Underinvestment and Capital Misallocation Under Sovereign Risk*

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Abstract

Capital and its sectoral allocation affect default incentives. Under general assumptions, default risk is decreasing in the total stock of capital and increasing in the share of capital allocated to non-tradable production. This implies that when competitive households make all investment decisions capital has two externalities: a capital-stock externality and a portfolio externality. These hamper the ability of a benevolent government to make optimal borrowing and default decisions and are exacerbated during periods of distress. Competitive equilibria feature underinvestment, larger non-traded sectors, more default, and lower debt and consumption than a centralized planner's allocation.

Keywords: Sovereign default, Underinvestment, Investment externalities.

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1 Introduction

Output dynamics are at the core of the study of sovereign default risk. Default probabilities depend on expectations about future output and directly affect the borrowing terms that governments face. In environments with capital accumulation, future output depends on investment decisions made in advance and, if productivity is affected by sovereign default, expectations about future default also affect current investment decisions.[1]

This feedback between default risk and investment has important implications for the dynamics of output, capital accumulation, and the allocation of capital in different sectors. The effects of sovereign debt on investment have been widely studied by the literature on "debt overhang". Starting with the work of Krugman (1988) and Sachs (1989), and followed by Aguiar, Amador, and Gopinath (2009), this literature has highlighted how government debt depresses private investment. Regarding the feedback from investment to debt, Gordon and Guerron-Quintana (2018) and Arellano, Bai, and Mihalache (2018) study how investment and the sectoral allocation of capital affect default risk. These papers, however, study environments where a benevolent government makes all borrowing and investment decisions on behalf of households. In this paper I study the feedback between sovereign risk and capital accumulation in an environment with private domestic investment and endogenous default.

My main contribution is to show how the interactions between capital allocations and sovereign risk give rise to two pecuniary externalities: a *capital-stock externality*, which generates inefficient levels of investment, and a *portfolio externality*, which generates inefficient sectoral allocations of capital. These externalities are reminiscent to those studied by the literature on financial crises and macroprudential policies (e.g. Lorenzoni (2008); Bianchi (2011); Bianchi and Mendoza (2018); Bianchi and Mendoza (2020)). The models in this literature feature exogenous collateral constraints linked to market prices, which give rise to a pecuniary externality of private borrowing on future collateral prices. In contrast, I study an environment in which the economy's ability to borrow is endogenously restricted by the market price of government debt, which depends on

[1] In the data, default is accompanied by large declines in output and TFP. Identification of the effect of default on the latter, however, is elusive because low levels of either also increase default incentives. Herbert and Schreger (2017) use legal rulings from a case between private bond holders and the Argentinean government to identify causal effects of default on equity returns. They find that an increase in default probability causes a decline in the value of Argentinean equities, which favors the hypothesis that default carries output and productivity costs.

future default incentives. Kim and Zhang (2012) and Arce (2021) study how, in such an environment, private borrowing inefficiently increases aggregate borrowing costs because households do not internalize how their borrowing affects the government's default incentives. The externalities that I study instead arise from households not internalizing the effect of their capital allocations on the price of sovereign debt. I show that they make borrowing costs inefficiently high for any given aggregate borrowing level, even when it is optimally chosen by a benevolent government. These pecuniary externalities generate aggregate underinvestment, larger non-traded sectors, more default events, and lower levels of debt and consumption relative to an economy where borrowing and investment are centralized.

First, I develop two two-period models of sovereign default to study these externalities. I prove that default incentives are decreasing in the aggregate stock of capital and increasing in the share of capital in the non-traded sector. Both results are consistent with the intuition that capital increases production possibilities and, thus, the ability to repay debt in the future.

The result about the aggregate stock of capital relies only upon the assumption that the cost of default on productivity is positive and weakly increasing, which is a standard assumption in the literature.[2] Capital improves both the value of defaulting and the value of repaying the debt, but the marginal effect on the latter is larger because capital is less productive in default. This implies that the default set shrinks when capital increases.

I study the effect of the sectoral allocation of capital on sovereign risk in an environment with a fixed stock of capital that has to be split between a traded and a non-traded sector. The share of capital in the non-traded sector unambiguously increases default risk as long as traded and non-traded goods are "complementary enough". A certain degree of complementarity is required because the portfolio allocation of capital has an income and a substitution effect on default incentives that counteract each other. The income effect relates to the intuition mentioned above: reducing the share of capital in the traded sector reduces the ability to service foreign debt (as the non-traded good cannot be exported for this purpose). The substitution effect follows from the fact that default changes the composition of the consumption bundle: it decreases consumption of non-traded

[2] Exogenous costs of default with these properties have been used in the quantitative literature because they allow models to generate countercyclical trade balances and default rates, which are observed in the data. Mendoza and Yue (2012) develop a general equilibrium model with production that endogenously generates a cost of default on TFP with these properties.

goods (through lower productivity) and increases that of traded goods (through not servicing the debt). Increasing the share of capital in the non-traded sector unambiguously increases the cost of a potential default. While this, in a sense, could “buy” the sovereign some commitment and reduce default incentives, the shrinkage of the traded sector implies a less balanced consumption bundle in repayment. When the goods are perfect substitutes the sovereign can use all of its traded production to service the debt and substitute this foregone consumption with non-traded goods. When they are complements the temptation to balance the consumption bundle by defaulting increases as the traded sector shrinks, dampening the substitution effect. With enough complementarity this increasing temptation to balance the bundle dominates the gain in commitment from a larger default penalty and default incentives unambiguously increase with a larger non-traded sector.

I then develop a quantitative sovereign default model with short-term debt, capital accumulation, and production in two sectors; which is a standard extension of the two-period models. The main innovation is that I solve for a competitive equilibrium in which households make all investment decisions and compare it to a constrained efficient equilibrium that arises from solving the problem of a benevolent central planner. Both externalities studied in the two-period models arise in this quantitative version and their behavior is consistent with the theoretical results described above.

Under a standard calibration the decentralized equilibrium features aggregate underinvestment and a higher share of capital in the non-traded sector. The constrained efficient allocation can be decentralized with appropriate state-contingent capital subsidies and I study their cyclical behavior. These subsidies are proportional to borrowing, countercyclical, and positively correlated with spreads, indicating that the externalities are amplified in crises. I analyze three simpler subsidy rules: a fixed subsidy, subsidies proportional to spreads, and subsidies proportional to borrowing. These rules yield positive but smaller welfare gains than the efficient allocation. I explore alternative parameterizations with a lower elasticity of substitution and a higher cost of default. While quantitatively different, the main theoretical results hold in both cases. Finally, I consider the same model with long-term debt where the private capital externalities interact with the known debt dilution problem in this class of models. Interestingly, for the chosen calibration the capital externalities limit debt dilution and, as opposed to the case with short-term debt, households in the economy are better off with the decentralized allocation.

Related literature.—This paper builds on the sovereign debt literature following Eaton and Gersovitz (1981). Aguiar and Gopinath (2006) and Arellano (2008) developed quantitative models to study the relationship between default risk and output fluctuations. Gordon and Guerron-Quintana (2018) study an environment with capital accumulation in a single traded sector, and Arellano, Bai, and Mihalache (2018) study an environment with capital and production in traded and non-traded sectors. My quantitative analysis builds on the models in these two papers, but differs in two key dimensions. First, they describe how their quantitative solutions can be decentralized using appropriate subsidies, but do not compute the decentralized equilibrium without them. I compute both equilibria and show that they yield different quantitative results for the same standard calibration. This analysis is important because it implies that parameters that bring the centralized model close to the data may not do so for the decentralized version and *vice versa*.[3] Moreover, this implies that the externalities pose a meaningful burden—in the form of higher spreads and more frequent defaults—and I show that this can be partially alleviated with simple subsidy rules that implement second-best outcomes. Second, I consider separate capital stocks for each sector, while Arellano, Bai, and Mihalache (2018) consider one aggregate stock that is freely allocated within the same period. This is a key distinction because with separate capital stocks the relative size of each sector is a dynamic choice, which has implications for future default incentives.

This paper is also related to the literature that studies disagreement between governments and households in environments without commitment. Aguiar and Amador (2011) study an environment that emphasizes political economy and contracting frictions. The government can default on its debt and expropriate capital from foreigners, which gives rise to slow growth driven by low foreign investment. In my environment slow capital accumulation results from the externalities of capital on borrowing terms. Building on Cole and Kehoe (2000), Galli (2021) studies an economy in which low private investment can be the result of self-fulfilling beliefs about default risk. I make timing assumptions that allow me to rule out the sources of multiplicity introduced by these two papers, which highlights that the externalities I study are orthogonal to theirs. Seoane and Yurdagul (2022) study an environment with production in one-sector, endogenous default risk, and private corporate investment. In their environment there is a similar externality from aggregate investment

[3] I calibrate to the decentralized equilibrium since it is conceptually closer to the targeted data. Both Gordon and Guerron-Quintana (2018) and Arellano, Bai, and Mihalache (2018) calibrate using the centralized equilibrium.

on default risk. My theoretical results regarding the capital-stock externality are complementary to their findings and their quantitative findings are consistent with mine.

Layout.—Section 2 presents the two-period models and the main theoretical results. Section 3 presents the quantitative analysis with an infinite-horizon model. Section 4 concludes.

2 Two-period models

Each of these models highlight one of the following externalities from private investment on sovereign risk: a *capital-stock externality* and a *portfolio externality*. Both models share the environment laid out below and only differ in the production technology for the final consumption good.

There is a small-open economy populated by a measure one of identical households, competitive firms, and a benevolent government. Households have preferences for consumption of a final good in each of the two periods represented by $U(c_0, c_1) = u(c_0) + \beta \mathbb{E}_0 [u(c_1)]$, where u is strictly increasing, concave and invertible, and $\beta \in (0, 1)$ is a discount factor. All goods are produced by competitive firms using capital.

The only source of uncertainty is a productivity shock $z \in \mathbb{R}_+$, which is realized at the beginning of period 1 and has CDF $J(z)$. Productivity in the initial period is normalized to $z_0 = 1$. Households own all the capital and firms in the economy, but do not have access to foreign borrowing. The benevolent government can borrow on behalf of the households in international financial markets. At the beginning of period 0, the budget constraint of the government is $T_0 = q(x_1) B_1 - B_0$, where B_0 is legacy debt that cannot be defaulted on, T_0 is a lump-sum transfer to the households, B_1 is non-contingent defaultable debt that matures in period 1, and $q(x_1)$ is the price schedule for B_1. The vector x_1 contains all payoff-relevant variables that are observable to the lenders when they purchase the debt. Lenders are competitive, risk-neutral, have deep pockets, and have access to a risk-free bond that pays interest rate r^*.

At the beginning of period 1, the government observes z and can choose to repay B_1 by levying a lump-sum tax $-T_1 = B_1$ to the households. Alternatively, the government can default, in which case no tax is levied but real resources are lost in the form of a productivity penalty. Productivity in default is characterized by a function $z_D(z) \leq z$, which is differentiable, has $\frac{\partial z_D}{\partial z} \leq 1$ for all z, and $\lim_{z \to 0} [z - z_D(z)] = 0$. Also, there is a $\bar{z} > 0$ such that for $z \geq \bar{z}$ the inequalities are strict.[4]

The timing of events in period 0 is as follows. First, given B_0 the government chooses B_1 to maximize the lifetime utility of households subject to its budget constraint. The government takes into account how its choices affect household behavior and prices. Then, households observe B_1 and make their decisions. Finally, lenders observe x_1 and purchase the debt for an actuarially fair price

$$q(x_1) = \frac{\int_0^\infty [1 - d(x_1, z)] \, dJ(z)}{1 + r^*} \tag{1}$$

where d is the government's default decision at the beginning of period 1.

Timing and multiplicity.—The above timing assumption rules out multiplicities of equilibria studied by Cole and Kehoe (2000) and by Galli (2021). In Cole and Kehoe (2000) lenders first offer a price schedule and then the government chooses whether to issue B_1 and repay B_0 or to default. For certain regions of the state space this allows for two equilibria: one in which optimistic lenders offer a generous price schedule and the government repays and one in which pesimistic lenders refuse to purchase B_1 and the government defaults. In Galli (2021) lenders observe the amount of debt issued and offer a price before investment is chosen. Lenders' beliefs about investment can be self-fulfilling due to the effect of fiscal policy on households. In my environment multiplicity *a la* Cole-Kehoe is ruled out because lenders price B_1 after the government chooses it and commits to pay B_0; multiplicity *a la* Galli is ruled out because lenders price B_1 after the capital allocation has been chosen.

2.1 Model 1: Capital-stock externality

This model shows how default incentives are decreasing in the stock of capital. The equilibrium allocation is inefficient because households do not internalize the effect of capital on default risk. Relative to the constrained efficient allocation (chosen by a benevolent planner without commitment) the equilibrium allocation features underinvestment.

Technology.—The final good is produced by a competitive firm with technology $y_t = F(z_t, K_t)$, where z_t and K_t are productivity and the aggregate stock of capital, respectively. Function F is

[4] These properties are satisfied by commonly used functions in the literature that feature an exogenous cost of defaulting that is not symmetric and increasing in z. These properties assure that default happens in "bad times" and not in "good times" (see Arellano (2008)), which is a feature of the data.

continuously differentiable, strictly increasing, strictly concave in K, weakly convex in z, and has a positive cross derivative $F_{zK} \geq 0$. The firm rents capital from the households in each period for a rate r_t. Since the firm behaves competitively, the rental rate in each period is $r_t = F_K(z_t, K_t)$.

Households.—At the beginning of period 0 a representative household is endowed with k_0 units of capital that will fully depreciate by the end of the period. The household observes B_1 and chooses consumption c_0 and how much capital to store for the next period k_1 subject to its budget constraint:

$$\begin{aligned} & \max_{c_0, k_1} \{u(c_0) + \beta \mathbb{E}[u(c_1)]\} \\ s.t. \quad & c_0 + k_1 \leq r_0 k_0 + \Pi_0 + T_0 \\ & c_1 = r_1 k_1 + \Pi_1 + T_1 \\ & K_1 = \Gamma_H(B_1) \end{aligned} \tag{2}$$

where Π_t are profits made by the firm and $\Gamma_H(B_1)$ are the household's beliefs about the law of motion of aggregate capital. In period 1 the household consumes all of its available income.

Government.—At the beginning of period 1, the government observes $x_1 = (K_1, B_1)$ and the realization of z and decides whether to repay or default. The default set $\mathcal{D}(x_1) = [0, z^*(x_1))$ is characterized by a cutoff value $z^*(x_1)$ such that

$$F(z^*(x_1), K_1) - B_1 = F(z_D(z^*(x_1)), K_1) \tag{3}$$

where the left-hand-side is consumption under repayment and the right-hand-side is consumption under default. The problem of the government at the beginning of period 0 is

$$\begin{aligned} & \max_{B_1} \left\{ u(c_0) + \beta \int_0^{z^*(x_1)} u(F(z_D(z), K_1))\, dJ(z) \right. \\ & \qquad \left. + \beta \int_{z^*(x_1)}^{\infty} u(F(z, K_1) - B_1)\, dJ(z) \right\} \\ s.t. \quad & c_0 = F(1, K_0) - K_1 + q(x_1) B_1 - B_0 \\ & K_1 = k^*(B_1) \end{aligned} \tag{4}$$

where $k^*(B_1)$ is the capital policy function of the household's problem in (2). The government understands how B_1 affects the aggregate capital allocation; however, as shown below, the lump-sum transfer is insufficient to induce optimal household behavior.

2.1.1 Equilibrium and efficiency

An *equilibrium* is policy functions for the household $c_0(B_1)$, $k^*(B_1)$, household beliefs $\Gamma_H(B_1)$, a quantity of debt issued B_1^*, and a price schedule $q(x)$ such that: (i) given q and $k^*(B_1)$, B_1^* solves the government's problem (4); (ii) given Γ_H, the policy functions $c_0(B_1)$ and $k^*(B_1)$ solve the household's problem (2) for any B_1; (iii) beliefs are consistent $\Gamma_H(B_1) = k^*(B_1)$; (iv) the price q satisfies

$$q(x) = \frac{1 - J(z^*(x))}{1 + r^*}. \tag{5}$$

An *equilibrium allocation* is $\tilde{x} = \left(k^*\left(B_1^*\right), B_1^*\right)$. To characterize the *constrained efficient allocation* consider a benevolent central planner with the ability to choose x_1 at the beginning of period 0 and the ability to default at the beginning of period 1 after observing z. The planner's default set is also characterized by the cutoff $z^*(x_1)$ defined in (3). The problem of the planner at the beginning of period 0 is

$$\begin{aligned} \max_{x_1} \Bigg\{ & u(c_0) + \beta \int_0^{z^*(x_1)} u(F(z_D(z), K_1))\, dJ(z) \\ & + \beta \int_{z^*(x_1)}^{\infty} u(F(z, K_1) - B_1)\, dJ(z) \Bigg\} \\ s.t. \quad & c_0 = F(1, K_0) - K_1 + q(x_1) B_1 - B_0 \end{aligned} \tag{6}$$

which is different from the government's problem (4) because the planner chooses both B_1 and K_1. The *constrained efficient allocation* is $\hat{x}_1$ that solves the planner's problem.

2.1.2 Underinvestment

To simplify notation I use "tilde" variables $\tilde{y}$ for variables (or functions) associated with (or evaluated at) the competitive equilibrium, and "hat" variables $\hat{y}$ for the constrained efficient allocation.

The Euler equation of a representative household (2) is:

$$u'(\tilde{c}_0) = \mathbb{E}\left[\beta u'(\tilde{c}_1)\tilde{r}_1\right] \tag{7}$$

which equates the marginal expected return of capital in $t = 1$ to its marginal cost. The planner's Euler equation for capital is:

$$u'(\hat{c}_0)\left[1 - \widehat{\frac{\partial q}{\partial K_1}}\hat{B}_1\right] = \mathbb{E}\left[\beta u'(\hat{c}_1)\hat{r}_1\right] \tag{8}$$

which introduces another trade off in period 0.[5] An additional unit of capital K_1 has two effects on consumption in $t = 0$. It directly reduces c_0 because the resource constraint is binding; but it also affects default incentives in $t = 1$ and, thus, changes the price of newly issued debt:

$$\frac{\partial q}{\partial K_1} = -\frac{j\left(z^*(x_1)\right)}{1+r^*}\frac{\partial z^*(x_1)}{\partial K_1} \tag{9}$$

where j is the PDF of z and $\frac{\partial z^*(x)}{\partial K}$ is the derivative of the default cutoff with respect to K.

Proposition 1. The default set is shrinking in K_1. That is, $\frac{\partial z^*(x_1)}{\partial K_1} \leq 0$.

Proof: See Appendix A.□

The proof consists of taking the full derivative of equation (3) and using the assumptions on F and z_D to determine the sign of $\frac{\partial z^*(x_1)}{\partial K_1}$. Consider the Cobb-Douglas case $F(z, K) = zK^{\alpha}$ with $\alpha \in (0, 1)$. Fully differentiating (3) with respect to K_1 we get

$$\frac{\partial z^*(x_1)}{\partial K_1} = -\frac{\left[z^*(x_1) - z_D\left(z^*(x_1)\right)\right]}{\left[1 - \frac{\partial z_D(z^*(x_1))}{\partial z}\right]}\frac{\alpha}{K_1} \leq 0$$

where the inequality follows from productivity being lower in default $z_D(z) \leq z$ and from the penalty being increasing in productivity $\frac{\partial z_D(z^*(x_1))}{\partial z} \leq 1$.

Intuitively, capital increases both the value of repayment and default because production possibilities increase. However, the positive effect on the value of repayment is larger because the marginal product in default is lower. More capital increases both sides of (3), but increases the re-

[5]Note that there is no rental rate of capital in the planner's problem. Here $\hat{r}_1 = F_K\left(z, \hat{K}_1\right)$ denotes the marginal product of capital evaluated at z and the planner's choice $\hat{K}_1$, which makes the comparison of equations (7) and (8) straightforward.

payment side more because $z \geq z_D$. A lower z^* decreases both sides of the equation, but decreases the repayment side more because $\frac{\partial z_D}{\partial z} \leq 1$. Then, for the equation to hold z^* must decrease as K increases.

Proposition 1 implies that q is an increasing function of K. There is a trade off between less consumption from setting resources aside for investment and more consumption from a higher ability to borrow that households do not internalize. Under the constrained efficient allocation, the household's Euler equation would be

$$u'(\hat{c}_0) \geq \mathbb{E}\left[\beta u'(\hat{c}_1)\hat{r}_1\right]$$

which is inconsistent with optimal behavior. This illustrates the disagreement between the households and the benevolent government, which is more severe when the desire to borrow is high and when default incentives are more sensitive to capital.

2.2 Model 2: Portfolio externality

The final consumption good is an aggregate of traded and non-traded intermediates. Debt is denominated in terms of the traded good, which is the nummeraire, and the final good is non-traded. Intermediates are produced using capital that has to be installed in each sector one period in advance. The sectoral allocation of capital affects default incentives because default—which only liberates traded resources—has a non-homothetic effect on final consumption. In contrast with the model presented in the previous section the aggregate stock of capital is fixed, which highlights the independent role of its sectoral allocation.

Technology.—The final good is produced by a competitive firm that aggregates traded and non-traded intermediates c_T and c_N, respectively, using technology $Y = F(c_N, c_T)$. The production function F is strictly increasing and strictly concave in both arguments, has positive cross derivatives, and has constant returns to scale. The intermediate goods are produced by competitive firms using Cobb-Douglas production technologies $y_{i,t} = zf(K_{i,t})$, where $i \in \{N, T\}$, $f(K) = K^\alpha$, $0 < \alpha < 1$, and productivity z is the same in both sectors. Intermediate firms rent capital from households at a rate $r_{i,t}$. The resource constraints of the economy are $c_{N,t} = y_{N,t}$, $c_{T,t} = y_{T,t} + T_t$, and $c_t = Y_t$.

Households.—Households own a fixed stock of capital $\bar{k}$ that does not depreciate and cannot be increased, sold, or consumed. Capital can be allocated in either of the two sectors $k_{N,t} + k_{T,t} = \bar{k}$, but this has to be made one period in advance. I normalize $\bar{k} = 1$ to simplify notation, but all the results hold for any $\bar{k} > 0$. Let λ_t be the share of a representative household's capital stock in the traded sector in period t and let Λ_t be the corresponding aggregate share. Households start period 0 with some given λ_0 and choose their portfolio λ_1 to maximize their lifetime utility taking all prices as given. The budget constraint of a representative household in period 0 is $P_0 c_0 = (1-\lambda_0) r_{N,0} + \lambda_0 r_{T,0} + \Pi_0 + T_0$, where P_0 is the relative price of the final good, $r_{N,0}$ and $r_{T,0}$ are the rental rates of capital in each sector, and Π_0 are profits from all firms. In period 1, the household consumes all available income $P_1 c_1 = (1-\lambda_1) r_{N,1} + \lambda_1 r_{T,1} + \Pi_1 + T_1$. The problem of a representative household is then:

$$\max_{\lambda_1} \{u(c_0) + \beta \mathbb{E}[u(c_1)]\} \tag{10}$$

subject to the budget constraints in both periods and to the household's beliefs about the law of motion of the aggregate capital allocation $\Lambda_1 = \Gamma_H(B_1)$.

Government.—At the beginning of period 1 the government observes $x_1 = (\Lambda_1, B_1)$ and z and decides whether to repay or default. The default set $\mathcal{D}(x_1) = [0, z^*(x_1))$ is characterized by a cutoff value $z^*(x_1)$ such that

$$V^D(z^*(x_1), \Lambda_1) = V^P(z^*(x_1), x_1) \tag{11}$$

where the values of default and repayment are

$$V^D(z,\Lambda) = u(F(z_D(z) f(1-\Lambda), z_D(z) f(\Lambda))) \tag{12}$$

$$V^P(z,x) = u(F(zf(1-\Lambda), zf(\Lambda) - B)) \tag{13}$$

respectively, for any (z,x). Equations (12) and (13) highlight the trade-off that the government faces when making its default decision: on one hand, consumption of traded goods increases by not exporting B but, on the other, production of both intermediates decreases. The problem of the government at the beginning of period 0 is:

$$\max_{B_1}\left\{u\left(c_0\right)+\beta\int_0^{z^*(x_1)}V^D\left(z_D\left(z\right),\Lambda_1\right)dJ\left(z\right)\right.$$
$$\left.+\beta\int_{z^*(x_1)}^{\infty}V^P\left(z,x_1\right)dJ\left(z\right)\right\} \tag{14}$$
$$s.t.\quad c_0=F\left(z_0 f\left(1-\Lambda_0\right),z_0 f\left(\Lambda_0\right)+q\left(x_1\right)B_1-B_0\right)$$
$$\Lambda_1=\lambda^*\left(B_1\right)$$

where $\lambda^*\left(B_1\right)$ is the policy function of the household's problem in (10).

2.2.1 Equilibrium and efficiency

The equilibrium definition is analogous to that in Model 1.[6] An *equilibrium allocation* is $\tilde{x}=\left(\lambda^*\left(B_1^*\right),B_1^*\right)$ and the *constrained efficient allocation* is $\hat{x}_1$ that solves the problem of a benevolent planner in period 0:

$$\max_{x_1}\left\{u\left(c_0\right)+\beta\int_0^{z^*(x_1)}V^D\left(z_D\left(z\right),\Lambda_1\right)dJ\left(z\right)\right.$$
$$\left.+\beta\int_{z^*(x_1)}^{\infty}V^P\left(z,x_1\right)dJ\left(z\right)\right\} \tag{15}$$
$$s.t.\quad c_0=F\left(z_0 f\left(1-\Lambda_0\right),z_0 f\left(\Lambda_0\right)+q\left(x_1\right)B_1-B_0\right)$$

which, as in Model 1, is different from the government's problem (14) because the planner chooses both B_1 and Λ_1 directly.

[6]An *equilibrium* is policy functions for the household $c_0\left(B_1\right)$, $\lambda^*\left(B_1\right)$, household beliefs $\Gamma_H\left(B_1\right)$, a quantity of debt issued B_1^*, a price schedule for bonds $q\left(x\right)$, and price functions $P\left(x,z\right)$, $r_N\left(x,z\right)$, $r_T\left(x,z\right)$ and $p_N\left(x,z\right)$, such that: (i) given all price schedules, B_1^* solves the government's problem (14); (ii) given Γ_H and prices, the policy functions $c_0\left(B\right)$ and $\lambda^*\left(B\right)$ solve the household's problem (10) for any B; (iii) beliefs are consistent $\Gamma_H\left(B\right)=\lambda^*\left(B\right)$; (iv) the price q satisfies $q\left(x\right)=\frac{1-J\left(z^*\left(x\right)\right)}{1+r^*}$ with $x=\left(\Lambda,B\right)$ and z^* as defined in (11); and (v) all markets for goods and capital clear.

2.2.2 Misallocation

As with Model 1, I use "tildes" for the competitive equilibrium and "hats" for the efficient allocation. The Euler equation associated with the problem of a representative household (10) is:

$$0 = \mathbb{E}\left[\beta u'\left(\tilde{c}_1\right)\left(\tilde{R}_{N,1} - \tilde{R}_{T,1}\right)\right] \tag{16}$$

where $\tilde{R}_{i,1} = \tilde{r}_{i,1}/\tilde{P}_1$ for $i = T, N$. This resembles a no-arbitrage condition: households allocate capital in each sector in a way such that the expected discounted marginal returns are equated. The planner's Euler equation for the sectoral allocation of capital Λ is:

$$u'\left(\hat{c}_0\right)\frac{\partial \hat{q}}{\partial \Lambda_1}\frac{\hat{B}_1}{\hat{P}_0} = \mathbb{E}\left[\beta u'\left(\hat{c}_1\right)\left(\hat{R}_{N,1} - \hat{R}_{T,1}\right)\right] \tag{17}$$

which illustrates the additional trade off for the planner in period 0.[7] On one hand, Λ_1 affects the aggregate capital portfolio and expected income for period 1, and, on the other, it affects the price of B_1 through its effect on default incentives.

Proposition 2. If the elasticity of substitution η between traded and non-traded intermediates is $\eta < 1$, then the default set is shrinking in Λ_1. That is, $\frac{\partial z^*(x_1)}{\partial \Lambda_1} \leq 0$.

Proof: See Appendix A.□

As with Proposition 1, the proof consists of taking the full derivative of equation (11) and using the properties of F and z_D to determine the sign of $\frac{\partial z^*(x_1)}{\partial \Lambda_1}$. The assumption of $\eta < 1$ is a sufficient condition for the result to hold and is in line with parameterizations and estimates used in the international macroeconomics literature.

To understand the role of this assumption, first note that Λ_1 has two effects on default incentives: an *income* and a *substitution effect*. The *income effect* refers to the fact that the ability to service the debt increases with Λ_1 because debt is denominated in terms of the traded good. Since the non-traded good cannot be shipped to pay for the debt, higher debt implies that the traded good is relatively scarcer. Thus, a higher Λ_1 reduces default incentives as repaying becomes less painful.

[7] As in Model 1, the only relative price that the planner faces is q. To ease exposition, denote $\hat{R}_{N,1} = \frac{\hat{p}_{N,1}}{\hat{P}_1} z_1 f'\left(1 - \Lambda_1\right)$, $\hat{R}_{T,1} = \frac{z_1 f'\left(\Lambda_1\right)}{\hat{P}_1}$, $\hat{p}_{N,t} = \frac{\partial F}{\partial c_N} / \frac{\partial F}{\partial c_T}$, $\hat{P}_t = 1/\frac{\partial F}{\partial c_T}$. These variables are akin to their decentralized counterparts because there is no "static inefficiency" in this model. Given the same (z_1, x_1) the planner would choose the same $c_{N,1}$ and $c_{T,1}$ as the decentralized economy.

For the *substitution effect*, note that at z^* default unambiguously reduces c_N and increases c_T. As Λ_1 decreases, the potential loss of c_N from default increases with the size of this sector. For high values of Λ_1 the pain from reducing consumption of an already small non-traded sector could be large given the concavity of the production functions. Reducing Λ_1 gets the government (or the planner) some commitment by increasing the potential cost of default. In summary, higher Λ_1 reduces default incentives through the *income effect* and could potentially increase them through the *substitution effect*. High complementarity of c_N and c_T ensures that the *substitution effect* never dominates.

Consider the extreme case in which c_N and c_T are perfect substitutes. Then, F is a linear combination of c_N and c_T and equation (11) becomes:

$$\omega z_D\left(z^*\right) f\left(1-\Lambda_1\right)+(1-\omega)\, z_D\left(z^*\right) f\left(\Lambda_1\right)=\omega z^* f\left(1-\Lambda_1\right)+(1-\omega)\left[z^* f\left(\Lambda_1\right)-B_1\right]$$

with $\omega \in (0,1)$. Taking the full derivative with respect to Λ_1 and letting $\omega = 0.5$ for simplicity we get:

$$\frac{\partial z^*}{\partial \Lambda}=-\frac{\left[z^*-z_D\left(z^*\right)\right]\left[f'\left(\Lambda_1\right)-f'\left(1-\Lambda_1\right)\right]}{\left[1-\frac{\partial z_D\left(z^*\right)}{\partial z}\right]\left[f\left(1-\Lambda_1\right)+f\left(\Lambda_1\right)\right]} \tag{18}$$

where the denominator is clearly positive since $\frac{z_D}{\partial z} \leq 1$. From concavity of f, it follows that for large enough values of Λ the numerator is negative, which implies that default incentives increase as Λ increases. This is because the marginal product of capital in the non-traded sector is so large that the marginal decrease in the cost of default from an increase in Λ—the *substitution effect*—overwhelms the marginal increase in the ability to pay—the *income effect*. The more complementary c_N and c_T are, the less overwhelming the substitution effect becomes. This is because unbalanced bundles are less efficient than balanced ones. A sufficient condition for the income effect to always dominate is an elasticity of substitution that is less than 1 (see the proof in Appendix A).

Proposition 2 implies that q is an increasing function of Λ (for $\eta < 1$). This implies a trade-off between increasing non-traded consumption in period 1 and increasing traded consumption in period 0 through higher borrowing. Under the constrained efficient allocation, the household's

Euler equation would be

$$0 \leq \mathbb{E}\left[\beta u'\left(\hat{c}_1\right)\left(\hat{R}_{N,1} - \hat{R}_{T,1}\right)\right]$$

which is inconsistent with optimal behavior. From the household's point of view, there are excess returns to capital in the non-traded sector, which implies that households underinvest in the traded sector and overinvest in the non-traded. As in Model 1, the disagreement is more severe when the desire to borrow is high and when default incentives are more sensitive.

3 Quantitative analysis

I now extend the environment from Section 2 to an infinite-horizon model that features both externalities. The model builds on the existing literature that follows the seminal work of Eaton and Gersovitz (1981) and is closely related to the models in Arellano, Bai, and Mihalache (2018) and Gordon and Guerron-Quintana (2018). The main innovation is to contrast an economy with decentralized investment with an economy where all allocations are chosen by a central planner. The planner's allocation can be decentralized with state-contingent subsidies to capital in each sector. I analyze the cyclical properties of these state-contingent subsidies and evaluate the implementation of simpler subsidy rules.

3.1 Environment

There is a small-open economy populated by a measure one of identical households and a benevolent government. Households own all capital and firms in the economy but lack access to foreign borrowing. The government borrows on behalf of the households by issuing sort-term non-contingent debt in international financial markets and lacks commitment to repay it.

Preferences and technology.—Households have preferences for streams of consumption of a final non-traded good represented by $U\left(\{c_t\}_{t=0}^{\infty}\right) = \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u\left(c_t\right)\right]$, where $0 < \beta < 1$ is a discount factor and $u\left(c\right) = \frac{c^{1-\sigma}}{1-\sigma}$. The final good is produced by a competitive firm using technology $F\left(c_N, c_T\right) = \left[\omega^{\frac{1}{\eta}} c_N^{\frac{\eta-1}{\eta}} + \left(1-\omega\right)^{\frac{1}{\eta}} c_T^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$, where c_N and c_T are traded and non-traded intermediate goods and η is the elasticity of substitution. All prices are denominated in terms of the intermediate traded good. The relative price of the non-traded intermediate is $p_N = \left(\frac{\omega}{1-\omega}\frac{c_T}{c_N}\right)^{\frac{1}{\eta}}$ and

$P = \left[\omega p_N^{1-\eta} + (1-\omega)\right]^{\frac{1}{1-\eta}}$ is the price index of the final good. Intermediate goods $j \in \{N,T\}$ are produced by competitive firms using technology $y_j = zK_j^{\alpha}$, where $\alpha \in (0,1)$, K_j is capital in sector j, and z is a productivity shock. Productivity follows an AR(1) process $\log z_t = \rho \log z_{t-1} + \epsilon_t$, where $\rho \in (0,1)$ is a persistence parameter and $\epsilon_t \sim N\left(0, \sigma_z^2\right)$. There are two stocks of capital in the economy (one for each sector) which depreciate at a rate δ. Capital is owned by the households and rented to the firms for a rate r_j, where $r_N = p_N \alpha z K_N^{\alpha-1}$ and $r_T = \alpha z K_T^{\alpha-1}$. The final good is purchased by the households and can be used for consumption and investment. Households make their investment goods and the cost, in units of the final good, of producing i_j units of the investment good j is $i_j + \Psi\left(i_j, k_j\right)$, where $\Psi\left(i_i, k_i\right) = \frac{\phi}{2}\frac{(i_i)^2}{k_i}$.[8] The shadow price of investment good j in terms of consumption is $P_{k,j} = 1 + \phi\frac{I_j}{K_j}$. The budget constraint of a representative household is:

$$P_t\left(c_t + \sum_{j\in N,T}\left[i_{j,t} + \Psi\left(i_{j,t}, k_{j,t}\right)\right]\right) = \sum_{j\in N,T}\left(r_{j,t}k_{j,t}\right) + \Pi_t + T_t \tag{19}$$

where T_t is a lump-sum transfer from the government and Π_t are the profits made by all firms in the economy. The law of motion of capital in sector j is

$$k_{j,t+1} = i_{j,t} + (1-\delta)\,k_{j,t} \quad j \in \{N,T\} \tag{20}$$

Government debt and default.—The government is benevolent and can issue short-term, non-contingent debt. At the beginning of each period the government observes the state of the economy and, if it is in good financial standing, decides whether to repay or default. If the government repays it gets to issue new debt B_{t+1} for a price q_t. Debt is purchased by risk-neutral competitive lenders with deep pockets and access to a risk-free bond with interest rate r^*. The government's budget constraint in repayment is $T_t = q_t B_{t+1} - B_t$, where T_t is a lump-sum transfer (or tax) of the traded good to the households. If the government defaults, then $T_t = 0$ and it gets excluded from financial markets. When the government is in autarky, it gets readmitted to financial markets with probability θ and zero debt. Also, when the government is in default productivity is $z_D(z) = z - \max\left\{0, d_0 z + d_1 z^2\right\}$, with $d_0 < 0 < d_1$.[9]

[8] Capital cannot be imported or exported directly. This assumption captures the idea that productive capital has a significant non-traded component, usually in the form of construction or land.

[9] This formulation of the cost of default follows Chatterjee and Eyigungor (2012) and Gordon and Guerron-

Timing within a period.—At the beginning of each period z is realized. Then, the government observes the state of the economy and decides whether to repay or default. If the government repays then it chooses borrowing to maximize household utility subject to its budget constraint, taking as given the price schedule q_t and how households will respond to policy. The government can commit to policy within the same period. Households then observe the government's policy and make their investment decisions. Finally, lenders observe borrowing and investment decisions and purchase the government debt.

3.2 Recursive formulation

The aggregate state of the economy is (x,z), where $x = (B,K)$ and $K = (K_N,K_T)$. Denote the government policy as $g = (d,B',T)$, where d is the current default state, B' is debt chosen for the next period, and T is the lump-sum transfer.

Static production equilibrium.—A *static production equilibrium* is prices r_N, r_T, p_N and P, and intermediate consumption allocations c_N and c_T such that, given (g,K',x,z) (i) the market for the final good clears, (ii) the market for the non-traded intermediate good clears, (iii) the markets for capital clear, and (iv) the balance of payments $c_T + B = y_T + q(x',z)B'$ holds ($q(x',z)$ is defined below).

Households.—Households take all prices as given. The individual state of a representative household is $k = (k_N,k_T)$. The value of a household when the government is in good financial standing is:

$$H^P(g,x;k,z) = \max_{c,i_N,i_T,k'} \left\{u(c) + \beta\mathbb{E}\left[d'H^D(K';k',z')\,|z\right] \right. \\ \left. +\beta\mathbb{E}\left[(1-d')H^P(g',x';k',z')\,|z\right]\right\} \tag{21}$$

where the maximization problem is subject to the household's budget constraint (19) in repayment, the laws of motion for capital (20), household's beliefs about the evolution of aggregate capital stocks in repayment $K' = \Gamma_K^P(g,x,z)$, and households beliefs about future government policy $g' =$

Quintana (2018). Note that, except for differentiability, this function for productivity in default satisfies all of the assumptions in Section 2.

$\Gamma_g(x', z')$. When the government is in default, the value of a representative household is:

$$H^D(K;k,z) = \max_{c,i_N,i_T,k'} \left\{u(c) + \beta(1-\theta)\mathbb{E}\left[H^D(K';k',z')|z\right]\right. \tag{22}$$
$$\left. + \beta\theta\mathbb{E}\left[(1-d')H^P(g',x';k',z') + d'H^D(K';k',z')|z\right]\right\}$$

where the maximization problem is subject to the household's budget constraint (19) in default, the laws of motion for capital (20), household's beliefs about the evolution of aggregate capital stocks in default $K' = \Gamma_K^D(g,x,z)$, and households beliefs about future government policy $g' = \Gamma_g(x',z')$ with $x' = (0, K')$.

Government.—The value of the government at the beginning of a period in good financial standing is

$$G(x,z) = \max_{d\in\{0,1\}} \left\{dG^D(K,z) + (1-d)G^P(x,z)\right\}. \tag{23}$$

The value of default is

$$G^D(K,z) = u\left(c^D(K;K,z)\right) + \beta(1-\theta)\mathbb{E}\left[G^D(K',z')|z\right]$$
$$+ \beta\theta\mathbb{E}\left[G(x',z')\right]$$

where $x' = (0, K')$, $K' = k^D(K;K,z)$, and k^D and c^D are the household's policy functions for capital and consumption in default. The value of repaying is

$$G^P(x,z) = \max_{T,B'} u\left(c^P(g,x;K,z)\right) + \beta\mathbb{E}\left[G(x',z')|z\right] \tag{24}$$
$$s.t. \quad T = q(x',z)B' - B$$
$$x' = \left(B', k^P(g,x;K,z)\right)$$

where k^P and c^P are the household's policy functions for capital and consumption in repayment, and q is the price schedule of B'. The government's policy function is $g(x,z) = (d(x,z), B(x,z), T(x,z))$. Since lenders are risk neutral, the price q is actuarially fair:

$$q(x',z) = \mathbb{E}\left[\frac{1-d(x',z')}{1+r^*}\right] \tag{25}$$

where d denotes the lenders' beliefs about the default policy in the next period. The dependence on x' follows from the timing assumption (i.e. the auction happens after all investment and borrowing decisions have been made).

Competitive equilibrium.—A *competitive equilibrium* is value and policy functions for the household, value and policy functions for the government, household beliefs, prices r_N, r_T, p_N and P, intermediate consumption allocations c_N and c_T, and a price schedule $\tilde{q}$, such that: (i) given all prices and government policy functions, the value and policy functions for the household solve the problems in (21) and (21) for $g = g(x,z)$; (ii) given all prices and household's policy functions, the value and policy functions for the government solve the problems in (23) and (24); (iii) household beliefs are consistent $\Gamma_K^P = k^P(g,x;K,z)$, $\Gamma_K^D = k^D(K;K,z)$, $\Gamma_g = g(x,z)$; (iv) the price schedule $\tilde{q}$ satisfies equation (25) with $d' = d(x,z)$; and (v) prices r_N, r_T, p_N and P, and intermediate consumption allocations c_N and c_T are a static production equilibrium for all (g,K',x,z).

The functions $\tilde{K}^P(x,z) = k^P(g(x,z),x;K,z)$ and $\tilde{K}^D(K,z) = k^D(K;K,z)$ describe the evolution of capital along the competitive equilibrium path.

3.3 Efficiency and decentralization

Consider now a benevolent social planner that can choose all allocations in the economy. The value of the planner in good financial standing is

$$V(x,z) = \max_{d\in\{0,1\}} \left\{ dV^D(K,z) + (1-d)V^P(x,z) \right\}.$$

Denote $\hat{d}(x,z)$ as the default policy function that solves the above maximization problem. The value of default is:

$$V^D(K,z) = \max_{c,I_N,I_T,K'} \left\{ u(c) + \beta(1-\theta)\mathbb{E}\left[V^D(K',z')\right] + \beta\theta\mathbb{E}\left[V(x',z')\right] \right\}$$

where the maximization problem is subject to the laws of motion for capital $K_j' = I_j + (1-\delta)K_j$ for $j = N,T$, and the resource constraints in default $c + \sum_{j\in N,T}\left[I_j + \Psi\left(I_j,K_j\right)\right] \le F(c_N,c_T)$, $c_N = z_D(z)K_N^{\alpha}$ and $c_T = z_D(z)K_T^{\alpha}$. Denote $\hat{K}^D(K,z)$ as the planner's policy function for capital in default. The value of repayment is

$$V^P(x,z) = \max_{c,I_N,I_T,x'} \{u(c) + \beta\mathbb{E}[V(x',z')]\}$$

where the maximization problem is subject to the laws of motion for capital, and the resource constraints in repayment $c + \sum_{j\in N,T}\left[I_j + \Psi\left(I_j, K_j\right)\right] \le F(c_N, c_T)$, $c_N = zK_N^\alpha$ and $c_T = zK_T^\alpha + \hat{q}(x',z)B' - B$. Here, $\hat{q}$ is the price schedule for bonds issued by the planner. Denote $\hat{K}^P(x,z)$ and $\hat{B}(x,z)$ as the capital and debt policy functions for the planner in repayment.

A *constrained efficient equilibrium* is value and policy functions for the planner and a price schedule $\hat{q}$ such that: (i) given $\hat{q}$, the value and policy functions solve the planner's problem; and (ii) the price schedule $\hat{q}$ satisfies equation (25) with $d' = \hat{d}(x,z)$.

Discussion.—As with the two-period models, capital allocations in the competitive equilibrium are inefficient because households fail to internalize how these affect future default incentives and, thus, present borrowing costs. The Euler equations of a representative household when the government is in good financial standing are:

$$u'(\tilde{c})\tilde{P}_{k,j} = \beta\mathbb{E}\left[\tilde{d}'u'(\tilde{c}')\tilde{R}'_j|z\right] + \beta\mathbb{E}\left[\left(1-\tilde{d}'\right)u'(\tilde{c}')\tilde{R}'_j|z\right] \quad j\in\{N,T\} \tag{26}$$

where $\tilde{R}_j$ is the return to capital in sector j in terms of the final consumption good; $\tilde{c}$ is the household's policy function for consumption; and $\tilde{d}$ is the government's policy function for default.[10] To ease exposition, primes indicate variables dependent on the state in the next period. Tildes denote prices evaluated at the states induced by the functions $\tilde{d}$, $\tilde{K}^D$, $\tilde{K}^P$, and $\tilde{B}$ defined above, as well as other competitive equilibrium policy functions.[11]

Using similar notation, the Euler equations for capital from the planner's problem in repayment can be written as

$$u'(\hat{c})\left[\hat{P}_{k,j} - \frac{\partial\hat{q}}{\partial K'_j}\frac{\hat{B}'}{\hat{P}}\right] = \beta\mathbb{E}\left[\hat{d}'u'(\hat{c})\hat{R}_j|z\right] + \beta\mathbb{E}\left[\left(1-\hat{d}'\right)u'(\hat{c})\hat{R}_j|z\right] \quad j\in\{N,T\} \tag{27}$$

[10]The return to capital in sector j is $\tilde{R}_j = \frac{\tilde{r}_j}{\tilde{P}} + (1-\delta)\tilde{P}_{k,j} - \tilde{\Psi}_{2,j}$ where $\tilde{\Psi}_{2,j}$ is the derivative of the adjustment cost function with respect to its second argument.

[11]For instance $\tilde{r}_N$ is really a function of the aggregate state and government policy. If $\tilde{d} = 1$ then $\tilde{r}_N(g,x,z) = p_N(g,x,z)\alpha z_D(z)K_N^{\alpha-1}$, with $p_N((1,0,0),x,z) = \left(\frac{\omega}{1-\omega}\frac{z_D(z)K_T^\alpha}{z_D(z)K_N^\alpha}\right)^{\frac{1}{\eta}}$.

where prices have the same functional form described above but the hat indicates that they are evaluated at allocations induced by the planner's policy functions.

The terms $-\frac{\partial \hat{q}}{\partial K'_j}\frac{\hat{B}'}{\hat{P}^P}$ on the left-hand-side of equation (27) indicate how resources borrowed by the planner change with investment. This margin is ignored by the households because they take the evolution of aggregate capital as given. The magnitude of the disagreement depends on the planner's desire to borrow (i.e. the optimal borrowing choice) given the state, on the real exchange rate (defined as $1/P$), and on the sensitivity of the planner's price schedule $\hat{q}$ to investment. Note that, absent default risk, $\hat{q}$ would be constant and the disagreement would vanish.

Proposition 3. (*State-contingent subsidies*) The constrained efficient equilibrium can be implemented as a competitive equilibrium with state-contingent subsidies to future capital equal to $\tau_{j,t} = \tau_j(x,z) = \frac{\partial \hat{q}(\hat{x}',z)}{\partial K'_j}\frac{\hat{B}'}{\hat{P}(x,z)}$ in repayment and $\tau_j = 0$ in default, where $\hat{x}' = \left(\hat{B}'(x,z), \hat{K}^P(x,z)\right)$ is the planner's policy for capital and debt when the state is (x,z). As is clear in the proof below, these subsidies are denominated in units of the final good for convenience of exposition.

Proof: With subsidies $\tau_{j,t}$ to each future stock of capital $k_{j,t+1}$ the household's budget constraint in repayment is now

$$P_t\left(c_t + \sum_{j\in N,T}\left[i_{j,t} + \Psi\left(i_{j,t}, k_{j,t}\right) - \tau_{j,t}k_{j,t+1}\right]\right) = \sum_{j\in N,T}\left(r_{j,t}k_{j,t}\right) + \Pi_t + T_t.$$

Then, the left-hand-side of equation (26) becomes $u'(\tilde{c})\left[\tilde{P}_{k,j} - \tau_j(x,z)\right]$, which makes the household's Euler equation identical to the planner's (27) when $\tau_j(x,z) = \frac{\partial \hat{q}(\hat{x}',z)}{\partial K'_j}\frac{\hat{B}'}{\hat{P}(x,z)}$.$\square$

3.4 Computation and calibration

I use value function iteration to solve for both the competitive and the constrained efficient equilibrium. Following Hatchondo, Martinez, and Sapriza (2010), I compute the limit of the finite-horizon version of the economy. In the constrained efficient case I jointly solve for investment and borrowing decisions using a non-linear optimization routine. In the competitive case I solve for capital such that the household's Euler equations hold for a given borrowing level. To find the optimal borrowing level I use a non-linear optimization routine where the objective function takes into account how debt affects the solution to the household's Euler equations. I approximate value functions and the price schedule using linear interpolation, and compute expectations over the productivity shock using a Gauss-Legendre quadrature.

A period in the model is one quarter. The calibration strategy follows closely Gordon and Guerron-Quintana (2018). There are two sets of parameters: one with values taken from the literature and another chosen to match some empirical moments for Argentina in the decentralized economy. The calibration is summarized in Table 1.

Table 1: Parameter values

Parameter	Value	Parameter	Value	Parameter	Value
σ	2	r^*	0.01	ρ	0.95
η	0.83	ω	0.6	σ_z	0.017
α	0.36	δ	0.05	θ	0.053
Parameter	**Value**	**Moment**	**Data**	**Decentralized (targeted)**	**Planner**
β	0.87	$\frac{B}{GDP}$	0.33	0.34	0.46
ϕ	38.2	$\frac{\sigma_i}{\sigma_{GDP}}$	2.65	2.67	2.99
d_0	-0.28	Av (spread)	0.08	0.07	0.10
d_1	0.35	Std (spread)	0.04	0.12	0.12

To compute the model moments I draw 100 samples of 1,400 periods and drop the first 1,000. Spreads are computed as $r_t - r_a^*$, where $1+r_t = \left[\frac{1}{q_t}\right]^4$ and $r_a^* = (1+r^*)^4 - 1$ is the annualized risk-free rate.

The risk-free interest rate is $r^* = 0.01$ and the CRRA parameter is $\sigma = 2$, which are standard values in business cycle and sovereign default studies. The elasticity of substitution between traded and non-traded goods is $\eta = 0.83$ and the share of non-traded is $\omega = 0.6$; both of which I take from Bianchi (2011). Values for η used in the literature range between 0.4 and 0.83.[12] The capital depreciation rate is $\delta = 0.05$, which is a standard value in the literature. Following Gordon and Guerron-Quintana (2018), I set the capital share to $\alpha = 0.36$ and the parameters governing the stochastic process for productivity to $\rho = 0.95$ and $\sigma_z = 0.017$. The probability of reentry $\theta = 0.053$ is set so that the average exclusion period after default is 4.7 years, which is the average duration documented by Gelos, Sahay, and Sandleirs (2011).

The discount factor β, the productivity loss parameters d_0 and d_1, and the capital adjustment cost parameter ϕ are set to jointly match an average debt-to-GDP ratio of 0.33, a relative volatility of total investment to GDP of 2.65, average spreads of 8 percent, and a standard deviation of spreads of 4 percent.[13] The lower part of Table 1 reports these moments for the decentralized

[12] See Stockman and Tesar (1995), Mendoza (2005), and Bianchi (2011). Subsection 3.7 below shows how the main results are unchanged with an elasticity of $\eta = 0.4$.

[13] Spreads in the model $spr = (1+r)^4 - (1+r^*)^4$ are calculated using the quarterly yield r implied by $q = 1/(1+r)$.

equilibrium (which are targeted) and for the planner's problem (which are not targeted). The planner can sustain higher levels of debt and spreads, which shows how the inability to coordinate borrowing and investment in the decentralized case limits the economy's ability to borrow. As a result, investment is more volatile in the centralized economy since it responds both to productivity and borrowing conditions. Spreads are more volatile in the centralized case because the planner is able to avoid default events by adjusting both borrowing and investment (see the first column in Table 2).

Table 2: Untargeted moments

	Pr (default)	$\frac{\sigma_c}{\sigma_{GDP}}$	σ_{GDP}	$\sigma_{tb/GDP}$	σ_{RER}	$\rho_{y,r-r^*}$	$\rho_{y,tb/y}$	$\rho_{y,RER}$
Data	0.03	1.23	4.82	2.34	3.03	-0.79	-0.68	0.05
Decentralized	0.012	1.37	2.64	2.71	5.05	-0.32	-0.21	-0.21
Planner	0.003	1.94	2.38	3.92	6.76	-0.41	-0.36	-0.38

To compute the model moments I draw 100 samples of 1,400 periods and drop the first 1,000. Spreads are computed as $r_t - r_a^*$, where $1 + r_t = \left[\frac{1}{q_t}\right]^4$ and $r_a^* = (1 + r^*)^4 - 1$ is the annualized risk-free rate.

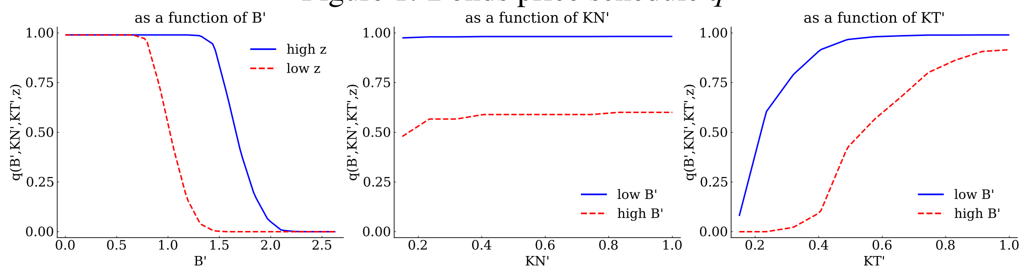
Table 2 reports other moments that are not targeted in the calibration exercise. Overall, the model's cyclical behavior is in line with the data. Spreads and the trade balance are countercyclical, and consumption is more volatile than GDP. The real exchange rate in the model is more volatile than GDP and countercyclical, while it is less volatile and uncorrelated with GDP in the data.[14]

3.5 Underinvestment and sectoral misallocation

Similar to the two-period models, equations (26) and (27) show that the signs of the derivatives of $\hat{q}$ determine whether the competitive equilibrium features over- or under-investment in each sector. Figure 1 illustrates how $\hat{q}$ is decreasing in borrowing B' (left panel), and increasing in both types of capital K_N' (middle panel) and K_T' (right panel). Moreover, $\hat{q}$ is much more sensitive to capital in the traded sector than to capital in the non-traded sector and steeper with higher borrowing.

[14] It is a well known fact that standard models of international business cycles do a poor job in replicating the cyclical behavior of exchange rates. Fitting this behavior is beyond the scope of this paper.

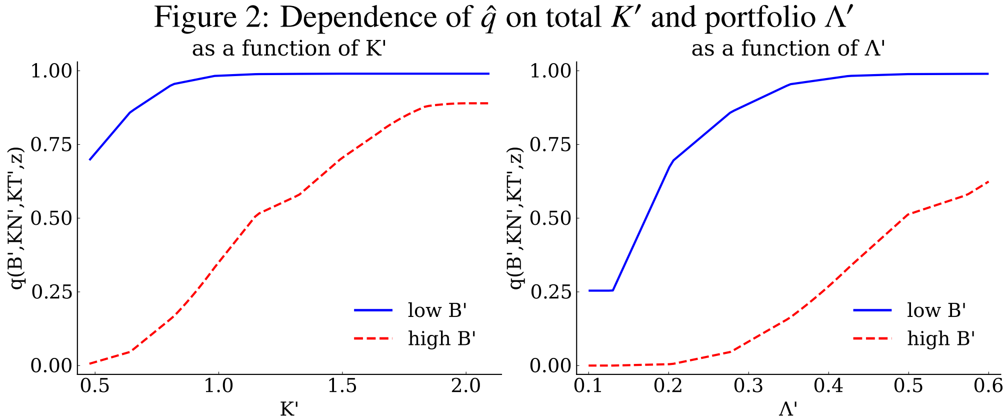
Figure 1: Bonds price schedule $\hat{q}$



In the left panel, the shock is set to its average plus and minus one standard deviation in the solid-blue and dashed-red lines, respectively. In the middle and right panel the shock is set to its average. In the middle panel, K_T' is set to its average in the ergodic distribution. In the right panel, K_N' is set to its average. Finally, B' is set to its average in the solid-blue lines, and to its average plus one standard deviation in the dashed-red lines.

To understand why this is the case it is useful to borrow some intuition from Propositions 1 and 2. All else constant, an increase in K_T' increases the aggregate stock of capital and the share in the traded sector, both of which lower default incentives. In contrast, an increase in K_N' increases the aggregate stock of capital but reduces the share in the traded sector. Propositions 1 and 2 imply that these have opposite effects on default incentives, which explains why $\hat{q}$ is "flatter" in K_N' and more sensitive to K_T'. To make the above point clearer, Figure 2 shows how $\hat{q}$ depends on the total stock of capital $K' = K_N' + K_T'$ while keeping the capital portfolio Λ' constant (left panel), and how it depends on the portfolio choice Λ' while keeping the aggregate stock constant (right panel).

Figure 2: Dependence of $\hat{q}$ on total K' and portfolio Λ'



In both graphs, the shock is set to its mean and the price $\hat{q}(x', z)$ is interpolated in order to evaluate it at $\hat{q}(B', \Lambda' K', (1-\Lambda')K', 1)$ for certain values for K' and Λ'. On the left panel, Λ' is set to its average in the ergodic distribution. On the right panel, $K' = K_N' + K_T'$ is set to its average. Finally, B' is set to its average in the solid-blue lines, and to its average plus one standard deviation in the dashed-red lines.

The capital externalities in this model have the same qualitative properties as in the two-period models from Section 2. Table 3 illustrates both inefficiencies in simulations of the model by comparing the average values of different variables for the planner and the decentralized economy over a long time series.

Table 3: Underinvestment and misallocation

	$K_N + K_T$	$\Lambda = \frac{K_T}{K_N+K_T}$	B	c
	(1)	(2)	(3)	(4)
Decentralized	1.05	0.496	0.71	1.42
Planner	1.19	0.498	0.99	1.47

To compute the above moments, I draw a long time series of 11,000 periods and drop the first 1,000. The reported numbers are the averages for each variable along the 10,000 periods except for those regarding borrowing, which are the averages conditional on being in good standing.

Columns (1) and (2) show how the planner accumulates more capital and allocates more of it in the traded sector. Columns (3) and (4) show how the planner's investment decisions allow it to sustain higher levels of debt and consumption.

3.6 Capital subsidies and welfare

Table 4 shows the cyclical properties of the state-contingent subsidies that implement the centralized allocation as defined in Proposition 3. Columns (1) through (4) report the average, standard deviation, and correlations with GDP and spreads, respectively.

Table 4: State-contingent subsidies over the business cycle

	μ_{τ_j}	σ_{τ_j}	$\rho_{\tau_j,GDP}$	$\rho_{\tau_j,r-r^*}$
	(1)	(2)	(3)	(4)
$\tau_{N,t}$	0.01	0.05	-0.08	0.53
$\tau_{T,t}$	0.12	0.10	-0.07	0.47

To compute these moments I simulate of 11,000 periods and drop the first 1,000. Subsidies are calculated to implement the centralized allocation following the formula in Proposition 3.

The average state-contingent subsidy to capital in the traded sector $\tau_{T,t}$ is 12 percent, while the one for the non-traded sector $\tau_{N,t}$ is barely 1. This difference in magnitude is a result of $\hat{q}$ being much more sensitive to capital in the traded sector, as illustrated in Figure 1. The volatility of $\tau_{N,t}$ is also lower, showing that $\hat{q}$ is "flatter" in $K_{N,t+1}$ over the entire ergodic distribution. Both subsidies are countercyclical and positively correlated with spreads, which shows that the disagreement between the government and the households is more severe in periods of distress.

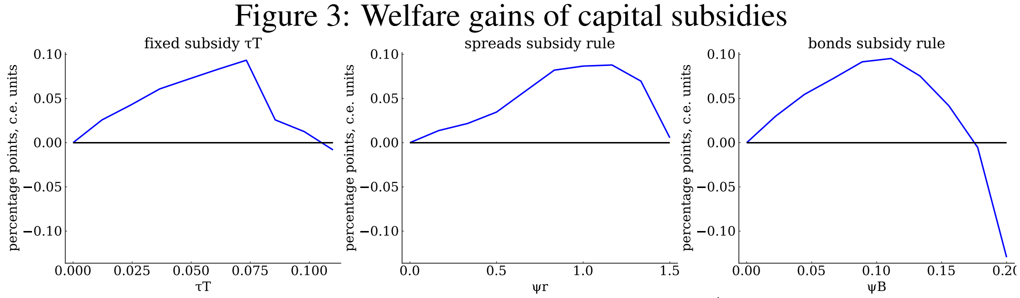
Let $\lambda^{\tau}(x, z)$ be the welfare gains of implementing a policy regime indexed by τ starting from a state (x, z) expressed in consumption equivalent units. These are defined as

$$\lambda^{\tau}(x, z) = 100 * \left[\left(\frac{H(x, z; \tau)}{H(x, z)} \right)^{\frac{1}{1-\sigma}} - 1 \right] \tag{28}$$

where $H(x, z; \tau)$ is the value of the household under regime τ, and $H(x, z)$ is the value of a representative household in the decentralized economy given the government's default choice. The average welfare gains of implementing the planner's allocation are 0.79 percent.[15]

Implementing the planner's allocation requires knowing the state of the economy at every point in time and the shape of the price schedule, which may be challenging in practice. Instead, I consider three simple subsidy rules with a zero subsidy to non-traded capital and a positive subsidy to traded capital.[16] The first is a fixed subsidy $\tau_{T,t} = \tau_T$; the second is a subsidy proportional to spreads $\tau_{T,t} = \psi_r \left((1 + r_t)^4 - (1 + r^*)^4 \right)$; and the third is a subsidy proportional to borrowing $\tau_{T,t} = \psi_B B_{t+1}$.

Figure 3: Welfare gains of capital subsidies



Welfare gains are the average of 1,000 draws of $\lambda^{\tau}(x, z) = 100 * \left[\left(\frac{H(x,z;\tau)}{H(x,z)} \right)^{\frac{1}{1-\sigma}} - 1 \right]$ over the ergodic distribution.

Figure 3 shows the welfare gains of implementing these rules for different values of τ_T, ψ_r, and ψ_B. The best rule in all three cases yields welfare gains of around 0.10 percent, which is roughly one-eight of the welfare gains from implementing the planner's allocation.

3.7 Sensitivity analysis

Table 3.7 presents the main results for three alternative parameterizations: a lower elasticity of substitution, a higher default penalty, and a model with long-term debt. The takeaway of the

[15] I compute the average of $\lambda^{\tau}(x, z)$ over 1,000 draws from the ergodic distribution. Each draw of (x, z) is the final value after simulating the economy for 1,001 periods.

[16] Rules considering positive subsidies for both stocks yield similar results. This is because the difference in magnitude of both optimal subsidies implies that welfare gains are much larger from subsidies to traded capital.

first two is that the main results continue to hold: the decentralized economies present aggregate underinvestment and sectoral misallocation of capital. The third illustrates how these capital externalities interact in a non-trivial way with debt dilution, which is an additional inefficiency present in sovereign debt models with long-term debt.

The first case considers a value for the elasticity of subsitution of $\eta = 0.4$. The parameters β, d_0, d_1, and ϕ are recalibrated to target the same moments as in the benchmark. In this case the average subsidy for capital in the non-traded sector is negative, indicating a stronger portfolio externality. The capital-stock externality seems to be smaller, indicated by the lower average subsidy to traded capital and by the lower difference in the aggregate capital stock. Overall, welfare gains from implementing the centralized allocation are positive but lower than in the benchmark. In the second case the default penalty is much larger, which implies less frequent defaults and larger stocks of debt. Interestingly, both externalities worsen because of the higher levels of debt achieved in the simulation. As indicated by Proposition 3, the severity of the disagreement is proportional to the desired bond issuance and to the sensitivity of the price schedule, which increases with borrowing.

Table 5: Sensitivity to elasticity of substitution and default cost

		Pr (def.)	$K_N + K_T$	$\Lambda = \frac{K_T}{K_N+K_T}$	B	τ_N	τ_T	λ^*
$\eta = 0.4$	dec.	0.0006	1.01	0.494	0.57	-0.02	0.09	0.23
	planner	0.0017	1.05	0.496	0.66			
$d_0 = -0.26$	dec.	0.0083	1.04	0.496	0.79	0.01	0.13	2.73
$d_1 = 0.35$	planner	0.0003	1.23	0.499	1.67			
long-term debt	dec.	0.0214	0.82	0.496	0.73	0.003	0.02	-0.25
	planner	0.0259	0.79	0.498	1.09			

The third case considers long-term debt. Following Chatterjee and Eyigungor (2012), I assume that every period a fraction $\gamma = 0.05$ of the debt matures and coupons $\kappa = 0.03$ are due on the remaining stock. The law of motion of debt is $B_{t+1} = i_{B,t} + (1-\gamma) B_t$, where $i_{B,t}$ is debt issuance, and debt service is $(\gamma + (1-\gamma)\kappa) B_t$. The parameters β, d_0, d_1, and ϕ are recalibrated to target the same moments as in the benchmark. As discussed by Hatchondo, Martinez, and Sosa-Padilla (2016), Aguiar, Amador, Hopenhayn, and Werning (2019), and Hatchondo, Martinez, and Roch (2020), the centralized allocation in this case is not constrained efficient because the planner cannot commit to not dilute the debt in the future. Default events are more frequent in this case, which depresses the overall stocks of capital.[17] In fact, the planner accumulates more debt and less

[17] As discussed by Hatchondo and Martinez (2009), models with long-term debt generate higher default rates for any given level of spreads.

capital than the decentralized economy. As in the benchmark, the capital externalities limit the government's ability to borrow but in this case the government's desired borrowing is inefficiently high. This implies that the centralized allocation need not be preferred to the decentralized one. For this calibration there are in fact welfare losses from implementing the centralized allocation, which suggests that the capital externalities may discipline debt dilution in some cases. Characterizing the efficient allocation in this environment with long-term debt (in the spirit of Hatchondo, Martinez, and Roch (2020)) is an exciting future avenue of research beyond the scope of this paper.

4 Conclusion

This paper studied how capital and its allocation in different sectors affect default incentives. Under general conditions, the aggregate stock of capital reduces default incentives and the share of capital in the non-traded sector increases them. Two externalities of private investment arise from these results: the *capital-stock externality* and the *portfolio externality*.

I used a quantitative sovereign default model with production and private investment to study the cyclical behavior of both externalities. The insights from the two-period models hold in model simulations. Relative to the problem of a benevolent social planner, the competitive equilibrium features underinvestment, a larger non-traded sector, more default events, and lower levels of debt and consumption. I show that the planner's allocation can be implemented as a competitive equilibrium with state-contingent capital subsidies. Subsidy rules that are simpler to implement, such as fixed subsidies and subsidies proportional to borrowing and spreads, yield positive but smaller welfare gains.

The insights from this paper can be extended to richer production settings with private dynamic decisions. For example, frictional labor markets in which labor allocations persist through several periods would feature similar externalities from labor allocations. Another interesting extension is to study how large endowments of natural resources affect the size and behavior of the portfolio externality through the classic Dutch disease mechanisms.

References

Aguiar, Mark and Manuel Amador. 2011. "Growth in the shadow of expropriation." *Quarterly Journal of Economics* 126:651–697. 4

Aguiar, Mark, Manuel Amador, and Gita Gopinath. 2009. "Investment Cycles and Sovereign Debt Overhang." *Review of Economic Studies* 76 (1):1–31. 1

Aguiar, Mark, Manuel Amador, Hugo Hopenhayn, and Ivan Werning. 2019. "Take the Short Route: Equilibrium Default and Debt Maturity." *Econometrica* 87 (2):423–462. 27

Aguiar, Mark and Gita Gopinath. 2006. "Defaultable Debt, Interest Rates and the Current Account." *Journal of International Economics* 69 (1):64–83. 4

Arce, Fernando. 2021. "Private Overborrowing Under Sovereign Risk." Working Paper WP 2022-17, Federal Reserve Bank of Chicago. 2

Arellano, Cristina. 2008. "Default Risk and Income Fluctuations in Emerging Economies." *American Economic Review* 98 (3):690–712. 4, 6

Arellano, Cristina, Yan Bai, and Gabriel Mihalache. 2018. "Default risk, sectoral reallocation, and persistent recessions." *Journal of International Economics* 112:182–199. 1, 4, 15

Bianchi, Javier. 2011. "Overborrowing and Systemic Externalities in the Buisness cycle." *American Economic Review* 101:3400–3426. 1, 22

Bianchi, Javier and Enrique G. Mendoza. 2018. "Optimal Time-Consistent Macroprudential Policy." *Journal of Political Economy* 126 (2):588–634. 1

———. 2020. "A Fisherian approach to financial crises: Lessons from the Sudden Stops literature." *Review of Economic Dynamics* 37:S254–S283. The twenty-fifth anniversary of "Frontiers of Business Cycle Research". 1

Chatterjee, Satyajit and Burcu Eyigungor. 2012. "Maturity, Indebtedness, and Default Risk." *American Economic Review* 102 (6):2674–2699. 16, 27

Cole, Harold L. and Timothy J. Kehoe. 2000. "Self-Fulfilling Debt Crises." *Review of Economic Studies* 67:91–116. 4, 6

Eaton, Jonathan and Mark Gersovitz. 1981. "Debt with Potential Repudiation: Theoretical and Empirical Analysis." *The Review of Economic Studies* 48 (2):289–309. 4, 15

Galli, Carlo. 2021. "Self-fulfilling debt crises, fiscal policy and investment." *Journal of International Economics* 131:103475. 4, 6

Gelos, R. Gaston, Ratna Sahay, and Guido Sandleirs. 2011. "Sovereign borrowing by developing economies: What determines market access?" *Journal of International Economics* 83:243–254. 22

Gordon, Grey and Pablo A. Guerron-Quintana. 2018. "Dynamics of investment, debt, and Default." *Review of Economic Dynamics* 28:71–95. 1, 4, 15, 16, 22

Hatchondo, Juan Carlos and Leonardo Martinez. 2009. "Long-Duration Bonds and Sovereign Defaults." *Journal of International Economics* 79 (1):117–125. 27

Hatchondo, Juan Carlos, Leonardo Martinez, and Francisco Roch. 2020. "Constrained Efficient Borrowing with Sovereign Default Risk." Working Paper 2020/227, International Monetary Fund. 27, 28

Hatchondo, Juan Carlos, Leonardo Martinez, and Horacio Sapriza. 2010. "Quantitative properties of sovereign default models: Solution methods matter." *Review of Economic Dynamics* 13 (4):919–933. 21

Hatchondo, Juan Carlos, Leonardo Martinez, and Cesar Sosa-Padilla. 2016. "Debt Dilution and Sovereign Default Risk." *Journal of Political Economy* 124 (5):1383–1422. 27

Herbert, Benjamin and Jesse Schreger. 2017. "The Costs of Sovereign Default: Evidence from Argentina." *American Economic Review* 107 (10):3119–3145. 1

Kim, Yun Jung and Jing Zhang. 2012. "Decentralized borrowing and centralized default." *Journal of International Economics* 88 (1):121–133. 2

Krugman, Paul. 1988. "Financing vs. forgiving a debt overhang." *Journal of Development Economics* 29 (3):253–268. 1

Lorenzoni, Guido. 2008. "Inefficient Credit Booms." *The Review of Economic Studies* 75 (3):809–833. 1

Mendoza, Enrique G. 2005. "Real Exchange Rate Volatility and the Price of Nontradable Goods in Economies Prone to Sudden Stops." *Economia: Journal of the Latin American and Caribbean Economic Association* 6 (1):103–135. 22

Mendoza, Enrique G. and Vivian Z. Yue. 2012. "A General Equilibrium Model of Sovereign Default and Buisness Cycles." *Quarterly Journal of Economics* 127:889–946. 2

Sachs, Jeffrey D. 1989. "The Debt Overhang of Developing Countries." In *Debt, Growth and Stabilization: Essays in Memory of Carlos Dias Alejandro*, edited by J. de Macedo and R. Findlay. Oxford: Blackwell, 81–102. 1

Seoane, Hernan and Emircan Yurdagul. 2022. "Sovereign Debt, Default, and the Investment Externality." *mimeo* . 4

Stockman, Alan C. and Linda L. Tesar. 1995. "Tastes and Technology in a Two-Country Model of the Buisness Cycle: Explaining International Comovements." *American Economic Review* 85 (1):168–185. 22

A Proofs of Section 2

A.1 Proof of Proposition 1

Proposition 1. The default set is shrinking in K_1. That is, $\frac{\partial z^*(x_1)}{\partial K_1} \leq 0$.

Proof: Taking the full derivative of equation (3) and rearranging terms we get

$$\frac{\partial z^*(x_1)}{\partial K_1} = -\frac{\frac{\partial F(z^*(x_1),K_1)}{\partial K} - \frac{\partial F(z_D(z^*(x_1)),K_1)}{\partial K_1}}{\frac{\partial F(z^*(x_1),K_1)}{\partial z} - \frac{\partial F(z_D(z^*(x_1)),K_1)}{\partial z}\frac{\partial z_D(z^*(x_1))}{\partial z}}$$

where both the numerator and denominator are positive.

For the numerator, note that, by assumption, the cross derivative is $F_{zK} \geq 0$. This implies that $\frac{\partial F}{\partial K}$ is weakly increasing in z. Since $z_D(z) \leq z$ for all z then we get that $\frac{\partial F(z^*(x_1),K_1)}{\partial K} - \frac{\partial F(z_D(z^*(x_1)),K_1)}{\partial K_1} \geq 0$ and, thus, the numerator is positive.

For the denominator, note that by assumption F is weakly convex in z, so by a similar argument $\frac{\partial F(z^*(x_1),K_1)}{\partial z} - \frac{\partial F(z_D(z^*(x_1)),K_1)}{\partial z} \geq 0$. In addition, $\frac{\partial z_D(z^*(x_1))}{\partial z} \leq 1$ so we get that the denominator is also positive.□

A.2 Proof of Proposition 2

Proposition 2 holds for any given $x_1 = (\Lambda_1, B_1)$. For convenience of notation, I will refer to c_N^D and c_T^D as consumption of the non-traded and traded goods, respectively, in default at $z = z^*(x_1)$. Similarly, c_N^P and c_T^P as consumption of the non-traded and traded goods, respectively, in repayment at $z = z^*(x_1)$. The following two lemmas are used throughout the proof of Proposition 2.

Lemma 1: $c_N^D \leq c_N^P$ and $c_T^D \geq c_T^P$.

Proof: First, note that since N is non-traded $y_N = c_N$, so we get $c_N^D \leq c_N^P$ from $z_D(z) \leq z$. Then, note that at z^* we have $F\left(c_N^P, c_T^P\right) = F\left(c_N^D, c_T^D\right)$, since F is increasing in both arguments then it must be that $c_T^D \geq c_T^P$ at z^*.□

Lemma 2: $\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_T} \geq \frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_T}$ and $\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_N} \leq \frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_N}$.

Proof: Note that:

$$\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_T} \geq \frac{\partial F\left(c_N^P, c_T^D\right)}{\partial c_T} \geq \frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_T}$$

$$\frac{\partial F\left(c_N^P,c_T^P\right)}{\partial c_N}\leq\frac{\partial F\left(c_N^D,c_T^P\right)}{\partial c_N}\leq\frac{\partial F\left(c_N^D,c_T^D\right)}{\partial c_N}$$

where the first inequality follows from Lemma 1 and concavity of F, and the second also follows from Lemma 1 and positive cross derivatives.□

Proposition 2. If the elasticity of substitution between traded and non-traded intermediates is $\eta<1$, then the default set is shrinking in Λ_1. That is, $\frac{\partial z^*(x_1)}{\partial\Lambda_1}\leq 0$.

Proof: Taking the full derivative of equation (11) and rearranging terms we get

$$\frac{\partial z^*}{\partial\Lambda_1}=-\frac{\frac{\partial V^P(z^*(x_1),x_1)}{\partial\Lambda}-\frac{\partial V^D(z^*(x_1),\Lambda_1)}{\partial\Lambda}}{\frac{\partial V^P(z^*(x_1),x_1)}{\partial z}-\frac{\partial V^D(z^*(x_1),\Lambda_1)}{\partial z}}\tag{29}$$

where $V^D(z,\Lambda)=u\left(F\left(z_D(z)f(1-\Lambda),z_D(z)f(\Lambda)\right)\right)$ and $V^P(z,x)=u\left(F\left(zf(1-\Lambda),zf(\Lambda)-B\right)\right)$. Lemma 3 below establishes that the denominator is positive. Lemma 4 below establishes that the numerator is positive. Both results imply that $\frac{\partial z^*}{\partial\Lambda_1}\leq 0$.□

Lemma 3. $\frac{\partial V^P(z^*(x_1),x_1)}{\partial z}-\frac{\partial V^D(z^*(x_1),\Lambda_1)}{\partial z}\geq 0$.

Proof: Note that V^D and V^P are increasing in z

$$\frac{\partial V^D}{\partial z}=u'\left(c^D\right)\left[\frac{\partial F}{\partial c_N}f(1-\Lambda)+\frac{\partial F}{\partial c_T}f(\Lambda)\right]\frac{\partial z_D}{\partial z}\geq 0$$
$$\frac{\partial V^P}{\partial z}=u'\left(c^P\right)\left[\frac{\partial F}{\partial c_N}f(1-\Lambda)+\frac{\partial F}{\partial c_T}f(\Lambda)\right]>0$$

for all (z,Λ,B). From the assumption that $\lim_{z\to 0}\left[z-z_D(z)\right]=0$ it follows that, for any $B>0$ and any $\Lambda\in(0,1)$, there exists a z_- such that $V^D(z_-,\Lambda)>V^P(z_-,\Lambda,B)$. That is, for any positive level of debt, there is a value for productivity low enough such that it is more convenient to default. Similarly, note that since $\frac{\partial z_D}{\partial z}<1$ and $z_D(z)<z$ for $z>\bar{z}$, then there exists $z_+<\infty$ such that $V^D(z_+,\Lambda)<V^P(z_+,\Lambda,B)$. Then, by the intermediate value theorem there is z^* such that $V^D(z^*,\Lambda)=V^P(z^*,\Lambda,B)$. Since both V^D and $V^P$are increasing and V^P is strictly increasing, then z^* is unique. Note that for $z<z^*$ we have $V^D(z,\Lambda)>V^P(z,\Lambda,B)$ and for $z>z^*$ we have $V^D(z,\Lambda)<V^P(z,\Lambda,B)$, then at z^* we get $\frac{\partial V^P(z^*(x_1),x_1)}{\partial z}>\frac{\partial V^D(z^*(x_1),\Lambda_1)}{\partial z}$.□

Lemma 4. If the elasticity of substitution between traded and non-traded intermediates is $\eta<1$, then $\frac{\partial V^P(z^*(x_1),x_1)}{\partial\Lambda}-\frac{\partial V^D(z^*(x_1),\Lambda_1)}{\partial\Lambda}\geq 0$.

Proof: The derivative of V^P with respect to Λ is:

$$\frac{\partial V^P(z,x)}{\partial \Lambda} = u'\left(c^P\right)\left[\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_T} z f'(\Lambda) - \frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_N} z f'(1-\Lambda)\right]$$

where $c^P = F\left(c_N^P, c_T^P\right)$, $c_N^P = zf(1-\Lambda)$, and $c_T^P = zf(\Lambda) - B$. Similarly, the derivative of V^D with respect to Λ is

$$\frac{\partial V^D(z,x)}{\partial \Lambda} = u'\left(c^D\right)\left[\frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_T} z_D(z) f'(\Lambda) - \frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_N} z_D(z) f'(1-\Lambda)\right]$$

where $c^D = F\left(c_N^D, c_T^D\right)$, $c_N^D = z_D(z) f(1-\Lambda)$, and $c_T^D = z_D(z) f(\Lambda)$. Let $x_1 = (\Lambda_1, B_1)$, note that at $(z^*(x_1), x_1)$ we have in general that $c^P \leq c^D \implies u'\left(c^P\right) \geq u'\left(c^D\right) \geq 0$, so subtracting and rearranging we get:

$$\begin{aligned}\frac{\partial V^P(z^*(x_1), x_1)}{\partial \Lambda} - \frac{\partial V^D(z^*(x_1), \Lambda_1)}{\partial \Lambda} \geq\; & u'\left(c^P\right)\left[\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_T} z - \frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_T} z_D(z)\right] f'(\Lambda) \\ & - u'\left(c^P\right)\left[\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_N} z - \frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_N} z_D(z)\right] f'(1-\Lambda)\end{aligned} \tag{30}$$

where $f' > 0$. This expression is the general version of the numerator in equation (18), which is the special case of perfect substitutes—where $\frac{\partial F(c_N,c_T)}{\partial c_T} = 1-\omega$ and $\frac{\partial F(c_N,c_T)}{\partial c_N} = \omega$.

For the first term, note that

$$\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_T} z - \frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_T} z_D(z) \geq \frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_T} z - \frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_T} z \geq 0$$

where the first inequality follows from $z \geq z_D(z)$ and the second inequality follows from Lemma 2.

For the second term, first recall that $f(k) = k^\alpha$, so $f'(1-\Lambda) = \alpha \frac{f(1-\lambda)}{(1-\lambda)}$. Plugging in we get that the second term is

$$-\left[\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_N} c_N^P - \frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_N} c_N^D\right] \frac{\alpha}{(1-\lambda)} \tag{31}$$

where we have used the fact that consumption of the non-traded good equals production. Then, for the result to hold, it suffices to show that the term in the bracket of (31) is negative.

From Lemma 2 we have that $\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_N} \leq \frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_N}$, but from Lemma 1 we have that $c_N^P \geq c_N^D$. Intuitively, the argument uses the fact that, when the elasticity of substitution is less than 1, the marginal rate of substitution changes more than the ratio of consumption on the same isoquant curve. This implies that the effect of higher marginal product of c_N from the default choice dominates the effect of the lower quantity and, thus, the term in brackets in negative. The formal argument follows below.

Note that F is homogeneous of degree 0 from the constant-returns-to-scale assumption. Then, applying Euler's theorem for homogeneous functions and using the fact that $F\left(c_N^P, c_T^P\right) = F\left(c_N^D, c_T^D\right)$ at z^* we get:

$$\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_N} c_N^P + \frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_T} c_T^P = \frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_N} c_N^D + \frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_T} c_T^D$$

which can be rearranged as

$$\frac{\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_N} c_N^P}{\frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_N} c_N^D} = \frac{1 + \frac{\frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_T}}{\frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_N}} \chi^D}{1 + \frac{\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_T}}{\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_N}} \chi^P} \tag{32}$$

where $\chi^D = \frac{c_T^D}{c_N^D}$ and $\chi^P = \frac{c_T^P}{c_N^P}$ are the consumption ratios in default and repayment. Now, note that since F is homogeneous of degree 1, its derivatives are homogeneous of degree 0. Then, we can define

$$MRS(\chi) = \frac{\frac{\partial F(1,\chi)}{\partial c_N}}{\frac{\partial F(1,\chi)}{\partial c_T}}$$

where the numerator is increasing and the denominator is decreasing (so MRS is increasing). Rewrite equation (32) as

$$\frac{\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_N} c_N^P}{\frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_N} c_N^D} = \frac{1 + e\left(\chi^D\right)}{1 + e\left(\chi^P\right)} \tag{33}$$

where $e(\chi) = \frac{\chi}{MRS(\chi)}$. The derivative of e is

$$
\begin{aligned}
e'(\chi) &= \frac{d(\chi) MRS(\chi) - \chi d(MRS(\chi))}{MRS(\chi) MRS(\chi)} \\
&= \frac{\chi d(MRS(\chi))}{MRS(\chi) MRS(\chi)} \left[\frac{d(\chi)}{\chi} \frac{MRS(\chi)}{d(MRS(\chi))} - 1 \right] \\
&= \frac{\chi d(MRS(\chi))}{MRS(\chi) MRS(\chi)} [\eta - 1] < 0
\end{aligned}
$$

where the inequality follows from $\eta < 1$ and from the observation that MRS is increasing. Note that Lemma 1 implies $\chi^D \geq \chi^P$, so we get that

$$
\frac{\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_N} c_N^P}{\frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_N} c_N^D} = \frac{1 + e\left(\chi^D\right)}{1 + e\left(\chi^P\right)} \leq 1
$$

which implies that $\frac{\partial F\left(c_N^P, c_T^P\right)}{\partial c_N} c_N^P - \frac{\partial F\left(c_N^D, c_T^D\right)}{\partial c_N} c_N^D \leq 0.\square$