$$SEP(p(\sigma x.WN(x)) + p(\sigma x.WIJ(x)))$$

we avoid this problem.

ANSWERS TO EXERCISES

CHAPTER ONE

Exercise 1

(a) If φ is not true in M, then for some g: $M \not\models \varphi$. Then (by the lemma) for all g: $M \not\models \varphi$, hence φ is false in M. Similarly, if φ in not false in M, by the same argument φ is true in M. Hence φ cannot be both not true in M and not false in M. So φ is either true or false in M.

(b) We know that for some $d \in D$: $d \in i(P)$ and for some $d' \in D$: $d' \notin i(P)$. Hence we know that for some g (a g where $g(x) = d$) $M \models P(x)[g]$ and that for some g' (a g' where $g'(x) = d'$) $M \not\models P(x)[g]$. The first tells us that $P(x)$ is not true, the second that $P(x)$ is not false, so $P(x)$ is undefined. Of course, the same argument applies to $\neg P(x)$.

(c) Take any assignment g. $M \models P(x) \vee \neg P(x)[g]$ iff $M \models P(x)[g]$ or $M \models \neg P(x)[g]$ iff $g(x) \in i(P)$ or $g(x) \notin i(P)$. The latter is of course true (for any g). So for all g: $M \models P(x) \vee \neg P(x)[g]$, so $P(x) \vee \neg P(x)$ is true in M (in any M, for that matter). A similar argument tells you that $P(x) \wedge \neg P(x)$ is false in M, because there is no g such that $g(x) \in i(P)$ and $g(x) \notin i(P)$.

So you see that, although $P(x)$ and $\neg P(x)$ themselves are undefined, $P(x) \vee \neg P(x)$ and $P(x) \wedge \neg P(x)$ are not.

Exercise 2

$\varphi[x]$ and $\forall x\varphi[x]$ are logically equivalent if for every M: $M \models \varphi[x]$ iff $M \models \forall x\varphi[x]$. $M \models \varphi[x]$ iff for every g: $M \models \varphi[x][g]$ iff for every g and every $d \in D$: $M \models \varphi[x][g_x^d]$ if for every g: $M \models \forall x\varphi[x][g]$ iff $M \models \forall x\varphi[x]$.

Exercise 3

1. Every atomic formula has an equal number of left and right brackets.
2. Assume φ has an equal number of left and right brackets. $\neg\varphi$

doesn't introduce any brackets, so $\neg\varphi$ has the same number of left and right brackets. The same holds for $\exists x\varphi$.

3. Assume that each of φ and ψ has an equal number of left and right brackets. $(\varphi \wedge \psi)$ introduces one left and one right bracket, so $(\varphi \wedge \psi)$ has an equal number of left and right brackets as well.

Exercise 4

(a) Suppose Δ is not deductively closed. This means that for some φ: $\Delta \vdash \varphi$ but $\varphi \notin \Delta$. But then $\Delta \cup \{\varphi\}$ is consistent, and since $\Delta \subseteq \Delta \cup \{\varphi\}$ and $\Delta \neq \Delta \cup \{\varphi\}$, Δ is not maximally consistent.

(b) Suppose that for some φ: $\Delta \nvdash \varphi$ and $\Delta \nvdash \neg\varphi$. Since $\Delta \nvdash \neg\varphi$, $\Delta \cup \{\varphi\}$ is consistent. But then, because Δ is maximally consistent, $\varphi \in \Delta$. If $\varphi \in \Delta$ then $\Delta \vdash \varphi$. Contradiction.

Exercise 5

Soundness

(1) if $\Delta \vdash \varphi$ then $\Delta \models \varphi$

(2) if Δ has a model then Δ is consistent

(2) $\rightarrow$ (1)

Suppose $\Delta \nvDash \varphi$. If $\Delta \nvDash \varphi$, Then for some model M: $M \models \Delta$ and $M \nvDash \varphi$, hence $\Delta \cup \{\neg\varphi\}$ has a model. Then (by 2) $\Delta \cup \{\neg\varphi\}$ is consistent, but then $\Delta \nvdash \varphi$.

(1) $\rightarrow$ (2)

Suppose Δ is inconsistent. Then $\Delta \vdash \bot$. Hence (by 1) $\Delta \models \bot$. But that means that Δ does not have a model.

Completeness

(1) if $\Delta \models \varphi$ then $\Delta \vdash \varphi$

(2) if Δ is consistent then Δ has a model

(2) $\rightarrow$ (1)

Suppose $\Delta \nvdash \varphi$. If $\Delta \nvdash \varphi$, then $\Delta \cup \{\neg\varphi\}$ is consistent. Then (by 2) $\Delta \cup \{\neg\varphi\}$ has a model, but then $\Delta \nvDash \varphi$.

(1) $\rightarrow$ (2)

Suppose Δ doesn't have a model. Then $\Delta \models \bot$. Hence (by 1) $\Delta \vdash \bot$, so Δ is inconsistent.

Exercise 6

Suppose every finite subset of Δ has a model. Then (by completeness) every finite subset of Δ is consistent. Then (by definition of consistency) Δ is consistent. Then (by completeness) Δ has a model.

Exercise 7

Fact 3: If φ were to define infinity then $\neg\varphi$ would define finiteness.

Exercise 8

Fact 4: Our set of sentences $\{\varphi_1, \varphi_2, \ldots\}$ defines infinity.

Exercise 9

$2 + 2 = 4$ means $SS0 + SS0 = SSSS0$.
$SS0 + SS0 =$ (axiom 4) $S(SS0 + S0) =$ (axiom 4) $SS(SS0 + 0) =$ (axiom 3) $SSSS0$.

Exercise 10

The statements expressing that $\subseteq$ is reflective and transitive are just logical tautologies. Antisymmetry follows from logic and extensionality. If $\forall x[x \in A \rightarrow x \in B]$ and $\forall x[x \in B \rightarrow x \in A]$ then $\forall x[x \in A \leftrightarrow x \in B]$, hence with extensionality $A = B$.

Exercise 11

Given set A. Comprehension and extensionality tell you that the unique set $\{a \in A\colon a \neq a\}$ exists, this is of course $\emptyset$.

Exercise 12

ZF is a first order theory. If ZF is consistent, then any subset of ZF is obviously also consistent, so ZF $-$ {F} is consistent.

Exercise 13

Well, nothing because, as I said, we can prove it from the other axioms. But that is not what I meant, suppose we leave out both substitution and separation, what happens? The theory is not going to be inconsistent, because the same argument as above applies. You are just for a lot of sets unable to prove in the resulting theory that they exist.

CHAPTER TWO

Exercise 1

(a) $\langle \mathbf{E}, < \rangle$ and $\langle \mathbf{0}, < \rangle$ are substructures of $\langle \mathbf{N}, < \rangle$. $\langle \mathbf{E}, <, + \rangle$ is a substructure of $\langle \mathbf{N}, <, + \rangle$, But not $\langle \mathbf{0}, <, + \rangle$, because $\mathbf{0}$ is not closed under $+$. Also not $\langle \mathbf{E}, < \rangle$ and $\langle \mathbf{0}, < \rangle$, because they are of the wrong type.

(b) Say that the type of **A** is a subtype of the type of **B** if **B** has relations, operations, special elements corresponding to all relations, operations and special elements of **A**, but maybe more. Let **B**′ be the result of disregarding in **B** all the relations, operations and special elements that do not correspond to any in **A**. Then **A** is a generalized substructure of **B** iff **A** is a substructure of **B**′.

Besides the ones that are already substructures, $\langle \mathbf{E}, < \rangle$ is a generalized substructure of $\langle \mathbf{N}, <, + \rangle$.

Exercise 2

We know already that $\lambda x.g(f(x))$ is a function from A into C. Suppose $R_A(a, a')$. Then, because f is a homomorphism, $R_B(f(a), f(a'))$, and because g is a homomorphism,

$$R_C(g(f(a)), g(f(a'))).$$

$$g(f(a *_A a'))$$

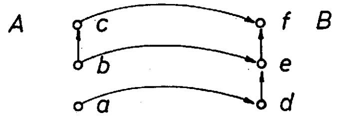
$$= g(f(a) *_B f(a')) \text{ (because } f \text{ is a homomorphism)}$$

$$= g(f(a)) *_C g(f(a')) \text{ (because } g \text{ is a homomorphism).}$$

Of course, $g(f(a)) = a$, for the special elements.

Exercise 3

1. h is an isomorphism between A and $h(A)$.

2. 

There is no relation between a and the other elements in A, so, trivially, the indicated function preserves the relation. The function is one–one, so indeed it is a bijective homomorphism. But not an isomorphism, because the relation between d and e is not preserved in A.

Exercise 4

(A) Relational structures
- (a) not a homomorphism, so it is none of the others either.
- (b) only a homomorphism.
- (c) again only a homomorphism
- (d) epimorphism
- (e) embedding
- (f) embedding
- (g) isomorphism
- (h) isomorphism
- (i) automorphism (the trivial automorphism, given by identity)
- (j) a non-trivial automorphism
- (k) isomorphism. Strictly speaking not an automorphism, because the first structure is $\langle A, \leqslant \rangle$ and the second $\langle A, \geqslant \rangle$. Of course, up to duality those two structures are the same, so in this sense it is an automorphism.
- (l) homomorphism

(B) Algebras.
- (a) of course also not a homomorphism.
- (b) homomorphism

(c) not a homomorphism: $b \wedge c = a$, $f(b \wedge c) = f$, but $f(b) \wedge f(c) = g \wedge g = g$

(d) not a homomorphism: for the same reason as under (c).

Remark: note the difference between (c) and (d): (c) does preserve $\vee$ (it is called a join-homomorphism), but not $\wedge$. (d) preserves neither $\vee$ nor $\wedge$.

(e) embedding

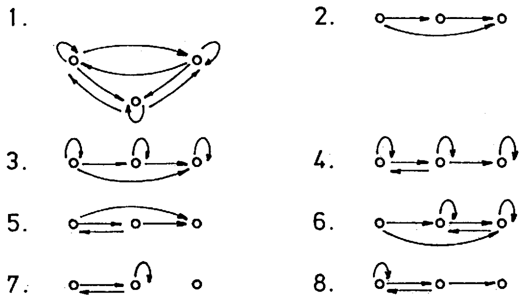
(f) not a homomorphism $c \vee d = e$, but $f(c \vee d) = l$ and $f(c) \vee f(d) = k$.

(g) and (h) are isomorphisms; (i) and (j) are automorphisms; (k) is again an isomorphism (and an automorphism up to duality).

(l) homomorphism

Exercise 5

(a)



(b) Suppose for some a: $R(a, a)$, then of course $R(a, a)$ and $R(a, a)$, and hence for some a, b $(a = b)$: $R(a, b)$ and $R(b, a)$.

(c) Suppose $R(a, b)$ and $R(b, a)$. Then, by transitivity, $R(a, a)$, which contradicts irreflexivity.

(d)

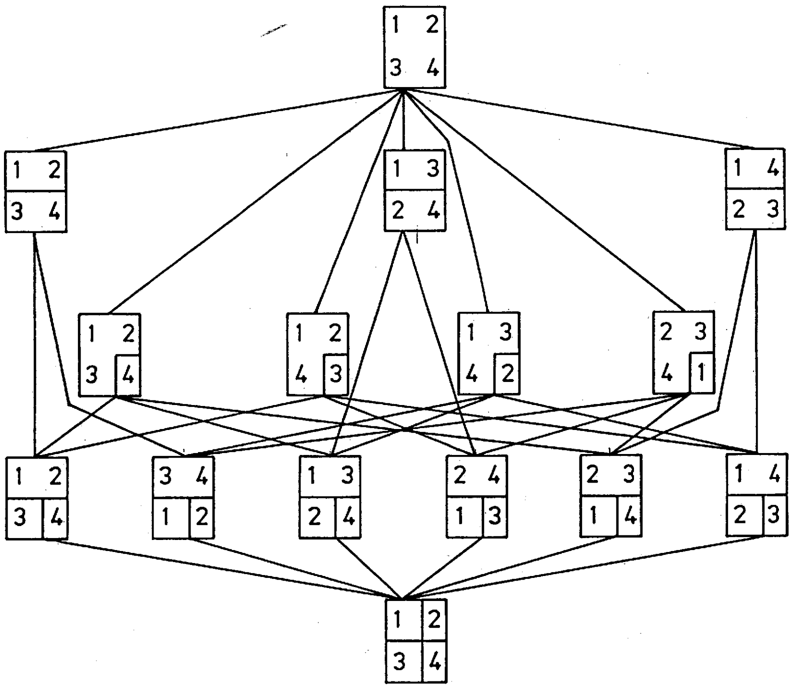
Exercise 6

1. $\leqslant$ is reflexive, $a \geqslant a$ iff $a \leqslant a$, so $\geqslant$ is reflexive.

2. Assume $a \geqslant b$ and $b \geqslant c$. Then we know that $c \leqslant b$ and $b \leqslant a$. $\leqslant$ is transitive, so $c \leqslant a$, hence $a \geqslant c$.

3. Assume $a \geqslant b$ and $b \geqslant a$. Then $b \leqslant a$ and $a \leqslant b$ and hence $a = b$.

Exercise 7



Exercise 8

1. Suppose $b \in [a]_{\approx}$. i.e. $b \in \{c: c \approx a\}$, so $b \approx a$. Then $a \approx b$ (because $\approx$ is symmetric), hence $a \in [b]_{\approx}$. Now, if $c \in [a]_{\approx}$, then because $a \in [b]_{\approx}$ and $\approx$ is transitive, $c \in [b]_{\approx}$. The same, the other way round, of course: if $c \in [b]_{\approx}$, then $c \in [a]_{\approx}$, because $b \in [a]_{\approx}$ and $\approx$ is transitive.

2. Since $\approx$ is reflexive, $a \approx a$, so $a \in [a]_{\approx}$ (for any a).

3. Suppose that $[a]_{\approx} \cap [b]_{\approx} \neq \emptyset$, say, $c \in [a]_{\approx}$ and $c \in [b]_{\approx}$. Then, by (1) $[a]_{\approx} = [c]_{\approx}$ and $[b]_{\approx} = [c]_{\approx}$, and hence $[a]_{\approx} = [b]_{\approx}$.

4. Suppose $a \in A$. Then $a \in [a]_{\approx}$ and hence $a \in \cup \{[b]_{\approx}: b \in A\}$. Suppose $a \in \cup \{[b]_{\approx}: b \in A\}$. Then for some $[b]_{\approx}$ $(b \in A)$: $a \in [b]_{\approx}$. $[b]_{\approx} = \{a \in A: a \approx b\}$, hence by definition of $[b]_{\approx}$ $a \in A$.

Exercise 9

Suppose $X \leqslant Y$ and $Y \leqslant X$. Then for some $a \in X$, $b \in Y$: aRb and for some $b' \in Y$, $a' \in X$: $b'Ra'$. We then know the following:

aRb; bRb' (by def. of Y), $b'Ra'$, $a'Ra$ (by def. of X), hence, by transitivity, bRa, but that means (by def. of X) that $b \in X$. Hence $X \cap Y \neq \emptyset$, and thus $X = Y$ (partition).

Exercise 10

Proof of the lemma. (a) Suppose $\exists c$: $c \leqslant b$ and $d(a) = d(c)$. Obviously $d(b) = d(c) + |\{d: c < d \leqslant b\}|$, so $d(c) \leqslant d(b)$ and hence $d(a) \leqslant d(b)$.

(b) Suppose $d(a) \leqslant d(b)$. Clearly the only interesting situation is the one where a and b are on different branches, where they are incomparable. So suppose that to be the case and say, $d(a) = m$ and $d(b) = n$, with $m \leqslant n$. So, $|\{c: c \leqslant b\}| = n - 1$. But, of course, we can just count back in $\{c: c \leqslant b\}$, until we find an element $c \in \{c: c \leqslant b\}$, such that $|\{d: d \leqslant c\}| = m - 1$: $c \leqslant b$ and $d(c) = d(a)$.

Proof of the theorem.

Transitivity. Suppose $X \leqslant Y$ and $Y \leqslant Z$. Then for some $a \in X$, $b \in Y$, $a \leqslant b$ and for some $b' \in Y$, $c \in Z$: $b' \leqslant c$. We know that $a \leqslant b$, hence $d(a) \leqslant d(b)$; we know that $d(b) = d(b')$, hence we know that $d(a) \leqslant d(b')$. Then, by the lemma, we know that there is an $a' \in X$, such that $a' \leqslant b'$. By transitivity, $a' \leqslant c$, hence indeed $X \leqslant Z$.

Antisymmetry.

Suppose $X \leqslant Y$ and $Y \leqslant X$. That means that for some $a \in X$ $b \in Y$ $a \leqslant b$ and for some $b' \in Y$ $a' \in X$ $b' \leqslant a'$. We know (by def.) that $d(b) = d(b')$ and $d(a) = d(a')$. Since $a \leqslant b$ and $b' \leqslant a'$, it then follows that $d(a) \leqslant d(b)$ and $d(b) = d(b') \leqslant d(a') = d(a')$, so $d(b) \leqslant d(a)$, hence $d(a) = d(b)$, and thus, by definition of X, $b \in X$. Hence, $X \cap Y \neq \emptyset$ and hence $X = Y$.

Linearity: $X \leqslant Y$ or $Y \leqslant X$.

Suppose $X \nleqslant Y$. Then for all $a \in X$, all $b \in Y$: $a \nleqslant b$. We know X and Y are non-empty, so take $a \in X$ and $b \in Y$.

$a \nleqslant b$. Now suppose that $d(a) \leqslant d(b)$. Then for some $a' \in X$: $a' \leqslant b$ (by the lemma), contradicting the assumption. So we know that $d(b) < d(a)$. But then, again by the lemma, we know that for some $b' \in Y$: $b' < a$, and hence we know that $Y \leqslant X$.

Exercise 11

We know that in a semitree for every a and b, there is a unique latest point, earlier than both a and b, call this $\bigcirc(a, b)$, the origin of a and b. If $x \leqslant y$, let us define $d(y, x)$, the distance from y to x as: $d(y, x) = |\{z: x \leqslant z \text{ and } z \leqslant y\}| - 1$.

Since $\bigcirc(a, b) \leqslant a$ and $\bigcirc(a, b) \leqslant b$, $d(a, \bigcirc(a, b))$ and $d(b, \bigcirc(a, b))$ are both well defined. So we can define equidistance as:

$$\approx := \lambda a \lambda b.\ d(a, \bigcirc(a, b)) = d(b, \bigcirc(a, b))$$

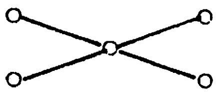
This is obviously going to give us the right result.

Exercise 12

Assume that for some b: $a \leqslant b$ and $a' \leqslant b$. Look at $\{c: c \leqslant b\}$. This is, by definition a linear order, and since both a and a' are in it, it follows that $a \leqslant a'$ or $a' \leqslant a$.

Exercise 13

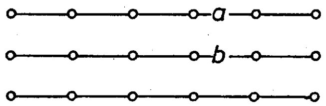
(a)



(b) Take any two minimal elements a and b. Either $a \leqslant b$ or $b \leqslant a$ (by linearity). But $a < b$ is false, because b is minimal, and similarity $b < a$ is false, hence $a = b$.

Exercise 14

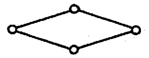
(a) The following poset is an example:



So, the posit can be a collection of linear orders.

(b) $\forall x \forall y \exists z[(z \leqslant x \wedge z \leqslant y) \vee (x \leqslant z \wedge y \leqslant z)]$. This excludes the previous example: take any a and b: they have a common predecessor c or a common successor, say a predecessor c. Then, by no branching to the future $a \leqslant b$ or $b \leqslant a$. Similarly, if they have a common successor, you use no branching to the past.

(c)



(d) A treelike structure is a strict partial order, which is not branching to the past and where every two elements have a common predecessor.

Exercise 15

(a) Because of the following possibility:

a has a direct successor, but the branch between a and b may be dense.

(b) $\langle T, <\rangle$ is discreet iff

$$\forall a \forall b[a < b \rightarrow \exists c[a < c \leqslant b \wedge \neg \exists d[a < d < c]]$$

Exercise 16

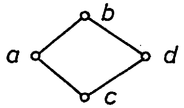
(a) T itself is a linear order in T, so it has a minimal element.

(b) I. Suppose T is not discreet to the future. Then there is an element a that has successors, but no direct successor. Look at $\{b: a < b\}$, the set of all a's successors. This set doesn't have a minimal element (because that would be a's direct successor. This contradicts well foundedness.

II. The ordinal numbers ordered by $<$ are a good example: this structure is linear, continuing to the future and well founded. **N** with a copy of **N** behind it is another example. These structures are well founded all right, but not discreet, because ω in the ordinal numbers and the second 0 in $\mathbf{N} * \mathbf{N}$ don't have a direct predecessor.

Exercise 17

Intuitively we want $\{a, b, d\}$ and $\{a, c, d\}$ to be intervals in a structure like:



They are chains all right, but not convex sets: not everything between a and d is in $\{a, b, d\}$.

Exercise 18

T_1 and T_2 are chains (because T is already linear). Suppose $t < t' < t''$ and $t, t'' \in T_1$. Since the cut is a bipartition, either $t' \in T_1$ or $t' \in T_2$. Suppose $t' \in T_2$: then $t'' \in T_1$ and $t' \in T_2$ but not $t'' < t'$, contradicting the definition of cut. So $t' \in T_1$, and T_1 is convex. The same argument for T_2.

Exercise 19

Suppose $\langle T_1, T_2\rangle$ determines a transition. Then either T_1 has a maximum and T_2 no minimum or the other way around. Suppose T_1 has a maximum m. m does have successors (because T_2 is non-empty), but doesn't have a direct successor (because T_2 does not have a minimum). So T is not discreet to the future. A similar argument holds for the other case.

Suppose T is not discreet; say, not discreet to the future. That means that there is some element t that has successors, but no direct successor. Look at $\langle\{t': t' \leqslant t\}, \{t': t < t'\}\rangle$. This is a cut that determines a transition (t is the maximum of the left part, the right part doesn't have a minimum).

The same argument if T is not discreet to the past.

Exercise 20

The simplest way to go is to simply use the definitions of tree and treelike structure.

A *diabolo* is a structure $\langle A, \leqslant, m\rangle$ where

1. $\langle\{a \in A\colon m \leqslant a\}, \leqslant, m\rangle$ is a tree

2. $\{\{a \in A\colon a \leqslant m\}, \geqslant, m\rangle$ is a tree

3. $\forall a \in A\colon a \leqslant m$ or $m \leqslant a$.

A *diabolo-like structure* is a structure $\langle A, \leqslant, m\rangle$ where

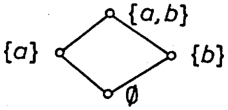
1. $\langle\{a \in A\colon m \leqslant a\}, \leqslant, m\rangle$ is a tree-like structure with origin m

2. $\langle\{a \in A\colon a \leqslant m\}, \geqslant, m\rangle$ is a tree-like structure with origin m

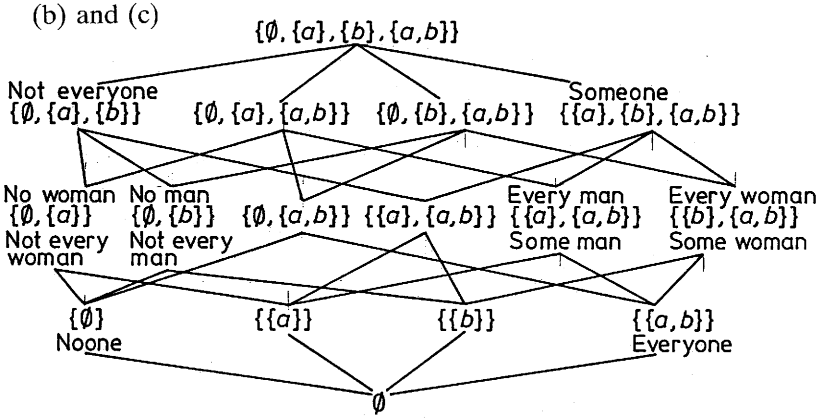
3. $\forall a \in A\colon a \leqslant m$ or $m \leqslant a$

Exercise 21

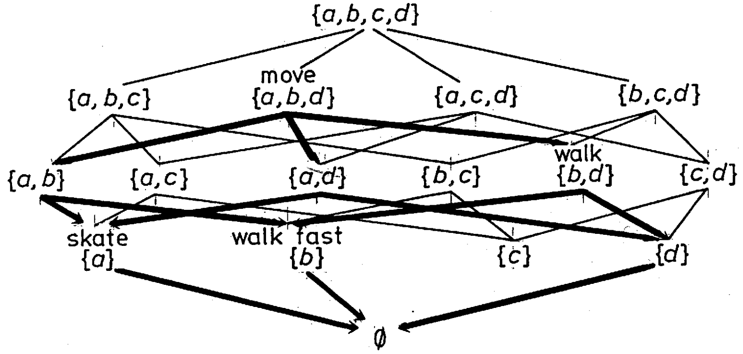
(a)



(b) and (c)



(d)



(e) $[\![\text{No woman}]\!] = \{\emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, d\}\}$

No woman moves entails *No woman walks* entails *No woman walks fast*
No woman moves entails *No woman skates*

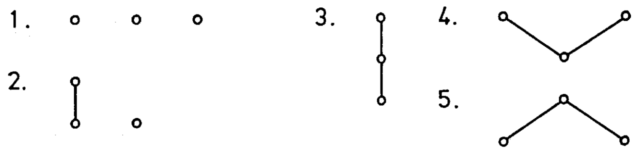
Downward entailing:

if $X \in [\![\text{No woman}]\!]$ and $Y \subseteq X$ then $Y \in [\![\text{No woman}]\!]$

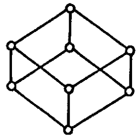
See the graph under (d).
Other example: *at most three* men

Exercise 22

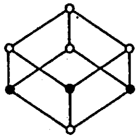
The posets with three elements are:



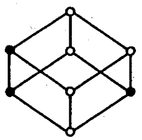
The poset with eight elements is:



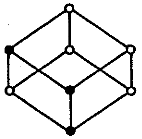
1.



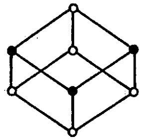
2.



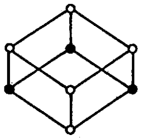
3.



4.



5.



CHAPTER THREE

Exercise 1

(a) $PP\varphi \rightarrow P\varphi$ is true on all transitive frames.

Let F be transitive and assume that in some point t $PP\varphi$ is true. That means that for some $t' < t$ $P\varphi$ is true, and hence for some $t'' < t'$ φ is true. Since $\leqslant$ is transitive, it follows that $t'' < t$, and hence $P\varphi$ is true in t.

(b) If F is not transitive we can give a counterexample to $PP\varphi \rightarrow P\varphi$. Let F be non-transitive, say $t'' < t'$ and $t' < t$ but not $t'' < t$. Assign 1

to p in t'', and 0 to p in t' (and to all t' such that $t' < t$). PPp is true in t, but Pp is not true in t. So $PPp \rightarrow Pp$ is false in t, and hence $PP\varphi \rightarrow P\varphi$ is not true on the frame.

Exercise 2

$F\varphi \rightarrow FF\varphi$ expresses density.

(a) $F\varphi \rightarrow FF\varphi$ is true on all dense frames.

Let F be a dense frame. Assume $F\varphi$ is true in t. That means that for some $t' > t$ φ is true in t'. Since F is dense, there is a t'' between t and t'. In t'' $F\varphi$ is true, hence in t $FF\varphi$ is true.

(b) Let F be a non-dense frame. Say, $t < t'$ and nothing is in between t and t'. Assign 1 to p in t', and 0 in all t'' later than t besides t'. Then Fp is true in t, but FFp is false in t. Hence $Fp \rightarrow FFp$ is false in t, hence $F\varphi \rightarrow FF\varphi$ is not true on the frame.

Exercise 3

(a) Let T be a tautology (for instance $p \vee \neg p$).

$$P\varphi := S(\varphi, T)$$

This says that there is a moment in the past where φ was true and at every later moment a tautology was true, which says just that there is a moment in the past where φ was true.

$$F\varphi := U(\varphi, T)$$

$$H\varphi := \neg P \neg \varphi; \quad G\varphi := \neg F \neg \varphi$$

(b) $$U(\varphi \wedge U(\neg \varphi \wedge \neg U(\varphi, T), \varphi), \neg \varphi)$$

At some moment t in the future φ is true and before that it was never true. At some moment t' later than t $\neg\varphi$ is true and between t and t' φ is true at every moment. After t' φ is never true again. In other words: after having been false uninterruptedly (from the present), φ will be uninterruptedly true for a while and then never be true again after.

Exercise 4

The semantics given for *before* is:

$$pBq(t_0) \text{ iff } \exists t_1 < t_0[q(t_1) \wedge \exists t_2 < t_1[p(t_2)]]$$

$pB(q \vee r)$ does not entail pBq: if there is a moment in the past where p is true and a later moment where either p or q is true, it doesn't follow that there is a moment in the past where p is true and a later moment where q is true.

The idea is: p before q means: there is a moment in the past where p is true and that moment is before every moment where q is true.

Two alternatives:

$$pBq(t_0) \text{ iff } \exists t_1 < t_0[p(t_1) \wedge \exists t_2 \colon t_1 < t_2 < t_0[q(t_2) \wedge \forall t[t < t_0 \wedge q(t) \rightarrow t_1 < t]]]$$

or

$$pBq(t_0) \text{ iff } \exists t_1 < t_0[p(t_1) \wedge \forall t_2[t_2 < t_0 \wedge q(t_2) \rightarrow t_1 < t_2]]$$

The first extends the given clause, the second changes it. They share the crucial point that the proposition in the before context is analyzed as being in the antecedent clause of a universal statement, which is a downward entailing context. The second definition makes the before clause indeed downward entailing, the first does not, for the same reason as the original clause was not downward entailing.

Take *John moved before Mary moved* and *John moved after Mary moved.*

Try to cancel the entailments/implicatures:

(a) John moved before Mary moved. In fact, John never moved.
(b) John moved before Mary moved. In fact, Mary never moved.
(c) John moved after Mary moved. In fact, John never moved.
(d) John moved after Mary moved. In fact, Mary never moved.

Of these sentences (a), (c) and (d) are clearly contradictions. But (b) doesn't seem to be a contradiction. Consequently, *Mary moved* is an implicature in (b), but an entailment in (d), and *John moved* is an entailment in both.

$$\neg(pAq)(t_0) \text{ iff } \forall t_1 < t_0[p(t_1) \rightarrow \forall t_2 < t_1[\neg q(t_2)]]$$

$$\neg(pBq)(t_0) \text{ iff } \forall t_1 < t_0[q(t_1) \rightarrow \exists t[t < t_0 \wedge p(t) \wedge t_1 \not< t]]$$

A model that makes pAq true but qBp false is:

t——t'——t''

p $\quad q \quad$ p

So, First John hit Mary, then she hit him, then he hit her again. If the judge asks me: Did John hit Mary after she hit him, I will certainly say yes. But if the judge asks me: Did Mary hit John before he hit her, I tend to say no, because John started.

CHAPTER FOUR

Exercise 1

The illusion of incompatibility arises if in this dense structure you only check the witness* condition from left to right, i.e. you look at z and z' where $z < z'$, and for those you can't satisfy witness*. But z' and z are such that z' doesn't precede z, and certainly no subperiod of z' precedes any subperiod of z, so there is no incompatibility.

Exercise 2

1. $<$ is a strict partial order:
Irreflexivity: assume $p < p$. Then for every $t \in p$: $t < t$.
Transitivity: assume $p < q$ and $q < r$. That means that for every $t \in p$ and every $t' \in q$: $t < t'$. Similarly for every $t' \in q$ and for every $t'' \in r$, $t' < t''$. Clearly then for every $t \in p$ and every $t' \in r$: $t < t'$, hence $p < r$.
2. $\sqsubseteq$ is a partial order: this is obvious because $\subseteq$ is a partial order.
3. Monotonicity: Assume $p < q$ and $r \sqsubseteq p$. $p < q$ means that every point in p is before every point in q. $r \sqsubseteq p$ means that $r \subseteq p$, hence clearly every point in r is before every point in q. The other one goes similarly.
4. Convexity. Let $p < q < r$ and $p \sqsubseteq s$ and $r \sqsubseteq s$. This means that p and r are subsets of s and every point in q is later than every point in p but earlier than every point in r. s is a period, hence it is convex, so every point in q is in s, hence $q \sqsubseteq s$.

5. Conjunction: this obviously follows from the second clause of period set.

Exercise 3

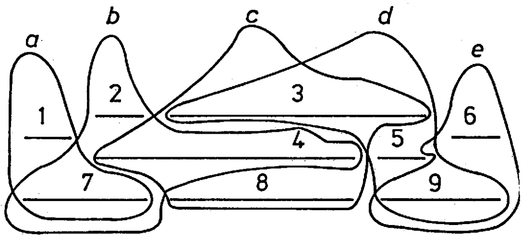
1. Assume $q \in [p)$ and $q \sqsubseteq r$. Then $p \sqsubseteq r$ and hence $r \in [p)$.
2. Assume $q, r \in [p)$. Then $p \sqsubseteq q$ and $p \sqsubseteq r$, hence $p \sqsubseteq q \sqcap r$, hence $q \sqcap r \in [p)$.
3. $p \in [p)$, hence $[p)$ is not empty, hence $[p)$ is consistent.

Exercise 4

1. Assume $p \bigcirc_1 q$. That means that for some e in p and some e' in q: $e \bigcirc e'$. The latter means that for some e'': $e'' \sqsubseteq e$ and $e'' \sqsubseteq e'$. Take $r = [e'']_{\approx}$. $r \sqsubseteq p$ and $r \sqsubseteq$ q, hence $p \bigcirc_2$ q.
2. Assume $p \bigcirc_2 q$. Then for some r: $r \sqsubseteq p$ and $r \sqsubseteq q$. Hence for some $e \in r$ and some $e' \in p$: $e \sqsubseteq e'$ and for some $e'' \in r$ and some $e''' \in q$: $e'' \sqsubseteq e'''$. Since $e, e'' \in q$: $e \approx e''$. Hence $e \sqsubseteq e'''$, so $e \sqsubseteq e'$ and $e' \in p$ and $e \sqsubseteq e'''$ and $e''' \in q$, hence $e' \bigcirc e'''$, hence $p \bigcirc_1 q$.

Exercise 5

(a)

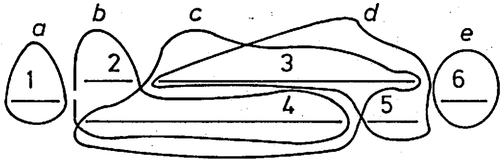


$a = \{1, 7\}$; $b = \{2, 4, 7\}$; $c = \{3, 4, 8\}$; $d = \{3, 5, 9\}$; $e = \{6, 9\}$

$a < b < c < d < e$

$\{a\}$ $\{a, b\}$ $\{b\}$ $\{b, c\}$ $\{c\}$ $\{c, d\}$ $\{d\}$ $\{d, e\}$ $\{e\}$

(b)



$a = \{1\}$; $b = \{2, 4\}$; $c = \{3, 4\}$; $d = \{3, 5\}$; $e = \{6\}$

$a < b < c < d < e$

Let's try to reconstruct the periods. What we would want is:

$\{a\}$ $\{b\}$ $\{b, c\}$ $\{c, d\}$ $\{d\}$ $\{e\}$

But this is not a period set, because $\{b, c\} \cap \{c, d\} = \{c\}$, but there is no period $\{c\}$.

CHAPTER FIVE

Exercise 1

q overlaps the beginning of p: $q \bigcirc p \wedge \neg \exists r[r \sqsubseteq p \wedge r < q]$. Similarly q' overlaps the end of p.

Exercise 2

Assume that the period structure is dense and assume that $B\varphi$ holds at p. That means that p can be split in p' where φ is false and p'' where φ is true, with $p' < p''$. Since the period structure is dense, we can split p' into q and q', with $q < q'$ and p'' into r and r' with $r < r'$. Consequently, φ is false at q' and true at r. But then $q' \sqcup r$ satisfies condition (1), hence $B\varphi$ does not hold at p.

$B\varphi$ can only hold at periods p that are the union of two atomic periods t and t' where at t φ is false and at t' φ is true. Clearly in that case condition 1 is satisfied and there is no subperiod for which that holds (clearly $B\varphi$ cannot be true in an atomic period).

Suppose that p is not the union of two atoms but satisfies Condition 1. Partition p in the part where φ is false and the part where φ is true.

There is either a jump, a transition or a gap between those periods (where a jump means that there is an atom overlapping the end of the part where φ is false and an atom overlapping the beginning of the part where φ is true). In all these cases you can find a subperiod of p satisfying Condition 1.

Exercise 3

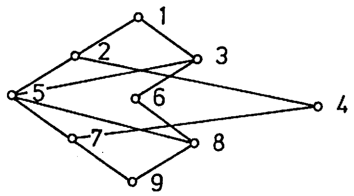
φ	$\neg\varphi$	$A\varphi$	$\neg A\varphi$	$\varphi\wedge\neg\varphi$	$\varphi\vee\neg\varphi$	$\varphi\wedge\neg A\varphi$	$\varphi\vee\neg A\varphi$	$A\varphi\wedge\neg A\varphi$	$A\varphi\vee\neg A\varphi$
1	0	1	0	0	1	0	1	0	1
0	1	0	1	0	1	0	1	0	1
*	*	0	1	*	*	*	1	0	1

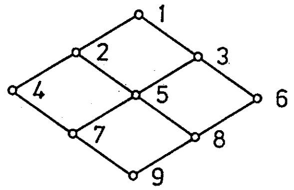
CHAPTER SIX

Exercise 1

(a) The first one is a lattice, the second isn't: there is not a unique smallest element that is the join of the two elements directly above the bottom.

(b)





It is easy to check that all the joins and meets are the same.

We have to show that for all $X\subseteq A$: $\bigvee X\in A$. Let $X\neq\emptyset$. Look at $\{y\in A\colon \forall x\in X[x\leqslant y]\}$. This is a subset of A, hence $\bigwedge\{y\in A\colon\ \forall x\in X[x\leqslant y\}\in A$. This is, of course, $\bigvee X$. Let $X=\emptyset$.

Since there is no $y\in\emptyset$, trivially $\forall y\in A\forall x\in\emptyset[x\leqslant y]$. Hence $\{y\in A\colon\ \forall x\in\emptyset[x\leqslant y]\}=A$. Hence $\bigwedge\{y\in A\colon\ \forall x\in\emptyset[x\leqslant y]\}=\bigwedge A$. So $\bigvee\emptyset=\bigwedge A$.

(c) This follows directly from the previous considerations: since A is complete $\bigvee A$ and $\bigwedge A$ are in A. $\bigvee A$ is the maximum of A, $\bigwedge A$ is the minimum of A.

(d) Suppose A has two distinct maximal elements a and b. Since A is a lattice $a\vee b$ exists and $a\leqslant a\vee b$ and $b\leqslant a\vee b$. But then a and b wouldn't be maximal.

(e) If A is a finite lattice then every subset X of A is finite. Then it follows by the theorem given that for every non-empty subset X of A $\bigvee X, \bigwedge X\in A$. This means that A has a minimum $\bigwedge A$ and a maximum $\bigvee A$. By the argumentation under (a) $\bigvee\emptyset=\bigwedge A$, similarly $\bigwedge\emptyset=\bigvee A$, hence $\bigvee\emptyset, \bigwedge\emptyset\in A$, so A is complete.

Exercise 2

Let $X\subseteq \text{pow}\,B$. Then obviously $\cup X, \cap X\subseteq B$, hence $\cup X, \cap X\in \text{pow}\,B$.

Exercise 3

The singleton sets.

Exercise 4

Define in terms of the intersection operation $\cap$ in the interval lattice the new operation $\wedge$:

Let i and i' be non-singleton intervals. Define:

$$\begin{aligned} i\wedge i' &:= i\cap i' \text{ if } i\cap i' \text{ is not a singleton}\\ &:= \emptyset \text{ if } i\cap i' \text{ is a singleton}\end{aligned}$$

Then eleiminate all singletons from the structure. The structure is a complete atomless lattice. To get a structure that is neither atomless, nor atomic, leave, say, one atom $\{t\}$ and modify the above definition replacing 'not a singleton' by 'not a singleton except $\{t\}$' twice.

Exercise 5

A.1. $\langle A, *\rangle$ is a semilattice, so $*$ is idempotent, commutative and associative. Let $\leqslant_\wedge$ be as given.

a. $\leqslant_\wedge$ is a partial order.

– $\leqslant_\wedge$ is reflexive: $a \leqslant_\wedge a$. This means $a * a = a$, which is idempotency.

– $\leqslant_\wedge$ is transitive. Assume $a \leqslant_\wedge b$ and $b \leqslant_\wedge c$. $a * b = a$ and $b * c = b$. Then $a * c = a * b * c$ (using the first) $= a * b$ (using the second) $= a$ (using the first).

– $\leqslant_\wedge$ is antisymmetric. Assume $a \leqslant_\wedge b$ and $b \leqslant_\wedge a$. This means $a * b = a$ and $b * a = b$, then, by commutativity $a = b$.

b. $*$ is meet in $\leqslant_\wedge$

– $a * b \leqslant_\wedge a$. This means $(a * b) * a = a * b$. $(a * b) * a = (b * a) * a = b * (a * a) = b * a$.

– Similarly, $a * b \leqslant_\wedge b$.

– Assume $c \leqslant_\wedge a$ and $c \leqslant_\wedge b$. Then $c * a = c$ and $c * b = c$. Then $c * (a * b) = (c * a) * b = c * b = c$. Hence $c \leqslant_\wedge a * b$, so indeed $a * b$ is the meet of a and b. So we have proved that $\langle A, \leqslant_\wedge\rangle$ is a meet semilattice.

2. The argument that $\langle A, \leqslant_\vee\rangle$ is a join semilattice goes in exactly the same way:

a. $\leqslant_\vee$ is a partial order.

– $\leqslant_\vee$ is reflexive: $a \leqslant_\vee a$. This means $a * a = a$, which is idempotency.

– $\leqslant_\vee$ is transitive. Assume $a \leqslant_\vee b$ and $b \leqslant_\vee c$. $a * b = b$ and $b * c = c$. Then $a * c = a * b * c$ (using the second) $= \mathrm{b} * \mathrm{c}$ (using the first) $= c$ (using the second).

– $\leqslant_\vee$ is antisymmetric. Assume $a \leqslant_\vee b$ and $b \leqslant_\vee a$. This means $a * b = b$ and $b * a = a$, then, by commutativity $a = b$.

b. $*$ is join in $\leqslant_\vee$.

– $a \leqslant_\vee a * b$. This means $a * (a * b) = a * b$. $a * (a * b) = (a * a) * b = a * b$.

– Similarly, $b \leqslant_\vee a * b$.

– Assume $a \leqslant_\vee c$ and $b \leqslant_\vee c$. Then $a * c = c$ and $b * c = c$. Then $(a * b) * c = a * (b * c) = a * c = c$, hence $a * b \leqslant_\vee c$, so indeed $a * b$ is the join of a and b. So we have proved that $\langle A, \leqslant_\vee\rangle$ is a join semilattice.

B. Let $\langle A, \leqslant\rangle$ be a join semilattice and $\langle A, \vee\rangle$ be the semilattice we get when we take join as an operation. We define in $\langle A, \vee\rangle$: $a \leqslant' b$ iff $a \vee b = b$.

Claim: $a \leqslant' b$ iff $a \leqslant b$. This means that we have to prove: $a \leqslant b$ iff $a \vee b = b$. This just follows from the fact that $\vee$ is join in $\langle A, \leqslant\rangle$.

Let $\langle A, *\rangle$ be a semilattice. Define $a \leqslant_\vee b := a * b = b$. Form $\langle A, \leqslant_\vee\rangle$. This is, as proved, a join semilattice. We form $\langle A, \vee\rangle$.

Claim: $a \vee b = a * b$. This follows obviously from the fact that we defined $\leqslant_\vee$ that way.

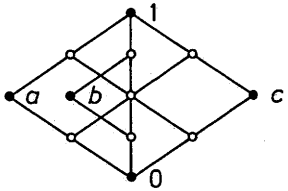
C. The only thing that does hold is that we can regard a join semilattice as a meet semilattice (by turning it upside down). $\langle A, \leqslant_\wedge\rangle$ is a meet semilattice and $\langle A, \leqslant_\vee\rangle$ a join semilattice, but they are not the same structure, in particular $\leqslant_\wedge \neq \leqslant_\vee$. In fact, $\leqslant_\wedge = \geqslant_\vee$.

Exercise 6

The definition of atomic says that every non-zero element has an atom below it. Atoms, however are defined as elements that have only 0 below them. The definition of atom, hence, presupposes that the structure has a zero, hence $\langle A, \leqslant\rangle$ has a zero. Then $\bigvee\emptyset \in A$ and hence A is complete.

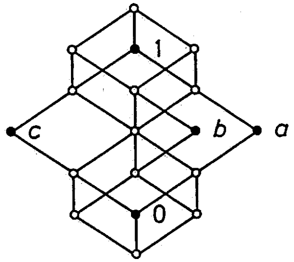
Exercise 7

(a) No. The diamond is a substructure of this structure:



It can be checked that this is a substructure by checking the joins and meets of a, b and c (1 and 0, resp.).

(b) Yes. It may look at first sight as if the structure under (a) is a substructure of this in mirror image, but if you check the distributive laws on a, b, c in the following structure, you will see that they hold here:



Exercise 8

If a has two complements b and c, then the restriction of the lattice to $\{a, b, c, 0, 1\}$ is the diamond, so the diamond is a sublattice, contradicting distributivity. We can also prove it algebraically (of course): assume: $a \wedge b = a \wedge c = 0$ and $a \vee b = a \vee c = 1$.

$$b \wedge c = b \wedge c$$

$$0 \vee (b \wedge c) = 0 \vee (b \wedge c)$$

$$(b \wedge a) \vee (b \wedge c) = (a \wedge c) \vee (b \wedge c)$$

$$b \wedge (a \vee c) = c \wedge (a \vee b) \text{ (by distributivity)}$$

$$b \wedge 1 = c \wedge 1$$

$$b = c$$

Exercise 9

(a) We know that $a \vee \neg a = \neg a \vee \neg\neg a = 1$ and we know that $a \wedge \neg a = \neg a \wedge \neg\neg a = 0$. By using the first:

$$(a \vee \neg a) \wedge (\neg\neg a \vee \neg a) = 1$$

and by distributivity $(a \wedge \neg\neg a) \vee \neg a = 1$. By using the second: $(a \wedge \neg a) \wedge (\neg\neg\, a \wedge \neg a) = 0$ and by associativity commutativity and idempotency $(a \wedge \neg\neg a) \wedge \neg a = 0$. Hence $(a \wedge \neg\neg a)$ is the complement of $\neg a$, so $(a \wedge \neg\neg a) = \neg\neg a$. Hence $\neg\neg a \leqslant a$.

Again by using the first: $(a \vee \neg a) \vee (\neg\neg a \vee \neg a) = 1$ and by ass. comm. and idem. $(a \vee \neg\neg a) \vee \neg a = 1$. Again using the second: $(a \wedge \neg a) \vee (\neg\neg a \wedge \neg a) = 0$ and by distributivity $(a \vee \neg\neg a) \wedge \neg a = 0$. Hence $(a \vee \neg\neg a)$ is also the complement of $\neg a$: $(a \vee \neg\neg a) = \neg\neg a$, hence $a \leqslant \neg\neg a$. Hence $a = \neg\neg a$.

(b) $(a \wedge b) \wedge (\neg a \vee \neg b) =$ (by distributing $(a \wedge b)$)

$$(a \wedge b \wedge \neg a) \vee (a \wedge b \wedge \neg b) = 0$$

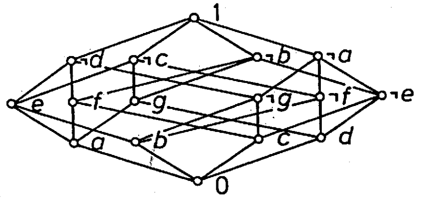
$(a \wedge b) \vee (\neg a \vee \neg\, b) =$ (by distributing $(\neg a \vee \neg b)$)

$$(a \vee \neg a \vee \neg b) \wedge (b \vee \neg a \vee \neg b) = 1$$

Hence $(\neg a \vee \neg b)$ is the complement of $(a \wedge b)$, so $a \wedge b = \neg(\neg a \vee \neg b)$. Thus $\neg(a \wedge b) = \neg\neg(\neg a \vee \neg b) = \neg a \vee \neg b$. The other one goes similarly.

(c) The first one is not complemented. In the picture of Exercise 7 b does not have a complement.

The second one is complemented:



Exercise 10

Assume $a \vee b = 1$.

$$(b \wedge \neg a) \wedge a = 0$$

$$(b \wedge \neg a) \vee a = (b \vee a) \wedge (\neg a \vee a) = b \vee a = 1$$

Hence $b \vee \neg a$ is the complement of a, so $b \vee \neg a = \neg a$, hence $b \leqslant \neg a$.

Assume $b \leqslant \neg\, a$. Then $b \wedge \neg a = b$. Hence $b \wedge a = (b \wedge \neg a) \wedge a = 0$. So $b \wedge a = 0$. We have proved:

$$a \vee b = 1 \text{ entails } b \leqslant \neg a \text{ entails } a \wedge b = 0$$

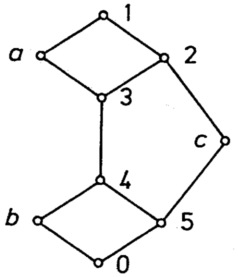
A mirror argument shows:

$$a \wedge b = 0 \text{ entails } \neg a \leqslant b \text{ entails } a \vee b = 1.$$

This proves that all of $a \wedge b = 0$; $a \vee b = 1$; $b \leqslant \neg a$; $a \leqslant \neg b$ are equivalent.

Exercise 11

(a)



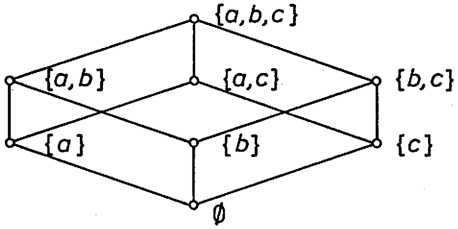
First: obviously none of $\{a\}$, $\{b\}$, $\{c\}$ generate the whole lattice. $\{a, b\}$ only generates itself.

$\{b, c\}$ generates $\{b, c, 0, 2\}$
$\{a, c\}$ generates $\{a, c, 1, 5\}$

So none of these generate the whole structure.

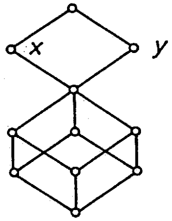
By the above considerations $\{a, b, c\}$ generates at least $\{a, b, c, 0, 1, 2, 5\}$. 3 is generated as the meet of a and 2; 4 is generated as the join of b and 5, hence indeed the whole structure is generated and as argued, minimally.

(b)



1. $\{\{a\}, \{b\}, \{c\}\}$
2. $\{\{a, b\}, \{a, c\}, \{b, c\}\}$
3. $\{\{a\}, \{b\}, \{a, c\}, \{b, c\}\}$
4. $\{\{a\}, \{c\}, \{a, b\}, \{b, c\}\}$
5. $\{\{b\}, \{c\}, \{a, b\}, \{a, c\}\}$

(c) Example



For the new sets of minimal generations you have to add x and y to the old sets.

(d) It has to be generated under the operations. In a Boolean lattice, the operations are $\wedge$ and $\vee$. You need minimally four elements: the four atoms (the case is similar to (b) above).

As a Boolean algebra the operations are $\wedge$, $\vee$ and $\neg$. To generate it as a Boolean algebra you only need two elements. In the example of Exercise 9c, e and f generate the Boolean algebra. They bring in $\neg e$ and $\neg f$. By taking meets this brings in 0, a, b and c. Taking complements brings in 1, $\neg a$, $\neg b$, $\neg c$. The join of e and f brings in $\neg d$, hence complements brings in d. The join of a and d brings in g, hence complements $\neg g$.

Exercise 12

$\approx$ is clearly an equivalence relation (because = is). So we have to show: if $a \approx a'$ and $b \approx b'$ then:

$$(a \wedge b) \approx (a' \wedge b')$$
$$(a \vee b) \approx (a' \vee b')$$

So assume $a \approx a'$ and $b \approx b'$. That means that $h(a) = h(a')$ and $h(b) = h(b')$. Since h is a homomorphism $h(a \wedge b) = h(a) \wedge h(b) = h(a') \wedge h(b') = h(a' \wedge b')$. Hence $a \wedge b \approx a' \wedge b'$. Similarly for $\vee$.

Exercise 13

$\approx$ is the relation $\lambda a \lambda a'.h(a) = h(b)$.

1. f clearly is a surjection.
2. f is an injection.

Assume $f([a]_\approx) = f([a']_\approx)$ that means that $h(a) = h(a')$, but then $a \approx a'$ and $[a]_\approx = [a']_\approx$.

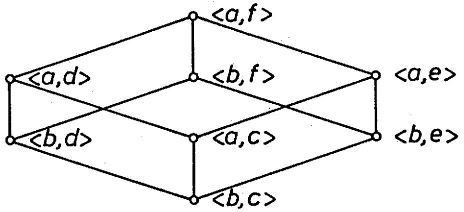
3. f is a homomorphism.

Since $\approx$ is a congruence relation: $[a \wedge a']_\approx = [a]_\approx \wedge [a']_\approx$. Hence

$$\begin{aligned} &f(([a]_\approx \wedge [a']_\approx) \\ &= f([a \wedge a']_\approx) = h(a \wedge a') = h(a) \wedge h(a') \\ &= f([a]_\approx) \wedge f([a']_\approx) \end{aligned}$$

Similarly for $\vee$.

Exercise 14



Exercise 15

(a) Let F be a filter in L. If $a, b \in F$ then $a \wedge b \in F$, by definition. If $a, b \in F$ then, since $a \leqslant a \vee b$, $a \vee b \in F$. Hence F is a sublattice of L.

Let $a, b \in F$ and assume $a \leqslant x \leqslant b$, then obviously $x \in F$, since F is a filter, so F is convex.

(b) Assume that F is a filter. Then:

if $a, b \in F$ then $a \wedge b \in F$ (filter)
if $a \wedge b \in F$ then $a, b \in F$ because $a \wedge b \leqslant a, b$

– Assume that F is a non-empty subset of A such that $a, b \in F$ iff $a \wedge b \in F$. Then obviously if $a, b \in F$ $a \wedge b \in F$.

Suppose $a \in F$ and $a \leqslant b$. Then $a = a \wedge b$, hence $a \wedge b \in F$, hence $b \in F$. So F is a filter.

This proves the equivalence of the first definition.

– If F is a filter we have proved that it is a sublattice of L. Let $a \in F$ and $b \in L$. $a \leqslant a \vee b$, hence, because F is a filter $a \vee b \in F$.

– Let F be a sublattice of L such that if $a \in F$ and $b \in L$ then $a \vee b \in F$. If $a, b \in F$, then because F is a sublattice and hence closed under $\wedge$,

$a \wedge b \in F$. Let $a \in F$ and $a \leqslant b$. Then $b = a \vee b$, $a \vee b \in F$, hence $b \in F$. So F is a filter.

This proves the equivalence of the second definition.

Non-empty subset I of L is an *ideal* in L iff

$$a \vee b \in I \text{ iff } a, b \in I.$$

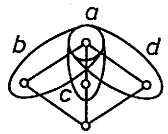
I is an ideal in L iff I is a sublattice of L such that:

$$\text{if } a \in I \text{ and } b \leqslant a \text{ then } a \wedge b \in I.$$

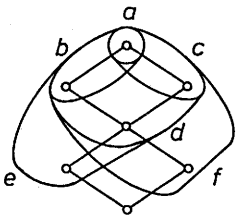
Exercise 16

If L is a finite lattice every subset of L, hence every filter in L is finite. Thus $F = \{a_1, \ldots, a_n\}$. Since F is closed under conjunction it has a minimal element $a_1 \wedge \cdots \wedge a_n$ and it is the principle filter generated by that minimal element.

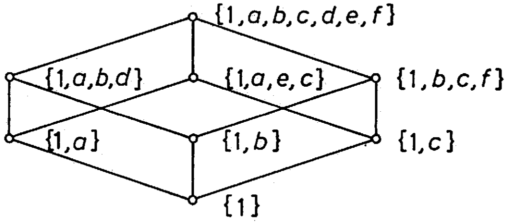
Exercise 17



Prime filters: none
Ultrafilters: b, c, d



Prime filters: b, c, e, f
Ultrafilters: e, f



$\{1, a, b, d\}$ $\{1, a, e, c\}$ $\{1, b, c, f\}$ are the prime filters and the ultrafilters.

Exercise 18

Let F be a prime filter. Hence if $a \vee b \in F$ then either $a \in F$ or $b \in F$.

Suppose $a, b \in L - F$. Then, because F is prime, $a \vee b \in L - F$. Suppose $a \in L - F$ and $b \leqslant a$. If $b \in F$, then $a \in F$, because F is a filter, hence $b \in L - F$. So $L - F$ is an ideal, and since F is a proper filter $L - F \neq L$, so $L - F$ is a proper ideal.

Assume that $a \wedge b \in L - F$ and assume that both $a \in F$ and $b \in F$. Then, because F is a filter $a \wedge b \in F$, contradiction. Hence either $a \in L - F$ or $b \in L - F$, hence $L - F$ is a prime ideal. The other side goes in the same way.

Exercise 19

Let F be a proper filter and assume that for every a: either $a \in F$ or $\neg a \in F$. Assume that $b \notin F$, then $\neg b \in F$.

Since $b \wedge \neg b = 0$ this means that b is incompatible with F, hence F is an ultrafilter.

Exercise 20

Let F be a prime filter and assume that $a \notin F$. $a \vee \neg a \in F$, hence, since F is prime, $\neg a \in F$. Hence F is an ultrafilter.

Exercise 21

There is no countable powerset Boolean algebra. If B is finite then pow B is finite, if B is countable, then pow B is uncountable.

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