Computational Methods in Physics (PHY4605) - **Arrays and Matrices**

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Overview

MATLAB can deal with data of any dimensions and sizes, known as arrays. An **array** is a collection of data values and can have any number of dimensions and sizes, which is an umbrella term that can be a scalar, vector, matrix or page. A **scalar** is a single value that does not have any dimension. Scalars are often used to represent constants, coefficients, or magnitudes of vectors. A **vector** is a one-dimensional array with either one row or one column. Vectors are often used to represent quantities that have both magnitude and direction, such as force or velocity. A **matrix** is a two-dimensional array with at least one row and one column. Matrices are often used to represent linear equations, transformations, or operations on data. A **page** is a slice of a three-dimensional array along the third dimension. Pages are useful for storing multiple matrices that have the same size and meaning.

Vectors

A vector is a special type of array, having only one row or one column. You can initialize vectors either explicitly:

```
a = [1, 2, 3]
```

```
a = 1×3
     1     2     3
```

```
b = [4, 5]
```

```
b = 1×2
     4     5
```

```
c = [a, -b]
```

```
c = 1×5
     1     2     3    -4    -5
```

```
d = [10; 12; 20]
```

```
d = 3×1
    10
    12
    20
```

You can do the same using the colon operator as well:

```
x = 1 : 10
```

```
x = 1×10
     1     2     3     4     5     6     7     8     9    10
```

```
x = 1 : 0.5 : 4
```

```
x = 1×7
    1.0000    1.5000    2.0000    2.5000    3.0000    3.5000    4.0000
```

```
x = 0 : -2 : -5
```

```
x = 1×3
     0    -2    -4
```

All of the vectors examined so far are row vectors. To generate the column vectors, you need to transpose such vectors:

```
y = [1 4 8 0 -1]'
```

```
y = 5×1
     1
     4
     8
     0
    -1
```

```
x'
```

```
ans = 3×1
     0
    -2
    -4
```

We can refer to particular elements of a vector by means of subscripts:

```
r = rand(1, 7)
```

```
r = 1×7
    0.6912    0.5646    0.1776    0.4704    0.7272    0.7639    0.8241
```

```
r(3)
```

```
ans =
0.6051
```

```
r(2:4)
```

```
ans = 1×3
    0.1661    0.6051    0.0993
```

```
r([1 7 2 6])
```

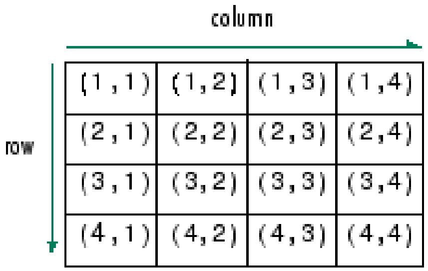
```
ans = 1×4
    0.5682    0.7205    0.1661    0.3675
```

Use an empty vector to remove elements from a vector:

```
r([1 7 2]) = []
```

```
Matrix index is out of range for deletion.
```

Matrices



A matrix may be thought of as a table consisting of rows and columns. You create a matrix just as you do a vector, except that a semicolon is used to indicate the end of a row:

```
a = [1, 2, 3; 4, 5, 6; 7, 8, 9; 11, 12, 13]
```

```
a = 4×3
     1     2     3
     4     5     6
     7     8     9
    11    12    13
```

A matrix may be transposed:

```
m = a'
```

```
m = 3×4
     1     4     7    11
     2     5     8    12
     3     6     9    13
```

A matrix can be constructed from column vectors of the same length:

```
x = 0 : 30 : 180;
t = [x' sin(x*pi/180)']
```

```
t = 7×2
         0         0
   30.0000    0.5000
   60.0000    0.8660
   90.0000    1.0000
  120.0000    0.8660
  150.0000    0.5000
```

```
  180.0000    0.0000
```

The colon operator provides for very efficient ways of handling matrices, for examples:

```
a(4, 2)
```

```
ans = 
12
```

```
a(2:3, 1:2)
```

```
ans = 2×2
     4     5
     7     8
```

```
m(1:2, 2:end)
```

```
ans = 2×3
     4     7    11
     5     8    12
```

```
a(1:2, 2:3) = 100*ones(2)
```

```
a = 4×3
     1   100   100
     4   100   100
     7     8     9
    11    12    13
```

On the right-hand side of an assignment:

```
b = a(:)
```

gives all the elements of a strung out by columns in one long column vector.

However, on the left-hand side of an assignment, `c(:)` reshapes a matrix and `c` must already exist. Then `a(:)` denotes a matrix with the same dimensions (shape) as `c`, but with new contents taken from the right-hand side:

```
c = NaN(6, 2)
```

```
c(:) = a
```

the contents of a are strung out into one long column and then fed into `c` by columns.

The same effect can be achieved using the `reshape` function. For example, the call to the function `reshape(A, 5, 2)` reshapes `A` into a 5-by-2 matrix. The new matrix must have at least 2 dimensions and the product of the dimension sizes must be the same as the number of elements of `A`. Another example:

```
A = 2 : 2 : 20
reshape(A, 5, 2)
```

Matrix Functions

There is a group of functions to generate 'elementary' matrices which are used in a number of applications. See `help elmat`. For example, the functions `NaN`, `zeros`, `ones` and `rand` generate matrices of empty values ('not a number' or `NaN`), `1`'s, `0`'s and random numbers, respectively. With a single argument n, they generate $n \times n$ (square) matrices. With two arguments n and m they generate $n \times m$ matrices.

```
NaNs = NaN(4)
```

```
NaNs = 4×4
   NaN   NaN   NaN   NaN
   NaN   NaN   NaN   NaN
   NaN   NaN   NaN   NaN
   NaN   NaN   NaN   NaN
```

```
Zeros = zeros(4)
```

```
Zeros = 4×4
     0     0     0     0
     0     0     0     0
     0     0     0     0
     0     0     0     0
```

```
Ones = ones(3)
```

```
Ones = 3×3
     1     1     1
     1     1     1
     1     1     1
```

```
Randoms = rand(3)
```

```
Randoms = 3×3
    0.6569    0.9983    0.7661
    0.1406    0.3656    0.6694
    0.4704    0.9515    0.4449
```

Many of the other functions operate on matrices column by column, e.g.:

```
a = randi([0, 1], 5, 'logical')
```

```
a = 5×5 logical array
   0   1   0   1   0
   1   0   0   0   1
   0   1   0   1   1
   0   1   1   1   0
   0   1   1   1   1
```

```
all(a)
```

```
ans = 1×5 logical array
   0   0   0   0   0
```

```
any(a)
```

```
ans = 1×5 logical array
   1   1   1   1   1
```

For each column of a where all the elements are `true` (non-zero) `all` returns `1`, otherwise it returns `0` (acts as an `AND` operator). To test if `all` the elements of `a` are `true`, use `all` twice. In this example, the statement:

```
all(all(a))
```

returns `0` because some of the elements of `a` are `0`. The same rule applies for `any` that acts as a logical `OR` operator instead of `AND`.

Here are some functions for manipulating matrices:

```
b = randi(100, [6, 8])
```

```
b = 6×8
    77    67    28    36    47     2    46   100
    52    50    11    87    69    27    76    10
     3    59    76    64    47    69    70    32
    85    57    16    23    68    54    66    67
    77    70    57    58    53    15    91    93
    25     8    59    24    76    80    45    17
```

```
diag(b)    % Extracts or creates a diagonal
```

```
ans = 6×1
    77
    50
    76
    23
    53
    80
```

```
fliplr(b)  % Flips from left to right
```

```
ans = 6×8
   100    46     2    47    36    28    67    77
    10    76    27    69    87    11    50    52
    32    70    69    47    64    76    59     3
    67    66    54    68    23    16    57    85
    93    91    15    53    58    57    70    77
    17    45    80    76    24    59     8    25
```

```
flipud(b)  % Flips from top to bottom
```

```
ans = 6×8
    25     8    59    24    76    80    45    17
    77    70    57    58    53    15    91    93
    85    57    16    23    68    54    66    67
     3    59    76    64    47    69    70    32
    52    50    11    87    69    27    76    10
    77    67    28    36    47     2    46   100
```

```
rot90(b)   % Rotates
```

```
ans = 8×6
   100    10    32    67    93    17
    46    76    70    66    91    45
     2    27    69    54    15    80
    47    69    47    68    53    76
    36    87    64    23    58    24
```

```
    28    11    76    16    57    59
    67    50    59    57    70     8
    77    52     3    85    77    25
```

```
tril(b)    % Extracts the lower triangular part
```

```
ans = 6×8
    77     0     0     0     0     0     0     0
    52    50     0     0     0     0     0     0
     3    59    76     0     0     0     0     0
    85    57    16    23     0     0     0     0
    77    70    57    58    53     0     0     0
    25     8    59    24    76    80     0     0
```

```
triu(b)    % Extracts the upper triangular part
```

```
ans = 6×8
    77    67    28    36    47     2    46   100
     0    50    11    87    69    27    76    10
     0     0    76    64    47    69    70    32
     0     0     0    23    68    54    66    67
     0     0     0     0    53    15    91    93
     0     0     0     0     0    80    45    17
```

While, here are some of MATLAB's more advanced matrix functions.

```
c = randi(100, 3)
det(c)  % Determinant
eig(c)  % Eigenvalue decomposition
expm(c) % Matrix exponential, i.e., e^A, where A is a matrix
inv(c)  % Inverse
lu(c)   % LU factorization (into lower and upper triangular matrices)
qr(c)   % Orthogonal factorization
svd(c)  % Singular value decomposition
```

Vectorization

MATLAB is optimized for operations involving matrices and vectors. The process of revising loop-based, scalar-oriented code to use MATLAB matrix and vector operations is called *vectorization*.

This code computes the sine of 10,000,001 values ranging from 0 to 10:

```
tic
i = 0;
for t = 0 : 0.000001 : 10
    i = i + 1;
    y(i) = sin(t);
end
toc
```

```
Elapsed time is 0.408910 seconds.
```

This is a vectorized version of the same code:

```
tic
t = 0 : 0.000001 : 10;
y = sin(t);
toc
```

```
Elapsed time is 0.082759 seconds.
```

Array operators perform the same operation for all elements in the data set. These types of operations are useful for repetitive calculations. For example, suppose you collect the volume (`V`) of various cones by recording their diameter (`D`) and height (`H`). If you collect the information for just one cone, you can calculate the volume for that single cone:

```
D = 10
H = 25
V = 1/12 * pi * (D^2) * H
```

Now, collect information on 10,000 cones. The vectors `D` and `H` each contain 10,000 elements, and you want to calculate 10,000 volumes. In most programming languages, you need to set up a loop similar to this MATLAB code:

```
D = randi(20, 1, 10000)
H = randi(30, 1, 10000)
for n = 1 : 10000
    V(n) = 1/12*pi*(D(n)^2)*H(n);
end
```

```
V
```

With MATLAB, you can perform the calculation for each element of a vector with similar syntax as the scalar case:

```
V = 1/12 * pi * (D.^2) .* H
```

Matrix Operations

Matrix addition and subtraction are defined in the same way as the equivalent array operations, i.e., element by element. Matrix multiplication, however, is quite different.

```
a = [1 2; 3 4]
```

```
a = 2×2
     1     2
     3     4
```

```
b = [5 6; 0 -1]
```

```
b = 2×2
     5     6
     0    -1
```

```
m1 = a * b
```

```
m1 = 2×2
     5     4
    15    14
```

```
m2 = a .* b
```

```
m2 = 2×2
     5    12
     0    -4
```

Note the important difference between the array operation `a .* b` and the matrix operation `a * b`.

The matrix operation A^2 means $A \times A$, where A must be a square matrix. The operator ^ is used for matrix exponentiation, e.g.:

```
p1 = a ^ 2
```

```
p1 = 2×2
     7    10
    15    22
```

```
p2 = a .^ 2
```

```
p2 = 2×2
     1     4
     9    16
```

`a ^ 2` is the same as `a * a`. Again, note the difference between the array operation `a .^ 2` and the matrix operation `a ^ 2`.

Linear Equations

A problem that often arises in scientific applications is the solution of a system of linear equations, e.g.:

$$3x_1 + 2x_2 - x_3 = 10$$
$$-x_1 + 3x_2 + 2x_3 = 5$$
$$x_1 - x_2 - x_3 = -1.$$

MATLAB was designed to solve a system like this directly and very easily, as we shall now see. If we define the matrix of coefficients, A, the vectors of unknowns, x, and the right-hand side, b, as:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ -1 & 3 & 2 \\ 1 & -1 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 5 \\ -1 \end{bmatrix}$$

we can write the above system of three equations in matrix form as:

$$\begin{bmatrix} 3 & 2 & -1 \\ -1 & 3 & 2 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ -1 \end{bmatrix}$$

or even more concisely as the single matrix equation $Ax = b$. The solution may then be written as $x = A^{-1}b$. The system can be solved by the following:

```
A = [3, 2, -1; -1, 3, 2; 1, -1, -1];
b = [10 5 -1]';
```

```
x = A \ b
```

```
x = 3×1
   -2.0000
    5.0000
   -6.0000
```

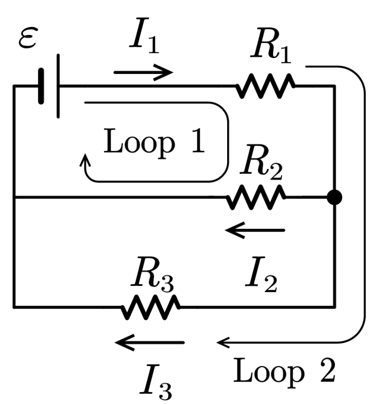
```
x = inv(A) * b
```

```
x = 3×1
   -2.0000
    5.0000
   -6.0000
```

which gives $x_1 = -2$, $x_2 = 5$, $x_3 = -6$. Notice that the left division operator (\) is used. You can think of the matrix operation `A \ b` as 'b divided by A', or as 'the inverse of A multiplied by b'.

Physics Pinpoint �: Kirchhoff's Rules

Kirchhoff's Rules are fundamental in analyzing electrical circuits. They allow us to calculate unknown currents, voltages, and resistances in complex circuits by formulating them as a system of linear equations.



Consider a simple circuit with three resistors ($R_1 = 4\,\Omega$, $R_2 = 6\,\Omega$ and $R_3 = 8\,\Omega$), connected in a combination of series and parallel, powered by a voltage source ($V = 12\text{ V}$). We need to calculate the currents flowing through each branch of the circuit, namely I_1, I_2 and I_3.

- Applying the junction rule at the node: $I_1 = I_2 + I_3 \Longrightarrow I_1 - I_2 - I_3 = 0$
- Applying the loop rule around the inner loop 1: $V - R_1 I_1 - R_2 I_2 = 0 \Longrightarrow 12 - 4I_1 - 6I_2 = 0 \;\Longrightarrow\; 4I_1 + 6I_2 + 0I_3 = 12$
- Applying the loop rule around the outer loop 2: $V - R_1 I_1 - R_3 I_3 = 0 \Longrightarrow 12 - 4I_1 - 8I_3 = 0 \;\Longrightarrow\; 4I_1 + 0I_2 + 8I_3 = 12$

Now, we have a system of linear equations:

$$\begin{bmatrix} I_1 & -I_2 & -I_3 \\ 4I_1 & 6I_2 & 0I_3 \\ 4I_1 & 0I_2 & 8I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 12 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -1 & -1 \\ 4 & 6 & 0 \\ 4 & 0 & 8 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 12 \end{bmatrix}$$

The system can be solved by the following:

```
A = [1, -1, -1
     4, 6, 0;
     4, 0, 8]
```

```
A = 3×3
     1    -1    -1
     4     6     0
     4     0     8
```

```
b = [0; 12; 12]
```

```
b = 3×1
     0
    12
    12
```

```
I = A \ b;
fprintf('I1 = %.2f A\n', I(1));
```

```
I1 = 1.62 A
```

```
fprintf('I2 = %.2f A\n', I(2));
```

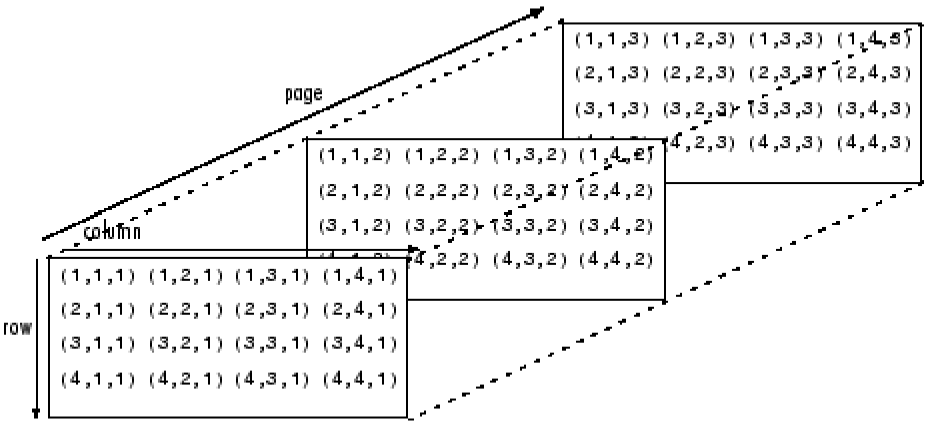
```
I2 = 0.92 A
```

```
fprintf('I3 = %.2f A\n', I(3));
```

```
I3 = 0.69 A
```

Multidimensional Arrays

Multidimensional arrays are an extension of 2-D matrices and use additional subscripts for indexing. A 3-D array, for example, uses three subscripts. The first two are just like a matrix, but the third dimension represents *pages* or *sheets* of elements.



A multidimensional array can be created by creating a 2-D matrix first, and then extending it. For example, first define a 3-by-3 matrix as the first page in a 3-D array:

```
A = [1 2 3; 4 5 6; 7 8 9]
```

Assign another 3-by-3 matrix to the index value 2 in the third dimension to add a second 'page'. The syntax `A(:,:,2)` uses a colon in the first and second dimensions to include all rows and all columns from the right-hand side of the assignment:

```
A(:,:,2) = [10 11 12; 13 14 15; 16 17 18]
```

The `cat` function can be a useful tool for building multidimensional arrays. For example, create a new 3-D array `B` by concatenating `A` with a third page. The first argument indicates which dimension to concatenate along:

```
B = cat(3,A,[3 2 1; 0 9 8; 5 3 7])
```

To access elements in a multidimensional array, use integer subscripts just as you would for vectors and matrices. For example, find the `(1,2,2)` element of `A`, which is in the first row, second column, and second page of `A`:

```
A(1,2,2)
```