wedge (a \vee c)) =$
$c \wedge (b \wedge (a \vee b)) \wedge (a \vee c) =$
$c \wedge b \wedge (a \vee c) = b \wedge (c \wedge (a \vee c)) = b \wedge c,$

hence

$$b \wedge c \leqslant (a \vee b) \wedge (a \vee c).$$

With the same Lemma 2, (a) and (b) give us 1.

2. The proof of this is similar.

A *distributive lattice* is a lattice $\langle A, \wedge, \vee \rangle$ satisfying:

distributivity:

either 1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
or 2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

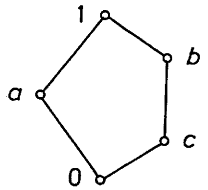
One of them is enough, the other will follow. We'll show that in one direction. Suppose we assume the first distributive axiom 1. Then:

$(a \vee b) \wedge (a \vee c) =$ [distributive 1]
$(a \wedge (a \vee c)) \vee (b \wedge (a \vee c)) =$ [absorption]
$a \vee (b \wedge (a \vee c)) =$ [distributive 1]
$a \vee (b \wedge a) \vee (b \wedge c) =$ [absorption]
$a \vee (b \wedge c)$

Given this, we know that the dual of a distributive lattice is a distributive lattice, hence, duality holds for distributive lattices.

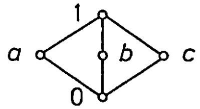
Typical lattices that are not distributive are:

The pentagon:



$c \vee (a \wedge b) = c$ but $(c \vee a) \wedge (c \vee b) = b$
$b \wedge (a \vee c) = b$ but $(b \wedge a) \vee (b \wedge c) = c$

and *the diamond*:



$b \vee (a \wedge c) = b$ but $(b \vee a) \wedge (b \vee c) = 1$
$b \wedge (a \vee c) = b$ but $(b \wedge a) \vee (b \wedge c) = 0$

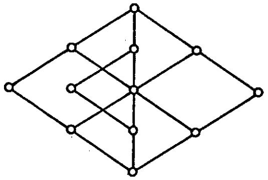
We call a sublattice of lattice $\langle A, \leqslant \rangle$ a pentagon (diamond) if it is isomorphic to the pentagon (diamond).

THEOREM. *A lattice $\langle A, \wedge, \vee \rangle$ is distributive iff no sublattice of $\langle A, \wedge, \vee \rangle$ is a pentagon or a diamond.*

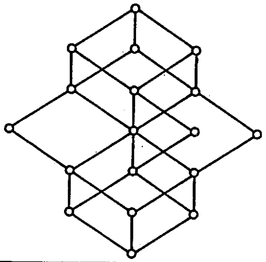
I won't prove this theorem here (see Grätzer's book for a proof). Note that, given this theorem, we can define a distributive lattice as a lattice $\langle A, \leqslant \rangle$ that contains no pentagon or diamond (as a sublattice!).

Examples of distributive lattices are the powerset lattices.

Exercise 7. (a) Is the following lattice a distributive lattice?



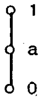
(b) Is the following lattice a distributive lattice?



Let $\langle A, \leqslant \rangle$ be a bounded lattice.

a is a *complement* of b iff $a \wedge b = 0$ and $a \vee b = 1$

Elements need not have complements in bounded lattices, nor need the complements be unique. For instance, in the following lattice, a does not have a complement:



In the pentagon, both b and c are complements of a (though a is the unique complement of b and the unique complement of c). In the diamond, both a and c are complements of b; the complements of a are b and c.

THEOREM. *In a bounded distributive lattice, an element can have at most one complement.*

Exercise 8. Prove this.

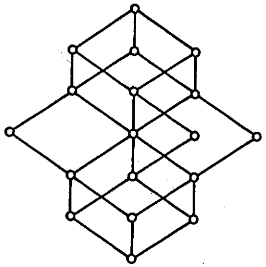
A *complemented lattice* is a bounded lattice where every element has a complement.

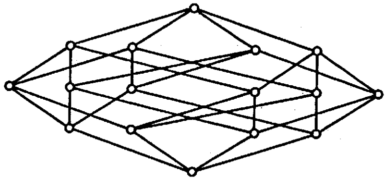
Hence, we know that in a complemented distributive lattice every element has a unique complement. We write the complement of a as $\neg a$.

Exercise 9. (a) Prove that in a complemented distributive lattice $\neg\neg a = a$.

(b) Prove that in a complemented distributive lattice, the de Morgan laws hold: $\neg(a \wedge b) = \neg a \vee \neg b$; $\neg(a \vee b) = \neg a \wedge \neg b$.

(c) Are the following structures complemented distributive lattices? If so, indicate the complements (start on the bottom, give a node a name n, and write name $\neg n$ at its complement, until every node has a name).





A *Boolean lattice* is a complemented distributive lattice.

A *Boolean algebra* is a structure $\langle A, \neg, \wedge, \vee, 0, 1 \rangle$ such that:

1. $\langle A, \wedge, \vee \rangle$ is a Boolean lattice.
2. The special elements 0 and 1 of the algebra are the 0 and 1 of $\langle A, \wedge, \vee \rangle$.

3. $\neg$ is a one-place operation that maps every element of A onto its complement.

So, a Boolean algebra is a structure $\langle A, \neg. \wedge, \vee, 0, 1\rangle$ such that:

1. $\wedge$ and $\vee$ satisfy idempotency, commutativity, associativity, absorption, distributivity.
2. $a \wedge 1 = a$; $a \wedge 0 = 0$
3. $a \wedge \neg a = 0$; $a \vee \neg a = 1$.

Examples of Boolean algebras are, of course, the powerset Boolean algebras, $\langle \text{pow}\, A, \cap, \cup, -, \emptyset, A\rangle$.

Exercise 10. Prove that in a Boolean algebra the following holds:

$$a \vee b = 1 \text{ iff } a \wedge b = 0 \text{ iff } b \leqslant \neg a \quad (a, b \neq 0, 1)$$

6.2. UNIVERSAL ALGEBRA

In this section we continue the universal algebra that we started in Chapter Two. The notions that we introduce will be discussed for lattices, but hold quite in general for algebras (that is, for most of these notions you can read algebra, where it says lattice).

Generated Lattices

Let $\langle A, \wedge, \vee\rangle$ be a lattice and let B be a non-empty set of sublattices of A. Then $\cap B$ is closed under $\wedge$ and $\vee$. Hence $\cap B$ is a sublattice of A iff it is non-empty.

Now take any non-empty set $X \subseteq A$. Look at the set of all sublattices of A that contain X: $\{B: B \subseteq A \text{ and } X \subseteq B\}$. Look at $\cap\{B: B \subseteq A \text{ and } X \subseteq B\}$. This set is non-empty, because X is a subset of it; it is closed under the lattice operations as we have said above, hence it is a sublattice of A. Let us cast this in terms of a definition:

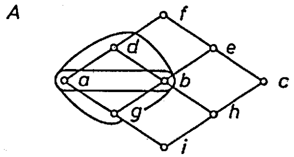
Let A be a lattice and let $X \subseteq A$ (X is non-empty).

the sublattice of A generated by X,

$$[X] = \cap\{B: B \subseteq A \text{ and } X \subseteq B\}$$

X is called the generating set of $[X]$. $[X]$ clearly is the smallest sublattice containing X.

Let $a \in A$. a is *generated by X* if $a \in [X]$.
An example:



$$[\{a, b\}] = \{a, b, g, d\}$$
$$[\{a, b, c\}] = A$$

If X is a non-empty subset of A, such that $[X] = A$, then X is also called a *set of generators for A*, and A is generated by X.

Clearly, lattices can have more than one set of generators, for instance A itself is of course always a set of generators for A. A set of generators X for A is a *minimal* set of generators for A, or a set of *independent* generators for A if no proper subset of A generates A. Another way of saying this is:

A set of generators B for A is *minimal* or *independent* iff

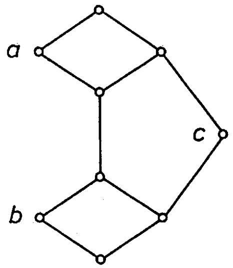
1. B generates A
2. $\forall b \in B$: $b \notin [B - \{b\}]$.

That is, if you leave an element of B out, you can't generate it (and hence, you can't generate all of A).

A is *minimally* or *independently* generated if it is generated by a minimal set of generators.

Lattices can have more than one set of minimal generators.

Exercise 11. (a) Consider the following lattice:



Show that $\{a, b, c\}$ is a minimal set of generators for this lattice.

(b) Give once more the diagram of pow$\{a, b, c\}$. This lattice has five sets of minimal generators. Which are those.

(c) Connect three new elements to the lattice under (b) such that the result is a distributive lattice that is no longer generated by the previous sets of generators. What are the new sets of minimal generators?

(d) This exercise is meant to stress the difference between a Boolean lattice and a Boolean algebra. Draw (or copy) the sixteen element powerset lattice.

Regard the structure as a Boolean lattice. How many elements do you need minimally to generate this lattice?

Now regard the structure as a Boolean Algebra. How many elements do you need minimally to generate this algebra?

A lattice is *finitely generated* iff it has a finite set of generators.

Look at lattice $\langle \text{pow}\,\mathbf{N}, \cap, \cup\rangle$. Let $S(\mathbf{N})$ be the set of singletons in pow **N**. What is the sublattice generated by $S(\mathbf{N})$? At first sight you might think that it is pow **N** itself. It is easy to fall into that trap. What you should realize is that *as a lattice* the operations on pow **N** are only the two-place intersection and union, and not their generalizations to arbitrary sets. We have seen that the set of all finite subsets of **N** is a sublattice of pow **N**. All singletons are in this sublattice, it is not hard to see that with iterations of $\cap$ and $\cup$ you stay within this sublattice, it is indeed this sublattice that is the generated sublattice.

Still, in some sense, pow **N** is generated by the singletons. Let's make this precise:

An *i-lattice* is a structure $\langle A, \wedge, \vee\rangle$, where:

> A is closed under $\bigwedge$ and $\bigvee$, the (infinitary) operations of meet and join.

If $\langle A, \wedge, \vee\rangle$ is a complete lattice, then it is of course also an *i*-lattice, by taking the general meets and joins (which exist because A is complete) as the operations, and *vice versa*, if $\langle A, \bigwedge, \bigvee\rangle$ is an *i*-lattice, then $\langle A, \wedge, \vee\rangle$ is complete.

Let $\langle A, \wedge, \vee\rangle$ be a complete lattice. Let $B \subseteq A$. The sublattice of A, *completely generated* by B, $[B]c$, is the complete lattice corresponding to the intersection of all sub-*i*-lattices of $\langle A, \bigwedge, \bigvee\rangle$ containing B. A is *completely generated* by B iff $[B]c = A$.

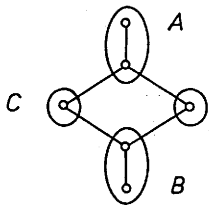
Under this definition, the set of finite subsets of N is not a sub-*i*-lattice of $\langle \mathbf{N}, \bigwedge, \bigvee\rangle$ (it's not a complete lattice), pow **N** is indeed minimally completely generated by the set of its singletons.

Homomorphisms

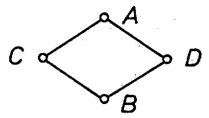
Let me summarize some things we have seen of homomorphisms, and some new things that are not that hard to see. $\approx$ now stands for congruence relations (because we're dealing with algebras).

1. The natural homomorphism h, given by: $h(a) = [a]_{\approx}$, is a homomorphism from A into $[A]_{\approx}$.

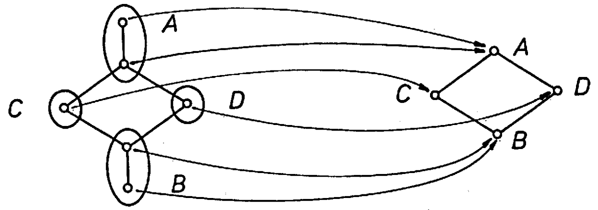
This can be seen in a picture. Suppose we have the following congruence classes:



this is isomorphic to:

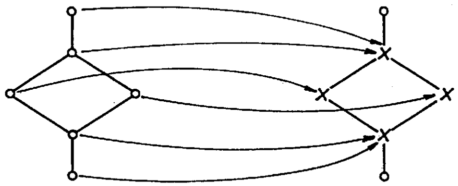


and the natural homomorphism is:



which is indeed a homomorphism.

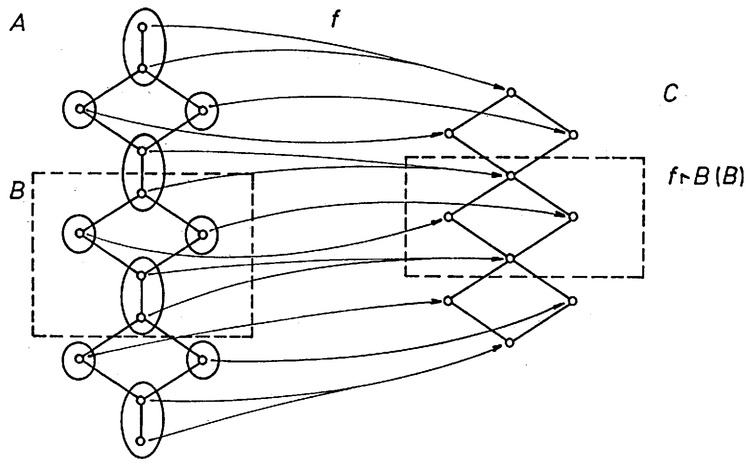
2. If $h\colon A \to B$ is a homomorphism, then $h(A)$ is a sublattice of B. Again a picture shows this:



3. Let B be a sublattice of A, and $\approx$ a congruence relation on A and $h\colon A \to C$ a homomorphism. Then:

> $\approx_B$ is a congruence relation on B and
> $h \restriction B\colon B \to C$ is a homomorphism from B into C.

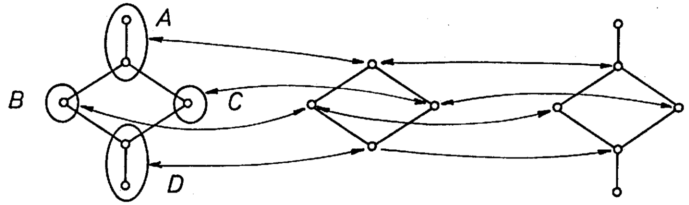
Look at the following example:



$B \subseteq A$ and clearly $\approx \restriction B$ is a congruence relation on B. Moreover, $f \restriction B$ maps B onto $f \restriction B(B)$ which, as can be checked, is a substructure of C.

4. The composition of two homomorphisms is a homomorphism.

5. Let $\langle B, \wedge, \vee \rangle$ be a sublattice of A and $\approx$ a congruence relation on A. Then $\langle [B]_{\approx}, \wedge, \vee \rangle$ is isomorphic to a sublattice of A. A picture:



6. Let B be a sublattice of A, $\approx$ a congruence relation on A and assume that every congruence class of $[A]_{\approx}$ contains an element of B. Then $[A]_{\approx}$ and $[B]_{\approx}$ are isomorphic.

Exercise 12. Let h: $A \to B$ be a homomorphism. Define:

$$\approx := \lambda a \lambda a'.h(a) = h(a')$$

Show that $\approx$ is a congruence relation on A.

The homomorphism theorem tells us that homomorphisms and congruence relations, and hence homomorphisms and partitions are in some sense two sides of the same coin.

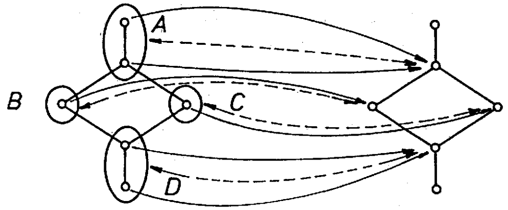
HOMOMORPHISM THEOREM. *Let h: $A \to B$ be a homomorphism, and let $\approx$ be the congruence relation defined in exercise* 12. *Look at the following function f:*

$$f([a]_{\approx}) = h(a)$$

f is an isomorphism *between $[A]_{\approx}$ and $h(A)$.*

Exercise 13. Prove this.

Again a picture:



f, the function indicated by the dotted lines, is an isomorphism

The homomorphism theorem tells us that the homomorphic image of A under h is isomorphic to a partition on A. The statement under (5) above tells us that this partition of A is isomorphic to a sublattice of A, hence the homomorphic image of A under f is isomorphic to a sublattice of A.

Let A be a lattice of some type. Identities are preserved under sublattices (by definition of sublattice), so any sublattice of A will be of the same type. So we see, that the homomorphic image of A under h will be of the same type as A.

Summarizing, if we have a lattice of a certain type, then we know, by definition of sublattice, that any sublattice of that lattice will be a lattice of that same type. If we have a homomorphism of a lattice of a certain type to another algebra, then the homomorphic image is of the same type. Another way of saying this is that equational classes of lattices are closed under the formation of sublattices and homomorphic images.

Here is yet another kind of lattice.
Let $\mathbf{A} = \langle A, \wedge, \vee\rangle$ and $\mathbf{B} = \langle B, \wedge, \vee\rangle$ be two lattices.

The *direct product* of **A** and **B**,

$$\mathbf{A} \times \mathbf{B} = \langle A \times B, \wedge, \vee\rangle$$

where $\wedge$ and $\vee$ are defined component wise (pointwise):

$$\langle a, b\rangle \wedge \langle a', b'\rangle = \langle a \wedge a', b \wedge b'\rangle$$
$$\langle a, b\rangle \vee \langle a', b'\rangle = \langle a \vee a', b \vee b'\rangle$$

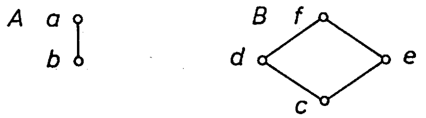
We can extend this to the product of a class of lattices.
Some facts about direct products:

- $\mathbf{A} \times \mathbf{B}$ is a lattice.
- If **A** and **B** are lattices of a certain type then $\mathbf{A} \times \mathbf{B}$ is a lattice of the same type. This means that equational classes of lattices are closed under the formation of direct products.

In fact, we have:

THEOREM. *K is an equational class of lattices iff K is closed under the formation of subalgebras, homomorphic images and direct products.*
Proof. See Grätzer (1978b).

Exercise 14. Given the following two lattices A and B, draw the diagram of $A \times B$:



Free Lattices

We will now discuss the notion of the most general structure of a certain type. The notions to follow apply to algebras in general, but we will talk about them in the context of lattices (and join semilattices).

Let K be an equational class of lattices. We will look only at the following special case. Suppose we have an unordered set of n elements. Look at all the lattices in K that are generated by this set (i.e. the elements occur in those lattices as *incomparable* elements).

A lattice is freely generated by this set within this equational class if it is generated by that set and the following holds: an identity statement holds in this lattice iff it is an identity statement of the equational class, that is, iff it holds in all lattices of this type.

Let me illustrate the idea by giving some reasons to be interested in free structures. For this, we will make an excursion into the theory of plurality.

In Link (1983), the domain of individuals is endowed with a *sum* operator. The idea is that a plural noun phrase like *John and Mary* denotes a plural individual that is the sum of the singular individuals John and Mary. Plural individuals being sums of singular individuals, we can assume that the domain of individuals gets the structure of a complete* atomic* join semilattice, in which the individuals are the atoms*s, and the join operation represents sum.

However, if we don't impose any more constraints, we allow domains of individuals to have rather pathetic structures. In the first place, we would want to disallow structures that are not generated or completely generated by the atoms. We can give this the following form:

Let me define the notions here only for the case we are interested in:

> A complete* join semilattice is *atomistic** if every non-zero element is the join ($\bigvee$) of atom*s.

(Similarly a complete lattice is atomistic if every non-zero element is the join of atoms.) Clearly, then, an atomistic* join semilattice is completely generated by the atoms (and 0). (Note that an atomistic* join semilattice has a unique minimal set of generators, namely the set of atom*s if there is no 0, and the set of atoms and 0 if there is a 0.) So we should require the domain of individuals to be an atomistic* join semilattice.

We can introduce some simplifications by restricting our attention to structures that do not have a 0 (those are precisely the structures that we are interested in). We do that by restricting ourselves to complete* join semilattices that are completely generated by an *unordered* set.

THEOREM. *Let $\langle A, \vee\rangle$ be a complete* join semilattice, and X an unordered set of generators for A (that is, a set of which the members are not related by the partial order and that generate A under $\bigvee$). Then A is atomistic* and X is the set of atoms* of A.*

Proof. Because A is generated under $\bigvee$ by X: $\forall y \in A$: $\exists Y \subseteq X$: $Y \neq \emptyset$ and $y = \bigvee Y$.

Claim: $X = AT$

1. $X \subseteq AT$

Since the only operation is $\bigvee$, no element smaller than any of the $x \in X$ can be generated, so the elements in X are atom*s in A.

2. $AT \subseteq X$

Suppose $a \in AT$. For some $Y \subseteq X$: $a = \bigvee Y$. This can only be so if $Y = \{a\}$ and hence $a \in X$.

Since X is AT, and A is generated by X, we then know that every element of A is the sum of atom*s, so A is atomistic*.

Note that none of the join semilattices that have a 0 element can be generated by an unordered set.

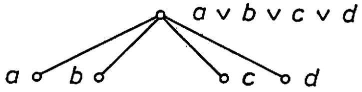
To get rid of the notion of completeness, let us introduce the structures we are interested directly with the infinitary operation. We call such structures *i*-join semilattices:

An *i-join semilattice* is a structure $\langle A, \bigvee\rangle$, where:

1. A is a non-empty set
2. $\bigvee$ assigns to every non-empty subset of A an element of A
3. The relation $\leqslant$, defined by: $x \leqslant y$ iff $\bigvee\{x, y\} = y$ is a partial order on A
4. $\bigvee$ assigns to every non-empty subset X of A the join of X under $\leqslant$

Thus the structures we are interested in here are *i*-join semilattices generated by an unordered set.

However, it is not enough to endow the domain of singular and plural individuals with this structure. The following structure is an *i*-join semilattice, generated by four individuals ($\{a, b, c, d\}$):

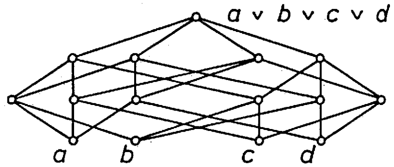


Note that $a \vee b$ here stands for the sum of a and b, that is, for *a and b*, so to speak.

This structure is pathological, because it violates our intuition about individuals that if a, b, c, d are distinct individuals, then the sum of a and b should be distinct from the sum of c and d. These sums are identified in the above structure.

If we want to endow the domain of individuals with a sum operator, then clearly we are interested in structures that do not make such identifications, where all intermediate sums are there.

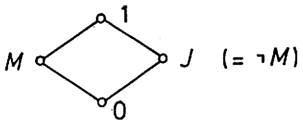
That is the idea behind free structures. Here is a free *i*-join semilattice generated by $\{a, b, c, d\}$:



(If we put the 0 back in place, we get a structure that is in fact a Boolean lattice (the sixteen element Boolean lattice).)

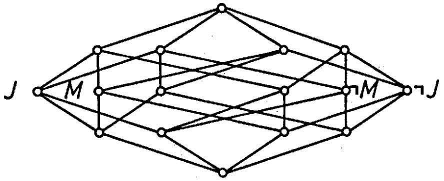
We see then that if, for the analysis of plurality, we want to assume a domain of singular and plural entities, we do not want structures that make unwanted identifications: we want free structures.

Another example can be found in Generalized Quantifier Theory. For a lot of purposes, it makes sense to interpret *John* as the set J of all properties that John has. At the level of these sets all the Boolean operations make sense: *John and Mary* is the set of properties that John and Mary share, *John or Mary* is the set of properties that at least one of John and Mary has, and *not John* is the set of properties that John doesn't have. Clearly, the structure that we want is a Boolean algebra. But not any Boolean algebra will do. If we have two individuals, John and Mary, the following would be a Boolean algebra generated by J and M:



Assuming that both John and Mary are individuals, *being an individual* is a property both in J and in M. But since we have identified J and $\neg M$, that would lead to the pathologic conclusion that *being an individual* is one of the properties in $\neg M$. That means that we have to assume that there is an individual not-Mary, i.e. that not-Mary is an individual like Mary herself. Clearly we don't want that.

Here we are interested in the *free* Boolean algebra with *two* generating individuals. (This point is made in Keenan and Faltz, 1985):



We see that in both cases it is our intuitive concept of individual that leads us to a free structure. This does not mean that we should always only be interested in free structures: at other levels, and for other purposes (say, at the level of properties or propositions), we may not want to limit ourselves to free structures.

Let us now give the definition of free lattices. As said, we will only look at the special case of lattices freely generated by a set. (For generalizations, like lattices generated by partial orders, etc. see Grätzer, 1978b.)

Let K be an equational class of lattices and X a non-empty unordered set.

$F_K(X)$ is a *free lattice* in K *generated by* X iff

1. $F_K(X) \in K$

i.e. a free lattice in K generated by X is an element of K.

2. $F_K(X) = [X]$

i.e. a free lattice in K generated by X is generated by X.

3. the elements of X are incomparable in $F_K(X)$.

4. For any lattice L and any function f: $X \to L$:
f can be extended to a *homomorphism*
f': $F_K(X) \to L$

Before I explain the impact of the last condition, let me first state some facts:

FACT: *If $F_K(X)$ is a free lattice generated by X, and L is any lattice in K and f a function from X into L, then there is exactly one homomorphism from $F_K(X)$ into L, extending f.*

Proof. Suppose both h and h' are such homomorphisms extending f. Assume that $h \neq h'$. Then for some $a \in F_K(X)$: $h(a) \neq h'(a)$. Since

$F_K(X)$ is generated by X, every element is generated as the result of applying some sequence of the operations, say p, to elements of X, i.e. for some $x_1, \ldots, x_n \in X$: $a = p(x_1, \ldots, x_n)$. Thus $h(p(x_1, \ldots, x_n)) \neq h'(p(x_1, \ldots, x_n))$, and thus (because h and h' are homomorphisms): $p(h(x_1), \ldots, h(x_n)) \neq p(h'(x_1), \ldots, h'(x_n))$ (where p is the same sequence of operations, but in L). This is only possible if for some $x_i \in X$: $h(x_i) \neq h'(x_i)$. But this contradicts the fact that for all $x \in X$:

$$h(x) = h'(x) = f(x).$$

A consequence of this fact is:

THEOREM. *Let $F_K(X)$ and $F_k(X)'$ be free lattices generated by X. Then $F_K(X)$ and $F_K(X)'$ are isomorphic.*

Proof. Suppose $F_K(X)$ and $F_K(X)'$ are both free. The identity function is a bijection between the sets of generators. Let h be the unique homomorphism extending identity to $F_K(X) \to F_K(X)'$.

Claim: h is a bijection between the two (and thus the structures are isomorphic).

1. h is onto. Let $b \in F_K(X)'$. Then for some sequence of operations p (in $F_K(X)'$) and $x_1, \ldots, x_n \in X$, $b = p(x_1, \ldots, x_n)$. Since for every $x \in X$: $h(x) = x$, this means that: $b = p(h(x_1), \ldots, h(x_n))$, and since h is a homomorphism it follows that $b = h(p(x_1, \ldots, x_n))$ (where p is the same sequence of operations, but now in $F_K(X)$. Thus we have indeed shown that h is onto.

2. h is one–one. Let $h(a) = h(b)$. Again, for some sequences of operations p, q and elements $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$:

$$a = p(x_1, \ldots, x_n) \text{ and } b = q(y_1, \ldots, y_m), \quad \text{so}$$
$$h(p(x_1, \ldots, x_n)) = h(q(y_1, \ldots, y_m)).$$

This means that: (h is a homomorphism)

$$p(h(x_1), \ldots, h(x_n)) = q(h(y_1), \ldots, h(y_m)).$$

Since h is the identity function, this means that:

$$p(x_1, \ldots, x_n) = q(y_1, \ldots, y_n), \text{ and thus that } a = b.$$

Clearly then, if X and Y have the same cardinality, the free lattice generated by X will be isomorphic to the free lattice generated by Y.

We write $F_K(n)$ for a free lattice in K with n generators (where n is any cardinal number).

The fundamental theorem for free lattices is:

THEOREM. *$F_K(X)$ exists iff there is a lattice $L \in K$ such that $X \subseteq L$ and all elements of X are incomparable in L.*

The proof of this theorem is far from trivial and goes beyond the scope of our discussion here (see Grätzer (1978b) for a proof). We can draw some consequences from it, though. Let us define:

An equivalence class is *trivial* iff it contains one element lattices only.

Here are some relevant facts:

– For any non-trivial equivalence class K of lattices, the two element lattice $\{0, 1\} \in K$.
– For any non-trivial equivalence class K of lattices and any non-empty set X, the characteristic function lattice of X, $\{0, 1\}^X \in K$.

The above facts mean that for any particular type of lattices, defined through equations, we can endow the two element lattice and any characteristic function lattice with that structure (so they are distributive lattices, Boolean lattices, etc.)

Given that we can choose X as big as we want, we see that for any non-trivial equivalence class we can always find a lattice in K that is big enough to have all elements of X incomparable in it. This means that the fundamental theorem for free lattices entails:

THEOREM. *For every non-trivial equivalence class K and every n: there is a unique (up to isomorphism) free lattice in K with n generators.*

Let us now see how the definition of free lattices, and in particular the difficult last clause guarantee that the free lattice on n generators $F_K(n)$ is the most general lattice in K with n generators.

That this is so can be seen as follows. Let $F_K(n)$ be generated by X. Take any lattice L in K with n generators. Let Y be the set of generators of L. The last condition of the definition of free lattice guarantees that any such L is *the homomorphic image* of $F_K(n)$. Namely, take a bijection f between X and Y. It can be extended to a homomorphism f' from $F_K(X)$ into L (by the last clause of the definition of free lattice). Suppose that some element $b \in L$ is not in the homomorphic image of f'. Since L is *generated* by Y, we know that for some $y_1, \ldots, y_n \in Y$, b is the result of a series of applications of the operations starting with $y_1, \ldots, y_n$. As above, letting p be the composition of that series of operations, we can write this as $b = p(y_1, \ldots, y_n)$. Hence, for some $x_1, \ldots, x_n \in X$:

$$b = p(f'(x_1), \ldots, f'(x_n)).$$

Now look at this same complex operation p but now in $F_K(X)$, in particular, look at $p(x_1, \ldots, x_n)$. This will be some element of $F_K(X)$, say a. Since f' is a homomorphism, i.e. it preserves the operations, it has to preserve p. This means that $f'(p(x_1, \ldots, x_n)) = p(f'(x_1), \ldots, f'(x_n))$. But that means that $f'(a) = b$, and hence b is in the homomorphic image of $F_K(X)$ under f' after all. So indeed every lattice in K with n generators is a homomorphic image of $F_K(n)$.

The homomorphic image of a lattice A is isomorphic to a partition of A, which itself is isomorphic to a sublattice of A. Hence, L is isomorphic to a sublattice of $F_K(n)$.

So every lattice in K with n generators can be embedded in $F_K(n)$. (Also, clearly every lattice with less than n generators can be embedded in $F_K(n)$: take a function from X onto the set of generators of L and repeat the above argument.)

Let us look at the two examples that we have discussed. Here is a fact about Boolean algebras.

FACT: *The free Boolean algebra on n generators has $2^{(2^n)}$ elements if n is finite.*

Let us call a lattice *completely free* if for some cardinality α it is a free lattice *completely* generated by α generators. Then the above fact generalizes to:

FACT: *The completely free Boolean algebra on α generators has 2^{2^α} elements.*

This is a very interesting fact. Remember that the powerset of any set X of n elements has 2^n elements. Consequently powpow X has 2^{2^n} elements as well. Let's call this set the powpower set of X and the powpower Boolean algebra. So we know that the free Boolean algebra on n elements and the powpower Boolean algebra of a set X of n elements have the same cardinality. It is not hard to check that powpow X is also a Boolean algebra with n generators: the sets $\{Y: x \in Y\}$, such that $x \in X$ will be the generators for powpow X. But that means that powpow X is the homomorphic image of the free Boolean algebra with n generators. Since both algebras have the same cardinality, this homomorphism is an isomorphism, so:

The free Boolean algebras are (up to isomorphism) the powpowerset Boolean algebras.

So our example of the sixteen element Boolean algebra was indeed the free Boolean algebra on two generators.

We will now look at free i-join semilattices generated by an unordered set. Let me stress here the following. The notion of an i-join semilattice on n generators that is free within the class of all i-join semilattices is a restricted concept. These structures are free in the sense that every i-join semilattice with a set of n (or less) incomparable generators can be embedded in them. It is not true, though, that for every i-join semilattice, there is some free i-join semilattice on some set of incomparable generators in which it can be embedded. The reason is that many i-join semilattices do not have sets of incomparable generators at all. In fact, apart from the 1 element lattice, no i-join semilattice that is a *lattice* has such a set (under the join operation, that is!). However, the free i-join semilattices generated from unordered sets are precisely the structures that we are interested in here.

What is the free i-join semilattice generated by X? It is a structure that is generated by X under $\bigvee$ and that does not make unwanted identifications. In the case of these particular structures it is rather easy to see what property they should satisfy to be free. We saw above that what we want to insure is that if individuals a, b and c are distinct, then the sums $a \vee b$, $a \vee c$ and $b \vee c$ should all be distinct. In general:

The *free* i-join semilattice generated by X is the i-join semilattice generated by X satisfying: for all $Y, Z \subseteq X$:

Distinctness: If $Y \neq Z$ then $\bigvee Y \neq \bigvee Z$.

We know that every i-join semilattice is atomistic*, i.e. every element is the sum of a set of atom*s. Distinctness then tells us that an i-join semilattice is free iff every element is the sum of one and only one set of atom*s.

The following theorem tells us in another way what the free i-join semilattices are:

THEOREM. *Every free i-join semilattice can be turned into a complete atomic Boolean algebra by adding a* 0 *element and every complete atomic Boolean algebra can be turned into a free i-join semilattice by deleting the* 0.

I will show that the theorem holds between free i-join semilattices and atomic i-Boolean algebras. Since it doesn't make any difference here, I will in fact call the latter complete atomic Boolean algebras.

Proof. Let $B = \langle B, \bigwedge, \bigvee, \neg, 0, 1\rangle$ be a complete atomic Boolean algebra. Look at $A = \langle B - \{0\}, \bigvee \restriction B - \{0\}\rangle$ with set of atoms AT.

We claim: A is a free i-join semilattice generated by AT.

It is not hard to see that A is an i-join semilattice under $\bigvee \restriction B - \{0\}$. If we can prove that B is atomistic, then it follows that A is atomistic*. In fact, we will prove a more general theorem:

Let B be a complete Boolean algebra.

THEOREM. *B is atomic iff B is atomistic.*

Proof: Let AT be the set of atoms in B. For every $b \in B$: let $b^* = \{a \in AT: a \leqslant b\}$.

Suppose B is not atomic, then there is a $b \in B$, $b \neq 0$ and $b^* = \emptyset$. Since B is complete $\bigvee b^*$ exists, and because $b^* = \emptyset$, $\bigvee b^* = 0$, hence $\bigvee b^* \neq b$ so B is not atomistic.

Suppose B is atomic, let $b \in B$. Because B is complete $\bigvee b^*$ exists. Since for every $a \in b^*$: $a \leqslant b$, it follows that $\bigvee b^* \leqslant b$.

So we have to show that $b \leqslant \bigvee b^*$.

Suppose $b \nleqslant \bigvee b^*$. Then we know (by Exercise 10) that $b \wedge \neg \bigvee b^* \neq 0$. Since B is atomic, that means that there is an atom $a \leqslant b \wedge \neg \bigvee b^*$. So $a \leqslant b$ and $a \leqslant \neg \bigvee b^*$. $a \leqslant b$, hence $a \in b^*$, hence $a \leqslant \bigvee b^*$. But then $a \leqslant \bigvee b^* \wedge \neg \bigvee b^*$, so $a = 0$. Contradiction, hence $b \leqslant \bigvee b^*$.

Thus $b = \bigvee b^*$. Hence B is atomistic.

We have proved that B is atomistic, hence A is atomistic*, so A is generated by AT. Distinctness follows from the following fact:

FACT: *If a is an atom then for every $b \in B$: $(a \leqslant b) \vee (a \leqslant \neg b)$.* Namely: $a = a \wedge (b \vee \neg b) =$ (distributivity) $(a \wedge b) \vee (a \wedge \neg b)$. Since a is an atom it can only be the join of 0 and itself. So either:

$a \wedge b = a$ and $a \wedge \neg b = a$, which is impossible, or:

$a \wedge b = a$ and $a \wedge \neg b = 0$, hence $a \leqslant b$

or

$a \wedge b = 0$ and $a \wedge \neg b = a$, hence $a \leqslant \neg b$

Since for every x:

$$\neg x = \bigvee\{a \in AT: a \leqslant \neg x\}$$

$$\neg x = \bigvee\{a \in AT\colon a \leqslant \neg x\}$$

the above fact implies:

$$\neg x = \bigvee\{a \in AT\colon a \wedge x = 0\}.$$

Let $X \subseteq AT$. If $a \in X$ then $a \leqslant \bigvee X$. If $a \notin X$ then for every $x \in X$: $a \wedge x = 0$. Since $\neg a = \bigvee\{b \in AT\colon b \wedge a = 0\}$ what follows is that $\bigvee X \leqslant \neg a$. Hence $a \leqslant \neg\bigvee X$, hence $a \not\leqslant \bigvee X$. So if $a \leqslant \bigvee X$ then $a \in X$. Thus, $a \in X$ iff $a \leqslant \bigvee X$, consequently $X = (\bigvee X)^*$, i.e. $X = \{a \in AT\colon a \leqslant \bigvee X\}$. This implies distinctness: if $\bigvee X = \bigvee Y$ then $(\bigvee X)^* = (\bigvee Y)^*$ then $X = Y$.

So we have proved that we can turn a complete atomic Boolean algebra into a free i-join semilattice by removing the 0.

We prove the other direction.

Let A be a free i-join semilattice generated by X. Let us define:

$a \bigcirc b$, a and b overlap, iff $\exists c[c \leqslant a$ and $c \leqslant b]$

Define

$$B = \langle A \cup \{0\}, \bigvee, \bigwedge, \neg, 0, \bigvee A\rangle$$

where:

1. $0 \notin$ A
2. $\bigwedge := \bigwedge\emptyset = \bigvee A$
   $\bigwedge Y = 0$ if either $0 \in Y$ or Y contains two elements that don't overlap
   $\bigwedge Y = \bigvee\{z \in A\colon \forall y \in Y[z \leqslant y]\}$ otherwise
3. $\bigvee := \bigvee\emptyset = 0$
   $\bigvee\{0\} = 0$
   $\bigvee Y = \bigvee_A(Y - \{0\})$ otherwise
4. $\neg := \neg\bigvee A = 0$
   $\neg 0 = \bigvee A$
   $\neg a = \bigvee\{x \in X\colon \neg(x \bigcirc a)\}$ otherwise

We have defined the operations $\bigwedge$ and $\bigvee$ for every subset of B. It is clear from the definition of $\bigwedge$ and $\bigvee$ (and the fact that A is an i-join semilattice) that $\bigwedge$ and $\bigvee$ are join and meet on B, and similarly it is clear that B is bounded by $\bigvee A$ and 0. So B is a complete bounded lattice.

Now, the elements of X don't overlap and every element is the join of elements in X (because A is generated by X), hence every (non-zero) element in B has an element on X below it, and the elements in X have only themselves and 0 below them; moreover, this holds only for the elements of X (because X generates A), so indeed X is the set of atoms in B. So B is a complete atomic bounded lattice.

We have to show that B is distributive and complemented. Let's prove the following lemma:

LEMMA. *A has the following property*:

$$\text{if } a \leqslant b \vee c \text{ then } a \leqslant b \text{ or } a \leqslant c \text{ or}$$
$$\exists b' \leqslant b \exists c' \leqslant c\colon a = b' \vee c'$$

Proof. The following holds:

$$(b \vee c)^* = b^* \cup c^*$$

Namely: $\bigvee(b^* \cup c^*) = b \vee c = \bigvee(b \vee c)^*$. Distinctness tells us that there is only one set for which that holds.

Assume $a \leqslant b \vee c$. That means $a^* \subseteq (b \vee c)^*$. So $a^* \subseteq b^* \cup c^*$. Consider:

$$a^* \cap b^* \text{ and } a^* \cap c^*.$$

If

$$a^* \cap b^* = \emptyset \text{ then } a^* \subseteq c^* \text{ and hence } a \leqslant c$$

If

$$a^* \cap c^* = \emptyset \text{ then } a^* \subseteq b^* \text{ and hence } a \leqslant b$$

If neither are empty then:

$$a^* = (a^* \cap b^*) \cup (a^* \cap c^*) \text{ and hence}$$

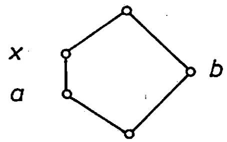
$$a = \bigvee(a^* \cap b^*) \vee \bigvee(a^* \cap c^*) \text{ and } \bigvee(a^* \cap b^*) \leqslant b \text{ and}$$

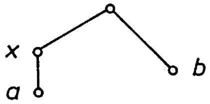
$$\bigvee(a^* \cap c^*) \leqslant c.$$

This completes the proof of the lemma.

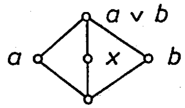
Now we can prove that B is distributive.

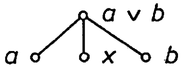
If B is not distributive, then B contains the pentagon or the diamond. Assume the pentagon. Then A contains either one of the following two structures:





In either case A violates the lemma: $x \leqslant a \vee b$ but not $x \leqslant a$ and not $x \leqslant b$, nor is x the sum of some part of a and some part of b. Exactly the same argument can be made for the diamond:





So we have proved that B is distributive.

Finally we have to show that B is complemented. This is trivial for 0 and $\bigvee A$, so assume $a \neq 0$, $\bigvee A$.

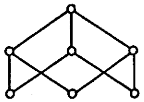
$$\neg a = \bigvee\{x \in X: \neg(x \bigcirc a)\}$$
$$a \wedge \neg a = a \wedge \bigvee\{x \in X: \neg(x \bigcirc a)\}$$

Since a does not overlap with any of the elements in this set, it does not overlap with $\neg a$, hence $a \wedge \neg a = 0$.

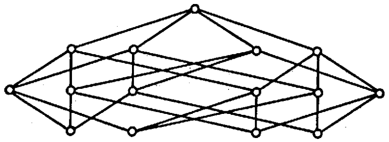
$$a \vee \neg a = a \vee \bigvee\{x \in X: \neg(x \bigcirc a)\}.$$
$$(a \vee \neg a)^* = a^* \cup (\neg a)^* = X, \text{ hence } a \vee \neg a = \bigvee A.$$

This completes the proof.

So the free *i*-join semilattices are exactly the complete atomic Boolean algebras with the bottom element removed. This tells us that the free general join semilattices on three and four generators are indeed:



and



Another example of a free lattice is the one given in Exercise 7: it is the free distributive lattice on three generators. If you want to see

more examples of free lattices: see Grätzer (1978b). (For instance, on p. 283 you will find an approximation of the structure of the free lattice on three elements).

6.3. FILTERS AND IDEALS

Let L be a lattice.

A *filter* in L is a non-empty subset $F \subseteq L$ such that:

1. if $a \in F$ and $a \leqslant b$ then $b \in F$
2. if $a, b \in F$ then $a \wedge b \in F$

An *ideal in* L is a non-empty subset $I \subseteq L$ such that:

1. if $b \in I$ and $a \leqslant b$ then $a \in I$
2. if $a, b \in I$ then $a \vee b \in I$

So a filter is closed under $\leqslant$ and $\wedge$; the dual notion, an ideal is closed under $\geqslant$ and $\vee$. If the lattice has a 1 and/or a 0 then $\{1\}$ is a filter and every filter contains 1; similarly $\{0\}$ is an ideal and every ideal contains 0. The lattice L itself is both a filter and an ideal.

Exercise 15. (a) Prove that a filter (ideal) in L is a convex sublattice of L.

(b) Prove that the following alternative definitions of filters are equivalent to the one given:

– Non-empty subset F of L is a *filter* in L iff

$$a \wedge b \in F \text{ iff } a, b \in F$$

– F is a *filter* in L iff

1. F is a sublattice of L
2. if $a \in F$ and $b \in L$ then $a \vee b \in F$

Give the dual alternative definitions for ideals.

THEOREM. *Let* $\mathbf{F}$ *be a set of filters in* L. *If* $\cap\mathbf{F}$ *is non-empty, then* $\cap\mathbf{F}$ *is a filter in* L.

Proof. Filters in L are convex sublattices of L. The non-empty intersection of convex sublattices is again a convex sublattice of L. Suppose

$a, b \in \cap \mathbf{F}$. Then both a and b are in every $F \in \mathbf{F}$, and hence so is $a \wedge b$, so $a \wedge b \in \cap \mathbf{F}$. Let $a \in \cap \mathbf{F}$ and $a \leqslant b$. Again, a is in every $F \in \mathbf{F}$, so b is in every such F, hence $b \in \cap \mathbf{F}$.

Let X be a non-empty subset of L. Given the above, we can define:

The *filter generated by* X, $[X)$, is the intersection of all filters extending X.

The *ideal generated by* X, $(X]$ is defined similarly.

In case $X = \{a\}$ we write $[a)$ for the filter generated by $\{a\}$ $(=[\{a\}))$:

The *principal filter generated by* a, $[a) = \{b: a \leqslant b\}$.

The *principal ideal generated by* a, $(a] = \{b: b \leqslant a\}$.

It is not hard to check that $\{b: a \leqslant b\}$ is indeed the smallest filter extending $\{a\}$, in fact, it will follow from the following more general theorem:

THEOREM. $[X) = \{a: \exists x_1, \ldots, x_n \in X: x_1 \wedge \cdots \wedge x_n \leqslant a)$.

Proof. Let $F = \{a: \exists x_1, \ldots, x_n \in X: x_1 \wedge \cdots \wedge x_n \leqslant a)$.

1. $X \subseteq F$. That is obvious, for every $x \in X$, $x \leqslant x$, so $x \in F$.

2. F is a filter. Let $a, b \in F$. That means that for some $x_1, \ldots, x_n \in X$, and some $y_1, \ldots, y_n \in X$: $x_1 \wedge \cdots \wedge x_n \leqslant a$ and $y_1 \wedge \cdots \wedge y_m \leqslant b$. But then $x_1 \wedge \cdots \wedge x_n \wedge y_1 \wedge \cdots \wedge y_m \leqslant a \wedge b$, hence $a \wedge b \in F$.

If $a \wedge b \in F$, then for some $x_1, \ldots, x_n \in X$: $x_1 \wedge \cdots \wedge x_n \leqslant a \wedge b$, but then $x_1 \wedge \cdots \wedge x_n \leqslant a$ and $x_1 \wedge \cdots \wedge x_n \leqslant b$, hence $a \in F$ and $\mathrm{b} \in \mathrm{F}$. So, using the alternative definition of filter from the exercise, F is a filter.

We now know that F is a filter extending X, hence $[X) \subseteq F$.

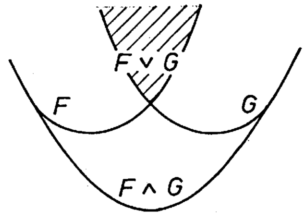
3. Suppose that G is a filter and $X \subseteq G$. Then for every $x_1, \ldots, x_n \in X$: $x_1 \wedge \cdots \wedge x_n \in G$ (because G is a filter) and hence for any $a, x_1, \ldots, x_n$ such that $x_1 \wedge \cdots \wedge x_n \leqslant a$, $a \in G$ (again, because G is a filter). Consequently, $F \subseteq G$. So F is a subset of any filter extending X, hence $F \subseteq [X)$.

So now we know that $F = [X)$.

Let $F(L)$ be the set of all filters in L and $F_0(L)$ be $F(L) \cup \{\emptyset\}$. Let us define: $[\emptyset) = L$

$$F \wedge G := [F \cup G)$$
$$F \vee G := [F \cap G)$$

In a picture:



Then $\langle F(L), \wedge, \vee \rangle$ is a lattice and $\langle F_0(L), \wedge, \vee \rangle$ is a complete lattice. If L has a maximal element, then $F(L)$ is also a complete lattice. Note that this lattice is a superset lattice, rather than a subset lattice: if there is a maximal element 1 in the lattice, the minimal filter $\{1\}$ is the maximal element of this lattice. With this, we can give the following representation theorem:

THEOREM. *Every lattice L can be embedded in $F(L)$ and in $F_0(L)$.*

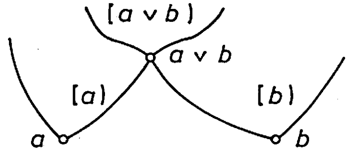
Proof. Let us define function $*: L \to F(L)$ (and hence into $F_0(L)$):

$$a^* = [a)$$

1. $*$ is an injection. Suppose $a \neq b$, that is, either $a \nleqslant b$ or $b \nleqslant a$, say $a \nleqslant b$. Then $b \notin [a)$, hence $[a) \not\supseteq [b)$,

2.
$$\begin{aligned}(a \vee b)^* &= [a \vee b) \\ &= [\{c: a \leqslant c\} \cap \{c: b \leqslant c\}) \\ &= [\{c: a \leqslant c\}) \vee [\{c: b \leqslant c\}) \\ &= [a) \vee [b) = a^* \vee b^*\end{aligned}$$

In a picture:



$$\begin{aligned}(a \wedge b)^* &= [a \wedge b) \\ &= [\{c: a \leqslant c\} \cup \{c: b \leqslant c\}) \\ &= [\{c: a \leqslant c)) \wedge [\{c: b \leqslant c\}) \\ &= [a) \wedge [b) = a^* \wedge b^*\end{aligned}$$

This representation theorem should be distinguished from the one that we have given for partial orders earlier. There we proved that any partial order can be embedded in a set theoretic partial order. In the present case we have only proved that every lattice can be embedded in the set of its filters, its filter structure, by proving that every lattice is isomorphic to the set of its principal filters. Now, in the case of partial orders there was no problem in further proving that the filter (or dually, ideal) structure of a poset is a subposet of a set theoretic poset. In the case of lattices, we would have to prove that every filter structure of a lattice is a *sublattice* of a set theoretic lattice, of a powerset lattice.

That this is not the case can be seen very easily. Any powerset lattice is distributive, hence we cannot embed the pentagon or diamond in it. Since we can embed (by the above theorem) the pentagon in its filter structure, we cannot embed its filter structure in any powerset lattice, else, by composition of the embeddings, we would have an embedding of the pentagon in a powerset lattice after all.

This shows that we cannot hope to have set theoretic representations of lattices in general. But, as we will see there are interesting classes of lattices for which we do have such representations.

Exercise 16. Prove that if L is a finite lattice, ever filter (ideal) is principal.

A filter F in L is *proper* iff $F \neq L$ (i.e. F is a proper subset of L). Similarly, an ideal I is *proper* iff $I \neq L$.

If L has a 0 and/or 1, then a filter F is proper iff $0 \notin F$ and an ideal I is proper iff $1 \notin I$.

Let L be a lattice with a 0.

Let us define:

a and b are *incompatible* iff $a \wedge b = 0$
X is *incompatible* iff $\exists x_1, \ldots, x_n \in X$: $x_1 \wedge \cdots \wedge x_n = 0$
a is *incompatible with* X iff $X \cup \{a\}$ is incompatible.
In all cases, compatible otherwise.

This terminology has, of course, an informational motivation: we think of 0 as the 'necessarily false proposition': a and b are incompatible iff $a \wedge b$ is the necessarily false proposition.

Let $X \subseteq L$.

X has the *finite intersection property* (*fip*) iff every finite subset of X is compatible.

This definition should remind you of the definition of consistency (a theory is consistent if every finite subtheory is consistent) and the compactness theorem: if a theory has a model, then every finite subtheory has a model.

THEOREM. *$[X)$ is a proper filter iff X has the fip.*

Proof. Obviously, if X doesn't have the fip, $0 \in [X)$. If X has the fip, then the definition of $[X)$ prevents 0 from ending up in $[X)$.

This tells us that a subset of L can be extended to a proper filter iff it has the fip.

A proper filter F is a *prime filter* iff

if $a \vee b \in F$ then either $a \in F$ or $b \in F$

A proper filter F is a *maximally* proper filter iff every proper filter that extends F coincides with F.

A proper filter F is an *ultrafilter* iff

$\forall a \in A$: either $a \in F$ or a is incompatible with F.

The following holds:

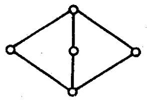
F is a maximally proper filter iff F is an ultrafilter.

Proof. (a) Suppose F is an ultrafilter and suppose $F \subseteq G$, $b \in G$ but $b \notin F$. Since F is an ultrafilter, it then follows that b is incompatible with F, i.e. $\exists a_1, \ldots, a_n \in F$: $a_1 \wedge \cdots \wedge a_n \wedge b = 0$. Since F is a filter, this means that for some $a \in F$: $a \wedge b = 0$. $F \subseteq G$, so $a \in G$, but then, because G is a filter, $0 \in G$, and G is not proper.

(b) Suppose F is a maximally proper filter, $b \notin F$ and b is compatible with F. The latter means (because F is a filter): $\forall a \in F$: $a \wedge b \neq 0$. Look at $F \cup \{b\}$. Every finite subset of this is compatible, hence $F \cup \{b\}$ has the fip, so $[F \cup \{b\})$ is a proper filter extending F. But F was maximal.

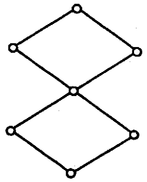
It is not the case, for lattices in general (although it is for some special cases that we will see later) that every ultrafilter is a prime filter, nor that every prime filter is an ultrafilter. This is shown in the next exercise.

Exercise 17. (a) Consider the following lattice:



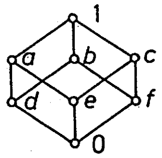
Draw all the proper filters in this diagram. Indicate which filters are prime and which filters are ultrafilters.

(b) Consider the following lattice:



Draw all the proper filters in this lattice. Indicate which filters are prime and which filters are ultrafilters.

(e) Consider the following lattice:



Draw the diagram of the set of all filters ordered by subset of this lattice. Which of the filters are prime? Which of the filters are ultrafilters?

Let us introduce one more notion, for powerset lattices, this time.

Let D be a set and $d \in D$. Look at the powerset lattice pow D.

The *principal ultrafilter generated by* d is the proper filter:

$$[\{d\}) = \{X \in \text{pow } D: d \in X\}$$

Of course, not every principal filter in pow D is a principal filter generated by some element of D (for instance $[\{d, d'\})$ is a principal filter). It is not hard to see that $[\{d\})$ is an ultrafilter. Suppose that $[\{d\}) \subseteq F$, where F is a proper filter in pow D and $[\{d\}) \neq F$. That means: for some $X \subseteq D$: $X \in F$ but $X \notin [\{d\})$, i.e. $d \notin X$. That means that $\{d\} \cap X = \emptyset$. F is a filter and both $\{d\}$ and X are in F, hence $\emptyset \in F$. But then F is not proper.

It is important to realize that in lattices in general the prime filters and the maximally proper filters do not coincide. With the Maximal Chain Principle we have the following theorem:

Let L be a lattice.

THEOREM (AC). *Every proper filter in L can be extended to an ultrafilter in L.*

Proof. Let F be a proper filter. Look at the set of proper filters extending F. Since, $\{F\}$ is a chain in this set, it can be extended to a maximal chain, C, in this set. $\cup C$ is a proper filter extending F. $\cup C$ is a filter: if $a, b \in \cup C$, then a and b are in some element in the chain, that element is a filter, so $a \wedge b$ is in it, hence it is in $\cup C$. If $a \wedge b \in \cup C$, it is in some element in the chain, so a and b are in that element, so they are in $\cup C$. Finally, $0 \notin \cup C$. If 0 were in $\cup C$, it would have to be in one of the elements of the chain, but all those elements are proper. So $\cup C \in C$, and since C is a maximal chain, $\cup C$ is hence a maximally proper filter extending F.

So we know that we can extend a proper filter to an ultrafilter, but that may not be enough for our purposes. For instance, in information applications, we are interested in using filters as information states or situations. (Sets of partial information, closed under conjunction and entailment.) Ultrafilters, then, can play the role of total information states or possible worlds. But certainly, for a total information state we want to require that it has the disjunction property:

If $a \vee b \in T$ then $a \in T$ or $b \in T$.

That is, we want total information states to be prime filters. On the other hand, of total information, we also want to require that it is indeed total:

for any a: either $a \in T$ or a is incompatible with T.

That is, if we analyze incompatibility as we have done before, a total information state should be a prime ultrafilter. But, for lattices in general, we cannot prove that every proper filter can be extended to a prime ultrafilter, so we cannot prove that every information state can be extended to total information.

It is very easy to make mistakes here, by not realizing that the standard theorems that you find in the textbooks may only apply to special cases (and that they may have built that into the terminology, for instance in the definition of the notion of ultrafilter). When you change things, for the application at hand, you will have to check whether those theorems still hold for your changed system.

For instance, you may be interested in a different notion of compatibility. We have seen an example of that earlier when we gave the representation theorem for period structures.

Those structures were not lattices (but partial lattices), there was no 0, and we defined: a and b are incompatible iff $a \sqcap b$ is not defined. Also there we could show that every compatible filter can be extended to a maximally compatible filter. But maximally compatible filters missed some crucial properties for them to be regarded as total filters. For that reason we restricted ourselves to witnessed structures, where the maximal filters did have the required totality.

One could also want to go the other way. We may not want to restrict ourselves to structures where there is one and only one necessarily false proposition (0), but we may want to have a set of necessarily false propositions, and define a compatible filter (a filter of compatible information) as a filter that has a non-empty intersection with that set.

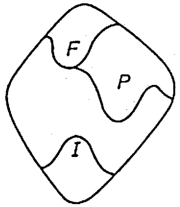
Again, in such structures, it won't be that hard to prove that every compatible filter in the new sense can be extended to a maximally compatible filter, but even if we have restricted ourselves to structures where the maximally *proper* filters are all *total* in the required sense, we will have to make sure separately that the maximally compatible filters are total in that sense. This is because, on the new definition of compatibility, maximally proper filters are bound to be incompatible (they are only required not to contain the minimal element, so they will quite likely contain necessarily false propositions that are not the minimal necessarily false proposition).

Such structures, by the way, arise naturally in partially valued semantics, like the one we discussed in Chapter Three. If we take the proposition expressed by a sentence as the pair consisting of the set of all information states that verify it and the set of all information states that falsify it, then $p \wedge \neg p$ and $q \wedge \neg q$ will be verified on no information state; they are necessarily false, but they need not express the same proposition since (because of partiality), the sets of states that falsify them need not coincide.

Exercise 18. An ideal I in L is *prime* iff if $a \wedge b \in I$ then $a \in I$ or $b \in I$. Show that filter F in L is a prime filter iff $L - F$ is a prime ideal.

Let us turn to distributive lattices. We will see that in distributive lattices every ultrafilter is a prime filter. Instead of proving this directly, and, combining this with what we have proved before, concluding that every proper filter can be extended to a prime filter, we will prove something more general (and useful).

STONE'S THEOREM (AC). *Let L be a distributive lattice, let F be a filter in L and I be a ideal in L such that $F \cap I = \emptyset$. Then there is a prime filter P extending F such that $P \cap I = \emptyset$.*
In a picture:

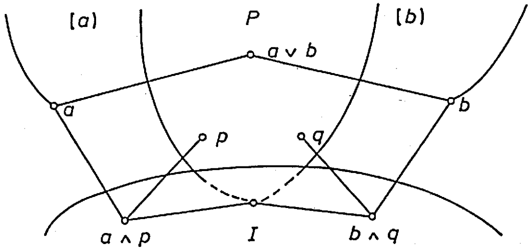


Proof. Look at the partial order of all filters extending F, disjoint with I. Zorn's lemma, in the form that we have used it several times now, tells us that there is a maximal element in this order, which is a filter P extending F and that is *maximally* such that it is disjoint with I. (This is crucial, we are not looking at maximally proper filters, but at filters that are maximally disjoint with I (but, of course, when $I = \{0\}$ or $\emptyset$ these notions coincide).)

Here is the new part:

Claim: P is a prime filter.

Proof. Suppose P is not prime. Then for some $a, b \in L$: $a \vee b \in P$, but $a, b \notin P$. Look at the filters $[P \wedge [a))$ and $[P \wedge [b))$ (where $\wedge$ is the operation on filters defined before, i.e. $F \wedge G = [F \cup G))$. Since P is a filter maximally disjoint with I, $[P \wedge [a)) \cap I \neq \emptyset$ and $[P \wedge [b)) \cap I \neq \emptyset$. This means that for some $p \in P$: $p \wedge a \in I$ and for some $q \in P$: $q \wedge b \in I$. Because I is an ideal, this means that $(p \wedge a) \vee (q \wedge b) \in I$. The results up to now are summarized in the following picture:



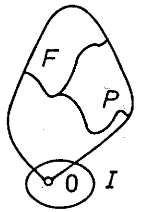
So we know that $(a \wedge p) \vee (b \wedge q) \in I$.
Now we rewrite this with the distributive laws:

$$\begin{aligned}(a \wedge p) \vee (b \wedge q) &= ((a \wedge p) \vee b) \wedge ((a \wedge p) \vee q) \\ &= (a \vee b) \wedge (p \vee b) \wedge (a \vee q) \wedge (p \vee q)\end{aligned}$$

By assumption, $a \vee b \in P$; $p \in P$, so because P is a filter, $p \vee b \in P$; for the same reason, $a \vee q \in P$, and obviously $p \vee q \in P$. That means, again because P is a filter, that their conjunction is in P, so $(a \wedge p) \vee (b \wedge q) \in P$. But then $P \cap I \neq \emptyset$. So we finally have a contradiction, hence P is a prime filter. This ends the proof of the theorem.

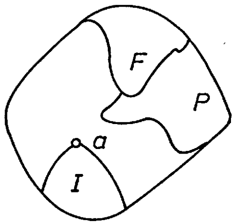
Here are some interesting corollaries. Let L be a distributive lattice.

COROLLARY 1. *Every proper filter can be extended to a prime filter.*



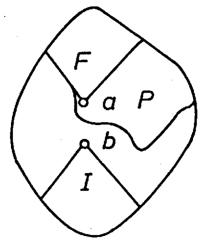
Proof. Let F be a proper filter. If L has a minimum 0, then $(0]$ is an ideal $\{0\}$. F is proper, this means, $F \cap (0] = \emptyset$. By Stone's theorem, there is a prime filter P extending F such that $P \cap (0] = \emptyset$, so P is proper as well. If L doesn't have a minimum, set $(\emptyset] = \emptyset$. Of course, this is officially not an ideal, but it is an element of $I_0(L)$, and the argument goes through for any $I \in I_0(L)$ as much as for any I in $I(L)$. The argument is further exactly the same as above.

COROLLARY 2. *Let F be a filter and $a \notin F$. Then there is a prime filter P extending F such that $a \notin P$.*



Proof. Let F be a filter. If $a \notin F$ then $F \cap (a] = \emptyset$. Hence there is a prime filter P with $P \cap (a] = \emptyset$, so there is a prime filter P not containing a.

COROLLARY 3. *Let $a, b \in L$, $a \neq b$. Then there is a prime filter containing exactly one of a and b.*



Proof. Either $[a) \cap (b] = \emptyset$ or $(a] \cap [b) = \emptyset$. In the first case there is a prime filter extending $[a)$ disjoint with $(b]$, and hence not containing b, in the second case there is a prime filter extending $[b)$ not containing a. In either case there is a prime filter containing exactly one of them.

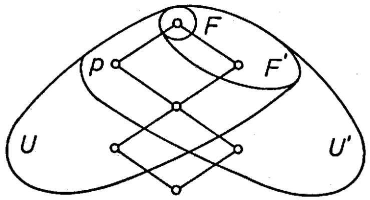
COROLLARY 4. *Every filter F in L is the intersection of all prime filters extending F.*

Proof. Let $F' = \bigcap\{P\colon F \subseteq P$ and P is a prime filter$\}$. Clearly $F \subseteq F'$. Suppose $F' \neq F$. Then for some a: $a \in F' - F$. $a \notin F$, hence there is a prime filter $P \supseteq F$ such that $a \notin P$. But then $F' \not\subseteq P$. Contradiction.

COROLLARY 5. *Every ultrafilter is a prime filter.*

Proof. In the proof of Stone's theorem, we proved that for an arbitrary filter F with an empty intersection with I, every extension of F that maximally has an empty intersection with I is prime. The situation of Corollary 1, where $I = \{0\}$ or $\emptyset$ is a special case of this, hence for an arbitrary proper filter, every maximally proper extension is prime. This, of course, tells us that all ultrafilters are prime.

Again these corollaries have to be taken with care. For instance, Corollary 2 does not mean that if $a \notin F$ there is an *ultrafilter* U extending F, such that $a \notin U$. For instance, look at the distributive lattice we discussed in a previous exercise:



In this lattice F' is the prime filter which is maximally such that it doesn't contain p. But the ultrafilters are U and U'.

Now let L be a Boolean lattice. Here we have the following results. We already know that for an ultrafilter U in a lattice, for all $a \in L$: either $a \in U$ or a is incompatible with U. If L is a Boolean lattice, then for every a: $a \wedge \neg a = 0$, $a \vee \neg a = 1$. So:

THEOREM. *If F is a proper filter then: for all a: $a \vee \neg a \in F$ for no a: $a \wedge \neg a \in F$.*

Proof. This is obvious.

THEOREM. *If F is an ultrafilter then for all a: either $a \in F$ or $\neg a \in F$.*

Proof. This follows, because every ultrafilter is prime and for every a: $a \vee \neg a \in F$.

But also the inverse holds:

THEOREM. *If F is a proper filter and for all a: either $a \in F$ or $\neg a \in F$ then F is an ultrafilter.*

Exercise 19. Prove this.

THEOREM. *In a Boolean lattice, the ultrafilters are exactly the prime filters.*

Exercise 20. Prove this.

Let us go on with the main representation theorems for distributive lattices and Boolean algebras. Let me repeat:

A powerset lattice is a lattice $\langle \text{pow}\, A, \cap, \cup \rangle$ for some set A. A powerset Boolean algebra is a Boolean algebra $\langle \text{pow}\, A, -, \cap, \cup, \emptyset, A \rangle$ for some set A.

We know already that any lattice can be represented in its filter structure, but that didn't help us much, because we couldn't embed the filter structure of an arbitrary lattice in a powerset lattice.

Prime filters play the central role in the representation of distributive lattices (and hence ultrafilters in the representation of Boolean algebras). The proof will be familiar from the proof for witnessed period structures (except that the present one is easier).

REPRESENTATION THEOREM FOR DISTRIBUTIVE LATTICES. *Every distributive lattice can be represented in a powerset lattice.*

Proof. Let L be a distributive lattice. $P(L)$ is the set of all prime filters in L. Of course, $P(L) \subseteq \text{pow}\, L$.

Let $a \in L$. The principal prime filter generated by a, a^*, is the set: $a^* = \{P \in P(L): a \in P\}$.

The principal prime filter structure generated by L, L^* is the structure $\langle \{a^*: a \in L\}, \cap, \cup \rangle$.

Clearly L^* is a subset of powpow L. That L^* is closed under union and disjunction will be proved below. So $\langle L^*, \cap, \cup \rangle$ is a sublattice of powpow L.

We will prove: L and L^* are isomorphic (in that we will indeed prove that L^* is closed under union and intersection). Clearly if we can prove that, we have proved our theorem.

Let us define function $*$ from L into L^*, by defining: $(a)^* = a^*$. So $*$ maps every element of L onto the principal prime filter generated by a.

1. $*$ is a surjection. This is obvious.
2. $*$ is an injection.

Suppose $a \neq b$. Then by Corollary 3 we know that there is a prime filter containing exactly one of a and b, say, $a \in P$, but $b \notin P$. Then $P \in a^*$, but $P \notin b^*$, hence $a^* \neq b^*$.

3. $$(a \wedge b)^* = \{P: a \wedge b \in P\} = \text{(by definition of filter)}$$ $$\{P: a \in P \text{ and } b \in P\}$$ $$= \{P: a \in P\} \cap \{P: b \in P\} = a^* \cap b^*$$
4. $$(a \vee b)^* = \{P: a \vee b \in P\} = \text{(by definition of prime filter)}$$ $$\{P: \text{either } a \in P \text{ or } b \in P\}$$ $$= \{P: a \in P\} \cup \{P: b \in P\} = a^* \cup b^*$$

This completes the proof.

REPRESENTATION THEOREM FOR BOOLEAN ALGEBRAS. *Every Boolean algebra can be represented in a powerset Boolean algebra.*

Proof. We prove that L is isomorphic with $\langle L^*, -, \cap, \cup, \emptyset, P(L)\rangle$. The proof stays exactly the same, except that the following clauses are added:

4. $0^* = \{P: 0 \in P\} =$ (prime filters are proper) $\emptyset$

5. $1^* = \{P: 1 \in P\} =$ (every filter contains 1) $P(L)$

4. $(\neg a)^* = \{P: \neg a \in P\}$

(every prime filter is an ultrafilter, so either $a \in P$ or $\neg a \in P$)

$$= \{P: a \notin P\} = P(L) - \{P: a \in P\} = P(L) - a^*$$

Which Boolean algebras can be represented *as* powerset Boolean algebras?

We know already of one case: the completely free Boolean algebras coincide with the powpowerset Boolean algebras (up to isomorphism).

Further, we know that every powerset Boolean algebra is a complete atomic Boolean algebra. With the following representation theorem we know that, up to isomorphism, the powerset Boolean algebras are exactly the complete atomic Boolean algebras.

REPRESENTATION THEOREM FOR COMPLETE ATOMIC BOOLEAN ALGEBRAS. *Every complete atomic Boolean algebra can be represented as a powerset Boolean algebra.*

Proof. Let B be a complete atomic Boolean algebra. AT is the set of all atoms in B.

This time we will represent the elements of B through sets of atoms. We define for every $b \in B$:

$$b^* = \{a \in AT: a \leq b\}$$

Let $B^* = \langle\{b^*: b \in B\}, \cap, \cup, -, \emptyset, AT\rangle$. We prove that B and B^* are isomorphic.

1. $*$ Is a surjection. As usual, this is obvious.

2. $*$ Is an injection.

Suppose $b^* = c^*$. That means that $b^* \subseteq c^*$ and $c^* \subseteq b^*$. $b^* \subseteq c^*$ means that

$$\{a \in AT: a \leq b\} \subseteq \{a \in AT: a \leq c\}, \text{ i.e.}$$
$$\forall a: \text{ if } a \in AT \text{ and } a \leq b \text{ then } a \leq c.$$

This means that

$$\bigvee\{a \in AT: a \leq b\} \leq c,$$

that is, $\bigvee b^* \leq c$. We have proved in the previous section that B is atomistic, so $\bigvee b^* = b$, so $b \leq c$. In the same way we show that $c \leq b$, hence $b = c$.

3. $0^* = \emptyset$; $1^* = \{a \in AT: a \leq 1\} = AT$

4.
$$\begin{aligned}(b \wedge c)^* &= \{a \in AT: a \leq b \wedge c\}\\ &= \{a \in AT: a \leq b \text{ and } a \leq c\}\\ &= \{a \in AT: a \leq b\} \cap \{a \in AT: a \leq c\}\\ &= b^* \cap c^*\end{aligned}$$

5. $(b \vee c)^* = \{a \in AT: a \leq b \vee c\} =$

(because a is an atom: if atom $a \leq b \vee c$ then $a \leq b$ or $a \leq c$, we proved that in the previous section)

$$\begin{aligned}&\{a \in AT: a \leq b \text{ or } a \leq c\}\\ &= \{a \in AT: a \leq b\} \cup \{a \in AT: a \leq c\}\\ &= b^* \cup c^*\end{aligned}$$

6. $(\neg b)^* = \{a \in AT: a \leq \neg b\} =$

(because, as we proved in the previous section, if a is an atom then either $a \leq b$ or $a \leq \neg b$)

$$\begin{aligned}&\{a \in AT: a \not\leq b\} = AT - \{a \in AT: a \leq b\}\\ &= AT - b^*\end{aligned}$$

So indeed we have proved that B and B^* are isomorphic.

Since $B^* \subseteq \text{pow}\, B$, this gives us a second proof that every complete atomic Boolean algebra can be represented in a powerset Boolean algebra. We have more information this time, however.

$$B^* = \{b^*: b \in B\} = \{\{a \in AT: a \leq b\}: b \in B\}.$$

Clearly if $X \in B^*$ then $X \subseteq AT$. Suppose $X \subseteq AT$. Then $\bigvee X \in B$ (because B is complete). Then, by definition of $*$,

$$\{a \in AT: a \leqslant \bigvee X\} \in B^*,$$

but clearly,

$$\{a \in AT: a \leqslant \bigvee X\} = X$$

(we proved that in the previous section), hence $X \in B^*$. This means that $B^* = \text{pow}\, AT$. This concludes our proof.

Now we have shown that any complete atomic Boolean algebra is isomorphic to the powerset Boolean algebra of its atoms. Hence, up to isomorphism the powerset Boolean algebras are exactly the complete atomic Boolean algebra.

COROLLARY. *Every finite Boolean algebra is isomorphic to a powerset Boolean algebra.*

Finite Boolean algebras are complete and atomic.

Let a powerset i-join semilattice minus $\emptyset$ be the structure $\langle \text{pow}(X) - \{\emptyset\}, \cup \rangle$ for some set X. It follows from that the free i-join semilattices are exactly the powerset i-join semilattices.

To summarize our findings:

- The finite Boolean algebras are the finite powerset Boolean algebras.
- The complete atomic Boolean algebras are the powerset Boolean algebras.
- The completely free Boolean algebras are the powpowerset Boolean algebras.
- Hence the finite free Boolean algebras are the finite powpowerset Boolean algebras.
- The free i-join semilattices are exactly the powerset i-join semilattices minus $\emptyset$.

A consequence is that there is no complete atomic countable Boolean algebra.

Exercise 21. Prove this.

Here is an example of a Boolean algebra that is not a complete atomic Boolean algebra:

Let X be a set. $A \subseteq X$ is *co-finite* iff $X - A$ is finite.

Let $S(X)$ be the set of all finite and co-finite subsets of X. Then $\langle S(X), -, \cap, \cup, \emptyset, X\rangle$ is a Boolean algebra. If X is countable, then $S(X)$ is countable, hence $S(X)$ cannot be a complete atomic Boolean algebra, $S(X)$ is not isomorphic to any powerset Boolean algebra. Look at $S(\mathbf{N})$. $S(\mathbf{N})$ is not complete.

$$\{\{2\}, \{4\}, \{6\}, \ldots\} \subseteq S(\mathbf{N}), \text{ but } \bigcup\{\{2\}, \{4\}, \{6\}, \ldots\} = \mathbf{E}$$

(the set of even numbers) and $\mathbf{E} \notin S(\mathbf{N})$.

A last theorem without proof:

THEOREM. *Up to isomorphism there is exactly one countable atomless Boolean algebra.*

We can construct the latter algebra in the following way. Take the Lindenbaum Algebra of propositional logic: the set of congruence classes of formulas of propositional logic (with a countable set of atomic formulas) under logical equivalence:

$$\langle [P]_{\leftrightarrow}, -, \cap, \cup, [\bot]_{\leftrightarrow}, [\top]_{\leftrightarrow}\rangle.$$

It is easily checked that this is a Boolean algebra. Moreover, it is countable: there are certainly not *more* blocks than formulas and since every atomic formula is in a *different* block, there are not any fewer blocks either. So it is a countable Boolean algebra.

Is it atomic? Or weaker: are there any atoms? Suppose there is an atom. Then there would have to be a formula φ, which is not a contradiction, such that for every ψ: *φ logically entails ψ or φ logically entails $\neg\psi$.*

We know that there are no such formulas in propositional logic, hence the Lindenbaum algebra is *atomless*. Thus the Lindenbaum algebra of propositional logic is the countable atomless Boolean algebra.

By using this example, we can also prove directly that the countable atomless Boolean algebra is not complete. If it were complete, then for every set of formulas of propositional logic $\{\varphi_1, \ldots, \varphi_n\}$, there would have to be a *formula* of propositional logic ψ such that: $\psi \leftrightarrow \varphi_1 \wedge \cdots \wedge \varphi_n$. This is clearly not the case for propositional logic, so the Lindenbaum algebra of propositional logic is not complete, hence the countable atomless Boolean algebra is not complete.