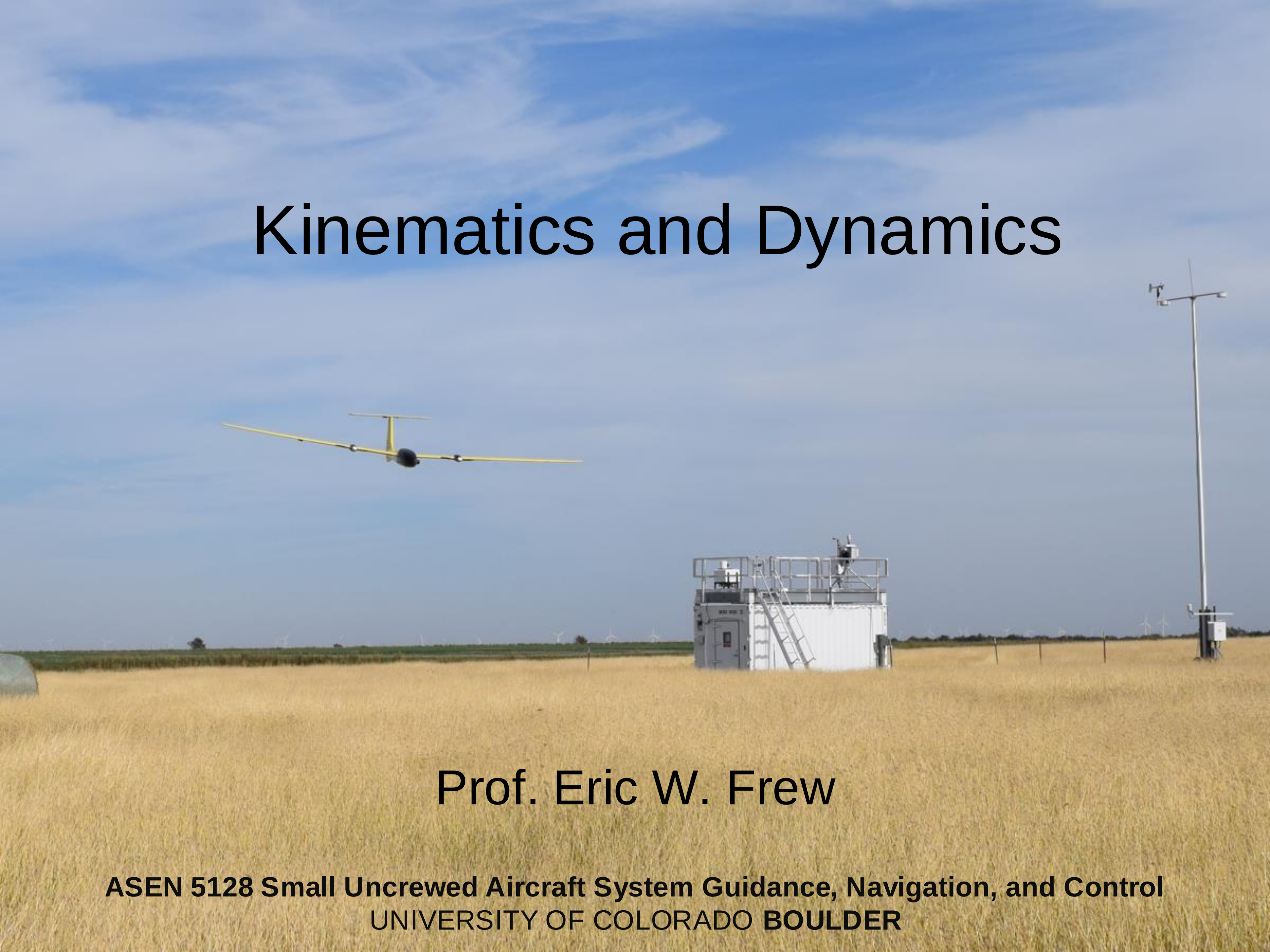
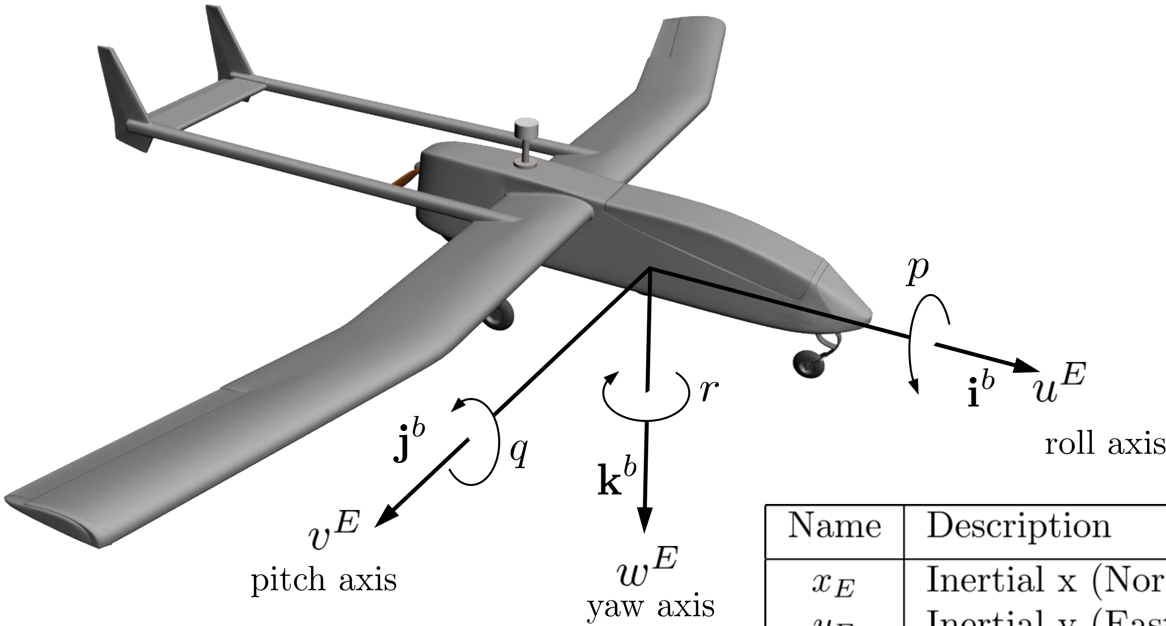


Kinematics and Dynamics

Prof. Eric W. Frew

Aircraft State Variables



Name	Description
x_E	Inertial x (North) position
y_E	Inertial y (East) position
z_E	Inertial z (Down) position
ϕ	Roll angle
θ	Pitch angle
ψ	Yaw angle
u^E	Inertial velocity vector component along $\mathbf{i}_B$
v^E	Inertial velocity vector component along $\mathbf{j}_B$
w^E	Inertial velocity vector component along $\mathbf{k}_B$
p	Angular velocity vector component along $\mathbf{i}_B$ (Roll rate)
q	Angular velocity vector component along $\mathbf{j}_B$ (Pitch rate)
r	Angular velocity vector component along $\mathbf{k}_B$ (Yaw rate)

Very important definition: (u^E,v^E,w^E) are variables used to describe components of inertial velocity expressed in the body coordinate frame.

Translational Kinematics

Rotation matrix

$$\frac{d}{dt}\begin{pmatrix} x_E \\ y_E \\ z_E \end{pmatrix} = R_B^E \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

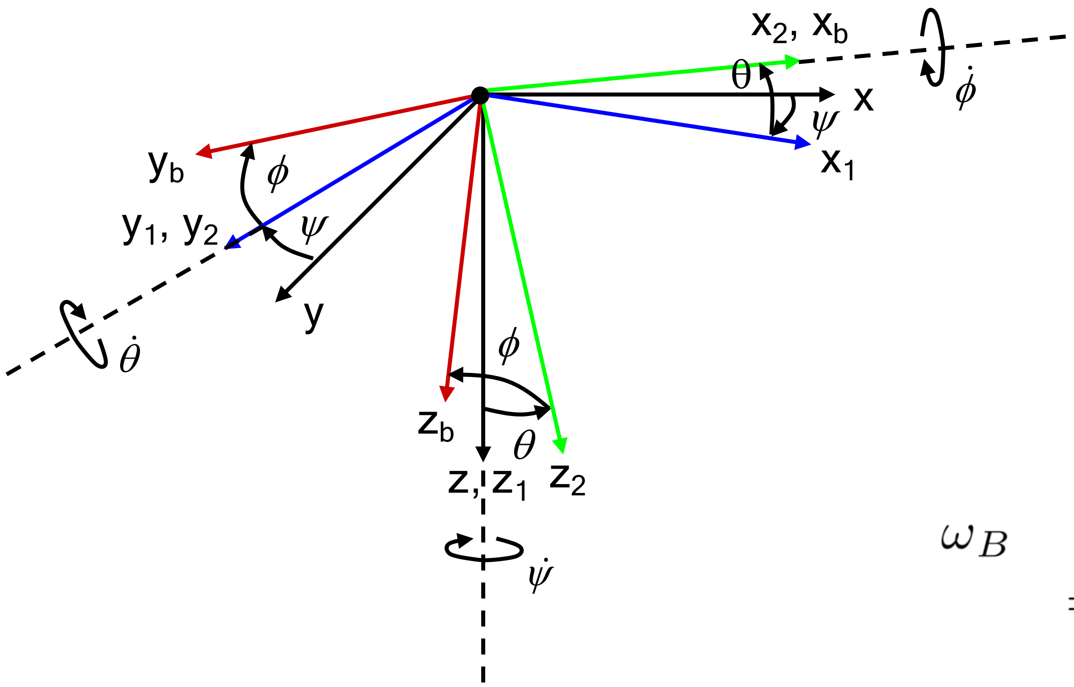
Inertial velocity in inertial coordinates

Inertial velocity in body coordinates

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} \mathrm{c}_\theta \mathrm{c}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{c}_\psi - \mathrm{c}_\phi \mathrm{s}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{c}_\psi + \mathrm{s}_\phi \mathrm{s}_\psi \\ \mathrm{c}_\theta \mathrm{s}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{s}_\psi + \mathrm{c}_\phi \mathrm{c}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{s}_\psi - \mathrm{s}_\phi \mathrm{c}_\psi \\ -\mathrm{s}_\theta & \mathrm{s}_\phi \mathrm{c}_\theta & \mathrm{c}_\phi \mathrm{c}_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

Vector form: $\dfrac{d\mathbf{p}_E}{dt} = \mathbf{V}_E^E = R_B^E \cdot \mathbf{V}_B^E$

Rotational Kinematics



Angular velocity of the aircraft (body coordinate system) with respect to the inertial coordinate system, expressed in body coordinates.

$$\omega_B = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{aligned} \omega_B \quad &= \dot\psi \mathbf{z} + \dot\theta \mathbf{y}_1 + \dot\phi \mathbf{x}_2 \\ &= \dot\psi \mathbf{k}_E + \dot\theta \mathbf{j}_1 + \dot\phi \mathbf{i}_2 \\ &= \dot\psi \mathbf{k}_1 + \dot\theta \mathbf{j}_2 + \dot\phi \mathbf{i}_b \end{aligned}$$

$$\omega_B = R(\phi)^B_{v2} R(\theta)^{v2}_{v1} \begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix} + R(\phi)^b_{v2} \begin{pmatrix} 0 \\ \dot\theta \\ 0 \end{pmatrix} + \begin{pmatrix} \dot\phi \\ 0 \\ 0 \end{pmatrix}$$

Rotational Kinematics

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \dot\phi \\ 0 \\ 0 \end{pmatrix} + \mathcal{R}^b_{v2}(\phi) \begin{pmatrix} 0 \\ \dot\theta \\ 0 \end{pmatrix} + \mathcal{R}^b_{v2}(\phi)\mathcal{R}^{v2}_{v1}(\theta) \begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix}$$

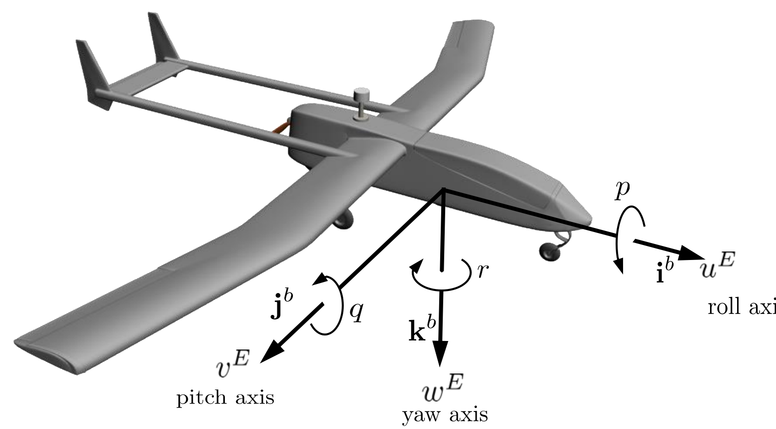
$$= \begin{pmatrix} \dot\phi \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 0 \\ \dot\theta \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{pmatrix} \begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix}$$

Not rotation matrices (not orthonormal)

Inverting gives:

$$\begin{pmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

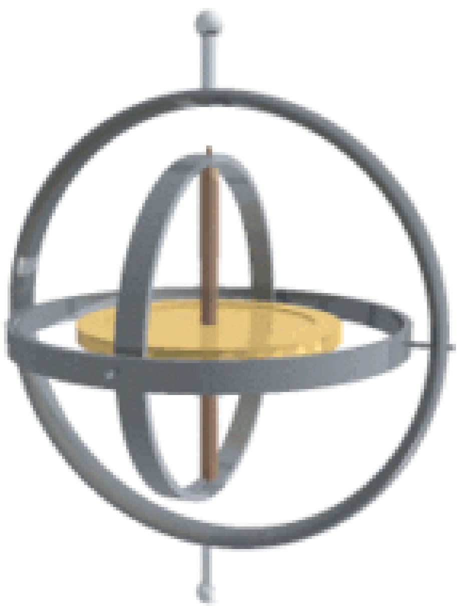


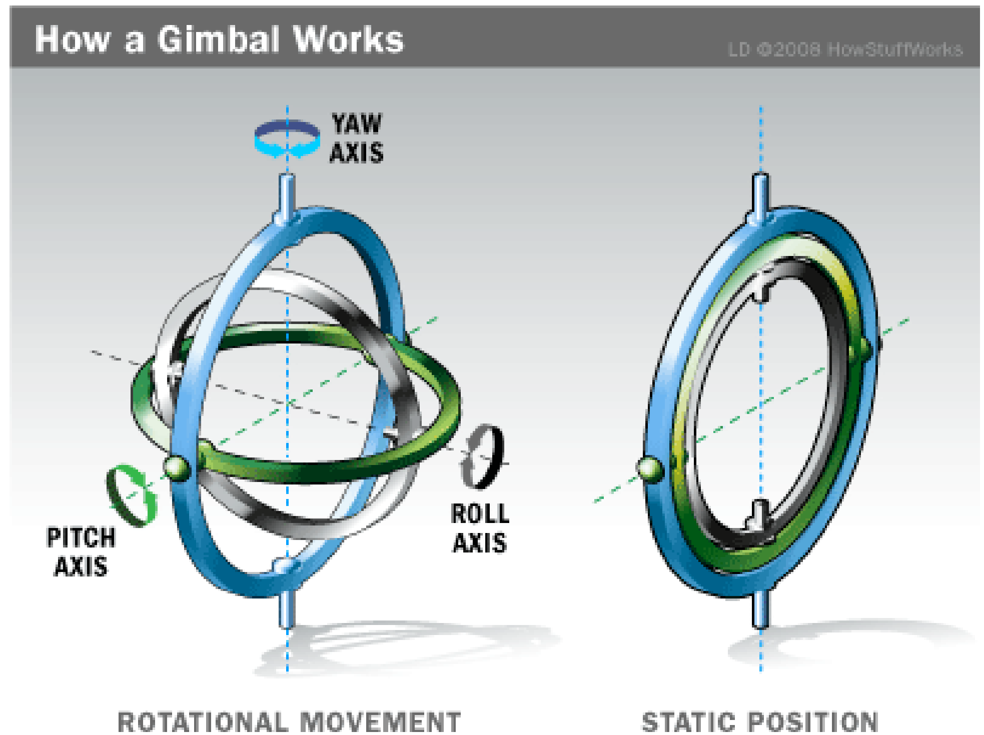
Vector form: $\boxed{\dot{\mathbf{o}}_{B/E}} = \mathbf{T} \cdot \omega_B$

My notation

Gimbal Lock

Rotational kinematic equations breakdown when $\theta = 90^o$, a condition known as gimbal lock because older sensors based on spinning gyros (gimbals) would actually lock up at that configuration.





http://en.wikipedia.org/wiki/Gimbal

http://science.howstuffworks.com/gimbal1.htm

Gimbal Lock

Rotational kinematic equations breakdown when $\theta = 90^o$, a condition known as gimbal lock because older sensors based on spinning gyros (gimbals) would actually lock up at that configuration.

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\tan\theta, \sec\theta \to \infty \text{ as } \theta \to 90^\circ$$

$$R_B^E(\phi,\theta,\psi) = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix}$$

$$R_B^E(\phi,\theta = 90^\circ,\psi) = \begin{pmatrix} 0 & s_\phi c_\psi - c_\phi s_\psi & c_\phi c_\psi + s_\phi s_\psi \\ 0 & s_\phi s_\psi + c_\phi c_\psi & c_\phi s_\psi - s_\phi c_\psi \\ -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sin(\phi-\psi) & \cos(\phi-\psi) \\ 0 & \cos(\phi-\psi) & -\sin(\phi-\psi) \\ -1 & 0 & 0 \end{pmatrix}$$

State Equations

Six of the 12 state equations for the aircraft come from the kinematic equations relating positions and velocities:

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

The remaining six equations will come from applying Newton's 2nd law to the translational and rotational motion of the aircraft.

State Equations

Six of the 12 state equations for the aircraft come from the kinematic equations relating positions and velocities:

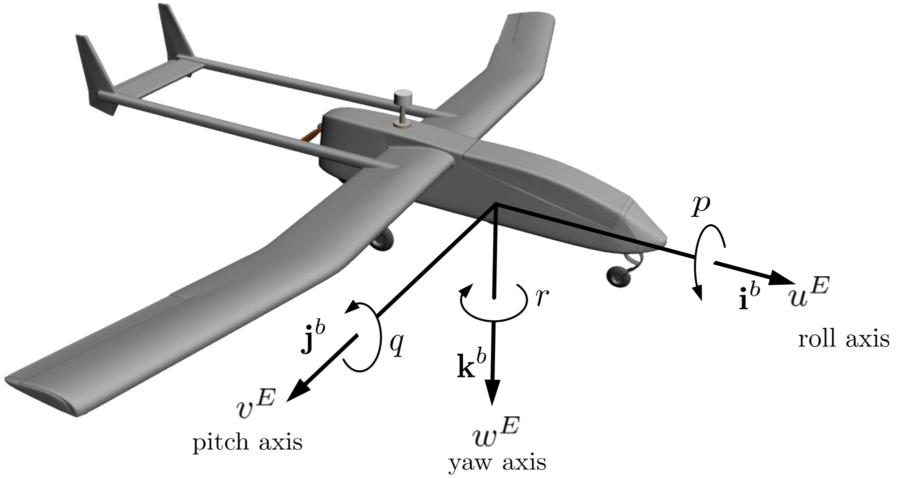
$$\frac{d\mathbf{p}_E}{dt} = R_B^E \cdot \mathbf{V}_B^E$$

“Orientation vector” not in a coordinate system
(Also referred to as “attitude” or just “Euler angles”)

$$\dot{\mathbf{o}}_{B/E} = \mathbf{T} \cdot \omega_B$$

The remaining six equations will come from applying Newton’s 2nd law to the translational and rotational motion of the aircraft.

Translational Dynamics



Newton's 2nd Law:

$$m\frac{d\mathbf{V}^E}{dt} = \mathbf{f}$$

- $\mathbf{f}$ is the sum of all external forces
- m is the mass of the aircraft

$$m\frac{d\mathbf{V}_B^E}{dt} = \mathbf{f}_B$$

$$\frac{d\mathbf{V}_B^E}{dt} = \dot{\mathbf{V}}_B^E + \tilde{\omega}_B\mathbf{V}_B^E$$

$$m\left(\dot{\mathbf{V}}_B^E + \tilde{\omega}_B\mathbf{V}_B^E\right) = \mathbf{f}_B$$

Translational Dynamics

$$m\left(\dot{\mathbf{V}}_B^E + \tilde{\omega}_B \mathbf{V}_B^E\right) = \mathbf{f}_B$$

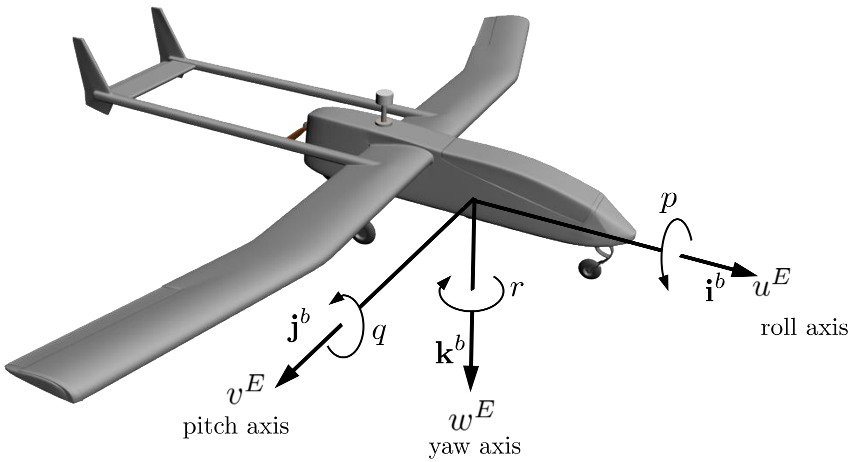
$$\dot{\mathbf{V}}_B^E = -\tilde{\omega}_B \mathbf{V}_B^E + \frac{\mathbf{f}_B}{m}$$

$$\mathbf{V}_B^E = \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix} \qquad \omega_B = \begin{pmatrix} p \\ q \\ r \end{pmatrix} \qquad \mathbf{f}_B = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

Since $\dot{\mathbf{V}}_B^E = \begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix}$ we have that

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + \frac{1}{m}\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

Rotational Dynamics



Newton's 2nd Law:

$$\frac{d\mathbf{h}}{dt} = \mathbf{G}$$

- $\mathbf{h}$ is the angular momentum vector
- $\mathbf{G}$ is the sum of all external moments

Expressed in the body frame,

$$\frac{d\mathbf{h}_B}{dt} = \dot{\mathbf{h}}_B + \tilde{\omega}_B \mathbf{h}_B = \mathbf{G}_B$$

Rotational Dynamics

For a rigid body, angular momentum is defined as the product of the inertia matrix and the angular velocity vector:

$$\mathbf{h} = \mathbf{I}\omega$$

$$\mathbf{I} = \begin{pmatrix} \int (y^2 + z^2)\, d\mathrm{m} & -\int xy\, d\mathrm{m} & -\int xz\, d\mathrm{m} \\ -\int xy\, d\mathrm{m} & \int (x^2 + z^2)\, d\mathrm{m} & -\int yz\, d\mathrm{m} \\ -\int xz\, d\mathrm{m} & -\int yz\, d\mathrm{m} & \int (x^2 + y^2)\, d\mathrm{m} \end{pmatrix}$$

$$= \begin{pmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{pmatrix}$$

Diagonal elements are called moments of inertia. Off-diagonal elements are called products of inertia.

$\mathbf{I}$ determined from mass properties in CAD program or measured experimentally using a bifilar pendulum.

Rotational Dynamics

$$\dot{\mathbf{h}}_B + \tilde{\omega}_B \mathbf{h}_B = \mathbf{G}_B$$

$$\dot{(\mathbf{I}_B \omega_B)} + \tilde{\omega}_B \left(\mathbf{I}_B \omega_B \right) = \mathbf{G}_B$$

$$\dot{\mathbf{I}}_B \omega_B + \mathbf{I}_B \dot{\omega}_B + \tilde{\omega}_B \left(\mathbf{I}_B \omega_B \right) = \mathbf{G}_B$$

Because $\mathbf{I}_B$ is unchanging in the body frame, $\dot{\mathbf{I}}_B = 0$ and

$$\mathbf{I}_B \dot{\omega}_B + \tilde{\omega}_B \left(\mathbf{I}_B \omega_B \right) = \mathbf{G}_B$$

Rearranging we get

$$\dot{\omega}_B = \mathbf{I}_B^{-1} \left[-\tilde{\omega}_B \left(\mathbf{I}_B \omega_B \right) + \mathbf{G}_B \right]$$

where $$\dot{\omega}_B = \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} \qquad \mathbf{G}_B = \begin{pmatrix} L \\ M \\ N \end{pmatrix}$$

Rotational Dynamics

If the aircraft is symmetric about the $\mathbf{i}_B - \mathbf{k}_B$ plane, then $I_{xy} = I_{yz} = 0$ and

$$\mathbf{I}_B = \begin{pmatrix} I_x & 0 & -I_{xz} \\ 0 & I_y & 0 \\ -I_{xz} & 0 & I_z \end{pmatrix}$$

This symmetry assumption helps simplify the analysis. The inverse of $\mathbf{I}_B$ becomes

$$\mathbf{I}_B^{-1} = \begin{pmatrix} \frac{I_z}{\Gamma} & 0 & \frac{I_{xz}}{\Gamma} \\ 0 & \frac{1}{I_y} & 0 \\ \frac{I_{xz}}{\Gamma} & 0 & \frac{I_x}{\Gamma} \end{pmatrix} \qquad \Gamma = I_x I_z - I_{xz}^2$$

Rotational Dynamics

$$\dot{\omega}_B = \mathbf{I}_B^{-1}\left[-\tilde{\omega}_B\left(\mathbf{I}_B\omega_B\right) + \mathbf{G}_B\right]$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) \\ \Gamma_7 pq - \Gamma_1 qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \dfrac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

$$\Gamma_1 = \frac{I_{xz}\left(I_x - I_y + I_z\right)}{\Gamma} \qquad \Gamma_4 = \frac{I_{xz}}{\Gamma} \qquad \Gamma_7 = \frac{I_x\left(I_x - I_y\right) + I_{xz}^2}{\Gamma}$$

$$\Gamma_2 = \frac{I_z\left(I_z - I_y\right) + I_{xz}^2}{\Gamma} \qquad \Gamma_5 = \frac{I_z - I_x}{I_y} \qquad \Gamma_8 = \frac{I_x}{\Gamma}$$

$$\Gamma_3 = \frac{I_z}{\Gamma} \qquad \Gamma_6 = \frac{I_{xz}}{I_y} \qquad \Gamma = I_x I_z - I_{xz}^2$$

Dynamic Equations

The remaining six equations for the aircraft come from the dynamic equations relating derivatives of velocities and forces and moments:

$$\dot{\mathbf{V}}_B^E = -\tilde{\omega}_B \mathbf{V}_B^E + \frac{\mathbf{f}_B}{m}$$

$$\dot{\omega}_B = \mathbf{I}_B^{-1} \left[-\tilde{\omega}_B \left(\mathbf{I}_B \omega_B \right) + \mathbf{G}_B \right]$$

Equation of Motion Summary

$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi - c_\phi s_\psi & c_\phi s_\theta c_\psi + s_\phi s_\psi \\ c_\theta s_\psi & s_\phi s_\theta s_\psi + c_\phi c_\psi & c_\phi s_\theta s_\psi - s_\phi c_\psi \\ -s_\theta & s_\phi c_\theta & c_\phi c_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + \frac{1}{m} \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$

System of 12 first-order ODE’s

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) \\ \Gamma_7 pq - \Gamma_1 qr \end{pmatrix} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \frac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

Equation of Motion in Vector Form

$$\dot{\mathbf{p}}^E = R_B^E \cdot \mathbf{V}_B^E$$

$$\dot{\mathbf{o}}_{B/E} = \mathbf{T} \cdot \omega_B$$

$$\dot{\mathbf{V}}_B^E = -\tilde{\omega}_B \mathbf{V}_B^E + \frac{\mathbf{f}_B}{m}$$

$$\dot{\omega}_B = \mathbf{I}_B^{-1} \left[-\tilde{\omega}_B \left(\mathbf{I}_B \omega_B \right) + \mathbf{G}_B \right]$$

Equations of Motion in Vector Form

$$\mathbf{x} = \begin{pmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{pmatrix}$$

Aircraft dynamics (equations of motion) can be described as a function of a single aircraft state vector $\mathbf{x}$.

$$\dot{\mathbf{x}} = f_{EOM}\left(\mathbf{x}, (\mathbf{f}_B, \mathbf{G}_B)^T\right)$$

For now the "inputs" to the equations of motion are the forces and moments acting on the aircraft. The next step is to define them as functions of the aircraft state and the aircraft control surfaces.