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The Logical Analysis of Plurals and Mass Terms: A Lattice-theoretical Approach*

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I. Introduction

The weekly *Magazine* of the German newspaper *Frankfurter Allgemeine Zeitung* regularly issues Marcel Proust's famous questionnaire which is answered each time by a different personality of West German public life. One of those recently questioned was Rudolf Augstein, editor of *Der Spiegel;* his reply to the question: "Which property do you appreciate most with your friends?" was

(1) "that they are few."

Clearly, this is not a property of any one of Augstein's friends; yet, even apart from the *esprit* it was designed to display the answer has a straightforward interpretation. The phrase (1) predicates something *collectively* of a *group* of objects, here: Augstein's friend's.

As it is well known, collective predication is a rather pervasive phenomenon in natural language, as the following sample of sentences shows:

(2) The children built the raft.
(3) The Romans built the bridge.
(4) Tom, Dick, and Harry carried the piano upstairs[1].
(5) The playing cards are scattered all over the floor.
(6) The members of the committee will come together today.
(7) Mary and Sue are room-mates.
(8) The girls hated each other.

There is a striking similarity between collective predication and predication involving mass nouns.

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[1] Massey's example; see Massey 1976

(9) a. The children gather around their teacher.
    b. The water gathers in big pools.

Moreover, a characteristic feature of mass terms, their *cumulative reference property*[2], can be imitated by plurals.

(10) a. If a is water and b is water then the sum of a and b is water.
     b. If the animals in this camp are horses, and the animals in that camp are horses, then the animals in both camps are horses.

All this has been observed and discussed in the literature although the noted parallelism has perhaps not been stressed too much[3]. As it can be seen from Pelletier's 1979 volume, however, there is much disagreement about the proper way of attacking the logical problems posed by plurals and mass terms. From a semantic point of view the basic question is: what do mass terms and plural expressions denote? Some have thought that in order to be able to give a satisfactory answer to this question it is necessary to give up or at last extend the underlying set theory and to define new kinds of objects, for instance *ensembles* (Bunt 1979) or *Kollektionen* (Blau 1979). I think, however, that we can retain the usual set-theoretic metalanguage and simply enrich the structure of our models as to account for properties like cumulative reference. On my view, such properties are also not secured by *defining* some plural or mass term denotations out of others through set-theoretic manipulations; they all should be recognized as simply being there. What we rather should try to discover, then, is the network of the various relations which they enter and through which they are tied together. In the case of group and mass objects this picture naturally leads to the notion of lattice structure[4], an idea which is, again, not new: it is inherent in mereological predicate logic and the Calculus of Individuals as developed by Leonard/Goodman 1940 and Goodman/Quine 1947. However, its possible use in the present context has perhaps been obscured by reductionist ontological considerations which are, in my opinion, quite alien to the purpose of logically analyzing the inference structures of natural language[5]. Our guide in ontological matters has to be

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[2] see Quine 1960, p. 91; Bunt 1979

[3] The main source for mass terms is Pelletier 1979; furthermore, see Bennett 1980, Bunt 1979, and Ter Meulen 1980, 1981. For the treatment of plurals and collective terms, see Massey 1976, Burge 1977, Blau 1979, Hoepelman/Rohrer 1980, and Scha 1981. The parallelism referred to is explicitly expressed in Bunt 1979.

[4] Recently, Keenan/Faltz 1978 and Keenan 1981 advanced a "Boolean approach" to the semantics of natural language. I feel very sympathetic with this enterprise which, unfortunately, I became aware of only a year ago (January 1981). It is reassuring to see similar techniques be successfully applied in other areas of semantics, too. I have to defer a concrete evaluation of these ideas to another occasion.

[5] On this point I agree with Ter Meulen 1981, I think. But I do not follow her in the conclusions she draws from this observation. The inherent lattice structure is independent of the philosophical motives that gave rise to the construction of mereological systems. For the role of nominal mass noun denotations see the remarks below, also footnore 11.

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language itself, it seems to me. So if we have, for instance, two expressions a and b that refer to entities occupying the same place at the same time but have different sets of predicates applying to them, then the entities referred to are simply not the same. From this it follows that my ring and the gold making up my ring are different entities; they are, however, connected by what I shall call the *constitution relation*: There is exactly one portion of matter making up my ring at a time. A constitution relation C has been explicitly introduced into the discussion by Parsons 1979. Sharing his intuitions on this point I shall provide a similar 2-place relation, '$\rhd$', for *constitutes* or *makes up*. Its semantic counterpart in the theory to be presented below, the "materialization" function h, lies at the heart of my reconstruction of the ontology of plurals and mass terms: individuals are created by linguistic expressions involving different structures even if the portion of matter making them up is the same. Consider the example from Blau 1979 (imagine that there is a deck of playing cards on the table):

(11) a. the cards
     b. the deck of cards

While the portions of matter denoted by (11 a) and (11 b) are the same, I consider the individuals as being distinct[6]. (11 a) refers to the pure collection of objects, and in many contexts (11 b), too, refers just to this collection. In general, however, the introduction of a collective term like (11 b) is indicative of connotations being added enough for it to refer to a different individual; for instance, a committee is not just the collection of its members, etc. Note, by the way, that the transition to an intension function would be of no help here. There might be two different committees which necessarily consist of exactly the same members[7].

It might be thought, then, that collective predication is just the context in which pairs of expressions like (11 a, b) refer to collections and thus are co-referential. This is not so, however, as can be seen from the following example. Imagine that there are several decks of cards, a blue one, a green one, etc., lying on a table[8]. Then the two following sentences do not mean the same although *number consecutively* is a collective predicate (the German word is *durchnumerieren*).

(12) a. The cards on the table are numbered consecutively.
     b. The decks of cards on the table are numbered consecutively.

By contrast, *spatio-temporal* collective predicates do refer to the pure collection, or, as I conceive it, the portion of matter making up the individual in

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[6] I guess that in German, with *die Karten* vs. *das Kartenspiel*, the point might come out more clearly.

[7] The point was apparently made first by David Kaplan as Bennett 1980 reports.

[8] This is the situation originally analyzed by Blau in his 1979 paper.

question. Examples are *be-on-the-table, occupy*, etc. I call such predicates *invariant*. So the following a) sentences are indeed equivalent to their corresponding b) sentences.

(13) a. The cards are on the table.
   b. The decks of cards are on the table.
(14) a. The stars that presently make up the Pleiades galactic cluster occupy an area that measures 700 cubic light years[9].
   b. The Pleiades galactic cluster occupies an area etc.

In the following I shall distinguish between pure plural individuals involved in (12) and collections in the portions-of-matter sense referred to in (13) and (14). The former I call (individual) *sums* or *plural objects;* they respect levels of "linguistic comprehension" as shown by (12). By contrast, collections do not, they typically merge those levels. Sums and collections are similar, however, in that they both are just individuals, as concrete as the individuals which serve to define them, and of the same logical type as these. The latter feature is important because there is no systematic type ambiguity inherent in predicates like *carry, build, demolish, defend*, etc. As to the question of concreteness of sums of concrete objects, I agree with the intuitions of those who say that an aggregate of objects like a heap of playing cards which can be shuffled, burned, etc., is simply not an abstract entity like a set (see Burge 1977, Blau 1979). What is more important, however, is the fact that the set approach to plural objects[10] does not carry over to the case of mass terms, thus missing the structural analogy between the two cases. Inherent in the notions of a set is atomicity which is not present in the linguistic behaviour of mass terms.

Before I go on to present my own approach I want to say something about what Ter Meulen calls *nominal mass terms*. Typical examples are stuff names like *gold* in sentences like *gold has the atomic number 79*. But there is also the time-honored sentence *water is widespread* in which the term *water* has apparently a somewhat different status. It seems to refer to the concrete "scattered individual" that you just find everywhere, hence Quine's analysis in terms of mereology. In this sense the sentence should be synonymous to *the water (on earth) is widespread*. Of the same type is the use of *gold* in *America's gold is stored in Fort Knox*. Here, again, a concrete object is referred to by *America's gold*, namely the material fusion of all quantities of the US gold reserve. So there can be no doubt that some notion of fusion is needed to account for definite descriptions involving predicative mass terms (the μ-operator defined below does just this). Genuine stuff names, however, are something else. Substances are abstract entities and cannot be defined in

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[9] This is Burge's example, see Burge 1977.

[10] This approach is the traditional one. In one form or another it can be found, for instance, in Bennett 1974, Hausser 1974, v. Stechow 1980, Hoepelman/Rohrer 1980, Scha 1981.

terms of their concrete manifestations. The question, then, is of what kind the connections are that are intuitively felt between substances and their quantities. Take water, for instance. A quantity is water if it displays the internal structure of water, that is H_2O. But this relation is not a logical one. Or else we might look for substance properties which carry over to the quantities of the substance in question. Water is a liquid and yet, all concrete water might be frozen. So we have to go over to dispositional properties, getting more and more involved into our knowledge of the physical world ... What I am getting at is that *nominal mass terms do not seem to have a proper logic*. Be this as it may, this issue is completely independent of the lattice structure that governs the behaviour of predicative mass terms and plural expressions; it is only this structure that I want address myself to in the present paper[11].

Now what I am going to propose, then, is basically the following. First of all, let us take seriously the morphological change in pluralization, which is present in many natural languages, and introduce an operator, '*', working on 1-place predicates P, which generates all the individual sums of members of the extensions of P. Such a starred predicate now has the same cumulative reference property as a mass predicate, it is closed under sum formation: any sum of parts which are $*P$ is again $*P$. This property gives rise to the introduction of a Boolean structure on the domain of discourse, E: technically, E becomes an atomic Boolean algebra which is taken to be complete so that every subset of E possesses a sum. Now let $\| \cdot \|$ be the denotation function in a model, and $\| P \|$ the extension of P. Then $\| *P \|$, the extension of $*P$, can be defined in terms of $\| P \|$ as the complete join-subsemilattice[12] in E generated

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[11] Contributions to the problem of substance names can be found in Pelletier 1979 (in particular, Parsons 1979), Bunt 1979, and Ter Meulen 1980, 1981. Let me comment on the latter work, which is formulated in a Montague framework. The few remarks I made here will make it evident that I fully agree with ter Meulen in that nominal mass nouns cannot be *reduced* to predicative mass nouns. But for this very reason I fail to see any cogent argument for the kind of denotation ter Meulen wants to assign to these terms at a given reference point (i.e. intension functions denoting in each world the set of concrete quantities of the substance in question). As it turns out, the arguments she puts forth in Ter Meulen 1981 really lend support only to the first, the critical, point (viz. that reduction is impossible). But what she then goes on to call a nominal mass noun's "extensional reference to an intensional object" (viz. the intension function referred to above) seems to me both syntactically and semantically misguided. For the inevitable doubling of syntactic rules is certainly unwelcome, to begin with. But what is more, those intension functions, even when lifted to still another intensional level as ter Meulen wants to have it, are simply not well motivated as substance name denotations. The statement, for instance, that two fictional substances can be differentiated (op. cit., p. 438) is not compatible with the principle of rigid designation introduced earlier (op. cit., p. 424). More generally, there are no rules that could justify intuitively valid inferences from contexts involving nominal mass nouns to contexts with their corresponding predicative terms – it is my view, anyway, that such inferences are not based on pure logic alone. I conclude from this that the problem of nominal mass nouns is best approached in a spirit of logical abstinence. Nominal mass nouns denote abstract entities, to be sure, and as such they are names of individuals just like *John, Munich*, and the rest. Beyond this minimal account things become notoriously vague.

[12] For this concept see, for instance, Grätzer 1971.

by $\|P\|$. This construction is the mathematical expression of the closure property referred to above. The set A of atoms of E consists of the "singular objects", like this card, that deck of cards, etc. Among them are all the different portions of matter, like the water making up this ice cube.

Now, let a and b denote two atoms in A. Then there are two more individuals to be called below $a + b$ and $a \oplus b$. $a + b$ is still a singular object in A, the *material fusion of a and b;* $a \oplus b$ is the *individual sum* or *plural object* of a and b. The theory is such that $a + b$ constitutes, but is not identical with, $a \oplus b$. This looks like a wild platonistic caprice strongly calling for Occam's Razor. Language, however, seems to function that way. Take for a, b two rings recently made out of some old Egyptian gold. Then *the rings,* $a \oplus b$, are new, *the stuff,* $a + b$, is old.

Sums are partially ordered through the intrinsic ordering relation '$\leq_i$' on E which is expressed in the object language by the 2-place predicate 'Π'. It is called the *individual part relation* (*i-part* relation, for short) and satisfies the biconditional

(15) $a \Pi b \ \leftrightarrow \ a \oplus b = b$.

The semantic counterpart of (15) is the well-known Boolean relation (where $\sqcup_i$ denotes the join operation on E):

(16) $\|a\| \leq_i \|b\|$ iff $\|a\| \sqcup_i \|b\| = \|b\|$

Now if P is a 1-place predicate and $*P$ the corresponding plural predicate let $\circledast P$ (the *proper plural predicate* of P) be true of exactly the non-atomic sums in the extension of $*P$. Then we can define the *sum* and the *proper sum* of the P's, $\sigma x P x$ and $\sigma^* x P x$, respectively, as the supremum of all objects that are $*P$ and $\circledast P$, respectively ('ι' is the description operator):

(17) a. $\sigma x P x \ := \ \iota x \big({}^*Px \wedge \bigwedge y(^*Py \rightarrow y \Pi x) \big)$
     b. $\sigma^* x P x \ := \ \iota x \big(\circledast Px \wedge \bigwedge y(^*Py \rightarrow y \Pi x) \big)$

$\sigma^* x P x$ carries the presupposition that there are at least two P's; in this case $\sigma x P x$ and $\sigma^* x P x$ coincide.

The *material fusion* of the P's, denoted by $\mu x P x$, becomes that object which constitutes the sum of the P's:

(18) $\mu x P x \ := \ \iota x (x \rhd \sigma x P x)$

The μ-operator has the effect of merging levels of comprehension, as I called it above. Thus we will have $\mu x P x = \mu x Q x$, if P stands for *is a card of one of the card decks,* and Q for *is a card deck.* $\sigma x P x$ and $\sigma x Q x$ are not the same, however. If we have two decks of Bridge cards, for instance, $\sigma x P x$ contains 104 atoms whereas $\sigma x Q x$ contains only 2 atoms.

There is a second ordering relation, called the *material part (m-part) relation* and denoted by '$\top$'. It establishes a partial order on portions of matter, but only a precorder, called $\leq_m$, on the whole domain of individuals. Objects

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which are m-parts of one another are *materially equivalent* in that they have the same portion of matter constituting them. If a is an i-part of b then a is a m-part of b; symbolically:

(19) $a \Pi b \rightarrow a \top b$

In order to explain the meaning of 'T' more precisely let me supply the remaining concepts of the model structure to be defined below. In addition to the domain of individuals, E, there is a set D which is endowed with a join operation '$\sqcup$' making D into a complete, but not necessarily atomic, join-semilattice. D is partially ordered by its intrinsic ordering relation, $\leq$, defined by

(20) $x \leq y$ iff $x \sqcup y = y$

D represents the set of all individual portions of matter in the model, and as such, it is considered as a subset of A, the set of atoms of E. The relation (20) stands for a material part-whole relation.[13] It can be used to order the individuals of E materially. For this I postulate for every model the existence of a semilattice homomorphism h from $E \backslash \{0\}$ to D such that h is the identity function on D and commutes with the supremum operator. Then h is order-preserving, i.e. we have

(21) $x \leq_i y$ $\Rightarrow$ $h(x) \leq h(y)$

for all $x, y \in E$ with $x \neq 0$. I can now define the relation $\leq_m$ mentioned above by

(22) $x \leq_m y$ iff $h(x) \leq h(y)$ $(x, y \in E \backslash \{0\})$

$\leq_m$ is the semantic counterpart of 'T'. Obviously, $\leq_i$ is coarser than $\leq_m$, a fact that is syntactically expressed by (19). In its intended interpretation, then, which is formally implemented into the model structure, $a \top b$ can be read as follows: "the portion of matter constituting a is m-part of the portion of matter constituting b".

One-place predicates are interpreted as subsets of $E \backslash \{0\}$. (Here, the null element 0 is assigned the role of a dummy object to which no predicate applies and which can take care of denotation gaps arising in our description theory; for details see Link 1979, 5.2.) If P is a mass term the extension $\|P\|$ of P in a given model should be a set of portions of matter which is closed under joins. So $\|P\|$ becomes a subsemilattice of D. This completes the informal description of the *Boolean model structure* $\mathfrak{B} = \langle E, A, D, h \rangle$ and of the LPM-*model* $M = \langle \mathfrak{B}, \|\cdot\| \rangle$. For precise definitions see below.

If P is a predicate we can introduce its *mass term correspondent*, called $^m P$; if $\|P\|$ is the extension of P in a model, define

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[13] Le me point out that the notions of *material part* and *portion-of-matter* are only a paradigm case of fixing intuitions for *homogeneous reference*. The structural properties actually used (non-atomicity, cumulative reference) are also found to govern the behavior of abstract entities like events. Implications for modes of temporal reference are discussed in Hoepelman/Rohrer 1980.

(23) $\|^mP\| := \{x \in D \mid x \leq \sup h[\|P\|]\}$

If P is already a predicative mass term we have, by this definition, $\|P\| \subseteq \|^mP\|$. Mass term correspondents occur in natural language.[14] Consider

(24) a.   There is apple in the salad.
     b. $\forall x (^mPx \wedge Qx)$

(24 b) is a formalization of (24 a), with P for *is an apple*, ^mP for *is apple*, and Q for *is in the salad*. From this we can infer, intuitively as well as formally:

(25) a.   There are apple parts in the salad.
     b. $\forall x \forall y (*Px \wedge yTx \wedge Qy)$

We have, in fact, the following theorem (cf. (T. 14) below):

(26) $^mPa \rightarrow \forall y (*Py \wedge aTy)$

Returning now to the model structure consider once again the extension of a 1-place predicate P. It can contain atoms as well as proper i-sums. This general case of a *mixed extension* is exemplified by predicates like *carry the piano*: think of Tom, Dick, and Harry (together), Obelix (all by himself), etc. Common nouns and intransitive verbs like *die*, however, seem to admit only atoms to their extension. I call such predicates *distributive*:

(27) $Distr(P) \leftrightarrow \wedge x (Px \rightarrow At x)$

(here the predicate At stands for the property of being an atom in the model). To illustrate take the intuitively valid inference from a) to b) in (28).

(28) a.   John, Paul, George, and Ringo are pop stars.
     b.   Paul is a pop star.

This inference can be formally represented if we consider *pop star* as a distributive predicate P in the sense of (27). In this case the extension of $*P$ is closed under non-zero i-parts (see theorem (T. 10) below), so every atom of an i-sum which is $*P$ is itself $*P$, hence it is a P. In symbolic form the inference (28) looks like this.

(29) a. $*P(a \oplus b \oplus c \oplus d)$
     b. $Distr(P)$
     c. $b \Pi a \oplus b \oplus c \oplus d$
     d. $*Pb$
     e. Pb

In a similar way we have the following valid inference with the distributive predicate Q for *die* (P stands for *animal*):

(30) a.   The animals died. So every animal died.
     b. $*Q(\sigma *xPx) \Rightarrow \wedge x (Px \rightarrow Qx)$

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[14] see already Quine 1960, also Bunt 1979, Hoepelman/Rohrer 1980.

For non-distributive predicates we have, of course, no such result, witness *carry*: $a \oplus b$ might be in the extension of *carry* while a or b alone is not. Notice that these predicates enter formalizations unstarred. To illustrate,

(31) a. The children gather around their teacher
 b. $Q(\sigma^* x Px)$ (with Q for *gather* etc.).

I think there is no harm in accepting this systematic difference: i.e. distributive predicates working on plural terms have to be starred, all the other predicates must not be. For distributivity seems to be a lexical feature. If we have a formal translation procedure the predicates have to be subcategorized accordingly.

There is one more operator, '^{T}', in the logic LPM to be presented below. $^T P$, for a predicate P, is to be read as *partakes in P*. I introduce this operator in order to be able to distinguish between the plural terms *the children* and *all the children*. It seems to me that in *all the children built the raft* it is claimed that every child took part in the action whereas in *the children built the raft* it is only said that the children somehow managed to build the raft collectively without presupposing an active role in the action for every single child. This intuition enters the formalization of the two phrases given below. I want to stress, however, that the operator '^{T}' can only be partially characterized in view of the essentially pragmatic nature of its intended interpretation. So let me formulate just the following meaning postulates for '^{T}' which seem to be plausible:

(32) a. $\bigwedge x (^T Px \rightarrow \bigvee y (x \Pi y \wedge Py))$
 b. $Distr(P) \rightarrow \bigwedge x (^T Px \leftrightarrow Px)$

I conclude the informal part of the paper with some more formalizations of natural language sentences involving plurals and mass terms. Most of the principles governing their logic that have been mentioned in the literature come out valid in the system LPM below. Some of them are instantiated here, like Massey's plurality principles of symmetry, expansion, and contraction (see Massey 1976). What I did not treat in LPM, however, are any "downward" closure properties that are somehow felt to be present in the behavior of mass terms ("a part of water is still water"). Such principles can be added when a careful linguistic analysis has succeeded in giving them a form that takes care, in particular, of the *problem of minimal parts* (is every m-part of lemonade really lemonade again?). For some discussion of downward closure properties see, e.g., Bunt 1979 and Hoepelman/Rohrer 1980.

(33) a. A child built the raft. Px: x is a child
 b. $\bigvee x (Px \wedge Qx)$; Qx: x built the raft
(34) a. Children built the raft.
 b. $\bigvee x (^\circledast Px \wedge Qx)$; Px, Qx: dto.
(35) a. The child built the raft.
 b. $\bigvee y (y = \iota x Px \wedge Qy)$; Px, Qx: dto.

(36) a. The children built the raft.
 b. $\vee \dot{y}(y = \sigma^* x Px \wedge Qy)$; Px, Qx: dto.

(37) a. Every child saw the raft.
 b. $\wedge x(Px \rightarrow Qx)$; Px: dto.
                                 Qx: x saw the raft

(38) a. All the children built the raft.
 b. $\wedge y(y = \sigma^* x Px \rightarrow \tilde{Q}y)$; Px, Qx as in (33)
                                         $\tilde{Q} := \lambda x(Qx \wedge \wedge z(z\, \Pi\, x \rightarrow {}^{\mathrm{T}}Qz))$

(39) a. Tom and Dick carried the piano upstairs.
 b. $P(a \oplus b)$; a: Tom, b: Dick,
                           Px: x carried the piano upstairs

(40) a. Tom and Dick carried the piano upstairs, so Dick and Tom carried it upstairs.
 b. $P(a \oplus b) \Rightarrow P(b \oplus a)$ *(symmetry)*

(41) a. John and Paul are pop stars and George is a pop star, so John, Paul, and George are pop stars.
 b. $\circledast P(a \oplus b) \wedge Pc \Rightarrow \circledast P(a \oplus b \oplus c)$ *(expansion)*

(42) a. John, Paul, George, and Ringo are pop stars, so Paul and Ringo are pop stars.   (see (28))
 b. $\circledast P(a \oplus b \oplus c \oplus d) \Rightarrow \circledast P(b \oplus d)$ *(contraction)*

(43) a. (All) water is wet.
 b. $\wedge x(Px \rightarrow Qx)$; Px: x is (a quantity of) water

(44) a. The water of the Rhine is dirty.
 b. $Q(\mu x Px)$; Px: x is (a quantity of) Rhine water

(45) a. This ice cube is water.
 b. $P\iota x(x \rhd a)$; a: this ice cube
                           Px: as in (43)

(46) a. The gold in Smith's ring is old, but Smith's ring is not old.
 b. $Q\iota x(Px \wedge x \rhd a) \wedge \neg Qa$; Px: x is gold
                                 Qx: x is old
                                 a: Smith's ring

II. The Logic of Plurals and Mass Terms (LPM)

LPM is a first order predicate calculus with the usual logical constants '$\neg$', '$\vee$', '$\wedge$', '$\rightarrow$', '$\leftrightarrow$', '$\vee$', '$\wedge$', the description operator denoted by 'ι', and the abstraction operator 'λ'.[15] The syntactic variables are, for formulas of LPM, 'ϕ', 'ψ', 'χ'; for individual terms, 'a', 'b', 'c'; for variables, 'x', 'y', 'z'; for (1-place) predicates, 'P', 'Q'. These symbols can also appear with primes and indices. For metalinguistic (definitional) identity the symbol '$\doteq$' ('$:=$') will be used.

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[15] For an outline of such systems ("PL1IKA") see Link 1979; notice that, in LPM, identity of individual terms need not be taken as primitive, but can be defined in terms of 'Π'.

The set of 1-place predicate constants contains two specified subclasses: the set MT of *predicative mass terms* and the set DP of *distributive predicates*. MT and DP are taken to be disjoint sets.

As special *primitive symbols* of LPM we have a 1-place predicate symbol 'E', three 2-place predicate constants, 'Π', 'T', '$\rhd$', and two operators on 1-place predicates, '$*$' and '$^{\mathsf{T}}$'. The intended interpretations are the following. $\mathbf{E}a$ stands for "*a exists*"; $a\,\Pi\,b$ for "*a is an individual part* (i-part) of *b*"; $a\,\mathsf{T}\,b$ for "*a is a material part* (m-part) of *b*", $a\rhd b$ for "*a constitutes* or *makes up b*"; $*P$ for "the *plural predicate* of *P*"; $^{\mathsf{T}}P$ for "*partakes in P*".

Now I introduce a number of *defined expressions*.

1. *General first order abbreviations.* Let P, Q be 1-place predicates, R a 2-place predicate, and a an individual term.

(D.1) $P \subset Q \quad \leftrightarrow \quad \wedge x (Px \to Qx)$

(D.2) $P \equiv Q \quad \leftrightarrow \quad P \subset Q \wedge Q \subset P$

(D.3) $P \oplus Q \quad := \quad \lambda x (Px \vee Qx)$

(D.4) $I_a \qquad := \quad \lambda x (x = a)$

Furthermore, the usual definitions for the following formulas involving R are assumed: $Refl(R)$ ("R is reflexive"), $Trans(R)$ ("R is transitive"), $Sym(R)$ ("R is symmetric"), $Antisym(R)$ ("R is antisymmetric"). We then have, with $PrO(R)$ for "R is a preordering relation", $PO(R)$ for "R is a partial ordering relation", and $Equ(R)$ for "R is an equivalence relation":

(D.5) $PrO(R) \quad \leftrightarrow \quad Refl(R) \wedge Trans(R)$

(D.6) $PO(R) \quad \leftrightarrow \quad Refl(R) \wedge Trans(R) \wedge Antisym(R)$

(D.7) $Equ(R) \quad \leftrightarrow \quad Refl(R) \wedge Trans(R) \wedge Sym(R)$

2. *Defined predicate constants.* With two individual terms a, b, let $a = b$ stand for "*a equals b*", $a \sim b$ for "*a is m-equivalent to b*", $At\,a$ for "*a is an atom*", and $Mp\,a$ for "*a is a portion of matter*". Define

(D.8) $a = b \quad \leftrightarrow \quad a\,\Pi\,b \wedge b\,\Pi\,a$

(D.9) $a \sim b \quad \leftrightarrow \quad a\,\mathsf{T}\,b \wedge b\,\mathsf{T}\,a$

(D.10) $At\,a \quad \leftrightarrow \quad \wedge x (x\,\Pi\,a \to x = a)$

(D.11) $Mp\,a \quad \leftrightarrow \quad a \rhd a$

3. *Defined predicate operators.* Let P be a 1-place predicate and a an individual term. The *proper plural predicate* $^{\circledast}P$ of P and the *mass term correspondent to P*, ^mP, are then defined as follows.

(D.12) $^{\circledast}Pa \quad \leftrightarrow \quad *Pa \wedge \neg At\,a$

(D.13) $^mPa \quad \leftrightarrow \quad \vee y (*Py \wedge a\,\mathsf{T}\,\imath z(z \rhd y))$

4. *Defined individual terms.* With 1-place predicates P and individual terms a, b, define the *i-sum* of the P's, the *proper i-sum* of the P's, the *i-sum* of a and b, the *material fusion* of the P's, and the *material fusion* of a and b, respectively:

$$(D.14) \quad \sigma x P x \quad := \quad \iota x \big(*Px \wedge \bigwedge y(*Py \to y\,\Pi\,x)\big)$$

$$(D.15) \quad \sigma^* x P x \quad := \quad \iota x \big(\circledast Px \wedge \bigwedge y(*Py \to y\,\Pi\,x)\big)$$

$$(D.16) \quad a \oplus b \quad := \quad \sigma x (I_a \oplus I_b) x$$

$$(D.17) \quad \mu x P x \quad := \quad \iota x (x \rhd \sigma x P x)$$

$$(D.18) \quad a + b \quad := \quad \mu x (I_a \oplus I_b) x$$

5. *Special abbreviative formulas.* Let P be a 1-place predicate.

$$(D.19) \quad Distr(P) \quad \leftrightarrow \quad \bigwedge x(Px \to At\,x)$$

$$(D.20) \quad M(P) \quad \leftrightarrow \quad \bigwedge x(Px \to Mp\,x)$$

$$(D.21) \quad Inv(P) \quad \leftrightarrow \quad \bigwedge x \bigwedge y \big(x \sim y \to (Px \leftrightarrow Py)\big)$$

Distr (P) is to be read as "*P* is *distributive*" (i.e. *P* is true of atomic individuals only), $M(P)$ as "*P* is a *material predicate*" (i.e. it is true of portions of matter only); $Inv(P)$ means: "*P* is an *invariant* predicate" (i.e. it is closed under substitution of m-equivalent terms).

6. *Meaning postulates for* 'T'.

$$(MP.1) \quad \bigwedge x({}^{T}Px \to \bigvee y(x\,\Pi\,y \wedge Py))$$

$$(MP.2) \quad Distr(P) \to \bigwedge x({}^{T}Px \leftrightarrow Px)$$

i.e. for all x, if x partakes in P, then x is an i-part of a y which is P (MP.1); if P is distributive, then for all x, x partakes in P just in case x is P (MP.2).

The semantics of LPM. I shall employ the usual set-theoretic terminology (see, for instance, Eberle 1970, Link 1979), with '$\subseteq$' for set inclusion and '$f[Y]$' for the f-image $\{f(x) \mid x \in Y\}$ of $Y \subseteq X$ under the function $f: X \to Z$. Moreover, some elementary notions of lattice theory are needed (see, e.g., Grätzer 1971, Sikorski 1969 for their definition) *partial ordering relation, partially ordered set ("poset"), 1-element, 0-element, atom, semilattice, join-semilattice, lattice, sublattice, ideal, principal ideal, complete lattice, Boolean lattice (algebra), atomic lattice, (semi) lattice homomorphism.* The supremum of a subset Y of a poset X, relative to its ordering relation $\leq$, if it exists, will be denoted by $\sup_{\leq} Y$, the principal ideal generated by an element $x \in X$, where X is a lattice, by $(x]$. So we have $(x] = \{y \in X \mid y \leq x\}$. If X is a complete (semi)lattice, and $Y \subseteq X$, then $[Y]$ denotes the complete sub(semi)lattice generated by Y.

I now provide a model-theoretic interpretation of the formal system LPM.

(D.22) A *Boolean model structure with homogeneous kernel ("boosk")* is a quadruple $\mathfrak{B} \doteq \langle E, A, D, h \rangle$ such that
1. E is a complete atomic (c.a.) Boolean algebra, with join operation $\sqcup_i$ and the intrinsic ordering relation $\leq_i$;
2. $A \subseteq E$ is the set of atoms in E;
3. $D \subseteq A$ is a complete join-semilattice with join $\sqcup$ and ordering relation $\leq$;
4. $h: E \backslash \{0\} \to D$ is a semilattice homomorphism such that

(i) $h \upharpoonright D \doteq id_D$, i.e. $h(x) \doteq x$ for all $x \in D$;

(ii) $h(\sup_{\leq_i} B) \doteq \sup h[B]$ for all $B \subseteq E \backslash \{\mathbf{0}\}$.

It follows from this definition that we have, in particular, for all $x, y \in E \backslash \{\mathbf{0}\}$:

(47) $h(x \sqcup_i y) \quad \doteq \quad h(x) \sqcup h(y)$

(48) $x \leq_i y \qquad \Rightarrow \quad h(x) \leq h(y)$

The homomorphism h induces a second ordering relation on $E \backslash \{\mathbf{0}\}$ which will be called the material part relation on $E \backslash \{\mathbf{0}\}$ and denoted by $\leq_m$:

(D.23) $x \leq_m y$ iff $h(x) \leq h(y)$ $(x, y \in E \backslash \{\mathbf{0}\})$

$\leq_m$ is only a preordering relation. It gives rise to an equivalence relation, defined by

(D.24) $x \sim_m y$ iff $x \leq_m y$ and $y \leq_m x$ $(x, y \in E \backslash \{\mathbf{0}\})$,

and thus to a partition of $E \backslash \{\mathbf{0}\}$ into equivalence classes each containing all the individuals that are made up by the same portion of matter.

In the following definition of a model for LPM I am going to interpret only 1-place predicates, for simplicity (except for the special 2-place predicate constants). All the other many-place predicates receive their usual first order interpretation.

(D.25) A *model for* LPM is an ordered pair $M \doteq \langle \mathfrak{B}, \| \cdot \| \rangle$ such that

1. $\mathfrak{B} \doteq \langle E, A, D, h \rangle$ is a boosk (E is the *domain of individuals* in M, A the *set of atoms* in M, D the *set of portions of matter* in M, and h the "*materialization function*" in M);

2. $\| \cdot \|$ is a first order assignment of denotations to the primitive expressions of LPM such that

(i) $\|a\| \in E$ if a is an individual constant;

(ii) $\|P\| \subseteq E \backslash \{\mathbf{0}\}$ if P is a 1-place predicate constant;

(iii) $\|P\| \subseteq A$ if $P \in \mathrm{DP}$;

(iv) $\|P\|$ is a complete subsemilattice of D if $P \in \mathrm{MT}$;

(v) $\|^{\mathrm{T}}P\| \subseteq \{x \in E \mid x \neq \mathbf{0} \ \& \ \exists y \in \|P\| \text{ such that } x \leq_i y\}$;

(vi) $\|^{\mathrm{T}}P\| \doteq \|P\|$ if $P \in \mathrm{DP}$.

The model is such that predicates are interpreted in the non-zero elements of E (condition (ii)). Conditions (iii), (iv) garantee the special properties of the distributive and mass predicates, respectively. (v) and (vi) are designed to validate the meaning postulates (MP.1) and (MP.2) above, respectively.

Let $M = \langle \mathfrak{B}, \| \cdot \| \rangle$ be a model for LPM. The usual first order semantical recursion rules are assumed, where quantification is taken to run over the non-zero elements of the domain of individuals only. The *truth* and *denotation conditions* for the special primitive symbols of LPM are defined as follows ('1' stands for 'true'; bivalence is assumed):

(D.26) $\|Pa\| \doteq 1$ iff $\|a\| \in \|P\|$;

(D.27) $\|\imath x Px\| \doteq d$ iff $d \neq \mathbf{0}$ and $\|P\| \doteq \{d\}$,

and $\mathbf{0}$ otherwise;

(D.28) $\|\mathbf{E}a\| \doteq 1$ iff $\|a\| \neq \mathbf{0}$

(D.29) $\|a \Pi b\| \doteq 1$ iff $\|a\| \neq \mathbf{0}$ and $\|a\| \leq_i \|b\|$;

(D.30) $\|a \top b\| \doteq 1$ iff $\|a\|, \|b\| \neq \mathbf{0}$ and

$h(\|a\|) \leq h(\|b\|)$;

(D.31) $\|a \rhd b\| \doteq 1$ iff $\|b\| \neq \mathbf{0}$ and $\|a\| \doteq h(\|b\|)$;

(D.32) $\|*P\| \doteq \mathbb{[}\|P\|\mathbb{]}_i$ (the complete $\sqcup_i$-subsemilattice generated by $\|P\|$);

A number of semantical facts can now be derived that give truth and denotation conditions for all the defined symbols of LPM.

(49) $\|P \subset Q\| \doteq 1$ iff $\|P\| \subseteq \|Q\|$

(50) $\|P \equiv Q\| \doteq 1$ iff $\|P\| \doteq \|Q\|$

(51) $\|P \oplus Q\| \doteq \|P\| \cup \|Q\|$

(52) $\|I_a\| \doteq \{\|a\|\}$

(53) $\|a = b\| \doteq 1$ iff $\|a\|, \|b\| \neq \mathbf{0}$ and $\|a\| \doteq \|b\|$

(54) $\|a \sim b\| \doteq 1$ iff $\|a\|, \|b\| \neq \mathbf{0}$ and $h(\|a\|) \doteq h(\|b\|)$

(55) $\|At\, a\| \doteq 1$ iff $\|a\| \in A$

(56) $\|Mpa\| \doteq 1$ iff $\|a\| \in D$

(57) $\|*P\| \doteq \{x \in E \mid \exists X \subseteq \|P\| \ \& \ X \neq \emptyset \text{ s.t. } x \doteq \sup_i X\}$

(58) $\|\circledast P\| \doteq \|*P\| \backslash A$

(59) $\|{}^m P\| \doteq (\sup_{\leq} h[\mathbb{[}\|P\|\mathbb{]}] \doteq \{x \in D \mid x \leq \sup_{\leq} h[\mathbb{[}\|P\|\mathbb{]}]\} \doteq$

$\doteq \{x \in D \mid \exists y \in \|*P\| \text{ such that } x \leq h(y)\}$

(60) $\|\sigma x Px\| \doteq \sup_i \|P\|$, with $\sup_i \emptyset \doteq \mathbf{0}$

(61) $\|\sigma^* x Px\| \doteq \|\sigma x Px\|$ if $\|P\|$ has ≥ 2 elements, and

$\mathbf{0}$ otherwise

(62) $\|a \oplus b\| \doteq \|a\| \sqcup_i \|b\|$

(63) $\|\mu x Px\| \doteq \sup_m h[\mathbb{[}\|P\|\mathbb{]}]$ if $\|P\| \neq \emptyset$, and $\mathbf{0}$ otherwise

(64) $\|a + b\| \doteq h(\|a\|) \sqcup h(\|b\|)$ if $\|a\| \neq \mathbf{0} \neq \|b\|$

Let the notion of *truth* in a given model be defined in the usual way. A formula is *valid* if it is true in every model.

I now give, in a loose order, a list of theorems of LPM that come out valid under the above interpretation. I may mention, in particular, the *homogeneous reference properties* (T.11), (T.12), and the *existence* and *identity criteria* for the σ- and μ-terms, (T.16)–(T.21).

(T.1) $a = b \leftrightarrow a \Pi b \wedge b \Pi a$

(T.2) $a \Pi b \rightarrow a \top b$

(T.3) $a = b \rightarrow a \top b \wedge b \top a$

(T.4) $Equ(=) \wedge Equ(\sim)$

(T.5) $PO(\Pi) \wedge PrO(\top)$

(T.6) $Distr(P) \rightarrow \bigwedge x(Px \rightarrow \neg {}^{\circledR}Px)$

(T.7) $P \subset {}^*P$

(T.8) $\bigwedge x(At\,x \rightarrow (Px \leftrightarrow {}^*Px))$

(T.9) $\bigwedge x({}^*Px \rightarrow x \Pi \sigma xPx \wedge \bigvee y(Py \wedge y \Pi x))$

(T.10) $Distr(P) \rightarrow \bigwedge x \bigwedge y({}^*Py \wedge x \Pi y \rightarrow {}^*Px)$

(T.11) $\bigwedge x \bigwedge y({}^*Px \wedge {}^*Py \rightarrow {}^*Px \oplus y)$

(T.12) $\bigwedge x \bigwedge y(Px \wedge Py \rightarrow Px + y)$ for $P \in MT$

(T.13) $M(P) \rightarrow P \subset {}^mP$

(T.14) $Mpa \rightarrow ({}^mPa \leftrightarrow \bigvee y({}^*Py \wedge a \top y))$

(T.15) $\bigvee xPx \wedge P \subset Q \rightarrow {}^*Q(\sigma xPx)$

(T.16) $\bigvee xPx \leftrightarrow \mathbf{E}\,\sigma xPx$

(T.17) $\bigvee xPx \leftrightarrow \mathbf{E}\mu xPx$

(T.18) $\mathbf{E}\,\sigma {}^*xPx \leftrightarrow \bigvee x \bigvee y(Px \wedge Py \wedge x \neq y)$

(T.19) $P \equiv Q \rightarrow \sigma xPx = \sigma xQx$

(T.20) $Distr(P) \wedge Distr(Q) \rightarrow (\sigma xPx = \sigma xQx \leftrightarrow P \equiv Q)$

(T.21) $P \equiv Q \rightarrow \mu xPx = \mu xQx$

(T.22) $At\,a \rightarrow a = \sigma x(x = a)$

(T.23) $Mpa \rightarrow a = \mu x(x = a)$

(T.24) $\mu xPx = \iota x(Px \wedge \bigwedge y(Py \rightarrow y \top x))$ for $P \in MT$

(T.25) $a \sim b \rightarrow \mu x(x = a) = \mu x(x = b)$

(T.26) $\mu xPx = \iota x(x \rhd \mu xPx)$

(T.27) $Mp(\mu xPx)$

(T.28) $a \rhd b \leftrightarrow Mpa \wedge a \sim b$

(T.29) $\mu xPx \sim \sigma xPx$

(T.30) $\sigma xPx = \sigma xQx \rightarrow \mu xPx = \mu xQx$

(T.31) $\bigwedge x \bigwedge y(x \Pi y \leftrightarrow x \oplus y = y)$

(T.32) $\bigwedge x \bigwedge y(x \top y \leftrightarrow x + y = y)$

III. Applications to Montague Grammar

Let the basic *syntax* be as in PTQ.[16] The category CN of common noun phrases has to be subcategorized into MCN (mass noun phrases), SCN (singular count noun phrases), and PCN (plural count noun phrases). The quantifiers discussed are $a, \emptyset_{pl}, some, the, all\ the, every, all$; they are of category

T/CN with suitable subcategorizations (term formation is done compositionally). We have the obvious plural rule

(65) $\zeta \in P_{SCN} \;\; \Rightarrow \;\; \zeta_{pl} \in P_{PCN}$

where $\zeta \doteq hand$ becomes $\zeta_{pl} \doteq hands$, $\zeta \doteq child$, $\zeta_{pl} \doteq children$, etc.

The logic of PTQ, TITL, is extended as follows. TITL', or *extended TITL*, contains as new symbols all the special symbols of LPM. Additional *meaningful expressions* are: (i) $*\zeta$, $^{\circledR}\zeta$, $^{T}\zeta \in ME_\rho$, and $^{m}\zeta \in ME_\rho$ for $\tau = e$ ($\tau \in Type$, $\rho \doteq \tau t$ or $s\tau t$, $\zeta \in ME_\rho$); (ii) $\sigma u \phi$, $\sigma^* u \phi$, $u\phi \in ME_\tau$, and $\mu u \phi \in ME_e$ ($\tau \in Type$, $u \in Var_\tau$, $\phi \in ME_t$); (iii) $\alpha \Pi \beta \in ME_t$ and $\alpha \oplus \beta \in ME_\tau$ ($\tau \in Type$, $\alpha, \beta \in ME_\tau$); (iv) $\alpha \mathsf{T} \beta$, $\alpha \sim \beta$, $\alpha \rhd \beta \in ME_t$, and $\alpha + \beta \in ME_e$ ($\alpha, \beta \in ME_e$).

The *semantics* for TITL' has to specify the "boolification" of the sets of possible denotations since the i-sum operation applies to expressions of any type (m-sums are formed on level e only). Let E be a c.a. Boolean algebra, I, J sets, and $K = I \times J$. A *Boolean hierarchy of possible denotations over* E *with respect to* K is a family of sets $(D_{\tau, E, K})_{\tau \in Type}$ defined as in PTQ, but endowed with a c.a. Boolean structure for every D_τ as follows: (i) The sets $D_e = E$ and $D_t = 2$ are already c.a. Boolean algebras; (ii) $f, g \in D_{s\tau}$; then: $(f \sqcup_{s\tau} g)(k) := f(k) \sqcup_\tau g(k)$ ($k \in K$), similarly for the other operations; (iii) $f, g \in D_{\sigma\tau}$; then $(f \sqcup_{\sigma\tau} g)(x) := f(x) \sqcup_\tau g(x)$ ($x \in D_\sigma$), similarly for the other operations. The resulting structures are c.a. Boolean.

An *interpretation* for TITL' is a 7-tuple $\mathfrak{A} \doteq \langle E, A, D, h, I, J, \|\cdot\| \rangle$ such that (i) $\langle E, A, D, h \rangle$ is a boosk, (ii) $K = I \times J$ is as in PTQ, and (iii) $\|\cdot\|$ is the interpretation function that naturally emerges from a combination of TITL and LPM. As *rules of truth and denotation* provided by an appropriate recursive definition let me mention the following: (i) $\|*\zeta\| \doteq$ the characteristic function of $[\{x \in D_\tau | \| \zeta \|(x) \doteq 1\}]_{\sqcup_\tau}$ ($\zeta \in ME_{\tau t}$); (ii) $\|*\zeta\|(k) \doteq$ the characteristic function of $[\{x \in D_\tau | \| \zeta \|(k)(x) \doteq 1\}]_{\sqcup_\tau}$ ($\zeta \in ME_{s\tau t}$, $k \in K$); (iii) $\|^{\circledR}\zeta\| \doteq \|*\zeta\| \sqcap_{\tau t} \chi'_{A\tau}$, where $\zeta \in ME_{\tau t}$, A_τ the set of atoms in D_τ, $\chi_{A\tau}$ the characteristic function of A_τ, *and* "' the Boolean complement; (iv) $\|^{\circledR}\zeta\|(k) \doteq \|*\zeta\|(k) \sqcap_{\tau t} \chi'_{A\tau}$ ($\zeta \in ME_{s\tau t}$, $k \in K$); (v) $\|\sigma u \phi\| \doteq \sup_i \|\lambda u \phi\|$, with $\sup \emptyset \doteq \mathbf{0}$, and $\|\sigma^* u \phi\| \doteq \|\sigma u \phi\|$ if $\|\lambda u \phi\|$ has more than one element, and $\mathbf{0}$ otherwise; (vi) $\|\alpha \prod \beta\| \doteq 1$ iff $\|\alpha\| \leq_\tau \|\beta\|$ and $\|\alpha\| \neq \mathbf{0}$ ($\alpha, \beta \in ME_\tau$, and $\leq_\tau$ the intrinsic ordering relation of D_τ); (vii) $\|\alpha \oplus \beta\| \doteq \|\alpha\| \sqcup_\tau \|\beta\|$ ($\alpha, \beta \in ME_\tau$). The remaining basic symbols are characterized as in LPM.

The notion of distributivity introduced above can be extended to class-denoting expressions of any type. Call a $\zeta \in ME_{\tau t}$ *distributive* if it can be true of atoms of type τ only:

(66) $\mathrm{DISTR}(\zeta) \;\leftrightarrow\; \bigwedge x [\zeta(x) \;\rightarrow\; At(x)]$ ($\zeta \in ME_{\tau t}$)

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[16] Montague 1974a; the following notation is that of Link 1979 (which differs only slightly from the one used in PTQ) except that the interpretation function F is denoted by the "norm" symbol $\|\cdot\|$ here. I also ignore, for the present purpose, the individual concept language of PTQ.

Most of the basic count nouns like *child* are taken as distributive, similarly IV phrases like *die* or *see*. In accordance with what I did above the translation rule T4 involving distributive expressions has to be such that their translations always enter this rule under the star operator. While this seems to me the correct move from a technical point of view I am aware that it carries the empirical prediction that distributivity is a lexical property.

I now give *translations* for the quantifiers. Let U be the translation relation, and $\tilde{P} := \lambda y [P(y) \wedge \Lambda x [x \Pi y \rightarrow {}^{\mathrm{T}}P(x)]]$.

1. $a, \varnothing_{pl}, some$ U $\lambda Q \hat{P} \vee x [Q(x) \wedge P(x)]$
2. *the*   U $\lambda Q \hat{P} \vee y [Q(y) \wedge \Lambda x [Q(x) \rightarrow x \Pi y] \wedge P(y)]$
3. *all the*   U $\lambda Q \hat{P} \wedge y [y = \sigma^* x Q(x) \rightarrow \tilde{P}(y)]$
4. *every, all*   U $\lambda Q \hat{P} \wedge x [Q(x) \rightarrow P(x)]$

a applies only to singular count noun phrases and $\varnothing_{pl}$ to PCN phrases. The quantifiers *some* and *the*, with one and the same translation, apply to both singular and plural phrases. As can be seen from 5. through 8. below it is only the incoming CN phrase which differentiates between the appropriate singular and plural readings. While this is obvious with *some*, a comment on the transformations under 7. is in order. The conjunct $\Lambda x [Q(x) \rightarrow x \Pi y]$ in the translation of *the* asserts maximality which is needed for the intended sum formation in the extension of $*Q$. At the same time, however, it generalizes the usual uniqueness condition for definite descriptions. This comes out when *the* is applied to a singular count noun like *child*: The i-part x of y in 7. cannot be **0**, and y is an atom because of DISTR (*child'*); so x equals y. I think this is a nice instance of strict compositionality in times in which this principle has come under heavy fire in view of all sorts of recalcitrant data (see below). Finally, 9.–11. show that universal quantification is as in LPM where, again, the difference in meaning between *the children* and *all the children* is only partly characterized ($\tilde{P}$ being defined in terms of '$^{\mathrm{T}}$').

5. *a child*
    some child   U $\hat{P} \vee x [child'(x) \wedge P(x)]$
    some water   U $\hat{P} \vee x [water'(x) \wedge P(x)]$
6. *(some) children*   U $\hat{P} \vee x [\circledast child'(x) \wedge P(x)]$
7. *the child*   U $\hat{P} \vee y [child'(y) \wedge \Lambda x [child'(x) \rightarrow x \Pi y] \wedge P(y)]$
    $= \hat{P} \vee y [child'(y) \wedge \Lambda x [child'(x) \rightarrow x = y] \wedge P(y)]$
    $= \hat{P} \vee y [y = \iota x \, child'(x) \wedge P(y)]$
8. *the children*   U $\hat{P} \vee y [\circledast child'(y) \wedge \Lambda x [\circledast child'(x) \rightarrow x \Pi y] \wedge P(y)]$
    $= \hat{P} \vee y [y = \sigma^* x \, child'(x) \wedge P(y)]$
9. *all the children*   U $\hat{P} \wedge y [y = \sigma^* x \, child'(x) \rightarrow \tilde{P}(y)]$
10. *every child*   U $\hat{P} \wedge x [child'(x) \rightarrow P(x)]$
11. *all water*   U $\hat{P} \wedge x [water'(x) \rightarrow P(x)]$

I may mention here that the incorporation of numerals like *one*, *two*, *three* into the fragment is straightforward. The extension of *three men*, for instance, is the set of all i-sums in $\|men'\| \doteq \|^{\circledast}man'\|$ which contain exactly three atoms. Sentences like *three men went to mow a meadow* receive a translation of the form $\vee z\left[(three\ men)'(z) \wedge Q(z)\right]$.

The final topic I want to say something about is the vexing problem of relative clauses with more than one head noun[17]. Call those structures *hydrae*. Let me consider restrictive relative clauses only. First of all, there is a rather friendly type of hydra, like the following.

(67)   the German or Austrian John met yesterday
(68)   the cabinet member and mafioso who was deeply involved in the scandal

PTQ style relativization, i.e. CN modification, takes care of these pets yielding the obvious representations for (67) and (68), respectively:

(67′)   the ((German or Austrian) such that John met him yesterday)
$$\widehat{P}\vee y\left[\wedge x\left[German' \oplus Austrian'(x) \wedge \phi\left[x\right] \leftrightarrow x=y\right] \wedge P(y)\right]$$

(68′)   the ((Cabinet member and mafioso) such that he was deeply involved in the scandal
$$\widehat{P}\vee y\left[\wedge x\left[Cabinet\text{-}member' \odot mafioso'(x) \wedge \psi\left[x\right] \leftrightarrow x=y\right] \wedge P(y)\right]$$

Here, ϕ and ψ are the translations of the relative clauses of (67) and (68), respectively; furthermore, $\zeta \oplus \eta \doteq \lambda x\left[\zeta(x) \vee \eta(x)\right]$ is the Boolean join and $\zeta \odot \eta \doteq \lambda x\left[\zeta(x) \wedge \eta(x)\right]$ the Boolean meet of ζ and η. This case is simple because CN conjunction does not lead to plural structures, and there is only one determiner present.[18]

Next, let us conjoin two singular count nouns to form a plural phrase. Examples are *man and woman*, *boy and girl*, *husband and wife*, *landlord and tenant*. Such phrases can be true of a sum of two individuals one being of the first and the other being of the second kind. An appropriate translation, therefore, is of the form

(69) $(\zeta\ and\ \eta)$ U $\lambda z\vee x\vee y\left[\zeta'(x) \wedge \eta'(y) \wedge z = x \oplus y\right]$

With this sentences like *John and Mary are husband and wife* can be handled in a first approximation while the non-symmetric features of this sentence have to be taken care of by other clues. This is because the predicate (69), by itself, is necessarily symmetric reflecting the fact that pluralization has the

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[17] Thanks to P. Staudacher and A. v. Stechow bringing the severeness of the problem to my attention. See v. Stechow 1980 and Janssen 1981 for thorough and critical discussions of the issues involved.

[18] In German there seems to be a restriction to the effect that the two nouns have to agree in gender for this construction to be possible. The same article can then be related to both nouns.

force of group formation which typically gives rise to symmetric construc-tions.[19] Such is the following CN phrase:

(70)   boy and girl who dated each other

With (69), the standard PTQ rule yields for this:

(71) $\lambda z \vee x \vee y [boy'(x) \wedge girl'(y) \wedge z = x \oplus y \wedge dated\text{-}each\text{-}other'(z)]$

Now term formation with one and the same determiner for both head nouns is again standard if we decide to have it distributed over the two nouns in a kind of copying process. So, for instance, *a boy and a girl who meet* becomes

(72)   *a ((boy and girl) such that they meet)*   U
$\hat{P} \vee z [\vee x \vee y [boy'(x) \wedge girl'(y) \wedge z = x \oplus y \wedge meet'(z)] \wedge P(z)]$

In a further step we admit conjunction of plural nouns. This move doesn't really add any new difficulty so I can write down an example immediately. Let θ be the translation of the phrase *students and professors who had met in secret*:

(73) $\theta \doteq \lambda z \vee x \vee y [{*}student'(x) \wedge {*}professor'(y) \wedge z = x \oplus y$
$\wedge had\text{-}met\text{-}in\text{-}secret'(z)]$

Adding a determiner as before, say the definite article, we get the term

(74)   the students and the professors who had met in secret
U $\hat{P} \vee z [z = \sigma{*}x\theta(x) \wedge P(z)]$

The present conception of conjunction of two CN phrases ζ and η is such that the extension of (ζ *and* η) contains i-sums of objects which are of kind ζ and η, respectively. Unlike the Boolean join this carries the presupposition that both sets of objects be non-empty. I think this is as it should be. Notice that in this case the overall sum is the same as the one which is formed in terms of the Boolean join, i.e. we have $\sigma x (\zeta' \oplus \eta')(x) = \sigma x (\zeta$ *and* $\eta)'(x)$. What is not captured in either approach, however, is the proper pair reading which is dominant in sentences like *landlords and tenants who hate each other will always find something to argue about*. I find these constructions hard to treat at the moment though I am confident that they will finally lend them-selves to an analysis which is compatible with the present framework. What is needed here is some notion or ordered pair in the object language, apart from the symmetrical sum operator.[20] It is only with such an additional instrument, it seems to me, that we can, in a close-to-language treatment, attack the most dangerous hydra lurking in the realm of pair reading and branching quantification.[21]

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[19] For the pair reading see the remarks below.

[20] The device of "multiple quantification" introduced in Thomason 1977 might be relevant here. Thanks to N. Belnap bringing these unpublished notes to my attention.

[21] see Barwise 1979 for a lucid account of the latter phenomenon.

The last type of hydra I want to address myself to shows up in a sentence like the following

(75)   all of the students and some of the professors who had met in secret were arrested after the coup d'état.

From a purely syntactic point of view it might look natural to try a T-S-analysis here: first conjoin the two unrelativized terms *all the students* and *some of the professors* and then modify the conjunct by the relative clause. However, I agree with Janssen 1981 in his sceptical attitude towards the semantical soundness of this approach in general, and the present case, I think, is apt to confirm these doubts. Even a generalized quantifier account along the lines of Barwise/Cooper 1981 is not easy to give because the set of properties denoted by the subject term of (75) is not the obvious intersection of two property sets. The point is that the collective predicate *meet-in-secret* does not distribute over the conjuncts.

Now there is no reason to leave the approach taken so far, it seems to me. We already have a clear understanding of the meaning of the plural CN phrase *students and professors who had met in secret* which is expressed by the λ-term θ of (73). so in terms of this, what is the meaning of the above NP? Well, the set of properties that all the students of the group of students and professors who had met in secret and some of its professors share. This set can be formally represented in TITL'. From the above group of individuals we have to pick out again the students and the professors, respectively, which is done by means of the Boolean meet $\odot$; we can then write (with $\alpha \doteq *student'$ and $\beta \doteq *professor'$):

(76)   all the students and some of the professors who had met in secret
$$U \quad \hat{P}\big[\Lambda y\big[y = \sigma^*x(\alpha \odot \theta)(x) \to \tilde{P}(y)\big] \land \lor y\big[(\beta \odot \theta)(y) \land P(y)\big]\big]$$

In the syntax, then, we have to introduce the determiners *all* and *some* simultaneously. I hope that a rule to this effect does not look too outlandish to the eye of the syntactician.

Let me summarize what I consider to be the virtues of the present approach to plurals and mass terms. (1) The logic of plurals and the logic of mass terms share a common lattice structure the only difference being that the former leads to an atomic structure while the latter does not. (2) By means of the star operator pluralization and term formation involving plural constructions can be treated compositionally. (3) Plural terms *(the cards)* and collective terms *(the deck of cards)* are equivalent in that they are interchangeable in invariant contexts; this does not make them coreferential, however, in contrast to systems of the reductionist lot. (4) collective predication becomes possible in a unified way accounting for the fact that many predicates (e.g. *carry*) are not marked with respect to distributivity and can, therefore, have mixed extensions.

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