Forces and Moments

Prof. Eric W. Frew

Announcements

- Office Hours now set:

  Wednesdays 2-3 PM, AERO 269

Equation of Motion in Vector Form

$$\dot{\mathbf{p}}^E = R_B^E \cdot \mathbf{V}_B^E$$

$$\dot{\mathbf{o}}_{B/E} = \mathbf{T} \cdot \omega_B$$

$$\dot{\mathbf{V}}_B^E = -\tilde{\omega}_B \mathbf{V}_B^E + \frac{\boxed{\mathbf{f}_B}}{m}$$

$$\dot{\omega}_B = \mathbf{I}_B^{-1} \left[-\tilde{\omega}_B \left(\mathbf{I}_B \omega_B\right) + \boxed{\mathbf{G}_B}\right]$$

What are the forces and moments acting on the aircraft?

External Forces and Moments

$$\mathbf{f} = {}^{\mathrm{g}}\mathbf{f} + {}^{\mathrm{a}}\mathbf{f} + {}^{\mathrm{p}}\mathbf{f}$$

<u>g</u>ravity, <u>a</u>erodynamic, <u>p</u>ropulsion

$$\mathbf{G} = {}^{a}\mathbf{G}$$

Gravity Force

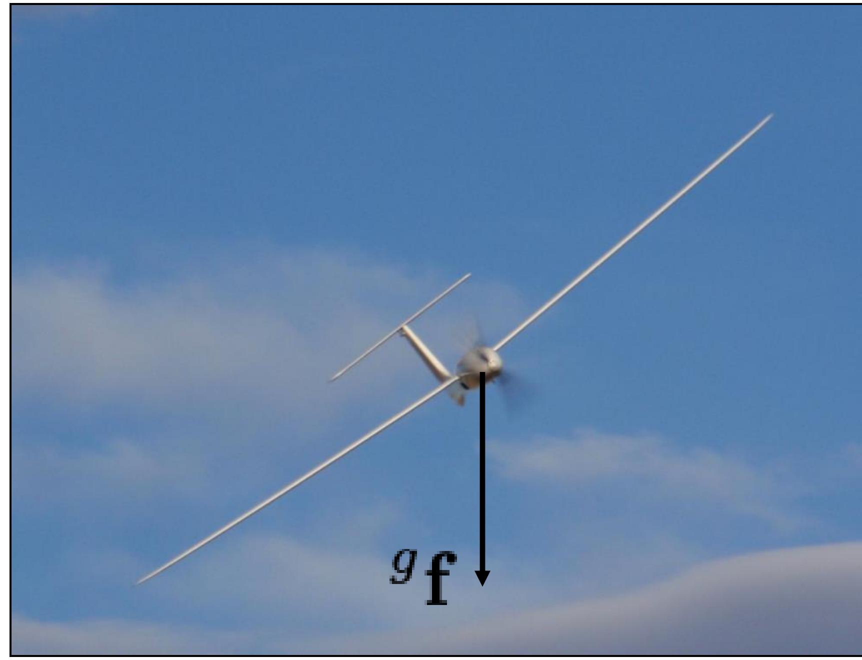
expressed in inertial frame

$$ {}^{g}\mathbf{f}_E = \begin{pmatrix} 0 \\ 0 \\ mg \end{pmatrix} $$

expressed in body frame

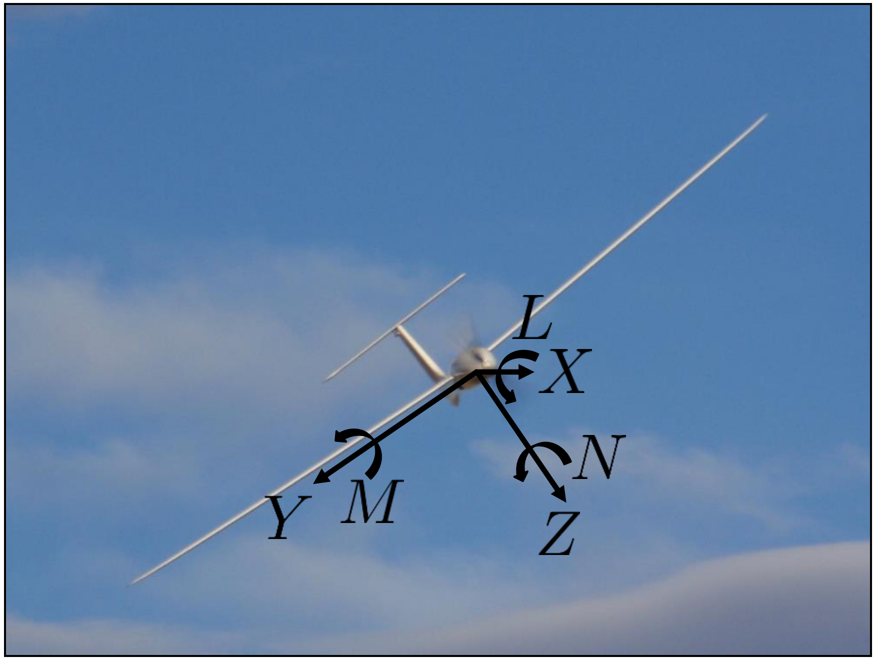
$$ {}^{g}\mathbf{f}_B = \mathbf{R}_E^B \begin{pmatrix} 0 \\ 0 \\ mg \end{pmatrix} $$

$$ = \begin{pmatrix} -mg\sin\theta \\ mg\cos\theta\sin\phi \\ mg\cos\theta\cos\phi \end{pmatrix} $$



Aerodynamic Forces and Moments

$$^{a}\mathbf{f}_B = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$



$$\mathbf{G}_B = {}^{a}\mathbf{G}_B = \begin{pmatrix} L \\ M \\ N \end{pmatrix}$$

The aerodynamic forces and moments are functions of the aircraft state, so more to come…

Propulsive Force

expressed in body frame

$$^{p}\mathbf{f}_B = \begin{pmatrix} T \\ 0 \\ 0 \end{pmatrix}$$

Assumes body frame is defined to be aligned with the thrust.
Not always true, but a good assumption in many cases.

Equation of Motion Summary

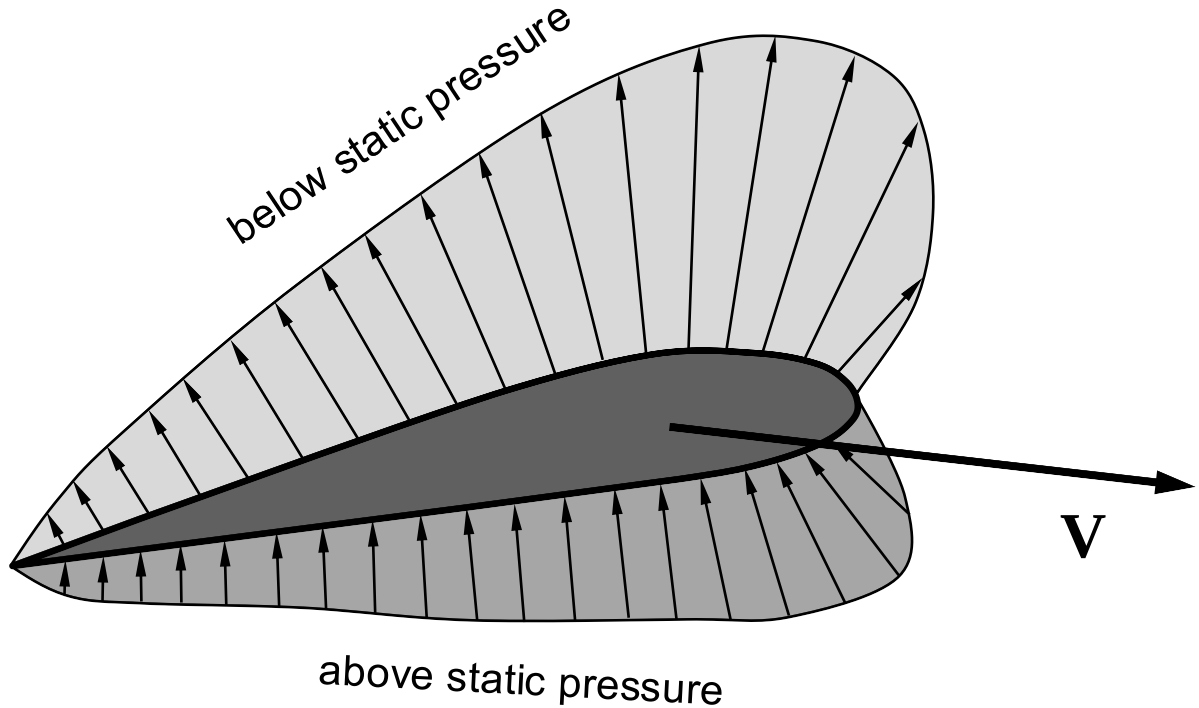
$$\begin{pmatrix} \dot{x}_E \\ \dot{y}_E \\ \dot{z}_E \end{pmatrix} = \begin{pmatrix} \mathrm{c}_\theta \mathrm{c}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{c}_\psi - \mathrm{c}_\phi \mathrm{s}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{c}_\psi + \mathrm{s}_\phi \mathrm{s}_\psi \\ \mathrm{c}_\theta \mathrm{s}_\psi & \mathrm{s}_\phi \mathrm{s}_\theta \mathrm{s}_\psi + \mathrm{c}_\phi \mathrm{c}_\psi & \mathrm{c}_\phi \mathrm{s}_\theta \mathrm{s}_\psi - \mathrm{s}_\phi \mathrm{c}_\psi \\ -\mathrm{s}_\theta & \mathrm{s}_\phi \mathrm{c}_\theta & \mathrm{c}_\phi \mathrm{c}_\theta \end{pmatrix} \begin{pmatrix} u^E \\ v^E \\ w^E \end{pmatrix}$$

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \sin\phi \tan\theta & \cos\phi \tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi \sec\theta & \cos\phi \sec\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

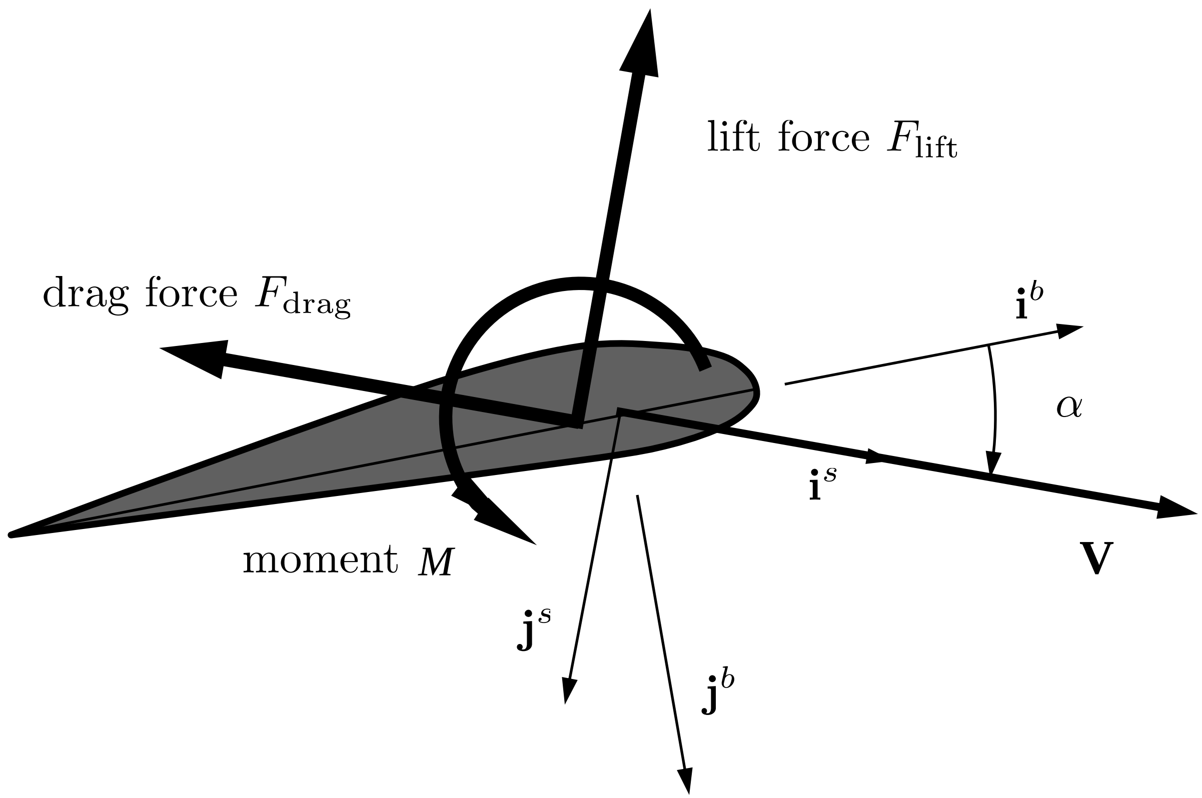
$$\begin{pmatrix} \dot{u}^E \\ \dot{v}^E \\ \dot{w}^E \end{pmatrix} = \begin{pmatrix} rv^E - qw^E \\ pw^E - ru^E \\ qu^E - pv^E \end{pmatrix} + g \begin{pmatrix} -\sin\theta \\ \cos\theta \sin\phi \\ \cos\theta \cos\phi \end{pmatrix} + \frac{1}{m} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \frac{1}{m} \begin{pmatrix} T \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} \Gamma_1 pq - \Gamma_2 qr \\ \Gamma_5 pr - \Gamma_6 (p^2 - r^2) \\ \Gamma_7 pq - \Gamma qr \quad {}_1 \end{pmatrix} + \begin{pmatrix} \Gamma_3 L + \Gamma_4 N \\ \frac{1}{I_y} M \\ \Gamma_4 L + \Gamma_8 N \end{pmatrix}$$

Airfoil Pressure Distribution



Aerodynamic Approximation



$$F_{\text{lift}} = \frac{1}{2}\rho V_a^2 S C_L$$

$$F_{\text{drag}} = \frac{1}{2}\rho V_a^2 S C_D$$

$$M = \frac{1}{2}\rho V_a^2 S c C_m$$

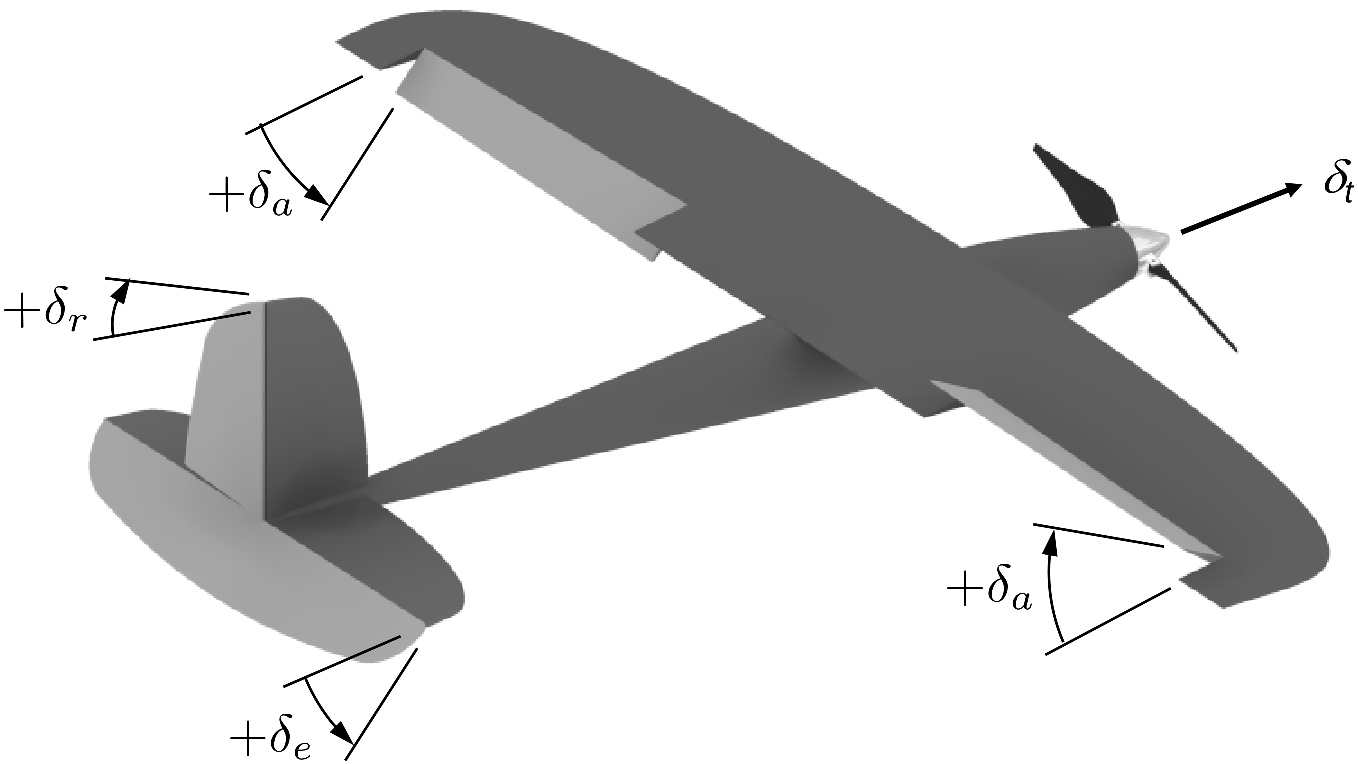
Pressure distribution over wing simplified to set of aerodynamic forces acting at quarter chord (aerodynamic center) of the wing and moments.

Contributions from multiple surfaces (wing, vertical tail, horizontal tail) combined into a single force vector and a single moment vector.

Control Surfaces - Conventional

$$\delta_a = \frac{1}{2}\left(\delta_{a\text{-left}} - \delta_{a\text{-right}}\right)$$

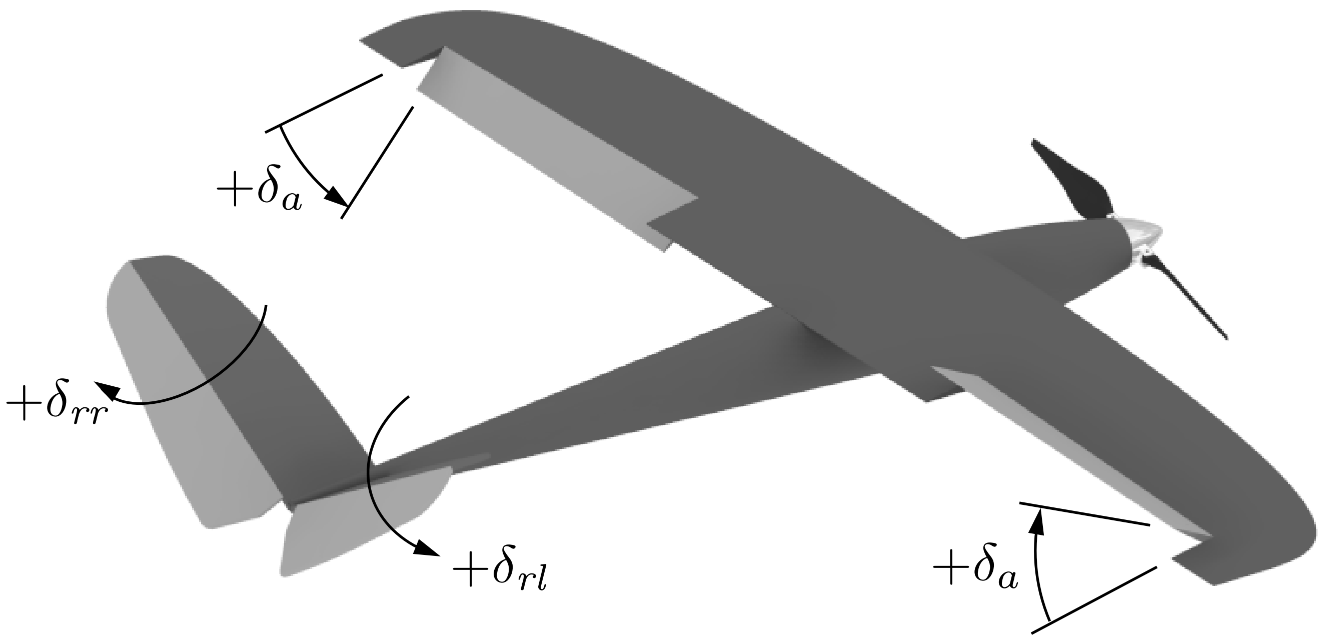
$+\delta_a$ results in positive roll rate p
$+\delta_e$ results in negative pitch rate q
$+\delta_r$ results in negative yaw rate r



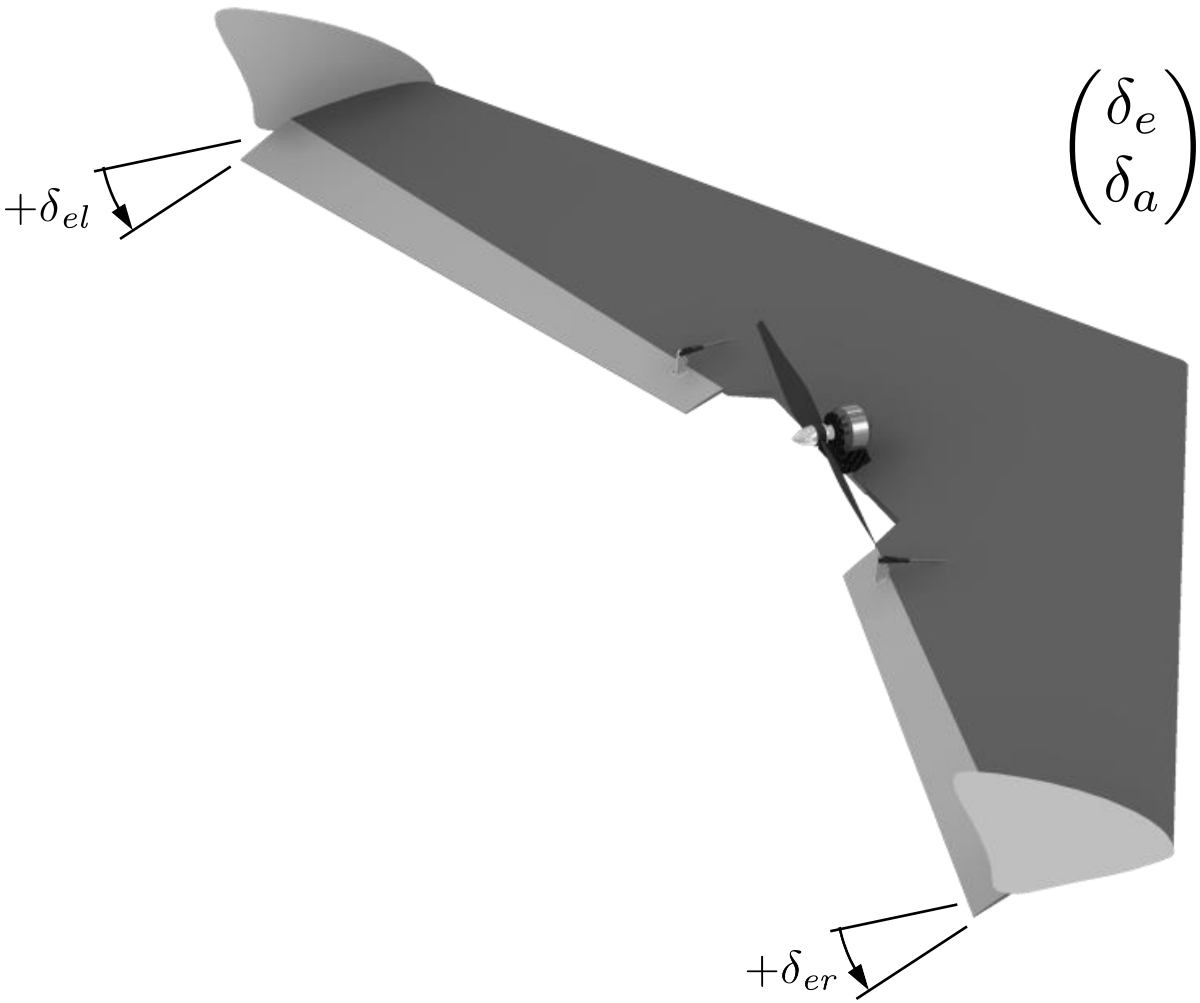
These are referred to as the "control surfaces" even though throttle is not an actual surface.

Control Surfaces – V-tail

$$\begin{pmatrix} \delta_e \\ \delta_r \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \delta_{rr} \\ \delta_{rl} \end{pmatrix}$$



Control Surfaces – Flying Wing



$$\begin{pmatrix}\delta_e \\ \delta_a\end{pmatrix} = \begin{pmatrix}1 & 1 \\ -1 & 1\end{pmatrix}\begin{pmatrix}\delta_{er} \\ \delta_{el}\end{pmatrix}$$

Aircraft Dynamics

- Aircraft dynamics and aerodynamics are commonly separated into two groups:
  - Longitudinal
    - Up-down, pitch plane, pitching motions
  - Lateral-directional
    - Side-to-side, turning motions (roll and yaw)

Longitudinal Aerodynamics

- Act in the $\mathbf{i}^b$-$\mathbf{k}^b$ plane, aka the pitch plane
- Heavily influenced by angle of attack
- Also influenced by pitch rate and elevator deflection

$$F_{\text{lift}} \approx \frac{1}{2}\rho V_a^2 S C_L(\alpha, q, \delta_e)$$

$$F_{\text{drag}} \approx \frac{1}{2}\rho V_a^2 S C_D(\alpha, q, \delta_e)$$

$$M \approx \frac{1}{2}\rho V_a^2 S c C_m(\alpha, q, \delta_e)$$

- These expressions now represent aerodynamic forces and moments acting on the aircraft (not just the wing)

Aerodynamic Approximation

$$F_{\text{lift}} \approx \frac{1}{2}\rho V_a^2 S C_L(\alpha, q, \delta_e)$$ nonlinear in general case

1st-order Taylor series approximation (linear):

$$F_{\text{lift}} = \frac{1}{2}\rho V_a^2 S \left[C_{L_0} + \frac{\partial C_L}{\partial \alpha}\alpha + \frac{\partial C_L}{\partial q} q + \frac{\partial C_L}{\partial \delta_e}\delta_e \right]$$

$$F_{\text{lift}} = \frac{1}{2}\rho V_a^2 S \left[C_{L_0} + C_{L_\alpha}\alpha + C_{L_q} \underbrace{\frac{c}{2V_a} q}_{=\hat{q}} + C_{L_{\delta_e}}\delta_e \right]$$

stability derivatives

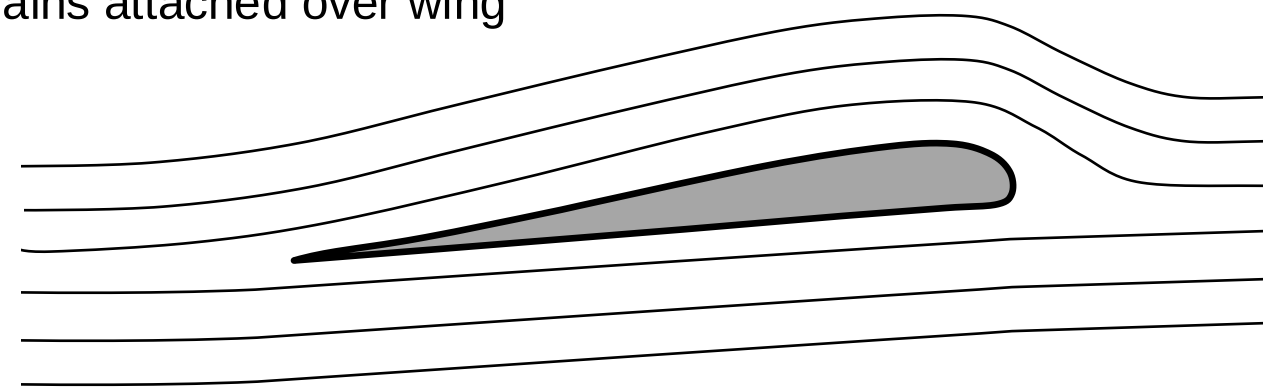
Coefficients C_{L_0}, $C_{L_\alpha} \triangleq \frac{\partial C_L}{\partial \alpha}$, $C_{L_q} \triangleq \frac{\partial C_L}{\partial \frac{qc}{2V_a}}$, and $C_{L_{\delta_e}} \triangleq \frac{\partial C_L}{\partial \delta_e}$ are dimensionless quantities

control derivative

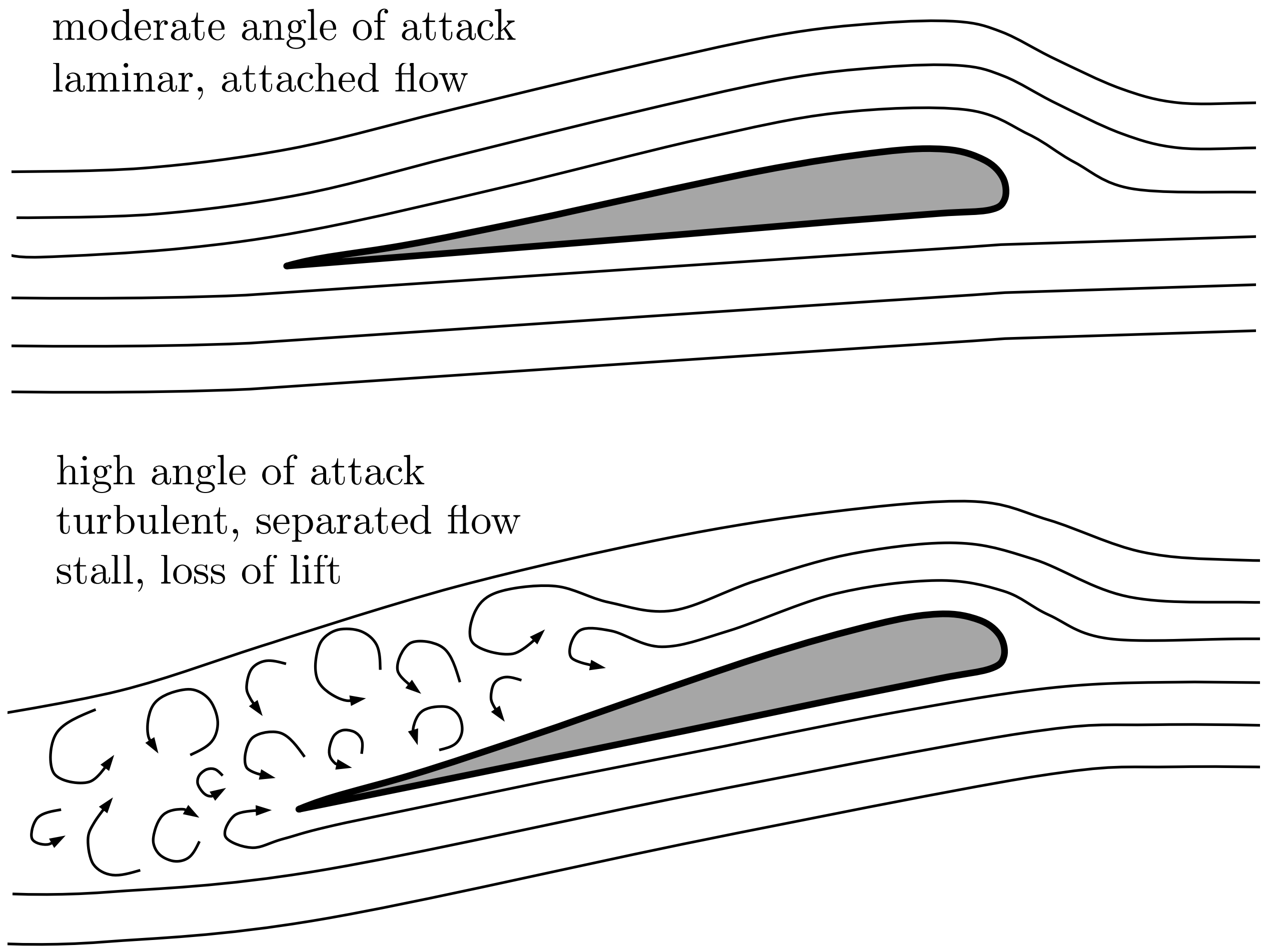
Linear Aerodynamic Model

$$F_{\text{lift}} = \frac{1}{2}\rho V_a^2 S\left[C_{L_0} + C_{L_\alpha}\alpha + C_{L_q}\frac{c}{2V_a}q + C_{L_{\delta_e}}\delta_e\right]$$

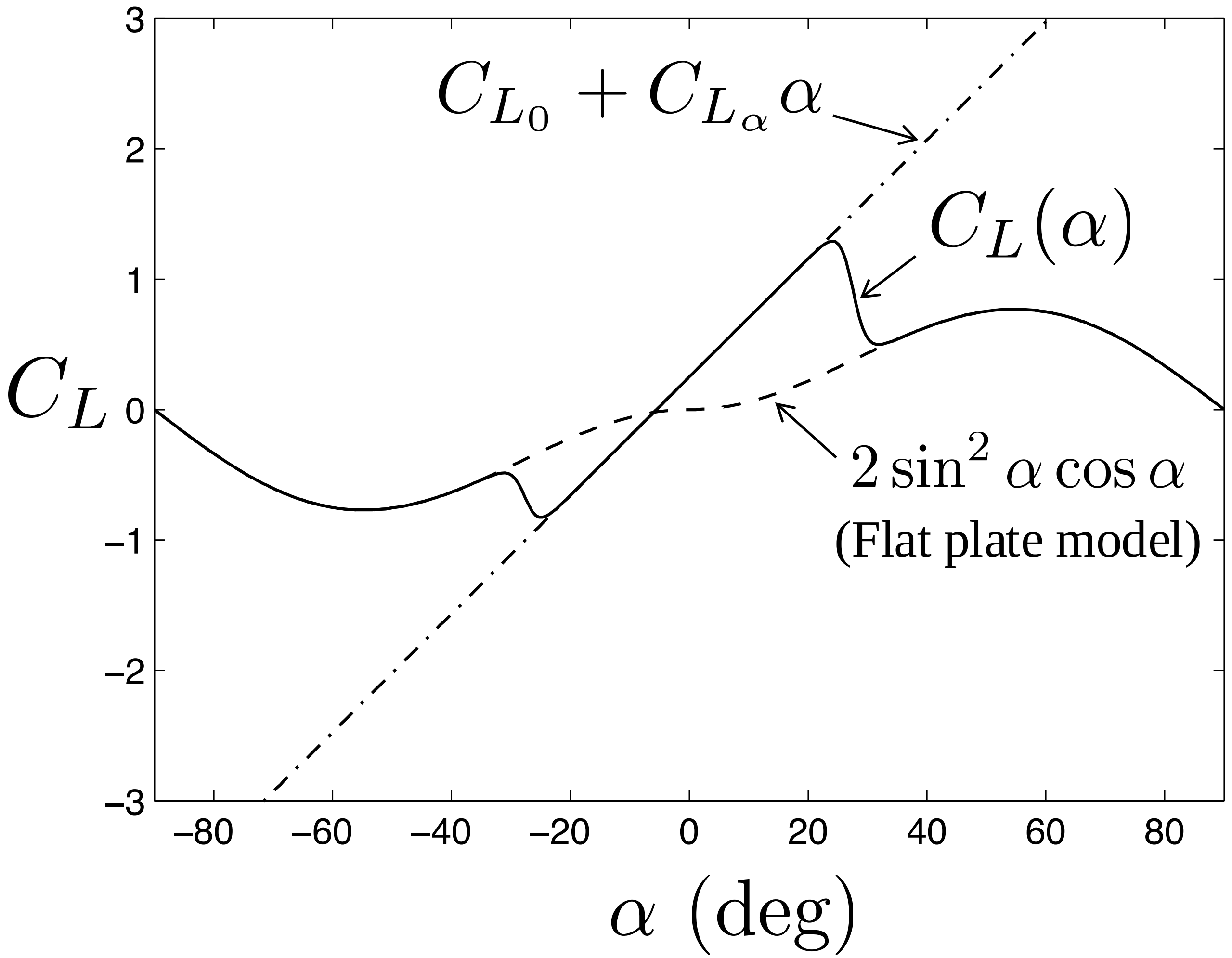
Linear aerodynamic model is valid for small angles of attack – flow remains attached over wing



Nonlinear Aerodynamics – Stall



Nonlinear Lift Model



Nonlinear Aerodynamic Model

$$F_{\text{lift}} = \frac{1}{2}\rho V_a^2 S \left[\underline{C_L(\alpha)} + C_{L_q}\frac{c}{2V_a}q + C_{L_{\delta_e}}\delta_e\right]$$

$$C_L(\alpha) = (1-\sigma(\alpha))\left[\underline{C_{L_0} + C_{L_\alpha}\alpha}\right] + \sigma(\alpha)\left[\underline{2\,\text{sign}(\alpha)\sin^2\alpha\cos\alpha}\right]$$

linear model

flat-plate model

$$\sigma(\alpha) = \frac{1 + e^{-M(\alpha-\alpha_0)} + e^{M(\alpha+\alpha_0)}}{\left(1+e^{-M(\alpha-\alpha_0)}\right)\left(1+e^{M(\alpha+\alpha_0)}\right)}$$

blending function

Why bother with blending function?

Nonlinear Aerodynamic Model

$$C_{L_\alpha} = \frac{\pi AR}{1 + \sqrt{1 + (AR/2)^2}}$$

Sensitivity of lift to angle of attack

$AR \triangleq b^2/S$ is the wing aspect ratio

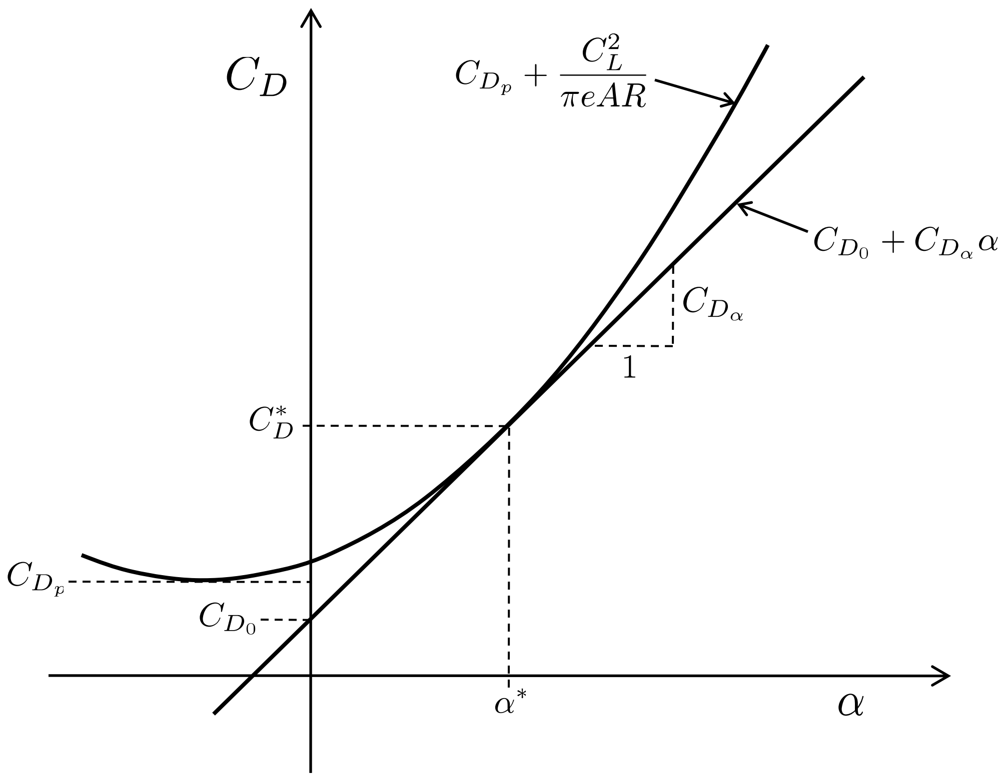
b is the wingspan

S is the wing area

What about effects from other surfaces?

Break

Drag vs. Angle of Attack



$$C_D(\alpha) = C_{D_p} + \frac{C_L(\alpha)^2}{\pi e AR}$$

parasitic drag

induced drag

e is the Oswald efficiency factor

Model of a wing and deviation from "ideal" elliptical shape

Drag Model for Aircraft

Also called “cambered wing” model

$$C_D = C_{D_{min}} + K\left(C_L - C_{L_{min}}\right)^2$$

Derived from CFD or experimental drag polar

$$K = \frac{1}{\pi e AR}$$

e	is the Oswald efficient factor
AR	is the aspect ratio

Linear Lift and Drag Models

$$C_L(\alpha) = C_{L_0} + C_{L_\alpha}\alpha$$
$$C_D(\alpha) = C_{D_0} + C_{D_\alpha}\alpha$$

Valid for small deviations of angle of attack from trim

Good for analysis, can do better for simulation

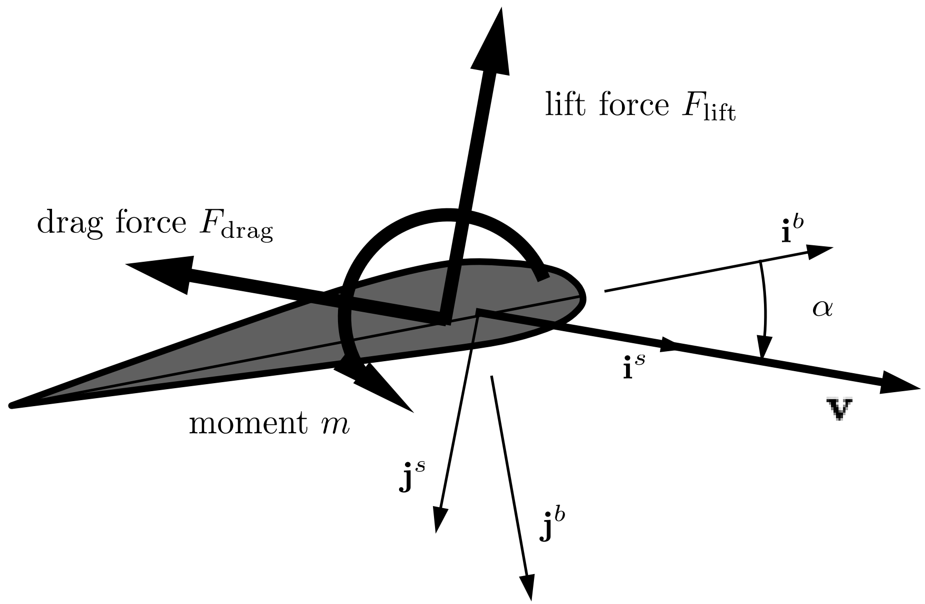
Nonlinear Lift and Drag Models

In this class we will use:

$$C_L(\alpha, q, \delta_e) = C_{L_0} + C_{L_\alpha}\alpha + C_{L_q}\frac{c}{2V_a}q + C_{L_{\delta_e}}\delta_e$$

$$C_D(\alpha, q, \delta_e) = C_{D_{min}} + K\left(C_L(\alpha, q, \delta_e) - C_{L_{min}}\right)^2$$

Longitudinal Forces – Body Frame



Lift and drag defined in stability (wind) frame, so rotate into body frame

$$\begin{pmatrix} X \\ Z \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} -F_{drag} \\ -F_{lift} \end{pmatrix}$$

Pitching Moment

$$M = \frac{1}{2}\rho V_a^2 Sc\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\frac{c}{2V_a}q + C_{m_{\delta_e}}\delta_e\right]$$

- Use linear model
- No rotation transformation necessary

Lateral Aerodynamics

$$
\begin{aligned}
Y &= \frac{1}{2}\rho V_a^2 S C_Y(\beta, p, r, \delta_a, \delta_r) \\
L &= \frac{1}{2}\rho V_a^2 S b C_l(\beta, p, r, \delta_a, \delta_r) \\
N &= \frac{1}{2}\rho V_a^2 S b C_n(\beta, p, r, \delta_a, \delta_r)
\end{aligned}
$$

Lateral Aerodynamics

$$
\begin{aligned}
Y &\approx \frac{1}{2}\rho V_a^2 S\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\frac{b}{2V_a}p + C_{Y_r}\frac{b}{2V_a}r + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r\right]\\
L &\approx \frac{1}{2}\rho V_a^2 Sb\left[C_{l_0} + C_{l_\beta}\beta + C_{l_p}\frac{b}{2V_a}p + C_{l_r}\frac{b}{2V_a}r + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r\right]\\
N &\approx \frac{1}{2}\rho V_a^2 Sb\left[C_{n_0} + C_{n_\beta}\beta + C_{n_p}\frac{b}{2V_a}p + C_{n_r}\frac{b}{2V_a}r + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r\right]
\end{aligned}
$$

For symmetric aircraft, $C_{Y_0} = C_{l_0} = C_{n_0} = 0$

Aerodynamic Coefficients

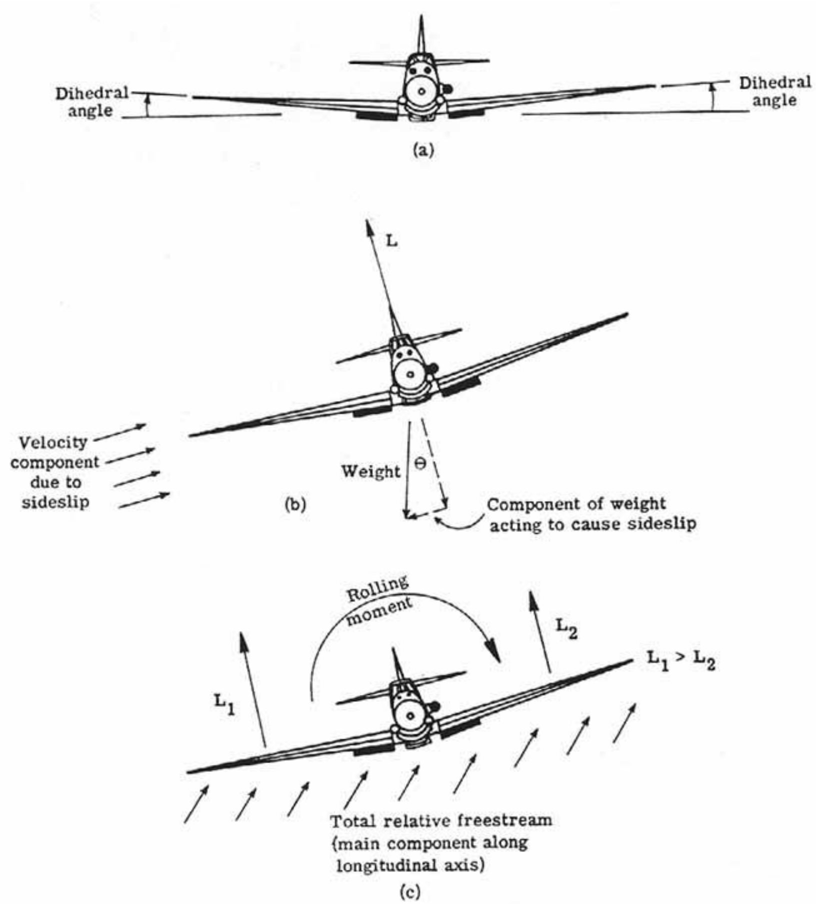
C_{m_α}, C_{ℓ_β}, C_{n_β}, C_{m_q}, C_{ℓ_p}, C_{n_r} are called the *stability derivatives* because their values determine the stability of the aircraft.

Static Stability Derivatives

- C_{m_α} - longitudinal static stability derivative. Must be ≤ 0 for stability: increase in α causes and downward pitching moment.
- C_{ℓ_β} - roll static stability derivative. Associated with dihedral in wings. Must be ≤ 0 for stability: positive roll ϕ causes a restoring moment.
- C_{n_β} - yaw static stability derivative. Weathercock stability derivative. Influenced by design of tail. Causes airframe to align with the wind vector. Must be ≥ 0 for stability: cocks airframe into wind driving β to zero.

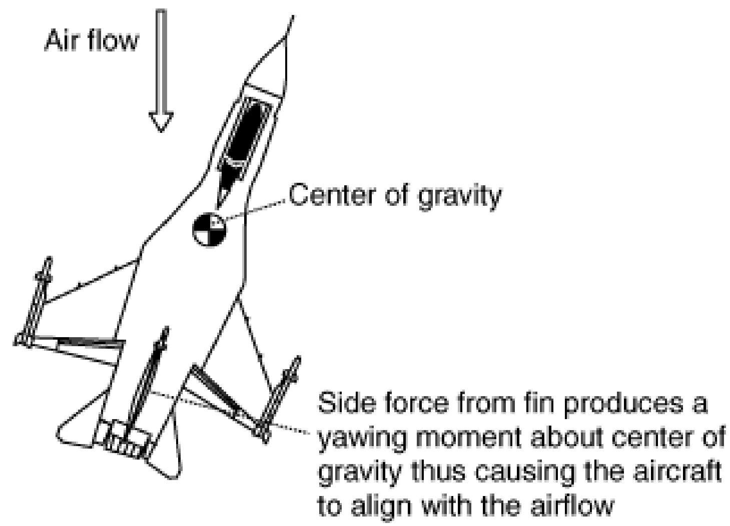
Static Stability Derivatives

Dihedral effect and C_{l_β}



http://history.nasa.gov/SP-367/f144.htm

Weathervaning and C_{n_β}



http://www.answers.com/topic/directional-stability

Aerodynamic Coefficients

Dynamic Stability Derivatives

- C_{m_q}, C_{ℓ_p}, C_{n_r} are known as the pitch damping derivative, roll damping derivative, and yaw damping derivative, respectively. The quantify the level of damping associated with angular motion of the airframe.

Control Derivatives

- $C_{m_{\delta_e}}$, $C_{\ell_{\delta_a}}$, and $C_{n_{\delta_r}}$ are the primary control derivatives and quantify the effect on the control surfaces on their primary intended axes of influence.
- $C_{\ell_{\delta_r}}$ and $C_{n_{\delta_a}}$ are the cross-control derivatives.

Propulsive Force

expressed in body frame

$$^{p}\mathbf{f}_B = \begin{pmatrix} T \\ 0 \\ 0 \end{pmatrix}$$

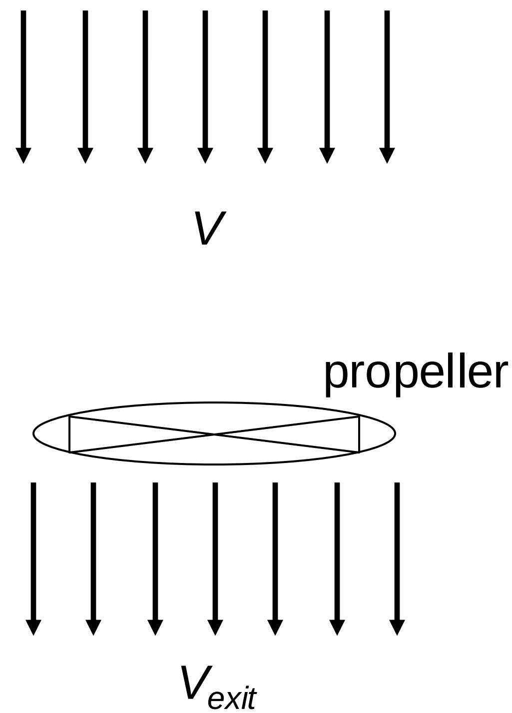
$$T = \frac{1}{2}\rho V_a^2 S C_T$$

Propeller Thrust

$$Q_d = C_{prop} S_{prop} V_{exit}$$

Quantity of air being discharged

$$T_{out} = \rho Q_d V_{exit}$$



$$T_{drag} = \rho Q_d V = \rho C_{prop} S_{Prop} V V_{exit}$$

momentum drag

$$T = T_{out} - T_{drag} = \rho C_{prop} S_{prop} V_{exit}(V_{exit} - V)$$

Exit velocity determined by throttle setting

$$V_{exit} = V + \delta_t (k_m - V)$$

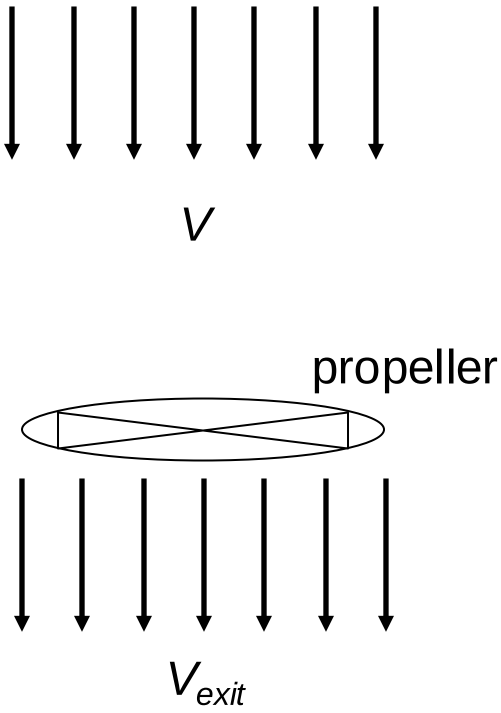
Km = motor constant

$$T = \rho S_{prop} C_{prop} \delta_t \left[V + \delta_t (k_m - V)\right] \left[k_m - V\right]$$

$$C_T = \frac{T}{1/2 \rho V^2 S} = 2 \frac{S_{prop}}{S} C_{prop} \frac{\delta_t}{V^2} \left[V + \delta_t (k_m - V)\right] \left[k_m - V\right]$$

Propeller Torque

$$\mathbf{G}_B^p = \begin{pmatrix} -k_{T_p} \left(k_\Omega \delta t\right)^2 \\ 0 \\ 0 \end{pmatrix}$$



Propeller Model: Alternative

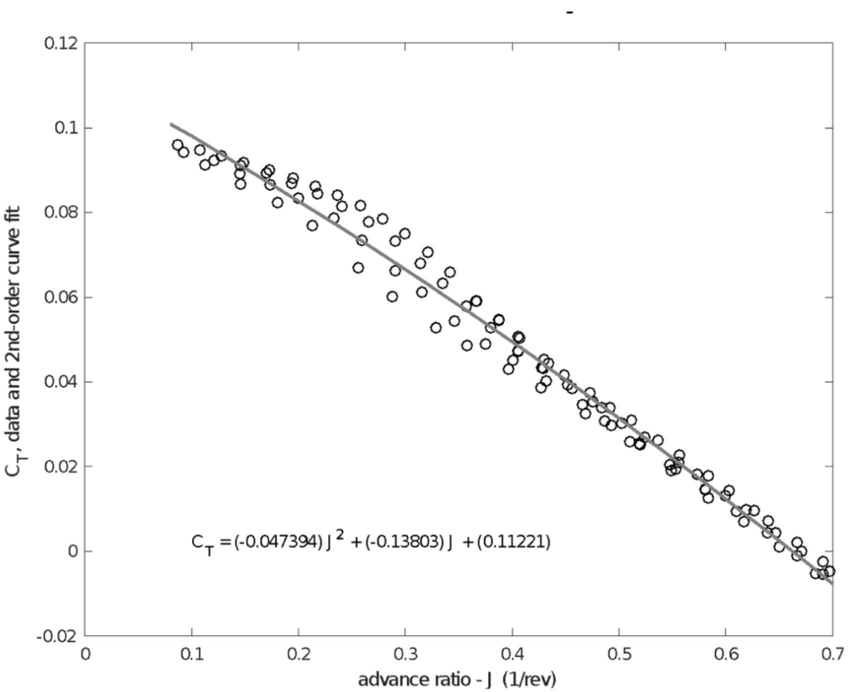
The thrust T_p and the torque Q_p along the propeller axis can be modeled as

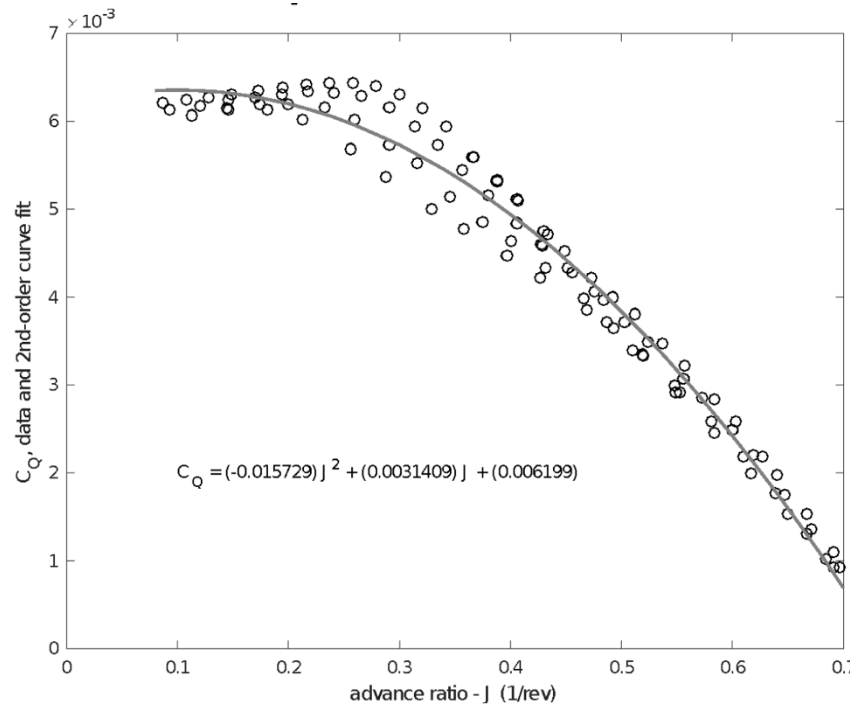
$$T_p = \rho n^2 D^4 C_T(J)$$
$$Q_p = \rho n^2 D^5 C_Q(J),$$

where

$$\rho \triangleq \text{density of air}$$
$$n \triangleq \text{ propeller speed (revolutions/sec)}$$
$$D \triangleq \text{ propeller diameter (m)}$$
$$C_T, C_Q \triangleq \text{ aerodynamic coefficients}$$
$$J \triangleq \frac{V_a}{nD} = \frac{2\pi V_a}{\Omega D}, \text{ advance ratio (unitless)}$$
$$\Omega \triangleq 2\pi n \text{ propeller speed (rad/sec)}$$

Propeller Model: Alternative





Experimental data indicates that

$$C_T(J) \approx C_{T2}J^2 + C_{T1}J + C_{T0}$$

$$C_Q(J) \approx C_{Q2}J^2 + C_{Q1}J + C_{Q0}$$

Equation of Motion in Vector Form

$$\dot{\mathbf{p}}^E = R_B^E \cdot \mathbf{V}_B^E$$

$$\dot{\mathbf{o}}_{B/E} = \mathbf{T} \cdot \omega_B$$

$$\dot{\mathbf{V}}_B^E = -\tilde{\omega}_B \mathbf{V}_B^E + \frac{\mathbf{f}_B}{m}$$

$$\dot{\omega}_B = \mathbf{I}_B^{-1} \left[-\tilde{\omega}_B \left(\mathbf{I}_B \omega_B \right) + \mathbf{G}_B \right]$$

Equation of Motion in Vector Form

$$\dot{\mathbf{p}}^E = R_B^E \cdot \mathbf{V}_B^E$$

$$\dot{\mathbf{o}}_{B/E} = \mathbf{T} \cdot \omega_B$$

$$\dot{\mathbf{V}}_B^E = -\tilde{\omega}_B \mathbf{V}_B^E + \frac{\mathbf{f}_B(\mathbf{x}, \mathbf{u})}{m}$$

$$\dot{\omega}_B = \mathbf{I}_B^{-1} \left[-\tilde{\omega}_B \left(\mathbf{I}_B \omega_B \right) + \mathbf{G}_B(\mathbf{x}, \mathbf{u}) \right]$$

Equations of Motion in Vector Form

$$\mathbf{x} = \begin{pmatrix} x_E \\ y_E \\ z_E \\ \phi \\ \theta \\ \psi \\ u^E \\ v^E \\ w^E \\ p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \mathbf{p}^E \\ \mathbf{o}_{B/E} \\ \mathbf{v}_B^E \\ \omega_B \end{pmatrix}$$

$$\dot{\mathbf{x}} = f_{EOM}\left(\mathbf{x}, \mathbf{u}\right)$$

$$\mathbf{u} = \left(\delta_a \ \delta_e \ \delta_r \ \delta_t\right)^T$$

Aircraft dynamics (equations of motion) still described by same aircraft state vector $\mathbf{x}$.