that no relaxing of the standards puts d' in the positive extension but d in the negative extension of *tall*, then indeed d got to be at least as tall as d'. This is what the clause for $\geqslant(P)(t, t')$ says. (We would need to add some more constraints to make sure that this relation is fully transitive. Think about how this could be done.)

We can give a similar analysis to the real comparative $>(P)(t, t')$, t is taller than t':

$$s \Vdash >(P)(t, t')[g] \text{ iff}$$

1. $\min(s, P, [\![t]\!]_g, [\![t']\!]_g) \Vdash P(t') \Rightarrow P(t)[g]$
2. $\min(s, P, [\![t]\!]_g, [\![t']\!]_g) \Vdash \mathbf{P}(P(t) \wedge \neg P(t'))[g]$

i.e. t is taller than t' iff t is at least as tall as t' and if you minimally move both t and t' into the gap of *tall*, it is possible to make t tall, but t' short.

In this way, then, Kamp is able to give the semantics of the comparative as a (modal) operation that builds the interpretation of the comparative relation *taller than* out of the interpretation of the adjective *tall*.

For a very interesting discussion of how a theory of vagueness of the sort described here can be extended to deal with puzzles of vagueness like the sorites paradox, see Pinkal (1984).

CONSTRUCTIONS WITH PARTIAL ORDERS

4.1. PERIOD STRUCTURES

In the previous two chapters, we have been interested in temporal structures, where the primitives are moments of time or instants. We will now be interested in structures where the basic units are periods (and in the next section, events). In particular, we will be interested in the relations between instant structures and period structures. For a thorough discussion of the topics in this chapter, see van Benthem (1983), Kamp (1979a) and Kamp (1979b). I will be following van Benthem's exposition rather closely here.

We have defined notions of convex sets and intervals earlier. Let's start the present discussion by looking at what period structures we can define using these notions. We start with a partial order of points of time $\langle T, < \rangle$.

Let $I(T)$ be the set of all convex sets in T. We define some natural relations and operations on $I(T)$.

Let $i, i' \in I(T)$

$i < i'$, *i completely precedes* i', iff $\forall t \in i \forall t' \in i' : t < t'$
$i \bigcirc i'$, *i overlaps* i' iff $i \cap i' \neq \emptyset$
$i \sqsubseteq i'$, i is *temporally included in* i' iff $i \subseteq i'$

On the present structures we can define overlap in terms of inclusion:

$$i \bigcirc i' \text{ iff } \exists i_0[i_0 \neq \emptyset \wedge i_0 \subseteq i \wedge i_0 \subseteq i']$$

So two periods overlap if they have a non-empty period in common. Also, on the present structures we can define inclusion in terms of overlap:

$$i \sqsubseteq i' \text{ iff } \forall i_0[\text{if } i_0 \bigcirc i \text{ then } i_0 \bigcirc i']$$

So i is part of i' if every period overlapping with i overlaps i'.

There are some disadvantages and some advantages of the present structure.

The advantages are that we can define uniformly union and intersection of periods. Note that $\sqsubseteq$ is a partial order (because $\subseteq$ is). The empty period $\emptyset$ is the minimal element of I under $\sqsubseteq$. We can define an operation of overlap:

$i \sqcap i'$, the *overlap of* i and $i' := i \cap i'$

We can generalize this, even, to:

Let $X \subseteq I$: $\sqcap X := \cap x$

I is not closed under union of course, $i \cup i'$ need not be a convex set. Still we can introduce some notion of union. Given a set $X \subseteq T$.

$c(X)$, *the convex closure* of
$X = \{t: \exists t' \in X \exists t'' \in X: t' \leq t \leq t''\}$

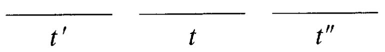
Define:

$i \sqcup i' := c(i \cup i')$
$\sqcup X := c(\cup X)$

So the union of two intervals is the minimal period that you get by filling their set theoretic union up to where you get a convex set. (So the union of yesterday and tomorrow is yesterday, today and tomorrow.)

I is closed under union (both of pairs and of sets of intervals). T is the maximal element of I (under $\sqsubseteq$). We haven't introduced the notions yet (see Chapter Six), but in fact, $\langle I, \sqsubseteq, \sqcap, \sqcup, \emptyset, T\rangle$ forms a complete atomic lattice under these operations (with the singleton periods as atoms).

Some comments on this structure. We have defined union and intersection. It is hard to see what complementation of periods should be. The reason is clear:



Suppose these three periods fill up T. The complement of t should presumably be everything that is not t, $t' \cup t''$, but that is not a period.

Note that the notion of union we have defined seems rather unintuitive, or useless. Maybe it is nice to know that we can define a well behaved notion of union, but it seems rather devoid of use, to have a notion that unites the period where the Great Pyramid was built with

the period where you read this text, by dragging along the whole period in between.

Of course, there is nothing wrong with this notion of union, if you want you can use it, but it doesn't seem to be that interesting to look at period structures as structures ordered by this operation.

The notions of precedence, overlap and inclusion seem more interesting as ordering relations on these structures. And here we note that the regular structure leads to some complexities with them.

Complete precedence intuitively should be a strict partial order. In fact, it is only a strict partial order if we leave out $\emptyset$. (On the present definition $\emptyset < \emptyset$, and, for any p and q: $p < \emptyset$ and $\emptyset < q$, hence $<$ is neither irreflexive, nor asymmetric.)

Similarly, overlap should be a reflexive and symmetric relation. In fact, it is, but again only if we leave out $\emptyset$. (Again on the present definition $\emptyset$ does not overlap $\emptyset$.) Given this, allowing the empty period as a period is a bit of a nuisance. The structure becomes more regular and intuitive with respect to the temporal orderings if we cut it out:

The *period structure* based on $\langle T, <\rangle$ is the structure $\langle P, <, \sqsubseteq, \bigcirc\rangle$, where P is the set of all *non-empty* convex sets in T, where $<$, complete precedence, $\sqsubseteq$, temporal inclusion and $\bigcirc$, overlap are defined as before.

We don't have an empty period now. Consequently, intersection is only a partial operation on P (only the intersection of overlapping periods is in P). Now the relations have the required properties.

Let $\langle T, <\rangle$ be a dense linear order. The period structure of T, $I(T)$ is the set of all non-empty intervals, with precedence, inclusion and overlap defined as before. By the definition of period as non-empty interval, this structure contains minimal periods or atoms: every singleton set $\{t\}$ with $t \in T$ is a minimal period. If we don't want such minimal periods, we can change the definition of period: redefine the period structure as the set of all non-empty, non-singleton intervals. We also have to redefine the notion of overlap: two periods overlap iff their intersection is non-empty and non-singleton. So we assume that for overlap two periods have to overlap in a period: overlap in a single point is not enough.

Now every period contains at least two instants, and hence, by density it contains infinitely many instants, and hence many subperiods: there are no minimal periods.

Let us now, before asking what properties these structures have, abstract away from the underlying moments of time and talk about period structures in general.

Let us assume that a period structure is a structure $\langle P, <, \sqsubseteq \rangle$, where P is a set of periods, $<$ is a strict partial order of precedence and $\sqsubseteq$ is a partial order of temporal inclusion. Overlap, $\bigcirc$, is defined as before in terms of temporal conclusion.

The question to be asked then is: are there other intuitive postulates that we may want to impose on period structures and are there principles that carve out interesting classes of period structures. Here is a list of possible conditions:

Conjunction:

$$x \bigcirc y \rightarrow \exists z[z \sqsubseteq x \wedge z \sqsubseteq y \wedge \forall u[u \sqsubseteq x \wedge u \sqsubseteq y \rightarrow u \sqsubseteq z]]$$

Let us introduce $x \sqcap y$ for the infimum of $\{x, y\}$ under $\sqsubseteq$ if there is one (this corresponds with the definition of $x \sqcap y$ that we gave before as $x \cap y$ (if that is non-empty). Then conjunction says:

$$x \bigcirc y \rightarrow \exists z[z = x \sqcap y]$$

In words: if x and y overlap, there is a period which is the overlap of x and y.

Let us similarly introduce a notion of union: $x \sqcup y$ is the supremum of $\{x, y\}$ under $\sqsubseteq$, if there is one. Then we can introduce a similar postulate for unions:

Disjunction:

$$\exists z[x \sqsubseteq z \wedge y \sqsubseteq z] \rightarrow \exists z[z = x \sqcup y]$$

We have cut the empty period out of our period structures, so it is not the case that all periods have a common part, but we may not want to be as restrictive for unions, we might want to say that any two periods are part of a bigger period:

Direction:

$$\forall x \forall y \exists z[x \sqsubseteq z \wedge y \sqsubseteq z]$$

The next question concerns the existence of minimal periods. In the structure we have defined above, the singletons are minimal, atomic periods; on the definition of interval that we mentioned (non-empty, non-singleton, dense intervals) there were no atomic periods. The condition of atomicity says that every chain of smaller and smaller subperiods ends in a minimal element, an atomic period:

Atomicity:

$$\forall x \exists y[y \sqsubseteq x \wedge \forall z[z \sqsubseteq y \rightarrow z = y]]$$

Let us look at some principles for precedence. When is a period structure linear? Intuitively, it is too strong to require linearity as we do for points: $t < t'$ or $t' < t$ or $t = t'$, because then we eliminate overlap. The obvious linearity requirement for periods is:

Linearity:

$$x < y \vee y < x \vee x \bigcirc y$$

Let us think about discreetness and density. Although on instant structures, these principles are incompatible, if we think what they should amount to on period structures, we see that there they are compatible. On period structures, discreetness will say that for every period x, if there is a period later (earlier) than x, then there is a period later (earlier) than x that is directly neighboring x. At instant structures, density says: however close to each other you take two instants, you find a third in the middle. For periods, this becomes: however small you take a period, you can divide it into smaller subperiods. In a period structure density does not conflict with discreetness but with atomicity: in a dense period structure there can be no atomic periods.

Intuitively, then, what we should say is that in a dense structure every period can be split into smaller periods. Let us introduce a new notion of union for this.

$x = y + z$, x is *the sum of* y and z iff
$x = y \sqcup z$ and $\forall u[u \sqsubseteq x \rightarrow u \bigcirc y \vee u \bigcirc z]$

This is a more intuitive notion of union than the one we have defined before. It is only defined for overlapping periods or neighboring periods: the intuition is that the sum of two overlapping or neighboring periods is the minimal period covering both. A period can be split into two smaller periods if it can be regarded as the sum of two neighboring periods (its left wing and its right wing):

Density:

$$\forall x \exists y \exists z[y < z \wedge x = y + z]$$

Let us consider now the interaction between $<$ and $\sqsubseteq$. A typical principle we would want is:

Separation:

$$x < y \rightarrow \neg x \bigcirc y$$

This principle is (given that $<$ is irreflexive) implied by the following equally plausible principles of monotonicity:

Monotonicity:

$$x < y \rightarrow \forall z[z \sqsubseteq x \rightarrow z < y]$$
$$x < y \rightarrow \forall z[z \sqsubseteq y \rightarrow x < z]$$

Monotonicity implies the following principle:

Transfer:

$$x < y \wedge y \bigcirc z \wedge z < u \rightarrow x < u$$

$y \bigcirc z$ tells us that for some v: $v \sqsubseteq y$ and $v \sqsubseteq z$. Since $v \sqsubseteq y$, it follows that $x < v$ with monotonicity; since $v \sqsubseteq z$ it follows that $v < u$, again with monotonicity. Hence $x < u$.

Another principle we may want is a principle that says that periods are not interrupted:

Convexity:

$$x < y < z \rightarrow \forall u[(x \sqsubseteq u \wedge z \sqsubseteq u) \rightarrow y \sqsubseteq u]$$

I will follow van Benthem in introducing two more principles:

Witness1:

$$\neg(x \sqsubseteq y) \rightarrow \exists z[z \sqsubseteq x \wedge \neg(z \bigcirc y)]$$

If a period x is not temporally included in a period y, then there is a part of x that doesn't overlap with y.

Witness2:

$$\neg(x < y) \rightarrow \exists u \sqsubseteq x \; \exists v \sqsubseteq y[\forall z \sqsubseteq u \; \forall w \sqsubseteq v[\neg(z < w)]]$$

Two periods that do not precede one another may have subperiods

that do, but they have to have subperiods that don't have such subperiods anymore.

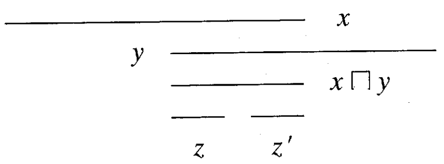
These two principles have a somewhat special status (they are called 'freedom' postulates in van Benthem 1983). They are not really independently motivated, but turn out to be needed in the representation theorem that will follow. I call these principles 'witness' principles, because their function is similar to that of witness constants for existential sentences that we saw in the completeness proof for first order logic. There we wanted to construct a model out of a set of sentences. But models have 'real' individuals in them. If one of the sentences in our set is of the form $\exists x\varphi$, we have to insure that there is at least one object in the model to witness the truth of that sentence. That is why we enriched this set of sentences with witness constants: to insure that every sentence is properly witnessed.

We will see a very similar problem in the representation theorem for our structures. We will try to represent period structures as constructed out of point structures, with points defined in terms of periods, but we will not get 'enough' points, we will loose certain distinctions, unless we assume that for every basic relation, that is $<$ and $\sqsubseteq$, *the absence of that relation* is witnessed by some period: we need the truth of negative statements $\neg(p < q)$ and $\neg(p \sqsubseteq q)$ to be witnessed by the actual positive presence of certain periods (else we will lose information).

Such principles you typically discover when you try to give a representation theorem for the structures you study: you find out that the construction you make only goes through if the structure you start out with is rich enough in this sense (we will see a similar situation in the representation of Boolean algebras later).

Exercise 1. Here is a putative argument that density and witness2 are incompatible. Witness2 says that if x is not before y then x and y have subperiods x' and y' such that for every subperiod of x' and every subperiod of y', the first doesn't precede the latter. Density says that every period can be split into two subperiods. Now suppose that $x \bigcirc y$ and look at $x \sqcap y$.

We know that $x \sqcap y$ can be split into z, z' with $z < z'$, and this goes all the way down: you can always find subperiods that can be split into a z and z' where $z < z'$. Since all these periods are part of both x and y, it seems that you can't find a subperiod of x (and y) and a subperiod of y (and x), such that all subperiods of the first do not precede all subperiods of the other, because we can always split into z and z' where z precedes z'.



Refute this argument.

Which of these principles should we take as defining characteristics of a period structure? Monotonicity is an absolute must. Conjunction will simplify the constructions we will discuss (though similar constructions can be given without), so we will assume that. Further there is good reason to assume convexity:

A *period structure* is a triple $\langle P, <, \sqsubseteq \rangle$ where

1. P is a non-empty set of periods
2. $<$ is a strict partial order of precedence.
3. $\sqsubseteq$ is a partial order of temporal inclusion
4. $<$ and $\sqsubseteq$ satisfy monotonicity
5. $\sqsubseteq$ satisfies conjunction
6. $<$ and $\sqsubseteq$ satisfy convexity

A *witnessed period structure* is a period structure satisfying both witness postulates.

Earlier we defined the period structure based on partial order $\langle T, < \rangle$, where periods were defined as non-empty convex sets. It is not hard to prove that this structure is a period structure. In fact, it satisfies more than just the axioms required for period structure, namely also: disjunction, direction, atomicity, witness1 and witness2.

We have noted, that if we change the definition of a period, (we assume density for the instant structure and define periods as non-empty, non-singleton convex sets) we can get period structures that satisfy density, rather than atomicity.

Let us, for this reason generalize the notion of a period structure based on $\langle T, < \rangle$, making it dependent on which definition of period we use:

Let $\langle T, < \rangle$ be a strict partial order.
A *period set* based on $\langle T, < \rangle$ is a set $P(T)$ such that:

1. every $p \in P$ is a non-empty convex subset of T
2. if $p, p' \in P$ and $p \cap p' \neq \emptyset$ then $p \cap p' \in P$

Let $\langle T, < \rangle$ be a partial order and $P(T)$ be a period set based on $\langle T, < \rangle$.
The *period structure based* on $P(T)$ is the structure $\langle P(T), <, \sqsubseteq \rangle$ where:

1. $p < p'$ iff $\forall t \in p\ \forall t' \in p' : t < t'$
2. $p \sqsubseteq p'$ iff $p \subseteq p'$

For any T and $P(T)$, we can prove that $P(T)$ is indeed a period structure.

Exercise 2. Prove this. I.e. check that $<$ is a strict partial order, $\sqsubseteq$ is a partial order and the principles of conjunction, monotonicity and convexity hold.

A *point generated period structure* is a period structure $\langle P, <, \sqsubseteq \rangle$ such that for some point structure $\langle T, < \rangle$ and some period set $P(T)$, $\langle P, <, \sqsubseteq \rangle$ is the period structure based on $P(T)$.

Point generated period structures stand to period structures as set theoretic partial orders stand to partial orders. They are easy to construct (they are set theoretic constructions out of a given base set (the partial order in this case)) and it is easy to see what their structure is. For that reason alone we should already be interested in representation theorems for period structures. There is an added reason here, however.

We are interested here in the question: suppose we start with a period structure as basis, i.e. suppose we take the notion of a period as primitive: can we get a point structure back; can we construct points out of periods (we are interested in that because soon we will try to construct periods out of events)?

What does it mean that we can construct points out of periods?

Suppose we have a period structure and suppose that we define in terms of this a notion of point (just like we did earlier for the notion of period in a point structure) and suppose we can prove that our period structure is isomorphic to a point generated period structure, more in particular to the period structure generated by the thus defined points. This would be a proof that our period structure can be represented as a point generated period structure, and it justifies the claim that the points that we have defined can be regarded as the points that underlie our period structure. Although points are defined as abstractions out of periods, they are for all semantic purposes indistinguishable from real (primitive) points and if the period definition under which we get the starting period structure back (up to isomorphism) is satisfying, we have indeed constructed a satisfying underlying point structure.

A representation theorem will tell you that given a period structure and a definition of points in that, you'll get a point structure such that some definition of period in that point structure gives you the original period structure back. It does not tell you more than that. It is possible, for instance that one point structure with one point definition and another point structure with another point definition gives you the same period structure. Similarly, if we choose different definitions of points in one and the same period structure, we may be able to represent this period structure as generated by different point structures. So you represent a period structure as point generated relative to some definition of point and, in terms of that, some definition of period.

Van Benthem discusses three methods of constructing point structures out of period structures. We will here only discuss the third method: the method of representation through maximal filters.

Representation through Maximal Filters

First let us say what filters are and how they are used here. We will talk about filters in detail when we discuss lattices. Let us define the notion of filter for the structures we are dealing with here.

Let $\langle P, <, \sqsubseteq \rangle$ a period structure.

A *filter* in P is a set $f \subseteq P$ such that:

1. if $p \in f$ and $p \sqsubseteq q$ then $q \in f$
2. if $p, q \in f$ then $p \sqcap q \in f$

We take clause (2) to mean that if $p, q \in f$ then $p \sqcap q$ exists and is in f.

I will give two ways here in which we can try to understand the idea

behind filter constructions: in terms of approximations, and in terms of sets of properties.

Let us take the following interpretation of what it means to be a period. Moments of time are idealized durationless points. We can only observe things that have duration, so periods are the closest we can come to those idealizations. On this view, periods are our approximations of such idealized points. Given this idea, the partial order of temporal inclusion gets the following informational interpretation: $p \sqsubseteq q$ means: every point that is approximated by q is also approximated by p, in other words, p is a sharper approximation than q.

On this view, atoms, if there are any, are the sharpest approximations of points: they cannot be refined any further. Since a period in general can still be sharpened in different ways, in general it approximates more than one point. Hence we can't identify points with those approximations.

The idea behind filter representations is that we can use certain *sets of approximations* as descriptions of points, and in the end as stand-ins for points. But not just any set of approximations will do. When can we say that a set of approximations describes a point? If we can describe a point by saying that it is approximated by period p, and p is a sharpening of q (q is a less sharp approximation of at least the same points as p) then we can describe that point by period q as well. If we can describe a point by saying that it is approximated by p and we can describe it by saying that it is approximated by q, then we can describe it by saying that it is approximated by $p \sqcap q$. Furthermore only periods p and q for which $p \sqcap q$ exists can be used in describing a point: in general, it doesn't make sense to try to use non-overlapping periods in describing one and the same point.

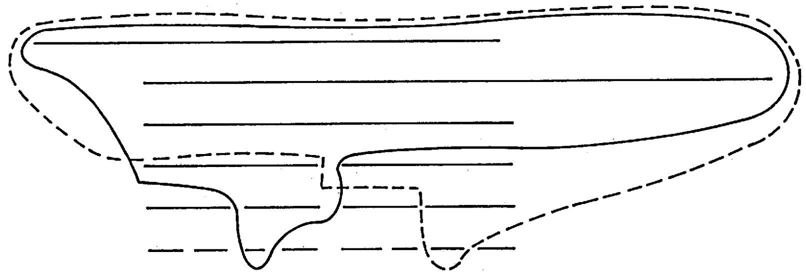
We see that the conditions we have imposed on filters are, under this interpretation, plausible conditions on descriptions of points.

Filters, however, are only partial descriptions of points. If we take the set of all non-empty filters on P, this set is partially ordered by $\subseteq$. Intuitively if $f \subseteq f'$ this means that f' is a more complete description of a point than f is, f' is a more detailed description than f is. The *maximal elements* in the partial order $\subseteq$ then give you the most complete descriptions of points you can get.

Conceptually the hardest thing about understanding such representation theorems is to switch interpretations. Points are intuitively small things, included in periods. Intuitively one's tendency would be to look for them *down* in the order $\sqsubseteq$. Filters go the other way: they are closed *upwards* under $\sqsubseteq$. Maximal filters are the biggest filters; how can they be points? What we do in filter representations is *reinterpret* the period structure order $\sqsubseteq$ and $\sqcap$ in a logical way: we think of the periods themselves as *propositions about points*. A filter then is a state of partial information about a point, closed under conjunction and logical consequence. The points are identified with the total information states about points.

Another way to think of it, is by taking periods to be *properties of points*.

We are used to switching between individual John and the set of all properties of John, j and $\lambda P.P(j)$. The latter is a maximally consistent filter of properties. If we take properties as primitives, we can define individuals directly as such maximally consistent filters. That is a good description of what we are doing here: we take periods as primitives and think of them as properties of points. Two periods are consistent (can be properties of the same point) if they overlap. Hence we can introduce points as maximally overlapping filters of periods. The following picture gives an example of a period structure with two maximal filters indicated:



Let $\langle P, <, \sqsubseteq\rangle$ be a period structure and let f be a filter in P.

f is *consistent* iff f is non-empty

Since f is closed under conjunction for all periods in it, it follows that a consistent filter consists of overlapping periods.

consistent filter f is *maximally consistent* iff $\forall f'$:
if $f \subseteq f'$ and f' is a consistent filter then $f = f'$

We know that maximally consistent filters abound if we assume the Maximal Chain Principle.

THEOREM (AC). *Every consistent filter can be extended to a maximally consistent filter.*

Proof. Let f be a consistent filter. Look at $\{f' : f \subseteq f'$ and f' is a consistent filter$\}$ ordered by subset. $\{f\}$ is a chain in this partial order, by the maximal chain principle it can be extended to a maximal chain m, m consists hence of consistent filters extending f and one another. Look at $\cup m$. We claim: $\cup m$ is a consistent filter.

(1) $\cup m$ is non-empty. This is obvious, because f is non-empty and $f \subseteq \cup m$.

(2) If $p \in \cup m$ and $p \sqsubseteq q$ then $q \in \cup m$. If $p \in \cup m$, then for some consistent filter f' in m: $p \in f'$, but then $q \in f'$, hence $q \in \cup m$.

(3) If $p, q \in \cup m$ then $p \sqcap q \in \cup m$. If $p, q \in \cup m$ then for some $f', f'' \in m$: $p \in f'$ and $q \in f''$. Since m is a chain, either $f' \subseteq f''$ or $f'' \subseteq f'$, say the first. Then $p, q \in f''$, hence $p \sqcap q \in f''$, hence $p \sqcap q \in \cup m$.

So indeed $\cup m$ is a consistent filter, hence $\cup m \in m$. Of course, $\cup m$ then is the maximal element of m, so $\cup m$ is indeed a maximally consistent filter.

Let's go on to the representation theorem.

A period structure can be *represented as* a point generated period structure if it is isomorphic to such a structure.

We will prove the following:

THEOREM. *Every witnessed period structure can be represented as a period structure, generated by the maximally consistent filters as points.*

Let $\langle P, <, \sqsubseteq\rangle$ be a witnessed period structure. As a first step, we will define a point structure. We define $\langle F, <\rangle$ as follows:

1. F is the set of all maximally consistent filters in P.
2. $f < f'$ iff $\exists p \in f\ \exists p' \in f' : p < p'$

THEOREM. *$\langle F, <\rangle$ is a strict partial order.*

Proof. (1) $<$ is transitive. Suppose $f < f'$ and $f' < f''$. Then, by definition of $<$ on F, for some $p \in f$, $p' \in f'$, $q' \in f'$ and $p'' \in f''$: $p < p'$ and $q' < p''$. Then, by monotonicity, $p < p' \sqcap q'$ and $p' \sqcap q' < p''$. Hence by transitivity of $<$ on P: $p < p''$. Hence, by definition of $<$ on F: $f < f''$.

(2) $<$ is irreflexive. Suppose $f < f$. Then for some $p, q \in f$: $p < q$. By definition of filter, $p \sqcap q$ exists, but this is incompatible with $p < q$. Hence $\neg f < f$.

So $\langle F, < \rangle$ is a point structure.

Let us now consider the following function * from P to pow F:

$$p^* = \{f \in F: p \in f\}$$

* Maps a period onto the set of all maximal filters that contain that period, in other words, it maps a period onto the set of all points that it approximates. Look at $\{p^*: p \in P\}$.

THEOREM. *$\{p^*: p \in P\}$ is a period set based on $\langle F, < \rangle$.*

Proof. (1) Every p^* is a non-empty subset of F. This holds for the following reason.

When we talked about partial orders we introduced the ideal generated by a: $(a] = \{b: b \leq a\}$. Here we introduce the dual notion: the *filter generated by* p, $[p) = \{q: p \sqsubseteq q\}$.

Exercise 3. Prove that $[p)$ is a consistent filter.

So we know that there is a consistent filter that contains p, namely $[p)$. Then we know, by the maximal filter theorem that we proved earlier, that there is a maximally consistent filter f_m containing p (because there is a maximally consistent filter f_m extending $[p)$). This filter $f_m \in p^*$.

(2) Every p^* is convex. Assume that $f_1 < f < f_2$ and $f_1, f_2 \in p^*$. Then for some $p_1 \in f_1$, some $p' \in f$, some $p_2 \in f_2$: $p_1 < p' < p_2$. Since f_1 and f_2 are filters, they are closed under conjunction, so both $p_1 \sqcap p$ and $p \sqcap p_2$ exist. With monotonicity, what follows is: $p_1 \sqcap p < p' < p \sqcap p_2$. Then convexity on the period structure tells us that $p' \sqsubseteq p$. $p' \in f$ and $p' \sqsubseteq p$, hence $p \in f$, hence $f \in p^*$.

(3) $\{p^*: p \in P\}$ is closed under intersection, i.e.: if p^* and q^* are in this set then so is $p^* \cap q^*$.

Any filter satisfies: $p \sqcap q \in f$ iff $p \in f$ and $q \in f$.

$$\begin{aligned}(p \sqcap q)^* &= \{f: p \sqcap q \in f\} = \{f: p \in f \text{ and } q \in f\} \\ &= \{f: p \in f\} \cap \{f: q \in f\} = p^* \cap q^*.\end{aligned}$$

This completes the proof that $\{p^*: p \in P\}$ is a period set based on $\langle F, < \rangle$.

Now look at the period structure based on $\{p^*: p \in P\}$: $\langle \{p^*: p \in P\}, <, \sqsubseteq \rangle$.

THEOREM. * *Is an isomorphism between* $\langle P, <, \sqsubseteq \rangle$ *and*

$$\langle \{p^*: p \in P\}, <, \sqsubseteq \}$$

Proof. (1) * is surjective. This is obvious.

(2) * is injective. Suppose $p \neq q$. Then either $p \not\sqsubseteq q$ or $q \not\sqsubseteq p$. Suppose $p \not\sqsubseteq q$. Then, by witness1, $\exists r \sqsubseteq p$: $\neg(r \bigcirc q)$. If $\neg(r \bigcirc q)$ then every maximally consistent filter containing r does not contain q (else the filter wouldn't be consistent). By the argument we have given earlier there is a maximally consistent filter containing r, and since $r \sqsubseteq p$, that filter contains p. So there is a maximally consistent filter containing p, but not q. Hence $p^* \not\subseteq q^*$ and hence $p^* \neq q^*$. The same applies if $q \not\sqsubseteq p$.

So we have proved that * is a bijection. Now we have to prove that it preserves and antipreserves the relations $\sqsubseteq$ and $<$.

(3a) Suppose $p \sqsubseteq q$, then, by definition of filter $p^* \subseteq q^*$.

(b) Suppose $p \not\sqsubseteq q$, then by the argument under (2) $p^* \not\subseteq q^*$.

(4a) Suppose $p < q$, then for every filter f containing p and every filter f' containing q, $f < f'$ (by definition of $<$ on the point structure), hence $p^* < q^*$ (by definition of $<$ on the period structure).

(b) Suppose $p \not< q$. Then by witness2 there are subperiods $p' \sqsubseteq p$ and $q' \sqsubseteq q$ such that no subperiod of p' precedes any subperiod of q'. Let f be a maximally consistent filter containing p' and f' be a maximally consistent filter containing q'. Then $f \not< f'$.

Namely, assume $f < f'$. Then for some $r \in f$ $r' \in f'$: $r < r'$. But since both p' and r are in f and both q' and r' are in f', both $p' \sqcap r$ and $q' \sqcap r'$ exist, and then by monotonicity $p' \sqcap r < q' \sqcap r'$. But $p' \sqcap r \sqsubseteq p'$ and $q' \sqcap r' \sqsubseteq q'$, so $p' \sqcap r$ and $q' \sqcap r'$ would be subperiods of p' and q', respectively, where the first precedes the second, contradicting witness2. So indeed $f \not< f'$.

Since $p \in f$ (because p' is and $p' \sqsubseteq p$) and $q \in f'$, $f \in p^*$ and $f' \in q^*$. But then, by definition of $<$ on the generated period structure: $p^* \not< q^*$.

This completes the proof.

The importance of this proof should be clear. For any witnessed period structure, we can define points as maximally consistent filters, get a point structure and find a structure of periods as sets of points that is isomorphic with the original one.

We cannot leave out the witness postulates.

If we check the proof, we see that we use witness1 to show that * is an injection: again, we use witness1 to show that if $p^* \subseteq q^*$ then $p \sqsubseteq q$, and we use witness2 to show that if $p^* < q^*$ then $p < q$.

It becomes interesting now to see how the conditions on the period structure and the point structure are related. Two examples:

A period structure is dense (in the period sense) iff the point structure generating it is dense (in the point sense).

A period structure is linear (again in the period sense) iff the point structure generating it is linear (in the point sense). For more discussion, see van Benthem (1983).

4.2. EVENT STRUCTURES

We have seen that for period structures, we had to make a choice, whether to take $\sqsubseteq$ or $\bigcirc$ (or both) as a primitive relation. But we also saw that it didn't make any difference which we chose, because we could give a plausible definition of the other. We now want to go one step further and construct periods out of events, and, as we will see, the choice of primitives seems to be less arbitrary there.

One motivation for constructing points out of periods and periods out of events lies in the conceptually highly abstract nature of the notion of a durationless moment of time.

Look at the notion of change. In a point structure we assume that our notion of change comes out of the fact that sentences (or propositions) happen to take different truth values at different moments of time. We could call this the film strip model of time: the present moment is the camera point, lighting up the present slide, the passing of time corresponds to lighting slide after slide; change simply means that at the next slide different things may be the case than at the previous one.

This is a static picture of change: change is reduced to difference at different moments of time. The intuition that the world is a dynamic system of changing processes will have to be reconstructed in this framework, and it is an open question whether such a reconstruction will or can do justice to our dynamic notion of change.

Another view on the relation between time and change would be to assume that the world, as we are connected to it, consists of changes: it is the dynamic notion of change that is primitive: it is changes that are the constituents of time.

Changes are events. We are able to locate a moment or period where the door was open before a moment or period where the door is closed, because we observed or are able to postulate an event of change leading from the state where the door was open to the state where the door was closed.

Think of our awareness of time. We are aware that time passes, even when our feeling is that nothing happens. But, of course, it is never true that nothing happens. In a totally stable environment, it is our heartbeat or breathing in terms of which in the end we will measure time. Can we say that in a really totally stable environment, where nothing whatsoever changes (think of a point structure where at every moment the same (non-temporal) propositions are true), can we really say that in such a temporal structure time passes?

Events are entities that inherently have duration. In this sense we cannot do without a temporal ordering even if we can do without points, because duration is a temporal concept. But we might be able to take events and their temporal relations as primitive and construct the temporal entities, periods and moments, as abstractions out of them.

We come to periods by looking at events that are cotemporal. We can order a period before another by pointing at an event that went on at the one but no longer at the other.

This approach – of taking events as basic and constructing periods and point out of them – goes back to Russell, Wiener and Whitehead and, in modern tense logic, has been revived by Hans Kamp and Johan van Benthem (in the references given above).

Here is where the choice of primitives becomes important. For period structures it seems very natural to define the notion of overlap in terms of inclusion: two periods overlap, if they have a common part. For events this seems much more problematic, though: here it would mean that we can only say that two events temporally overlap if there is a third event that goes on exactly at the period of overlap.

Of course, we can just postulate that, for any two overlapping events, there is an abstract third event: 'the state of overlap of those two events', but that makes our notion of event from the start more abstract than we may want it to be. This discussion is reflected in the different event structures that we find in the literature.

Kamp's event structures have precedence and overlap as primitive relations (temporal inclusion is defined). Since overlap is primitive, there is no requirement that there be an abstract event that is the overlap of two events.

Van Benthem's event structures have only precedence (and define the others in terms of that). So van Benthem's structures do have such abstract events (in fact, finding an event that is temporally included in both e and e' is here our way of expressing that e and e' overlap).

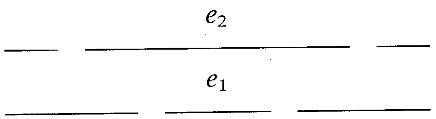
In fact, both choices have their problems.

Van Benthem makes the event structure richer, to get the period structures that he wants. Kamp has poorer event structures, but consequently the period structures that he gets are poorer than those that van Benthem gets. Here is the dilemma: on the one hand we do not want to postulate an event that is a temporal part of e and e' to characterize overlap; on the other hand, for periods that seems to be completely plausible: if two periods overlap, then obviously there is a period that is their overlap. Kamp gets the first, van Benthem gets the second and it is not clear how you can get both in a simple way.

Let me introduce van Benthem's event structures first.

An *event structure* is a structure $\langle E, <\rangle$, where $<$, the relation of temporal precedence, in terms of which $\sqsubseteq$ and $\bigcirc$ will be defined in a moment, is a strict partial order satisfying Conjunction (for events).

When is an event e_1 temporally included in an event e_2? If every event that precedes e_2 also precedes e_1 and every event that is preceded by e_2 is also preceded by e_1:

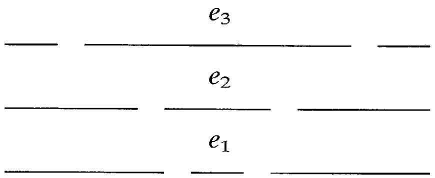


In a definition:

$$e_1 \sqsubseteq e_2 \text{ iff } \forall e[e < e_2 \rightarrow e < e_1] \quad \text{and} \\ \forall e[e_2 < e \rightarrow e_1 < e]$$

THEOREM. *In $\langle E, <, \sqsubseteq\rangle$, $\sqsubseteq$ is a pre-order and $<$ and $\sqsubseteq$ satisfy monotonicity, convexity and conjunction.*

Proof. That $\sqsubseteq$ is reflexive is obvious. Transitivity is easy to see in the following picture:

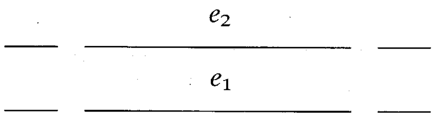


Monotonicity follows directly from the definition of $\sqsubseteq$.

Convexity: suppose $e_1 < e_2 < e_3$ and $e_1 \sqsubseteq e$ and $e_3 \sqsubseteq e$.

Take any $e' < e$, then $e' < e_1$, hence $e' < e_2$. Similarly any $e' > e$ will be after e_3 and hence after e_2. So $e_2 \sqsubseteq e$.

$\sqsubseteq$ is of course not antisymmetric:

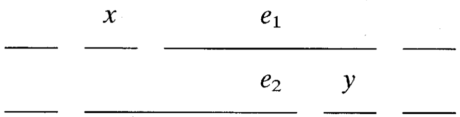


$e_1 \sqsubseteq e_2$ and $e_2 \sqsubseteq e_1$, but not $e_1 = e_2$. That is precisely what we want: cotemporal events need not be identical.

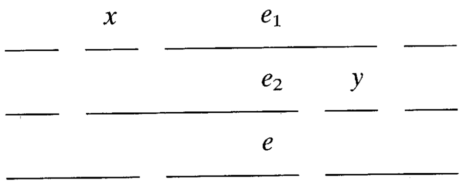
It is important to realize that all the relations we have discussed are temporal relations on events: $e_1 \sqsubseteq e_2$ means that e_1 is temporally included in e_2. We may want to put more structure on events, for instance a relation of part-of: we have the feeling that the Invasion of Normandy is not just temporally included in The Second World War, but is a constituting part of it. We cannot use $\sqsubseteq$ for that, so to express such things we have to introduce a new partial order $\preccurlyeq$, part of. This cannot be defined in terms of the temporal ones (at least not extensionally). An obvious axiom that we would have to impose is: if $e \preccurlyeq e'$ then $e \sqsubseteq e'$. I will come back to the relation between events and their parts later.

Now that we have $<$ and $\sqsubseteq$, we can define $\bigcirc$ on events in the usual way: $e_1 \bigcirc e_2$ iff $\exists e$: $e \sqsubseteq e_1 \wedge e \sqsubseteq e_2$.

To get a feeling of what these definitions give you, look at the following pictures:



In this case $x < e_1$ and $x \bigcirc e_2$ and $e_2 < y$ and $y \bigcirc e_1$. But e_1 does not overlap e_2, because there is no event temporally included in both. (So, by the way, this event structure is not linear.) For e_1 and e_2 to overlap we have to add a new event, as in:



Now e_1 and e_2 do overlap (because e is temporally included in both), and in fact, this structure is a linear structure.

On a period structure, temporal inclusion is a partial order. Of course, we already know how to construct a partial order out of a preorder:

The *period structure* generated by event structure $\langle E, < \rangle$ is: $\langle [E]_{\approx}, <, \sqsubseteq \rangle$ where:

1. $\approx \; = \lambda e \lambda e'.e \sqsubseteq e' \wedge e' \sqsubseteq e$
2. $p < p'$ iff $\exists e \in p \; \exists e' \in p' : e < e'$
3. $p \sqsubseteq p'$ iff $\exists e \in p \; \exists e' \in p' : e \sqsubseteq e'$
4. if $p \bigcirc p'$ then
   $p \sqcap p' = \{e : \exists e_1 \in p \; \exists e_2 \in p' : e \approx e_1 \sqcap e_2\}$

THEOREM. *The generated period structure is a period structure.*

Proof. We have proved before (in Chapter Two) that $\sqsubseteq$ is a partial order. It is not hard to see that $<$ is a strict partial order.

Irreflexivity: suppose $p < p'$. Then some for $e, e' \in p: e < e'$, but this is, of course, impossible, because (by definition of p) $e' \sqsubseteq e$, which would mean that $e < e$.

Transitivity: let $p < q < r$. This means that for some $e_1 \in p$, $e_2 \in q$, $e_3 \in q$, $e_4 \in r$: $e_1 < e_2$ and $e_3 < e_4$. Because $e_2 \sqsubseteq e_3$ and $e_3 \sqsubseteq e_2$, it follows that $e_1 < e_4$, and hence $p < r$.

Convexity. Assume: $p < q < r$. Then for some for $e \in p$, $e' \in q$, $e'' \in r$: $e < e' < e''$.

Assume further: $p \sqsubseteq s$ and $r \sqsubseteq s$. Then for some $e_1 \in p$, $e_2 \in r$, $e_3, e_4 \in s$: $e_1 \sqsubseteq e_3$ and $e_2 \sqsubseteq e_4$. By definition of periods it follows that:

$e \sqsubseteq e_3$ and $e'' \sqsubseteq e_4$. Once more by definition of periods: $e_3 \sqsubseteq e_4$ and $e_4 \sqsubseteq e_3$. hence $e'' \sqsubseteq e_3$.

Now we have proved that both $e \sqsubseteq e_3$ and $e'' \sqsubseteq e_3$, hence by convexity on E: $e' \sqsubseteq e_3$, and thus $q \sqsubseteq s$.

Monotonicity: assume	$p < q$	and $r \sqsubseteq p$. We show $r < q$:
	$p < q$	
hence:	$e_1 < e_2$	(for some $e_1 \in p$, $e_2 \in q$)
	$r \sqsubseteq p$	
hence:	$e_3 \sqsubseteq e_4$	(for some $e_3 \in r$, $e_4 \in p$)
hence:	$e_3 \sqsubseteq e_1$	(by definition of periods)
and:	$e_3 < e_2$	
so:	$r < q$	

The other side goes in the same way.

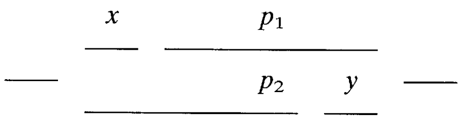
Conjunction. If we realize that if $e \approx e'$ and $e \bigcirc e''$ then $e \sqcap e'' \approx e' \sqcap e''$, then it should be clear that we can define for $p \bigcirc p'$, $p \sqcap p'$ in the above way and this will indeed give us conjunction on the periods.

Now that we have a period structure, we have two alternatives for defining overlap: we can lift it from the events and we can define it in terms of inclusion:

$$p \bigcirc_1 q \text{ iff } \exists e \in p \; \exists e' \in q : e \bigcirc e'$$
$$p \bigcirc_2 q \text{ iff } \exists r : r \sqsubseteq p \wedge r \sqsubseteq q$$

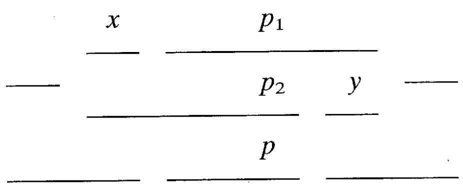
Exercise 2. Prove that the two notions coincide: $p \bigcirc_1 q$ iff $p \bigcirc_2 q$.

The period structures of the two event structures discussed above, are not that different from the event structures themselves:



and:

and:



Of these two structures, the first is neither witnessed1 nor witnessed2, the second is both witnessed1 and witnessed2. The first is not witnessed1, because every part of p_2 overlaps x (namely p_2 and x), but p_2 is not included in x.

It is not witnessed2 either: $p_2 \not< p_1$; p_1's parts are p_1 and y, p_2's parts are p_2 and x. Since $x < y$ and $x \sqsubseteq x$ and $y \sqsubseteq y$, both parts of p_2 have a part that is before some part of p_1.

None of this holds in the second structure: here it is no longer true that every part of p_2 overlaps x, so this counterexample to witness1 is removed; further p is a part of p_1 and of p_2 such that all parts of p do not precede any parts p, so this counterexample to witness2 is removed as well. So the second structure is a witnessed, linear period structure.

Exercise 5. (a) Number all the periods in the second structure and construct all the points. Now number all the points from left to right and reconstruct the period structure through sets of points.

(b) Number all the periods in the first structure and construct all the points. Argue that you cannot reconstruct the period structure through sets of these points.

Let us now look at Kamp's event structures.

An *event structure* is a triple $\langle E, <, \bigcirc\rangle$ where $<$ is a strict partial order, $\bigcirc$ is a reflexive symmetric relation, such that $<$ and $\bigcirc$ satisfy:

1. Separation: $e < e' \rightarrow \neg(e \bigcirc e')$
2. Transfer: $e < e' \wedge e' \bigcirc e'' \wedge e'' < e''' \rightarrow e < e'''$

$\sqsubseteq$ is defined as follows:

$$\begin{aligned} e_1 \sqsubseteq e_2 \text{ iff } & \forall e[e < e_2 \rightarrow e < e_1] \quad \text{and} \\ & \forall e[e_2 < e \rightarrow e_1 < e] \quad \text{and} \\ & \forall e[e \bigcirc e_1 \rightarrow e \bigcirc e_2] \end{aligned}$$

The *generated period* structure is the structure:

$\langle [E]_{\approx}, <, \bigcirc, \sqsubseteq\rangle$ where:

1. $\approx\; = \lambda e \lambda e'. e \sqsubseteq e' \wedge e' \sqsubseteq e$
2. $<, \bigcirc, \sqsubseteq$ are defined as before.

The generated period structure is a period structure, except for the fact that it doesn't satisfy conjunction: $<$ is a strict partial order, $\sqsubseteq$ is a partial order, and the structure satisfies monotonicity and convexity.

Let us define:

> a *quasi-filter* in P (the period structure) is a set $F \subseteq P$ such that: $\forall p, p' \in F: p \bigcirc p'$.

So a quasi-filter is a set of pairwise overlapping periods. A consistent quasi-filter is a non-empty quasi-filter, and a maximally consistent quasi-filter is a quasi-filter that cannot be extended any further to a quasi-filter.

The point structure generated by the period structure is the set of maximal quasi-filters ordered by $<$ in the same way as filters were ordered before. The point structure will indeed be a partial order (see Kamp, 1979a for a proof). We leave out the conjunction clause in the notion of period set. We can prove:

THEOREM. *Every linear period structure can be represented as a period structure, generated by the maximally consistent quasi-filters as points.*

I won't give the proof here in full detail. You can go through the stages of the proof for the van Benthem period structures and prove analogous things for the Kamp period structures. Let me just point out how linearity makes up for the witness postulates in the proof.

In the proof for van Benthem's period structures, we used witness1 to prove that * is an injection, and that * antipreserves inclusion; and we used witness2 to prove that * antipreserves precedence.

(1) Suppose $p \not\sqsubseteq q$. Then either for some r: $r < q$ and $r \not< p$, or for some r: $q < r$ and $p \not< r$, or for some r: $r \bigcirc p$ and $\neg(r \bigcirc q)$. The last case is easy: if $r \bigcirc p$ and $\neg(r \bigcirc q)$, then certainly there will be a maximally consistent quasi-filter containing r and p, this filter will not contain q, hence the set of all quasi-filters containing p won't be a subset of the set of all quasi-filters containing q.

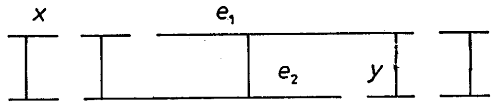
So suppose $q < r$ and $p \not< r$. Then, by linearity, $r < p$ or $r \bigcirc p$. In the latter case, again $r \bigcirc p$ and $\neg(r \bigcirc q)$, so let's assume $r < p$. Then, of course, $q < p$, and no quasi-filter will contain both. In all cases, indeed $p^* \not\subseteq q^*$.

This shows that here too * antipreserves $\not\subseteq$. We can use the same argument to show that * is an injection.

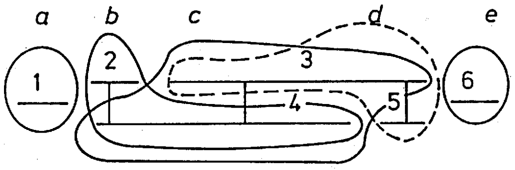
(2) Assume $p \not< q$. Then $q < p$ or $p \bigcirc q$. If we assume that $q < p$ then it is clear that $p^* \not< q^*$: take $f \in p^*$ and $f' \in q^*$: every period in f overlaps p, every period in f' overlaps q. If $q < p$, you certainly can't find a period r overlapping p and a period r' overlapping q, such that $r < r'$. Hence, no such filter containing p is before any such filter containing q, so indeed $p^* \not< q^*$.

If we assume $p \bigcirc q$, then there will be a filter containing both p and q, that filter is in p^* and in q^*, so indeed $p^* \not< q^*$.

Let us look at an example. We can use the same diagrams for events as before, except that we now have to mark overlap explicitly.



This time, e_1 and e_2 do overlap, so this is a linear Kamp structure. We get the period structure in the same way as before:



The points are:

$$a = \{1\};\ b = \{2, 4\};\ c = \{3, 4\};\ d = \{3, 5\};\ e = \{6\}$$

This structure is indeed a linear point structure in the given order, because $1 < 2$, $2 < 3$, $4 < 5$, $5 < 6$, hence:

$a \quad b \quad c \quad d \quad e$

The following period structure, with periods as sets of points is isomorphic with the one above:

$\{a\}$ $\{b\}$ $\{c, d\}$ $\{e\}$

$\{b, c\}$ $\{d\}$

You can check that the intervals precede and overlap in the right way.

Note that the representation theorem requires the period structure, and hence the underlying event structure, to be linearly ordered. We cannot rely on witness here, because the structures need not be witnessed, for instance, the above structure is not witnessed (at least not in van Benthem's sense). Linearity is a very strong condition.

Certainly if we want to construct an order out of the events that pass our eye, it won't be a linear order, because there are all sorts of events that we don't know how exactly they are temporally related.

This by itself need not lead us to nonlinear structures. Assuming that the events that pass our eye are part of the events that make up our world, we could argue that only the latter form a fully fledged linear event structure: we should not try to think that our partial information is total when it isn't: the proper way to look at this is to combine the analysis of partial information with the analysis of events: assume that the events that we have observed form a *partial event structure* that can still grow to different total event structures. A large part of the papers Kamp (1979a) and (1979b) is devoted to the issues of partiality. Kamp gives different arguments, as well, that linearity may be too strong. We can add to that that we might be interested in an event-based analysis of branching time, which won't be linear either.

Another disadvantage of Kamp's structures, that I mentioned before, is that the period structure is not very rich. Despite the problems that we have when we add conjunction at the event level, as van Benthem does, this does give us nice, rich period structures, and it seems that if we are interested in using these structures in an interval semantics, we would really like to have the period at which two events overlap in our model. Can we get the best of both worlds?

Here is a suggestion. I said earlier that the temporal relations that we have put on events should be distinguished from intensional relations like *part of*, in particular, we have to distinguish *part of*, for which I will use $\preccurlyeq$, from temporal inclusion $\sqsubseteq$. An event can be temporally included in another event without having anything more than an accidental relation to it. Not so for the parts of an event.

It is in terms of this relation that we will have to deal with semantic relations like the relation between John's running and John's moving. It is with respect to this relation that we may have to consider to allow semantically complex events in our event structure, like maybe, Johns singing and dancing.

Look at two overlapping events: solution A turning orange and solution B turning blue. Those events, of course, need not have any part in common. Further, we don't want to say that there is always any old unrelated third event going on exactly in the period of their overlap. Moreover, we are not going to recognize 'the state of e and e' overlapping' as a serious event.

Yet it does not seem to be far fetched to assume that if e and e' overlap, then there is a *part* of e and a *part* of e' that are completely cotemporal. It is, I think, not even far fetched to assume that there is a part of e and a part of e' that are maximally so. Once we realize that we have to put the parts of events in our model anyway, the idea that, for two overlapping events, there will be events that go on exactly at the period of overlap is, I think, no longer objectionable.

Let us put this in a definition:

An *event structure* is a tuple $\langle E, \preccurlyeq, <, \sqsubseteq, \bigcirc \rangle$ (of which only $\preccurlyeq$ and $<$ are primitives) where:

1. E is a non-empty set of events.
2. $\preccurlyeq$ is a partial order, the relation of part-of.
3. $<$ is a strict partial order.
4. $\sqsubseteq := \lambda e_1 \lambda e_2.\ \forall e[e_2 < e \rightarrow e_1 < e] \wedge \forall e[e < e_2 \rightarrow e < e_1]$
5. $e \preccurlyeq e' \rightarrow e \sqsubseteq e'$
6. $\bigcirc := \lambda e \lambda e'.\ \exists e_1[e_1 \sqsubseteq e \wedge e_1 \sqsubseteq e']$
7. Conjunction: if $e \bigcirc e'$ then
   $\exists e_1 \preccurlyeq e\ \exists e_2 \preccurlyeq e' \colon e_1 \sqsubseteq e_2 \wedge e_2 \sqsubseteq e_1 \wedge$
   $\forall e_3 \colon e_3 \sqsubseteq e \wedge e_3 \sqsubseteq e' \rightarrow e_3 \sqsubseteq e_1 \wedge e_3 \sqsubseteq e_2$

Note that we do not have an operation of $\sqcap$ on the event structure. In fact, in this perspective, there isn't a unique event that is the maximal temporal part of e and e' (and there shouldn't be!), because both e_1 and e_2 have that property. We now define the generated period structure as usual:

$$\langle [e]_{\approx}, <, \sqsubseteq, \bigcirc \rangle \text{ under } \approx\ = \lambda e \lambda e'.e \sqsubseteq e' \wedge e' \sqsubseteq e$$

It is easy to see that this is indeed a period structure satisfying monotonicity, convexity, *and conjunction*. Although there isn't a unique event, temporally including every event that is temporally included in both e and e', there is a unique period, $[e]_{\approx} \sqcap [e']_{\approx}$, namely $[e_1]_{\approx}$ (= $[e_2]_{\approx}$) that is temporally included in $[e]_{\approx}$ and in $[e']_{\approx}$ that temporally includes every period that is included in both e and e'.

From here, then, we can use van Benthem's construction.

INTERVALS, EVENTS AND CHANGE

5.1. INTERVAL SEMANTICS

Whereas the logic and model theory of instant tense logic is relatively well developed, well understood and elegant, such cannot be said of period tense logic and event logic.

There seems to be little agreement on what notions to take as basis for the semantics and what form the semantic clauses should take; the intuitions to base the whole on are more subtle and shaky; and deep logical results that might justify one approach over others are largely absent. In sum, interval semantics is a mine field, a field that is by far more complex than instant tense logic, definitely more fuzzy as well, but also, semantically by far more fascinating.

In order to indicate some reasons for why the subject is so much harder, I will briefly go over some of the different directions one could take.

In the first place, as I have mentioned before, one may not believe that a primitive notion 'truth relative to an interval' makes sense at all, whereas 'truth relative to an instant' is a clean and simple notion. This leads to what I have called reductionistic tense logic: 'truth relative to an interval' is a derived notion which can be reduced to 'truth relative to certain instants'.

I have given some reason earlier for scepticism about the viability of this approach, and I won't dwell on it here. Yet it should be mentioned that when we accept a notion of 'truth relative to an interval' as basic in the semantic recursion, this notion is inherently less clear than a notion of 'truth relative to an instant'. Though instants may be abstract entities, once we accept them the number of choices for truth conditions relative to them is relatively small (that is, not bigger than for truth conditions relative to possible worlds, or for that matter, relative to models *per se*).

As Vlach (1981) points out, with intervals there is from the start a fundamental unclarity concerning how to interpret 'truth relative to an interval'. Suppose *Brutus stabbed Caesar* is true relative to an interval.