

Problems

1. Find all primes p and q such that $3p^{q-1} + 1$ divides $11^p + 17^p$.
2. 10000 nonzero digits are written in a 100-by-100 table, one digit per cell. From left to right, each row forms a 100-digit integer. From top to bottom, each column forms a 100-digit integer. So the rows and columns form 200 integers (each with 100 digits), not necessarily distinct. Prove that if at least 199 of these 200 numbers are divisible by 2013, then all of them are divisible by 2013.
3. Let k be a fixed positive integer. Alberto and Beralto play the following game: Given an initial number N_0 and starting with Alberto, they take turns to perform the following operation: change the number n into a number m such that $m < n$ and m and n differ, in their base-2 representations, in exactly l consecutive digits for some l such that $1 \leq l \leq k$. If someone can't play, he loses.

We say a non-negative integer t is a winner number when the player who receives the number t has a winning strategy, that is, he can choose the next numbers in order to guarantee his own victory, regardless the options of the other player. Else, we call it a loser.

Prove that for every positive integer N , the total number of non-negative loser integers smaller than 2^N is $2^{N-1} - \lfloor \frac{\log(\min\{N, k\})}{\log 2} \rfloor - 1$.

4. A real polynomial of odd degree has all positive coefficients. Prove that there is a (possibly trivial) permutation of the coefficients such that the resulting polynomial has exactly one real zero.
5. Consider $m + 1$ horizontal and $n + 1$ vertical lines ($m, n \geq 4$) in the plane forming an $m \times n$ table. Consider a closed path on the segments of this table such that it does not intersect itself and also it passes through all $(m - 1)(n - 1)$ interior vertices (each vertex is an intersection point of two lines) and it doesn't pass through any of outer vertices. Suppose A is the number of vertices such that the path passes through them straight forward, B number of the table squares that only their two opposite sides are used in the path, and C number of the table squares that none of their sides is used in the path. Prove that

$$A = B - C + m + n - 1.$$

6. Suppose $ABCD$ is a parallelogram. Consider circles w_1 and w_2 such that w_1 is tangent to segments AB and AD and w_2 is tangent to segments BC and CD . Suppose that there exists a circle which is tangent to lines AD and DC and externally tangent to w_1 and w_2 . Prove that there exists a circle which is tangent to lines AB and BC and also externally tangent to circles w_1 and w_2 .
7. For which positive integers k can we construct a sequence a_0, a_1, a_2, \dots , where for each i , we have $a_{i+1} = a_i + k$ or $a_{i+1} = a_i - k$ or $a_{i+1} = a_i \times k$ or $a_{i+1} = a_i/k$?
8. Determine if the following statement is true: given any non-negative $\lambda_{i,j}$, $1 \leq i < j \leq n$, there always exists nonnegative reals a_i , $1 \leq i \leq n$ such that $|a_i - a_j| \geq \lambda_{i,j}$ for all $1 \leq i < j \leq n$, and

$$\sum_{i=1}^n a_i \leq \sum_{1 \leq i < j \leq n} \lambda_{i,j}$$