$a \not\leqslant b$. Then $a \notin (b]$. Since $a \in (a]$ it follows that $(a] \neq (b]$. So indeed f is an injection.

2. If $a \leqslant b$ then $(a] \subseteq (b]$. This is obvious. If $(a] \subseteq (b]$ then $\{c\colon c \leqslant a\} \subseteq \{c\colon c \leqslant b\}$, hence $a \in (b]$, thus $a \leqslant b$.

So f is a homomorphism, in fact, an isomorphism between $\langle A, \leqslant\rangle$ and $\langle f(A), \subseteq\rangle$. Hence, indeed f is an embedding of $\langle a, \leqslant\rangle$ in $\langle \text{pow}\, A, \subseteq\rangle$.

Exercise 22. Draw the graph of the set theoretic poset with eight elements. Show that every poset with three elements can be embedded into it.

So set theoretic posets are the most general posets. There are no posets that have such a complicated structure that you can't find a set theoretic poset in which you cannot carve it out. Another way of stating the result is: you can get any poset, by starting with a set theoretic poset and leaving out lines and/or element with their connecting lines.

SEMANTICS WITH PARTIAL ORDERS

3.1. INSTANT TENSE LOGIC

I will start with Priorian tense logic. (For more thorough discussion: see Prior, 1967; van Benthem, 1983; and Burgess, 1984.) Priorian tense logic is based on three assumptions. The first is that tenses (like the past tense) are sentential operators. The second is that tenses are (implicitly) quantifiers over times. The third is that times are instants.

All these assumptions have been challenged. For instance, Bach (1979) treats tenses as VP operators, Enç (1981) as V operators. Partee (1973) and Partee (1984) discuss the deictic and anaphoric rather than quantificational uses of the past tense. Many analyses of tense adopt some version of Reichenbach's (1947) notion of reference time, which is lacking in Priorian tense logic. Finally, there is interval semantics (and event semantics), which we will discuss in a later chapter.

A discussion of these various alternatives falls outside the scope of this book.

Let me say here that, if I think that it is still worthwhile talking about Priorian tense logic, it is not because I think that the criticism that led to the alternatives is not well founded. On the contrary, I think that in particular Prior's identification of the past tense with an existential quantifier over times is misguided, and may even have had some negative influence on the development of semantic thinking about tense. I will be interested in something else here, though.

It will be part of any semantics for temporal expressions to determine what temporal property a given temporal expression expresses. I.e., whatever your framework, if you give a semantics for, say, *until*, you will have to specify what exactly the temporal relation is that is expressed by *until*.

Priorian tense logic, as a simple example of a temporal language, is very well suited to a study of the questions of how to express such temporal properties in a temporal language and what temporal properties can be expressed. So we will use Priorian tense logic as a simple example to get a grip on these problems.

We are concerned with a language L of propositional logic and four sentential tense operators P, F, G, H.

> A *frame* for L is a structure $F = \langle T, < \rangle$, with $<$ a binary relation on T. Though at first we will put no conditions on $<$, I will still write $<$.
>
> A *model* for L is a pair $M = \langle F, i \rangle$ with F a frame for L and i an interpretation function. $i{:}ATFORM \times T \to \{0, 1\}$.

So i assigns to a propositional formula and a moment of time a truth value. I will write $i_t(p)$ for $i(p, t)$.

Let M be a model $\langle T, <, i \rangle$, $t \in T$, we define $[\![\varphi]\!]_{M,t}$, the truth value of φ at t in M. (I will skip index M)

$$\begin{array}{ll}
[\![p]\!]_t = i_t(p) & \text{for atomic formulas } p \\
[\![\neg\varphi]\!]_t = 1 \text{ iff } [\![\varphi]\!]_t = 0; & 0 \text{ otherwise} \\
[\![\varphi \wedge \psi]\!]_t = 1 \text{ iff } [\![\varphi]\!]_t =]\psi]\!]_t = 1; & 0 \text{ otherwise} \\
[\![P\varphi]\!]_t = 1 \text{ iff } \exists t' < t{:}\ [\![\varphi]\!]_{t'} = 1; & 0 \text{ otherwise} \\
[\![H\varphi]\!]_t = 1 \text{ iff } \forall t' < t{:}\ [\![\varphi]\!]_{t'} = 1; & 0 \text{ otherwise} \\
[\![F\varphi]\!]_t = 1 \text{ iff } \exists t' > t{:}\ [\![\varphi]\!]_{t'} = 1; & 0 \text{ otherwise} \\
[\![G\varphi]\!]_t = 1 \text{ iff } \forall t' > t{:}\ [\![\varphi]\!]_{t'} = 1; & 0 \text{ otherwise}
\end{array}$$

With this we define:

> $M \models \varphi$, φ is true in model M iff $\forall t \in T$: φ is true at t in M
> $F \models \varphi$, φ is true on frame F iff for every i: φ is true in $\langle F, i \rangle$.

Let **F** be a class of frames.

> $\Delta \models_F \varphi$, Δ/φ is valid in **F** iff for every frame F in **F** and every i, and every $t \in T$: if Δ is true in $\langle F, i \rangle$ at t, then φ is true in $\langle F, i \rangle$ at t.

So Δ/φ is valid iff Δ/φ is valid in the class of all frames.

Let's first talk a little bit about tense logical definability and tense logical theories. A property is tense logically definable if there is a tense logical formula that defines it (that is true on all frames that have that property and false on all frames that don't).

The basic completeness proof for tense logic tells us that an inference is valid (in the class of all frames) iff it is derivable from the following axiom system, called minimal tense logic.

Minimal tense logic has the following axioms and rules:

A. the axioms and rules of propositional logic.

B. rules: if $\vdash \varphi$ then $\vdash H\varphi$ and $\vdash G\varphi$

1. $G(\varphi \to \psi) \to (G\varphi \to G\psi)$

(If it will always be the case that if φ then ψ and φ will always be the case, then ψ will always be the case.)

2. $H(\varphi \to \psi) \to (H\varphi \to H\psi)$
3. $PG\varphi \to \varphi$

(If 'φ will always be the case' is true at some moment in the past, φ is true now.)

4. $FH\varphi \to \varphi$

In minimal tense logic the obvious equivalences $P\varphi \leftrightarrow \neg H \neg \varphi$ and $F\varphi \leftrightarrow \neg G \neg \varphi$ are provable.

We can also say that (1)–(4) define the class of all tense logical structures.

What logicians are interested in here is what properties of the accessibility relation tense logical formulas express (and what properties can or cannot be expressed with tense logical formulas). The first results are that certain very simple properties of the accessibility relation are not tense logical definable:

> *Irreflexivity and asymmetry of $<$ are not tense logical definable.*

So there is no tense logical formula that expresses that the temporal order is asymmetric. This result would be an extreme burden for tense logic if we did not have the following result: The class of *all* frames and the class of *all asymmetric* frames are tense logically indistinguishable:

> *Δ/φ is valid in the class of all frames iff Δ/φ is valid in the class of all asymmetric frames.*

This means that minimal tense logic is complete for the class of all asymmetric frames as well.

Whenever I say from now on that a formula defines a certain property, I mean that it defines that property relative to the class of all asymmetric frames (in other words, that it holds on all asymmetric frames that have that property, and that it doesn't hold on all asymmetric frames that don't have that property). A totally equivalent way of going would be to redefine a tense logical frame as a structure $\langle T, < \rangle$, where $<$ is an asymmetric relation on T.

5. $G\varphi \rightarrow GG\varphi$

(If φ will always be the case then it will always be the case that 'φ will always be the case' is true.)

Statement (5) expresses that the temporal order is transitive. In fact, (5) defines transitivity of the temporal order relative to the class of all frames. Given the assumption about asymmetry made above, (1)–(5) gives us the theory of strict partial orders. Statement (5) is equivalent to $PP\varphi \rightarrow P\varphi$.

Exercise 1. Check that $PP\varphi \rightarrow P\varphi$ (where φ is an atomic formula) has to be true on all transitive frames and that for a non-transitive frame, you can always find an interpretation function such that the formula is not true in the resulting model (i.e. that there is some moment in that model at which it is not true).

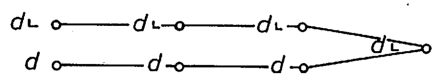
Expressions (6) and (7) express the fact that the order is not branching to the future, respectively the past:

6. $F\varphi \rightarrow G(\varphi \vee P\varphi \vee F\varphi)$

(If φ will at some time be the case then for every moment in the future either φ is true there or 'φ was once the case' is true there or 'ϕ will be the case sometime' is true there.)

7. $P\varphi \rightarrow H(\varphi \vee P\varphi \vee F\varphi)$

The antecedent of (6) is only true in a moment of time if there is a later moment. If time branches to the future from t, you can always give a counterexample for the consequent:



Adding (6) and (7) to (1)–(5), then gives us the theory of linear orders. (The theory of non-branching partial orders and the theory of linear orders are tense logically indistinguishable.)

Statements (8) and (9) express that the order is continuing to the future, resp. the past:

8. $G\varphi \rightarrow F\varphi$

(If φ will always be the case it will at some time be the case.)

9. $H\varphi \rightarrow P\varphi$

Axiom (10) expresses density:

10. $F\varphi \rightarrow FF\varphi$

(If φ will at some time be the case then it will at sometime be the case that 'φ will be the case sometime' is true.)

Exercise 2. Prove this, i.e. show that (10) holds at every dense frame and show that you can always turn a non-dense frame into a model where at some time (10) is false.

On linear orders (11) and (12) express discreetness (i.e. (11) and (12) define discreetness relative to the class of all linear orders):

11. $P(\varphi \vee \neg\varphi) \rightarrow (HF\varphi \rightarrow (\varphi \vee F\varphi))$

(If there is a past moment and it was always the case that 'φ will at some time be the case' was true then either φ is the case now or at sometime in the future.)

12. $F(\varphi \vee \neg\varphi) \rightarrow (GP\varphi \rightarrow (\varphi \vee P\varphi))$

That we can always give a counterexample to these sentences on a non-discreet structure is not hard to see. Let us consider (11). Take a frame and a point t, which has predecessors, but no immediate predecessor. $P(\varphi \vee \neg\varphi)$ is true at t. Make φ true at all t's predecessors, but false at t and from t onwards. $\langle\{t' \in T: t' < t\}, \{t' \in T: t \leqslant t'\}\rangle$ forms a transition with t the approximated element. In other words, there is an infinite chain of elements earlier than t, approximating t. Since at all those elements φ is true, $HF\varphi$ is true at t. But, $\varphi \vee F\varphi$ is false at t. The other way around, assume that T is discreet. Either a point t doesn't have a predecessor. Then $P(\varphi \vee \neg\varphi)$ is false there, and (11) is true there. Or t does have a predecessor, and hence a direct predecessor $t-1$. $P(\varphi \vee \neg\varphi)$ is true at t. Assume that also $HF\varphi$ is true at t. This means that F is true at $t-1$. Clearly, this can only be the case if either φ is true at t, or at some successor of t, so $\varphi \vee F\varphi$ is true at t.

On linear orders (13) expresses wellfoundedness:

13. $H(H\varphi \rightarrow \varphi) \rightarrow H\phi$

(I won't try to gloss this.)

Again, it is not hard to give a counterexample on a non-well founded frame. Take **Z**, and assume that on the negative numbers φ is alternatively true and false. Since, at no moment in the past of 0, $H\varphi$ is true, at every such moment, $H\varphi \rightarrow \varphi$ is true, hence $H(H\varphi \rightarrow \varphi)$ is true at 0. But $H\varphi$ is false at 0. The other side is interesting. Assume that T is well founded. Suppose t is the minimal element. Then $H\varphi$ is true, so (13) is true at t. Suppose t does have predecessors and suppose that $H(H\varphi \rightarrow \varphi)$ is true at t. Here are the possible cases. Either there is some predecessor of t where $H\varphi$ is true but φ is false, then $H\varphi \rightarrow \varphi$ is false there and $H(H\varphi \rightarrow \varphi)$ is false at t, so (13) is true at t. So let us assume that $H\varphi \rightarrow \varphi$ is true for all predecessors of t. This can mean two things again. The one case is where some predecessor t' of t has a predecessor t'' where φ is false, hence $H\varphi$ is false in t'. This can only mean that at t'' $H\varphi$ is false as well (because by assumption $H\varphi \rightarrow \varphi$ is true there). This pushes us back to a predecessor of t'' where φ is false, and again $H\varphi$ has to be false there. Since T is well founded, this process necessarily stops at the minimal element. So, in order for $H\varphi$ to be false in t', $H\varphi$ will have to be false at the minimal element. However, there $H\varphi$ is true, in other words, this case can never occur. The only possibility left, then, is that for every predecessor of t, both $H\varphi$ and φ are true. Then $H(H\varphi \rightarrow \varphi)$ is true at t, but, of course, $H\varphi$ is also true at t, so (13) is true. So, indeed, (13) is valid.

Axioms (14) and (15) express continuity:

14. $(F\varphi \wedge FG\neg\varphi) \rightarrow F(HF\varphi \wedge G\neg\varphi)$

15. $(P\varphi \wedge PH\neg\varphi) \rightarrow P(GP\varphi \wedge H\neg\varphi)$

Axiom (14) expresses roughly that if at some stage in the future φ changes from true to false, then there is a point t such that φ is true up to t and false from t onwards. This point, is (of course) the transition point.

We see that tense logical definability does not coincide with first order definability. Certain first order properties of the accessibility relation (like irreflexivity) are not tense logically definable; certain

second order properties are tense logically definable. The relation between tense logic (or modal logic) and first order and second order logic, including the question of to what extent there are procedures to determine which properties tense logical formulas express is studied in correspondence theory (see van Benthem 1983, van Benthem 1984).

3.2. ALGEBRAIC SEMANTICS, FUNCTIONAL COMPLETENESS AND EXPRESSIBILITY

In Priorian tense logic, tenses are sentential operators. Prior gives four basic tenses, P, F, H and G (of which two can be defined in terms of others). The question we will talk about now is what other tense operators can be defined in this theory. For that we need to define what a tense operator is. This is easier to do if we give an *algebraic* reformulation of the tense logical semantics we have given before. To give the proper setting for that I will first do the same for propositional logic and first order predicate logic.

In propositional logic, a model is an interpretation function that assigns to every atomic sentence (propositional constant) a truth value 0 or 1. The truth definition then specifies the compositional semantics of the connectives. The connectives are taken to be syncategorematic, they are not assigned an interpretation directly, but only indirectly through the truth definition. If we give the semantics of propositional logic an algebraic form, we can directly interpret the connectives.

The algebraic form is based on the idea that the connectives are operators mapping n-tuples of formulas onto formulas, and hence should be interpreted compositionally as operations mapping the interpretations of their arguments on the interpretation of the value.

Given this, we can take a model for propositional logic to be a tuple $\langle\{0,1\}, \neg, \wedge, i\rangle$ where i is as usual an interpretation for the atomic formulas and $\neg$ and $\wedge$, the interpretations of the connectives $\neg$ and $\wedge$, are the truth functions:

$\neg$: $\{0,1\} \rightarrow \{0,1\}$ such that:

$\neg(0) = 1;\ \neg(1) = 0$

$\wedge$: $\{0,1\} \times \{0,1\} \rightarrow \{0,1\}$ such that:

$\wedge(1,1) = 1;\ \wedge(0,1) = \wedge(1,0) = \wedge(0,0) = 0$

(If we add the obvious truth function for disjunction, it becomes clear that on the algebraic approach we are directly interpreting our propositional language in a Boolean algebra.)

In general, we define a truth function:

an *n-place truth function* is a function $f: \{0, 1\}^n \rightarrow \{0, 1\}$.

All propositional operators are interpreted as truth functions. Now the question of *functional completeness* arises: suppose we have a language with connectives $\{O_1, \ldots, O_n\}$ and their interpretation as truth functions $f_1, \ldots, f_n$. Can we define all other possible truth functions in that language, i.e. can we express all other truth functions. If so, then the system of operators is functionally complete.

As you probably know, the following systems of connectives are functionally complete (for every n, every n-place truth function can be defined in terms of them): $\{\neg, \wedge\}$,$\{\neg, \vee\}$, $\{\neg, \rightarrow\}$. Let ! stand for the truth function

$$!(1, 1) = !(1, 0) = !(0, 1) = 0;\ !(0, 0) = 1 \text{ (neither \ldots nor)},$$

{!} is functionally complete (if you want practice, define $\neg$ and $\wedge$ in terms of !).

Let us do the same for first order predicate logic. Here the interpretation of formulas is relativized to assignment functions. We cannot define the truth value of a formula, but only the truth value of a formula relative to an assignment function.

So we took the interpretation of a formula relative to an assignment function in a model to be a truth value. Generalizing, we can then say that the interpretation of a formula is a function from assignment functions to truth values: Let G be the set of all assignment functions.

For every $g \in G$: $[\![\varphi]\!]_g \in \{0, 1\}$

Hence we can define $[\![\varphi]\!]: G \rightarrow \{0, 1\}$ as that function such that for every g: $[\![\varphi]\!](g) = [\![\varphi]\!]_g$.

So the interpretation of an expression is that function that assigns to every assignment the extension of that expression relative to that assignment.

Since for every assignment the extension of a formula relative to that assignment is a truth value, the interpretation of a formula becomes a function from assignments to truth values.

Since those functions are the characteristic functions of sets of assignment functions, this is equivalent to saying that formulas are interpreted as sets of assignment functions. So the interpretation of φ in M, $[\![\varphi]\!]_M$, is the set of all assignment functions relative to which φ is true in M.

Given that formulas are interpreted as sets of assignment functions, the n-place (sentential) operations of our first order language, $\neg, \wedge, \forall x_1, \forall x_2, \ldots, \exists x_1, \exists x_2, \ldots$ are functions from n-tuples of sets of assignment functions to sets of assignment functions.

This suggests, that we could give a semantics to predicate logic by interpreting every formula of predicate logic as an element of the pow(G_M) (the set of all sets of assignment functions on M), and find operations on sets of assignment functions, corresponding to the logical operations.

Let M be a model for predicate logic. G the set of all assignment functions on M. We define the interpretation of L in M, $[\![\ ]\!]_M$ as follows: (I leave out the index for M)

Constants and variables:

$$[\![c]\!](g) = i(c) \quad \text{(i.e. } [\![c]\!] = \lambda g.i(c))$$
$$[\![P]\!](g) = i(P)$$
$$[\![x]\!](g) = g(x) \quad \text{(i.e. } [\![x]\!] = \lambda g.g(x))$$

Formulas:

$$[\![P(t)]\!] = \{g{:}[\![t]\!](g) \in i(P)\}$$
$$[\![\neg\varphi]\!] = G - [\![\varphi]\!]$$
$$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$$
$$[\![\varphi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$$

We define: h varies from g at most with respect to variable x:

$$h\langle x\rangle g := \forall y{:}\ y \neq x \rightarrow h(y) = g(y)$$
$$[\![\forall x_n \varphi]\!] = \{g\text{: for all } h\text{: if } h\langle x_n\rangle g \text{ then } h \in [\![\varphi]\!]\}$$
$$[\![\exists x_n \varphi]\!] = \{g\text{: for some } h\text{: } h\langle x_n\rangle g \text{ and } h \in [\![\varphi]\!]\}$$

(The algebraic structure that we get by considering these operations, $\langle \text{pow}\, G, -, \cap, \cup, \exists_n, \forall_n\rangle_{n\in\omega}$ is called a (power set) cylindric algebra.)

We define:

φ is *true relative to g in M*: iff $g \in [\![\varphi]\!]_M$

All the other notions are now defined as usual.

This semantics is totally equivalent to the one we have given in Chapter One. The difference is that it brings out the compositional nature of the interpretation more directly (for instance, the quantifier $\forall x_n$ is now an operation mapping sets of assignment functions onto sets of assignment functions. (We could even go on and say: $\exists$ is a function from natural numbers to such operations, $\exists(n) = \exists x_n$; $\exists = \lambda n.\{g$: for some h: $h\langle x_n\rangle g$ and $h \in p\}$.)

We could now define:

> An *n-place quantifier* on M is a function from $(\text{pow}G)^n$ to $\text{pow}G$.

We could call a system of operators functionally complete if every quantifier can be expressed. The system of operators that we have in predicate logic is not functionally complete. A quantifier can be expressed if it is first order definable. A simple cardinality argument shows that there are (many) quantifiers that are not first order definable.

Suppose the domain of our model is infinite. Then there are uncountably many distinct quantifiers on M. But our first order language is only capable to express at most countably many distinct quantifiers, because there are only countably many formulas. So we cannot expect countable languages with finitary operations to be functionally complete, and – as we have seen – we shift our attention to the question of which quantifiers are expressible.

Now let us go to tense logic. We have the same tense logical language as before. The notion of model is defined almost as before. Earlier, model was a structure $\langle T, <, i\rangle$ where $i{:}ATFORM \times T \to \{0, 1\}$. The interpretation of a formula at a moment of time in such a model is a truth value. We make the same step as in predicate logic and let the interpretation of a formula be a function from moments of time to truth values: i.e. $i(\varphi)$: $T \to \{0, 1\}$ is the function $\lambda t.i_t(\varphi)$. Again taking characteristic functions, we see that formulas are interpreted as sets of moments of time:

> $i(\varphi)$ is the set of moments of time at which φ is true

Given that formulas are interpreted as sets of moments of time in T, the n-place (sentential) operations in a tense logical language are functions from n-tuples of sets of moments of time to sets of moments of time.

Now we can define the notion of model:

> A *model* for L is a pair $\langle F, i\rangle$ where $F = \langle T, <\rangle$ is a frame for L and $i{:}ATFORM \to \text{pow } T$

So indeed the interpretation function assigns to every atomic formula a set of moments of time.

We can now define the category of interpretations of tense logical connectives:

> An *n-place tense* in F is a function $f{:}(\text{pow } T)^n \to \text{pow } T$

We interpret all operators as tenses:

The *interpretation* for L in M, $[\![\ ]\!]_M$ is defined by:

$$[\![p]\!]_M = i(p) \text{ for all atomic formulas.}$$

The operators of L then are the unique tenses such that for all φ and ψ:

$$\begin{aligned}
&[\![\neg\varphi]\!]_M = T - [\![\varphi]\!]_M\\
&[\![\varphi \wedge \psi]\!]_M = [\![\phi]\!]_M \cap [\![\psi]\!]_M\\
&[\![\varphi \vee \psi]\!]_M = [\![\varphi]\!]_M \cup [\![\psi]\!]_M\\
&[\![P\varphi]\!]_M = \{t \in T{:}\ \exists t' \in T[t' < t \text{ and } t' \in [\![\varphi]\!]_M]\}\\
&[\![H\varphi]\!]_M = \{t \in T{:}\ \forall t' \in T[\text{if } t' < t \text{ then } t' \in [\![\varphi]\!]_M]\}\\
&[\![F\varphi]\!]_M = \{t \in T{:}\ \exists t' \in T[t' > t \text{ and } t' \in [\![\varphi]\!]_M]\}\\
&[\![G\varphi]\!]_M = \{t \in T{:}\ \forall t' \in T[\text{if } t' > t \text{ then } t' \in [\![\varphi]\!]_M]\}
\end{aligned}$$

Again, in this way we get a power set algebra of propositions $\langle \text{pow } T, -, \cap, \cup, P, F, G, H\rangle$ which is a Boolean algebra with some additional operations.

We can now define:

> φ *is true on* t in model M iff $t \in [\![\varphi]\!]_M$

It is not hard to see that the present semantics is equivalent to the earlier one.

We could now ask the question: can we find for tense logic a system of operators, such that every tense can be defined in terms of them. For the same reason as for predicate logic, the answer to this is negative. If T is countable, the number of distinct tenses in T is uncountable. With any finite system of operators we can only define at most countably many distinct tenses, so there is no chance that systems of tenses can be capable of expressing all tenses. Another way of saying this is that the language is not able to express every tense logical proposition. To push this discussion a little bit further, I will give a different way of doing tense logic.

Instead of taking a propositional language and sententional operations, we take a first order language L', where we interpret all the variables as moments of time, we have a special relation symbol $<$ in the language, and instead of a set of propositional symbols, we now take a set of one place predicates.

A model for this language L' is a first order model $\langle T, <, i \rangle$ with a special relation $<$. The interpretation of the language is totally standard for a first order language.

The idea behind this interpretation is the following: instead of translating time-dependent sentence *it rains* in a propositional language as a propositional constant and interpreting it relative to a moment of time, we make the time-dependency explicit in the language, we translate it into a predicate logical formula with a free time variable *it rains*(t), which we can read as 'it rains at t'.

Where in the old interpretation we could ask whether *it rains* is true at moment t, we now ask whether *it rains*(t) is true relative to an assignment g such that $g(t) = t$. (If we go to predicate logical tense logic, by adding individuals, our language becomes a two sorted language.)

Basically what happens in this language is that we make the quantification over times explicit in the language. (The same is done for Montague's intensional logic IL in Gallin's (1975) two sorted type theory TY2. Montague's IL does not explicitly quantify or abstract over possible worlds, there are no expressions of type s; TY2 makes this quantification and abstraction explicit by introducing variables over possible worlds (type s) in the language.)

This way of doing things is clearly as expressive as propositional tense logic, because we can translate the latter in to it: here is a translation †:

We associate with every atomic formula p of our propositional language a one place predicate of times P of our quantificational language. We define:

$$(p)^{\dagger}[n] = P(t_n)$$
$$(\neg\varphi)^{\dagger}[n] = \neg(\varphi)^{\dagger}[n]$$
$$(\varphi \wedge \psi)^{\dagger}[n] = (\varphi)^{\dagger}[n] \wedge (\psi)^{\dagger}[n]$$

$$(P\varphi)^{\dagger}[n] = \exists t_{n+1}[t_{n+1} < t_n \wedge (\varphi)^{\dagger}[n+1]]$$
etc.
$$(\varphi)^{\dagger} = (\varphi)^{\dagger}[0]$$

Examples:

$$(p \wedge q)^{\dagger} = (p \wedge q)[0] = (p)^{\dagger}[0] \wedge (q)^{\dagger}[0]$$
$$= P(t_0) \wedge Q(t_0)$$
$$(Pp)^{\dagger}[0] = \exists t_1[t_1 < t_0 \wedge (p)^{\dagger}[1]]$$
$$= \exists t_1[t_1 < t_0 \wedge P(t_1)]$$
$$(PPp)^{\dagger}[0] = \exists t_1[t_1 < t_0 \wedge (Pp)^{\dagger}[1]]$$
$$= \exists t_1[t_1 < t_0 \wedge \exists t_2[t_2 < t_1 \wedge (p)^{\dagger}[2]]$$
$$= \exists t_1[t_1 < t_0 \wedge \exists t_2[t_2 < t_1 \wedge P(t_2)]]$$

Let us call, for propositional tense logic L, $L^{\dagger}$ the set of L' translations of L under †. Once we know this, we can compare the expressive power of systems of tense logical operators in propositional tense logic relative to what is expressible in this quantificational language L'.

We can call a system of tense operators of L functionally complete if every tense that is first order definable in L' is definable with a formula in $L^{\dagger}$ (where L is the language that has $\neg$, $\wedge$ and those operators).

Similarly, a system of tense operators is functionally complete with respect to a class of models M, if it is able to express every tense that is first order definable in L' on that class of models.

Most of the classical results about the expressibility of propositional tense logic are due to Hans Kamp. He proved in his thesis (Kamp, 1968) that Prior's system $\{P, F, G, H\}$ is not functionally complete in the latter sense but that the following two two-place operations *Since* and *Until* are functionally complete with respect to continuous linear orders (while Prior's system is not):

$$[\![S(p,q)]\!]_t = 1 \quad \text{iff}$$
$$\exists t' < t[\ [\![p]\!]_{t'} = 1 \quad \text{and} \quad \forall t''[t' < t'' < t \rightarrow [\![q]\!]_{t''} = 1]]$$
$$[\![U(p,q)]\!]_t = 1 \quad \text{iff}$$
$$\exists t' > t[\ [\![p]\!]_{t'} = 1 \quad \text{and} \quad \forall t''[t < t'' < t' \rightarrow [\![q]\!]_{t''} = 1]]$$

Exercise 3. (a) Define $P\varphi$, $F\varphi$, $G\varphi$ and $H\varphi$ in terms of S and U.
(b) Let T be a tautology.
What does $U(\varphi \wedge U(\neg\varphi \wedge \neg U(\varphi, T), \varphi), \neg\varphi)$ express?

Kamp's result is very interesting. It means that if we take continuous linear orders as our model of time (which is for a lot of purposes a very plausible choice), then every first order expressible proposition is expressible with *since* and *until* (and the connectives). This shows that, if a language, say, Hopi, has *since* and *until*, we can express in that language everything about continous linear orders that can be expressed in a first order way. So, if Whorf is right, that the model of time underlying English is the model of time in modern physics (the reals) (see Whorf, 1956), we can, without risk of cultural relativity, explain to Hopi speakers our idea of time, at least, as long as it concerns first order properties.

Another lack of expressibility of Priorian tense logic that Kamp discovered (Kamp, 1970), is the impossibility of expressing temporal indexicals like *now*.

To express *now* in our first order tense logic, we have to add a special constant (say t_0), that is interpreted as the present moment.

Before Kamp's work it was believed that *now* is superfluous (*it rains now* simply means *it rains*). So, a now operator N, would be given the semantics:

$$[\![N\varphi]\!]_t = 1 \quad \text{iff } [\![\varphi]\!]_t = 1$$

Kamp argues that, to treat *now* adequately, we have to add a special element (the moment of speech) to our temporal structures, (i.e. $\langle T, <, t_0, i\rangle$) and define:

$$[\![N\varphi]\!]_t = 1 \quad \text{iff } [\![\varphi]\!]_{t_0} = 1$$

Kamp's result is more surprising than those that grew up with indexicality would expect (i.e. the thought that indexicals like *now* could be eliminated, are expressible by sentences without *now*, may seem rather naive nowadays).

Kamp showed that as long as we consider *propositional* tense logic, *now* is indeed superfluous: every statement that has *now* in it is equivalent to some statement that does not. However, as soon as we go to a temporal *predicate logic*, the situation changes. Kamps's classical example sentences concern the difference between:

A child was born that would become king.
$P(\exists x[C(x) \wedge B(x) \wedge F(K(x))])$
A child was born that will become king.
$P(\exists x[C(x) \wedge B(x) \wedge NF(K(x))])$

The latter sentence is not reducible to a sentence without *now*.

What *now* does is to shift the moment of evaluation back to the present moment. After Kamp's paper, several shift (and cycle, and whatnot) operations have been studied in tense logic (see van Benthem, 1977) and for other indexicals (see especially Kaplan, 1978; and Stalnaker, 1978). The definition of such operators tends to become very complicated and their number tends to grow with every new phenomenon that cannot be expressed with the previous operators.

For that reason, more and more people take a language with quantification over times as basis, and just define the operators they need in them (see, for instance, Hinrichs, 1985).

For the simple cases this has the disadvantage of length ($\exists t < t_0$ Walks(j, t) is longer than $W(j)$).

For the longer cases that has the advantage of readability and keeping track of the interpretation.

(The same argument is used by Groenendijk and Stokhof, 1982, 1985 to advocate TY2 over IL in the analysis of questions, and is very obvious if you translate Stalnaker's (1978) indexical operations into TY2.)

3.3. SOME LINGUISTIC CONSIDERATIONS CONCERNING INSTANTS

I want to do three things in this section. First I want to make the obvious point that, if we take instants as the basis of our ontology, we cannot do without the notion of an interval and I want to talk about when the use of intervals still counts as instant tense logic. Secondly, I want to talk briefly about the question of whether there are linguistic arguments in favor or against taking time as discreet or dense. Thirdly, I want to give some examples of problems you may encounter in giving an instant semantics for temporal constructions.

The point that if we have moments of time we need intervals of those as well is obvious if we look at temporal adverbs:

> Last year, I won the prize; the year before, Bill did; the year before that, Sue did. I won the prize, in February last year.

In an instant semantics, truth at a moment of time is the basis of the semantics. Even if we assume that in the beginning of the example, the instants are years, we want to relate *last year* with *in February last year* and we can't take both *last year* and *in February last year* as instants in the same structure. The obvious thing to do is to take *last year* to be an interval of moments of time and impose a temporal measurement on the instant structure, measuring time in years and another temporal measurement, measuring time in months, where the second measurement is a refinement of the first (so the interval *in February last year* is part of *last year*).

We are going beyond instant tense logic if we change our basic semantic notion from truth relative to a moment of time to truth relative to an interval of time, and give a new semantic definition of the latter notion. We stay within instant tense logic if we define what is the case at an interval (like *last year*) in terms of what is true at the moments of time that constitute that interval.

For instance, we could interpret the example in the following way.

Let us assume that a model for tense logic consists of a linear order of moments of time and a system of temporal measurements, (interval-partitions of T) $p_1, p_2, \ldots$ and let's only consider standard measurements, so every p_{n+1} is a refinement of p_n, where every block in p_n is replaced by the obvious number of blocks (so every year is refined into twelve months, the months into the right number of days, the days into 24 hours, etc.)

We can then let the interpretation function assign to a temporal adverb like *last year* or *in February last year* the right interval relative to the right measure.

If we now want our semantics to tell us what happened last year, we go beyond instant tense logic if we directly assign truth values to formulas relative to those intervals, but we stay within instant tense logic if we define what happened last year in terms of what happened at the *moments of time* constituting last year.

For instance, we could say: $[\![\textit{last year}(\varphi)]\!]_t = 1$ iff $\exists t' \in [\![\textit{last year}]\!]_t$: $[\![\varphi]\!]_{t'} = 1$. If we do the same for *in February last year*, this would imply directly that if I won the prize last year in February, I won the prize last year.

An interval semantics like this is a reductionistic interval semantics. The position that interval semantics should be reductionistic is taken by Pavel Tichy (in Tichy, 1985). Whether this is the right model for tense is rather dubious.

One problem is that if truth at an interval is to be reduced to truth at its moments of time, then you are forced to make a decision about which moments count and which don't.

One problem you then stumble on directly is the problem of vagueness: if a solution is changing from red to orange in a certain interval, then at the beginning of the interval it is red, at the end it is orange (presumably), but what happens in the middle: are we really able to partition the interval into the initial interval where it is red, directly followed by the end interval where it is orange?

Vagueness is of course a problem for anyone, and it has to be seen that an interval semantics does better on this account. But vagueness comes in directly at the place where the reduction is supposed to be taking place. Take an adverbial phrase like *from nine to ten* and we look at the sentence:

> John drank beer from nine to ten.

What should this mean? At some time in $[\![\textit{from nine to ten}]\!]$ John drank beer? At all times in $[\![\textit{from nine to ten}]\!]$ John drank beer? At most times in $[\![\textit{from nine to ten}]\!]$ John drank beer? This is just not clear, and as long as it isn't, it is not clear that a reduction is possible at all.

Another problem comes in with the notion of moment of time itself.

The instants that our temporal measurements are imposed upon and that truth is relativized to are abstract, durationless points: since points are timeless, define time, it is senseless to say that they take time. Our language consists of atomic formulas and temporal operators. Given the way the interpretation works, atomic formulas get a durationless interpretation: an atomic formula is simply true or false at a moment of time. Whatever the atomic formula expresses doesn't take time.

The question to ask then is: does English have atomic formulas? Can we seriously decompose *John is walking* into an atomic formula that expresses something that doesn't take time combined with temporal operators?

One view we could take on this (while staying within the instant time framework) is the following. It is not the abstract underlying time structure that is semantically crucial, but the system of temporal measurements. We shouldn't ask just 'what is a moment of time', because that is a context dependent question.

We can assume that the context determines how precisely we are measuring time; it chooses in the hierarchy of temporal measurements one measurement that is taken as 'time as finely grained as this context requires it to be'. The elements of that measurement are then regarded as moments *in that context*.

The idea then is that something can be a durationless *moment* of time in the present context, but become an interval that has duration after all, when we shift the context to a more finegrained one.

As far as I know the semantic details of an interpretation in such a context-dependent refinement sequence of measurements have never been worked out (although the idea of using context dependent granularity can be found at several places in the literature in various contexts, see for instance Lewis 1979, Landman 1986, Link 1987, Oversteegen 1989). I won't try here to work it out.

I do want to use it, though, to make some remarks about the structure of time.

Let us ask the question: are there linguistic arguments in favor of or against the claim that time is discreet rather than dense?

There is no doubt that in our temporal discourse we refer to discreet orders of time: counting time in minutes is doing just that. This, of course, only shows that whatever the structure of time, we can and do impose discreet temporal measurements on it. But what about the temporal order that we start with? We do constantly use expressions like:

> John can never sit still. The one moment, he is repairing the radio, the next moment he is doing the dishes, the moment after that he paints the wall.

Since we talk here about 'the next moment', this might lead to a discreet structure of time.

This is not a very good argument, though. In the first place, moments in the above discourse clearly have to be understood as intervals; at least, certainly not as durationless moments of time, because certainly the activities that are placed in moments here take time. This shows

that, if we take the word 'moment' seriously in the preceding discourse, we have to rely on a contextual refinement model, where one discreet order is taken as basic in the context, but where the moments of that structure can be refined into intervals by changing the context.

The relevant *linguistic* observation to be made here is that there is no linguistic reason to assume that such contextual refinement cannot always be done: yesterday afternoon between twelve and two I was sleeping; I woke up when you called me; in fact, at the moment you called me I woke up; more precisely, at the moment that the telephone first rang, I woke up; in fact, I woke up at the moment the telephone first started ringing; to be more precise, I woke up when the first sound of the first ring started, etc.

It is inherent to the notion of refinement, in as far as it plays a role in semantics, that we can go on to refine *indefinitely* (of course, we will in fact only go on as long as it is sensible).

Although there may be points after which refinement is no longer practically or even physically possible (these would be points where our measurement systems are not finegrained enough to measure), we cannot take those to be the minimal elements. Such a choice of minimal elements is linguistically arbitrary (as arbitrary as the assumption that sentences that are too long to fit in our memory space are ungrammatical).

If there is no linguistic reason to assume a cut-off point, then linguistically it is not there (this argument can be found at a lot of places in the literature for various phenomena related to vagueness and refinement, for instance, in the discussion of minimal parts of mass nouns like *water*, see Chapter Seven).

In fact, I think that counterfactuals give us linguistic reason to assume the contrary. Suppose that up to the minimal point we can measure, my waking up and your calling coincide. There is nothing incomprehensible about my saying: but had we have a better measurement instrument, we would have found out after all that I woke up before you called me.

If there are minimal intervals of time (of the form $\{t\}$) we cannot reach them by refining the context: every moment that we do reach can be split into smaller moments by linguistic means.

This has consequences for the discussion of discreetness versus density of time.

Although in every context we can take a discreet order as basis, we can always refine it by going to a more fine grained context. This means that the whole context dependent model of time (the model that contains all these refinements) has to be dense: if we take all refinements to be measurements on a linear order, the fact that we can always refine the context forces this linear order to be dense.

Another moral can be drawn from this. Although we may believe that linguistic structures should be finite, that can only mean that *in a context* we have a finite structure. If we have indefinite contextual refinement, the whole context dependent model certainly cannot be finite, and if a structure and its refinement are thought of as part of the same overall structure, and the overall structure is what we are semantically interested in, then also there, we can't restrict ourselves to finite structures.

You sometimes see the claim that models that are relevant for semantics should be finite, and I think it is good to be rather suspicious of that, certainly in the case of temporal phenomena. The above considerations advise us to critically investigate such claims, and separate the cases where something 'deep and subtle' is meant, from the cases that are just 'arbitrary limit' theories, and those from the cases where the proposal just happens not to work for infinite structures.

Exercise 4. BEFORE and AFTER
A programmed instruction

The content of this exercise was inspired by a discussion with Richard Larson, several years ago. Some of the points discussed here go back as far as Anscombe (1964). For more discussion of the semantics/pragmatics of *before* and *after*, see Heinämäki (1977), Hinrichs (1981), Partee (1984), Oversteegen (1989) and references in those works.

We are concerned here with the semantics of *before* and *after*, which we will treat as two place past tense operators: (so we will only be concerned with: John came before Mary came). Moreover, here we will only be interested in the case whare the arguments of *before* and *after* express pointlike events. pBq will stand for 'p before q'; pAq for 'p after q'. I will use the language in which quantification over times is made explicit to define these operations.

Our first guess for the semantics is based on the intuition that *before* and *after* are the obvious inverse of each other: 'p after q' is the same as 'q before p'.

This leads to the following definition: (where p and q are simple past tense sentences):

$$pAq(t_0) \text{ iff } \exists t_1 < t_0[p(t_1) \wedge \exists t_2 < t_1[q(t_2)]]$$
$$pBq(t_0) \text{ iff } \exists t_1 < t_0[p(t_1) \wedge \exists t_2[t_1 < t_2 < t_0 \wedge q(t_2)]]$$

Clearly these definitions capture the above intuitions, i.e. John came before Mary came iff Mary came after John came.

Here is the problem: in contrast with what we expected there is a semantic difference between *before* and *after*: *before* allows negative polarity items (in its second argument) while *after* does not (in none of its arguments):

John read the book before anyone else ever did.
*Anyone else ever read the book before John did.
*John read the book after anyone else ever did.
*Anyone else ever read the book after John did.

Other contexts where negative polarity items are allowed are: in negative contexts:

It's not the case that John ever went to Paris.

but not in positive contexts:

*John ever went to Paris.

under *at most NP*:

At most three boys ever went to Paris.

but not under *at least NP*:

*At least three boys ever went to Paris.

in relative clauses with a universal head:

Every boy who ever went to Paris came back unhappy.

but not in relative clauses with an existential head:

*Some boy who ever went to Paris came back unhappy.

and many more (see Ladusaw, 1980).

Ladusaw observed the connection between negative polarity items and downward entailing contexts: the contexts that allow negative polarity items are typically the contexts that are downward entailing (the connection is a very strong one although there are some problematic cases, see Linebarger (1987), Kadmon and Landman (1990)). A context C is downward entailing if it reverses entailment: if A entails B then $C(B)$ entails $C(A)$.

You can test it in the above examples:

Every boy who lives in London or Paris is happy. entails
Every boy who lives in London is happy.
At most three boys move. entails
At most three boys walk. but
Some boy who moves is happy. does not entail
Some boy who walks is happy.

We will assume Ladusaw's analysis. If we now check whether the semantics we have given to *before* is downward entailing, we get a negative result.

* Check this.

Here is the main exercise: * give a new semantics for pBq that makes the second argument downward entailing. Read on after you have done this.

If you have *replaced* the second part of the clause, there is a good chance that your definition is indeed downward entailing. If you have *added* something to it, there is a good chance that your definition is not downward entailing.

To see which one is right, we should go to the evidence. If *before* is downward entailing the following inferences should be valid:

John came before Bill or Mary came. entails
John came before Bill came.
John moved before Mary moved. entails
John moved before Mary jumped in the gorge.

The first case seems unproblematic. We might have some doubts about the second one. It may seem rather weird that we can *draw* this inference if we are talking about John and Mary on the dance floor.

We could draw the conclusion from this that *before* should not be downward entailing and that we have a counterexample to Ladusaw.

However, we should be careful. The fact that the inference is weird doesn't mean that it is invalid. This issue clearly has to do with the question whether p *before* q entails that q took place or *presupposes* or *implicates* that q took place.

Entailments cannot be canceled, implicatures can.

* Argue that p *before* q entails that p took place but implicates rather than entails that q took place;

* argue that p *after* q entails both that p took place and that q took place.

From this we can draw the conclusion that semantically *before* is indeed downward entailing and that seeming counterexamples have a perfectly natural pragmatic explanation.

* If your semantic analysis wasn't strictly downward entailing, repair it now, before reading on.

The analysis we have now is:

$$pAq(t_0) \text{ iff } \exists t_1 < t_0[p(t_1) \wedge \exists t_2 < t_1[q(t_2)]]$$
$$pBq(t_0) \text{ iff } \exists t_1 < t_0[p(t_1) \wedge \forall t_2[t_2 < t_0 \wedge q(t_2) \rightarrow t_1 < t_2]]$$

This analysis makes *before*, but not *after* downward entailing on its second argument.

Now we should check the correctness of the analysis. A first test is to see what happens under negation:

It's not the case that John came before Mary came
It's not the case that John came after Mary came

* Write the negations of the semantic clauses that you have got, and check with your intuitions about the above sentences whether they are adequately captured by those clauses.

You'll probably find no problem there, this test is just good to do, sometimes it reveals embarrasing aspects of your definitions.

The next step is serious. You are now claiming an asymmetry between the semantics of *before* and *after*. This goes against what we would think at first sight. You get the negative polarity predictions in accordance with the theory, but now you have to sell your analysis and defend the semantic assymmetries that it postulates.

On the analysis that you have now, pBq does not entail qAp, but this is not that serious, because together with its presupposition it does entail qAp. But qAp does not entail pBq, and this is a crucial aspect of your analysis, you have to *argue* now that it shouldn't.

Technically, you have to make a model where your clauses tell you that qAp is true but pBq is not.

* Make a minimal model that does that.

Now you have to convince yourself (and others) that this counter-example also intuitively is a good counterexample.

To sell this, you have to invent a story, or a situation. Here is an example: the above model describes the sad case where q is *John hit Mary*, and p is *Mary hit John*. The model describes the facts. Now imagine the judge asking you: "Did John hit Mary after she hit him?" and after that she asks: "Did Mary hit John before he hit her?"

If your intuitions go like mine, then you'll answer yes the first time and no the second (or rather "yes, your Honor" and "no, your Honor"), and you have found your example.

At this moment you have an analysis that you can defend. Now you can go on and look at the predictions that your semantics of *before* makes for other situations. For instance, you can look at a situation where for quite some time John and Mary were hitting each other, while varying who starts the hitting, and ask the same questions, and see whether it makes a difference; you can look at what happens with: *John hit Mary before and after she hit him* (of course, you don't have an analysis for this, but you might want to get one). If your analysis gets into problems here you may have to modify it, but it is now strong enough to first look for other factors that may interfere here (for instance, strictly speaking pBq quantifies over all time; that is not a problem, because like all quantification, we clearly have to assume that the temporal quantification is restricted to a temporal interval given by the context. If in some cases, *before* seems not to behave in the way your analysis predicts, you might want to look into the possibility that a context shift to a sub-interval has taken place).

And from there on a world of research opens. We've assumed that p and q are pointwise. This case is not much discussed in the literature. What is discussed a lot (see the mentioned references) are cases where p and q are extended in time. One question that is immediately raised there is whether p *before* q means that p *started* before q *started*, or

whether p *ended* before q *started*. These questions interact with the aspectual class nature of p and q (i.e. whether they are states, activities, accomplishments, or whether they are durative or not).

Discussion of this goes beyond the present simple setting of the problems in instant semantics, but it becomes interesting to see how and whether an analysis like the one developed here can be fitted with one that deals with such interval and aspectual problems.

3.4. INFORMATION STRUCTURES

Up to now we have considered theories, like tense logic, that were extensions of classical logic and that were based on the same classical semantic conception: the basis of the semantic recursion forms the notion of truth (in a possible world, at a moment of time, etc.).

We will now look at some theories that take a more epistemic view, in which the semantics does not specify conditions for *when* a sentence is true or false in a situation, but rather conditions for when a situation or information state justifies calling a sentence true or false, when a sentence is true or false *on the basis of* a situation or information state.

Although it may seem that what we will be doing here, define truth/falsehood on the basis of information state s, is completely analogous to what we have been doing before (define truth/falsehood at a moment of time), the difference should not be underestimated: in the semantic theories we will be considering, partiality plays a central role, and the fact that information states are partial means that, even if the semantic clauses look the same, the logic, and with that the predictions about which inferences are valid, will differ (and, as we will see, you will have to play tricks to get classical logic back), and this difference reflects a different conception of semantics (evidence conditions or assertability conditions versus objective truth conditions), or, if you want, a different conception of truth. I will discuss Intuitionistic logic in this section as an example of semantics relative to information structures. The discussion will be continued in the next section with an application of information structures to the analysis of vagueness.

Intuitionistic Logic

(For more discussion and history, see van Dalen, 1986; Dummett, 1973, 1977; and Kripke, 1965.)

Intuitionism was born out of a strong dissatisfaction with mathematical arguments proving the existence of a mathematical object, without there being any mathematical means of constructing it.

According to Brouwer, the initiator of Intuitionism, mathematics is a mental activity, based on our mathematical intuition (we could say, structural insight); since mathematical entities are the products of this activity, mathematical operations are abstract mental operations with which we can construct mathematical entities.

On this view, a proof that there is a mathematical entity that cannot be constructed is simply nonsensical: mathematical reasoning is, in view of what mathematics is, constructive reasoning, and a semantics validating non-constructive proofs should be rejected.

The semantics then should reflect the idea that mathematical reasoning is constructive.

In the abstract, this can be given the following form: a mathematical argument is constructively valid if all mathematical evidence that establishes the premises also establishes the conclusion.

In the context of natural language semantics two remarks should be made. One could say: why this interest in constructive reasoning: our daily life reasoning is non-constructive. We conclude from the fact that the lock is broken and the compact-disk player is gone that someone broke into the house, without being able to specify who.

This is true, but doesn't show the point. Suppose that with the facts given above no evidence *imaginable* (by us) could ever show of any particular person that he broke into the house, suppose that we wouldn't be able to imagine what such evidence would look like: are we then still allowed to conclude that someone broke into the house?

Of course, we *can* imagine what evidence would show a particular person to have broken into the house (in fact, we hope to find it, although we may fail to) and the constructivist only claims that we should understand the meaning of 'someone broke into the house' in terms of this capacity.

Note that also in classical logic, the above inference (from 'the lock is broken' and 'the compact-disc player is gone' to 'someone broke into the house') is not regarded as *logically* valid. The constructivist point, then, is that the conclusion 'someone broke into the house' only logically follows from premises (that is, in virtue of their meaning) that themselves can only be established by evidence that establishes the conclusion as well.

The question, then, whether natural language semantics can or should be constructive, is not answered (negatively) by cases like this.

A second, related point has to do with inferences that are classically valid, but intuitionistically invalid. Don't people have intuitions that (at least for predicates that are not vague, let's put those aside for the moment, because that is not what the dispute is about) something either is a cow or isn't? And don't people have the intuition that if it is not the case that every man is mortal, then some man is not mortal?

Again, the fact that such inferences in general are not intuitionistically valid, does not mean that these particular instances could not be valid. For instance, the law of excluded middle does hold for decidable predicates, and the other inference also holds if in the universe of discourse, the relevant predicates are decidable.

It is not clear that, apart from vagueness, natural language examples like the above could not fall within a class for which the logical principles mentioned hold unproblematically.

Suppose that we have a predicate *plunk* for which we cannot imagine what would count as evidence for saying that something is a plunk or not, would we still be willing to say that something is either a plunk or is not?

From a constructivist point of view, the validity of 'something either is a cow or isn't' need not be regarded as a counterexample, but can be taken as an indication of the constructive nature of the meanings of our lexical items.

The choice between constructivism and realism is a deep and subtle issue, relating ultimately to the nature of our semantic domains: whether we can regard those as constructions built by our conceptual apparatus or not. What I tried to argue above was that such choices are not settled by superficial and crude arguments like the ones that I discussed.

Besides its foundational importance, semantically, intuitionistic logic is interesting, for one thing, because its analysis of indicative conditional sentences is much more satisfying than the material implication of classical logic.

Before we go to the interpretation, it might be helpful to give a brief account of what intuitionistic logic is, and how it differs from classical logic. To do this, it is probably best to give the proof theory of intuitionistic logic in the form of a natural deduction system.

The natural deduction system for intuitionistic logic consists of the following inference rules for *introduction* (I) and *elimination* (E) of the logical connectives in a proof:

$I\wedge$: $\dfrac{\varphi, \psi}{\varphi \wedge \psi}$ $\qquad E\wedge$: $\dfrac{\varphi \wedge \psi}{\varphi} \quad \dfrac{\varphi \wedge \psi}{\psi}$

$I\vee$: $\dfrac{\phi}{\phi \vee \psi} \quad \dfrac{\psi}{\varphi \vee \psi}$ $\qquad E\vee$: $\dfrac{\varphi \vee \psi \quad \begin{bmatrix} \varphi & \psi \\ \chi & \chi \end{bmatrix}}{\chi}$

$I\rightarrow$: $\dfrac{\left[\begin{array}{l} \varphi \\ \psi \end{array}\right.}{\varphi \rightarrow \psi}$ $\qquad E\rightarrow$: $\dfrac{\begin{array}{l}\varphi \rightarrow \psi \text{ (\textit{Modus ponens})} \\ \varphi\end{array}}{\psi}$

$I\neg$: $\dfrac{\left[\begin{array}{l} \varphi \\ \bot \end{array}\right.}{\neg\varphi}$ $\qquad E\neg$: $\dfrac{\varphi, \neg\varphi}{\bot} \quad \dfrac{\bot}{\psi}$ (*Ex falso*)

So I $\wedge$ means that if you find ϕ and ψ in a proof, you can conclude $\varphi \wedge \psi$ in that proof, $I\rightarrow$ means that if on the assumption of φ you can prove ψ, then without that assumption you can prove $\varphi \rightarrow \psi$. A boxed part in a proof is a *subproof*, or an *embedded* proof.

Besides these rules, there is a *repetition* rule: if you find ϕ at some level in a proof (i.e. in a proof, or some embedded proof), then you can freely repeat φ at that level and in all subproofs that are embedded from that level. This is a monotonicity requirement.

If we were to leave out the *Ex Falso* (*sequitur quodlibet*) rule, we get a logical system that is called Minimal Logic.

The rules of Intuitionistic logic are all part of the natural deduction system for classical logic. In fact, only one rule of the classical system is missing, namely the rule for double negation:

$$\frac{\neg\neg\varphi}{\varphi}$$

The quantifier rules (which I won't give) are also the same as those for classical logic, so as far as the natural deduction system goes, the fact

that the double negation rule is missing is the only difference with classical logic.

This has some consequences:

–: $\varphi \vee \neg\varphi$, the law of excluded middle is not intuitionistically provable.
–: $\neg\neg\varphi \rightarrow \varphi$, the principle of double negation is not provable (the other side is provable).
–: $\neg(\varphi \wedge \psi) \rightarrow (\neg\varphi \vee \neg\psi)$, one of the de Morgan laws is not provable.
–: $\neg\forall x\varphi \rightarrow \exists x\neg\varphi$, one of the quantifier interchange principles is not provable (the others are).
–: we do have the equivalence between $\neg\varphi$ and $\varphi \rightarrow \bot$, this is a common way to define negation in intuitionistic logic.

Let me indicate why the law of excluded middle is not provable, by giving its proof in classical logic, that is, by using the double negation rule as well:

(1)	$\neg(p \vee \neg p)$	Assumption
(2)	p	Assumption
(3)	$p \vee \neg p$	Introduction of $\vee$ on (2)
(4)	$\bot$	Elimination of $\neg$ on (1) and (3)
(5)	$\neg p$	Introduction of $\neg$ on (2) and (4)
(6)	$p \vee \neg p$	Introduction of $\vee$ on (5)
(7)	$\bot$	Elimination of $\neg$ on (1) and (6)
(8)	$\neg\neg(p \vee \neg p)$	Introduction of $\neg$ on (1) and (7)
(9)	$(p \vee \neg p)$	Double negation rule on (8)

We see here that intuitionistic logic gives us a proof of (8). To conclude (9) from this, we have to use the double negation rule of classical logic, which is not available in intuitionistic logic. There is no way to derive (9) in intuitionistic logic.

As far as conditionals go, as I said above, the intuitionistic analysis is much more attractive than the material conditional of classical logic.

Besides *Modus Ponens* and *Modus Tollens*, we do keep the inferences:

$$(\varphi \vee \psi) \rightarrow \chi \vdash (\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi)$$
$$\varphi \rightarrow (\psi \wedge \chi) \vdash (\varphi \rightarrow \psi) \wedge (\varphi \rightarrow \chi)$$
$$\varphi \rightarrow \psi \vdash \neg(\varphi \wedge \neg\psi)$$

But we don't have the classical troublemakers:

$$\nvdash (\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$$
$$\neg(\varphi \rightarrow \psi) \nvdash \varphi \wedge \neg\psi$$
$$(\varphi \wedge \psi) \rightarrow \chi \nvdash (\varphi \rightarrow \chi) \vee (\psi \rightarrow \chi)$$
$$\varphi \rightarrow (\psi \vee \chi) \nvdash (\varphi \rightarrow \psi) \vee (\varphi \rightarrow \chi)$$

The models that we will be concerned with are information structures.

They consist of a set S of possible information states, partially ordered by a relation of information growth. $s \leqslant s'$ then means that s' contains all the information s does and maybe more.

The branches through s then represent the possible ways in which the information in s can still grow.

We assign truth values 0 or 1 to sentences relative to information states.

We interpret assigning 1 to φ in s as: in s we have enough information to prove φ, or in s we can establish φ.

Given the interpretation of $\leqslant$ as information growth, it is clear that we should impose a monotonicity or stability requirement on this: if s contains enough information to prove φ (we assign 1 to φ in s), then so does every information state where we have even better information (then we assign 1 to every extension s' of s).

It is very important to keep in mind the following: we do *not* interpret 'assigning 0 to φ in s' as 's contains enough information to prove $\neg\varphi$' but as 's does not contain enough information to prove φ'.

From this it is obvious that we do not want to impose a similar monotonicity requirement for 0: we may not be able to prove φ now (s assigns 0 to φ), but we may be later (there may be an extension s' where φ is assigned 1), so on the paths to better information, 0s may change into 1s.

Only if it is no longer possible to extend the information to states where we can prove φ (i.e. only if s only allows extensions where φ is assigned 0) we can say that we have proved $\neg\varphi$.

We will give two ways of doing the semantics, with Kripke models and with Beth models. These two ways correspond to two different opinions that we can take on what the information structures represent.

Think of growth of information as going through successive information states as time passes.

On Kripke's interpretation, the information structure represents just in what possible ways the information can still be extended, regardless of how much time it takes us to get there. Extension of information represents going through successive information states, proving more and more things, eliminating alternatives on the way:

> $s \leqslant s'$ then means: we can proceed from s to better information s', and we don't care how long it may take us.

On Beth's interpretation the information structure represents in what possible ways the information can be extended in a *finite amount of time*. So we interpret $\leqslant$ in a slightly different way:

> $s \leqslant s'$ here means that from s we could, if we want, *actually* proceeed to s', because it is interpreted as: s' is an extension of s that we can reach from s in a finite amount of time.

These different interpretations lead to different semantic clauses. A proof is a finite thing and hence we can only say that we have a proof of φ in s if it takes us a finite amount of time to work out the details, to write down the actual proof. An infinite proof is no proof, we can't say that we have proved φ is it takes us an infinite amount of time to work out the proof.

This is the intuitionistic notion of proof, and we can see that in Kripke's a-temporal approach we will have to put much stronger requirements on what counts as a proof of φ than in Beth's approach.

Suppose that we have come to a stage s where we know that at every branch extending s we will reach a stage where we have written down the actual proof of φ (we reach a state where φ is assigned 1).

On Beth's metaphor, we can now in s announce our proof: we may not have worked through all the details, it may take us some time to actually write it down, but we know that, whatever else happens, we can produce the proof in a finite amount of time.

On Kripke's metaphor we cannot. We have no guarantee that the extensions where we have the actual proof can be reached within a finite amount of time: we have at most an infinite proof, and that just is not a proof for us. Since, for Kripke, there is no guarantee that if $s \leqslant s'$ we can reach s' in a finite amount of time, we can only claim to have a proof of φ in s if in fact in s itself we have the proof.

We see that the two metaphors, just looking at what extensions are possible vs. looking at what extensions we can get to in time, have to lead to a different semantics, because they do agree on what it is a semantics for: they both want to capture that fact that we haven't proved something if it takes us forever to actually construct the proof.

Both semantics do agree on the analysis of negation and conditionals.

A proof for $\neg\varphi$ is a *construction* for deriving a contradiction from every proof for φ. Since in no information state we have enough information to prove a contradiction (no possible mathematical evidence can prove a contradiction), we are guaranteed to have such a construction in s if in s it is no longer possible (conceivable) to find later information states where we have a proof for φ (i.e. if in s we don't have a mathematical argument against φ, and can't conceive of better information proving φ).

A proof for $\varphi \to \psi$ is a construction, a method, for turning every proof of φ into a proof of ψ. Again, we are guaranteed to have such a method if at every possible later stage of evidence where we have enough evidence to prove φ, we also have enough evidence to prove ψ.

Let us, with these interpretations in mind give the semantics. (I don't want to go at this point into the discussion of partially defined objects, and their existence and identity conditions. For that reason I will assume that the predicate logical language does not contain individual constants or identity. For discussion of this in intuitionistic logic, see Scott (1979) and van Dalen's paper.)

Kripke Semantics for Propositional Logic P

A *Kripke frame* for P is a partial order $\langle S, \leqslant\rangle$.

A *Kripke model* for P is a triple $\langle S, \leqslant, i\rangle$ where $\langle S, \leqslant\rangle$ is a Kripke frame and i, the interpretation function is a function

$$i: ATFORM \times S \to \{0, 1\}$$

satisfying the following monotonicity condition:

$$\text{if } s \leqslant s' \text{ and } i(p, s) = 1 \text{ then } i(p, s') = 1$$

Let M be a Kripke model $\langle S, \leqslant, i\rangle$ and $s \in S$. We define the notion $s \Vdash_M \varphi$, φ is true *on the basis of* s (I leave out the subscript M where possible; $s \nVdash_M \varphi$ is interpreted as φ is *false on the basis of* s).

$s \Vdash p$ iff $i(p, s) = 1$
$s \Vdash \neg\varphi$ iff $\forall s' \geqslant s$: $s' \nVdash \varphi$
$s \Vdash \varphi \wedge \psi$ iff $s \Vdash \varphi$ and $s \Vdash \psi$
$s \Vdash \varphi \vee \psi$ iff $s \Vdash \varphi$ or $s \Vdash \psi$

(So we don't accept a proof for $\varphi \vee \psi$ that doesn't tell us which one of them we have proved.)

$$s \Vdash \varphi \to \psi \text{ iff } \forall s' \geqslant s\text{: if } s' \Vdash \varphi \text{ then } s' \Vdash \psi$$

(The algebraic structure that we get by letting the proposition expressed by φ be the set of all information states on the basis of which φ is true, and considering those propositions and operations corresponding to the connectives, is called a Heyting Algebra.) The notion of validity is as usual:

$\varphi_1, \ldots, \varphi_n \Vdash \varphi$ iff for every model $M = \langle S, \leqslant, i\rangle$ for every $s \in S$: if $s \Vdash_M \varphi_1, \ldots, s \Vdash_M \varphi_n$ then $s \Vdash_M \varphi$

Kripke Semantics for Predicate Logic L

A *Kripke frame* for L is a triple $\langle S, \leqslant, D\rangle$ where $\langle S, \leqslant\rangle$ is a partial order, and where D, the *domain function*, assigns to every $s \in S$ a non-empty set D_s. D satisfies the following condition of growing domains:

$$\text{if } s \leqslant s' \text{ then } D_s \subseteq D_{s'}$$

A *Kripke model* for L is a quadruple $\langle S, \leqslant, D, i\rangle$, where $\langle S, \leqslant, D\rangle$ is a Kripke frame and i is an interpretation function for the predicates, such that:

1. $i(P^n, s) \subseteq D_s^n$
2. if $s \leqslant s'$ then $i(P^n, s) \subseteq i(P^n, s')$ (the predicates grow monotonically as well).

Assignments are functions g: $VAR \to \cup_{s \in S} D_s$.

We define $s \Vdash \varphi[g]$.

We get the propositional clauses by inserting gs in the right places; I won't repeat them here, and only give the relevant predicate logical clauses:

$s \Vdash P(x_1, \ldots, x_n)[g]$ iff $\langle g(x_1), \ldots, g(x_n)\rangle \in i(P, s)$
$s \Vdash \exists x\varphi[g]$ iff for some $d \in D_s$: $s \Vdash \varphi\ [g_x^d]$
$s \Vdash \forall x\varphi[g]$ iff $\forall s' \geqslant s$: for all $d \in D_{s'}$: $s' \Vdash \varphi\ [g_x^d]$

Beth Semantics for P

A *Beth frame* for P is a *tree* $\langle S, s_0, \leqslant\rangle$

A *Beth model* for P is a quadruple $\langle S, s_0, \leqslant, i\rangle$, where $\langle S, s_0, \leqslant\rangle$ is a Beth frame and i: $ATFORM \times S \to \{0, 1\}$ satisfying monotonicity:

$$\text{if } s \leqslant s' \text{ and } i(p, s) = 1 \text{ then } i(p, s') = 1$$

Again we define $s \Vdash \varphi$.

This time we can say that we have proved p in s if, whatever will happen, at a certain moment we have a (worked out) proof of p.

In other words, on every branch extending s, p will become true at some moment, in yet other words (using the notion of *bar*, defined in Chapter Two), there is a *bar* B for s, such that p is true in every $s' \in B$.

All the elements of B are possible extensions of s, so we can reach them from s in a finite amount of time. Note that monotonicity tells us that for every branch, from the moment(s) where B intersects that branch onwards (maybe even earlier) p will be true.

$s \Vdash p$ iff there is a bar B for s: $\forall s' \in B$: $i(p, s') = 1$
$s \Vdash \neg\varphi$ iff $\forall s' \geqslant s$: $s' \nVdash \varphi$
$s \Vdash \varphi \wedge \psi$ iff $s \Vdash \varphi$ and $s \Vdash \psi$
$s \Vdash \varphi \vee \psi$ iff there is a bar B for s: $\forall s' \in B$:
$s \Vdash \varphi$ or $s \Vdash \psi$

(So here the requirement is that we have a proof of $\varphi \vee \psi$ if within a finite amount of time we can figure out for which one of them we have a proof.)

$$s \Vdash \varphi \to \psi \text{ iff } \forall s' \geqslant s\text{: if } s' \Vdash \varphi \text{ then } s' \Vdash \psi$$

$\varphi_1, \ldots, \varphi_n \Vdash \varphi$ iff for every model $M = \langle S, \leqslant, s_0, i\rangle$:

$$\text{if } s_0 \Vdash \varphi_1, \ldots, s_0 \Vdash \varphi_n \text{ then } s_0 \Vdash \varphi$$

Beth Semantics for L

The main difference in the models here is that all information states have the same domain.

A *Beth frame* for L is a quadruple $\langle S, \leqslant, s_0, D\rangle$, where $\langle S, \leqslant, s_0\rangle$ is a tree and D is a nonempty set.

A *Beth model* for L is a quintuple $\langle S, \leqslant, s_0, D, i\rangle$ where $\langle S, \leqslant, s_0, D\rangle$ is a Beth frame for L and i is an interpretation function for the predicates satisfying:

1. $i(P^n, s) \subseteq D^n$
2. if $s \leqslant s'$ then $i(P^n, s) \subseteq i(P^n, s')$

Again, I only give the new clauses:

$s \Vdash P(x_1, \ldots, x_n)[g]$ iff there is a bar B for s: $\forall s' \in B$:
$\langle g(x_1), \ldots, g(x_n)\rangle \in i(P, s')$

$s \Vdash \exists x\varphi[g]$ iff there is a bar B for s: $\forall s' \in B$:
there is a $d \in D$: $s' \Vdash \varphi\ [g_x^d]$
$s \Vdash \forall x\varphi[g]$ iff for every $d \in D$: $s \Vdash \varphi\ [g_x^d]$

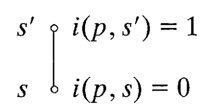
(This clause is simpler than the Kripke clause, because we don't quantify over extensions; in Beth semantics the present clause is equivalent to: $\forall s' \geqslant s$: for every $d \in D$: $s' \Vdash \varphi[g_x^d]$).

Kripke showed that intuitionistic logic is complete both with respect to the Kripke semantics and with respect to the Beth semantics.

In fact, intuitionistic logic is relatively insensitive to what structures you use and what semantics you give on them. If you start out with Kripke frames (just partial orders) and give a Beth semantics, the logic stays the same as well.

(This is quite generally the case in theories of partial information: often there are different ways of giving the semantic clauses that give the same general logic. This doesn't mean that there is no way to choose between them. For instance, it doesn't mean that if we add other operators to the language the theories will make the same predictions about sentences where the old operators interact with the new ones.)

Let me as an illustration give a counterexample to the law of excluded middle. A Kripke model on which $p \vee \neg p$ is false can be very simple:



It is easy to see that $s \nVdash p \vee \neg p$.

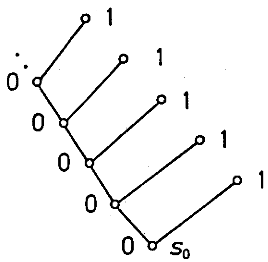
$s \Vdash p \vee \neg p$ iff $s \Vdash p$ or $s \Vdash \neg p$ iff $i(p, s) = 1$ (which is not the case) or $\forall s' \geqslant s$: $i(p, s') = 0$, i.e. $i(p, s) = 0$ and $i(p, s') = 0$ (which is also not the case).

The above model is of course also a Beth model (it is a tree), but as a Beth model it is not a counterexample to the law of excluded middle.

$s \Vdash p \vee \neg p$ iff there is a bar B over s where in every s' either p is assigned 1 or in every extension of s' p is assigned 0. $\{s'\}$ is a bar over s and in s' p is assigned 1, so indeed $s \Vdash p \vee \neg p$.

You have a Beth counterexample if there is a branch where at every stage in time there is an extension where p is assigned 1 and there is an extension where p is assigned 0: since 1 is monotonic, this means that p is 0 in every state on the branch, but every state on the branch has an extension not on the branch where p is 1. The branch cannot have a maximal element, because that would be a state that does not have both a 0 and a 1 extension, so the branch is infinite.

Such a model is a 'Beth comb':



In this model $s_0 \nVdash p \vee \neg p$ because there is no bar B for s_0 where for every $s' \in B$: $i(p, s') = 1$ or $\forall s'' \geqslant s'$: $i(p, s') = 0$.

Such a bar would have to intersect the branch of 0s somewhere, but then it is not the case that in all information states in it p is assigned 1; further there is no s on the branch that has only states as extensions that assign 0 to p, so also the second condition does not hold; hence there is no such bar and indeed $s \nVdash p \vee \neg p$.

3.5. PARTIAL INFORMATION AND VAGUENESS

In intuitionistic logic we interpreted the situation that φ is assigned 0 in s as 'in s we don't (yet) know whether φ'. Our interpretation function i is a total function: it tells you for every atomic statement p and state s whether in s we know that p or whether we don't know that p. Negation, on the other hand is interpreted in a strong way: $\neg p$ is true in s means that in s we know that p is not the case. We thus see a crucial mismatch between the notions $i(p, s) = 0$ and $s \Vdash \neg p$.

In intuitionistic logic, there is, thus, an asymmetry between verifying p and falsifying p: to verify p in s, you just have to have a proof of p available in s; to falsify p in s, you have to have in s a construction showing that you will never be able to construct a proof of p.

The notion of falsification is, in this way, ultimately dependent on the notion of verification.

For other purposes, we may be interested in a theory that has a more symmetric relation between verification and falsification, where we do not treat these relations as interdefinable.

Suppose that we give the following reinterpretation of the interpretation function: $i(p, s) = 1$ means: in s we know p (we have verified p in s); $i(p, s) = 0$ means: in s we know $\neg p$ (we have falsified p in s). In such a theory we would take both the notions of verification and falsification as primitives and we would have to specify recursively for complex formulas both their verification conditions and their falsification conditions.

The crucial difference is, that in such a theory we can no longer allow the interpretation function i to be a total function, i.e. let for every s: $i(p, s) = 1$ or $i(p, s) = 0$. The reason is that we have reinterpreted $i(p, s) = 0$. If i were a total function, then it would hold that for every s and p that we either know that p in s or we know that $\neg p$ in s. We have no way of representing the situation where on s we don't know yet whether p. In order to repair this, we have to assume that i is only a *partial* function, assigns not to every p a value (or assigns it a value $*$, undefined). Then we can represent:

$i(p, s) = 1$: in s we have verified p
$i(p, s) = 0$: in s we have falsified p
$i(p, s) = *$: in s we have neither verified nor falsified p.

This change of interpretation, thus, leads to a *three valued semantics*.

Three valued semantics seems to be very well suited to deal with problems of vagueness.

Think of a vague predicate like *tall*. Clearly, in a certain context, some things will be definitely tall, others definitely not tall, but yet others may be borderline cases.

If we assume that predicates like *tall* are partial predicates in that context, that the interpretation function assigns a *positive extension* of things that are clearly tall and a *negative extension* of things that are clearly not tall, and a gap of things that are borderline cases, we are able to capture that intuition.

Now, such assignment of extensions will take place relative to a certain standard of precision, and – as will become clear – we will be interested in comparing what happens with predicates relative to different standards of precision. In particular, we will be interested to see what happens when we relax or tighten our standards of precision.

We can model these ideas by reinterpreting information states as *precision states*, states where a certain standard of precision is assumed. We then can assume an ordering on precision states of precisification.

Let me give content to these notions in the course of giving a semantics for a vague language. The following discussion is by and large based on Kamp (1975), Fine (1975), although there are some differences in details.

We will consider two languages in this section: L is a standard first order language. L_M is a modal extension of that, that I will introduce later. We will look at L first.

A *vagueness frame* for L is a tuple $F = \langle S, \leqslant, D\rangle$ where:

1. S, the set of *precision states* is partially ordered by $\leqslant$, the relation of *precisification*.
2. D is the domain of individuals (I will assume, for simplicity one domain for all precision states).
3. Let's call the maximal elements in $\langle S, \leqslant\rangle$ the *total precision states*, and indicate them with t.
   Completability: $\forall s \in S \exists t \in S$: $s \leqslant t$

Let me comment on these structures.

The idea is going to be the following: in a state s, we make a decision, relative to the standard of precision in s, on the positive and negative extensions of the predicates.

The states extending s in F are the possible ways (according to F) in which the standard of precision in s could be *relaxed*. This means that if $s \leqslant s'$, objects that according to our standards in s cannot be taken to fall either in the positive extension or the negative extension of a predicate, might possible do so in s', where those standards are relaxed.

Thus, if d falls in the gap of P at s, the statement 'there is an

extension of s where d has P' is interpreted as: 'according to my standard of precision in s, $P(d)$ is undetermined, but if you force me to make a decision (and hence force me to relax my standards), then it is possible for me to relax my standards in such a way (by going to s') that that decision falls out positively'.

Going downwards from s to s', such that $s' \leqslant s$, thus involves tightening of the standards: objects may be removed from the positive extension of a predicate by tightening our standards.

The condition of completability says that for every precision state, there is in F a way of loosening our standards completely, so that everything will fall under either the positive or negative extension of a predicate. I will come back to it, after I introduce the models:

A *vagueness model* for L is a tuple $M = \langle F, i, i^+, i^-\rangle$, where:

1. $F = \langle S, \leqslant, D\rangle$ is a vagueness frame for L.
2. i interprets the individual constants: for every $c \in CON$: $i(c) \in D$.
3. i^+ assigns to every P^n and s the *positive extension* of P^n in s: $i^+(P^n, s) \subseteq D^n$;
   i^- assigns to every P^n and s the *negative extension* of P^n in s: $i^-(P^n, s) \subseteq D^n$
   and: $i^+(P^n, s) \cap i^-(P^n, s) = \emptyset$
4. Both i^+ and i^- are monotonic:
   Monotonicity: if $s \leqslant s'$ then $i^+(P^n, s) \subseteq i^+(P^n, s')$
   if $s \leqslant s'$ then $i^-(P^n, s) \subseteq i^-(P^n, s')$
5. The interpretations are total on total states:
   Totality: for total states t: $i^+(P^n, t) \cup i^-(P^n, t) = D^n$

So indeed our predicates are allowed to have gaps on (non-total) precision states: only on total states does the union of the positive and the negative extension of a predicate necessarily exhaust the whole domain. The positive extension and the negative extension of a predicate plausibly don't overlap. Moreover, each is monotonic: when we relax our standards more objects may count as tall and more as not tall, but the ones that we already decided were tall (or not tall) relative to out tight standards for tallness, stay tall, when we relax those standards.

It will be important for the discussion later to note that our model M can encode certain semantic relations. It is possible, for instance, that in s both d and d' are in the gap of P, but nevertheless, in going to extensions of s, d will always be put into the positive extension of P *before* d'. Similarly, d can be in the gap of both P and Q at s, while no extension of P and Q allows d to be in the positive extension of P and of Q at the same time. This also works downwards: it is also possible that M encodes that every way of removing d and d' from the positive extension of P will remove d' first.

Again for later discussion, we may be interested in restricting ourselves to models for English: those possible models that satisfy certain *meaning postulates* for English (like: for every model for English M and every s: if d is in the positive extension of *boy* in s then d is in the positive extension of *male* in s). Let us call models that satisfy these meaning postulates *background contexts*.

If we have a certain precision state s in a background context M, there are in principle different ways of changing the standard of precision in (s, M).

1. *Extension:* we go from s to $s' \geq s$, but stay in the same background context M (and similarly we can go down).
2. *Refinement:* we stay in s, but change the background context. I.e. we go from M to M' where, say, $\langle S_M, \leq_M, D\rangle \subseteq \langle S_M{}', \leq_{M'}, D'\rangle$. This would be a model that preserves the structure, but makes more *fine grained* distinctions.

These distinctions are related to the discussion of refinement earlier in this chapter. The distinction is important for the notion of completability. We are assuming, for *semantic reasons*, that within a background context, we can always extend a precision state to a total precision state. This will be important, because we will semantically relate the vagueness of a predicate to the possible ways in which the vagueness can be removed.

However, it would be very dubious, if this committed us to the opinion that vagueness can always be removed. We are not making that commitment, though. We are assuming that, *within a background context*, we could temporarily decide to remove all vagueness by going to a total state. But that state is only total relative to that background context: if we *refine* our standards of precision, that state may very well become partial in the new background context. The sense in which vagueness may be essential is that such *refinement* may always be possible (see Landman (1986) for similar discussion in the context of epistemic modals).

We will restrict ourselves to one background context here.

Let's go on to the semantics.

Assignment function g assigns object in D to the variables. Our three-valued semantics will specify two relations:

$s \Vdash_M \varphi[g]$ — φ is *true in s relative to g* (in M)
$s \dashv\vert_M \varphi[g]$ — φ is *false in s relative to g* (in M)

If neither one of these relations holds, then φ is undetermined in s relative to g (in M). We drop subscript M.

First we define the interpretation of the terms as usual:

$$[\![c]\!]_{M,g} = i(c); \qquad [\![x]\!]_{M,g} = g(x)$$

1. $s \Vdash P(t_1, \ldots, t_n)[g]$ iff $\langle [\![t_1]\!]_g, \ldots, [\![t_n]\!]_g\rangle \in i^+(P, s)$
   $s \dashv\vert\, P(t_1, \ldots, t_n)[g]$ iff $\langle [\![t_1]\!]_g, \ldots, [\![t_n]\!]_g\rangle \in i^-(P, s)$
2. $s \Vdash \neg\varphi[g]$ iff $s \dashv\vert\, \varphi[g]$
   $s \dashv\vert\, \neg\varphi[g]$ iff $s \Vdash \varphi[g]$
3. $s \Vdash \varphi \wedge \psi[g]$ iff $s \Vdash \varphi[g]$ and $s \Vdash \psi[g]$
   $s \dashv\vert\, \varphi \wedge \psi[g]$ iff $s \dashv\vert\, \varphi[g]$ or $s \dashv\vert\, \psi[g]$
4. $s \Vdash \varphi \vee \psi[g]$ iff $s \Vdash \varphi[g]$ or $s \Vdash \psi[g]$
   $s \dashv\vert\, \varphi \vee \psi[g]$ iff $s \dashv\vert\, \varphi[g]$ and $s \dashv\vert\, \psi[g]$
5. $s \Vdash \varphi \rightarrow \psi[g]$ iff $s \dashv\vert\, \varphi[g]$ or $s \Vdash \psi[g]$
   $s \dashv\vert\, \varphi \rightarrow \psi[g]$ iff $s \Vdash \varphi[g]$ and $s \dashv\vert\, \psi[g]$
6. $s \Vdash \exists x\varphi[g]$ iff for some $d \in D$: $s \Vdash \varphi[g_x^d]$
   $s \dashv\vert\, \exists x\varphi[g]$ iff for every $d \in D$: $s \dashv\vert\, \varphi[g_x^d]$
7. $s \Vdash \forall x\varphi[g]$ iff for every $d \in D$: $s \Vdash \varphi[g_x^d]$
   $s \dashv\vert\, \forall x\varphi[g]$ iff for some $d \in D$: $s \dashv\vert\, \varphi[g_x^d]$

This semantics gives us a standard strong Kleene three valued logic. Note that indeed, the notions of verification and falsification play a symmetric role here. Unlike in intuitionistic logic, negation switches us simply from verification to falsification and back.

(If we want to look at this semantics algebraically, we can't just take the proposition expressed by φ be the set of states on the basis of which φ is true, because falsity is not definable as not-truth. We can take propositions to be pairs $\langle X, Y\rangle$ where X is the set of states establishing φ and Y is the set of states refuting φ. On such propositions we can define the operators as: $\neg\langle X, Y\rangle = \langle Y, X\rangle$, $\langle X, Y\rangle \wedge \langle Z, V\rangle =$

$\langle X \cap Z, Y \cup V\rangle$, $\langle X, Y\rangle \vee \langle Z, V\rangle = \langle X \cup Z, Y \cap V\rangle$. The resulting structure is called a de Morgan lattice, or also a Kleene algebra.)

We define now as usual for *sentences* φ:

$$\varphi \text{ is true in } s,\ s \Vdash \varphi \text{ iff for every } g\text{: } s \Vdash \varphi\ [g]$$
$$\varphi \text{ is false in } s,\ s \dashv\!\mid \varphi \text{ iff for every } g\text{: } s \dashv\!\mid \varphi\ [g]$$

With this we can then go on the define *three valued entailment and validity*:

$$\Delta \Vdash \varphi \text{ iff for every } M \text{ and } s\text{: if for every } \delta \in \Delta\text{: } s \Vdash_M \delta \text{ then } s \Vdash_M \varphi$$

We should now ask: does the semantics as we have given it here make the right predictions?

Note that this semantics makes a three way distinction: either d has P or d has $\neg P$ or d is in the gap of P in s. It doesn't distinguish in s between two objects d and d' that are both borderline cases of P, but d is more of a P than d'.

One way of trying to accommodate this is to assign to every object a *degree* to which it falls under P, that is, we could replace i^+ and i^- by a function that assigns to every object a value in the real interval between 0 and 1. This is the approach of Fuzzy Logic.

The next task then is to try and find out what the fuzzy value of a complex formula should be, given the fuzzy values of the parts. We won't go into this, because the problems that I will mention apply to Fuzzy logic as strongly as to the above three valued logic. (See Fine and Kamp's papers for a thorough discussion.)

Another problem is the following: we have a theory in which predicates are vague, there is a gap between the good guys and the bad guys, but the line between the good guys and the borderline cases is sharp. This is the problem of higher order vagueness. For discussion, see Fine (1975).

The main problem with the above semantics (and its fuzzy alternatives) is a problem of compositionality: when we look at what the truth values of complex formulas containing vague parts can look like on such approaches, we see that we tend to lose all logic.

For instance, suppose John is a borderline case of a bald man. John is in the gap of *Bald*, so both *Bald*(j) and $\neg$*Bald*(j) are undefined (have an intermediate value) in s. But then $B(j) \wedge \neg B(j)$ is undefined (has an intermediate value) as well: in other words, John will be in the gap of $\lambda x.B(x) \wedge \neg B(x)$, which seems to be as much a contradictory predicate for vague predicates as for sharp ones: there is no reason to expect d to be a borderline case of that predicate.

The general moral that Kamp and Fine draw is that the extension $(+, \text{gap}, -)$ of a complex predicate cannot be a function just of the (three valued or fuzzy) extensions of their parts. (I.e. in the context of vagueness, the connectives are not extensional.)

The reason is, if d is in the gap of P and in the gap of Q, then whether d will be in the gap of a complex predicate built with P and Q depends not only on the extensions of P and Q, *but also on the logical, semantic relations between* P and Q.

To give an example of Fine's: suppose that a blob is on the border between *Red* and *Pink* and that it is also a borderline case of *Small*. (In terms of degrees, let's assume that the degree to which it is *Pink* is the same as the degree to which it is *Small*.)

Intuitively, we probably would want to say that it is a borderline case of a small red blob, but is it a borderline case of a pink red blob?

Extensional theories predict that both sentences, *it is a small red blob* and *it is a pink red blob* should get the same value (because d has the same value for *Small* and for *Pink*) but intuitively they should not. The reason for it is, of course clear: *Pink* and *Red* are contraries: something can't be both *Pink* and *Red*, while *Small* and *Red* are logically independent predicates. The extensional approach does not take the logical relations between the predicates into account.

Why isn't our blob a borderline pink red blob, while it is a borderline small red blob?

On Kamp and Fine's analysis, to be sharp is to draw a sharp border between the good and the bad. The thing that makes a vague predicate vague is not that we couldn't draw a sharp border (we can do anything if we want, and in fact, in practice, we often do), but rather that there are different ways of drawing such a border, between which we cannot make a principled choice.

Take John and Bill. They are both equally bald and they are both borderline case of *Bald*. By making the predicate *Bald* sharp, we could count both among the bald, both among the non-bald, one among the bald the other among the non-bald, and we don't have any means of telling which we should prefer.

We can take this as the crucial idea of the analysis: vague predicates don't draw a sharp border, but they are *potentially sharp*, the reason that they are vague is that they allow different ways of drawing a border. This means that we can (semantically) measure the vagueness of a predicate in terms of the possible ways in which it can be made precise: this will predict that *being a red blob*, *being a pink blob*, *being a small blob*, and *being a small red blob* are vague, because for all these predicates, there are different ways of filling their truth value gap.

But this doesn't hold for the predicate *Being a pink red blob*. The semantic constraints on *Pink* and *Red*, making them contraries, tell us that on every way the vagueness can be removed, the line between the pink red blobs and the non pink red blobs will be drawn in exactly the same place.

How can we account for this formally?

We want to measure the vagueness of a sentence in terms of the different possible ways in which it can be made precise. The notion of truth/falsity in s that we have defined do not do that, they are local.

We need to define a new notion of truth/falsity in s in terms of the possible ways that s can be made total. A way to do that is to use *supervaluations*.

In s we have a partial valuation (a partial specification of truth/falsity). A supervaluation is a total valuation extending that. In our model, a supervaluation of s is a total extension of s.

Let's define a notion of *supertruth/falsity* in s:

φ is *supertrue* in s, $s \models \varphi$ iff $\forall t \geqslant s$: $t \Vdash \varphi$
φ is *superfalse* in s iff $\forall t \geqslant s$: $t \dashv\!| \, \varphi$

And we define validity $\Delta \Vdash \varphi$ in the standard way, but not in terms of $\Vdash$, but in terms of supertruth, $\models$:

Δ *superentails* φ iff for every M and s:
if for every $\delta \in \Delta$: $s \models \delta$ then $s \models \varphi$

The technique of supervaluations was developed by Van Fraassen (see Van Fraassen, 1971). In fact, we have used the idea already in our treatment of truth/falsity in predicate logic.

Remember that we couldn't define truth for formulas, so we defined truth relative to an assignment and after that defined truth as truth relative to all assignments. The result was that the semantics became bivalid for sentences, and that the logic became classical logic, even for

formulas: $P(x) \wedge \neg P(x)$ is a contradiction, even though $P(x)$ and $\neg P(x)$ are neither true nor false.

Exactly the same happens here. Take our blob b. *Pink*(b) is not supertrue in s, nor superfalse: in some total extensions of s, b will be put in the positive extension of *Pink*, in others in the negative extension. The same holds for *Red*(b) and *Small*(b). Also *Red*$(b) \wedge$ *Small*(b) is neither supertrue nor superfalse, because on some total extensions, b will be put both in *Red* and in *Small*, but in some others in *not-Red* or in *not-Small*.

But things are different for *Red*$(b) \wedge$ *Pink*(b). Assuming a meaning postulate for English that says that for no M and s and d: $d \in i^+(red, s)$ and $d \in i^+(pink, s)$, no total extension of s will put b both in *Red* and in *Pink*, so every total extension will put b in the negative extension of $\lambda x.Red(x) \wedge Pink(x)$. Consequently, *Red*$(b) \wedge$ *Pink*(b) is superfalse.

It is not hard to see that, by using the notion of supertruth, the logic of vagueness becomes classical logic. This is because we have made the logic of sharp predicates (i.e. the logic in the precise extensions) classical logic (i.e. we can think of total precision states as classical predicate logical models).

A consequence of this is that also for vague predicates the law of excluded middle holds: if p is a borderline case of *bald*, then still it is logically valid that $B(p) \vee \neg B(p)$. This may not be very intuitive (although Kamp and Fine defend it). But, I think that this is independent of the insight of the analysis.

We can try to change the logic at the total extensions to, say, intuitionistic logic (assume that total information states (total with respect to the precisification relation) are intuitionistic models): then supervaluations would give vague and sharp predicates intuitionistic logic.

Supervaluations, thus, give us a technique for maintaing that, though the semantics of vague predicates and sharp predicates is not the same, yet their logic, the inferences that can be made with either, is the same.

Interesting though this may be, it is not clear that we actually want this. Up to now, we have only used the extensions of s in the final definition of supertruth. But in fact, there may be reasons to use those directly in the semantics of our connectives and other expressions.

In the first place, Kamp's (1975) starting point in giving a semantics for vagueness is the semantics of adjectives and comparatives. Kamp wants to follow the lead of natural language and define the interpretation of the comparative *taller* in terms of the interpretation of the adjective *tall* and the meaning of the comparative. However, if we try to define the truth value of *a is taller than b* in s in terms of a comparative operation and only the interpretation of *tall* relative to s, we will never be able to do that: s does not make any distinction between two objects d and d' that are both in the positive extension of *tall* at s, neither does it make a distinction between d and d' if both are in the gap (or both in the negative extension).

Yet that clearly doesn't mean that such objects are equally tall in s. If we want to express the comparison in this theory, it has to be modal: it has to see what happens at other states, besides s.

Kamp and Partee, in unpublished work, give another example concerning conditionals. Let's assume that we have the following meaning postulates for English:

For every M and s and d:

$$\begin{aligned} &d \in i^+(Boy, s) \text{ iff } d \in i^+(Child, s) \cap i^+(male, s)\\ &d \in i^-(Boy, s) \text{ iff } d \in i^-(Child, s) \cup i^-(male, s)\\ &d \in i^+(Man, s) \text{ iff } d \in i^+(Adult, s) \cap i^+(male, s)\\ &d \in i^-(Man, s) \text{ iff } d \in i^-(Adult, s) \cup i^-(male, s)\end{aligned}$$

Now suppose we have in s Bob, who is a male adolescent, i.e. Bob is definitely male, but in the gap of *boy* and in the gap of *adult* at s. Now look at the following conditional sentences:

(1) If Bob is a child, then Bob is a boy.

(2) If Bob is a child, then Bob is a man.

Of these two sentences, intuitively the first is true, but the second is false. In our language L we would represent them respectively as:

(1′) $Child(b) \rightarrow Boy(b)$

(2′) $Child(b) \rightarrow Man(b)$

But if we evaluate (1′) and (2′) on s, we see that:

$$\begin{aligned} &s \Vdash (1') \text{ iff } b \in i^-(Child, s) \text{ or } b \in i^+(Boy, s)\\ &s \Vdash (2') \text{ iff } b \in i^-(Child, s) \text{ or } b \in i^+(Man, s)\end{aligned}$$

Since, given our meaning postulates, Bob is in the gap of both these predicates in s, both (1′) and (2′), and hence (1) and (2) are undefined in s.

If we want to capture the intuition that (1) is true and (2) is false on s, it seems that we are forced to give a modal analysis of the conditional.

The modal extension of L, L_M is meant to deal with both these problems. We add in L_M three clauses to the language L:

(1) if $\varphi, \psi \in FORM$ then $(\varphi \Rightarrow \psi) \in FORM$

(2) if $\varphi \in FORM$ then $\mathbf{M}\varphi$, $\mathbf{P}\varphi \in FORM$

(3) if $t, t' \in TERM$, $P \in PRED^1$ then $\geqslant(P)(t, t') \in FORM$

This means: t is at least as P as t'.

A *vagueness model* for L_M is a vagueness model $M = \langle S, \leqslant, i, i^+, i^- \rangle$ for L that satisfies the following additional condition:

Limit assumption:

for every s and $P \in PRED^1$ and $d, d' \in D$: there is a unique $s' \leqslant s$, $s' = \min(s, P, d, d')$, such that d and d' are in the gap of P at s', and there is no s'' such that $s'' \neq s'$ and $s' < s'' \leqslant s$ and d and d' are in the gap of P in s''.

s' is the state that you get from s, if you remove d and d' minimally from P in s (this may involve doing other things as well, like removing other objects).

So we assume that there is a unique way of minimally removing both d and d' from P in s. We do not have to make this assumption, but it simplifies things. It is similar to the Limit Assumption of Stalnaker's semantics for counterfactuals (see Stalnaker, 1968; Lewis, 1973).

The semantics for the modal language stays the same, except that we add:

1. $s \Vdash (\varphi \Rightarrow \psi)[g]$ iff for no $s' \geqslant s$:
   $s' \Vdash \varphi[g]$ and $s' \dashv\vdash \psi[g]$

   $s \dashv\vdash (\varphi \Rightarrow \psi)[g]$ iff for some $s' \geqslant s$:
   $s' \Vdash \varphi[g]$ and $s' \dashv\vdash \psi[g]$

So $(\varphi \Rightarrow \psi)$ is true in s if it is not possible to extend s to a state where φ is true but ψ is false. Note that this entails that for every total state extending s where φ is true, ψ is true as well.

2. $s \Vdash \mathbf{M}\varphi[g]$ iff for no $s' \geqslant s$: $s' \dashv\!\mid \varphi[g]$
$s \dashv\!\mid \mathbf{M}\varphi[g]$ iff for some $s' \geqslant s$: $s' \dashv\!\mid \varphi[g]$

M is a necessity operator. $\mathbf{M}\varphi$ is true in s if there is no extension of s where φ is true. Note again that this entails that φ is true in every total extension of s.

3. $s \Vdash \mathbf{P}\varphi[g]$ iff for some $s' \geqslant s$: $s' \Vdash \varphi[g]$
$s \dashv\!\mid \mathbf{P}\varphi[g]$ iff for no $s' \geqslant s$: $s' \Vdash \varphi[g]$

P is a possibility operator: $\mathbf{P}\varphi$ is true in s if there is a possible extension of s where φ is true.

4. $s \Vdash\ \geqslant (P)(t, t')[g]$ iff
$$\min(s, P, [\![t]\!]_g, [\![t']\!]_g) \Vdash P(t') \Rightarrow P(t)[g]$$
$s \dashv\!\mid\ \geqslant (P)(t, t')[g]$ iff
$$\min(s, P, [\![t]\!]_g, [\![t']\!]_g) \dashv\!\mid P(t') \Rightarrow P(t)[g]$$

I will discuss this clause shortly.

Restricted to propositional logic, the above structures (apart from the limit assumption) and semantics is exactly the data semantics given in Veltman (1985). The notion of entailment, $\Delta \Vdash \varphi$, that we have defined before, gives us Veltman's Data logic. In data semantics the states are interpreted as information states and the modal connectives model the epistemic use of conditionals and modals *must* and *may*. See Veltman (1985), Landman (1986), and Landman (1990) for discussion.

Some facts: if φ is an L-sentence then:

φ is supertrue in s iff $s \Vdash \mathbf{M}\varphi$
φ is superfalse in s iff $s \Vdash \mathbf{M}\neg\varphi$

Consequently, $\models \varphi$ iff $\Vdash \mathbf{M}\varphi$.

Let's go back to Bob, our male adolescent in s. We make the same assumptions about s as before. But let's now represent the conditionals with our model implication:

(1″) $Child(b) \Rightarrow Boy(b)$

(2″) $Child(b) \Rightarrow Man(b)$

(1″) is true on s iff there is no extension of s where b is put in the positive extension of *Child*, but in the negative extension of *Boy*. Given the meaning postulates that we have assumed, there is indeed no such extension: Bob is in the positive extension of *Male* in s, hence, by monotonicity in the positive extension of *Male* in every extension of s. In every extension where he is in the positive extension of *Child*, the meaning postulate tells us that he is in the positive extension of *Boy* there. Hence indeed (1″) is true on s.

For (2″) let us make the plausible additional assumption that it is possible to relax our standards in s to make Bob a child:

$$s \Vdash \mathbf{P}(Child(b))$$

Then it is easy to see that (2″) is false in s: our additional assumption says that in some extension of s, Bob is in the positive extension of *Child*. In every such extension Bob is male (by monotonicity), hence in the negative extension of *Man* (by the meaning postulate).

A similar example. Let's assume that every object in the domain is male according to s: $s \Vdash \forall x Male(x)$. In such a situation, if we read (3) as (3′) it will be undefined in s, but if we read it as (3″) it will be true:

(3) Every child is a boy.

(3′) $\forall x[Child(x) \rightarrow Boy(x)]$
(3″) $\forall x[Child(x) \Rightarrow Boy(x)]$

Let's now look at the comparatives: $\geqslant (P)(t, t')$, t is at least as P as t'. Let P be *Tall*. Kamp (1975) discusses the possibility of analyzing this like:

$$s \Vdash P(t') \Rightarrow P(t)$$

i.e. no extension of s makes t' tall but t short.

This works quite well if both t and t' are borderline cases of *Tall*: then it says that for any way of relaxing your standards for *tallness* from s: if you decide there that t' is tall, then you can't make t not tall there.

The problem arises if both t and t' are in the positive (or negative) extension of *tall*. In that case trivially:

$$s \Vdash P(t') \Rightarrow P(t)$$

Kamp's idea (though he doesn't formulate it this way) is: treat this comparative not like an *indicative* conditional, but like a *counterfactual* conditional: if you were to tighten your standards in s minimally so that both t and t' would end up in the gap of *tall*, and there it is the case that no relaxing of the standards puts d' in the positive extension but d in the negative extension of *tall*, then indeed d got to be at least as tall as d'. This is what the clause for $\geqslant(P)(t, t')$ says. (We would need to add some more constraints to make sure that this relation is fully transitive. Think about how this could be done.)

We can give a similar analysis to the real comparative $>(P)(t, t')$, t is taller than t':

$$s \Vdash >(P)(t, t')[g] \text{ iff}$$

1. $\min(s, P, [\![t]\!]_g, [\![t']\!]_g) \Vdash P(t') \Rightarrow P(t)[g]$
2. $\min(s, P, [\![t]\!]_g, [\![t']\!]_g) \Vdash \mathbf{P}(P(t) \wedge \neg P(t'))[g]$

i.e. t is taller than t' iff t is at least as tall as t' and if you minimally move both t and t' into the gap of *tall*, it is possible to make t tall, but t' short.

In this way, then, Kamp is able to give the semantics of the comparative as a (modal) operation that builds the interpretation of the comparative relation *taller than* out of the interpretation of the adjective *tall*.

For a very interesting discussion of how a theory of vagueness of the sort described here can be extended to deal with puzzles of vagueness like the sorites paradox, see Pinkal (1984).

CONSTRUCTIONS WITH PARTIAL ORDERS

4.1. PERIOD STRUCTURES

In the previous two chapters, we have been interested in temporal structures, where the primitives are moments of time or instants. We will now be interested in structures where the basic units are periods (and in the next section, events). In particular, we will be interested in the relations between instant structures and period structures. For a thorough discussion of the topics in this chapter, see van Benthem (1983), Kamp (1979a) and Kamp (1979b). I will be following van Benthem's exposition rather closely here.

We have defined notions of convex sets and intervals earlier. Let's start the present discussion by looking at what period structures we can define using these notions. We start with a partial order of points of time $\langle T, <\rangle$.

Let $I(T)$ be the set of all convex sets in T. We define some natural relations and operations on $I(T)$.

Let $i, i' \in I(T)$

$i < i'$, *i completely precedes* i', iff $\forall t \in i \forall t' \in i' : t < t'$
$i \bigcirc i'$, *i overlaps* i' iff $i \cap i' \neq \emptyset$
$i \sqsubseteq i'$, i is *temporally included in* i' iff $i \subseteq i'$

On the present structures we can define overlap in terms of inclusion:

$$i \bigcirc i' \text{ iff } \exists i_0[i_0 \neq \emptyset \wedge i_0 \subseteq i \wedge i_0 \subseteq i']$$

So two periods overlap if they have a non-empty period in common. Also, on the present structures we can define inclusion in terms of overlap:

$$i \sqsubseteq i' \text{ iff } \forall i_0[\text{if } i_0 \bigcirc i \text{ then } i_0 \bigcirc i']$$

So i is part of i' if every period overlapping with i overlaps i'.

There are some disadvantages and some advantages of the present structure.