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THE CHINESE POSTMAN PROBLEM FOR MIXED NETWORKS*

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The Chinese postman problem is to find a least cost way to traverse each arc of a network at least once and to return to the vertex from which you started. Diverse problems such as the routing of road crews, police patrol scheduling, garbage collection and the programming of computer map printers can be modelled as Chinese postman problems.

This paper surveys available solution techniques for the Chinese postman problem for totally undirected networks (when all streets are two-way streets) and for totally directed networks (when all streets are one-way streets). A known solution technique for networks with both directed and undirected arcs (both one-way and two-way streets) in which the degree of each vertex is an even number is also reviewed.

A solution technique for these mixed networks in which some vertices have odd degree is presented. This technique is based on the before mentioned technique and requires the solution of a minimum cost flow problem on a network that is an extension of the original network. Some of the additional arcs in this network have gain factors (i.e., the flow leaving these arcs equals the flow entering times the gain factor) and the flows are required to be integer valued.

(NETWORKS/GRAPHS; TRANSPORTATION; MATHEMATICS—COMBINATORICS)

1. Introduction to the Chinese Postman Problem

Let N be any connected network with vertex set V and arc set A. Let $\$(x, y)$ denote the cost (or time or distance) of traversing arc (x, y). The postman problem is the problem of finding the least cost way for a postman to traverse each arc of network at least once and to return to his starting vertex. This problem is often called the Chinese postman problem since the first modern work on it appeared in a Chinese journal [7].

The solution of this problem is important not only to postmen who must cover every street in their district but for police patrols, road sweepers, indefatigable tourists, and repair crews. Happily Edmonds and Johnson [2] have presented solutions for the postman problem when (a) the network consists entirely of undirected arcs (i.e. all streets are two-way streets), (b) the network consists entirely of directed arcs (all streets are one-way streets) and (c) when arcs are mixed (i.e. both one-way and two-way streets) but when the degree of each vertex is an even number. (The degree of a vertex is the number of arcs incident to it.)

No solution for the remaining case for mixed networks with some odd-degree vertices has been found. Papadimitriou [9] has shown that this unsolved case is a difficult problem and is NP complete. This paper rephrases the unsolved case of the Chinese postman problem as a integer minimum cost flow with gains problem that can be solved by integer linear programming. Though not elegant, it is at least a feasible solution method worth considering in the light of Papadimitriou's findings.

§2 sketches the solution techniques for the three solved cases of the postman problem mentioned above. §3 rephrases the unsolved case of the postman problem as a flow with gains problem. §4 describes how the solution techniques for all the cases of the postman problem can be modified when arc costs are not equal in both directions and when the number of times an arc can be repeated is limited.

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2. The Solved Cases of the Postman Problem

First, let's consider the postman problem on a network consisting entirely of undirected (two-way) arcs. Observe that in any route taken by the postman, he must enter and exit a vertex an equal number of times. Hence, if each arc that is repeated by the postman is duplicated in the network—once for each time that the postman repeated it—then the resulting network will have only even vertex degrees. Consequently, the solution to the postman problem for the undirected case lies in finding which arcs to duplicate so that the resulting network has only even vertex degrees. Edmonds and Johnson [2] solved this as follows: First the least cost path from each odd degree vertex to each other odd degree vertex is found. (Since the network is undirected, these costs are symmetric.) Next, the odd degree vertices are matched off. (There are an even number of odd degree vertices and hence no odd degree vertex will be left unmatched.) The cost of each match is the cost of the shortest path between the two pair-off odd degree vertices. The matching with the minimum total cost is selected, and the arcs in the shortest paths in this matching are the arcs that the postman must repeat in his optimum routing. Also see Minieka [8, p. 238] for a detailed description of this solution.

Secondly, let's consider the postman problem on a network consisting entirely of directed (one-way) arcs. Observe that in any route taken by the postman, he must again enter and exit a vertex an equal number of times. Let $\mathrm{In}(x)$ denote the number of arcs directed into vertex x, and let $\mathrm{Out}(x)$ denote the number of vertices directed out of vertex x. Hence, if each arc that is repeated by the postman is duplicated—once for each time that he repeats this arc—then in the resulting enlarged graph, $\mathrm{In}(x) = \mathrm{Out}(x)$ for each vertex x. Such a network is called symmetric. Thus, the solution to the postman problem for directed graphs consists in deciding how to repeat the arc so that the resulting network is symmetric.

Edmonds and Johnson [2] solved the postman problem for directed graphs as follows: If $\mathrm{In}(x) > \mathrm{Out}(x)$, then let vertex x be a source with a supply of $\mathrm{In}(x) - \mathrm{Out}(x)$ units. If $\mathrm{In}(x) < \mathrm{Out}(x)$, then let vertex x be a sink with a demand of $\mathrm{Out}(x) - \mathrm{In}(x)$ units. Total demand equals total supply. Let all arcs have infinite capacity. Find a minimum cost flow from the sources to the sinks. Since all supplies and demands are integers, the resulting minimum cost flow will be all integers. The postman must repeat arc (x, y) once for each flow unit that traverses arc (x, y). Hence, the directed postman problem is solved by finding a minimum cost flow.

Next, let's consider a mixed network—both directed and undirected arcs. In this situation, we must decide how many times each directed arc must be repeated, how many times each undirected arc will be repeated in each direction and if an undirected arc is not repeated, then in which direction will it be traversed. Consequently the mixed network postman problem is more complicated.

Edmonds and Johnson [2] solved the mixed network postman problem for networks in which every vertex has even degree as follows:

Arbitrarily assign a direction to each undirected arc. Now, the graph is totally directed and the minimum cost flow solution technique described above for the directed network postman problem can be employed. However, before this flow is found, some additional arcs must be added to the network. Specifically, for each undirected arc (x, y) there must be an arc (x, y) and an arc (y, x), each with cost $\$(x, y)$ and with infinite capacity. This allows repeats of undirected arcs in either direction. Moreover, if undirected arc (x, y) was arbitrarily directed from x into y, then an arc (y, x) with zero cost and maximum capacity of 2 must be added to the network. Thus, three new arcs between x and y are created. If 2 flow units are sent through a free arc, then reverse the arbitrary direction assigned to this undirected arc. Thus the free arcs are a device to correct injudicious arbitrary directions.

The minimum cost flow is found in this expanded graph. Since the vertices all have even degree, all supplies and demands will be multiples of 2. Since all arc capacities are 2 or infinite, all arc flows will be multiples of 2, and the flow units travel in pairs. If two flow units traverse a free arc, then its arbitrary direction is reversed. If flow units traverse an arc with non-zero cost, then this arc is repeated once for each flow unit.

The resulting directed network with corrected directions for the undirected arcs and with repeated arcs will be symmetric and yield an optimum route for the postman.

Note that this solution technique for mixed networks relies upon the fact that flow units will travel in pairs. If the vertex has an odd degree, then after the arbitrary directions are placed on the undirected arcs, this vertex will have an odd valued supply or demand. If there are odd valued supplies and demands, there is no guarantee that the flow units will travel in pairs. Hence, if 1 flow unit traverses a free arc, the flow cannot be related to the postman problem. Hence, the Edmonds and Johnson technique for even-degree mixed networks cannot be extended to general mixed networks.

3. The Mixed Postman Problem as a Flow with Gains Problem

The key to solving the postman problem on a mixed network is the following:

LEMMA 1. *In any optimum postman route, if an undirected arc (x, y) is traversed in both directions, then arc (x, y) is traversed exactly two times.*

PROOF. The proof is achieved through contradiction: Suppose that undirected arc (x, y) is traversed three or more times in some optimum postman route and not always in the same direction. For example, suppose that the first crossing is from x to y, the second crossing is from x to y and the third crossing is from y to x. Between the first and second crossing, the postman must traverse some path $P_1(y,x)$ from y to x. Between the second and third crossing, the postman must traverse some path $P_2(y, y)$ from y to y.

Consider the following rearrangement of the postman's route: (x, y), $P_2(y, y)$, $P_1(y,x)$. This rearrangement starts and ends at x and contains both $P_1(y,x)$ and $P_2(y, y)$ and is shorter than the original route since (x, y) is crossed only once. Thus, the new route is shorter than the original route contradicting the supposition that the original route was optimum. Consequently, no optimum route will cross (x, y) three times in this way.

A similar contradiction results for the other arrangements of the directions for the three crossings of (x, y).

From the lemma, it follows that the postman problem for mixed networks is solved by deciding for each undirected arc (x, y) whether it will be crossed (a) only from x to y, (b) only from y to x, or (c) exactly once in each direction. Once this decision has been made optimally for each undirected arc, then the optimum number of times to repeat each arc is found by the minimum cost flow technique for directed networks given above. (However, for case (c), arc (x, y) must be assigned an infinite cost in each direction so that it will not be repeated.)

With this as motivation, consider the following: Let the demand at vertex x be defined as $\mathrm{Out}(x) - \mathrm{In}(x)$. Negative demands are supplies. To each undirected arc arbitrarily assign a direction, say from x to y. Consonant with this direction from x to y, increase the demand at x by one unit, but do *not* make any changes in the demand at y.

If case (a) is the correct decision for arc (x, y), then the demand at y must be decreased by 1 unit. If it is case (b), then the demand at x must be decreased by 2 units and the demand at y increased by 1 unit. If it is (c), the demand at x must be decreased by one unit and the demand at y left unchanged. To summarize:

TABLE 1

Case	Demand Change at x	Demand Change at y	Total
(a)	0	-1	-1
(b)	-2	1	-1
(c)	-1	0	-1

Since the original demand increase at x was $+1$, the net change in demand for each case is $+1 - 1 = 0$.

Enlarge the network for each undirected arc (x, y) by adding two more vertices and five more directed arcs as shown in Figure 1. Unless otherwise stated, each arc capacity is infinite and its gain factor is $+1$. Source s has a supply of 1 unit. If this unit travels arc (s, x), then the demand at x is decreased by 1 unit which corresponds to case (c) in Table 1. If this unit travels arc (s, v), then 2 units arrive at v; one unit must travel arc (v, x) and the other must travel arc (v, y). This results in two units being supplied at x and one more unit being demanded at y, which corresponds to case (b). Lastly, if the unit at s travels arc (s, y), then the demand at y is decreased by 1, which corresponds to case (a).

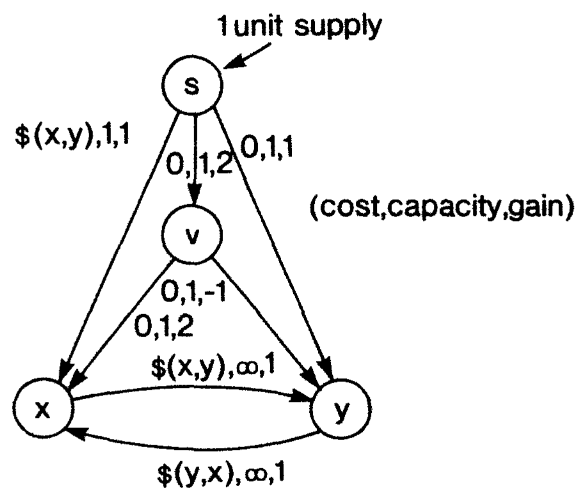

FIGURE 1. Network Enlargement for Undirected Arc (x, y).

Consequently, the mixed postman problem can be solved in the following way:

Step 1. Arbitrarily assign a direction to each undirected arc. For each vertex x, determine $\text{In}(x)$ and $\text{Out}(x)$, but don't count any previously undirected arcs in the determination of $\text{In}(x)$. (Thus, the undirected arcs after being directed only go into the calculation of the out degrees.) Next, for each vertex x, calculate $\text{Out}(x) - \text{In}(x)$, which is the demand at vertex x. If the demand is positive vertex x is a sink; if the demand is negative, vertex x is a source.

Step 2. For each undirected arc (x, y), enlarge the network by adding the configuration shown in Fig. 1. Let each new vertex s be a source with a supply of 1 unit. The reader can easily verify that total demand now equals total supply in the enlarged network. Let each directed arc (x, y) have infinite capacity and cost equal to its original cost $\$(x, y)$.

Step 3. Find a minimum cost integer-valued flow with gains in which all sources discharge all their supplies and all sink demands are satisfied.

If flow units traverse an arc joining two original vertices u and v, then the postman must repeat this arc once for each such flow unit in the direction taken by these flow units.

If the flow unit for source s for undirected arc (x, y), traverses (s, y), then the arbitrary direction for arc (x, y) is left unchanged. If this flow unit traverses arc (s, v) then this direction is reversed. If this flow unit traverses arc (s, x), then arc (x, y) is traversed once in each direction.

Thus, the minimum cost integer flow tells the postman how to optimally direct and repeat (if necessary) the arcs in his network.

PROOF. It is easily verified that each integer flow corresponds to a postman routing and vice versa. Moreover, the cost of each flow equals the total cost of the arcs repeated by the postman. Hence, a minimum cost integer valued flow corresponds to a minimum cost postman route.

Initially Jewell [5] and Maurras [6] devised algorithms for the minimum cost flow with gains problem. Jewell's algorithm is also presented in Christofides [1] and Minieka [8, p. 151]. This problem is a special case of the generalized network problem studied extensively by Glover, Hultz, Klingman and Stutz [4] and by Elam, Glover, and Klingman [3]. Unfortunately, none of these algorithms guarantee that the optimal flow values will be integer. How to modify these flows with gains algorithms for solving the highly structured integer flow with gains problem generated by the postman problem remains an open question. Hopefully this paper will stimulate research in this direction.

Of course, any flow with gains problem is a linear programming problem and consequently, integer requirements can be imposed by using cutting plane techniques. At least integer linear programming presents an available if not elegant solution technique for the mixed postman problem. However, Papadimitriou [9] showed that this problem is NP-complete and consequently we dare not hope for any very elegant solution technique.

4. Windy Postman Problem

Up to now, the cost of traversing an undirected arc was assumed to be the same in both directions. This is hardly a good assumption when one direction might be uphill and the other downhill, when one direction might be with the wind and the other against the wind or when fares are different depending on direction. (Recently the stand-by air fares between New York and London have been quite different depending upon direction.)

Can the mixed postman problem be solved when cost depends upon direction? Happily, yes. The solution procedure outlined in §3 should be amended so that the cheapest direction is always chosen for the arbitrary direction. Suppose $\$(y,x) > \$(x,y)$. The costs in Figure 1 should be altered as follows:

$$\$(s,x) = \$(y,x) + [\$(y,x) - \$(x,y)] > 0.$$

$$\$(s,v) = \$(y,x) - \$(x,y) > 0.$$

In this way, all costs remain positive and Jewell's initialization technique for the flow with gains problem remains feasible.

Can the undirected postman problem be solved when cost depends upon direction? Unfortunately, in this case, the solution technique of Edmonds and Johnson can no longer be applied since it requires the flexibility of using the arcs in either direction. When selecting the paths to be repeated by the postman by finding a minimum cost matching, they reserve the option to use these paths in either direction. For instance, matching vertex a to b might necessitate that the path that matches c to d be used only in a costly direction. Fortunately, this case can be treated as a special case of the mixed postman problem since the solution techniques for the mixed postman problem do not require that any arcs be directed.

Lastly, it was shown in Lemma 1 that if the postman traverses arc (x, y) more than twice, then it is always in the same direction. Suppose for reasons of avoiding boredom or overexposure, the postman wishes to limit the number of times he repeats arc (x, y) to $r(x, y) \geqslant 0$. This situation is easily incorporated into the various solution algorithms simply by changing the infinite capacity of arc (x, y) to the finite value $r(x, y)$. In the totally undirected case, it is obvious that the postman will never repeat an arc twice (otherwise, there would be a better minimum cost matching of the odd degree vertices). Hence, we need only consider the case when no repeats are allowed. This is equivalent to giving arc (x, y) an infinite cost so that it will never be included in any least cost path joining two odd degree vertices let alone in any minimum cost matching of odd degree vertices.

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