Introduction to Algebraic Geometry

Alexander Duncan and Eric Walker

Algebraic Geometry (AG) is the science of encoding (resp. decoding) questions about complex manifolds into (resp. from) questions about commutative algebra. Below its surface, AG weaves a profound yet beautiful tapestry connecting number theory and complex geometry. On its surface, AG is very natural and very simple: given a system of polynomial equations with coefficients in some field K, I wish to know the solution set of this system over some field $k \subseteq K$. If this set is infinite and over $\mathbb{C}$, the best we can do is roughly sketch the real component of this solution set. The power of doing this is introduced to you at a very early age. If you ever drill calculus or teach college algebra, you will become familiar with taking some simple optimization problem, encode the data in the problem's instructions into one or more polynomials over $\mathbb{R}$, and then use derivatives or some special form to find the needed min or max. Let's generalize that process for just a moment:

1. Get data from problem
2. Encode problem into a polynomial when possible to do so
3. Graph as best you can
4. Look at graph for "interesting points" (such as local extrema or singularities)
5. Interpret these interesting points in terms of the original problem

1 Motivation for a Theory

Another important aspect of AG is its ability to provide concise generalizations for results in commutative algebra. Being able to impose a condition on a ring by imposing a condition on a complex manifold associated to said ring leads to wonderfully clever trades in difficulty of proofs. I think a proof is "difficult" when the reader must make a series of nontrivial leaps or guesses. In this sense, many proofs early on in a number theory course are very difficult because they involve the creation of complicated functions seemingly from scratch. The next exercise gives a controllable example of AG attempting to explain the inherent difficulty of making a helpful guess from scratch.

Example 1.1. It's a good exercise in "the path of least resistance" to show

$$\mathbb{C}[x,y]/(y-x^2-1) \cong \mathbb{C}[x]$$

as $\mathbb{C}$-algebras. It's best to naturally guess a hom

$$\varphi : \mathbb{C}[x,y] \to \mathbb{C}[x]$$

and then see if the kernel is the ideal $(y-x^2-1)$. Coming up with the right hom on the first try isn't impossible. We are compelled to make sure $y \mapsto x^2+1$ so that $(y-x^2-1) \subseteq \ker\varphi$. Next, we choose $x \mapsto x$ because its a safe guess. It's easy to see that this gives the needed isomorphism whence the reverse inclusion is checked. It would be nice if we had a *canonical* way of obtaining φ without having to guess and check. Geometry gives us a rough idea of what the "best" first guess should be. Consider figure 1 below and imagine projecting the quadratic curve vertically down. This is just coordinate projection and so we obtain a holomorphic map from one complex manifold to the other. These curves are complex manifolds but we have only graphed their real components in Desmos. This mapping is obviously a bijection which we will call f. The inverse is $f^{-1}(z) = (z, z^2+1)$ which is polynomial, hence holomorphic, in both coordinates and so f is bi-holomorphic. Back to commutative algebra, there is a way to obtain an isomorphism of algebras ψ from f. The trick is to identify the point $(z,0)$ on the line $y=0$ as the maximal ideal $(x-z)$ of $\mathbb{C}[x,y]/(y)$ and identify (z,w) on the quadratic as the maximal ideal $(x-z, y-w) = (x-z, x^2+1-w)$ of $\mathbb{C}[x,y]/(y-x^2-1)$ and then let f become the pullback of ψ. That is,

$$\psi^{-1}(x-z, y-w) = (x-z)$$

Here, f is trying to tell you that ψ can just kill off the vertical coordinate and leave the horizontal coordinate alone (provided you believe in these identifications which came from thin air). This completely ignores the

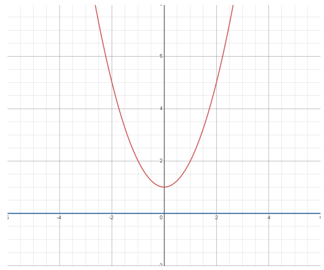


Figure 1: The quadratic curve projects onto the line $y = 0$

equation $y = x^2 + 1$. This is a consequence of f being projection from one curve to another *without ever leaving the plane* (the plane is just $\mathbb{C}^2$ for now). The most natural ring to associate to this line in the plane is

$$\mathbb{C}[x, y]/(y)$$

which is *much* easier to use than $\mathbb{C}[x]$ for defining ψ. So we actually have

$$\psi : \mathbb{C}[x, y]/(y) \to \mathbb{C}[x, y]/(y - x^2 - 1)$$

and from our data mining of f we've discovered the required relations $\psi(x) = x$ (leave the horizontal coordinate alone) and $\psi(y) = 0$ (kill-off the vertical coordinate). This only leaves one possible hom which must be defined as

$$\psi(h(x, y)) = h(x, 0). \tag{1}$$

The input h in the definition of ψ is actually a choice of representative for a particular element of $\mathbb{C}[x, y]/(y)$.

Exercise 1.2 (★★★☆☆). Check that the ψ defined in line (1) is well-defined. Also check that ψ is an isomorphism of $\mathbb{C}$-algebras. Finally, mention how all of this shows $\mathbb{C}[x, y]/(y - x^2 - 1) \cong \mathbb{C}[x]$.

On one hand, we gave ψ without any algebraic guesswork. On the other hand, I gave us some completely unjustified identifications between points and maximal ideals. You might be thinking that the difficulty of guessing φ was just moved to the difficulty of knowing there were unjustified identifications of maximal ideals and points. Sure, but this isn't quite as difficult as guessing. In order for the above example to be a coherent mathematical argument, however, I would need to prove that there is a bijection between the set of all maximal ideals of $\mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_m)$ and the solution set for the system consisting of the f_i. If no such bijection exists, then $\varphi^{-1}(x - z, y - w) = (x - z)$ only implies that $\varphi(x - z)$ lands *somewhere* in the ideal $(x - z, y - w)$, i.e. we can't know for sure that $\varphi(x) = x$. Another inherent problem with the last example is that the pullback of a maximal ideal isn't always maximal. In our case, every ideal we needed was generated by linear polynomials and we got lucky that pullbacks of linear polynomials just so happened to also be linear. Is there any hope of formalizing the ideas in example 1.1? Is there a rigorous mathematical way of specifying $\mathbb{C}$-algebra homomorphisms from graphs in Desmos? Who would even think to identify maximal ideals of rings with points on a manifold?

2 Hilbert and The New Frontier

The construction of a smooth (resp. complex) manifold is intrinsically tied to its smooth (resp. holomorphic) functions which send that manifold to $\mathbb{R}$ (resp. $\mathbb{C}$). This is a consequence of the local nature of these manifolds. At the turn of the 20th century, Hilbert paved the way for a similar theory to be developed for quotients of finitely generated polynomial rings. The main content of this theory between complex geometry and commutative algebra is the Hilbert Nullstellensatz Theorem which is used to create a bijection between maximal ideals of $\mathbb{C}[x_1, ..., x_n]$ and points of $\mathbb{C}^n$.

Closed sets in a chart of a smooth manifold can always be written in terms of vanishing coordinates of a smooth function. The same goes for closed sets on a complex manifold but with holomorphic functions. We will need an analogous notion of vanishing for elements of a quotient of a polynomial ring. Hence we say a set X in the set $\mathbb{C}^n$ is closed if X is the zero locus of a set of polynomials in $S = \mathbb{C}[x_1, ..., x_n]$.

The next exercise sketches a proof that this is a topology on the set $\mathbb{C}^n$. It is called the Zariski topology and it is crucial that the reader complete this exercise because it introduces important techniques and notation. When $\mathbb{C}^n$ is given the Zariski topology we refer to the resulting space as affine n-space and denote it by $\mathbb{A}^n$. When we wish to denote $\mathbb{C}^n$ in its usual metric topology we will continue to write $\mathbb{C}^n$.

Exercise 2.1 (★★★☆☆)**.** Let k be a field. For an ideal I of $S = k[x_1, ..., x_n]$ define $Z(I)$ to be the zero locus of I in k^n. That is,

$$Z(I) = \{P \in k^n \mid f(P) = 0 \text{ for all } f \in I\}$$

A set X in k^n is Zariski closed if $X = Z(I)$ (this is the formal way of defining what has already been mentioned).

1. Let $\{I_\alpha\}_\alpha$ be a collection of ideals in S. Show

$$Z\left(\bigoplus_\alpha I_\alpha\right) = \bigcap_\alpha Z(I_\alpha)$$

2. Let I and J be ideals of S. Show

$$Z(IJ) = Z(I) \cup Z(J)$$

3. Show $\mathbb{A}^n$ and $\varnothing$ are of the form $Z(f)$ for appropriate $f \in S$.
4. Define $\mathbb{A}^n_k$ to be k^n with the above notion of Zariski closed sets. Conclude that $\mathbb{A}^n_k$ is a topological space. We call this space affine n-space over k. When $k = \mathbb{C}$ we let $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{C}}$.

We can give The Hilbert Nullstellensatz theorem after three more definitions. The third definition is given as an exercise.

Definition 2.2. Let I be an ideal of S. Define the radical of I denoted $\sqrt{I}$ to be the ideal of S which is generated by polynomials f for which some positive power of f is in I.

Definition 2.3. For a closed set X in $\mathbb{A}^n_k$ define $I(X)$ to be the ideal of $S = k[x_1, ..., x_n]$ generated by all polynomials whose vanishing set includes X.

Exercise 2.4 (★☆☆☆☆)**.** A nonempty closed set X in $\mathbb{A}^n_k$ is irreducible if whenever $X = Z(I) \cup Z(J)$ it is the case that $I \subseteq J$ or $J \subseteq I$. Show that X is irreducible if and only if X is the zero set of a prime ideal.

Theorem 2.5 (Affine Nullstellensatz)**.** *Let k be algebraically closed and let $\mathfrak{a}$ be an ideal of $k[x_1, ..., x_n]$. If $f \in I(Z(\mathfrak{a}))$ then there exists a $r \geq 1$ such that $f^r \in \mathfrak{a}$.*

Proof. See Atiyah-Macdonald. □

Corollary 2.6 (Fundamental Theorem of Affine Space)**.** *Let k be algebraically closed. The operations $Z(-)$ and $I(-)$ satisfy the following:*

1. *Let X be a closed set in $\mathbb{A}^n_k$. Then $Z(I(X)) = X$.*
2. *Let J be an ideal of S. Then $IZ(J) = \sqrt{J}$.*
3. *$Z(-)$ is a bijection from the set of prime ideals of $k[x_1, ..., x_n]$ to the set of irreducible closed subsets of $\mathbb{A}^n_k$. Its inverse function is $I(-)$*
4. *The bijection in (3) restricts to a bijection from $\mathbb{A}^n_k$ to the set of maximal ideals of $k[x_1, ..., x_n]$.*

Remark 2.7. The algebraically closed condition is necessary because of examples like the following. Let $k = \mathbb{R}$. Consider the closed set $X = Z(x^2 + 1)$ in $\mathbb{A}^1_{\mathbb{R}}$. It's well known that X is empty. But $(x^2 + 1)$ is a maximal ideal.

With this theorem, techniques like the ones in example 1.1 can now be carried out in a rigorous mathematical way. This gives us a completely new way to tackle difficult questions from commutative algebra. For our first example of this, we will show how the Nullstellensatz makes quick work of difficult dimension theory questions from commutative algebra. This would be a good time to review some commutative algebra fundamentals.

By a ring, we will always mean a commutative ring with unity. A ring R is noetherian if all ideals of the ring are finitely generated as R-modules.

Definition 2.8. Let R be a ring (recall our conventions). A chain of prime ideals is a strict linear finite inclusion of prime ideals. The length of the chain

$$P_0 \subsetneq P_2 \subsetneq ... \subsetneq P_n$$

is n, the number of inclusions (not the number of ideals). The Krull dimension of R is the supremum of the set of all possible lengths of chains of prime ideals in R.

The Krull dimension of $S = k[x_1, ..., x_n]$ is n but a purely commutative algebra proof of this is a commitment. The first step uses a fundamental of commutative algebra called localization which sends the given ring to a new ring where all ideals except for one previously chosen prime ideal are the unit ideal. This new ring is called a local ring meaning it has only one maximal ideal, $\mathfrak{m}$. The next step is to have a conversation about why $V = \mathfrak{m}/\mathfrak{m}^2$ is a k-vector space. Finally one must show

$$\dim S \leq \dim_k V$$

which requires a fundamental result called Nakayama's lemma as well as a discussion of primary ideals and their generators. We will provide the theory of localization in great detail in the next section and primary ideals will be discussed in sufficient detail in section four. Using AG, we can prove $\dim S = n$ in the case that $k = \mathbb{C}$ with far less work than what's described above.

Definition 2.9. An affine algebraic variety (AAV) over k is an irreducible closed subset of $\mathbb{A}^n_k$. A quasi-affine algebraic variety (QAAV) is an open subset of an AAV in the subspace topology. For X an AAV, define the coordinate ring of X denoted $A[X]$ to be any k-algebra isomorphic to $k[x_1, ..., x_n]/I(X)$. The dimension of X is the length of the longest chain of irreducible subsets of X.

The quadratic curve and the x-axis are both AAV's over $\mathbb{C}$ in $\mathbb{A}^2$. Once we define morphisms of AAV's, we will see the x-axis in $\mathbb{A}^2$ is isomorphic to $\mathbb{A}^1$ and hence the quadratic curve is also isomorphic to $\mathbb{A}^1$. As another example, let

$$X = \mathbb{A}^2 - \{\text{origin}\}$$

By definition, X is a QAAV. Let U and V be the complements of the x and y axis, respectively, in $\mathbb{A}^2$. Then $X = U \cup V$. Furthermore, each of U and V have a natural bijection with the AAV $Z(yx - 1)$ in $\mathbb{A}^3$. You can see how this example suggests that AAV's could play the role of charts for QAAV's. We have not defined the notion of coordinate rings for QAAV's (because we need localization) but the hyperbola and U and V all share isomorphic coordinate rings.

Example 2.10. The longest chain of irreducible subsets of $\mathbb{A}^n_k$ is

$$Z(x_1, ..., x_n) \subsetneq Z(x_1, ..., x_{n-1}) \subsetneq ... \subsetneq Z(0)$$

which has length n. To verify that this chain is indeed the longest possible, notice that these are linear subsets of $\mathbb{A}^n_k$ meaning this chain determines a chain of vector spaces in $\mathbb{C}^n$.

Proposition 2.11. *Let X be an AAV over an algebraically closed field. Then* $\dim X = \dim A[X]$.

Proof. By the Nullstellensatz, prime ideals of $A[X]$ correspond to irreducible closed subsets of X. This gives a bijection

$$\{\text{chains of prime ideals of } A[X]\} \iff \{\text{chains of irreducible subsets of } X\}$$

and the proposition follows from 2.10. $\square$

AS mentioned earlier, AAV's play the same roll as charts on a manifold. They are primarily used to cover closed subsets of projective space which we will now define. Unless otherwise stated, let

$$S = k[x_0, ..., x_r]$$

where k is any field.

Definition 2.12. A ring R is $\mathbb{N}$-graded (or just graded provided no confusion may arise) if there is an isomorphism of additive abelian groups

$$\varphi : R \to \bigoplus_{n=0}^{\infty} G_n$$

satisfying the following: if $\varphi(x) \in G_a$ and $\varphi(y) \in G_b$ then $xy \in G_{a+b}$. Writing $R = \bigoplus_{n=0}^{\infty} R_0$ the abelian group R_n is called the degree n piece of R and a nonzero element of R_n is called a homogeneous element of degree n. The additive identity of R is not homogeneous. We say an ideal I of R is homogeneous if it satisfies any of the equivalent conditions in the following proposition.

Proposition 2.13. *If an ideal I of a homogeneous ring R. The following are equivalent:*

1. *I is generated by homogeneous elements*
2. *There is an isomorphism of R-modules*

$$I \cong \bigoplus_{n=0}^{\infty} I_n$$

3. *The quotient R/I is a graded ring with grading*

$$R/I \cong \bigoplus_{n=0}^{\infty} R_n/I_n$$

Proof. See Atiyah-Macdonald □

Exercise 2.14 (★☆☆☆☆)**.** Prove that S is graded via the usual notion of degree of polynomials. Is this the only grading one can have on S?

We will now define projective space. Let $\mathbb{P}_k^r$ denote the set of points of the form

$$[a_0 : a_1 : ... : a_r]$$

for $a_i \in k$ provided at least one of the a_i are non zero modulo the relation

$$[a_0 : a_1 : ... : a_r] = [b_0 : b_1 : ... : b_r]$$

iff there is a nonzero $\lambda \in K$ for which $\lambda a_i = b_i$ for all $1 \leq i \leq r$. A choice of representative for an equivalence class under this relation is called a homogeneous coordinate. We topologize $\mathbb{P}_k^r$ by defining a closed set as a set of the form $Z(I)$ for I a homogeneous ideal in S along with the set $\mathbb{P}_k^r$. We require such I to be missing at least one of the x_i.

Exercise 2.15 (★★★☆☆)**.** Check that this indeed produces a topology on $\mathbb{P}_k^r$. Then, show that $\mathbb{P}_k^r$ is in bijection with the set of lines in $\mathbb{A}_k^{r+1}$ through the origin. Recall that a line in affine r space over k is the solution set of $r-1$ linear equations in r variables with coefficients over k.

Definition 2.16. The space $\mathbb{P}_k^r$ is called projective r-space over k. When $k = \mathbb{C}$ we omit the k unless stated otherwise. A projective algebraic variety (PAV) over k is an irreducible closed subset of $\mathbb{P}_k^r$ in the subspace topology. The coordinate ring of a PAV X is

$$A[X] = S/I(X)$$

where $I(X)$ denotes the homogeneous polynomials of S which vanish on X. A quasi-projective algebraic variety (QPAV) is an open subset of a PAV in the subspace topology. The dimension of a PAV or a QPAV is the length of its largest chain of irreducible closed subsets.

There is a homogeneous Nullstellensatz. We will state it now for completeness and its proof is usually given side by side with the affine Nullstellensatz.

Theorem 2.17 (Projective Nullstellensatz)**.** *Let k be algebraically closed and let $\mathfrak{a}$ be a homogeneous ideal of S which is missing at least one of the x_i. If $f \in I(Z(\mathfrak{a}))$ then there exists a $r \geq 1$ such that $f^r \in \mathfrak{a}$.*

Proof. See Atiyah-Macdonald. □

Corollary 2.18 (Fundamental Theorem of Projective Space)**.** *Let k be algebraically closed. The operations $Z(-)$ and $I(-)$ satisfy the following:*

1. *Let X be a closed set in $\mathbb{P}_k^r$. Then $Z(I(X)) = X$.*
2. *Let J be a homogeneous ideal of S. Then $IZ(J) = \sqrt{J}$.*
3. *$Z(-)$ is a bijection from the set of homogeneous prime ideals of $k[x_1, ..., x_n]$ which are missing at least one x_i to the set of irreducible closed subsets of $\mathbb{P}_k^r$. Its inverse function is $I(-)$.*
4. *The bijection in (3) restricts to a bijection from $\mathbb{P}_k^r$ to the set of homogeneous maximal ideals of $k[x_1, ..., x_n]$ which are missing at least one x_i.*

The parity of the two fundamental theorems allows for an open cover of any PAV by a finite collection of AAV's.

Theorem 2.19. *Let U_i denote the set of points of $\mathbb{P}_k^r$ for which the i^{th} coordinate is nonzero. Then the U_i form an open cover of $\mathbb{P}_k^r$ and there are homeomorphisms*

$$U_i \cong \mathbb{A}_k^r$$

which restrict to homeomorphisms between AAV's and QPAV's. In particular, every PAV has an open cover of QPAV's which are homeomorphic to various AAV's.

Proof. Define

$$\varphi_i : U_i \to \mathbb{A}_k^r$$

by

$$\varphi_i\left(\left[\frac{a_0}{a_i} : ... : \frac{a_{i-1}}{a_i} : 1 : \frac{a_{i+1}}{a_i} : ... : \frac{a_r}{a_i}\right]\right) = \left(\frac{a_0}{a_i}, ..., \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, ..., \frac{a_r}{a_i}\right)$$

Checking that the U_i form an open cover of $\mathbb{P}_k^r$ and that the φ_i are homeomorphisms is left to the following exercise. Let X be a PAV in $\mathbb{P}_k^r$. Then any $Y_i = X \cap U_i$ is a QAAV in X. Let ψ_i be restriction of φ_i to Y_i. Since Y_i is closed in U_i, we see ψ_i is a homeomorphism from a QAAV to a AAV. □

Exercise 2.20 (★★☆☆☆)**.** Show that the U_i in the proof of theorem 2.19 are open in $\mathbb{P}_k^r$. Show that the φ_i are homeomorphisms.

Example 2.21. The PAV $\mathbb{P}^1$ is covered by $U_i \cong \mathbb{A}^1$, $i = 0, 1$. More specifically

$$U_0 = \mathbb{P}^1 - \{[0:1]\} \quad U_1 = \mathbb{P}^1 - \{[1:0]\}$$

The point $[0 : 1]$ (resp. $[1 : 0]$) is called "a point at infinity" with respect to U_0 (resp. U_1). Its absence is due to the fact that it can't ever be reached, in some sense, via traveling a finite distance in $\mathbb{A}^1$. Taking an affine chart U_i on $\mathbb{P}^r$ gives "a hyperplane at infinity", i.e. after moving an infinite distance away from the origin in $U_i \cong \mathbb{A}^r$ one would find a copy of $\mathbb{A}^{r-1}$ missing from U_i.

Exercise 2.22 (★★★☆☆)**.** Let X be a PAV in $\mathbb{P}_k^r$ with its usual affine algebraic cover U_i and charts $\varphi_i : U_i \to \mathbb{A}^r$.

1. Explain why the φ_i send irreducible closed subsets of X to irreducible closed subsets of $\mathbb{A}^r$.
2. Show that if Y is an irreducible closed subset of X then $Y_i = U_i \cap X$ are AAV's of the same dimension as Y.

3. Conclude that $\dim X + 1 = \dim A[X]$.

Exercise 2.23 (★★★★☆)**.** Let k be algebraically closed. Let $f \in S = k[x_0, x_1, x_2]$ be a homogeneous and irreducible polynomial of degree d.

1. Explain why $X = Z(f)$ is a PAV. Use the previous exercise to conclude that X is a projective algebraic curve, i.e. a PAV of dimension one.
2. Show that any copy of $\mathbb{P}^1$ in $\mathbb{P}^2$ intersects $Z(f)$ in at most d distinct points.
3. A line $\mathbb{P}^1$ in $\mathbb{P}^2$ is said to be general if it intersects X in exactly d points. Let $f = x_2x_1^2 - x_0^3$ and assume this polynomial is irreducible. Give an example of a general line and a non-general line. The examples of lines you give will need to be defined via charts.
4. Let f be a linear polynomial. Remark on why $Z(f)$ is homeomorphic to $\mathbb{P}^1$. Show that the only non-general line relative to this f is $Z(f)$.

3 The Category of Varieties over a field k

Up to this point, we have only been considering homeomorphisms between algebraic varieties. Unfortunately these maps are not specific enough to compare intricacies of algebraic varieties. For example the projective line $\mathbb{P}^1$ and the curve

$$x_2x_1^2 = x_0^3$$

in $\mathbb{P}^2$ have charts into $\mathbb{A}^1$ which are homeomorphic to $\mathbb{A}^1$ but exercise 2.23 suggests that these algebraic varieties differ. However, exercise 2.23 can't very well be trusted because of the misleading nature of embeddings. That said, these curves are indeed different and one can prove this by checking that the coordinate rings of these algebraic varieties are not isomorphic. Unfortunately, the word "different" doesn't have a meaning yet because we haven't defined what it means for varieties to be "isomorphic". In this section, we will do just that but the discussion must start with a reformulation of our intuitive ideas about what a variety should be.

Hopefully someone has wondered why we use the word "algebraic" next to every definition involving the word variety. The reason is that all of these objects were originally defined *as subsets* of either $\mathbb{A}^r_k$ or $\mathbb{P}^r_k$. We do not have any examples of algebraic varieties which appear *outside* of $\mathbb{A}^r_k$ or $\mathbb{P}^r_k$ because the definitions of AAV and PAV do not allow for this. Without allowing for varieties to exist outside of these cases, we would be starving ourselves of intricate underlying relationships between AG and other areas of math. The first major example of this will be encountered in the section on rational maps which, among other results, will prove that structure preserving maps between curves are in correspondence to field extensions. The proof of this requires a discussion of curves whose points are in bijection with a certain class of discrete valuation rings. This connection wouldn't be possible if we only considered varieties as objects that are born of $\mathbb{A}^r_k$ or $\mathbb{P}^r_k$. Therefore, we must define a precursor of a variety independently of affine or projective space. This is accomplished by introducing locally ringed spaces. The road to these objects requires a deep understanding of sheaves and their morphisms. We have abbreviated the terminology as much as possible so that we may transition to varieties more quickly. To avoid terseness, we pad with examples throughout.

Definition 3.1. Let X be a topological space. A pre-sheaf of rings on X is an assignment

$$U \mapsto \mathcal{F}(U)$$

for all open $U \subset X$ where the various $F(U)$ are rings satisfying:

1. The empty set is sent to the zero ring
2. Inclusions of open sets $V \subseteq U$ are sent to ring homomorphisms

$$\mathcal{F}(U) \to \mathcal{F}(V)$$

which are called restriction maps with notation $s|_V$ denoting the image of s in $\mathcal{F}(V)$

3. The identity map on open sets is sent to the identity map on rings
4. If $V \subseteq U \subseteq W$ then for $s \in \mathcal{F}(W)$
$$(s|_U)|_V = s|_V$$

Elements of $\mathcal{F}(U)$ are called sections over U. The key to understanding pre-sheaves is to understand what "restriction" means in the context of how the sheaf is defined. We will practice with a few simple but important examples.

Example 3.2. Let $X = \mathbb{R}^2$ in its usual topology. Define $\mathcal{F}(U)$ to be the set of continuous functions from U into $\mathbb{R}$. Define the maps assigned to inclusions $V \subseteq U$ as literal function restriction. We must check that $\mathcal{F}(U)$ is a ring. This follows from the fact that continuity behaves well under addition and multiplication. Furthermore, we may define $\mathcal{F}(U) = 0$ when U is the empty set. The last axiom of a pre-sheaf, which is sometimes called the composition rule, is automatically satisfied because the needed ring homomorphisms are defined by literal function restriction and this always satisfies the composition rule. Thus $\mathcal{F}$ is a presheaf and restriction in the context of $\mathcal{F}$ is good ol' function restriction.

Example 3.3. Let $X = \mathbb{R}^2$ again. Let $\mathbb{Z}$ denote the presheaf of rings on X which sends all nonempty open sets to $\mathbb{Z}$ and sends the empty set to $0 \in \mathbb{Z}$. Define restriction maps to be the identity map when possible and let all other restriction maps be the trivial map. This is routinely checked to be a presheaf of rings on X. Let P be the origin in X. Define a presheaf of rings $\mathcal{G}$ on X which agrees with the presheaf $\mathbb{Z}$ whenever $P \in U$ and is trivial otherwise. Restriction maps follow the same convention as they did for the presheaf $\mathbb{Z}$. The first presheaf in this example is called the constant presheaf of rings at $\mathbb{Z}$ and the other presheaf is called the skyscraper presheaf of $\mathbb{Z}$ over P. Over the constant presheaf, restriction means "do nothing". Whereas in the skyscraper presheaf, restriction means "report $\mathbb{Z}$ if P is present and report zero otherwise".

Definition 3.4. Let X be a topological space. A sheaf of rings on X is a pre-sheaf of rings $\mathcal{F}$ satisfying the following: if (U_i, s_i) is an open cover of X with $s_i \in \mathcal{F}(U_i)$ which agree on intersections of the U_i, then there is a unique section $s \in \mathcal{F}(X)$ satisfying
$$s|_{U_i} = s_i$$
for all i.

This last axiom is called the gluing axiom and for good reason. When $\mathcal{F}$ is a sheaf of rings, we may "glue" sections together whenever the sections agree on an overlap, i.e. when
$$s|_{U\cap V} = t|_{U\cap V}$$
for $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$.

Exercise 3.5 (★★☆☆☆). There are two examples of presheaves of rings in **Example 3.3** and one example in **3.2**. Which, if any, of these three presheaves are also sheaves of rings?

Exercise 3.6 (★★★☆☆). Let $X = \{P, Q\}$ in the discrete topology. Construct a presheaf of rings which is not a sheaf of rings.

The goal now is to construct a sheaf of rings for an arbitrary algebraic variety which gives us the ability to utilize the most fundamental commutative algebra technique: localization of rings. We will now briefly review localization. The exercises for the rest of this section are exceptionally punishing if avoided now. That said, most consist of routine fact checking from definitions.

Definition 3.7. Let R be a ring. Let S be a multiplicatively closed subset of R which includes 1. Define the set
$$S^{-1}R = \left\{ \frac{r}{s} \mid r \in R \text{ and } s \in S \right\} / \sim$$
wherein $\frac{r}{s} = \frac{a}{b}$ if and only if $0 = u(rb - sa)$ for some $u \in S$

Exercise 3.8 (★★☆☆☆)**.** The set $S^{-1}R$ is a ring under the usual notions of addition and multiplication of fractions.

Let P be a prime ideal of R. Define

$$R_P = (R - P)^{-1}R$$

which we call the localization of R at P (check that $R - P$ is multiplicatively closed with 1 if P is prime).

Exercise 3.9 (★☆☆☆☆)**.** Compute the localization of $\mathbf{Z}$ with respect to the multiplicatively closed subset $S = \mathbf{Z} \setminus \{0\}$.

Exercise 3.10 (★☆☆☆☆)**.** Let p be a prime integer. Compute $\mathbf{Z}_{(p)}$, the localization of $\mathbf{Z}$ at the prime ideal (p).

Exercise 3.11 (★★★☆☆)**.** Given a ring R and an element $f \in R$, we denote R_f the localization of R with respect to the multiplicatively closed subset $S = \{1, f, f^2, f^3, \ldots\}$.

1. Let p be a prime integer. Compute $\mathbf{Z}_p$. How does this compare to $\mathbf{Z}_{(p)}$?
2. Show that R_f is canonically isomorphic to $R[f^{-1}] := R[t]/(tf - 1)$.

Definition 3.12. A local ring is a ring A with precisely one maximal ideal $\mathfrak{m}$. We refer to local rings as the pair $(A, \mathfrak{m})$.

Exercise 3.13 (★★☆☆☆)**.** Let P be a prime ideal of the ring R. Let $\mathfrak{m}$ denote the ideal of R_P generated by the image of P in R_P. Then I is not the unit ideal of R_P if and only if I is contained in $\mathfrak{m}$. Conclude that $(R_P, \mathfrak{m})$ is a local ring.

Let X be an AAV in $\mathbb{A}^r_k$. Let U be open in X. Let

$$s : U \to k$$

be a continuous function when k is given the topology of $\mathbb{A}^1_k$. Then s is regular at $P \in U$ if s agrees with a rational function $\frac{f}{g}$ for $f, g \in k[x_1, ..., x_r]$ in some neighborhood V of P such that g never vanishes over V. The function s is regular over U if s is regular at every point of U. Define a presheaf of rings $\mathcal{O}_X$ on X which assigns a nonempty U to the set of functions which are regular over U. Let the empty set be assigned the zero ring. Restriction is defined to be literal function restriction. Once its checked that the $\mathcal{O}_X(U)$ are indeed rings, the other axioms of a presheaf are a quick consequence of the fact that restriction is defined to be literal function restriction.

Exercise 3.14 (★★☆☆☆)**.** Check that $\mathcal{O}_X(U)$ is a ring.

Proposition 3.15. *Define $\mathcal{O}_X$ on X as above. Then $\mathcal{O}_X$ is a sheaf of rings on X.*

Proof. Let $s \in \mathcal{O}_X(U)$ and $t \in \mathcal{O}_X(V)$ agree over $U \cap V$. Let q be the result of gluing s and t. Then q is continuous. Suppose $P \in U \cup V$. Let W be a neighborhood of P in $U \cup V$ for which

$$s(x) = \frac{f}{g} \quad t(x) = \frac{a}{b}$$

for polynomials f, g, a, b with g and b never vanishing over U. We may also assume s and t agree over U or just take more intersections as necessary. These functions agree with q over W by construction. Since P was arbitrary, we obtain an open over of $U \cup V$ for which q restricts to a rational function over any set in this cover. For unicity, we need only use the well known fact that if r_1 and r_2 are rational functions without singularities over an open set U in X which agree on U, then they are equal on U. □

We must also define a sheaf of rings on PAV's. Let X be a PAV in $\mathbb{P}^r_k$. Let U be open in X. Let

$$s : U \to k$$

be a continuous function when k is given the structure of $\mathbb{A}^1_k$. Then s is regular at $P \in U$ if s agrees with a rational function $\frac{f}{g}$ for homogeneous $f, g \in k[x_1, ..., x_r]$ of the same degree in some neighborhood V of P such that g never vanishes over V. All other conventions for $\mathcal{O}_X$ are the same as the affine case. The proof of the previous proposition goes through just fine for the projective case of $\mathcal{O}_X$ despite the additional conditions on homogeneity and degree of polynomials. Without these conditions, regular functions on a PAV wouldn't be well defined: When s is regular at $P = [a_0, ..., a_r]$ there is some neighborhood U of P such that for any $Q \in U$ and $\lambda \neq 0$

$$s(\lambda Q) = \frac{f(\lambda Q)}{g(\lambda Q)} = \lambda^{\deg f - \deg g} \cdot \frac{f(Q)}{g(Q)} = 1 \cdot s(Q)$$

This precisely says in some neighborhood of P, the regular function s takes values independently of the choice of homogeneous coordinates.

We extend the affine (resp. projective) definition of $\mathcal{O}_X$ to QAAV's (resp. QPAV's). The following result is absolutely crucial. The proofs of which are found in chapter I.3 of Hartshorne.

Theorem 3.16. *Let X be an AAV. Then*

1. $A[X] = \mathcal{O}_X(X)$
2. *Let $P \in X$. Define $\mathcal{O}_{X,P}$ to be the set of regular functions at P modulo agreement in a neighborhood of P. Then $\mathcal{O}_{X,P}$ is a local ring whose maximal ideal $\mathfrak{m}_P$ consists of equivalence classes which can be represented by a regular function which vanishes at P.*
3. *Let $\mathfrak{m}$ be the maximal ideal of $A[X]$ which corresponds to P. There is an isomorphism of rings $\mathcal{O}_{X,P} \cong A[X]_{\mathfrak{m}_P}$*

Theorem 3.17. *Let X be a PAV. Then*

1. $A[X] = \mathcal{O}_X(X)$
2. *Let $P \in X$. Define $\mathcal{O}_{X,P}$ to be the set of regular functions at P modulo agreement in a neighborhood of P. Then $\mathcal{O}_{X,P}$ is a graded local ring whose homogeneous maximal ideal $\mathfrak{m}_P$ consists of equivalence classes which are represented by regular functions which vanish at P.*
3. *Let $\mathfrak{m}$ be the homogeneous maximal ideal of $A[X]$ which corresponds to P. There is an isomorphism of graded rings $\mathcal{O}_{X,P} \cong A[X]_{\mathfrak{m}_P}$.*

Example 3.18. We will now distinguish the affine line from the vanishing of $y^2 - x^3$ in $\mathbb{A}^2$ which we denote by X. Let P be a point on the affine line. Then

$$\mathcal{O}_{\mathbb{A},P} \cong A[X]_{\mathfrak{m}_P} \cong \mathbb{C}[x]_{(x-p)}$$

which has Krull dimension one and if $Q = (0, 0)$ is the cusp of X

$$\mathcal{O}_{X,Q} \cong A[X]_{\mathfrak{m}_Q} \cong (\mathbb{C}[x, y]/(y^2 - x^3))_{(x)} \cong (\mathbb{C}[x, y]/(y^2 - x^3))_{(x,y)} \cong \mathbb{C}[t^2, t^3]_{(t^2,t^3)}$$

which has Krull dimension two. Thus the local rings on these two curves are distinct because every local ring on the affine line is dimension one since P was arbitrary. Later on, we will learn that these dimensions are actually tracking dimensions of tangent spaces to corresponding points.

It's a natural construction to obtain the underlying space of a variety from its corresponding prime ideal. However, this does little to explain how to obtain a sheaf of rings from this construction. Given a prime ideal $I \subseteq S = k[x_1, ..., x_n]$ let $X = Z(I)$. We know $\mathcal{O}_X(X) \cong S/I$. Every open subset of X is of the form $U \cap X$ for some open U of $\mathbb{A}^n_k$. Let $\mathcal{O}$ denote the structure sheaf of $\mathbb{A}^n_k$. Let $r : \mathcal{O}(\mathbb{A}^n_k) \to \mathcal{O}(U)$ denote restriction. Define

$$\mathcal{O}_X(U \cap X) = \mathcal{O}(U)/r(I)$$

Note that $r(I)$ is just the set of polynomials of S which vanish on X and are restricted to U.

Example 3.19. Let $f(x,y) = x$. Let $I = (f)$ in $S = \mathbb{C}[x,y]$. Let $X = Z(I)$ which is indeed a variety, hence a line in the plane. Let P denote the origin and let $U = \mathbb{A}^2 - P$. Then

$$\mathcal{O}_X(U \cap X) = \mathcal{O}(U)/r(I) \cong \mathcal{O}(U)/(f_U)$$

The ring $\mathcal{O}(U)$ is the set of all regular function on U, i.e. those functions which are locally rational. Thus

$$\mathcal{O}(U) \cong \mathbb{C}\,[x,y,u,v]\,/(xu-1, yv-1)$$

The open set U is sometimes called a distinguished set and, in some sense, the coordinate ring of any distinguished set is obtained by letting one or more variables become units. Hence we may write

$$\mathcal{O}(U) \cong \mathbb{C}\left[x, y, \frac{1}{x}, \frac{1}{y}\right]$$

so that x and y are units but, for example, $x-1$ and $y-1$ are not units.

Recall that the original goal was to define a variety without ever mentioning affine or projective space. For this to be possible, we need a "holding tank" of sorts for these more abstract varieties. This is where the definition of locally ringed space comes into play. We briefly mention that the process of taking equivalence classes to form $\mathcal{O}_{X,P}$ on an algebraic variety X works for arbitrary presheaves of rings on any space. The ring $\mathcal{O}_{X,P}$ is called the stalk at P. However, stalks aren't always local rings, as the next definition implies.

Definition 3.20. A locally ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ whose associated stalks $\mathcal{O}_{X,P}$ are local rings.

The previous two theorems tell us that algebraic varieties are indeed locally ringed spaces. Let $(X, \mathcal{O}_X)$ be a locally ringed space. What conditions should we impose on $(X, \mathcal{O}_X)$ in order to obtain a variety? First, X should be homeomorphic to an algebraic variety Y. Second, the local rings of both spaces should all line up via that homeomorphism. Turns out, the correct definition of isomorphisms of varieties is a bit special because its a homeomorphism f which actually induces an isomorphism of local rings via the pullback map $s \mapsto s \circ f$. There is no analogous situation for arbitrary locally ringed spaces. Hence we must decouple the homeomorphism from the induced maps on local rings when defining structure preserving maps between locally ringed spaces.

Definition 3.21. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally ringed spaces. Let $f : X \to Y$ be a continuous function. The pushdown of f is the sheaf of rings $f_*\mathcal{O}_X$ on Y defined by

$$f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$$

Its easy to check that the pushdown is indeed a sheaf of rings on Y. Note that this definition wouldn't make sense without the continuity assumption on f. Furthermore, $f_*\mathcal{O}_X$ has nothing to do with $\mathcal{O}_Y$. The point of using the pushdown is to give two sheaves of rings on the same space, namely Y in our case. This allows us to compare sheaves.

Definition 3.22. A morphism of locally ringed spaces is a pair $(f, f^\sharp)$ where $f : X \to Y$ is continuous and $f^\sharp$ is an indexing of homomorphisms of rings,

$$f^\sharp(U) : \mathcal{O}_Y(U) \to f_*\mathcal{O}_X(U)$$

for U open in Y such that for all open $V \subseteq U$ the diagram

$$\begin{array}{ccc}
\mathcal{O}_Y(U) & \xrightarrow{f^\sharp(U)} & f_*\mathcal{O}_X(U) \\
\downarrow & & \downarrow \\
\mathcal{O}_Y(V) & \xrightarrow{f^\sharp(V)} & f_*\mathcal{O}_X(V)
\end{array}$$

commutes (the vertical maps are restriction). In addition, we require the limit of every $f^\sharp(U)$ to be a local homomorphism, i.e. the induced map

$$f^\sharp_P : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$$

must satisfy

$$(f_P^\sharp)^{-1}(\mathfrak{m}_P) = \mathfrak{m}_{f(P)}$$

An isomorphism of locally ringed spaces is a morphism $(f, f^\sharp)$ such that f is a homeomorphism and every $f^\sharp(U)$ is an isomorphism.

The induced map in the previous definition comes from a construction which we have not seen yet. Given a point in a locally ringed space $P \in X$ there are two ways to obtain the local ring $\mathcal{O}_{X,P}$. The first way is via taking the set of all functions which are regular at P modulo agreement over a neighborhood of P. The other way comes from taking the direct limit. We will introduce a special case of the direct limit now. Given a sequence of ring homomorphisms

$$A_0 \to A_1 \to ...$$

$$d_i : A_i \to A_{i+1}$$

define $A = \lim_i A_i$ to be the unique ring satisfying the following universal mapping property: if $\varphi : B \to A_i$ then there is a unique $\psi : B \to A$ such that

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & A_j \arrow[d, "d_j"] \\ & A \end{tikzcd}$$

commutes. Existence is not obvious but the construction can be found on wikipedia. Uniqueness comes from a simple but clever use of the universal property. Given a different linear sequence (Bi, d_i) with maps $A_i \to B_i$ there is a unique map $A \to B$ where B is the direct limit of the B_i.

Fix P in a locally ringed space X. Let

$$U_0 \supseteq U_1 \supseteq ...$$

be a sequence of inclusions of open sets with $P \in U_i$ for all $i \geq 0$. Taking restriction maps gives

$$\mathcal{O}_X(U_0) \to \mathcal{O}_X(U_1) \to ...$$

meaning there is a direct limit to associate to P provided this sequence contains all other linear sequences of inclusion of open sets. Turns out, $\mathcal{O}_{X,P}$ is this direct limit. Let

$$(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

be a morphism of locally ringed spaces. Fix $P \in X$. Let U_i denote a sequence of inclusions of open sets in Y such that $f(P) \in U_i$ for all $i \geq 0$. Then there are ring homs

$$f^\sharp(U_i) : \mathcal{O}_Y(U_i) \to f_*\mathcal{O}_X(U_i)$$

which give rise to a unique ring hom $f_P^\sharp$ via taking the direct limit, i.e.

$$f_P^\sharp : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$$

This is a direct consequence of the fact that

$$f^{-1}(U_0) \supseteq f^{-1}(U_1) \supseteq ...$$

and all of these open sets contain P. With this, the definition of a morphism is simply asking for two things:

1. taking $f^\sharp$ of a section must commute with restriction
2. the unique ring homs $f_P^\sharp$ must satisfy $(f_P^\sharp)^{-1}(\mathfrak{m}_P) = \mathfrak{m}_{f(P)}$

The second condition is what was meant by "we require the limit of every $f^\sharp(U)$ to be a local homomorphism". This condition is awkward to include but necessary because we needed a canonical way of constructing ring homs between stalks and the direct limit is the industry standard.

After 3 pages of definitions, we can finally define the category of varieties.

Definition 3.23. A variety over k is a locally ringed space which is isomorphic to an AAV, PAV, QAAV, or QPAV over k. An affine variety (resp. projective variety) is any locally ringed space which is isomorphic to an AAV (resp. a PAV). A morphism of varieties is a morphism of locally ringed spaces.

Exercise 3.24 (★★★☆☆)**.** Prove that compositions of morphisms of locally ringed spaces are morphisms of locally ringed spaces.

Exercise 3.25 (★★☆☆☆)**.** Let X and Y affine varieties over k. Let $\varphi : A[Y] \to A[X]$ be a k-algebra hom. Define

$$f : X \to Y \quad f(P) = \varphi^{-1}(I(P))$$

and define

$$f^\sharp(U) : \mathcal{O}_Y(U) \to f_*\mathcal{O}_X(U) \quad f^\sharp(U)(s) = s \circ f$$

Show that $(f, f^\sharp)$ is a morphism of varieties. Conclude that giving a k-algebra hom is equivalent to giving a morphism of varieties.

Remark 3.26. In the previous exercise, $f^\sharp$ is defined by the classic pullback map from topology, i.e. $s \mapsto s \circ f$. This is not the only possible $f^\sharp$ that one can define from a continuous f (recall that f and $f^\sharp$ were decoupled in the definition of a morphism). However, this particular $f^\sharp$ is special because it gives the needed equivalence between algebra homs and morphisms of varieties in exer. 3.21. For an arbitrary morphism $(g, g^\sharp)$ of locally ringed spaces $g^\sharp$ does not have to be $s \mapsto s \circ g$. This is quite common in examples outside of the category of varieties.

4 Rational Maps

At this point we've encountered different varieties, and we have a certain kind of map between two varieties, called a morphism, which you can recall is a continuous map $X \to Y$ where pulling back a regular function on Y gives you a regular function on X.

In the following sections we're going to introduce a different kind of map, called a rational map, between varieties, which will allow us a little bit more nuance. The basic idea is that a rational map will only be a partial function – we won't map all of X to Y, in the same way that the "function" $1/x : \mathbf{R} \to \mathbf{R}$ doesn't actually have domain $\mathbf{R}$. But because of the rigidness of varieties, rational maps are still going to give us a lot of information even if they're only defined on part of a variety. This nuance is important for later results we might see, like the classification of varieties or for resolutions of singularities, just to name-drop two important ideas so that you believe we have some reason to care. Otherwise, why worry about anything other than honest morphisms between varieties, right?

Before we get started, let's consider two salient ideas:

1. The nonempty open subsets of a(n irreducible)[1] variety are dense (*big*!).
2. If you have a morphism, then it's determined by a nonempty open subset. In other words:

Lemma 4.1. *If $\varphi : X \to Y$ and $\psi : X \to Y$ are two morphisms of varieties X and Y such that there is a nonempty open subset $U \subseteq X$ such that $\varphi|_U = \psi|_U$, then $\varphi = \psi$.*

Exercise 4.2 (★★☆☆☆)**.** Prove **Lemma 4.1**. Hint: one such proof could take the following steps:

0. Since Y is a variety, it lives inside of $\mathbf{P}^n$ for some n. Without loss of generality, we can compose φ and ψ with the inclusion map $Y \hookrightarrow \mathbf{P}^n$, so it suffices to prove **Lemma 4.1** in the case where $Y = \mathbf{P}^n$.
1. Show the map $\varphi \times \psi : X \to \mathbf{P}^n \times \mathbf{P}^n$, $x \mapsto (\varphi(x), \psi(x))$ is continuous.
2. Show that the diagonal $\Delta = \{(p, p)\} \subseteq \mathbf{P}^n \times \mathbf{P}^n$ is closed.
3. Using the hypotheses of the lemma, where does $\varphi \times \psi(U)$ live inside of $\mathbf{P}^n \times \mathbf{P}^n$?
4. Use the fact that $U \subseteq X$ is dense and Δ is closed to state where $\varphi \times \psi(X)$ lives in $\mathbf{P}^n \times \mathbf{P}^n$.

[1] Our varieties are, by definition, irreducible, but as you may know that's not a universal convention. So sometimes I'll specifically point out the irreducibility condition whenever it's relevant to the claim I'm making. Not always though because the parentheticals get cumbersome.

5. Interpret step 4 as the conclusion to the lemma.

One additional way to motivate rational maps is that they're trying to ask and answer the converse to number 2 above. If we do have a function that's only defined on a dense open subset, can it extend to a map on the whole space? The answer is actually "no, not always," but even if not, rational maps suffice to say a lot.

Now let's begin by actually defining a rational map.

Definition 4.3. Let X and Y be two varieties. A **rational map** from X to Y is a pair of an open subset $U \subseteq X$ and a morphism $\varphi_U : U \to Y$. We write $\langle U, \varphi_U \rangle$ for this pair. We also put an equivalence relation on the set of pairs $\langle U, \varphi_U \rangle$, where $\langle U, \varphi_U \rangle$ is equivalent to $\langle V, \varphi_V \rangle$ if φ_U and φ_V agree in the only place they could, $U \cap V$. Sometimes you'll see the notation $\varphi : X \dashrightarrow Y$ to denote a rational map.

Right off the bat there's some work to be done with this definition.

Exercise 4.4 (★☆☆☆☆). Check that the relation we just described on the pairs $\langle U, \varphi_U \rangle$ is indeed an equivalence relation; that is, it is symmetric, reflexive, and transitive. Hint: use **Lemma 4.1**.

Okay, even once you've done **Exercise 4.4**, this is still a pretty esoteric definition right now. A rational map $X \dashrightarrow Y$ is a class of an open subset $U \subseteq X$ and a morphism $\varphi_U : U \to Y$, which means we're not dealing with functions on the nose but equivalence classes of both subsets and morphisms. Yuck! Fortunately though, we can still say a lot. First let's just cook up some examples, without worrying about the equivalence relation yet.

Example 4.5. Any morphism $\varphi : X \to Y$ is a rational map. One representative of the equivalence class is simply $\langle X, \varphi \rangle$.

Example 4.6. Let's see a more interesting example. Let $\mathbf{A}^2$ have coordinates (x, y) and we'll cook up a rational map $\mathbf{A}^2 \dashrightarrow \mathbf{A}^1$ that is not a morphism. It'll be the map $(x, y) \mapsto y/x$, with open subset $D(x) = \{(x, y) \in \mathbf{A}^2 \mid x \neq 0\}$.

Remark 4.7. Let's turn our attention to the equivalence relation now, and see how it actually helps simplify the picture, rather than being a cumbersome piece of data to carry around.

It actually is the case that I could be a bit sloppy in **Example 4.6** and get away with simply saying $\mathbf{A}^2 \dashrightarrow \mathbf{A}^1$ is $(x, y) \mapsto y/x$. In other words, I could omit the open subset, even though the definition requires us to carry it around. The reason for that is because $D(x)$ is the largest open subset that permits $(x, y) \mapsto y/x$ to be a morphism. Extending it any further introduces 0s in the denominator!

But now you might ask, what about a smaller open subset? And this is where the equivalence relation kicks in, and simplifies the picture. If we did something rather silly and chose a different open subset like $D(xy) = \{(x, y) \in \mathbf{A}^2 \mid xy \neq 0\}$, then $D(x)$ and $D(xy)$ are two open subsets where $(x, y) \mapsto y/x$ is defined – and must agree with itself! – on their intersection, which is just $D(xy)$ since $D(xy) \subseteq D(x)$.

And really $D(xy)$ was an arbitrary example; the same argument works for any open subset where the morphism $(x, y) \mapsto y/x$ is defined. So in other words, if we fix the morphism part of a rational map, then we may permit ourselves to occasionally be sloppy and omit the open subset because it's implied to be maximal.

Definition 4.8. Given a rational map $\varphi : X \dashrightarrow Y$, the largest open subset on which φ is defined is called the **domain of** φ, and the complement of the domain is called the **locus of indeterminancy**.

Here's some more examples.

Example 4.9. Consider the rational map $\mathbf{P}^2 \dashrightarrow \mathbf{P}^1$ defined by $[x : y : z] \mapsto [x : y]$.

Exercise 4.10 (★☆☆☆☆). What is the locus of indeterminancy of the rational map in **Example 4.9**? Can that rational map extend to a honest morphism $\mathbf{P}^2 \to \mathbf{P}^1$?

Example 4.11. Here's an example where we can see a bit of a complication re-arise with regards to the equivalence classes. What happens if we don't a priori fix one defining equation for the morphism? It turns out that the equivalence class doesn't just carry the data of the domain of the rational map, but also the data of it as a morphism too. Consider the variety $V(xw - yz) \subseteq \mathbf{A}^4 = \{(x, y, z, w)\}$. Let's define a rational map $V(xw - yz) \dashrightarrow \mathbf{A}^1$. It'll be defined to be

$$(x, y, z, w) \mapsto \begin{cases} x/y & \text{if } y \neq 0 \\ z/w & \text{if } w \neq 0. \end{cases}$$

Now notice that when both $y \neq 0$ and $w \neq 0$, we get $x/y = z/w$, because on our variety, $xw - yz = 0$ and therefore $x/y = z/w$ via cross multiplication. So this is indeed a well-defined rational map with domain $\{(x, y, z, w) \mid y \neq 0 \text{ or } w \neq 0\}$. This means that our $V(xw - yz) \dashrightarrow \mathbf{A}^1$ is a rational map that can't be described as a single morphism – it must be defined piecewisely, and the equivalence relation allows us to see that if you define it as $\langle D(y), x/y\rangle$ and I define it as $\langle D(w), z/w\rangle$, we are each defining the same rational map (whence the same equivalence class).

5 The Category of Varieties with Dominant Rational Maps

☡ **Warning! 5.1.** Without the soon-to-be-defined notion of "dominant" rational maps, we won't be able to get off the ground. The attempt to define a category given by "the objects are varieties and the arrows are rational maps" *doesn't* actually form a category! That's because we can't necessarily compose rational maps, since they're not defined everywhere. It could be the case that the image of the first rational map might lie in the locus of indeterminancy of the second.

Exercise 5.2 (★☆☆☆☆). Consider the morphism $\varphi : \mathbf{A}^2 \to \mathbf{A}^3$ defined by $(x, y) \mapsto (x^2, xy, y^2)$ thought of as a rational map. Consider the rational map $\psi : \mathbf{A}^3 \dashrightarrow \mathbf{A}^1$ defined by $(a, b, c) \mapsto \frac{1}{ac-b^2}$. Explain why the composition $\psi \circ \varphi : \mathbf{A}^2 \dashrightarrow \mathbf{A}^1$ is not defined.

Fortunately all hope is not lost, as long as we impose one more condition on our rational maps.

Definition 5.3. We say that a rational map $\varphi : X \dashrightarrow Y$ is **dominant** if for some (and hence every) pair $\langle U, \varphi_U\rangle$, the image of φ_U is dense in Y.

Exercise 5.4 (★★☆☆☆). Explain why you can compose *dominant* rational maps.

Now our title has been justified. The category we get contains (irreducible) varieties as objects and dominant rational maps as arrows. And,

Definition 5.5. When we have an isomorphism in this category, which is simply to say we have a rational map $\varphi : X \dashrightarrow Y$ with an inverse $\psi : Y \dashrightarrow X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$, we call φ (and ψ) a **birational map** and call X and Y **birationally equivalent** (or sometimes just **birational** to each other).

Exercise 5.6 (★★★★☆). In the last section we listed a lot of examples of rational maps.

- Any morphism $X \to Y$ (**Example 4.5**)
- $\mathbf{A}^2 \dashrightarrow \mathbf{A}^1$, $(x, y) \mapsto y/x$ (**Example 4.6**)
- $\mathbf{P}^2 \dashrightarrow \mathbf{P}^1$, $[x : y : z] \mapsto [x : y]$ (**Example 4.9**)
- $V(xw - yz) \dashrightarrow \mathbf{A}^1$, $(x, y, z, w) \mapsto x/y$ or z/w (**Example 4.11**)

Of those that were dominant (which ones were those?), which of them are birational maps? What are their inverses?

Example 5.7. Depending on how much you do of **Exercise 5.6**, you may (or may not!) now have several examples of birational maps. Here's at least one more for your benefit. Consider the variety $V(y^2-x^3) \subseteq \mathbf{A}^2$. We'll show that $V(y^2-x^3)$ is birationally equivalent to $\mathbf{A}^1$ (which we'll give coordinate t). In one direction, we define $\varphi : \mathbf{A}^1 \dashrightarrow V(y^2-x^3)$ by $t \mapsto (t^2, t^3)$, and in the other direction, define $\psi : V(y^2-x^3) \dashrightarrow \mathbf{A}^1$ by $(x, y) \mapsto y/x$. Indeed, see that

$$\psi \circ \varphi(t) = \psi(t^2, t^3) = \frac{t^3}{t^2} = t, \text{ and}$$
$$\varphi \circ \psi(x, y) = \varphi\left(\frac{y}{x}\right) = \left(\left(\frac{y}{x}\right)^2, \left(\frac{y}{x}\right)^3\right) = \left(\frac{y^2}{x^2}, \frac{y^3}{x^3}\right) = \left(\frac{x^3}{x^2}, \frac{y^3}{y^2}\right) = (x, y).$$

Therefore $V(y^2-x^3)$ and $\mathbf{A}^1$ are birationally equivalent, as desired.

Exercise 5.8 (★★★☆☆)**.** If you're avoiding **Exercise 5.6** (don't!), but at the very least, you should at least do one example. Show that $V(x^3+y^3-xy)$ and $\mathbf{A}^1$ are birationally equivalent.

Remark 5.9. We're knee deep in the mathematics currently, but let me just point out that heuristically, **Exercise 5.8** is believable because when we draw $V(x^3+y^3-xy)$, it looks like an $\mathbf{A}^1$ with a knot in it! This is the intuition that you want to have kinda half-stewing in the back of your mind, and it's something that textbooks tend to omit unless they draw lots of pictures. Hopefully the speaker for these notes (whether that's me or someone else) is self-aware enough to draw these pictures for you. If not, ask me about them! In fact, this intuition gets to be made precise by the time we get to **Corollary 5.20**; for now, let me give you the slogan to have in mind that jives with the story the pictures tell: two varieties are birational if and only if they have isomorphic (dense) open subsets.

Definition 5.10. We call a variety **rational** if it is birationally equivalent to $\mathbf{P}^n$ for some n. I'm not super fond of this; it may take more words to say "birationally equivalent to some $\mathbf{P}^n$" but it's explicit. Ho hum.

Exercise 5.11 (★★☆☆☆)**.** Show that $\mathbf{P}^n \times \mathbf{P}^m$ is rational by constructing an explicit birational map $\mathbf{P}^n \times \mathbf{P}^m \dashrightarrow \mathbf{P}^{n+m}$. As a corollary, show that if X and Y are rational, then $X \times Y$ is rational. Hint: this is an exercise in combinatorics.

We've seen some specific examples of rational maps and how they may not be morphisms, but now we have a notion of birational equivalence, just like we have a notion of isomorphism of varieties. Do these coincide; i.e., if two varieties are birationally equivalent, need they be isomorphic?

Exercise 5.12 (☆☆☆☆☆)**.** *Certainly* the converse is true; if X and Y are isomorphic then they are birationally equivalent. Why?

It turns out that the answer to our question is "no." (– which is good! We're not retreading old ground.) There are varieties that are birationally equivalent but not isomorphic.

Exercise 5.13 (★★☆☆☆)**.** Prove that $\mathbf{P}^1$ and $\mathbf{A}^1$ are birationally equivalent, but not isomorphic.

Also, now that we have a category, another thing you might have on your mind is: have we seen this category before? In the same way that, for example, finite dimensional $\mathbf{R}$-vector spaces and matrices over $\mathbf{R}$ describe the same thing in different languages, maybe this category of varieties and dominant rational maps is equivalent to a different category we can describe? The answer is yes! (In fact, this question and its affirmative answer will pop up many times throughout algebraic geometry; see the remark after **Theorem 5.14** for just how big a deal this is.)

Theorem 5.14. *There is an equivalence of categories*

$$\{\textit{varieties and domainant rational maps}\} \leftrightarrow \{\textit{finitely generated field extensions of } \mathbf{C}\}.$$

Remark 5.15. First, a small comment: We know how to define varieties over algebraically closed fields other than $\mathbf{C}$. In this setting, **Theorem 5.14** remains true, replacing $\mathbf{C}$ by your field.

Next, a big comment: **Theorem 5.14** is remarkable! An equivalence of categories is an incredibly strong thing; it's basically the magic that makes algebraic geometry happen. What it means is that any question you would want to ask or answer in one category can be asked and answered in the equivalent category. A question that is hard in one category might be easy in the other, and just the ability to translate itself can motivate questions. This is exactly the same magic that we saw on day 1 when we took a $\mathbf{C}$-algebra like $\mathbf{C}[x,y]/(y-x^2)$ and turned its maximal ideals into points on the parabola $y = x^2$!

Proof of ***Theorem 5.14***. To show this, we need to produce:
- a field extension, if we're given a variety,
- a map of fields, if we're given a dominant rational map, and
- a way to do both of these things in reverse.

Let's start by producing a field extension, given a variety. That part is easy because we've already done it in a past talk. Let X be a variety; we get a field by taking the function field $K(X)$. Recall

$$K(X) = \{\langle U, f\rangle \text{ a rational function on } X \mid U \subseteq X \text{ and } f \text{ is regular on } U\}$$

modulo the relation $\langle U, f\rangle \sim \langle V, g\rangle$ if $f = g$ on $U \cap V$.

☡ **Warning! 5.16.** This is the first time we've reintroduced rational *functions* in these notes. Keep them distinct from rational *maps*! Their definitions are dangerously similar! In fact, that's not a coincidence:

Exercise 5.17 (★★☆☆☆). Definition/sanity check: show that a rational function in $K(X)$ is the same as a rational map $X \dashrightarrow \mathbf{A}^1$.

The function field $K(X)$ is a field extension of $\mathbf{C}$, so we are done with this part. In fact, let us note here that we're really showing an equivalence of categories

$$\{\text{varieties and domainant rational maps}\} \leftrightarrow \{\text{function fields } K(-)\},$$

and that the latter is in fact the finitely generated field extensions of $\mathbf{C}$.

Moving on, given a dominant rational map $\varphi : X \dashrightarrow Y$ which we'll write $\langle U, \varphi_U\rangle$, we need to produce a map of fields. We'll actually do so "contravariently," which means that our output will be a map which goes $K(Y) \to K(X)$ – the direction has been reversed. Here's how we get it. Let $\langle V, f\rangle \in K(Y)$ be a rational function, where $V \subseteq Y$ and f is regular on V. Since $\langle U, \varphi_U\rangle$ is dominant, $\varphi_U(U)$ is dense in Y, so $\varphi_U^{-1}(V)$ is a nonempty open subset of X. Since φ_U is a morphism, $f \circ \varphi_U$ is regular on $\varphi_U^{-1}(V)$, and hence it defines an equivalence class $\langle \varphi_U^{-1}(V), f \circ \varphi_U\rangle \in K(X)$. So that's our map; given a dominant rational map $\langle U, \varphi_U\rangle : X \dashrightarrow Y$, we get a map $K(Y) \to K(X)$ defined by

$$\langle V, f\rangle \mapsto \langle \varphi_U^{-1}(V), f \circ \varphi_U\rangle.$$

Exercise 5.18 (★★★☆☆). Do the due diligence of checking this is well-defined on equivalence classes. It's not particularly enlightening in my opinion though.

Okay, now we need to do each of these constructions in reverse. Again, if we have a finitely generated field extension of $\mathbf{C}$, then it is a function field $K(X)$ for some variety X.

What about a map of $\mathbf{C}$-algebras $\eta : K(Y) \to K(X)$; how do we get a rational map $X \dashrightarrow Y$? The variety Y has an affine cover, so without loss of generality, let Y be affine, since we just need to define this map on a local open subset and we can shrink to an affine one. Y has a coordinate ring $A(Y) = \mathbf{C}[y_1, \ldots, y_n]/I(Y)$. Using η, we see that $\eta(y_1), \ldots, \eta(y_n) \in K(X)$, so there exists $U \subseteq X$ such that $\eta(y_i)$ is regular on U. Thus η induces an injective $\mathbf{C}$-algebra homomorphism $A(Y) \hookrightarrow \mathcal{O}(U)$. This corresponds to an injective morphism of varieties $U \hookrightarrow Y$, hence a dominant rational map $X \dashrightarrow Y$. □

Remark 5.19. There are a couple of things we black boxed in the proof of **Theorem 5.14** above. Feel free to check them if you like.

Corollary 5.20 (Corollary to **Theorem 5.14**.)**.** *The following are equivalent.*

1. *X and Y are birationally equivalent.*
2. *There exists open subsets $U \subseteq X$ and $V \subseteq Y$ such that $U \cong V$ as varieties.*
3. *$K(X) \cong K(Y)$ as field extensions over $\mathbf{C}$.*

Exercise 5.21 (★★☆☆☆)**.** Prove **Corollary 5.20** by showing (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Note that only (1) $\Rightarrow$ (2) needs any work since the other two implications are essentially the theorem (confirm this for yourself).

Exercise 5.22 (★★★☆☆)**.** Let's allow **Theorem 5.14** to suggest some mathematics to us. Consider $X = \mathbf{P}^1$ with homogeneous coordinates $[x : y]$, but you may reduce to the affine cover where $y = 1$, so we have local coordinate x. Consider $Y = \mathbf{A}^1$ with coordinate t. Find a dominant rational map $\mathbf{P}^1 \dashrightarrow \mathbf{A}^1$ corresponding to the morphism of function fields $K(\mathbf{A}^1) \to K(\mathbf{P}^1)$ given by $t \mapsto x^2$.

Example 5.23. Here's another application of **Theorem 5.14**. We'll show that $\mathbf{P}^2$ is birationally equivalent to $X = V(xy - zw) \subseteq \mathbf{P}^3 = \{[x : y : z : w]\}$. But we won't actually produce a birational map $\mathbf{P}^2 \dashrightarrow X$ with birational inverse! Instead, we'll compute the function fields $K(X)$ and $K(\mathbf{P}^2)$, then apply **Theorem 5.14**.

To compute $K(X)$, first reduce to the affine subset of X where $w \neq 0$. This is valid since rational functions are equivalence classes defined on open subsets. So in coordinate language we may set $w = 1$ and identify this affine subset as sitting inside $\mathbf{A}^3 = \{(x, y, z)\}$. Here, the coordinate ring of the variety X is

$$A(X) = \mathbf{C}[x, y, z]/I(X) = \mathbf{C}[x, y, z]/(xy - z).$$

This ring is isomorphic to $\mathbf{C}[x, y]$ via the map $f(x, y) \mapsto f(x, y, xy)$. The field of fractions of $\mathbf{C}[x, y]$ is $\mathbf{C}(x, y)$, and thus $K(X) = \mathbf{C}(x, y)$. And also we know $K(\mathbf{P}^2)$ is $\mathbf{C}(x, y)$ as well, so since $K(X) \cong K(\mathbf{P}^2)$, X and $\mathbf{P}^2$ are birationally equivalent (even though we never described a birational map!).

Remark 5.24. It requires a few commutative algebra facts to properly prove, so we won't do so here unless you twist my arm later, but it's a neat fact that any variety X with $\dim(X) = r$ is birational to a hyperplane in $\mathbf{P}^{r+1}$. Recall that a hyperplane is defined by the vanishing of a single equation.

6 Blow Ups and Singularities: The Reason for Birational Maps

Now we're getting into (one of) the big kahunas. This section of the notes might be skipped and returned to at a later date because it's technical and we might not need it anytime soon. But if we're now at the stage where we're needing/wanting it, or if you're reading it of your own volition, be not afraid! (Although those three words should carry all the connotations you'd imagine – if a biblical wheel-of-eyes old testament angel screams discordantly at you not to fear blow ups, would you?)

https://twitter.com/reactjpg/status/1330804563796926464/photo/1

So, what is a blow up, and why should we care? The true answer, which we'll cycle back to towards the end, is that blow ups give us a way to "fix" singularities. What does this mean? The idea is: say we have a variety which is singular at some places. As an example to keep in the back of your mind, consider the cuspidal cubic $V(y^2 - x^3)$, which is singular at the origin. We'd like a way to understand "basically all" of our variety by comparing it to something similar which isn't singular and is therefore nicer. Isomorphisms of varieties can't do this for us – any singular variety *must* be isomorphic to another singular variety – but birational equivalence is *exactly* the tool we need! We already know our slogan: varieties are birational when they have isomorphic open subsets, and this is exactly the way in which we can compare "basically all" of our singular variety to one that is smooth! A birational equivalence between, for example, $V(y^2 - x^3)$ and $\mathbf{A}^1$, is the precise way to see that the cuspidal cubic is basically the affine line, up to the cusp point, which gets unkinked. Blow ups are a systematic way to fix your singularities, no matter what horrendous variety you might start with!

Let's ease our way into this topic not by jumping right in with a definition, but instead a motivating example.

Example 6.1. Let $\mathbf{A}^2 \dashrightarrow \mathbf{A}^1$ be a rational map defined by $(x, y) \mapsto y/x$. We have seen back in **Example 4.6** that this is a rational map with domain $D(x) = \{(x, y) \in \mathbf{A}^2 \mid x \neq 0\}$. Suppose we're a bit miffed about that and want to extend this rational map to a larger domain. One thing we can do to extend this map is to replace $\mathbf{A}^1$ with $\mathbf{P}^1$; that is, we get a map $(x, y) \mapsto [x : y]$. This is, honest to goodness, how you might naturally want to extend the map $(x, y) \mapsto y/x$, since its outputs are slopes of lines through the origin, and that's what determines points in $\mathbf{P}^1$. In fact, the extension $(x, y) \mapsto [x : y]$ is now clearly defined everywhere except $(0, 0) \in \mathbf{A}^2$. Naturally, we'd ask: how does this construction compare to our original rational map?

We answer this by considering the graph of our extension. We'll denote it by Γ and recall that

$$\Gamma = \{(x, y) \times [x : y]\} \subseteq \mathbf{A}^2 \times \mathbf{P}^1.$$

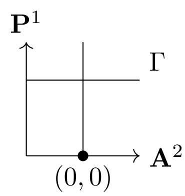
(I'm writing $(x, y) \times [x : y]$ because it's a lot more tractable than $((x, y), [x : y])$.) We'll visualize this graph as we do with graphs in other contexts in math: draw an axis for the input (in this case $\mathbf{A}^2$), an axis for the output ($\mathbf{P}^1$), and Γ is a picture sitting in the $\mathbf{A}^2 \times \mathbf{P}^1$-plane, just like a graph $(x, f(x))$ would in the xy-plane. I'd love to actually draw this for you, but I don't have a great way to do so that's helpful until we do the work to unpack Γ by hand.

Fortunately, we don't have to actually draw Γ for what I'm about to argue (though I'll give you a really crappy picture at the end and you can tell me if it helps). We're just going to think about what happens when we take Γ and project it to the $\mathbf{A}^2$-factor. When we're away from $(0, 0) \in \mathbf{A}^2$, the projection $\pi : \Gamma \to \mathbf{A}^2$ is an isomorphism, because like I said, (x, y) determines a slope y/x (which is ∞ if $x = 0$ and $y \neq 0$) and hence a point in $\mathbf{P}^1$, so π has an inverse $(x, y) \mapsto (x, y) \times [x : y]$, where $[x : y]$ represents a single well-defined unique slope, the slope of the line from the origin to (x, y).

But what happens above $(0, 0) \in \mathbf{A}^2$? It turns out that we get a whole copy of $\mathbf{P}^1$ inside of Γ above $(0, 0)$! How can we verify that claim? Consider a line $y = mx$ which passes through the origin in $\mathbf{A}^2$. Every point (x, y) on this line, minus the origin, gets sent to $(x, y) \times m$, to abuse notation and use m to denote $[x : y]$, since it's supposed to represent our slope. If we take the closure of this line, meaning to now include the origin, all those points will still be sent to $(x, y) \times m$. But yet m was arbitrary, so the origin gets sent to $(0, 0) \times [x : y]$ for any $[x; y] \in \mathbf{P}^1$.

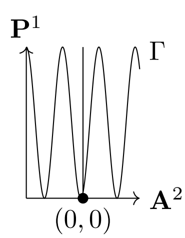
So what's going on here? We understand the extension by understanding Γ, and we understand Γ by understanding its projection $\pi : \Gamma \to \mathbf{A}^2$. The projection π is an isomorphism outside of $(0, 0)$ and contracts a whole copy of $\mathbf{P}^1$ to $(0, 0)$. This gives us the idea of "blowing up" the origin in $\mathbf{A}^2$.

Here's that picture. It's "honest" but because of that not particularly enlightening without the accompanying explanation.



See that away from $(0, 0)$, Γ is a horizontal "line," which is supposed to look like the same "line" given by the $\mathbf{A}^2$-axis, hence an isomorphism of varieties there. And at $(0, 0)$, Γ has a vertical "line," which is supposed to look like the entire $\mathbf{P}^1$-axis. So we've taken $\mathbf{A}^2$, left it completely undisturbed away from $(0, 0)$, and taken $(0, 0)$ and stretched it into a whole copy of $\mathbf{P}^1$ (i.e., blown it up).

Remark 6.2. As a quick aside, there is one lie in the picture of Γ above. Writing the preimage of $\mathbf{A}^2 \setminus \{(0, 0)\}$ as a horizontal line helps highlight the fact that it's isomorphic to $\mathbf{A}^2 \setminus \{(0, 0)\}$, but it does suggest that the $\mathbf{P}^1$ factor is always constant, which is definitely not true. Different $(x, y) \in \mathbf{A}^2$ are going to have different slopes $[x : y]$. A slightly more honest picture would be something like



which highlights the fact that as you vary the element $(x, y) \in \mathbf{A}^2$, you'll get different slopes from the origin to (x, y) and hence different values $[x : y]$ in the $\mathbf{P}^1$-factor. But adding this honesty doesn't really contribute much to our understanding of how a blow up actually works, so we'll go back to the picture that lies a little bit from now on.

Believe it or not (well, believe it; it's true!), the construction we've just described in **Example 6.1** is actually general!

Definition 6.3. The **blow up of $\mathbf{A}^n$ at the origin** is the closed subset $\Gamma \subseteq \mathbf{A}^n \times \mathbf{P}^{n-1}$ defined by

$$\Gamma = \left\{x_i y_j = x_j y_i \mid i, j \in \{1, \ldots, n\}, \mathbf{A}^n = \{(x_1, \ldots, x_n)\}, \mathbf{P}^{n-1} = \{[y_1 : \cdots : y_n]\}\right\}.$$

> **Exercise 6.4** (★☆☆☆☆)**.** Do the typechecking necessary to verify that **Example 6.1** satisfies the definition of a blow up in **Definition 6.3**. It's just a matter of relabeling a couple of things in our example to match the notation of the definition.

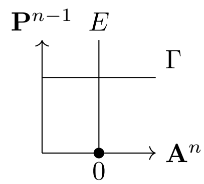
Remark 6.5. There is a natural map $\varphi : \Gamma \to \mathbf{A}^n$ defined by the composition $\Gamma \hookrightarrow \mathbf{A}^n \times \mathbf{P}^{n-1} \xrightarrow{\pi} \mathbf{A}^n$. The results we saw in **Example 6.1** hold:

1. If $P \in \mathbf{A}^n$ is not the origin, then $\varphi^{-1}(P)$ is a single point.
2. The preimage of the origin, $\varphi^{-1}(0)$, which is called the exceptional divisor, is all of $\mathbf{P}^{n-1}$.
3. The points of $\varphi^{-1}(0)$ are in bijection with the set of lines through 0 in $\mathbf{A}^n$.

4. Γ is irreducible.

To check (4), since we didn't in **Example 6.1**, observe that Γ is the union of $\Gamma \setminus \varphi^{-1}(0)$ and $\varphi^{-1}(0)$. The first piece is isomorphic to $\mathbf{A}^n \setminus 0$, hence irreducible. But observe that $\varphi^{-1}(0)$ is in the closure of a subset $L \subseteq \Gamma \setminus \varphi^{-1}(0)$ (this subset is $L = \{x_i = a_i t, y_i = a_i \mid t \in \mathbf{A}^1 \setminus 0, a_i \in \mathbf{C} \text{ not all } 0\}$). Hence $\Gamma \setminus \varphi^{-1}(0)$ is dense in Γ, and thus Γ is irreducible.

It's the same picture as before! We'll write E for the exceptional divisor $\varphi^{-1}(0)$. Also note that Γ is still the entire graph, while E is only the "vertical line" $\varphi^{-1}(0)$.



Okay, fantastic; all blow ups of $0 \in \mathbf{A}^n$ are captured by our toy example! But there are two wrinkles, one not important and one important. The first wrinkle is: what if we want to blow up at a different point $P \in \mathbf{A}^n$? The reason this one is not important is because you just have to first make a linear change of coordinates sending P to 0, and then do a blow up at 0, which we already know how to do. The second wrinkle, which isn't quite so trivial, is: what if we want to blow up a different variety other than $\mathbf{A}^n$?

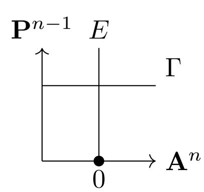
Definition 6.6. Let X be a closed subvariety of $\mathbf{A}^n$ passing through 0 (or not, remember we can do a change of coordinates). The **blow up of** X **at the origin** is $\widetilde{X} = \overline{\varphi^{-1}(X \setminus 0)} \subseteq \Gamma$, where $\varphi : \Gamma \to \mathbf{A}^n$ is defined in **Remark 6.5**. By abuse of notation, we'll write $\varphi : \widetilde{X} \to X$ for $\varphi|_{\widetilde{X}}$.

Remark 6.7. Just like we've been seeing for $\mathbf{A}^n$, if we blow up a variety X, then $\varphi : \widetilde{X} \to X$ induces an isomorphism $\widetilde{X} \setminus \varphi^{-1}(0) \cong X \setminus 0$. By **Corollary 5.20**, $\widetilde{X}$ and X are birationally equivalent. One other comment to make is that right now, the definition of the blow up $\widetilde{X}$ appears to depend on how X has been embedded in $\mathbf{A}^n$, but this is not the case; blowing up is actually intrinsic to the variety X and doesn't depend on an embedding.

In addition to the exceptional divisor, we have a few more names for the parts of a blow up:

Definition 6.8. Given any birational map $\varphi : X \dashrightarrow Y$, the locus where it is not an isomorphism is called the **exceptional locus**, E. If $V \subseteq Y$, then $\varphi^{-1}(V)$ is called the **total transform of** V. If we write $Z = \varphi(E)$, then for any $V \not\subseteq Z$, we call $\overline{\varphi^{-1}(V \setminus Z)}$ the **strict transform of** V.

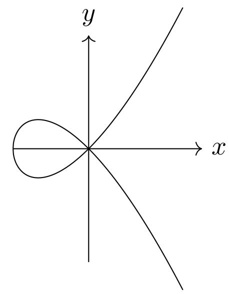
So in our ongoing picture:



given our birational map $\Gamma \dashrightarrow \mathbf{A}^n$, we have that E is the exceptional locus, and letting $V = \mathbf{A}^n$, the total transform of V is Γ and the strict transform of V is $\overline{\Gamma \setminus E}$.

Example 6.9. Let's do one explicit example of the blow up of a variety together. Just as was the case for $\mathbf{A}^n$, you should imagine a blow up as stretching out or pulling apart your variety near the origin according to the different directions of lines through the origin. Let's try to mesh that intuition with the mathematics as we work through this example.

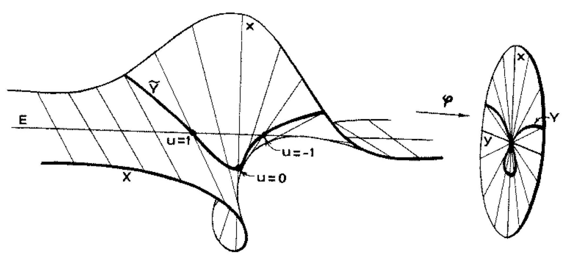
Let $X = V(y^2 - x^2(x+1))$. In $\mathbf{A}^2$, we can picture X as the curve



Let's write $[t : u]$ for the coordinates of $\mathbf{P}^1$. We've already seen that $\Gamma \subseteq \mathbf{A}^2 \times \mathbf{P}^1$, the blow up of $\mathbf{A}^2$ at $(0,0)$, is given by $xu = ty$, either through **Definition 6.3** directly, or through your translation in **Exercise 6.4**.

The total transform of X is determined by the simultaneous system $y^2 = x^2(x+1)$ and $xu = ty$ in $\mathbf{A}^2 \times \mathbf{P}^1$. We'll describe this space using two charts on $\mathbf{P}^1$: the canonical ones where $t \neq 0$ and where $u \neq 0$. In the chart $t \neq 0$, let $t = 1$ and treat u as an affine parameter. Our system becomes $y^2 = x^2(x+1)$ and $y = xu$, so we can substitute xu for y to see that $x^2u^2 - x^2(x+1) = 0$ in $\mathbf{A}^3 = \{(x,y,u)\}$. This equation factors, giving us two irreducible components. The first is defined by $x = y = 0$ and u free. This is the exceptional curve E. The second is $u^2 = x+1$ and $y = xu$. This is the blow up $\widetilde{X}$. One way to conceptualize $\widetilde{X}$ is that we've stretched out $\mathbf{A}^2$ at the origin, which has the effect of parametrizing the curve so that it "sort of" no longer self intersects (if by "sort of" we mean on the strict transform), at the cost of introducing that exceptional curve E. You can see this a bit more precisely by realizing that $\widetilde{X} \cap E = \{u = 1, u = -1\}$, which correspond to the slopes of the two branches of X at the origin.

Here's a picture from Hartshorne to help you visualize what's going on here.



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Exercise 6.10 (★★☆☆☆). What happens on the chart $u \neq 0$? Can you draw a picture?

Exercise 6.11 (★★★☆☆). Ready to do a blow up yourself? Let Y be the cuspidal cubic defined by $V(y^2 - x^3) \subseteq \mathbf{A}^2$. Blow up $(0,0)$. Let E be the exceptional curve and let $\widetilde{Y}$ be the strict transform of Y. Show that E meets $\widetilde{Y}$ in one point and that $\widetilde{Y} \cong \mathbf{A}^1$. Draw plenty of pictures!

Exercise 6.12 (★★★☆☆). Here's another blow up. Let Y be the cone $V(x^2 + y^2 - z^2) \subseteq \mathbf{A}^3$. Blow up $(0,0,0)$. Describe it. Draw pictures!

Exercise 6.13 (★★★☆☆). What about blowing up $(0,0)$ in $V(x^3 + y^3 - xy)$?

Exercise 6.14 (★★★★☆). Can you generalize the blow up of a variety in $\mathbf{A}^2$ at the origin? Let's say that the variety is defined as the vanishing of a single irreducible polynomial $f \in \mathbf{C}[x,y]$. What can you say about the blow up in relation to f? The exceptional divisor? The strict transform? Okay, now what about $V(f_1, \ldots, f_s)$?

Now we finally (modulo the fact that I *will not* prove this for you) reach one of the name-dropped reasons to care that I gave in the introduction.

Theorem 6.15 (Hironaka 1964)**.** *Let X be a variety over $\mathbf{C}$, possibly singular at some points. There exists a nonsingular variety Y and a proper, birational map $Y \dashrightarrow X$. This is called a resolution of the singularities of X. You can construct Y explicitly by repeatedly blowing up along nonsingular subvarieties.*

Remark 6.16. The proof of this is hard[2], which is again why I'm not proving this for you. But look at how incredible this theorem is! You hand me any variety you like, with the grossest defining equations or the nastiest singularities you can muster, and no matter what, we can just blow up a finite number of times to get a birationally equivalent nonsingular variety. Wow! Also, for what it's worth, the problem is still open[3] in characteristic p; the proof requires the fact that $\operatorname{char} \mathbf{C} = 0$.

I also didn't define proper here; it's a bit technical but we may see it eventually. If you're just looking for a reason why that word is there, it's essentially so that the resolution of singularities isn't something dumb like just taking Y to be the subvariety of nonsingular points. If you've seen proper maps in topology, it's got the same vibes in algebraic geometry, but the definition is more technical (as are all topological notions) since our spaces aren't Hausdorff.

7 Examples and Motivations for Scheme Theory

This section focuses heavily on fundamental examples that help one understand how to think about varieties in the neo-classical way. Some examples also serve to explain the limitations of using varieties to solve problems from other areas of math. This will foreshadow another enlargement of the category of varieties into the category of schemes over a base field.

Varieties, as we have currently defined them, have three "blind spots". These blind spots are as follows:

1. The ability to identify collections of varieties as fibers of a morphism
2. The ability to calculate tangent lines from a purely algebraic setting
3. Solving number theory problems

To understand the first blind spot, let $X = Z(zy = x^2)$ in $\mathbb{A}^3$ over any field. Project this space to the y-axis in $\mathbb{A}^3$. The fibers of this morphism are defined to be the pullbacks of singletons. All of which are indeed varieties (copies of $Z(ay = x^2)$, in fact) except for the fiber over the origin. As a set, this fiber is the y-axis but it's given by the vanishing of z and x^2 so it isn't a variety. Lowering the restrictions on what it takes to be a variety would allow us to consider this non-reduced object as a deformation of the morphism's fibers. The second blind spot will be explored in the exercises at the end of this chapter.

This last blind-spot is primarily motivated by the Weil conjectures which were proved by Grothendieck and his student Deligne (for which the latter was awarded the Fields Medal). These conjectures were highly related to the Riemann-zeta hypothesis; one can think of them colloquially as "the Riemann hypothesis in characteristic p." The most famous accomplishment of AG related to number theory is the proof of Fermat's last theorem. The theorem states there are no nontrivial integer solutions to $x^n + y^n = z^n$ when $n \geq 3$. The proof was first given by contradiction by Andrew Wiles in 1994. First, Wiles proved a special case of the Modularity Conjecture which roughly states that smooth curves called elliptic curves are in one to one correspondence with functions on the upper half complex plane called modular forms. The special case concerned semistable elliptic curves. If any of the equations of Fermat's last theorem had a nontrivial solution, one could construct a semistable elliptic curve over a finite field of characteristic p with integer solutions

[2] at least the one Hironaka gave; wikipedia says there's modern proofs which are 1/10 the size of Hironaka's and approachable in an introductory graduate course. I guess I trust that but I didn't personally check and regardless, it's still a digression we don't need at this time

[3] we'll discuss in person

modulo p. This curve should correspond to a modular form but the integer solutions make this impossible so one reaches a contradiction. The construction of the needed elliptic curve requires more technology than locally ringed spaces. Hence varieties aren't general enough to handle problems of this caliber.

Recall that a variety is a pair $(X, \mathcal{O}_X)$ consisting of a locally ringed space X and its structure sheaf of rings $\mathcal{O}_X$ which is isomorphic (as a locally ringed space) to a pair $(Y, \mathcal{O}_Y)$ where Y is a QAAV or a QPAV and $\mathcal{O}_Y$ is its sheaf of rings of regular functions.

Let X denote the origin in $\mathbb{A}^3$. Consider the vanishing set of $I = (x, y, z)^2$ which is X. However, I is not prime. We call $Z(I)$ a non-reduced point. In general, when a closed subset Y is given by an ideal which is not a prime ideal, we say Y is non-reduced. Otherwise, we say Y is reduced. Any non-reduced algebraic set Y has a canonical reduced algebraic set associated to it. In our current setting, if Y is irreducible then $Y = Z(I^n)$ for some prime ideal I and some $n \geq 2$. Thus $Y = Z(I)$ as sets. If Y is not irreducible then Y is the union of two or more distinct varieties $Y_i = Z(I_i)$ which are also closed in Y. Hence there are n_i for which

$$S/I(Y) \cong S/(I_1^{n_1}...I_m^{n_m})$$

It's sometimes helpful to mentally picture non-reduced varieties as being "thick" but still occupying the same exact amount of space that one would expect the corresponding reduced variety to occupy. Note that the global sections of an affine (resp. projective) algebraic set Y are an integral domain iff Y is an affine (resp. projective) variety iff Y is an irreducible space which is also reduced.

There is more subtle information we can obtain by studying the geometry of non-radical ideals.

Example 7.1. Let I be an ideal of $S = k[x_1, ..., x_n]$. Then $Z(I)$ determines a closed subset of $\mathbb{A}_k^n$. We know that $Z(I)$ must be a union of AAV's of the form $Z(P)$ where P is a prime ideal of S. We wish to calculate a maximal list of distinct prime ideals P_i for which

$$Z(I) = Z(P_1) \cup ... \cup Z(P_m)$$

If I is a radical ideal, the Nullstellensatz guarantees that this occurs precisely when

$$I = P_1 \cap ... \cap P_m$$

where the P_i all contain I but are minimal in the following sense: if $I \subseteq Q \subseteq P_i$ for a prime ideal Q then $Q = P_i$. Provided all of the P_i are pairwise distinct primes, this list of prime ideals is unique. Let $I = (x^2, y)$ in $S = k[x, y]$. This ideal is not a radical ideal since its vanishing set is the origin which is already married to the radical ideal (x, y). However, this ignores some important geometric data. Let P denote the origin. As sets, $Z(I)$ is equal to P. As locally ringed spaces, we can distinguish $Z(I)$ from P by stating that $Z(I)$ has $\{P\}$ as its underlying set but comes equipped with a horizontal tangent line. The tangent line represents the surviving first derivative of x^2 in the horizontal direction. To elaborate: let $f = 3 + 14x - 6y + 5x^2 + 2xy + x^3$. Say I write down f on a piece of paper that I keep hidden from you. Then, I write down f modulo I

$$f \equiv 3 + 14x$$

on a new piece of paper which I then show you. What information about f can you deduce from only the second paper? You know the constant term is $f(0, 0)$ which isn't killed off by I so you deduce that the constant term is 3. The term $14x$ also survived giving

$$14 = \partial f/\partial x(0, 0)$$

However, any data concerning the partial with respect to y has been lost. Therefore, I killed off everything except position and velocity in the horizontal direction. Thus we may think of $Z(I)$ as the origin along with a horizontal tangent line. Given a polynomial f, $Z(I)$ scans f and returns its constant term (position) and the x coefficient (velocity in the horizontal direction). This object is by no means a variety but it still conveys fundamental geometric properties.

Example 7.2. Let $I = (x^2, xy, y^2)$. Again, I is contained in (x, y) and $Z(I)$ is just the origin. We see constant terms have survived along with the first x-partial. Yet y is not included in I so the first y-partial also survived. Hence $Z(I)$ may be thought of as the origin along with all possible tangent lines through the origin. Given a polynomial f, $Z(I)$ scans f and returns its constant term and a vector in the plane with coordinates given by the coefficients of x and y.

Example 7.3. Let $I = (x) \cap (x^2, xy, y^2)$. Looking at the vanishing sets of these ideals, we see $Z(I)$ is the y-axis in the plane together with an embedded component at the origin. This embedded component comes equipped with the ability to read off all first order derivatives evaluated at the origin.

By now, the reader should be getting the feeling that the Nullstellensatz doesn't give us the whole picture. If you re-read the fundamental theorems which follow from it, Hilbert's Nullstellensatz is a statement about set inclusion. We need fundamental theorems which are deep enough to pick up the subtle geometric information which survives quotients by non-radical ideals as demonstrated in the previous three examples.

Exercise 7.4 (★☆☆☆☆)**.** Give an ideal I of $S = \mathbb{C}[x_1, ..., x_n]$ such that $Z(I)$ has one embedded component of dimension r for each $1 \leq r \leq n-1$.

Exercise 7.5 (★★☆☆☆)**.** Let $I = (x) \cap (x^2, xy, y^2)$. Describe the geometric object in the plane associated to this ideal.

8 Affine Schemes

Now that we have some motivation for an algebro-geometric object that is an enhancement of varieties, we should say what it is, and check that it fixes the issues from the previous section. We'll dive right in to the definition but it will be given in two parts with some examples in between.

Definition 8.1 (Part 1)**.** Let R be a ring. An **affine scheme** $\operatorname{Spec} R$, also called the **(prime) spectrum of** R, is a certain locally ringed space. Recalling **Definition 3.20**, this means an affine scheme is a topological space with a sheaf of rings whose stalks at every point in the space are local rings.

The topological space is defined to be $\operatorname{Spec} R \coloneqq \{\mathfrak{p} \mid \mathfrak{p} \subseteq R \text{ is a prime ideal}\}$, with the Zariski topology on this set, where closed sets are of the form $V(\mathfrak{a}) \coloneqq \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subseteq \mathfrak{p}\}$.

We will relegate the definition of the structure sheaf of an affine scheme to part 2 of this definition. In the mean time, we will elucidate the topology of affine schemes.

Remark 8.2. For a variety, a set of polynomials generates an ideal and a Zariski closed set consists of the points in n-space where those polynomials (and thus the ideal) vanish. Although we've lost the familiarity of polynomials as a result of the abstraction to affine schemes, it is still possible to talk about the vanishing of ideals. On a variety X over k, for an ideal $\mathfrak{a}$ of the ring $A[X]$ to vanish at a point P in X means that if $f \in \mathfrak{a}$ then $f(p) = 0$. However, the zero element of the equation $f(P) = 0$ comes from the field k. A priori, affine schemes do not have a "one size fits all" base field (some do, and they are called k-schemes but it's an additional assumption). Grothendieck was the first to consider giving each point of $\operatorname{Spec} R$ its own personal base field, namely $k(\mathfrak{p}) = R_{\mathfrak{p}}/(\mathfrak{p})$ which is sometimes called the residue field at $\mathfrak{p}$. With this idea in hand,

for $\mathfrak{a}$ to vanish at the point $\mathfrak{p}$ means that if $f \in \mathfrak{a}$ then the image of f in $k(\mathfrak{p})$ is zero

Note that this occurs iff the image of $\mathfrak{a}$ in $k(\mathfrak{p})$ is the zero ideal iff the image of $\mathfrak{a}$ in $R_{\mathfrak{p}}$ is contained in the maximal ideal of this local ring iff $\mathfrak{a} \subseteq \mathfrak{p}$. In this sense and as explored in **Exercise 8.3**, the space is quite familiar.

Exercise 8.3 (★★★☆☆)**.** Here's the usual typechecking to do. Note the similarities to when we defined varieties. Let R be a ring.

1. If $\mathfrak{a}, \mathfrak{b} \subseteq R$ are two ideals, show that if $\mathfrak{a} \subseteq \mathfrak{b}$, then $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$.
2. Show that the Zariski topology defined by declaring $V(\mathfrak{a})$ closed is indeed a topology on $\operatorname{Spec} R$.
3. Show that if $E \subseteq R$ is a subset and $\mathfrak{a}$ is the ideal generated by E, then $V(\mathfrak{a})$ is equal to $V(E) \coloneqq \{\mathfrak{p} \in \operatorname{Spec} R \mid E \subseteq \mathfrak{p}\}$. This justifies the fact that we will often write, e.g., $V(f)$ for $V((f))$.
4. Show that $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

Exercise 8.4 (★☆☆☆☆)**.** Let $X = \operatorname{Spec} R$ be an affine scheme with the Zariski topology. Define the set $D(f) \coloneqq \{\mathfrak{p} \in \operatorname{Spec} R \mid f \in R, f \notin \mathfrak{p}\}$ called the distinguished open set at f.

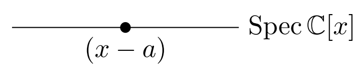
1. Show that X can be covered by a collection of distinguished open sets.
2. Show that $D(f) \cap D(g) = D(fg)$.
3. Show that the collection $\mathcal{B} \coloneqq \{D(f) \mid f \in R\}$ forms a topological basis for X with the Zariski topology. We call the collection $\mathcal{B}$ the standard basis of distinguished opens.

Note: you may also see the notation D_f or U_f for distinguished opens in other texts.

Exercise 8.5 (★★☆☆☆)**.** Show that the following are equivalent.
1. $D(f) \subseteq D(g)$.
2. $f^n \in (g)$ for some $n \geq 1$.
3. g is a unit in the ring R_f.

Example 8.6. We have not finished defining affine schemes, but let's see a few examples. Let k be a field; then $\operatorname{Spec} k$ is a point, since a field has exactly one prime ideal, (0).

Example 8.7. What is $\operatorname{Spec} \mathbb{C}[x]$? As a set, it's defined to be $\{\mathfrak{p} \subseteq \mathbb{C}[x] \mid \mathfrak{p} \text{ a prime ideal}\}$ and one can check that this set is $\{(x-a) \mid a \in \mathbb{C}\} \cup \{(0)\}$. As a set, since the maximal ideals are parameterized by a, $\operatorname{Spec} \mathbb{C}[x]$ is isomorphic to $\mathbb{C} \cup \{(0)\}$. The standard basis is $\mathcal{B} = \{D(f) \mid f \in \mathbb{C}[x]\}$, where each $D(f) = \{(x-a) \mid f \notin (x-a)\}$. Polynomials not in the ideal $(x-a)$ are polynomials which do not vanish at $a \in \mathbb{C}$. So the standard basis on $\operatorname{Spec} \mathbb{C}[x] \setminus \{(0)\}$ consists of the complements of singletons; i.e. the topology on the subspace of maximal ideals is the finite complement topology (aka the cofinite topology) on $\mathbb{C}$. We draw this part of $\operatorname{Spec} \mathbb{C}[x]$ as an affine line.

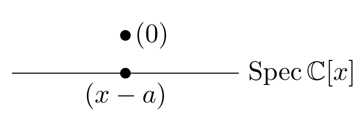


But what about the zero ideal? Something wild happens in general:

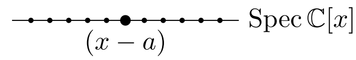
Remark 8.8. Let R be a ring where (0) is a prime ideal (this occurs if and only if R is an integral domain). Notice that $\overline{\{(0)\}}$ in the Zariski topology is all of $\operatorname{Spec} R$. To see this, the closure of $\{(0)\}$ is the smallest closed set containing $\{(0)\}$. But $V(\mathfrak{a})$ does not contain (0) for any $\mathfrak{a} \neq (0)$; indeed, $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subseteq \mathfrak{p}\}$ but for all nonzero $\mathfrak{a}$, we have $(0) \subseteq \mathfrak{a}$, not the other way around. So $\overline{\{(0)\}} = V(0) = \operatorname{Spec} R$. We call $\{(0)\}$ a generic point of the affine scheme. Note that this singleton set is not closed. Furthermore, it's dense in $\operatorname{Spec} R$. In contrast, the singleton sets consisting of maximal ideals are closed and so we call any one such element a closed point of $\operatorname{Spec} R$. Affine schemes are most often drawn using only their closed points. Then, the author makes some esoteric remark about how the generic points are everywhere and nowhere.

Exercise 8.9 (★★☆☆☆)**.** Prove the following: a point $\mathfrak{p} \in \operatorname{Spec} R$ is closed if and only if it is a maximal ideal.

Example 8.10. Returning to **Example 8.7**, the completed picture not only has $\mathbb{C}$ with the finite complement topology (just like the variety $\mathbf{A}^1$!), but it also has a generic point $\{(0)\}$ which is dense in $\operatorname{Spec} \mathbb{C}[x]$. Of course, it's pretty crazy to try and draw a single point which is dense in an uncountable set so different authors draw different pictures. Some draw $\{(0)\}$ floating above the scheme to represent its everywhere-ness:



Others draw $\{(0)\}$ as a sort of cloud or buckshot inside the rest of $\operatorname{Spec} R$, like how you might picture $\mathbf{Q}^n$ being dense in $\mathbf{R}^n$ (but remember that it is still just a single point).



Others still don't draw $\{(0)\}$ at all but just remember its existence. We will mostly stick to this third option after we've seen more examples of affine schemes. Although, you definitely should still remember its existence, because it's not a given; not every affine scheme $\operatorname{Spec} R$ contains the point (0) – only those where R is an integral domain do.

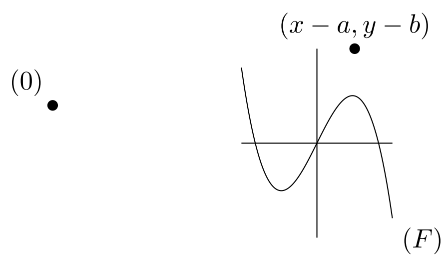
This also gels with the fact that we call $\operatorname{Spec} \mathbb{C}[x]$ the affine line and write $\mathbf{A}^1$, even though this isn't equal to the variety $\mathbf{A}^1$ as we defined varieties; the sets are different: $\mathbb{C}$ versus $\mathbb{C} \cup \{(0)\}$.

Example 8.11. What is $\operatorname{Spec} \mathbb{C}[x, y]$? We will write $\mathbf{A}^2$ to refer to this affine scheme even though the affine plane is not homeomorphic to the spectrum of its coordinate ring. That said, these spaces only differ by the inclusion of generic points.

From the definition and with a bit of algebra knowledge,

$$\begin{aligned}
\operatorname{Spec} \mathbb{C}[x, y] &\coloneqq \{\mathfrak{p} \subseteq \mathbb{C}[x, y] \text{ prime}\} \\
&= \{(x-a, y-b) \text{ the maximal ideals} \mid a, b \in \mathbb{C}\} \\
&\quad \cup \\
&\quad \{(F) \text{ the principal ideals generated by irreducible polynomials } F\} \\
&\quad \cup \\
&\quad \{(0)\}.
\end{aligned}$$

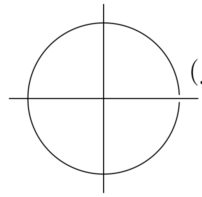
So as a space, all three of these types of prime ideals form points in our space. Again, we like to lie a bit with the picture and still draw $\mathbf{A}^2$ as the affine plane but we recognize the fact that in addition to the points (a, b) corresponding to $(x - a, y - b)$, the graph $F = 0$ also produces a *point* in $\mathbf{A}^2$ corresponding to (F). And we also have (0) as well. We might draw the following, where we've labeled three particular points in $\operatorname{Spec} \mathbb{C}[x, y]$:



Notice that (F) is a point in $X = \operatorname{Spec} \mathbb{C}[x, y]$, and also the picture tells us that it is a point which itself contains other points (a, b) such that $F(a, b) = 0$, corresponding to the algebraic fact that (F) is contained in some maximal ideals $(x - a, y - b)$. Furthermore, $\{(F)\}$ is dense in $Y = V((F))$ and thus generic over Y. Later, we will leverage this property to create the function field $k(Y) \subseteq \mathbb{C}(x, y)$ over Y. Once we have a structure sheaf defined for Y (and X) we will see that the stalks of Y are precisely the local valuation rings of $\mathbb{C}[x, y]/(F)$. By dimension considerations, a stalk at P in Y is a discrete valuation ring iff it is a regular local ring. Since DVR's come with a valuation $\nu : k(Y) \to \mathbb{Z}$ we will soon have a rigorous way of counting holes and poles of rational functions along F using purely algebraic data. No complex analysis required.

Here's an example of a basis open set in $\operatorname{Spec} \mathbb{C}[x, y]$. The polynomial $f(x) = x^2 + y^2 - 1$ is irreducible, hence a prime. In this case

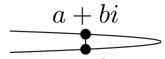
$$\begin{aligned}
D(f) &= \{\mathfrak{p} \mid f \notin \mathfrak{p}\} \\
&= \operatorname{Spec} R \setminus \{\mathfrak{p} \mid f \in \mathfrak{p}\} \\
&= \operatorname{Spec} R \setminus (f).
\end{aligned}$$



Open sets are large; they are essentially complements of zero sets. The non-Hausdorff Zariski topology strikes again! We should also mention that this example generalizes; we write $\mathbf{A}_k^d$ for the affine scheme $\operatorname{Spec} k[x_1, \ldots, x_d]$, and an omitted k typically means $k = \mathbb{C}$.

Example 8.12. What is $\operatorname{Spec} \mathbf{R}[x]$? In terms of closed points, we have maximal ideals $(x-a)$ for all $a \in \mathbf{R}$. But we also have irreducible polynomials in $\mathbf{R}[x]$ which are nonlinear. The ideal (x^2+px+q) is generated by an irreducible polynomial of degree 2 when our discriminant $p^2 - 4q$ is negative. Over $\mathbb{C}$, such polynomials would give rise to complex conjugate zeros, and vanish there.

In $\mathbf{R}[x]$, no higher degree polynomials are irreducible, since $\mathbf{R}(i)$ is algebraically closed and a degree 2 extension of $\mathbf{R}$. Thus, the affine scheme $\operatorname{Spec} \mathbf{R}[x]$ can be thought of as taking the affine line $\operatorname{Spec} \mathbb{C}[x]$ and identifying complex conjugates $a + bi \sim a - bi$. If $b = 0$, then the action is trivial, but otherwise, we have glued. You therefore might visualize $\operatorname{Spec} \mathbf{R}[x]$ as a folded and glued $\operatorname{Spec} \mathbb{C}[x]$.



Exercise 8.13 (★★☆☆☆). What is $\operatorname{Spec} \mathbf{Z}$? What are its closed and generic points? Draw a picture.

Exercise 8.14 (★★★★☆). What is $\operatorname{Spec} \mathbf{Z}[x]$? What are its closed and generic points? Draw a picture. Hint: try this yourself first, but if you get stuck, Mumford's *The Red Book of Varieties and Schemes* has a good picture, and you can see it and an explanation at this link:

`http://www.neverendingbooks.org/mumfords-treasure-map`

(The "to be continued" link is fascinating too.) Even if you don't fully get a picture by yourself first, this would still be a really fun exercise to explain to the group.

Exercise 8.15 (★★★★☆). Pick your favorite ring R. What is $\operatorname{Spec} R$? What are its closed and generic points? Draw the best picture you can.

Example 8.16. Let's compare $X = \operatorname{Spec} \mathbb{C}[x]/(x)$ and $Y = \operatorname{Spec} \mathbb{C}[x]/(x^2)$. We will see soon that the construction $\operatorname{Spec} -$ is functorial, so as sets $X = \operatorname{Spec} \mathbb{C}[x]/(x) \cong \operatorname{Spec} \mathbb{C} = \{(0)\}$. And the ring $\mathbb{C}[x]/(x^2)$ has a single prime ideal (x), so Y too is a one-point space.

But this isn't good! From **Section 7** both X and Y will embed as the origin in $\mathbf{A}^1$ but the scheme Y should be carrying first order differentiation information somehow. This leads us to finish the definition of an affine scheme.

Definition 8.17 (Part 2)**.** Let R be a ring and let $X = \operatorname{Spec} R$ be the topological space described in **Definition 8.1**. An affine scheme X comes equipped with its **structure sheaf** $\mathcal{O}_X$, making $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ into a locally ringed space. The structure sheaf is the unique sheaf defined on the standard basis to be $\mathcal{O}_X(D(f)) = R_f$. Given a point $x_0 \in X$, the stalk at x_0 is the local ring

$$\mathcal{O}_{X,x_0} \coloneqq \varinjlim_{x_0 \in U} \mathcal{O}_X(U).$$

Quick note: We will write $\operatorname{Spec} R$ when we are referring to the locally ringed space $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$. We will use $|\operatorname{Spec} R|$ to explicitly reference the underlying topological space. This is necessary to do because of commutative algebra terms like "noetherian" and "dimension" which are ambiguous on an affine scheme (at least for now).

Exercise 8.18 (★★★☆☆). Let $X = \operatorname{Spec} R$. Let $x_0 \in X$ and let $\mathfrak{p}$ be the corresponding prime ideal. Prove that $\mathcal{O}_{X,x_0} \cong R_{\mathfrak{p}}$ via the following steps:

1. Fix $x \notin \mathfrak{p}$. Show that $\frac{1}{x}$ is naturally an element of $\mathcal{O}_X(D(x))$.
2. Show that R_x is the direct limit of rings $\mathcal{O}_X(U)$ for the collection of U satisfying $D(x) \subseteq U$
3. Prove
$$\bigcap_{x \notin \mathfrak{p}} D(x) = \{x_0\}$$
4. Prove that $R_{\mathfrak{p}}$ is the direct limit of rings $\mathcal{O}_X(D(x))$ for the collection of $D(x)$ satisfying $x_0 \in D(x)$.
5. Use the fact that distinguished open sets form a basis of X to show that the collection in part 4 has a direct limit which is isomorphic to $\mathcal{O}_{X,x_0}$.

Example 8.19. Let's finish **Example 8.16**. While as spaces, $X \cong \{(0)\}$ and $Y = \{(x)\}$ are homeomorphic, what are their structure sheaves? Well, since they must be sheaves on a one-point space, that means that they are constant sheaves, and we can determine them by evaluating on that one point:

$$\begin{aligned}\mathcal{O}_X(X) = \mathcal{O}_X(D(1)) &= \left(\mathbb{C}[x]/(x)\right)_1 = \mathbb{C}[x]/(x), \text{ while}\\ \mathcal{O}_Y(Y) = \mathcal{O}_Y(D(1)) &= \left(\mathbb{C}[x]/(x^2)\right)_1 = \mathbb{C}[x]/(x^2).\end{aligned}$$

The structure sheaves are different!

Remark 8.20. We've already mentioned in **Section 7** and in **Example 8.16** that a scheme should be trying to tell me more geometric information that just vanishing sets - in particular, we want tangent information. By **Example 8.19**, the structure sheaves need to be the object that carries that information, so let's confirm for ourselves this to be the case.

We still need to show that $\operatorname{Spec} -$ is functorial, and when we do so, we will also show that it is contravariant. That is, the embedding of schemes $X \hookrightarrow \mathbf{A}^1$ and $Y \hookrightarrow \mathbf{A}^1$ corresponds to ring homomorphisms $\mathbb{C}[t] \to \mathbb{C}[x]/(x)$ and $\mathbb{C}[t] \to \mathbb{C}[x]/(x^2)$. The inclusion $X \hookrightarrow \mathbf{A}^1$ is the honest inclusion of the origin, but we will see that Y carries tangent information.

The embedding $Y \hookrightarrow \mathbf{A}^1$ corresponds to the ring homomorphism $\mathbb{C}[t] \to \mathbb{C}[x]/(x^2)$ defined by $t \mapsto x$. Let's take two points in $\mathbf{A}^1$, $\mathfrak{p}$ and $\mathfrak{q}$, and think about what happens as they get closer together. We can leave $\mathfrak{q} = (t)$ fixed, and let $\mathfrak{p} = (t-a)$ where $a \in \mathbb{C}$ is a parameter we will vary. The ideal of the union is $(t(t-a))$, and as a approaches 0, it is natural to identify the limit as (t^2), which is the ideal of $Y \subseteq \mathbf{A}^1$. Now, with this picture, it's natural to think of Y as a point with an abstract tangent vector, and as seen before, this lines up with tangent vectors algebraically too; if we want to differentiate, we expand in powers of x and ignore all terms of degree 2 and higher.

Example 8.21. What is $\operatorname{Spec} \mathbb{C}[x_1, \ldots, x_d]/(f)$ for a polynomial f? The prime ideals of $\mathbb{C}[x_1, \ldots, x_d]/(f)$ are precisely the prime ideals of $\mathbb{C}[x_1, \ldots, x_d]$ which vanish on f. In fact:

Exercise 8.22 (★★☆☆☆). Let R be a ring and $\mathfrak{a} \subseteq R$ any ideal. Show that an ideal $\mathfrak{p}$ is prime in R if and only if $\mathfrak{p}/\mathfrak{a}$ is prime in $R/\mathfrak{a}$.

Returning to the example, this means that as a set,

$$\operatorname{Spec}\mathbb{C}[x_1,\ldots,x_d]/(f) = \{\mathfrak{p} \subseteq \mathbb{C}[x_1,\ldots,x_d] \mid (f) \subseteq \mathfrak{p}\}.$$

By definition, this is the closed subspace $V(f)$ in $\mathbb{A}^d$; i.e. $\operatorname{Spec}\mathbb{C}[x_1,\ldots,x_d]/(f)$ is the hypersurface $f = 0$ along with two generic points. Similarly, for any quotient by an ideal $(f_1,\ldots,f_n) \subseteq \mathbb{C}[x_1,\ldots,x_d]$ the space is the system of equations $f_i = 0$ for $1 \leq i \leq n$ with a generic point coming from each generator along with the generic point corresponding to zero.

Since localization commutes with quotienting, for all $g \in \mathbb{C}[x_1,\ldots,x_d]$ we have the computation

$$\mathcal{O}_{\operatorname{Spec}\mathbb{C}[x_1,\ldots,x_d]/I}(D(g)) = \left(\mathbb{C}[x_1,\ldots,x_d]/I\right)_g \cong \mathbb{C}[x_1,\ldots,x_d]_g/I_g.$$

For a concrete example, the affine scheme $X = \operatorname{Spec}\mathbb{C}[x,y]/(xy)$ can be compared to $xy = 0$ in $\mathbb{A}^2_{\mathbb{C}}$. The stalk at, say, $(x, y-1)$ is the local ring

$$\left(\mathbb{C}[x,y]/(xy)\right)_{(x,y-1)} \cong \mathbb{C}[x,y]_{(x,y-1)}/(xy)_{(x,y-1)} \cong \mathbb{C}[x,y]_{(x,y-1)}/(x)_{(x,y-1)} \cong \mathbb{C}[y]_{(y-1)}$$

This corresponds to the geometric fact that the topology of this space near the point $(0,1)$ acts like that of $\operatorname{Spec}\mathbb{C}[y]$ near the point $(y-1)$. Contrast this with localization at the prime ideal (x,y):

$$\left(\mathbb{C}[x,y]/(xy)\right)_{(x,y)}$$

which has krull dimension 2. Unsurprisingly, all other stalks are isomorphic to $\mathbb{C}[x]$ localized at a maximal ideal making the origin the black sheep. This example illustrates one of the most important differences between affine schemes and varieties: one may take the spectrum of a stalk at P to obtain a new affine scheme which is naturally an open neighborhood of P. Therefore a point P in an affine scheme carries the structure of another affine scheme which is also an open set of X. In the category of varieties, this would be analogous to asking for the coordinate ring of a stalk which is unhelpful nonsense. This is because the definition of a variety was far too restrictive. Recall that all one-point varieties are isomorphic as locally ringed spaces. Meaning one can only understand the location of a point from its stalk. However, the spectrum of a local ring is guaranteed to contain exactly one closed point and potentially one or more non-closed points. The maximal ideal preserves the data of the location of the point on an affine scheme and the non-closed points carry the topological data needed to acquire an open set.

Remark 8.23. A structure sheaf should be thought of as a sheaf of functions over $\operatorname{Spec} R$ even though this isn't always literal. This is for three good reasons:

1. Your working mental model of sheaves in any setting should be functions, like holomorphic ones on a complex manifold. (At least, that's what mine is.)
2. Doing so satisfies a second blind-spot mentioned in **Section 7**: now any ring, not just integral domains, can serve as the ring of functions on some space.
3. Most of the time it's close enough, since we tend to work with polynomial rings modulo some ideal, and elements in these rings can be thought of as (equivalence classes of) functions.

Remark 8.24. Let's now begin to show that $\operatorname{Spec} -$ is a contravariant functor (in fact, we will show something more!). Its image will be the category **AffSch** of affine schemes, but while we have defined affine schemes as objects, we have not defined morphisms between them. Although, the careful among you will notice that in **Remark 8.20**, we have already given some morphisms between schemes; we embedded $X \hookrightarrow \mathbf{A}^1$ and $Y \hookrightarrow \mathbf{A}^1$. That's because we actually already have the definition. Since an affine scheme is a locally ringed space, morphisms of affine schemes are morphisms of locally ringed spaces as we defined them in **Definition 3.22**.

Exercise 8.25 (★★☆☆☆)**.** Check that $f : X \hookrightarrow \mathbf{A}^1$, $f^\sharp : \mathbb{C}[t] \to \mathbb{C}[x]/(x)$ and $g : Y \hookrightarrow \mathbf{A}^1$, $g^\sharp : \mathbb{C}[t] \to \mathbb{C}[x]/(x^2)$ from **Remark 8.20** are indeed morphisms of locally ringed spaces as defined in **Definition 3.22**.

Exercise 8.26 (★★★☆☆)**.** Let's first just show that Spec − is a functor, from the category **CRing** of commutative rings with ring homomorphisms to the category **AffSch** of affine schemes with morphisms of locally ringed spaces. Since, by definition, given a ring R, $\operatorname{Spec} R$ is an affine scheme, we just need to show that Spec − carries arrows to arrows, respects identities, and respects composition contravariantly.

1. Let $\varphi : R \to S$ be a ring homomorphism. Show that there is a continuous map of topological spaces $\operatorname{Spec}(\varphi) : \operatorname{Spec} S \to \operatorname{Spec} R$ given by preimage of prime ideals. Thus we've defined $\operatorname{Spec} - : \mathbf{CRing} \to \mathbf{Top}$.
2. Let $\mathfrak{p}$ be a prime in S. Show that φ descends to homomorphisms $\mathcal{O}_{\operatorname{Spec} R, \varphi^{-1}(\mathfrak{p})} \to \mathcal{O}_{\operatorname{Spec} S, \mathfrak{p}}$ of local rings. Conclude why we have a functor $\operatorname{Spec} - : \mathbf{CRing} \to \mathbf{AffSch}$. In particular, what is $\operatorname{Spec}(\varphi)^\sharp : \mathcal{O}_{\operatorname{Spec} R} \to \mathcal{O}_{\operatorname{Spec} S}$?
3. Let $\mathrm{id}_R : R \to R$ be the identity map for a ring R. Show that $\operatorname{Spec}(\mathrm{id}_R)$ is the identity map on $\operatorname{Spec} R$ in **Top**. Then show that $(\operatorname{Spec}(\mathrm{id}_R), \operatorname{Spec}(\mathrm{id}_R)^\sharp)$ is the identity map on $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ in **AffSch**.
4. Finish the proof by showing that Spec − respects composition. Let $R \xrightarrow{\varphi} S \xrightarrow{\psi} T$ be the composition of homomorphisms in **CRing**. Show that $\operatorname{Spec}(\psi \circ \varphi) = \operatorname{Spec}(\varphi) \circ \operatorname{Spec}(\psi)$, in **Top** and then in **AffSch**.

Remark 8.27. Now, not only is Spec − a contravariant functor from **CRing** to **AffSch**, but further, it is a contravariant equivalence of categories! We will construct an inverse functor **AffSch** → **CRing** and leave the categorical detail checking to the next exercise. The functor **AffSch** → **CRing** is defined to be

$$(X, \mathcal{O}_X) \quad \mapsto \quad \mathcal{O}_X(X)$$

and

$$(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y) \quad \mapsto \quad f^\sharp(Y) : \mathcal{O}_Y(Y) \to f_*\mathcal{O}_X(Y).$$

We also write $\Gamma(X, \mathcal{O}_X)$ for $\mathcal{O}_X(X)$. The functor Γ is called the global sections functor.

Exercise 8.28 (★★★☆☆)**.** Prove that Spec − and global sections $\Gamma(-, \mathcal{O}_-)$ form an equivalence of categories between **CRing** and **AffSch**. In particular:

1. Let R be a ring. Compute $\Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$.
2. Let $(X, \mathcal{O}_X)$ be an affine scheme. Compute $\operatorname{Spec}(\Gamma(X, \mathcal{O}_X))$.
3. What are the natural isomorphisms defining the equivalence?
4. Show further that this is an adjoint equivalence. That is, let R be a ring. Let $(X, \mathcal{O}_X)$ be an affine scheme. Show that there is a bijection of sets

$$\operatorname{Hom}_{\mathbf{AffSch}}(X, \operatorname{Spec} R) \cong \operatorname{Hom}_{\mathbf{CRing}}(R, \Gamma(X, \mathcal{O}_X)).$$

(Number 4 can be stated slightly more generally in terms of *schemes*, which we have yet to see. In the affine case here, it's a lot easier to prove.)

Remark 8.29. Note: other resources may instead show that $\operatorname{Spec} - : \mathbf{CRing} \to \mathbf{LRS}$, the category of locally ringed spaces, is a fully faithful functor. Then the category **AffSch** is defined to be the essential image of the functor Spec −, which thus by definition is essentially surjective on **AffSch**. This is an alternative characterization of an equivalence of categories, so in this sense it is not wrong to say that $\mathbf{AffSch} \coloneqq \mathbf{CRing}^{op}$ via the functor Spec −.

9 Schemes

In the previous section, we have been carrying around the adjective "affine" every time we make reference to a scheme. If an *affine* scheme is $\operatorname{Spec} R$, what is a scheme, in general? The quick answer is that it's built by gluing together affine schemes. The long answer is the following definition.

Definition 9.1. A **scheme** is a locally ringed space $(X, \mathcal{O}_X)$ which admits a covering by open sets U_i such that each $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to an affine scheme $(\operatorname{Spec} A_i, \mathcal{O}_{A_i})$.

In other words, a general scheme is locally affine; i.e., for each point $x \in X$, there is an open subset $U \subseteq X$ containing x such that $(U, \mathcal{O}_X|_U)$ is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.

Remark 9.2. This construction is not unfamiliar to you; in the same way that manifolds are built out of an atlas of $\mathbf{R}^n$s or $\mathbf{C}^n$s, a scheme is built out of an atlas of $\operatorname{Spec} A_i$s. Unlike a manifold, there is no restriction on the A_is other than that the affine schemes glue; for instance, the dimension can vary.

Example 9.3. An easy example of a nonaffine scheme is $\mathbf{A}^2_k$ minus the origin, since it defers the technicalities of gluing, since the charts are already glued. Write $X = \mathbf{A}^2_k \setminus \{(0,0)\}$.

We know that the affine plane is $\mathbf{A}^2_k = \operatorname{Spec} k[x,y]$, and inside of this scheme are distinguished open subsets $D(x)$ and $D(y)$ whose union is X. The sets $D(x)$ and $D(y)$ are in fact affine, since

$$\mathcal{O}_{\mathbf{A}^2_k}(D(x)) = k[x,y]_x \cong k[x,y,1/x],$$

so $D(x) \cong \operatorname{Spec} k[x,y,1/x]$, and similarly $D(y) \cong \operatorname{Spec} k[x,y,1/y]$.

A regular function f on X therefore restricts on $D(x)$ to some rational function $g(x,y)/x^n$ and restricts on $D(y)$ to another rational function $h(x,y)/y^m$. On $D(xy) = D(x) \cap D(y)$, one has the equality

$$\frac{g(x,y)}{x^n} = \frac{h(x,y)}{y^m}.$$

But since g and h are polynomials in x and y, that forces $n = m = 0$. Thus we can conclude that the sections of X are $\mathcal{O}_{\mathbf{A}^2_k}(X) \cong k[x,y]$, the polynomials. In other words, we get no extra sections by removing the origin from $\mathbf{A}^2_k$. For the complex analysts in the room: we just proved an algebro-geometric version of Hartogs' Lemma: every section on X extends to a section on $\mathbf{A}^2_k$.

But how does this show that X is not affine? If X were affine, i.e., if $(X, \mathcal{O}_{\mathbf{A}^2_k}|_X)$ were isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$, then we would know what R should be by taking global sections, which we already saw to be

$$\mathcal{O}_{\mathbf{A}^2_k}(X) \cong k[x,y].$$

So $R = k[x,y]$; i.e., X is supposed to be $\operatorname{Spec} k[x,y] \cong \mathbf{A}^2_k$. But the prime ideal $(x,y) \subseteq k[x,y]$ should be a point in $\operatorname{Spec} R$, and yet on X, $V(x) \cap V(y) = \varnothing$, so the prime ideal (x,y) has no corresponding point in X. We've reached a contradiction and therefore X is not affine!

Remark 9.4. Now we would like to see an example where we learn how to take affine schemes and glue them together, rather than start with the scheme and figure out the affine chart. So let's discuss how to glue affine schemes. The easiest way to unpack this is to first forget about the structure sheaf and just discuss what it means to glue two topological spaces.

Given two topological spaces X and Y with open subsets $U \subseteq X$ and $V \subseteq Y$ with a homeomorphism $U \cong V$, we can create a new space where we glue X to Y along $U \cong V$. As a space, it is defined to be

$$X \sqcup Y \big/ U \sim V.$$

X and Y can be naturally identified with open subsets of the quotient, and together cover it. With pictures, this is exactly gluing as you imagine it.

The structure sheaf complicates things, but it's not too bad until you get into the weeds. Suppose we have two affine schemes $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. To glue, we need a homeomorphism of open subsets $U \subseteq X$ and $V \subseteq Y$, call it $\varphi : U \to V$, and we also need an isomorphism of structure sheaves $\mathcal{O}_V \to \varphi_* \mathcal{O}_U$. Together this produces an isomorphism of schemes $(U, \mathcal{O}_X|_U) \cong (V, \mathcal{O}_Y|_V)$. The following *quite* technical exercise tells you how to glue along the isomorphism of schemes.

Exercise 9.5 (★★★★☆)**.** Suppose we are given a collection of schemes X_i, open subschemes $X_{ij} \subseteq X_i$ (with the notational convention that $X_{ii} = X_i$), and isomorphisms $f_{ij} : X_{ij} \to X_{ji}$ (with the notational convention $f_{ii} = \mathrm{id}_{X_i}$). Suppose further the collection of isomorphisms satisfies the following condition, which essentially says they agree on triple intersections:

$$f_{ik}|_{X_{ij} \cap X_{ik}} = f_{jk}|_{X_{ji} \cap X_{jk}} \circ f_{ij}|_{X_{ij} \cap X_{ik}}$$

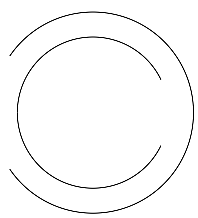
if $f_{ij}(X_{ik} \cap X_{ij}) \subseteq X_{jk}$. Show that there exists a unique scheme X along with open subsets isomorphic to the X_i which respects this gluing data. Hint: what is X as a set? What is the topology on X? In terms of your description of the open sets of X, what are the sections of this sheaf over each open set?

Example 9.6. The previous exercise is quite a lot to do, so let's make sure we learn gluing via example. Let $X = \operatorname{Spec} k[x]$ and let $Y = \operatorname{Spec} k[y]$. Additionally let $U = D(x) = \operatorname{Spec} k[x, 1/x] \subseteq X$ and let $V = D(y) = \operatorname{Spec} k[y, 1/y] \subseteq Y$. Hence X and Y are copies of $\mathbf{A}^1_k$ and U and V are the affine line minus its origin. We will glue X to Y along an isomorphism $U \cong V$, in two different ways.

In the first way, let the isomorphism $U \cong V$ be given by $k[x, 1/x] \cong k[y, 1/y]$ via $x \mapsto y$. The resulting scheme is the affine line with two origins.

Exercise 9.7 (★★★☆☆)**.** Show that the affine line with two origins in **Example 9.6** is not affine, by calculating its ring of global sections and performing a similar argument to **Example 9.3**.

Example 9.8. Let X, Y, U, and V be as in **Example 9.6**. In contrast to that example, we could define an isomorphism $U \cong V$ by $k[x, 1/x] \cong k[y, 1/y]$ via the map $x \mapsto 1/y$. The resulting picture is below.



This example is really important – it is the projective line over k, denoted $\mathbf{P}^1_k$. When k is algebraically closed, here is a description. On the first affine line, we have closed points $(x - a)$ for $a \in k$, and the generic point $\{(0)\}$. On the second affine line, we have closed points $(y - b)$ for $b \in k$ and another generic point. To glue, we connect $(x - a)$ to $(y - 1/a)$ for nonzero $a \in k$, and we glue the two generic points together.

To interpret this construction like the projective line as you've already seen it, the closed points are written in the form $[a : b]$ where not both a and b are zero, and we identify $[a : b]$ with $[\lambda a : \lambda b]$ for a $\lambda \in k^\times$. If $b \neq 0$, we can think of $[a : b]$ as $[a/b : 1]$ identifying this point as a/b in the first affine line, and symmetrically, if $a \neq 0$, we can identify $[a : b]$ with b/a in the second affine line.

Exercise 9.9 (★★★☆☆)**.** A very important exercise: show that $\mathbf{P}^1_k$ is not affine. Hint: compute the ring of global sections, which correspond to sections over $X = \operatorname{Spec} k[x]$ and $Y = \operatorname{Spec} k[y]$ that agree on $X \cap Y$. Explain why sections that agree on $X \cap Y$ have to be constant polynomials. Therefore $\mathcal{O}_{\mathbf{P}^1_k}(\mathbf{P}^1_k) = k$. Reach a contradiction if you assume $\mathbf{P}^1_k$ is affine.

Example 9.10. Let's generalize **Example 9.8**. We built $\mathbf{P}^1_k$ by taking points $[x_0 : x_1]$ defined up to scaling by a unit, and conceptualized these points as $[x_0/x_1 : 1]$ is $x_0/x_1 \in \mathbf{A}^1_k$ if $x_1 \neq 0$ and $[1 : x_1/x_0]$ is $x_1/x_0 \in \mathbf{A}^1_k$ if $x_0 \neq 0$, since while x_0 and x_1 are not uniquely defined for a point $[x_0 : x_1]$, their quotients are.

To build $\mathbf{P}^n_k$, we will glue together $n+1$ affine charts, indexed by $0 \leq i \leq n$. Let

$$U_i = \mathbf{A}^n_k = \operatorname{Spec} k[x_0/x_i, \ldots, x_n/x_i],$$

and glue $D(x_i/x_j) \subseteq U_j$ to $D(x_j/x_i) \subseteq U_i$ by describing the identification of rings

$$k\left[x_0/x_i, \ldots, x_n/x_i, \frac{1}{x_j/x_i}\right] \cong k\left[x_0/x_j, \ldots, x_n/x_j, \frac{1}{x_i/x_j}\right]$$

via $x_k/x_i \mapsto (x_k/x_j)/(x_i/x_j)$ and $x_k/x_j \mapsto (x_k/x_i)/(x_j/x_i)$.

Exercise 9.11 (★★★☆☆). Show that the only global section of $\mathbf{P}^n_k$ are constants, and therefore that $\mathbf{P}^n_k$ is not affine for any n. Show that if k is algebraically closed, then the closed points of $\mathbf{P}^n_k$ can be identified with points $[a_0 : \cdots : a_n]$ where not all a_i are zero, and $[a_0 : \cdots : a_n] \sim [\lambda a_0 : \cdots : \lambda a_n]$ for $\lambda \in k^\times$.

Exercise 9.12 (★★★★☆). Let $(X, \mathcal{O}_X)$ be a scheme. Let R be a ring. Show that there is a bijection

$$\operatorname{Hom}_{\mathbf{Sch}}((X, \mathcal{O}_X), \operatorname{Spec} R) \cong \operatorname{Hom}_{\mathbf{CRing}}(R, \Gamma(X, \mathcal{O}_X)).$$

Compare to **Exercise 8.28**, number 4. The difficulty here comes from compatibility with affine covers.

Exercise 9.13 (★★★☆☆). Fixing a ring R, one may consider it a sheaf on the one point space $*$. For any ringed space $(X, \mathcal{O}_X)$, show there is a bijection

$$\operatorname{Hom}_{\mathbf{CRing}}(R, \Gamma(X, \mathcal{O}_X)) \cong \operatorname{Hom}_{\mathbf{RS}}((X, \mathcal{O}_X), (*, R)).$$

A Category Theory I

You're diving into this appendix because you've heard people talk about categories, objects and arrows, hom sets, limits, diagrams, functors, adjoints, universal properties, etc., and you've finally decided it's time to learn what the hell all that terminology means and how you're supposed to think about it. In fact, maybe during STAG we've already been dropping some of these terms, and you might be side-eyeing the speaker: "Am I suppose to already know what that word means?" you might even ask. The answer is no, not really! It's okay if you don't yet. At least as far as STAG is concerned, when speakers start to speak categorically, they're really only doing so to give a sense of the fact that there's a greater context to, or perhaps an easier way to discuss, the topics they're focused on. They still should be explaining things concretely too, so if you tuned out to the category stuff, you wouldn't be missing anything (other than that greater context). But now we're at the stage where we'd like to appreciate that greater context, or maybe you're just fed up with people saying words and not being part of the cognoscenti. Regardless, here you are! First, some words of advice: the way you're learning category theory right now is the correct way! Almost every resource will agree with me on this. The "right" way to learn category theory is as you need it, and with some context from algebra or topology already bouncing around in the back of your mind, so that you have a well of examples to draw from. Take it from me, who tried to do it the wrong way by just cracking open a category theory textbook to page one (or worse still, reading nLab before I was ready). It didn't stick until I came back with a specific question and a well of examples. These notes are going to start from the basics, assuming no category theory, but in the spirit of the previous paragraph, we're going to have a specific question in mind. That question, at least for this section, is this: what does it mean when we say "there is a category of varieties over any fixed field k, and this category is equivalent to the category of finitely generated k-algebras," why should we care, and how do we prove it?

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You can start reading the appendix at any time, but this first section has connections to section 3, and less so 4 and 5.

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Also, I'd like to warn you before we get started: these notes have a lot more footnotes than I usually write. All of them can be ignored. They're there because even when you're already knee-deep in category theory, there's going to be greater context still, and since these notes are the other side of the looking glass, I want to keep giving tastes of that.

A.1 Categories and Examples

Definition A.1. A **category** $\mathcal{C}$ is a collection of the following data:

1. a collection of objects $\mathrm{obj}(\mathcal{C})$,
2. a collection of arrows (also called maps or morphisms) subject to the axioms:
   - each arrow has a domain and a codomain which are both in $\mathrm{obj}(\mathcal{C})$; one writes $f : X \to Y$ if f is the arrow, $X = \mathrm{dom}(f)$ is the domain of f, and $Y = \mathrm{codom}(f)$ is the codomain,
   - composition of arrows is defined when it must be,
   - composition is associative, and
   - for each object X, there is an identity arrow $\mathrm{id}_X : X \to X$, behaving as an identity should.

Remark A.2. That's a terse way to define categories, so I do want to unpack explicitly what each part of the definition means. First, when we say "composition of arrows is defined when it must be," that means that if you've got two arrows f and g such that $\mathrm{codom}(f) = \mathrm{dom}(g)$, then you can compose them to get a new arrow which goes from $\mathrm{dom}(f)$ to $\mathrm{codom}(g)$. In symbols, we can take two maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

and produce a new map, called $g \circ f$ or just gf (we will basically never multiply two maps so the latter notation is not confusing and is in fact preferable):

$$X \xrightarrow{gf} Z.$$

The next part of the definition to unpack is associativity of composition. It simply says that if you want to compose three arrows

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W,$$

then whether you compose the first two arrows and get

$$X \xrightarrow{gf} Z \xrightarrow{h} W$$

or you compose the second two arrows and get

$$X \xrightarrow{f} Y \xrightarrow{hg} W,$$

you're getting the same thing, which is in fact the multi-composite

$$X \xrightarrow{hgf} W,$$

which is well-defined by associativity. That's the whole reason to require associativity; it makes compositions of multiple maps well-defined. The final part of the definition to unpack is what an identity arrow does. When we say it behaves how you should expect an identity to behave, that means, explicitly, if $\mathrm{id}_X : X \to X$ is the identity arrow for X, then given any $f : X \to Y$ or any $g : Z \to X$, we get $f\,\mathrm{id}_X = f$ and $\mathrm{id}_X\, g = g$.

Example A.3. This is perhaps the most natural example of a category to take. It is the category **Set**, whose objects are sets and whose arrows are functions between sets. The reason that this is the most natural example is because all the axioms you need to check are obvious for sets and set functions; we already know that composition of functions is defined, that composition of functions is associative, and that you can define an identity function on any set. It's all the stuff you've done with sets and set functions since you were in the womb!

Remark A.4. Understanding the category **Set** in **Example A.3** is also why I don't feel guilty giving you a kinda terse definition of a category in **Definition A.1** and only unpacking it fully after-the-fact. The reason is because if your go-to example of a category is **Set** or something equally familiar, then of course I can compose associatively and I have identity maps. It's so well-known that we never really worry about it. In fact, hang tight, and eventually we'll see that pretty much all the categories we *actually* care about (to differentiate them from some pathological examples we're about to see) have an underlying structure of **Set**. We'll even make this explicit a little while later, after we develop some more vocabulary.

In the meantime, let me give you a few more definitions, and then some more examples.

Definition A.5. If $f : X \to Y$ is an arrow in $\mathcal{C}$, then we can write $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, meaning that f is an element of the **hom set** from X to Y.[4]

Definition A.6. An **isomorphism** in a category $\mathcal{C}$ is just an arrow that has an inverse; i.e., $f : X \to Y$ is an isomorphism if there exists $g : Y \to X$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$.

Example A.7. So let's see some pathological examples just to kinda get a taste for what exactly a category can be. Note that as far as algebraic geometry is concerned, these pathological examples don't matter. They're fun though, and if you like category theory like I do, the exercises I'm going to give in a moment are pretty neat, if a bit unnecessary. Skip ahead to **Example A.10** to get back to the actually important stuff. So here are some pathological examples:

- Any set or class can be viewed as a category whose objects are the elements and whose only morphisms are the identity morphisms (such a category is called **discrete**).
- The category **0** has no objects and no arrows.
- The category **1** has one object 0 and one arrow id_0.

[4] Technically there are some concerns about sets versus proper classes to worry about that I'm slipping under the rug (in fact, I defined categories nebulously as a "collection" of objects and arrows to avoid this). But for all the categories we'll actually care about, for every X and Y in $\mathrm{obj}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ will be a set. Such categories are called **locally small**.

- The category **2** has two objects, 0 and 1, and one non-identity arrow $0 \to 1$.
- The category **3** has three objects and three non-identity arrows: $0 \to 1 \to 2$ (small exercise: what is the third, even though it's not technically drawn here?).
- You can keep going and define a category **n** for every $n \in \mathbf{N}$.

These were all examples of the following exercise:

Exercise A.8 (★★☆☆☆)**.** Recall the definition of a poset. Prove that given any poset $(P, \leq)$, it can be viewed as a category whose objects are $x \in P$ and whose arrows are $\leq$, and vice versa. That is, show the axioms of each definition translate into each other. Show that if $|P| = n$ and the partial order is a total order, then you recover the category **n** above.

Up next, an exercise I consider a lot more fun and surprising, even though it's the same flavor as the previous one. As you read through the definition of a category, its axioms should have felt a lot like the axioms you've seen of algebraic structures in your algebra classes. That means we can do the following:

Exercise A.9 (★★☆☆☆)**.** Prove that given any monoid M, it can be viewed as a category with a single object $*$ and an arrow for every $m \in M$, and vice versa. Prove that given any group G, it can be viewed as a category with a single object $*$ and an isomorphism for every $g \in G$, and vice versa.

Example A.10. Okay, let's see some more examples of categories, this time ones that we actually care about. The moral of the story here is: a category is basically just a collection of objects with some sort of structure, and arrows are structure-preserving maps.

- **Top** is the category of topological spaces and continuous functions.
  - $\mathbf{Top}^*$ is the category of pointed topological spaces (spaces X with a distinguished $x \in X$) and continuous functions preserving base point.
  - $\mathbf{Man}^p$ is the category of differentiable manifolds and p-fold continous differentiable functions.
  - **Fib** is the category of fiber bundles and bundle maps.
- **Grp** is the category of groups and group homomorphisms (different from **Exercise A.9**, which is a category that represents a single fixed group).
  - **Ab** is the category of abelian groups and group homomorphisms.
- **Ring** is the category of rings and ring homomorphisms.
  - **CRing** is the category of commutative rings and ring homomorphisms.
  - R-**Mod** or $\mathbf{Mod}_R$ is the category of R-modules and R-module homomorphisms for a fixed ring R.
  - If $R = k$ is a field, then $\mathbf{Mod}_R = \mathbf{Vec}_k$, the category of k-vector spaces and k-linear maps.
  - $\mathbf{Ch}(R\text{-}\mathbf{Mod})$ is the category of chain complexes of R-modules and chain maps for a fixed ring R.
  - R-**Alg** or $\mathbf{Alg}_R$ is the category of R-algebras and R-algebra homomorphisms for a fixed ring R.
- Given any directed graph, we get a category whose objects are the vertices and whose arrows are the paths, with composition given by concatenation of paths. This doesn't really have a boldfont name like the above examples, since you can do it for any digraph you're handed. Note that this might feel like the pathological examples, but I *promise* it's very important!
- **Cat** is the category of small categories (meaning that both $\mathrm{obj}(\mathcal{C})$ and the collection of arrows are sets, not proper classes), with arrows yet to be defined. We avoid Russell's paradox because **Cat** itself is not small, so **Cat** is not an object of itself.[5]

Okay, so what are the arrows in **Cat**? Actually, that's a very category-theory-centric way to ask that question, because it's abstracted a layer higher than what we'll actually use it for. A better way to ask this question, one that lives more on the ground level, is: given two categories, how can I map one to the other,

[5]Yep, since category theory can be used as a foundation for mathematics, we've gotta worry about this just like we did with sets in undergrad. There really is a lot of content I'm purposefully avoiding by not discussing small/large/locally small categories, which deal with such worries. But, just like with sets in undergrad, it ultimately doesn't matter for us; the ways in which we'll use categories won't bump up against paradoxes, in the same way you've never encountered Russell's paradox on sets in your day-to-day math.

in a structure-preserving way? That is, how can I map one category to the other without disturbing the category axioms?

A.2 Functors and Examples

> **Exercise A.11** (★★☆☆☆)**.** Without reading **Definition A.12**, see if you can come up with the definition yourself; i.e., how can you map one category to the other while preserving the axioms in **Definition A.1**? Obviously this exercise can't be presented.

Definition A.12. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A **(covariant) functor** $F : \mathcal{C} \to \mathcal{D}$ is a mapping such that

1. if X is an object of $\mathcal{C}$, then $F(X)$ is an object of $\mathcal{D}$,
2. if $f : X \to Y$ is an arrow in $\mathcal{C}$, then $F(f) : F(X) \to F(Y)$ is an arrow in $\mathcal{D}$, and
   - for every object X in $\mathcal{C}$, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, and
   - for all arrows

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

$F(g \circ f) = F(g) \circ F(f)$.

In succinct terms, functors map objects to objects and arrows to arrows in a way that preserves identities and composition (which is exactly what it has to be if we're trying to preserve the category axioms).

Remark A.13. Sometimes there are constructions that would be functors as we have defined them in **Definition A.12** except for the fact that they turn arrows around. Explicitly, you could have a mapping $F : \mathcal{C} \to \mathcal{D}$ such that

1. if X is an object of $\mathcal{C}$, then $F(X)$ is an object of $\mathcal{D}$,
2. if $f : X \to Y$ is an arrow in $\mathcal{C}$, then $F(f) : F(Y) \to F(X)$ is an arrow in $\mathcal{D}$, and
   - for every object X in $\mathcal{C}$, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, and
   - for all arrows

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

$F(g \circ f) = F(f) \circ F(g)$.

The arrows and the compositions go the "wrong" direction. This happens often enough that we don't want to exclude such constructions. They're therefore called **contravariant** functors. Often, this distinction is suppressed, and both are simply called functors. One can also avoid the word contravariant entirely, by the following exercise.

> **Exercise A.14** (★☆☆☆☆)**.** Given any category $\mathcal{C}$, we can define the opposite category $\mathcal{C}^{op}$ with the same objects and the arrows reversed. Show the following are equivalent.
> 1. A contravariant functor $F : \mathcal{C} \to \mathcal{D}$,
> 2. a covariant functor $F : \mathcal{C}^{op} \to \mathcal{D}$, and
> 3. a covariant functor $F : \mathcal{C} \to \mathcal{D}^{op}$.

Example A.15. Okay, we definitely need to see some examples of some functors. First some easy ones:

- Given any category $\mathcal{C}$, the identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ maps every object to itself and every morphism to itself.
- Given any category $\mathcal{C}$, there is a functor $\mathcal{C} \to \mathcal{C}^{op}$ which maps every object to itself and swaps the domain and codomain of every morphism.
- Given any two categories $\mathcal{C}$ and $\mathcal{D}$ and a choice of object D in $\mathcal{D}$, there is a constant functor sending every object to D and every arrow to id_D.

Next, some useful ones. How much utility you get out of these examples is dictated by how familiar you already are with the players, like I said in the **Introduction**. As such, none of these examples are going to be explicitly algebraic geometry examples, because in STAG you're not expected to know those yet, although some of the best examples are in this realm. And I'm going to bombard you with examples; feel free to skim over ones that you haven't already seen in your algebra or topology classes.

- The dual of a k-vector space is a functor from $\mathbf{Vec}_k$ to $\mathbf{Vec}_k$.
- The tensor product of an R-module with a fixed R-module N, $- \otimes_R N$, is a functor from R-**Mod** to R-**Mod**. Similarly so is $N \otimes_R -$.
- The module of homomorphisms from an R-module to a fixed R-module N, $\mathrm{Hom}_R(-, N)$, is a functor from R-**Mod** to R-**Mod**. Similarly so is $\mathrm{Hom}_R(N, -)$, but unlike the previous example, one of these is covariant and one is contravariant. (Important exercise: which is which?)
- More generally, given any category $\mathcal{C}$ and any fixed object C of $\mathcal{C}$, the functors $\mathrm{Hom}_{\mathcal{C}}(-, C)$ and $\mathrm{Hom}_{\mathcal{C}}(C, -)$ map $\mathcal{C}$ to **Set**, one covariantly and one contravariantly.
- The direct sum of an R-module with a fixed R-module N, $- \oplus N$, is a functor from R-**Mod** to R-**Mod**. Similarly so is $N \oplus -$.
- The free product of a group with a fixed group H, $- * H$, is a functor from **Grp** to **Grp**. Similarly so is $H * -$.
- The direct product of an R-module with a fixed R-module N, $- \times N$, is a functor from R-**Mod** to R-**Mod**. Similarly so is $N \times -$.
- The direct product of a topological space with a fixed space Y, $- \times Y$, is a functor from **Top** to **Top**. Similarly so is $Y \times -$.
- If you're noticing a pattern, it's intentional! These were examples of what is called a bifunctor. A bifunctor on a category $\mathcal{C}$ to some category $\mathcal{D}$ is a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{D}$, interpreted in the only way that $\mathcal{C} \times \mathcal{C}$ makes sense.[6]
- The fundamental group is a functor from $\mathbf{Top}^*$ to **Grp**.
- The assignment sending a topological space X to $C(X, \mathbf{R})$, the real-valued continuous functions on X, is a functor from **Top** to $\mathbf{Alg}_{\mathbf{R}}$.
- Sending a differentiable manifold to its tangent bundle and a smooth map to its derivative is a functor from $\mathbf{Man}^p$ to **Fib**.
- Homology and cohomology are functors from $\mathbf{Ch}(R\text{-}\mathbf{Mod})$ to **Ab**.
- A commutative diagram is a functor from the following digraph category:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

  to a category $\mathcal{D}$. More generally, *every* diagram in a category $\mathcal{D}$ is a functor from a digraph category to $\mathcal{D}$. (That's why digraphs are important!)
- Assignments which forget structure are functors. For instance,
  - Let $\mathcal{C}$ be a category whose objects are sets with additional structure and whose arrows are functions preserving that structure. (**Top**, **Grp**, **Ring**, etc. Basically every category we care about.) There is a forgetful functor $U : \mathcal{C} \to \mathbf{Set}$ which sends an object to its underlying set and sends an arrow to itself, having forgotten it was structure-preserving. If a category has a forgetful functor to **Set**, we call that category **concrete**, and this is what we meant in **Remark A.4** when we said categories that we care about have an underlying **Set** structure.
  - There are also forgetful functors which do not forget all of the structure. For example, the obvious functors $U : \mathbf{Ring} \to \mathbf{Ab}$ or $U : \mathbf{Top}^* \to \mathbf{Top}$.
- There are also free functors. These are functors which start with a set and build an algebraic structure out of that set. For instance, $\mathbf{Set} \to \mathbf{Grp}$ defined by sending a set X to the free group generated by X, $F(X)$, is a functor, as is $\mathbf{Set} \to \mathbf{Alg}_k$ defined by X is sent to $k[X]$. There is a subtle relationship between free and forgetful functors, but it is not that they are inverses of one another.[7]

[6] Actually, once you know what these words mean, you can say $\mathcal{C} \times \mathcal{C}$ is just a product object in **Cat**. But like our comment on functors "just being arrows in **Cat**" earlier, when you say it that way, it's abstracted a level higher than what you actually use it for. Good god this rabbit hole goes ever deeper!

[7] They're what are called "adjoint," but we won't see that in these notes.

Remark A.16. Notice often we are sloppy when we describe a functor. We tend not to specify if it's covariant or contravariant unless it's important to what we're doing, and sometimes we omit the image of the arrows with the understanding that there's only one sensible way to define them. If you're really wanting to understand functors, it might be a useful exercise to fill in those gaps in the previous examples. After that, here's another exercise:

Exercise A.17 (★☆☆☆☆). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Prove the following two consequences of the functor axioms.

1. F transforms commutative diagrams in $\mathcal{C}$ to commutative diagrams in $\mathcal{D}$.
2. If $f : X \to Y$ is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$. Prove or give a counterexample to the converse.

Remark A.18. As you can tell from the fact that I tried to throw every example under the sun at you, functors are important. (Well, I also did that so that there's a greater possibility at least one of the things I listed is familiar to you.) Let me give you some philosophy as to why: The purpose of category theory, at least how I see it, isn't necessarily to be a foundation of mathematics, although that is a thing it does do. I see categories as zooming out and viewing algebraic and topological structures not on an individual level, but via the way they interact with other objects in their category, and understanding those objects via understanding the properties of the categories they live in.[8] That means that instead of studying some specific topological space X or a specific group G, we study all of the spaces or groups, and all of the maps between spaces/groups. Therefore functors are really valuable because they give us ways to relate that zoomed out picture between two different topics. The fundamental group, if you've seen it before, is a great example of this; it connects two seemingly unrelated categories, $\mathbf{Top}^*$ and **Grp**, and by doing so lets us say interesting things about both. This is also why our motivating question from the **Introduction** is such a good one to ask. It too is trying to relate two different categories, that of "varieties over a field k" and "finitely generated k-algebras." But this question is even more beautiful, because we're not just trying to relate the two categories via some functor; we're trying to say that those categories are *actually* one and the same![9]

A.3 Answering Our Motivating Question

Believe it or not, we actually don't need many moving parts to answer the question from the **Introduction**: just categories and functors. That's often the case with category theory; we can get lost in examples and fun constructions and abstraction, but for practical purposes you often don't need huge sledgehammers. Let's start by recalling the players in this story.

Definition A.19. Let k be an algebraically closed field. A(n) **(affine) variety over** k, X, is an irreducible algebraic set with a structure sheaf $\mathcal{O}_X$ of regular functions. Remember that an irreducible algebraic set is just the zero set $X = V(\mathfrak{p})$ for $\mathfrak{p} \subset k[x_1, \dots, x_d]$ a prime ideal. We'll define the structure sheaf $\mathcal{O}_X$ first on stalks, then on open subsets $U \subseteq X$. At $x \in X$, a stalk is $\mathcal{O}_{X,x} = \{f/g \mid f, g \in k[x_1, \dots, x_d], g(x) \neq 0\}$ and for $U \subseteq X$,

$$\mathcal{O}_X(U) = \bigcap_{x \in U} \mathcal{O}_{X,x} = \left\{ \frac{f}{g} \mid f, g \in k[x_1, \dots, x_d], g(x) \neq 0 \text{ for all } x \in U \right\}.$$

Example A.20. Here's a variety. Consider $\mathfrak{p} = (x^2 + y^2 - 1) \subseteq \mathbf{C}[x, y]$ a prime ideal, and let $X = V(\mathfrak{p})$. In $\mathbf{A}^2_{\mathbf{C}}$, this is the unit circle. The structure sheaf at $(0, 1) \in X$, for instance, is

$$\mathcal{O}_{X,(0,1)} = \left\{ \frac{f}{g} \mid f, g \in k[x, y], g(0, 1) \neq 0 \right\}.$$

So for instance, $(x + 1)/y$ is a germ in $\mathcal{O}_{X,(0,1)}$, while $(x + 1)/xy$ is not.

[8] This latter perspective isn't here yet, because we haven't given any definitions which describe properties of a category. We'll do so some other time.

[9] Up to an appropriate notion of sameness of categories, yet to come.

Definition A.21. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two varieties over k. A **morphism of varieties** is a continuous map of topological spaces $\varphi : X \to Y$ such that for all open $V \subseteq Y$ and all $f \in \mathcal{O}_Y(V)$, the composition $f\varphi : \varphi^{-1}(V) \to k$ is in $\mathcal{O}_X(\varphi^{-1}(V))$.

Remark A.22. I will draw the picture of pulling back the regular functions during the talk, but the reader should draw this picture for themselves. Consider it a small but valuable exercise; without it, this definition is opaque and feels artificial but with it you see that it is still adhering to the theme we've seen many times of a "structure-preserving function." Here's another exercise to do:

> **Exercise A.23 (★☆☆☆☆).** Check that the collection of varieties over k with morphisms of varieties forms a category. We will call this category $\mathbf{AffVar}_k$.

The category $\mathbf{AffVar}_k$ is our first player. Let's start to introduce our second player.

Definition A.24. A k**-algebra**, A, is a ring that is also an k-module, whose addition in both settings coincide, and the scalar multiplication commutes with ring multiplication. You can explicitly write out all those axioms, which is an ordeal, or you can take the following alternative definition, which is what I prefer.

Definition A.25. A k**-algebra**, A, is a ring along with a choice of ring homomorphism $k \to A$.

> **Exercise A.26 (★★★☆☆).** Show these two definitions are the same.

Remark A.27. I prefer the latter definition because it's a lot more tractable and immediate. Rather than checking:

- all the axioms for a ring (and implicitly this involves the axioms for
  - an additive abelian group and
  - a multiplicative monoid),
- all the axioms for a k-module (which involves
  - an additive abelian group and
  - k-linear scalar multiplication),
- that ring addition and module addition coincide, and
- that scalar multiplication commutes with ring multiplication,

you can instead just fix a ring A and a specified map $k \to A$, called the structure map. It's a lot easier.

Example A.28. An important example is the fact that $k[x_1, \dots, x_d]$ is a k-algebra. Since our ring maps must send $1 \mapsto 1$ and extend linearly, there is only one possible structure map $k \to k[x_1, \dots, x_d]$. It is the map uniquely determined by sending $1 \mapsto 1$, and by extension, it sends any $r \in k$ to the constant polynomial r. It is called the **free algebra** on $\{x_1, \dots, x_d\}$, meaning[10] that maps from $k[x_1, \dots, x_d] \to A$ for any k-algebra A are determined by the images of $x_1, \dots, x_d$. But I haven't told you what k-algebra maps are yet, so hold on.

> **Exercise A.29 (★★☆☆☆).** If you don't want to bother with **Exercise A.26**, you could still check that $k[x_1, \dots, x_d]$ is a k-algebra via the first definition of a k-algebra. This gives you a taste for how you might show the equivalence of definitions, without actually doing it. (Again though, I just straight up prefer to think of k-algebras in the second way, and if you want to do the same, by all means.)

So now let's actually define k-algebra maps. We'll do so in two different ways as well.

Definition A.30. If A and B are k-algebras, then a k**-algebra homomorphism** $\varphi : A \to B$ is a k-linear ring homomorphism. Explicitly, in terms of axiom checking one must have

[10] much as it does anytime you see the word "free" categorically, but I only gave you a taste of that when we mentioned free functors

1. $\varphi(ra) = r\varphi(a)$ for all $r \in k$ and $a \in A$,
2. $\varphi(a + a') = \varphi(a) + \varphi(a')$ for all $a, a' \in A$,
3. $\varphi(aa') = \varphi(a)\varphi(a')$ for all $a, a' \in A$, and
4. $\varphi(1) = 1$.

Fortunately this is a little more tractable than defining k-algebras was. Although, just like with the definition of a k-algebra, the second definition we'll give for a k-algebra homomorphism is fast and efficient.

Definition A.31. If A and B are k-algebras (meaning each of them has a structure map from k), then a **k-algebra homomorphism** $\varphi : A \to B$ is a map such that the following diagram commutes

$$\begin{tikzcd} & k \arrow[dl] \arrow[dr] & \\ A \arrow[rr, "\varphi"'] & & B \end{tikzcd}$$

where $k \to A$ and $k \to B$ are the structure maps of each.[11]

As we already saw back in **Example A.10**, the category of k-algebras and k-algebra homomorphisms (whichever way you choose to define it) is called $\mathbf{Alg}_k$. This isn't the exact player we need though; we want *finitely generated* k-algebras. Let's give a definition.

Definition A.32. A **finitely generated k-algebra** is a k-algebra A such that $A \cong k[x_1, \ldots, x_d]/I$ where $I \subseteq k[x_1, \ldots, x_d]$ is an ideal. Equivalently, there is a surjection $k[x_1, \ldots, x_d] \twoheadrightarrow A$ (so by the first isomorphism theorem, the two definitions are equivalent taking I to be the kernel of the surjection). You also might see them called **k-algebras of finite type**.

Remark A.33. If in $\mathbf{Alg}_k$ you restrict yourself to only the finitely generated k-algebras, you won't lose any maps. What I mean by this is that if $\varphi : A \to B$ is an arrow in $\mathbf{Alg}_k$, that is, a k-algebra homomorphism, and it so happens that both A and B are of finite type, then $\varphi : A \to B$ is a finite type k-algebra homomorphism. In other words, we get a subcategory $\mathbf{Alg}_k^{ft}$ only by restricting the objects we are allowed to consider; we get the maps for free. The precise way to say this is that the inclusion $\mathbf{Alg}_k^{ft} \hookrightarrow \mathbf{Alg}_k$ is a fully faithful functor, meaning the hom sets before and after applying the inclusion functor to two objects are in bijection. The previous sentence was going to go in a footnote but it turns out we'll actually need this notion, so stay tuned (**Definition A.35**).

So now we have our two players: $\mathbf{AffVar}_k$ and $\mathbf{Alg}_k^{ft}$. Our last tool before the final act – the proof – is a discussion of what it means for two categories to be equivalent.

Remark A.34. If you're naïve like me, then your first guess for defining what it means for $\mathcal{C}$ and $\mathcal{D}$ to be two equivalent categories is that there is a functor $F : \mathcal{C} \to \mathcal{D}$ which is an isomorphism, meaning there is an inverse functor $G : \mathcal{D} \to \mathcal{C}$ where F and G compose to give the identity functors on the respective categories. But this is *not* what it means to have an equivalence of categories! Although, it almost is. The issue, as is often the case, is the fact that an isomorphism of two objects is not the same as equality. In a category $\mathcal{C}$, we need to think of two objects X and Y as distinct even if there exists an isomorphism between them, even though in practice when we're actually dealing with explicit objects we call isomorphic ones "the same." An equivalence of categories is weaker than an honest isomorphism of categories, which will keep such objects X and Y distinct in its image; an equivalence of categories need not. An equivalence of categories could possibly smoosh some of these distinct objects into one isomorphism class or vice versa, introduce new objects connected to old ones via isomorphisms, but that's okay for the way we actually work with objects in a category, which is only up to isomorphism. For that reason, equivalence of categories is the much more natural way to say that two categories are "the same," since it jives with that notion. It's weaker than an isomorphism of categories, which is itself weaker than an equality of categories.

[11]This construction is called a coslice category, in this case of **CRing** under k. I'm not going to say any more than just dropping the name, but the fact that it is named should signal to you that this construction is important and could be seen in other settings.

So that's the justification and vibes of an equivalence of categories, but what is the definition? The actual definition you usually see is a bit technical, so we're going to give a slightly less technical but equivalent characterization.

Definition A.35. A functor $F : \mathcal{C} \to \mathcal{D}$ is an **equivalence of categories** if it is

- **fully faithful**; i.e., if C and C' are objects in $\mathcal{C}$, then $\mathrm{Hom}_{\mathcal{C}}(C, C') \cong \mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$ as sets, and
- **essentially surjective** (also called **dense**); i.e., for all D in $\mathrm{obj}(\mathcal{D})$, D is isomorphic to $F(C)$ for some C in $\mathrm{obj}(\mathcal{C})$.

Remark A.36. One upside to this characterization is that it's easy to check. One downside is that you don't see the existence of an "inverse up to isomorphism" functor that runs the other way, which you would see in the actual definition. We'll use **Definition A.35** to actually check an equivalence of categories between $\mathbf{AffVar}_k$ and $\mathbf{Alg}_k^{ft}$, but then we'll give a taste of the inverse construction.

Theorem A.37. *There is an equivalence of categories between* $\mathbf{AffVar}_k$ *and* $\mathbf{Alg}_k^{ft}$ *given by the contravariant functor* $\Gamma : \mathbf{AffVar}_k \to \mathbf{Alg}_k^{ft}$, $\Gamma(X, \mathcal{O}_X) = \mathcal{O}_X(X)$ *and* $\Gamma(\varphi) = \varphi^*$.

Proof. To see that Γ is fully faithful, let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two objects in $\mathbf{AffVar}_k$. We must show that

$$\mathrm{Hom}_{\mathbf{AffVar}_k}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) \cong \mathrm{Hom}_{\mathbf{Alg}_k^{ft}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$$

as sets, meaning we must show a bijection. By definition, the left-hand side contains maps $\varphi : X \to Y$ that pull back regular functions on Y to regular functions on X. Hence any $\varphi : X \to Y$ induces a map

$$\mathrm{Hom}_{\mathbf{AffVar}_k}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) \to \mathrm{Hom}_{\mathbf{Alg}_k^{ft}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$$

and we need to produce an inverse to show it is a bijection. Suppose we have an element on the right-hand side; i.e., a k-algebra homomorphism $h : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Since $\mathcal{O}_Y(Y)$ is of finite type, we know $\mathcal{O}_Y(Y) \cong k[x_1, \ldots, x_d]/I$, and further still, since $(Y, \mathcal{O}_Y)$ is an affine variety, $I = I(Y)$ is the ideal of Y (i.e., $Y = V(I(Y))$). The k-algebra map h is determined by the images $h(x_i) \in \mathcal{O}_X(X)$ for each $i \in \{1, \ldots, d\}$. Since these elements live in $\mathcal{O}_X(X)$, they are global rational functions, so they define an embedding $\psi : X \to \mathbf{A}_k^d$. We'll show this embedding actually factors through Y, producing a map of varieties $X \to Y$. To do this, we've gotta see that for any point $p \in X$, $\psi(p) \in Y = V(I(Y))$; it is enough to see that for all $f \in I(Y)$, $f(\psi(p)) = 0$. By construction,

$$f(\psi(p)) = h(f(x_1, \ldots, x_d))(p) = 0.$$

So for every $h : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$, we get a map of varieties $X \to Y$. This construction is the inverse to the map induced by φ,

$$\mathrm{Hom}_{\mathbf{AffVar}_k}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)) \to \mathrm{Hom}_{\mathbf{Alg}_k^{ft}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)),$$

but I'll leave that as an exercise for you to check. All that remains is to see that Γ is essentially surjective. Let A be any finitely generated k-algebra, so $f : k[x_1, \ldots, x_d] \twoheadrightarrow A$ and therefore $A \cong k[x_1, \ldots, x_d]/\ker f$. So $\ker f$ is a prime ideal. We can define a variety $X \coloneqq V(\ker f)$, and by construction, $\mathcal{O}_X(X) \cong k[x_1, \ldots, x_d]/I(X) = k[x_1, \ldots, x_d]/\ker f \cong A$. Therefore Γ is essentially surjective, as desired. ☐

Remark A.38. What is the inverse construction $\mathbf{Alg}_k^{ft} \to \mathbf{AffVar}_k$? We've actually already seen it; we more-or-less built it when we showed Γ is essentially surjective. Given a k-algebra of finite type, i.e., $A \cong k[x_1, \ldots, x_d]/I$, you can produce a variety over k by defining it to be $V(I)$.

B Category Theory II

Welcome back! When you come to this section, you likely have some category theory under your belt, and also some algebraic geometry as well. But as we only dipped our toes into category theory last time, there's still more treasure to plunder yet. We were able to prove a fundamental theorem in algebraic geometry only knowing categories, functors, and equivalences of categories (namely, that varieties are determined by their rings of regular functions and vice versa), so imagine what we'll be able to say with more tools in our toolbelt! This set of notes is going to follow the same principles that we established last time. We're going to develop some category theory, not in a vacuum, but in service of a question we want to answer. This time, our goal is a little more esoteric, but I will sales-pitch it like this: varieties, as we know, are extremely non-metric spaces – in fact, they are not even Hausdorff. But when we draw, for example, a curve in $\mathbf{A}^2$, that picture has tangent information, and we'd like a way to get at that information through algebraic geometry. What does it mean to have a differential form on a variety over a fixed algebraically closed field k? Or, for that matter, on a k-algebra A, since we know from STAG and the last category theory notes that these notions are categorically equivalent.

B.1 Universal Properties: Good Grief They're Everywhere

It's kinda hard to convey how true the title of this section is without just showing you. In fact, in lieu of defining a universal property in total generality, let me just show you tons of them.

Example B.1. Last time we mentioned free functors, like the functors $\mathbf{Set} \to \mathbf{Grp}$ defined by $X \mapsto \langle X \mid \varnothing \rangle$ or $\mathbf{Set} \to \mathbf{Alg}_k$ defined by $X \mapsto k[X]$. Both of these functors, and indeed any free functor, can be defined using a universal property. The notion is this: let $\mathcal{C}$ be a concrete category. We define a free functor $F : \mathbf{Set} \to \mathcal{C}$ to be a functor such that for all $A \in \operatorname{obj}(\mathcal{C})$ and any set map $X \to A$, there exists a unique $\mathcal{C}$-morphism $F(X) \to A$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & F(X) \arrow[dl, dashed, "\exists!"] \\ A & \end{tikzcd}$$

The notion to have in mind is that a free functor generalizes a basis of vector spaces (and indeed, defining a vector space by choosing a set of basis vectors is a free functor). Under this lens, the set X is a basis, the object $F(X)$ is the free object defined by that basis (a free group, a free k-algebra, a vector space, etc.), and the universal property says that free objects are characterized by the fact that you can define maps out of them just by declaring where the basis elements go. For instance, to define a k-algebra homomorphism $k[x_1, \ldots, x_d] \to A$ for any A, it suffices to define the images of the set $\{x_1, \ldots, x_d\}$, for then the image of any polynomial $f \in k[x_1, \ldots, x_d]$ can be deduced via k-linearity.

Exercise B.2 (★☆☆☆☆)**.** Suppose $\varphi : k[x, y] \to A$ is defined by $\varphi(x) = a_1$ and $\varphi(y) = a_2$. Use the universal property of free k-algebras to find the image of $f = x^2y + xy^2 - x - y + 2$ under φ.

Example B.3. Another functor we saw which can be built out of a universal property is the tensor product over a fixed ring R. We can define the tensor product $- \otimes_R - : \mathbf{Mod}_R \times \mathbf{Mod}_R \to \mathbf{Mod}_R$ in the following way. Let $M \times N \to P$ be any bilinear map. There is a unique R-homomorphism $M \otimes_R N \to P$ such that the following diagram commutes.

$$\begin{tikzcd} M \times N \arrow[r] \arrow[d] & M \otimes_R N \arrow[dl, dashed, "\exists!"] \\ P & \end{tikzcd}$$

In other words, the universal property says you can characterize homomorphisms out of tensor products $M \otimes_R N$ by bilinear maps out of $M \times N$; the two are in bijection.

Exercise B.4 (★★☆☆☆)**.** Prove that $R \otimes_R R$ is isomorphic to R via the map $\varphi : R \to R \otimes_R R$, $\varphi(r) = r \otimes 1$. Hint: use the bilinear map $R \times R \to R$ given by coordinate multiplication to construct an R-homomorphism inverse to φ.

Example B.5. Localization of a ring by a multiplicatively closed subset satisfies a universal property. Let R be a ring and let S be a multiplicatively closed subset of R containing 1. There is a natural map $R \to S^{-1}R$ defined by sending an element r to $r/1$. If $R \to T$ is a ring homomorphism for which every element of S is mapped to a unit, then there is a unique ring homomorphism $S^{-1}R \to T$ such that the following diagram commutes.

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S^{-1}R \arrow[dl, dashed, "\exists!"] \\ T & \end{tikzcd}$$

Exercise B.6 (★★★☆☆)**.** Let R be a ring. Let S and S' be multiplicatively closed subsets. Show that $SS' \coloneqq \{ss' \mid s \in S, s' \in S'\}$ is multiplicatively closed. Let $\overline{S}$ be the image of S in ${S'}^{-1}R$. Use the universal property of localization to show that $(SS')^{-1}R \cong \overline{S}^{-1}({S'}^{-1}R)$.

Example B.7. The symmetric algebra of an R-module M can be characterized via a universal property. Let $M \to A$ be an R-module homomorphism. There exists a unique R-algebra homomorphism $\mathrm{Sym}(M) \to A$ such that the following diagram commutes.

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & \mathrm{Sym}(M) \arrow[dl, dashed, "\exists!"] \\ A & \end{tikzcd}$$

Exercise B.8 (★★★☆☆)**.** Let V be a k-vector space.

1. Let $T^nV \coloneqq V^{\otimes n} = V \otimes V \otimes \cdots \otimes V$. Let
$$T(V) \coloneqq \bigoplus_{n=0}^{\infty} T^kV = k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots .$$
Let $S(V)$ be defined as the quotient of $T(V)$ by the ideal $(v \otimes w - w \otimes v \mid v, w \in V)$. Show that $S(V)$ satisfies the universal property of $\mathrm{Sym}(V)$.
2. If $B \subseteq V$ is a basis, show that the polynomial ring $k[B]$ satisfies the universal property of $\mathrm{Sym}(V)$.

Example B.9. The exterior algebra of an R-module M has the following universal property. Let A be an R-algebra and let $f : M \to A$ be an R-module homomorphism such that $f(m)f(m) = 0$ for all $m \in M$. There exists a unique R-algebra homomorphism $\Lambda(M) \to A$ such that the following diagram commutes.

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d, "f"'] & \Lambda(M) \arrow[dl, dashed, "\exists!"] \\ A & \end{tikzcd}$$

Exercise B.10 (★★★☆☆). Let V be a k-vector space. Let $T^nV \coloneqq V^{\otimes n} = V \otimes V \otimes \cdots \otimes V$. Let

$$T(V) \coloneqq \bigoplus_{n=0}^{\infty} T^kV = k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots .$$

Let $L(V)$ be defined as the quotient of $T(V)$ by the ideal $(v \otimes v \mid v \in V)$. Show that $L(V)$ satisfies the universal property of $\Lambda(V)$.

Example B.11. The Stone-Čech compactification of a topological space X is defined via a universal property. If K is a compact Hausdorff space and $f : X \to K$ is continuous, then there exists a compact Hausdorff space βX and a unique continuous map $\beta X \to K$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"'] & \beta X \arrow[dl, dashed, "\exists!"] \\ K & \end{tikzcd}$$

This construction works for any space X, but for those curious, there are nicer results if you impose hypotheses on X. For instance, $X \hookrightarrow \beta X$ is a homeomorphism onto its image if and only if X is Tychonoff, and $X \hookrightarrow \beta X$ is a homeomorphism to an open subspace if and only if X is locally compact and Hausdorff.

Exercise B.12 (★☆☆☆☆). Let X be compact and Hausdorff. Show that $\beta X \cong X$.

Exercise B.13 (★★★★☆). Let $C = \{f : X \to [0,1]\}$. Recall the notation $[0,1]^C \coloneqq \{\varphi : C \to [0,1]\}$. Consider the map $e : X \to [0,1]^C$ defined by $e(x) : f \mapsto f(x)$. Show e is a continuous map onto its image. Cite a theorem which says $[0,1]^C$ is compact. Since closed subspaces of compact spaces are compact, show that $\overline{e(X)} \subseteq [0,1]^C$ satisfies the universal property of the Stone-Čech compactification.

Example B.14. Quotients can be defined in terms of universal properties. We will focus on the setting of quotient spaces and quotient maps. Let X be a topological space and let $q : X \to X/\sim$ be the quotient map $x \mapsto [x]$ for an equivalence relation $\sim$ on X. If $g : X \to Z$ is a continuous map such that $x \sim y$ implies $g(x) = g(y)$ for all $x, y \in X$, then there exists a unique continuous map $X/\sim \to Z$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"'] & X/\sim \arrow[dl, dashed, "\exists!"] \\ Z & \end{tikzcd}$$

A general quotient map $q : X \to Y$ satisfies the condition that given a function $f : Y \to Z$, f is continuous if and only if $f \circ q$ is continuous. In other words, Y is equipped with the final topology with respect to q.

Exercise B.15 (★★☆☆☆). Suppose $q : X \to Y$ is a quotient map and Y is equipped with the final topology with respect to q. Show there exists an equivalence relation $\sim$ on X such that $Y \cong X/\sim$.

Example B.16. Products, in many settings, are defined via a universal property. Let X and Y be two objects of a category $\mathcal{C}$. The product $X \times Y$ is an object of $\mathcal{C}$ equipped with two morphisms $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ satisfying the following universal property. Given any other object Z with maps $f_1 : Z \to X$ and $f_2 : Z \to Y$, there exists a unique morphism $f : Z \to X \times Y$ such that the following diagram commutes.

$$\begin{tikzcd}
& Z \arrow[dl, "f_1"'] \arrow[d, dashed, "\exists! f"] \arrow[dr, "f_2"] & \\
X & X \times Y \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & Y
\end{tikzcd}$$

Exercise B.17 (★★★☆☆). Show that the following are all examples of the categorical notion of a product.

1. The Cartesian product of sets in **Set**.
2. The Cartesian product of topological spaces with the product topology in **Top**.
3. The direct product of groups in **Grp**.

Exercise B.18 (★★☆☆☆). Give a universal property definition for an infinite product of objects.

Example B.19. The fiber product[12] of two objects is defined via a universal property. Let $f : X \to Z$ and $g : Y \to Z$ be two morphisms in $\mathcal{C}$. The fiber product of f and g is an object $X \times_Z Y$ along with two morphisms $p_1 : X \times_Z Y \to X$ and $p_2 : X \times_Z Y \to Y$ such that the following diagram commutes.

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Z
\end{tikzcd}$$

The construction is universal in the sense that if there exists another object Q with morphisms $q_1 : Q \to X$ and $q_2 : Q \to Y$ such that $fq_1 = gq_2$, then there exists a unique morphism $u : Q \to X \times_Z Y$ such that $p_1 u = q_1$ and $p_2 u = q_2$. Diagrammatically, one has the following commutative diagram.

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\exists! u"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\
& X \times_Z Y \arrow[r, dashed, "p_2"] \arrow[d, dashed, "p_1"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

Exercise B.20 (★★☆☆☆). Show that in **Set**, given two functions $f : X \to Z$ and $g : Y \to Z$, the set $\{(x, y) \in X \times Y \mid f(x) = g(y)\}$ is the fiber product of f and g.

Exercise B.21 (★★★☆☆). Show that in **Set**, given a function $f : X \to Y$ and a subset inclusion $g : Y_0 \hookrightarrow Y$, the fiber product of f and g is the preimage $f^{-1}(Y_0)$. What are the two maps p_1 and p_2? This exercise justifies the name pullback for the fiber product.

Example B.22. Initial and terminal objects are defined via a universal property. An initial object I is an object such that for every X in $\mathrm{obj}(\mathcal{C})$, there exists a unique map $I \to X$. Dually, a terminal object T has a unique map $X \to T$.

[12] Also very frequently called a pullback but unfortunately there are two notions of a pullback in mathematics, and while they are related it's not helpful to conflate them when you first encounter them.

Exercise B.23 (★★★☆☆)**.** Let $\mathcal{C}$ be a category with a terminal object T. Show that $X \times_T Y$ is just the ordinary product $X \times Y$.

Exercise B.24 (★★★★☆)**.** Show the following are initial objects.

- $\varnothing$ in **Set**
- $\{*\}$ in **Top***
- $\{1\}$ in **Grp**
- 0 in **Ab**
- 0 in $\mathbf{Mod}_R$
- $\mathbf{Z}$ in **Ring**

Show the following are terminal objects.

- $\{*\}$ in **Set**
- $\{*\}$ in **Top***
- $\{1\}$ in **Grp**
- 0 in **Ab**
- 0 in $\mathbf{Mod}_R$
- 0 in **Ring** (if you consider it a ring)

When an object is both initial and terminal, we call it a zero object.

Example B.25. Given a map f, a kernel and its inclusion into the domain can be described via a universal property. Let $f : X \to Y$ be an arbitrary morphism in a category $\mathcal{C}$ for which zero maps exist (i.e., a category with a zero object; e.g., **Grp**, **Ab**, $\mathbf{Mod}_R$, $\mathbf{Vec}_k$, etc.). A kernel of f is an object K with a morphism $k : K \to X$ such that the following diagram commutes.

$$\begin{tikzcd} & K \arrow[dl, "k"'] \arrow[dr, "0"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

The kernel is unique, meaning if there is another pair of object K' and morphism $k' : K' \to X$ such that $fk' = 0$, then there is a unique morphism $u : K \to K'$ such that $ku = k'$.

$$\begin{tikzcd} & K' \arrow[ddl, bend right, "k'"'] \arrow[d, dashed, "\exists! u"] \arrow[ddr, bend left, "0"] & \\ & K \arrow[dl, "k"'] \arrow[dr, "0"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

Exercise B.26 (★★☆☆☆)**.** Let $f : A \to B$ be a homomorphism of R-algebras. Show that $K = \ker f$ and $k = \mathrm{incl} : \ker f \to A$ satisfy the universal property of a kernel.

Example B.27. The categorical concept of an equalizer is described via the following universal property. Let $f : X \to Y$ and $g : X \to Y$ be two morphisms in a category $\mathcal{C}$. An equalizer is an object E and a morphism $e : E \to X$ such that $fe = ge$ and is unique, meaning if $m : F \to X$ is any morphism such that $fm = gm$, then there exists a unique morphism $u : F \to E$ such that $eu = m$. In other words, the following diagram commutes.

$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ F \arrow[u, dashed, "\exists! u"] \arrow[ur, "m"'] & & \end{tikzcd}$$

Exercise B.28 (★★★☆☆)**.** Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$ for which zero maps exist. Show that the categorical notion of a kernel (K, k) of f is the same as an equalizer (E, e) of $f : X \to Y$ and $0 : X \to Y$.

Exercise B.29 (★★★☆☆)**.** Let $f : X \to Y$ and $g : X \to Y$ be two morphisms in a category $\mathcal{C}$ for which it makes sense to add or subtract morphisms; i.e., hom sets are abelian groups (e.g., **Ab**, $\mathbf{Mod}_R$, $\mathbf{Vec}_k$, etc.). Show that the categorical notion of an equalizer (E, e) of f and g is the same as the kernel of $f - g : X \to Y$.

Exercise B.30 (★★★★☆)**.** Let F be a presheaf on a topological space X. Let $U \subseteq X$ be any open set and let $\{U_i\}$ be an open cover of U. Show that F is a sheaf if and only if the following diagram is an equalizer.

$$F(U) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)$$

Example B.31. A categorical limit is defined via a universal property. Let J be a digraph category; i.e., its objects are vertices and its morphisms are directed edges. In other words, J is an abstract diagram, like a commutative square.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

When we have a functor $F : J \to \mathcal{C}$, we say that F is a diagram of shape J in $\mathcal{C}$. A limit of a diagram of shape J, F, is an universal cone $(L, \{\varphi\})$. A cone is an object L and a family of morphisms $\varphi_A : L \to F(A)$ for all A in obj(J) such that for every $f : X \to Y$, one has $F(f)\varphi_X = \varphi_Y$.

$$\begin{tikzcd} & L \arrow[dl, "\varphi_X"'] \arrow[dr, "\varphi_Y"] & \\ F(X) \arrow[rr, "F(f)"'] & & F(Y) \end{tikzcd}$$

A cone is universal if given another cone $(N, \{\psi\})$, there exists a unique morphism $u : N \to L$ such that $\varphi_A u = \psi_A$ for all A in obj(J).

$$\begin{tikzcd} & N \arrow[d, dashed, "\exists! u"] \arrow[ddl, bend right, "\psi_X"'] \arrow[ddr, bend left, "\psi_Y"] & \\ & L \arrow[dl, "\varphi_X"'] \arrow[dr, "\varphi_Y"] & \\ F(X) \arrow[rr, "F(f)"'] & & F(Y) \end{tikzcd}$$

Exercise B.32 (★★★★☆). Let J be a discrete digraph with two vertices; i.e., J is the following digraph.

$$\bullet \qquad \bullet$$

Let $F : J \to \mathcal{C}$ be a diagram of shape J in $\mathcal{C}$. Show that the limit of F is a product.

Exercise B.33 (★★★★☆). Let J be the following digraph.

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

Let $F : J \to \mathcal{C}$ be a diagram of shape J in $\mathcal{C}$. Show that the limit of F is a fiber product.

Exercise B.34 (★★★★☆). Let J be an empty digraph. Let $F : J \to \mathcal{C}$ be a diagram of shape J in $\mathcal{C}$. Show that the limit of F is a terminal object.

Exercise B.35 (★★★★☆). Let J be the following digraph.

$$\bullet \rightrightarrows \bullet$$

Let $F : J \to \mathcal{C}$ be a diagram of shape J in $\mathcal{C}$. Show that the limit of F is an equalizer.

Example B.36. A categorical colimit is defined via a universal property as well. Given any construction in category theory, the "co-" construction amounts to the dual – the same construction with arrows reversed. Thus a colimit of a diagram of shape J, F, is an universal cocone $(L, \{\varphi\})$; i.e., for all $f : X \to Y$ in J, the following diagram commutes.

$$\begin{tikzcd} F(X) \arrow[rr, "F(f)"] \arrow[dr, "\varphi_X"'] & & F(Y) \arrow[dl, "\varphi_Y"] \\ & L & \end{tikzcd}$$

Given another cocone $(N, \{\psi\})$, there exists a unique morphism $u : L \to N$ such that $u\varphi_A = \psi_A$ for all A in $\mathrm{obj}(J)$.

$$\begin{tikzcd} F(X) \arrow[rr, "F(f)"] \arrow[dr, "\varphi_X"] \arrow[ddr, bend right, "\psi_X"'] & & F(Y) \arrow[dl, "\varphi_Y"'] \arrow[ddl, bend left, "\psi_Y"] \\ & L \arrow[d, dashed, "\exists! u"] & \\ & N & \end{tikzcd}$$

Exercise B.37 (★★★★☆). Let X be an affine variety over k with structure sheaf $\mathcal{O}_X$ defined by $\mathcal{O}_X(U) \coloneqq \{f/g \mid f, g \in k[x_1, \ldots, x_d], g(x) \neq 0 \text{ for all } x \in U\}$ for any open $U \subseteq X$. Let $x \in X$ and define the stalk at x by $\mathcal{O}_{X,x} \coloneqq \{f/g \mid f, g \in k[x_1, \ldots, x_d], g(x) \neq 0\}$. Let $U_1 \supseteq U_2 \supseteq \cdots$ be a chain of nested subsets all containing x, defining the diagram below.

$$U_1 \longleftarrow U_2 \longleftarrow U_3 \longleftarrow \cdots$$

By restriction of sections to a subset, $\mathcal{O}_X-$ induces the following diagram.

$$\mathcal{O}_X(U_1) \longrightarrow \mathcal{O}_X(U_2) \longrightarrow \mathcal{O}_X(U_3) \longrightarrow \cdots$$

Show that $\mathcal{O}_{X,x} \cong \operatorname{colim} \mathcal{O}_X(U_i)$. Describe all morphisms in the universal cocone.

Remark B.38. When universal constructions exist, they are unique (up to a unique isomorphism, meaning if you have a different object satisfying the universal property, there is a unique isomorphism between the two). Notice that nothing in the universal properties we have constructed in this section requires them to exist.[13] In fact, to show existence, it is often the case that we build an explicit object and check that it satisfies the universal property, as you did in many of this section's exercises. Nevertheless, defining an object via a universal property is powerful; categorical arguments are often smoother than explicit, element-wise ones, at the cost of some "abstract nonsense."

B.2 Kähler Differentials: Differential Forms on a Variety/Algebra

Now that we've seen tons of examples of constructions defined via a universal property, we're prepared to use universal properties to define the thing we set out to define at the start: an algebraic construction that tells us differential information. This construction is called the relative Kähler differentials. In this section, we'll define them categorically, prove they exist via explicit constructions, and then see some results. Our story starts with the desire to define differentiation algebraically. If we think only in terms of formal symbol moving, then differentiation of polynomials in a k-algebra of finite type can be described explicitly. The tool is called a derivation δ.

Definition B.39. Let k be a ring.[14] Let A be a k-algebra; i.e., there is a structure map $\varphi : k \to A$. Let M be an A-module. A **k-linear derivation of A with coefficients in M** is a homomorphism of A-modules $\delta : A \to M$ satisfying the Leibniz rule. Explicitly, for all $f, g \in A$,

$$\delta(fg) = \delta(f)g + f\delta(g).$$

Equivalently, a derivation δ is a homomorphism of abelian groups satisfying the Leibniz rule and such that $\delta\varphi = 0$.

Exercise B.40 (★★☆☆☆). Show that these two definitions are equivalent.

Exercise B.41 (★★☆☆☆).
1. Show that as a consequence, a derivation δ must satisfy the power rule: $\delta(f^n) = nf^{n-1}\delta(f)$.
2. Show that as a consequence, a derivation is $\varphi(k)$-linear: $\delta(\varphi(r)f) = \varphi(r)\delta(f)$. In almost all contexts we care about, $\varphi : k \to A$ is injective, so we typically write r for $\varphi(r)$; i.e., $\delta(rf) = r\delta(f)$.

Definition B.42. We denote the set of all $\delta : A \to M$, k-linear derivations of A with coefficients in M, by $\mathrm{Der}_k(A; M) \subseteq \mathrm{Hom}_k(A, M)$.

[13] Of course all of the examples we have given above exist in the settings we used them in, but this is not guaranteed. Even very general constructions, like limits or colimits, need not exist in a given category always. When they do, the category is called complete or cocomplete (get it? :)).

[14] You might be worried that this is too general, since varieties need to be defined over an algebraically closed field k. We will see a generalization of varieties which alleviates this concern (in fact, in my eyes, varieties are artificially restricted by comparison), but until then, you are welcome to think of k as a field.

Exercise B.43 (★★☆☆☆). Show that $\mathrm{Der}_k(A; M)$ is an A-submodule via the action $(f \cdot \delta)(g) = f\delta(g)$.

Definition B.44. The **module of Kähler differentials of** A **over** k, $\Omega_{A/k}$, along with the **universal derivation** $d : A \to \Omega_{A/k}$, satisfy the following universal property. Let $\delta : A \to M$ be a k-linear derivation. There exists a unique A-module homomorphism $\Omega_{A/k} \to M$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "d"] \arrow[d, "\delta"'] & \Omega_{A/k} \arrow[dl, dashed, "\exists!"] \\ M & \end{tikzcd}$$

In other words, there is an isomorphism of A-modules

$$\mathrm{Hom}_A(\Omega_{A/k}, M) \cong \mathrm{Der}_k(A; M)$$

given by composition with the universal derivation $d : A \to \Omega_{A/k}$.

Exercise B.45 (★★★☆☆). Let $\varphi : k \to A$ be surjective. Show that $\Omega_{A/k} \cong 0$.

Exercise B.46 (★★★☆☆). Since it need not necessarily, let's show $\Omega_{A/k}$ exists (and not only in the case where φ is a surjection). We'll do it in two different ways.

1. Let K be the A-module generated by symbols df for all $f \in A$, and let $d : A \to K$ be $d(f) = df$, such that for each $f, g \in A$ and $r \in k$, we have
   - $d(f + g) = df + dg$,
   - $d(fg) = df \cdot g + f \cdot dg$, and
   - $d\varphi(r) = 0$.

   Show that K and d satisfy the universal property of $\Omega_{A/k}$.
2. Let $\mu : A \otimes_k A \to A$ be defined by $f \otimes g \mapsto fg$ on simple tensors. Let $I = \ker \mu$. Let $K' = I/I^2$ and let $d : A \to K'$ be defined by $d(f) = 1 \otimes f - f \otimes 1$. Show that K' and d satisfy the universal property of $\Omega_{A/k}$.

Remark B.47. The two explicit constructions of $\Omega_{A/k}$ in **Exercise B.46** above give us two interesting perspectives on how to think about Kähler differentials. Construction 1 is formal symbol moving; building the A-module K amounts to defining exactly the relations needed, and no more, that guarantee $d : A \to K$ is a derivation. On the other hand, construction 2 is first-order tangent information; the quotient $K' = I/I^2$ amounts to functions which vanish modulo functions vanishing at least to second order. You can think of taking a function's Taylor series and truncating it to get the first order differentiation. Of course, since these two constructions are both equivalent to $\Omega_{A/k}$ and hence to each other, there is a dictionary between the two, but at face value the different perspectives they give are valuable both algebraically and geometrically.

Exercise B.48 (★★☆☆☆). Let L/k be a separable field extension. Show that $\Omega_{L/k} \cong 0$ via the following hint: for any $a \in L$, choose a function $f \in k[x]$ such that $f(a) = 0$ and $f'(a) \neq 0$.

Exercise B.49 (★★☆☆☆). Let $A = k[x_1, \ldots, x_n]$. Show that $\Omega_{A/k} \cong A \cdot dx_1 \oplus \cdots \oplus A \cdot dx_n$.

Example B.50. Let's compute the Kähler differentials of a k-algebra of finite type, and we'll later see how this computation generalizes. Let $A = k[x,y]/(y - x^2)$. Every element of A can be represented as a polynomial

$$f = \sum_{i,j} c_{i,j} x^i y^j.$$

By explicit derivation computations, we have

$$df = \sum_{i,j} c_{i,j} d\left(x^i y^j\right) = \sum_{i,j} c_{i,j}\left(d\left(x^i\right) y^j + x^i d\left(y^j\right)\right) = \sum_{i,j} c_{i,j}\left(ix^{i-1} y^j dx + x^i j y^{j-1} dy\right),$$

so as above in **Exercise B.49**, $\Omega_{A/k}$ is generated as an A-module by dx and dy. Since $y - x^2 = 0$ in A, we should expect $d(y - x^2) = dy - 2xdx = 0$ in $\Omega_{A/k}$. Indeed, we have defined a morphism of A-modules

$$\Omega_{A/k} \to A \cdot dx \oplus A \cdot dy \twoheadrightarrow (A \cdot dx \oplus A \cdot dy)/(dy - 2xdx)$$

and one can check that the composition above is an isomorphism.

Exercise B.51 (★★☆☆☆)**.** Check it.

Remark B.52. The argument in **Example B.50** above is natural in that it should be unsurprising, but it is a bit janky and artificial. Currently to generalize it to an arbitrary k-algebra of finite type A, we have to make similar arguments about the relations of $\Omega_{A/k}$ as being derived from the relations of A, but we would prefer a way to describe $\Omega_{A/k}$ for any A without dealing with specific elements each time. The tools involved are two fundamental exact sequences, which we provide here without proof.

Theorem B.53 (The First Fundamental Exact Sequence)**.** *Let $k \to R \to S$ be ring maps. The following sequence of S-modules is exact.*

$$\Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to \Omega_{S/R} \to 0$$

Proof. See *The Stacks Project*, tag [00RS]. `https://stacks.math.columbia.edu/tag/00RS` □

Example B.54. Before seeing the second fundamental exact sequence, let's see how the first one can be used in practice (and how it motivates the second). Consider the ring maps $k \hookrightarrow k[x,y] \twoheadrightarrow k[x,y]/f$ for a polynomial $f \in k[x,y]$. Write A for $k[x,y]/f$ and our first fundamental exact sequence is the following.

$$\Omega_{k[x,y]/k} \otimes_{k[x,y]} A \to \Omega_{A/k} \to \Omega_{A/k[x,y]} \to 0$$

Since $k[x,y] \twoheadrightarrow A$ is a surjection, by **Exercise B.45** the last term $\Omega_{A/k[x,y]}$ is zero. So by the exactness of the sequence, we have a surjection $\Omega_{k[x,y]/k} \otimes_{k[x,y]} A \twoheadrightarrow \Omega_{A/k}$. **Exercise B.49** implies that we can simplify one of the tensor product factors as $\Omega_{k[x,y]/k} \cong k[x,y] \cdot dx \oplus k[x,y] \cdot dy$. Therefore $\Omega_{A/k}$ is a quotient of

$$(k[x,y] \cdot dx \oplus k[x,y] \cdot dy) \otimes_{k[x,y]} A \cong A \cdot dx \oplus A \cdot dy,$$

just like in **Example B.50**. If we knew the kernel of $A \cdot dx \oplus A \cdot dy \twoheadrightarrow \Omega_{A/k}$, then we would be able to say that $\Omega_{A/k}$ is the quotient of $A \cdot dx \oplus A \cdot dy$ by that kernel. So what is the kernel? Note that since $f = 0$ in A, the element $df \otimes 1$ in $\Omega_{k[x,y]/k} \otimes_{k[x,y]} A$ will map to 0 in $\Omega_{A/k}$. Thus one might hope to define a map of A-modules defined by $(f) \to \Omega_{k[x,y]/k} \otimes_{k[x,y]} A$. But there is a problem! It is unclear how to view (f) as an A-module; it is only a $k[x,y]$-module. The best we can do is base change to A via the tensor product $(f) \otimes_{k[x,y]} A \cong (f)/(f^2)$. Only now, one may verify that there is a well-defined map of A-modules giving rise to the following exact sequence.

$$(f)/(f^2) \to \Omega_{k[x,y]/k} \otimes_{k[x,y]} A \to \Omega_{A/k} \to 0$$

Remark B.55. The A-module $(f)/(f^2)$ in **Example B.54** above isn't necessarily the kernel. But all hope is not lost. In fact, the previous example is very enlightening; if we generalize it to the case of ring maps $k \to R \to S$ where $R \to S$ is a surjection, the first fundamental exact sequence reduces to the following.

$$\Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to 0$$

Repeating the argument from **Example B.54** in generality, we actually reach the second fundamental exact sequence.

Theorem B.56 (The Second Fundamental Exact Sequence)**.** *Let $R \to S$ be a surjective ring map of k-algebras. Let $I = \ker(R \to S)$. The following sequence of S-modules is exact.*

$$I/I^2 \xrightarrow{f \mapsto df \otimes 1} \Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to 0$$

Proof. See *The Stacks Project*, tag [00RU]. `https://stacks.math.columbia.edu/tag/00RU` □

Corollary B.57. *If A is a k-algebra of finite type, i.e., $A \cong k[x_1, \ldots, x_n]/(f_1, \ldots, f_s)$, then $\Omega_{A/k}$ is the cokernel of the following matrix.*

$$\begin{bmatrix} \frac{\partial f_i}{\partial x_j} \end{bmatrix}$$

Proof. Combine the second fundamental exact sequence where $R = k[x_1, \ldots, x_n]$ and $S = A$ with the observation that

$$df_i = \sum_j \frac{\partial f_i}{\partial x_j} dx_j.$$

□

Example B.58. Now our computation in **Example B.50** does indeed generalize. We have, for instance,

- if $A = k[x,y]/(x^2 - y^3)$, then $\Omega_{A/k} \cong (A \cdot dx \oplus A \cdot dy)/(2xdx - 3y^2dy)$.
- if $A' = k[x,y,z]/(xy, xz, yz)$, then $\Omega_{A'/k} \cong (A' \cdot dx \oplus A' \cdot dy \oplus A' \cdot dz)/(xdy + ydx, xdz + zdx, ydz + zdy)$.

Remark B.59. Let's use the fundamental exact sequences to help justify a statement we made in **Remark B.47** that the Kähler differentials tell us geometric differential information. We'll argue in the setting of locally ringed spaces. Suppose $(R, \mathfrak{m}, k)$ is a local ring R with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = k$, so it can be understood as in correspondence with a point x of some locally ringed space X. The resulting fundamental exact sequence gives a surjection $\varphi : \mathfrak{m}/\mathfrak{m}^2 \twoheadrightarrow \Omega_{R/k} \otimes_R k$. (Small exercise: check this!) In fact, φ is an isomorphism. To check this, we just need to check injectivity; by the fact that the Hom functor is left exact, it's equivalent to check that $\mathrm{Hom}_k(\Omega_{R/k} \otimes_R k, k) \to \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is surjective. I don't blame you if this feels like an approach you wouldn't naturally think to take. We do it because it turns out $\mathfrak{m}/\mathfrak{m}^2$ is what's called the cotangent space at x, and its k-vector space dual $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is the tangent space, so it's reasonable to look at. To show that $\mathrm{Hom}_k(\Omega_{R/k} \otimes_R k, k) \to \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is surjective, we need to show that any k-linear map $\psi : \mathfrak{m}/\mathfrak{m}^2 \to k$ lifts to a k-linear map $\Omega_{R/k} \otimes_R k \to k$. Define a map $R \to k$ by writing $r = a + b$ for $a \in k$ and $b \in \mathfrak{m}$, and (small exercise!) check that $x \mapsto \psi(b)$ is a well-defined derivation.

Exercise B.60 (★★★★☆)**.** Finish showing that $\mathrm{Hom}_k(\Omega_{R/k} \otimes_R k, k) \to \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is a surjection by an application of the universal property of $\Omega_{R/k}$.