Faculty of Sciences	Exam Linear Algebra
Vrije Universiteit Amsterdam	Thursday, March 29, 2018 (8:45-11:30)

Use of a basic calculator is allowed. This exam consists of 6 questions and a total of 36 points can be obtained. The grade is calculated as (number of points + 4)/4.

Please provide an argument or calculation at every question!

Question 1 [8 pnt]. Let

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}.$$

a) [3 pnt] A basis for the null space of A can be found by solving $Ax = 0$ and writing the solution in parametric vector form. To solve $Ax = 0$, we bring A in reduced echelon form:

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 \\ 0 & -6 & -18 \\ 0 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 \\ 0 & -1 & -3 \\ 0 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

We now express the leading variables x_1 and x_2 in terms of the free variable x_3, which gives $x_1 = -5x_3$ and $x_2 = -3x_3$. The general solution in parametric vector form is now given by $x = x_3 \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$, so a basis for the null space of A is given by $\left\{ \begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix} \right\}$.

Grading: 1 pnt for recognizing that $Ax = 0$ needs to be solved, 1 pnt for bringing A in reduced echelon form (and only 0.5 pnt if there is a mistake in the calculation), and 1 pnt for the correct answer.

b) [3 pnt] A basis for the column space of A is the set of pivot columns of A. From the echelon form of A above, we see that the first two columns of A are pivot columns. A basis is hence given by $\left\{ \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix} \right\}$.

Grading: 1 pnt for noting that a basis for the column space of A is the set of pivot columns of A, 1 pnt for identifying the pivot columns, and 1 pnt for the correct answer.

The matrix A can be viewed as a linear transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$.

c) [1 pnt] The transformation is not one-to-one, because the dimension of the null space of A is > 0.

Grading: 0.5 pnt for the correct answer and 0.5 pnt for a correct argument.

d) [1 pnt] The transformation is not onto $\mathbb{R}^3$ because the dimension of the range of A equals the dimension of the column space of A, which equals 2.

Grading: 0.5 pnt for the correct answer and 0.5 pnt for a correct argument.

Question 2 [7 pnt]. Let

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}.$$

a) [2 pnt] The determinant of A is easiest to compute by using a cofactor expansion along the last column:

$$det(A) = det \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} = det \begin{bmatrix} 1 & 5 \\ 0 & -2 \end{bmatrix} = -2.$$

Grading: 1 pnt for the correct approach (cofactor expansion) and 1 pnt for the correct answer (and only 0.5 pnt if there is a mistake in the calculation).

b) [1 pnt] A is invertible because invertibility is equivalent to $det(A) \neq 0$.

Grading: 0.5 pnt for the correct answer and 0.5 pnt for a correct argument.

c) [2 pnt] Yes: Let $b \in \mathbb{R}^3$ be given, then we can take $x = A^{-1}b$ because A is invertible.

Grading: 1 pnt for the correct answer and 1 pnt for a correct argument.

d) [2 pnt] Yes, because if x_1 and x_2 satisfy $Ax_1 = b$ and $Ax_2 = b$, then $A(x_1 - x_2) = 0$, but because A is one-to-one, the null space of A only contains the zero vector, so $x_1 = x_2$.

Grading: 1 pnt for the correct answer and 1 pnt for a correct argument.

Question 3 [7 pnt]. Let

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}.$$

a) [2 pnt] $det(A - \lambda I) = det \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} = (2-\lambda)(-6-\lambda) - 9 = \lambda^2 + 4\lambda - 21.$

Grading: 1 pnt for noting that we need to compute $det(A - \lambda I)$ and 1 pnt for the correct calculation.

b) [2 pnt] Substitute them for λ is the characteristic polynomial: this gives 0. Another way is to use the *abc*-formula.

Grading: 1 pnt for each eigenvalue.

c) [2 pnt] Solve $(A - 3I)x = 0$ and write x in parametric vector form. First calculate the reduced echelon form of $A - 3I$:

$$A - 3I = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}.$$

Since the system is homogenous, it follows that $x_1 = 3x_2$ so that the general solution in parametric vector form is given by

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

A basis for the eigenspace of $\lambda = 3$ is thus

$$\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}.$$

Grading: 0.5 for noting that $(A - 3I)x = 0$ needs to be solved, 0.5 for bringing $A - 3I$ in reduced echelon form, 0.5 pnt for giving the solution in parametric vector form, and 0.5 pnt for the correct answer.

d) [1 pnt] A does not have complex eigenvalues, becauses the characteristic polynomial has degree 2 and so has exactly two roots, but both roots are real-valued.

Grading: 0.5 pnt for the correct answer, and 0.5 pnt for a correct argument.

Question 4 [5 pnt] Let $y = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and $u = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$ be two vectors in $\mathbb{R}^3$.

a) [1 pnt] Is y orthogonal to u?

No: y and u are orthogonal iff $y \cdot u = 0$, but $y \cdot u = 2 \times 1 - 3 \times 4 + 1 \times 2 = -8 \neq 0$.

Grading: 0.5 pnt for noting that we need we calculate $y \cdot u = 0$ and 0.5 pnt for the correct calculation.

b) [2 pnt] Compute the orthogonal projection of y onto u.

The orthogonal projection of y onto u is given by

$$\frac{y \cdot u}{||u||^2} u = -\frac{8}{21} \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} = -\frac{1}{21} \begin{bmatrix} 8 \\ -32 \\ 16 \end{bmatrix}.$$

Grading: 1 pnt for noting that the orthogonal projection is given by $\frac{y \cdot u}{||u||^2} u$ and 1 pnt for the correct answer (and only 0.5 pnt if there is a mistake in the calculation).

c) [2 pnt] Let x be a vector in $\mathbb{R}^3$ that is orthogonal to y. Show that x is orthogonal to cy, for any scalar c.

We have to show that $x \cdot (cy) = 0$, but $x \cdot (cy) = c(x \cdot y) = 0$, because x is orthogonal to y.

Grading: 1 pnt for noting that we have to show that $x \cdot (cy) = 0$ and 1 pnt for a correct argument.

Question 5 [4 pnt] Let $u_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, and $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

a) [1 pnt] Show that $\{u_1, u_2\}$ is an orthogonal basis for $W = \text{span}\{u_1, u_2\}$.

$u_1 \cdot u_2 = 2 \times (-2) + 5 \times 1 - 1 \times 1 = 0.$

Grading: 1 pnt for correct calculation (0 pnt otherwise).

b) [2 pnt] Compute the lengths of u_1 and u_2 and the distance between u_1 and u_2.

$||u_1|| = \sqrt{4 + 25 + 1} = \sqrt{30}$, $||u_2|| = \sqrt{4 + 1 + 1} = \sqrt{6}$, and $||u_1 - u_2|| = \sqrt{(4^2 + 4^2 + 2^2} = \sqrt{36} = 6.$

Grading: 0.5 point for the length of u_1, 0.5 pnt for the length of u_2, and 1 pnt for the distance between u_1 and u_2.

c) [1 pnt] Determine an orthonormal basis for W.

Because $\{u_1, u_2\}$ is an orthogonal basis for W, an orthonormal basis for W is given by

$$\left\{ \frac{u_1}{||u_1||}, \frac{u_2}{||u_2||} \right\} = \left\{ \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Grading: 0.5 pnt for noting that we only need to normalize u_1 and u_2 and 0.5 pnt for the correct answer.

Question 6 [5 pnt] Determine if the following statements are true or false. Provide an argument for your answer.

a) [1 pnt] The columns of any 4×5 matrix are linearly dependent.

True. There are 5 columns in $\mathbb{R}^4$, so they cannot be independent (if they were, then $\mathbb{R}^4$ would be 5-dimensional).

b) [1 pnt] A linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^m$ with $n < m$ cannot be onto $\mathbb{R}^m$.

True. This is because $dim(range(T)) \leq n < m$.

c) [1 pnt] The null space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.

False. It is a subspace of $\mathbb{R}^n$ and so the statement is only true in the special case that $m = n$.

d) [1 pnt] If $\lambda + 7$ is a factor of the characteristic polynomial of A, then 7 is an eigenvalue of A.

False. It should be $\lambda = -7$.

e) [1 pnt] For every matrix M, the matrix MM^T is symmetric.

True. For every M it holds that $(MM^T)^T = MM^T$, so MM^T is symmetric.

Grading: For a)-e): 0.5 pnt for correct answer and 0.5 pnt for correct argument.