Derived algebraic geometry and jet schemes

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Abstract

To any scheme X one can define a jet scheme $J^m X$ whose points should be thought of as points in X along with tangent directions up to order m. This perspective lends itself well to the setting of derived algebraic geometry, where deformation theory is more tractable.

In this talk, we'll discuss the derived setting via some well-known constructions, then discuss how one can upgrade the classical jet construction to the derived realm. Time permitting, we'll see that smoothness forces derived jet spaces to be homotopically discrete, suggesting the construction is a tool that can measure degree of singularity.

1 Homological Algebra

If we want to motivate the computations and the philosophy of derived algebraic geometry, we can start in the more familiar setting of derived functors of modules. Fix a ring R and recall that for a (right exact) functor $F : \mathbf{Mod}_R \to \mathbf{Mod}_R$, computing a (left) derived functor $\mathbf{L}F(M)$ amounts to finding a projective resolution $P_\bullet \to M$ and computing $\mathbf{L}F(M) \coloneqq F(P_\bullet)$. For example, $F(\_) = \_ \otimes_R N$ is a right exact functor, and for any M, computing a flat/projective/free resolution $P_\bullet \to M$ allows us to compute $M \otimes_R^{\mathbf{L}} N \coloneqq P_\bullet \otimes_R N$. This is a complex whose homology groups $h_i(M \otimes_R^{\mathbf{L}} N)$ are the Tor groups $\mathrm{Tor}_i^R(M, N)$.

One example where this computation is necessary is in computing multiplicity of intersections of algebraic varieties. We know from Bézout's theorem that if two curves C and C' meet transversely in $\mathbf{P}^2$, their intersection contains mn points, where $\deg C = m$ and $\deg C' = n$. If you allow for non-transverse intersections, then you have to consider the scheme theoretic intersection, but even this may not be enough. If C and C' are subvarieties of $\mathbf{P}^n$, it is not guaranteed that the intersection number of $p \in C \cap C'$ is

$$\dim_{\mathbf{C}}(\mathcal{O}_{C,p} \otimes_{\mathcal{O}_{\mathbf{P}^n,p}} \mathcal{O}_{C',p}).$$

Instead, due to Serre, it is

$$\sum_i (-1)^i \dim \mathrm{Tor}_i^{\mathcal{O}_{\mathbf{P}^n,p}}(\mathcal{O}_{C,p}, \mathcal{O}_{C',p}).$$

Thus a driving mantra for derived methods is that a derived functor tells us the whole picture that, due to failures of exactness, an underived functor may miss. In particular, intersections and deformations are ripe for derived considerations.

It's worth mentioning now that everything mentioned above is working thanks to some abstract nonsense due to Grothendieck, which says that in abelian categories, of which $\mathbf{Mod}_R$ is one (and is in fact the prototypical one), one has enough projectives and can compute resolutions and take homology of complexes.

2 The Nonabelian Setting

Since deformations would be nicer in a derived setting, we can turn to the functor $\Omega_{\_/k}$, the module of Kähler differentials of a k-algebra for a fixed ring k. The Kähler differentials tell us deformation theory, and like tensor product, there are half exact sequences.

Unfortunately though, unlike $\mathbf{Mod}_R$, the category $\mathbf{Alg}_k$ is not an abelian category! We would be out of luck, if not for a great theorem due to Dold and Kan:

Theorem (Dold-Kan '58)**.** *There is an equivalence of categories*

$$\mathbf{Ch}_{\geq 0}(\mathbf{Mod}_R) \leftrightarrow s\mathbf{Mod}_R$$

and homology of chain complexes corresponds to simplicial homotopy.

Just to remind us: a simplicial object, say a simplicial R-module, is a sequence of R-modules

$$M_\bullet = (M_0, M_1, M_2, \ldots)$$

with, for each n, maps $d_i : M_n \to M_{n-1}$ and $s_i : M_n \to M_{n+1}$, $0 \leq i \leq n$, that satisfy identities analogous to face and degeneracy maps of simplicial complexes. In fact, you are welcome to think of them as embedding a face into a higher degree simplex or collapsing a simplex down to one of its faces.

As a basic example, given a module M, you can consider the constant simplicial module where $M_n = M$ for all n and every face and degeneracy map is id_M. In fact, this gives an embedding $\mathbf{Mod}_R \hookrightarrow s\mathbf{Mod}_R$. This embedding has a left adjoint which is $\pi_0(\_)$ ($H_0(\_)$ if you're thinking chain complexes, which is of course adjoint to considering a module as a complex concentrated in degree 0).

So we have a candidate for a nonabelian version of derived methods! It is simplicial constructions and homotopy groups; namely we need to consider $s\mathbf{Alg}_k$, and resolutions of objects in this setting. For the Kähler differentials, a lovely construction due to Illusie tells us that we can compute the derived notion of $\Omega_{\_/k}$, the cotangent complex $\mathbf{L}\Omega_{\_/k} \coloneqq \mathbf{L}_{\_/k}$, via a Kan extension

$$\begin{tikzcd} s\mathbf{Alg}_k \arrow[r, "\mathbf{L}_{\_/k}"] & s\mathbf{Mod}_R \\ s\mathbf{Poly}_k \arrow[u] \arrow[ur, "\Omega_{\_/k}"'] & \end{tikzcd}$$

Fix $R_\bullet \in \mathrm{obj}(s\mathbf{Alg}_k)$, and take a polynomial resolution $P_\bullet \to R_\bullet$, and then $\mathbf{L}_{R_\bullet/k} \coloneqq \Omega_{P_\bullet/k} \otimes_{P_\bullet} R_\bullet$.

To do this properly, one needs to work in an ∞-category setting, which is where people tend to get scared off, but for our purposes, we'll treat the ∞-category of $s\mathbf{Alg}_k$ as simply simplicial k-algebras that are only considered up to homotopy equivalence. These are called *animated* k-algebras, and form the ∞-category $a\mathbf{Alg}_k$. There are some technical details to check that ensure this is okay, and while we won't do them in this talk, it bears mentioning that this works since polynomial k-algebras generate k-algebras via two categorical constructions called filtered colimits and reflexive coequalizers. A filtered colimit is a colimit over a filtered category - you can think direct limits - and a reflexive coequalizer is the colimit of a diagram

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend right, "s"'] \end{tikzcd}$$

where s is a common section of f and g; i.e., it is a way to equate two maps with a common section. It suffices though, everywhere we discuss animated algebras, roughly to think simplicially, with the freedom to change any object to one with the same homotopy type. Also, given two animated algebras $A_\bullet$ and $B_\bullet$, instead of a hom-set, we have a mapping space

$$\mathrm{Maps}_{a\mathbf{Alg}_k}(A_\bullet, B_\bullet),$$

again only defined up to homotopy. You can compute it directly though by considering a polynomial resolution $P_\bullet \to A_\bullet$ and then calculating the simplicial set $\mathrm{Hom}_{s\mathbf{Alg}_k}(P_\bullet, B_\bullet)$.

What exactly is a polynomial resolution? In the same way that a projective resolution $\widehat{P}_\bullet \to M$ is a complex of projective modules such that $P_\bullet$ is quasi-isomorphic to $[M]$, a polynomial resolution $P_\bullet \to R$ is a simplicial object composed of polynomial terms such that $P_\bullet$ is homotopy equivalent to R.

3 Jets

Recall, we have fixed a ring k. We build, in the affine k-scheme setting, jets. For a fixed algebra R and $m \in \mathbf{N}$, we have a functor $\mathrm{Jet}^m_R : \mathbf{Alg}_k \to \mathbf{Set}$ defined by

$$\mathrm{Jet}^m_R(A) \coloneqq \mathrm{Hom}_{\mathbf{Alg}_k}(R, A[t]/t^{m+1}).$$

Under moderate hypotheses on R (e.g., finite type), Jet^m_R is (co)representable by an algebra we call $J^m R$; i.e.,

$$\mathrm{Hom}_{\mathbf{Alg}_k}(R, A[t]/t^{m+1}) \cong \mathrm{Hom}_{\mathbf{Alg}_k}(J^m R, A).$$

The picture to have in mind is, e.g., taking $A = k$, we see that a map $J^m R \to k$, equivalently a point $\operatorname{Spec} k$ in $\operatorname{Spec} J^m R$, is the same as a $\operatorname{Spec} k[t]/t^{m+1}$ in $\operatorname{Spec} R$, which can be conceptualized as a point plus tangent directions up to order m (hence the name jet!). Using the representability criterion, we can compute specific examples; for instance, when $R = k[x_i]/(f_j)$ for $1 \le i \le d$ and $1 \le j \le s$, then an exercise: $J^m R \cong k[x_i, x_i', \dots, x_i^{(m)}]/(f_j, f_j', \dots, f_j^{(m)})$.

This construction glues to give, for any scheme X, the mth order jet scheme $J^m X$. And intuitively, jet spaces enjoy connections to deformations. Indeed, given an étale morphism $f : X \to Y$, then we get $J^m X \cong J^m Y \times_Y X$, and additionally, $J^1 X \cong \operatorname{Spec} \operatorname{Sym} \Omega_{X/k}$.

Now, we know how to consider a derived notion of $\Omega_{\_/k}$, which begs the question: can we derive $J^1\_$? Or, of course, $J^m\_$?

4 Derived Jets

As we've laid the foundation, there are two natural ways one might want to define a derived notation of a jet space. The first is to ask for the representing object of a derived notion of the Jet^m_R functor. That is, returning to the affine setting, you could define a functor where, when handed an animated algebra $R_\bullet$, we have

$$\mathrm{Jet}^m_{R_\bullet}(A_\bullet) \coloneqq \mathrm{Maps}_{a\mathbf{Alg}_k}(R_\bullet, A_\bullet[t]/t^{m+1}),$$

where $A_\bullet[t]/t^{m+1} \coloneqq A_\bullet \otimes^{\mathbf{L}}_k k[t]/t^{m+1}$. This looks like the functor Jet^m_R, but using derived constructions. We can ask if this functor is representable in $a\mathbf{Alg}_k$.

On the other hand, the construction $J^m\_$ is functorial, so like the cotangent complex, we could compute the derived functor of $J^m\_$, call it $\mathbf{L}J^m\_$. Namely, given an animated algebra $R_\bullet$, we compute $\mathbf{L}J^m R_\bullet \coloneqq J^m P_\bullet$ for a polynomial resolution $P_\bullet \to R_\bullet$. This also feels compelling as we know $J^1 X \cong \operatorname{Spec}\operatorname{Sym}\Omega_{X/k}$, so both sides should derive in a similar way.

Theorem 1. *Both of these constructions are well-defined, and furthermore, they are equivalent. Namely, when $R_\bullet$ is an animated algebra of finite type, the functors $\mathrm{Jet}^m_{R_\bullet}$ are corepresentable by $\mathbf{L}J^m R_\bullet$:*

$$\mathrm{Maps}_{a\mathbf{Alg}_k}(R_\bullet, A_\bullet[t]/t^{m+1}) \simeq \mathrm{Maps}_{a\mathbf{Alg}_k}(\mathbf{L}J^m R_\bullet, A_\bullet).$$

Ideas of the proof. Showing these constructions, $\mathrm{Jet}^m_{R_\bullet}\_$ and $\mathbf{L}J^m\_$, are well-defined amounts to checking that they commute with aforementioned filtered colimits and reflexive coequalizers. They do, so they each have left Kan extensions. To show that they are equivalent, it's enough to show the constructions agree on (non-derived!) polynomial k-algebras, since Kan extensions are unique. But of course they do; applying the adjunction between $\pi_0(\_)$ and the embedding $\mathbf{Alg}_k \hookrightarrow s\mathbf{Alg}_k$ reduces the computation to checking that $\mathrm{Jet}^m_{\pi_0 R_\bullet}$ is corepresentable by $J^m \pi_0 R_\bullet$, but that's just the non-derived corepresentability. $\square$

Of course, once one has the affine construction, we'd like to glue. In the derived setting, a derived scheme is locally derived affine (i.e., $\operatorname{Spec} R_\bullet$). You don't have prime ideals to define $\operatorname{Spec} R_\bullet$, but instead this is a space $\operatorname{Spec} \pi_0 R_\bullet$ with animated structure sheaf $\mathcal{O}$ where $\pi_i \mathcal{O}$ is a quasi-coherent sheaf on $\operatorname{Spec} \pi_0 R_\bullet$. Gluing is done in such a way that is homotopy-invariant.

There are fiddly details to worry about, but fortunately derived jets do glue.

Theorem 2. *Let X be a quasi-compact quasi-separated flat locally of finite type derived scheme over k. Let $m \in \mathbf{N}$. There exists a derived k-scheme $\mathbf{L}J^m X$ such that*

$$\mathrm{Maps}_{d\mathbf{Sch}_k}(\operatorname{Spec} A_\bullet[t]/t^{m+1}, X) \simeq \mathrm{Maps}_{d\mathbf{Sch}_k}(\operatorname{Spec} A_\bullet, \mathbf{L}J^m X).$$

We can also check that some of the deformation results for jet schemes that held in the ordinary setting have derived analogs.

Theorem 3. *If X is a derived scheme over k, then* $\mathbf{L}J^1X \simeq \operatorname{Spec}\mathbf{L}\operatorname{Sym}\mathbf{L}_{X/k}$.

Idea. As previously stated, $J^1\_$ and $\operatorname{Sym}\Omega_{\_/k}$ agree on polynomial algebras/affine space, so by uniqueness of their derived constructions and by some niceness hypotheses which are satisfied in this case, the result is shown. $\square$

Theorem 4. *If $f : X \to Y$ is an étale morphism of derived schemes over k, then the natural morphism $\mathbf{L}J^mX \to \mathbf{L}J^mY \times^{\mathbf{L}}_Y X$ is an equivalence.*

Idea. Much like in the non-derived realm, the definition of étale (in particular an infinitesimal lifting property), plus the representability criterion, gives the desired result. $\square$

Theorem 5. *If X is a smooth discrete scheme over k, then $\mathbf{L}J^mX$ is discrete for all m.*

Proof. Since X is smooth, it has étale coordinates. Let $U \subseteq X$ be affine. The following diagram commutes.

$$\begin{tikzcd}
U \arrow[r, hook, "\text{ét}"] \arrow[d, "\text{ét}"'] & X \arrow[d] \\
\mathbf{A}^d \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

By Theorem 4, we have two equivalences

$$\mathbf{L}J^m\mathbf{A}^d \times^{\mathbf{L}}_{\mathbf{A}^d} U \leftarrow \mathbf{L}J^mU \to \mathbf{L}J^mX \times^{\mathbf{L}}_X U.$$

Affine space $\mathbf{A}^d = \operatorname{Spec} k[x_1, \ldots, x_d]$ is polynomial, so $\mathbf{L}J^m\mathbf{A}^d = J^m\mathbf{A}^d$ is discrete. Thus $\mathbf{L}J^m\mathbf{A}^d \times^{\mathbf{L}}_{\mathbf{A}^d} U$ is discrete, and thus so is $\mathbf{L}J^mX \times^{\mathbf{L}}_X U$.

But we'd like $\mathbf{L}J^mX$ to be discrete. Recalling that fiber product is the categorical notion of intersection, we get to employ some topology describing the homotopy of spaces and their intersection; via the Mayer-Vietoris long exact sequence [Eckmann-Hilton, '64], we have

$$\cdots \to \pi_{i+1}X \to \pi_i(\mathbf{L}J^mX \times^{\mathbf{L}}_X U) \to \pi_i(\mathbf{L}J^mX) \oplus \pi_iU \to \pi_iX \to \cdots$$

Let $i > 0$. As X, U, and $\mathbf{L}J^mX \times^{\mathbf{L}}_X U$ are discrete, plugging those in forces $\pi_i\mathbf{L}J^mX$ to be zero, as we needed to show. $\square$

Not every $\mathbf{L}J^mX$ is discrete though, which is exciting - it suggests the homotopy groups of the derived jets are in some sense detecting distance from smoothness! Our current work involves creating computable examples, but, much as in the homological algebra setting we began with, while one proves theorems using projective resolutions, to compute them explicitly one typically needs some control of complexity. That is a harder nut to crack in the simplicial setting (I know of no "Hilbert Syzygy Theorem," for instance), but we have some promising ongoing work.