Derived algebraic geometry and jet schemes

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Abstract

To any scheme X one can define a jet scheme $J^m X$ whose points should be thought of as points in X along with tangent directions up to order m. This perspective lends itself well to the setting of derived algebraic geometry, where deformation theory is more tractable.

In this talk, we'll discuss the derived setting via some well-known constructions, then discuss how one can upgrade the classical jet construction to the derived realm. Time permitting, we'll see that smoothness forces derived jet spaces to be homotopically discrete, suggesting the construction is a tool that can measure degree of singularity. Joint with Lance Edward Miller.

1 What is the derived setting?

As we may all know, in algebraic geometry, the objects one studies are commutative rings and schemes, which enjoy the wonderful relationship that schemes are locally defined by $\operatorname{Spec} R$, the space of prime ideals of a commutative ring R. What may be surprising is that, under certain settings, this field of view is too narrow.

For instance, Bézout's theorem is a statement about the intersection of two algebraic curves C and C'. When their degrees are m and n, and they intersect transversely, then their intersection contains mn points. We may be familiar with ways to weaken these hypotheses, like counting intersection with multiplicity, but one killing blow cannot be avoided: if the curves are completely coincident ($C = C'$), then not only do we fail to have mn intersection points, but even fundamentally the dimension of the intersection is incorrect.

Among other things, derived algebraic geometry serves to remedy this problem. In the same way that Grothendieck promoted varieties to schemes by allowing nilpotents, derived algebraic geometry promotes schemes to derived schemes, allowing so-called "homotopic" nilpotents. To do this, we create a new wonderful relationship that derived schemes are locally defined by $\operatorname{Spec} R_\bullet$, where $R_\bullet$ is not a ring but instead an *animated ring*. These objects form an ∞-category, and their construction is in some sense formal, but we will also see a concrete way to understand them.

Fix once and for all a ring k. The category $\mathbf{Alg}_k$ of k-algebras has a full subcategory $\mathbf{Poly}_k$ of polynomial algebras, which satisfy the property that $\operatorname{Hom}_{\mathbf{Alg}_k}(P, -)$ commutes with sifted colimits. Now, of all functors out of $\mathbf{Alg}_k$ and into an ∞-category $\mathcal{A}$, there is a full subcategory of those that commute with sifted colimits. We define the ∞-category of animated k-algebras $a\mathbf{Alg}_k$ to be that which satisfies

$$\operatorname{Fun}_{\text{sifted}}(a\mathbf{Alg}_k, \mathcal{A}) \cong \operatorname{Fun}(\mathbf{Poly}_k, \mathcal{A}).$$

While this is purely formal, one can get a concrete description of an animated k-algebra as follows. A simplicial k-algebra is a sequence of k-algebras $R_\bullet = (R_0, R_1, R_2, \ldots)$ with, for each n, maps $d_i : R_n \to R_{n-1}$ and $s_i : R_n \to R_{n+1}$, $0 \le i \le n$, which satisfy certain identities analogous to the face and degeneracy maps of simplicial complexes. Morphisms of simplicial k-algebras $f : R_\bullet \to S_\bullet$ are simply morphisms $f_n : R_n \to S_n$ for each n that commute with face and degeneracy maps.

$$\begin{tikzcd} R_n \arrow[r, "f_n"] \arrow[d, shift right, "d_i"'] & S_n \arrow[d, shift right, "d_i"'] \\ R_{n-1} \arrow[r, "f_{n-1}"'] \arrow[u, shift right, "s_i"'] & S_{n-1} \arrow[u, shift right, "s_i"'] \end{tikzcd}$$

This creates a category $s\mathbf{Alg}_k$ of simplicial k-algebras. For a basic example of one, let $R = (R, R, R, \ldots)$ and every face/degeneracy map id_R. This defines a "constant" simplicial k-algebra, and in fact determines an embedding $\mathbf{Alg}_k \to s\mathbf{Alg}_k$.

Given such objects we can compute their homotopy groups. We write $\pi_i R_\bullet$ for the ith homotopy group of $R_\bullet$, and to go from the category $s\mathbf{Alg}_k$ to the ∞-category $a\mathbf{Alg}_k$, we identify two simplicial rings $R_\bullet$ and $S_\bullet$ as the same if they have the same homotopy type; i.e., there is a map $f : R_\bullet \to S_\bullet$ which induces an isomorphism $\pi_i R_\bullet \to \pi_i S_\bullet$ for all i. Animated algebras are only defined up to their homotopy class.

As an example, R has homotopy groups $\pi_0 R = R$ and $\pi_i R = 0$ for $i > 0$. We call such objects discrete. In fact, the fully faithful embedding $\mathbf{Alg}_k \to a\mathbf{Alg}_k$ has π_0 as its left adjoint.

Certain operations on simplicial algebras need to be done in the derived way. For example, we should take $R_\bullet \otimes_k^{\mathbf{L}} S_\bullet$ when we compute a "tensor product." Recall, for R and S k-algebras, $R \otimes_k^{\mathbf{L}} S$ is a complex whose homotopy groups are the Tor groups. This is the right idea for several reasons, but one which we have already motivated is that the intersection of curves corresponds to a tensor product of local rings, and the multiplicity, as per Serre's intersection formula, is computed via an alternating sum of the dimension of Tor modules.

$$\mathcal{O}_{C\cap C',p} = \mathcal{O}_{C,p} \otimes_{\mathcal{O}_{\mathbf{P}^n,p}} \mathcal{O}_{C',p}$$
$$m(p) = \sum (-1)^i \dim \mathrm{Tor}_i^{\mathcal{O}_{\mathbf{P}^n}}(\mathcal{O}_{C,p}, \mathcal{O}_{C',p})$$

Similar to the homological setting, you can compute this derived tensor product $R \otimes_k^{\mathbf{L}} S$ by producing a "homotopy-acyclic" resolution of R, call it $P_\bullet$, and computing $P_\bullet \otimes_k S$ termwise. We will come back to this point.

This is in fact a special case of the following formalism. If $F : \mathbf{Alg}_k \to \mathcal{D}$ is a functor that preserves sifted colimits, then there is a unique functor $\mathbf{L}F : a\mathbf{Alg}_k \to a\mathcal{D}$ which satisfies $\mathbf{L}F|_{\mathbf{Poly}_k} = F|_{\mathbf{Poly}_k}$ and $\pi_0 \mathbf{L}F- = F\pi_0-$. This notation comes from the fact that this is a left Kan extension of F along the inclusion $\mathbf{Poly}_k \to \mathbf{Alg}_k$. Here's another example.

Consider the functor $\Omega_{-/k} : \mathbf{Alg}_k \to \mathbf{Mod}_k$ which, for each R, produces the module of Kähler differentials of R over k. Once again, take a resolution $P_\bullet \to R$, and the derived Kähler differentials are the cotangent complex $\mathbf{L}_{R/k} \coloneqq \Omega_{P_\bullet/k} \otimes_{P_\bullet}^{\mathbf{L}} R$.

For us, resolutions $P_\bullet \to R$, or generally $P_\bullet \to R_\bullet$, will be composed of polynomial k-algebras at every term n, and such that the induced maps $\pi_i P_\bullet \to \pi_i R_\bullet$ are all isomorphisms. One explicit, though intractable, way to construct a $P_\bullet$ given a R is to create

$$\cdots k[k[k[R]]] \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; k[k[R]] \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; k[R] \;\substack{\longrightarrow\\[-2pt]\longleftarrow\\[-2pt]\longrightarrow}\; R$$

Finally, as $a\mathbf{Alg}_k$ is an ∞-category, in lieu of homsets we have mapping spaces which we denote $\mathrm{Maps}_{a\mathbf{Alg}_k}(R_\bullet, S_\bullet)$. This is defined up to homotopy class, and just as with animated rings themselves, we can model the mapping space simplicially - in this case with the simplicial set $\mathrm{Hom}_{s\mathbf{Alg}_k}(P_\bullet, S_\bullet)$, where $P_\bullet \to R_\bullet$ is a polynomial resolution.

2 Jet spaces in the classical setting

A brief recap of the classic theory of jet spaces allows us to compare to the derived setting. For a fixed scheme X and $m \in \mathbf{N}$, we have a functor $\mathrm{Jet}_X^m : \mathbf{Alg}_k \to \mathbf{Set}$ defined by

$$\mathrm{Jet}_X^m(A) \coloneqq \mathrm{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} A[t]/t^{m+1}, X).$$

Under moderate hypotheses on X (e.g., locally of finite type), Jet_X^m is representable by a scheme we call $J^m X$; i.e.,

$$\mathrm{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} A[t]/t^{m+1}, X) \cong \mathrm{Hom}_{\mathbf{Sch}_k}(\operatorname{Spec} A, J^m X).$$

The picture to have in mind is, e.g., taking $A = k$, we see that a point $\operatorname{Spec} k$ in $J^m X$ is the same as $\operatorname{Spec} k[t]/t^{m+1}$ in X, which can be conceptualized as a point plus tangent directions up to order m (hence the name jet!). Using the representability criterion, we can compute specific examples; for instance, when $X = \operatorname{Spec} k[x_i]/(f_j)$ for $1 \le i \le d$ and $1 \le j \le s$, then $J^m X \cong \operatorname{Spec} k[x_i, x_i', \ldots, x_i^{(m)}]/(f_j, f_j', \ldots, f_j^{(m)})$, since the representability criterion is equivalently

$$\mathrm{Hom}_{\mathbf{Alg}_k}(k[x_i]/(f_j), A[t]/t^{m+1}) \cong \mathrm{Hom}_{\mathbf{Alg}_k}(k[x_i, x_i', \ldots, x_i^{(m)}]/(f_j, f_j', \ldots, f_j^{(m)}), A).$$

Jet spaces also enjoy some connections to deformations; for instance, $J^1X \cong \operatorname{Spec}\operatorname{Sym}\Omega_{X/k}$. Also, given an étale morphism $f : X \to Y$, then $J^mX \cong J^mY \times_Y X$.

3 Jet spaces in the derived setting

As we've laid the foundation, there are two natural ways one might want to define a derived notation of a jet space. The first is to ask for the representing object of a derived notion of the Jet^m_X functor. That is, working affinely with $X = \operatorname{Spec} R_\bullet$ so as to take advantage of the animated rings we've constructed, you could define a functor

$$\mathrm{Jet}^m_{R_\bullet}(A_\bullet) \coloneqq \mathrm{Maps}_{a\mathbf{Alg}_k}(R_\bullet, A_\bullet[t]/t^{m+1}),$$

where $A_\bullet[t]/t^{m+1} \coloneqq A_\bullet \otimes^{\mathbf{L}}_k k[t]/t^{m+1}$. This looks like the functor Jet^m_X, but using derived constructions. We can ask if this functor is representable in $a\mathbf{Alg}_k$.

On the other hand, like the cotangent complex, we could compute the derived functor of J^m-, call it $\mathbf{L}J^m-$. Namely, given an animated ring $R_\bullet$, $\mathbf{L}J^mR_\bullet \coloneqq J^mP_\bullet$ computed termwise for a polynomial resolution $P_\bullet \to R_\bullet$.

Theorem 1 (Miller,–,22)**.** *Both of these constructions are well-defined, and furthermore, they are equivalent. Namely, when $R_\bullet$ is an animated algebra of finite type, the functors $\mathrm{Jet}^m_{R_\bullet}$ are corepresentable by $\mathbf{L}J^mR_\bullet$.*

$$\mathrm{Maps}_{a\mathbf{Alg}_k}(R_\bullet, A_\bullet[t]/t^{m+1}) \cong \mathrm{Maps}_{a\mathbf{Alg}_k}(\mathbf{L}J^mR_\bullet, A_\bullet)$$

Ideas of the proof. Showing that these constructions are well-defined amounts to checking that they commute with sifted colimits. They do, so they have left Kan extensions.

The two candidates also agree on $\mathbf{Poly}_k$, since if $A_\bullet = Q$ is a discrete polynomial algebra, then the left hand side is

$$\begin{aligned}
\mathrm{Maps}_{a\mathbf{Alg}_k}(R_\bullet, Q[t]/t^{m+1}) &\simeq \mathrm{Hom}_{s\mathbf{Alg}_k}(P_\bullet, Q[t]/t^{m+1}) && \text{by modeling mapping spaces}\\
&\cong \mathrm{Hom}_{\mathbf{Alg}_k}(\pi_0P_\bullet, Q[t]/t^{m+1}) && \text{by adjunction}\\
&\cong \mathrm{Hom}_{\mathbf{Alg}_k}(J^m\pi_0P_\bullet, Q) && \text{by jet corepresentability,}
\end{aligned}$$

and the right hand side is, using the fact that $\mathbf{L}J^mR_\bullet \coloneqq J^mP_\bullet$ for a resolution $P_\bullet \to R_\bullet$,

$$\begin{aligned}
\mathrm{Maps}_{a\mathbf{Alg}_k}(\mathbf{L}J^mR_\bullet, Q) &\simeq \mathrm{Hom}_{s\mathbf{Alg}_k}(J^mP_\bullet, Q) && \text{by modeling (and jets of polynomials are polynomial)}\\
&\cong \mathrm{Hom}_{\mathbf{Alg}_k}(\pi_0J^mP_\bullet, Q) && \text{by adjunction.}
\end{aligned}$$

Now since $\pi_0J^mP_\bullet \cong J^m\pi_0P_\bullet$, the result is shown.

So we have two functors that agree on $\mathbf{Poly}_k$ and have Kan extensions to $a\mathbf{Alg}_k$. By uniqueness of the Kan extension, they are equivalent. □

While we now have derived jets in the affine setting, we'd also like to see if this construction glues. Indeed,

Theorem 2 (Miller,–,22)**.** *Let X be a quasicompact quasiseparated flat locally of finite type derived scheme over k. Let $m \in \mathbf{N}$. There exists a derived k-scheme $\mathbf{L}J^mX$ such that*

$$\mathrm{Maps}_{d\mathbf{Sch}_k}(\operatorname{Spec} A_\bullet[t]/t^{m+1}, X) \cong \mathrm{Maps}_{d\mathbf{Sch}_k}(\operatorname{Spec} A, \mathbf{L}J^mX).$$

Proof by example. Let X have a simple affine cover $\{\operatorname{Spec} A_{i,\bullet},\}_{i=1}^n$, and suppose $A_{i,\bullet}$ are polynomial.

To be glued together means that the following diagram is a homotopy coequalizer:

$$\prod_{i,j} \operatorname{Spec} A_{i,\bullet} \times^{\mathbf{L}} \operatorname{Spec} A_{j,\bullet} \rightrightarrows \prod_i \operatorname{Spec} A_{i,\bullet} \longrightarrow X$$

$$\prod_{i,j} \operatorname{Spec}(A_{i,\bullet} \otimes^{\mathbf{L}} A_{j,\bullet}) \rightrightarrows \prod_i \operatorname{Spec} A_{i,\bullet} \longrightarrow X$$

$$\prod_{i,j} \operatorname{Spec}(A_{i,\bullet} \otimes A_{j,\bullet}) \rightrightarrows \prod_i \operatorname{Spec} A_{i,\bullet} \longrightarrow X$$

Taking $\mathbf{L}J^m-$, everything is polynomial so we can compute J^m-, and it's easily checked that J^m commutes with the tensor product, since it does so classically at each level.

$$\prod_{i,j} \operatorname{Spec} \mathbf{L}J^m(A_{i,\bullet} \otimes A_{j,\bullet}) \rightrightarrows \prod_i \operatorname{Spec} \mathbf{L}J^m A_{i,\bullet}$$

$$\prod_{i,j} \operatorname{Spec}(\mathbf{L}J^m A_{i,\bullet} \otimes \mathbf{L}J^m A_{j,\bullet}) \rightrightarrows \prod_i \operatorname{Spec} \mathbf{L}J^m A_{i,\bullet}$$

$$\prod_{i,j} \operatorname{Spec}(\mathbf{L}J^m A_{i,\bullet} \otimes^{\mathbf{L}} \mathbf{L}J^m A_{j,\bullet}) \rightrightarrows \prod_i \operatorname{Spec} \mathbf{L}J^m A_{i,\bullet}$$

$$\prod_{i,j} \operatorname{Spec} \mathbf{L}J^m A_{i,\bullet} \times^{\mathbf{L}} \operatorname{Spec} \mathbf{L}J^m A_{j,\bullet} \rightrightarrows \prod_i \operatorname{Spec} \mathbf{L}J^m A_{i,\bullet}$$

Define $\mathbf{L}J^m X$ to be the homotopy coequalizer of this diagram. □

We also can check that some of the deformation results for jet schemes that held in the ordinary setting have derived analogs.

Theorem 3 (Miller,–,22). *If X is a derived scheme over k, then $\mathbf{L}J^1 X \cong \operatorname{Spec} \mathbf{L} \operatorname{Sym} \mathbf{L}_{X/k}$.*

Idea. Under nice enough conditions which are satisfied here, the composition of Kan extensions is a Kan extension. So $\mathbf{L}(\operatorname{Sym} \Omega_{-/k}) \cong \mathbf{L} \operatorname{Sym} \mathbf{L}_{-/k}$. As previously stated, J^1- and $\operatorname{Sym} \Omega_{-/k}$ agree on polynomial algebras, so by uniqueness of Kan extensions, the result is shown. □

Theorem 4 (Miller,–,22). *If $f : X \to Y$ is an étale morphism of derived schemes over k, then the morphism $\mathbf{L}J^m X \to \mathbf{L}J^m Y \times^{\mathbf{L}}_Y X$ is an equivalence.*

Idea. Much like in the nonderived realm, the definition of étale (in particular the infinitesimal lifting property), plus the representability criterion, gives the desired result. □

Theorem 5 (Miller,–,22). *If X is a smooth discrete scheme over k, then $\mathbf{L}J^m X$ is discrete for all m.*

Proof. Since X is smooth, it has étale coordinates. Let $U \subseteq X$ be affine. The following diagram commutes.

$$\begin{tikzcd} U \arrow[r, hook, "\text{ét}"] \arrow[d, "\text{ét}"'] & X \arrow[d] \\ \mathbf{A}^d \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

By Theorem 4, we have two equivalences

$$\mathbf{L}J^m\mathbf{A}^d \times^{\mathbf{L}}_{\mathbf{A}^d} U \leftarrow \mathbf{L}J^m U \to \mathbf{L}J^m X \times^{\mathbf{L}}_X U.$$

Affine space $\mathbf{A}^d = \operatorname{Spec} k[x_1, \ldots, x_d]$ is polynomial, so $\mathbf{L}J^m\mathbf{A}^d = J^m\mathbf{A}^d$ is discrete. Thus $\mathbf{L}J^m\mathbf{A}^d \times^{\mathbf{L}}_{\mathbf{A}^d} U$ is discrete, and thus so is $\mathbf{L}J^m X \times^{\mathbf{L}}_X U$.

We get to employ some topology; via the Mayer-Vietoris long exact sequence from the homotopy pullback, we have

$$\cdots \pi_{i+1}X \to \pi_i(\mathbf{L}J^m X \times^{\mathbf{L}}_X U) \to \pi_i(\mathbf{L}J^m X) \oplus \pi_i U \to \pi_i X \to \cdots$$

Let $i > 0$. As X, U, and $\mathbf{L}J^m X \times^{\mathbf{L}}_X U$ are discrete, plugging those in forces $\pi_i \mathbf{L}J^m X$ to be discrete, as we needed to show. □

Not every $\mathbf{L}J^m X$ is discrete though. Taking a slightly more restrictive model of derived algebraic geometry, [Bouaziz,18] showed that if $X = \operatorname{Spec} \mathbf{C}[x]/x^2$, then $\mathbf{L}J^m X$ has nontrivial π_1.

Time permitting, we can show an example of a derived scheme whose jets have nontrivial homotopy.

Example 6. Let $R_\bullet = k[x]/x \otimes^{\mathbf{L}}_{k[x]} k[x]/x$. We build a specific resolution via a construction called the *bar resolution.*

Resolve $P_\bullet \to k[x]/x$ via $P_n \coloneqq k[x]^{\otimes_k (n+1)} \otimes_k k[x]/x \cong k[x_0, \ldots, x_n]$. We write a tensor $g \otimes f_1 \otimes \cdots \otimes f_n \otimes h$ as $g[f_1 \mid \cdots \mid f_n]h$.

The resolution $P_\bullet \to k[x]/x$ grants us a weak equivalence between $R_\bullet$ and $k[x]/x \otimes_{k[x]} P_\bullet$. Furthermore, the terms in $k[x]/x \otimes_{k[x]} P_\bullet$ are polynomial, so this is a resolution of $R_\bullet$, not just a weak equivalence, and we can compute $\mathbf{L}J^m R_\bullet$ on it.

Observe $\mathbf{L}J^m R_\bullet = J^m(k[x]/x \otimes_{k[x]} P_\bullet) \cong J^m k[x_1, \ldots, x_n] \cong k[x_j^{(\ell)}]$ for $1 \le j \le n$ and $0 \le \ell \le m$. The elements can be written as sums of formal derivatives of bar elements $(g[f_1 \mid \cdots \mid f_n]h)^{(\ell)}$.

Computing the homotopy groups can be done explicitly if you know the face maps of the simplicial object. Or, in the case of π_0, you compute directly:

$$\pi_0 J^m R_\bullet \cong J^m \pi_0 R_\bullet \cong J^m k[x]/x^2 \cong k[x, x', \ldots, x^{(m)}]/(x^2, 2xx', \ldots, (x^2)^{(m)}).$$

To compute π_1, it is the cokernel of the Moore differential $\partial_2 = d_0 - d_1 + d_2 : J^m k[x_1, x_2] \to J^m k[x_1]$. A generating bar element for $J^m k[x_1, x_2]$ is $[x^a \mid x^b]^{(\ell)}$ for $0 \le \ell \le m$, and computing ∂_2 on this element, we get

$$\partial_2 \left([x^a \mid x^b]^{(\ell)} \right) = \begin{cases} -[x^{a+b}] & \text{if } a, b > 0, \\ 0 & \text{if } \ell = 0 \text{ and exactly one of } a \text{ and } b \text{ is } 0, \text{ or } \ell \neq 0 \text{ and } ab = 0, \\ 1 & \text{if } \ell = 0 \text{ and } a = b = 0. \end{cases}$$

The cokernel of this map is a free k-module of rank $m+1$, corresponding to $[x], [x]', \ldots, [x]^{(m)}$. Hence, the derived jets of a derived thick point have nontrivial homotopy.