Department of Computer Engineering

IE400-1 Term Project

Group 1

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Project Report

Team

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Question 1

We designed the mathematical model so that a parent's walk distance to a center is minimized. Let N be the total number of villages.

Parameters:

$d_{i,j} = Distance\ between\ village\ i\ and\ village\ j\ for\ i,j \in \{1,\ 2,\ \ldots,\ N\}$

Decision Variables:

$$X_{i,j} = \begin{cases} 1\ if\ \ village\ j\ is\ assigned\ to\ center\ village\ i \\ 0\ else \end{cases} \quad for\ i,j\ \in\ \{1, \ldots, N\}$$

$minD\ =\ the\ minumum\ distance\ a\ parent\ walks\ to\ a\ center$

$$C_i = \begin{cases} 1\ if\ \ village\ i\ is\ selected\ as\ center \\ 0\ else \end{cases} \quad for\ i\ \in \{1, \ldots, N\}$$

Model:

$min(\ \ minD\ )$

Subject To:

There are exactly 4 centers:

$$\sum_{i=1}^{N} C_i = 4$$

Determine if a village is selected as center:

$$\sum_{j=1}^{N} X_{i,j} \leq N \cdot C_i \quad for\ \forall i\ \in\ \{1, \ldots, N\}$$

Only a single center can be assigned to a village:

$$\sum_{i=1}^{N} X_{i,j} = 1 \quad for\ \forall j\ \in\ \{1, \ldots, N\}$$

Minimum distance a parent must walk:

$$X_{i,j} \cdot d_{i,j} \leq\ minD \quad for\ \forall i, \forall j = \{1, \ldots, N\}$$

Integrity constraints:

$minD \geq 0,$

Question 2

This model is essentially the same as question 1's model with one additional constraint to handle the blocked roads constraint required in the question.

Parameters:

Same as question 1 parameters with the addition of probabilities:

$P_{i,j} = the\ probability\ of\ the\ road\ being\ blocked\ between\ village\ i\ and\ j \quad for\ i, j \in \{1,\ 2,\ \ldots,\ N\}$

Decision Variables:

Same as question 1 decision variables.

Model:

Same as question 1 model.

Subject To:

Same as question 1 constraints with the addition of one more constraint that handles the probability of a road being blocked.

The road taken to the center must not be blocked with a probability higher than 0.60:

$X_{i,j} \cdot P_{i,j} \leq 0.60 \ \ for\ \forall i, \forall j = \{1, \ldots, N\}$

Question 3

This question is basically the "Travelling Salesman Problem". We used the MTZ formulation to avoid sub-tours. Let N be the number of villages.

Parameters:

S = 40 km/h: Speed of the snowplow

$d_{i,j} = Distance\ between\ village\ i\ and\ village\ j\ for\ i,j \in \{1,\ 2,\ \ldots,\ N\}$

$p_{i,j}\ =\ Probability\ of\ unusability\ of\ the\ road\ of\ village\ i\ to\ j\ for\ i,j\ \in \{1,2,\ldots,N\}$

Decision Variables:

$$x_{i,j} = \begin{cases} 1,\ if\ \ Santa\ used\ the\ road\ from\ i\ to\ j \\ 0,\ otherwise \end{cases} \quad for\ \ i,j\ \in \{1,2,\ldots,N\}$$

u_i: $Dummy\ variables\ to\ prevent\ subtours\ for\ i\ \in \{1,2,\ldots,N\}$

Model:

$$min \sum_{i=1}^{N} \sum_{j=1}^{N} x_{i,j}\ \cdot\ d_{i,j}$$

Subject To:

Each village entered once:

$$\sum_{i=1}^{N} x_{i,j}\ =\ 1 \quad for\ \forall j \in \{1,2,\ldots,N\}$$

Each village exited once:

$$\sum_{i=1}^{N} x_{j,i}\ =\ 1 \quad for\ \forall j \in \{1,2,\ldots,N\}$$

Avoid sub-tours by MTZ formulation:

$$u_i\ -\ u_j\ +\ 1\ \le\ (N\ -\ 1)\,(1\ -\ x_{i,j}) \quad for\ \forall i\ \neq\ 1,\ \forall j\ \neq\ 1.$$

Blocked roads:

$$x_{i,j}\ \cdot\ p_{i,j}\ \leqslant 0.60 \quad for\ \forall i\ \in \{1,2,\ldots,N\},\ \forall j\ \in \{1,2,\ldots,N\}$$

A village cannot be visited while Santa is there, diagonal must be zero:

$$\sum_{i=1}^{N} x_{i,i} = 0 \quad for\ \forall\ i \in \{1, 2, \ldots, N\}$$

Variable constraints (Integrity):

$$0 \leq x_{i,j} \leq 1,\ x_{i,j}\ integer\ \ for\ \forall i\ \in \{1, 2, \ldots, N\},\ \forall j\ \in \{1, 2, \ldots, N\}$$

$$u_1 = 1$$

$$2 \leq u_i \leq n \quad for\ \forall i \neq 1$$

Question 4

- **3-Index Solution**

This solution is basically, modified version of "Travelling Salesman Problem" from question 3. We apply a TSP model for each volunteer.

Let N be the number of villages. If an optimal solution exists, the total number of volunteers is at most N-1, because each village can be assigned to a single volunteer.

Parameters:

S = 40 km/h: Speed of a snowplow

T = 10 hour: Time limit set by Santa

$d_{i,j} = Distance\ between\ village\ i\ and\ village\ j\ for\ i,j \in \{1,\ 2,\ \ldots,\ N\}$

Decision Variables:

$$x_{i,j,k} = \begin{cases} 1, & if\ \ volunteer\ k\ goes\ from\ village\ i\ to\ village\ j \\ 0, & otherwise \end{cases}$$

$for\ i,j\ \in \{1,\ 2,\ \ldots,\ N\},\ k \in \{1, 2,\ \ldots,\ N-1\}$

$$y_k = \begin{cases} 1, & if\ \ volunteer\ k\ helps \\ 0, & otherwise \end{cases} \quad for\ k \in \{1, 2, \ldots, N-1\}$$

$u_{i,\ k}$: $dummy\ variables\ to\ prevent\ subtours$

$$z_{i,k} = \begin{cases} 1, & if\ \ village\ i\ is\ visited\ by\ volunteer\ k \\ 0, & otherwise \end{cases} \quad for\ i \in \{2, 3, \ldots, N\},\ k \in \{1, 2, \ldots, N-1\}$$

n_k: $Number\ of\ \ villages\ visited\ by\ volunteer\ k$

Model:

$$min \sum_{k=1}^{N-1} y_k$$

(It is exact number of volunteers that help)

S.t.

$$\sum_{i=1}^{N} \sum_{j=1}^{N} x_{i,j,k} \leq N \cdot y_k \; for \; k \in \{1, 2, \ldots, N-1\}$$

(If some of the $x_{i,j,k}$ is 1, then volunteer k helps. If volunteer k does not help, then all $x_{i,j,k}$ is 0)

$$\sum_{i=1}^{N} \sum_{j=1}^{N} x_{i,j,k} \cdot d_{i,j} \leq s \cdot t \; for \; k \in \{1, 2, \ldots, N-1\}$$

(Each volunteer has a time limit set by Santa)

$$\sum_{i=1}^{N} x_{i,1,k} = \sum_{j=1}^{N} x_{1,j,k} = y_k \; for \; k \in \{1, 2, \ldots, N-1\}$$

(If volunteer k helps, it goes from village 1 to some other village)

$$\sum_{i=1}^{N} x_{i,i,k} = 0 \; for \; k \in \{1, 2, \ldots, N-1\}$$

(No way from each village to itself)

$$\sum_{i=1}^{N} x_{i,j,k} = \sum_{i=1}^{N} x_{j,i,k} = z_{j,k} \; for \; j \in \{1, 2, \ldots, N\}, \; k \in \{1, 2, \ldots, N-1\}$$

(Volunteer k comes in to and goes out from village j once, if it visits village j)

$$u_{i,k} - u_{j,k} + N \cdot x_{i,j,k} \leq N - 1 \; for \; i,j \in \{2, 3, \ldots, N\}, \; k \in \{1, 2, \ldots, N-1\}$$

(MZT constraint to prevent subtour)

$$z_{i,k} \leq u_{i,k} \leq n_k - y_k \; for \; i \in \{2, 3, \ldots, N\}, \; k \in \{1, 2, \ldots, N-1\}$$

(If volunteer k does not help, $u_{i,k} = 0 \; for \; i \in \{2, 3, \ldots, N\} \, for \; volunteer \; k$. If it helps, $0 \leq u_{i,k} \leq n_k - 1$)

$$u_{i,k} \leq N \cdot z_{i,k} \; for \; i \in \{2, 3, \ldots, N\}, \; k \in \{1, 2, \ldots, N-1\}$$

($u_{i,k} \leq 0$, if volunteer k does not visit village i, which means $u_{i,k} = 0 \, for \; non \; visited \; villages$)

$$\sum_{i=2}^{N} z_{i,k} \geq y_k \ for\ k \in \{1, 2, \ldots, N-1\}$$

(If no village is visited by volunteer k, then volunteer k does not help. If volunteer k helps, at least 1 village is visited by volunteer k)

$$\sum_{k=1}^{N-1} z_{i,k} = 1 \ for\ i \in \{2, 3, \ldots, N\}$$

(Each village is visited once except the village 1)

$$n_k = \sum_{i=2}^{N} z_{i,k} + y_k \ for\ k \in \{1, 2, \ldots, N-1\}$$

(Number of visited villages)

$$y_{k-1} \geq y_k \ for\ k \in \{2, 3, \ldots, N-1\}$$

(First volunteers help firstly)

$$0 \leq x_{i,j,k} \leq 1 \ for\ i,j \in \{1, 2, \ldots, N\},\ k \in \{1, 2, \ldots, N-1\}$$

$$0 \leq y_k \leq 1 \ for\ k \in \{1, 2, \ldots, N-1\}$$

$$0 \leq z_{i,k} \leq 1 \ for\ i \in \{1, 2, \ldots, N\},\ k \in \{1, 2, \ldots, N-1\}$$

$$u_{i,k} \in Z^+ \cup \{0\} \ for\ i \in \{1, 2, \ldots, N\},\ k \in \{1, 2, \ldots, N-1\}$$

$$n_k \in Z^+ \cup \{0\} \ for\ k \in \{1, 2, \ldots, N-1\}$$

This model gives an optimal solution. However, since x variable is 3 dimensional and because of the number of constraints, the execution time is too long. Therefore, we needed to decrease the dimension. As a result, we implemented a 2 dimensional model.

- **2-Index Solution**

The idea is simple, we will not make a direct constraint on the number of leaving volunteers from the first node. Moreover, we will add a new constraint on the total distance a volunteer can cover. If a city j is visited, then the next city i would have the remaining distance after this city j is visited. The base case for distances is the first node. The formal definition of our model can be find below:

Parameters:

S = 40 km/h: Speed of a snowplow

T = 10 hour: Time limit set by Santa

D = S * T = 400 km (maximum distance a volunteer can travel)

$d_{i,j} = Distance\ between\ village\ i\ and\ village\ j\ for\ i, j \in \{1,\ 2,\ \ldots,\ N\}$

Decision Variables:

$$X_{i,j} = \begin{cases} 1\ if\ \ village\ j\ is\ visited\ from\ village\ i \\ 0\ else \end{cases} \quad for\ i,j\ \in\ \{1, \ldots, N\}$$

$RD_{i,j}\ =\ the\ remaining\ distance\ after\ village\ j\ is\ visited\ from\ village\ i$

$(\ RD\ stands\ for\ remianing\ distance)$

$for\ i,j\ \in\ \{1, \ldots, N\}$

$numVolunteer\ =\ the\ number\ of\ \ volunteers\ that\ are\ helping\ Santa$

Model:

$min(\ \ numVolunteers\ )$

Subject To:

We cannot travel to more than 1 village from any village except the starting village:

$$\sum_{j\ =\ 1}^{N} X_{i,j}\ =\ 1\ \ for\ \forall i\ \in\ \{2, \ldots, N\}$$

A village is left once except the starting village:

$$\sum_{i\ =\ 1}^{N} X_{i,j}\ =\ 1\ \ for\ \forall j\ \in\ \{2, \ldots, N\}$$

A village cannot visit itself:

$$\sum_{i=1}^{N} x_{i,i} = 0 \quad for\ \forall\ i \in \{1, 2, \ldots, N\}$$

The number of volunteers leaving the starting village must be equal to:

$$\sum_{j=1}^{N} X_{1,j} = numVolunteer$$

The number of volunteers entering the starting village must be equal to:

$$\sum_{i=1}^{N} X_{i,1} = numVolunteer$$

Remaining distance for each village must be smaller or equal to the distance limit for each volunteer:

$$RD_{i,j} \leq D \cdot X_{i,j} \ \ for\ \forall i, \forall j = \{1, \ldots, N\}$$

Base case for the remaining distances for the first visited villages:

$$D \cdot X_{1,j} - d_{1,j} \cdot X_{1,j} = RD_{1,j} \ \ for\ \forall j = \{1, \ldots, N\}$$

Updating remaining distances of the visited villages and eliminating sub-tours:

$$\sum_{j=1,\ j\ !=i}^{N} RD_{i,j} - \sum_{j=1,\ j\ !=i}^{N} RD_{j,i} + \sum_{j=1}^{N} X_{i,j} \cdot d_{i,j} = 0 \ \ for\ \forall i = \{2, \ldots, N\}$$

Integrity Constraints:

$$numVolunteer \geq 1,\ numVolunteer \leq N - 1,\ RD_{i,j} \geq 0\, for\ \forall i, \forall j \in \{1, \ldots, N\}$$