Introduction to Algebraic Geometry

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Algebraic Geometry (AG) is the science of encoding (resp. decoding) questions about complex manifolds into (resp. from) questions about commutative algebra. Below its surface, AG weaves a profound yet beautiful tapestry connecting number theory and complex geometry. On its surface, AG is very natural and very simple: given a system of polynomial equations with coefficients in some field K, I wish to know the solution set of this system over some field $k \subseteq K$. If this set is infinite and over $\mathbb{C}$, the best we can do is roughly sketch the real component of this solution set. The power of doing this is introduced to you at a very early age. If you ever drill calculus or teach college algebra, you will become familiar with taking some simple optimization problem, encode the data in the problem's instructions into one or more polynomials over $\mathbb{R}$, and then use derivatives or some special form to find the needed min or max. Let's generalize that process for just a moment:

1. Get data from problem
2. Encode problem into a polynomial when possible to do so
3. Graph as best you can
4. Look at graph for "interesting points" (such as local extrema or singularities)
5. Interpret these interesting points in terms of the original problem

1 Motivation for a Theory

Another important aspect of AG is its ability to provide concise generalizations for results in commutative algebra. Being able to impose a condition on a ring by imposing a condition on a complex manifold associated to said ring leads to wonderfully clever trades in difficulty of proofs. I think a proof is "difficult" when the reader must make a series of nontrivial leaps or guesses. In this sense, many proofs early on in a number theory course are very difficult because they involve the creation of complicated functions seemingly from scratch. The next exercise gives a controllable example of AG attempting to explain the inherent difficulty of making a helpful guess from scratch.

Example 1.1. It's a good exercise in "the path of least resistance" to show

$$\mathbb{C}[x,y]/(y-x^2-1) \cong \mathbb{C}[x]$$

as $\mathbb{C}$-algebras. It's best to naturally guess a hom

$$\varphi : \mathbb{C}[x,y] \to \mathbb{C}[x]$$

and then see if the kernel is the ideal $(y-x^2-1)$. Coming up with the right hom on the first try isn't impossible. We are compelled to make sure $y \mapsto x^2+1$ so that $(y-x^2-1) \subseteq \ker\varphi$. Next, we choose $x \mapsto x$ because its a safe guess. It's easy to see that this gives the needed isomorphism whence the reverse inclusion is checked. It would be nice if we had a *canonical* way of obtaining φ without having to guess and check. Geometry gives us a rough idea of what the "best" first guess should be. Consider figure 1 below and imagine projecting the quadratic curve vertically down. This is just coordinate projection and so we obtain a holomorphic map from one complex manifold to the other. These curves are complex manifolds but we have only graphed their real components in Desmos. This mapping is obviously a bijection which we will call f. The inverse is $f^{-1}(z) = (z, z^2+1)$ which is polynomial, hence holomorphic, in both coordinates and so f is bi-holomorphic. Back to commutative algebra, there is a way to obtain an isomorphism of algebras ψ from f. The trick is to identify the point $(z,0)$ on the line $y=0$ as the maximal ideal $(x-z)$ of $\mathbb{C}[x,y]/(y)$ and identify (z,w) on the quadratic as the maximal ideal $(x-z, y-w) = (x-z, x^2+1-w)$ of $\mathbb{C}[x,y]/(y-x^2-1)$ and then let f become the pullback of ψ. That is,

$$\psi^{-1}(x-z, y-w) = (x-z)$$

Here, f is trying to tell you that ψ can just kill off the vertical coordinate and leave the horizontal coordinate alone (provided you believe in these identifications which came from thin air). This completely ignores the

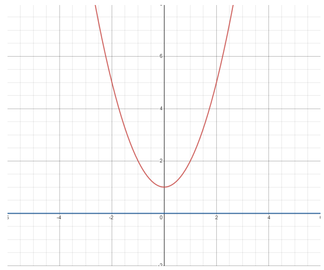


Figure 1: The quadratic curve projects onto the line $y = 0$

equation $y = x^2 + 1$. This is a consequence of f being projection from one curve to another *without ever leaving the plane* (the plane is just $\mathbb{C}^2$ for now). The most natural ring to associate to this line in the plane is

$$\mathbb{C}[x, y]/(y)$$

which is *much* easier to use than $\mathbb{C}[x]$ for defining ψ. So we actually have

$$\psi : \mathbb{C}[x, y]/(y) \to \mathbb{C}[x, y]/(y - x^2 - 1)$$

and from our data mining of f we've discovered the required relations $\psi(x) = x$ (leave the horizontal coordinate alone) and $\psi(y) = 0$ (kill-off the vertical coordinate). This only leaves one possible hom which must be defined as

$$\psi(h(x, y)) = h(x, 0). \tag{1}$$

The input h in the definition of ψ is actually a choice of representative for a particular element of $\mathbb{C}[x, y]/(y)$.

Exercise 1.2 (★★★☆☆)**.** Check that the ψ defined in line (1) is well-defined. Also check that ψ is an isomorphism of $\mathbb{C}$-algebras. Finally, mention how all of this shows $\mathbb{C}[x, y]/(y - x^2 - 1) \cong \mathbb{C}[x]$.

On one hand, we gave ψ without any algebraic guesswork. On the other hand, I gave us some completely unjustified identifications between points and maximal ideals. You might be thinking that the difficulty of guessing φ was just moved to the difficulty of knowing there were unjustified identifications of maximal ideals and points. Sure, but this isn't quite as difficult as guessing. In order for the above example to be a coherent mathematical argument, however, I would need to prove that there is a bijection between the set of all maximal ideals of $\mathbb{C}[x_1, ..., x_n]/(f_1, ..., f_m)$ and the solution set for the system consisting of the f_i. If no such bijection exists, then $\varphi^{-1}(x - z, y - w) = (x - z)$ only implies that $\varphi(x - z)$ lands *somewhere* in the ideal $(x - z, y - w)$, i.e. we can't know for sure that $\varphi(x) = x$. Another inherent problem with the last example is that the pullback of a maximal ideal isn't always maximal. In our case, every ideal we needed was generated by linear polynomials and we got lucky that pullbacks of linear polynomials just so happened to also be linear. Is there any hope of formalizing the ideas in example 1.1? Is there a rigorous mathematical way of specifying $\mathbb{C}$-algebra homomorphisms from graphs in Desmos? Who would even think to identify maximal ideals of rings with points on a manifold?

2 Hilbert and The New Frontier

The construction of a smooth (resp. complex) manifold is intrinsically tied to its smooth (resp. holomorphic) functions which send that manifold to $\mathbb{R}$ (resp. $\mathbb{C}$). This is a consequence of the local nature of these manifolds. At the turn of the 20th century, Hilbert paved the way for a similar theory to be developed for quotients of finitely generated polynomial rings. The main content of this theory between complex geometry and commutative algebra is the Hilbert Nullstellensatz Theorem which is used to create a bijection between maximal ideals of $\mathbb{C}[x_1, ..., x_n]$ and points of $\mathbb{C}^n$.

Closed sets in a chart of a smooth manifold can always be written in terms of vanishing coordinates of a smooth function. The same goes for closed sets on a complex manifold but with holomorphic functions. We will need an analogous notion of vanishing for elements of a quotient of a polynomial ring. Hence we say a set X in the set $\mathbb{C}^n$ is closed if X is the zero locus of a set of polynomials in $S = \mathbb{C}[x_1, ..., x_n]$.

The next exercise sketches a proof that this is a topology on the set $\mathbb{C}^n$. It is called the Zariski topology and it is crucial that the reader complete this exercise because it introduces important techniques and notation. When $\mathbb{C}^n$ is given the Zariski topology we refer to the resulting space as affine n-space and denote it by $\mathbb{A}^n$. When we wish to denote $\mathbb{C}^n$ in its usual metric topology we will continue to write $\mathbb{C}^n$.

Exercise 2.1 (★★★☆☆). Let k be a field. For an ideal I of $S = k[x_1, ..., x_n]$ define $Z(I)$ to be the zero locus of I in k^n. That is,

$$Z(I) = \{P \in k^n \mid f(P) = 0 \text{ for all } f \in S\}$$

A set X in k^n is Zariski closed if $X = Z(I)$ (this is the formal way of defining what has already been mentioned).

1. Let $\{I_\alpha\}_\alpha$ be a collection of ideals in S. Show

$$Z\left(\bigoplus_\alpha I_\alpha\right) = \bigcap_\alpha Z(I_\alpha)$$

2. Let I and J be ideals of S. Show

$$Z(IJ) = Z(I) \cup Z(J)$$

3. Show $\mathbb{A}^n$ and $\varnothing$ are of the form $Z(f)$ for appropriate $f \in S$.
4. Define $\mathbb{A}^n_k$ to be k^n with the above notion of Zariski closed sets. Conclude that $\mathbb{A}^n_k$ is a topological space. We call this space affine n-space over k. When $k = \mathbb{C}$ we let $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{C}}$.

We can give The Hilbert Nullstellensatz theorem after three more definitions. The third definition is given as an exercise.

Definition 2.2. Let I be an ideal of S. Define the radical of I denoted $\sqrt{I}$ to be the ideal of S which is generated by polynomials f for which some positive power of f is in I.

Definition 2.3. For a closed set X in $\mathbb{A}^n_k$ define $I(X)$ to be the ideal of $S = k[x_1, ..., x_n]$ generated by all polynomials whose vanishing set includes X.

Exercise 2.4 (★☆☆☆☆). A nonempty closed set X in $\mathbb{A}^n_k$ is irreducible if whenever $X = Z(I) \cup Z(J)$ it is the case that $I \subseteq J$ or $J \subseteq I$. Show that X is irreducible if and only if X is the zero set of a prime ideal.

Theorem 2.5 (Affine Nullstellensatz)**.** *Let k be algebraically closed and let $\mathfrak{a}$ be an ideal of $k[x_1, ..., x_n]$. If $f \in I(Z(\mathfrak{a}))$ then there exists a $r \geq 1$ such that $f^r \in \mathfrak{a}$.*

Proof. See Atiyah-Macdonald. □

Corollary 2.6 (Fundamental Theorem of Affine Space)**.** *Let k be algebraically closed. The operations $Z(-)$ and $I(-)$ satisfy the following:*

1. *Let X be a closed set in $\mathbb{A}^n_k$. Then $Z(I(X)) = X$.*
2. *Let J be an ideal of S. Then $IZ(J) = \sqrt{J}$.*
3. *$Z(-)$ is a bijection from the set of prime ideals of $k[x_1, ..., x_n]$ to the set of irreducible closed subsets of $\mathbb{A}^n_k$. Its inverse function is $I(-)$*
4. *The bijection in (3) restricts to a bijection from $\mathbb{A}^n_k$ to the set of maximal ideals of $k[x_1, ..., x_n]$.*

Remark 2.7. The algebraically closed condition is necessary because of examples like the following. Let $k = \mathbb{R}$. Consider the closed set $X = Z(x^2 + 1)$ in $\mathbb{A}^1_{\mathbb{R}}$. It's well known that X is empty. But $(x^2 + 1)$ is a maximal ideal.

With this theorem, techniques like the ones in example 1.1 can now be carried out in a rigorous mathematical way. This gives us a completely new way to tackle difficult questions from commutative algebra. For our first example of this, we will show how the Nullstellensatz makes quick work of difficult dimension theory questions from commutative algebra. This would be a good time to review some commutative algebra fundamentals.

By a ring, we will always mean a commutative ring with unity. A ring R is noetherian if all ideals of the ring are finitely generated as R-modules.

Definition 2.8. Let R be a ring (recall our conventions). A chain of prime ideals is a strict linear finite inclusion of prime ideals. The length of the chain

$$P_0 \subsetneq P_2 \subsetneq ... \subsetneq P_n$$

is n, the number of inclusions (not the number of ideals). The Krull dimension of R is the supremum of the set of all possible lengths of chains of prime ideals in R.

The Krull dimension of $S = k[x_1, ..., x_n]$ is n but a purely commutative algebra proof of this is a commitment. The first step uses a fundamental of commutative algebra called localization which sends the given ring to a new ring where all ideals except for one previously chosen prime ideal are the unit ideal. This new ring is called a local ring meaning it has only one maximal ideal, $\mathfrak{m}$. The next step is to have a conversation about why $V = \mathfrak{m}/\mathfrak{m}^2$ is a k-vector space. Finally one must show

$$\dim S \leq \dim_k V$$

which requires a fundamental result called Nakayama's lemma as well as a discussion of primary ideals and their generators. We will provide the theory of localization in great detail in the next section and primary ideals will be discussed in sufficient detail in section four. Using AG, we can prove $\dim S = n$ in the case that $k = \mathbb{C}$ with far less work than what's described above.

Definition 2.9. An affine algebraic variety (AAV) over k is an irreducible closed subset of $\mathbb{A}_k^n$. A quasi-affine algebraic variety (QAAV) is an open subset of an AAV in the subspace topology. For X an AAV, define the coordinate ring of X denoted $A[X]$ to be any k-algebra isomorphic to $k[x_1, ..., x_n]/I(X)$. The dimension of X is the length of the longest chain of irreducible subsets of X.

The quadratic curve and the x-axis are both AAV's over $\mathbb{C}$ in $\mathbb{A}^2$. Once we define morphisms of AAV's, we will see the x-axis in $\mathbb{A}^2$ is isomorphic to $\mathbb{A}^1$ and hence the quadratic curve is also isomorphic to $\mathbb{A}^1$. As another example, let

$$X = \mathbb{A}^2 - \{\text{origin}\}$$

By definition, X is a QAAV. Let U and V be the complements of the x and y axis, respectively, in $\mathbb{A}^2$. Then $X = U \cup V$. Furthermore, each of U and V have a natural bijection with the AAV $Z(yx - 1)$ in $\mathbb{A}^3$. You can see how this example suggests that AAV's could play the role of charts for QAAV's. We have not defined the notion of coordinate rings for QAAV's (because we need localization) but the hyperbola and U and V all share isomorphic coordinate rings.

Example 2.10. The longest chain of irreducible subsets of $\mathbb{A}_k^n$ is

$$Z(x_1, ..., x_n) \subsetneq Z(x_1, ..., x_{n-1}) \subsetneq ... \subsetneq Z(0)$$

which has length n. To verify that this chain is indeed the longest possible, notice that these are linear subsets of $\mathbb{A}_k^n$ meaning this chain determines a chain of vector spaces in $\mathbb{C}^n$.

Proposition 2.11. *Let X be an AAV over an algebraically closed field. Then* $\dim X = \dim A[X]$.

Proof. By the Nullstellensatz, prime ideals of $A[X]$ correspond to irreducible closed subsets of X. This gives a bijection

$$\{\text{chains of prime ideals of } A[X]\} \iff \{\text{chains of irreducible subsets of } X\}$$

and the proposition follows from 2.10. $\square$

AS mentioned earlier, AAV's play the same roll as charts on a manifold. They are primarily used to cover closed subsets of projective space which we will now define. Unless otherwise stated, let

$$S = k[x_0, ..., x_r]$$

where k is any field.

Definition 2.12. A ring R is $\mathbb{N}$-graded (or just graded provided no confusion may arise) if there is an isomorphism of additive abelian groups

$$\varphi : R \to \bigoplus_{n=0}^{\infty} G_n$$

satisfying the following: if $\varphi(x) \in G_a$ and $\varphi(y) \in G_b$ then $xy \in G_{a+b}$. Writing $R = \bigoplus_{n=0}^{\infty} R_0$ the abelian group R_n is called the degree n piece of R and a nonzero element of R_n is called a homogeneous element of degree n. The additive identity of R is not homogeneous. We say an ideal I of R is homogeneous if it satisfies any of the equivalent conditions in the following proposition.

Proposition 2.13. *If an ideal I of a homogeneous ring R. The following are equivalent:*

1. *I is generated by homogeneous elements of the same degree*
2. *There is an isomorphism of R-modules*

$$I \cong \bigoplus_{n=0}^{\infty} I_n$$

3. *The quotient R/I is a graded ring with grading*

$$R/I \cong \bigoplus_{n=0}^{\infty} R_n/I_n$$

Proof. See Atiyah-Macdonald □

> **Exercise 2.14 (★☆☆☆☆).** Prove that S is graded via the usual notion of degree of polynomials. Is this the only grading one can have on S?

We will now define projective space. Let $\mathbb{P}^r_k$ denote the set of points of the form

$$[a_0 : a_1 : ... : a_r]$$

for $a_i \in k$ provided at least one of the a_i are non zero modulo the relation

$$[a_0 : a_1 : ... : a_r] = [b_0 : b_1 : ... : b_r]$$

iff there is a nonzero $\lambda \in K$ for which $\lambda a_i = b_i$ for all $1 \leq i \leq r$. A choice of representative for an equivalence class under this relation is called a homogeneous coordinate. We topologize $\mathbb{P}^r_k$ by defining a closed set as a set of the form $Z(I)$ for I a homogeneous ideal in S along with the set $\mathbb{P}^r_k$. We require such I to be missing at least one of the x_i.

> **Exercise 2.15 (★★★☆☆).** Check that this indeed produces a topology on $\mathbb{P}^r_k$. Then, show that $\mathbb{P}^r_k$ is in bijection with the set of lines in $\mathbb{A}^{r+1}_k$ through the origin. Recall that a line in affine r space over k is the solution set of $r-1$ linear equations in r variables with coefficients over k.

Definition 2.16. The space $\mathbb{P}^r_k$ is called projective r-space over k. When $k = \mathbb{C}$ we omit the k unless stated otherwise. A projective algebraic variety (PAV) over k is an irreducible closed subset of $\mathbb{P}^r_k$ in the subspace topology. The coordinate ring of a PAV X is

$$A[X] = S/I(X)$$

where $I(X)$ denotes the homogeneous polynomials of S which vanish on X. A quasi-projective algebraic variety (QPAV) is an open subset of a PAV in the subspace topology. The dimension of a PAV or a QPAV is the length of its largest chain of irreducible closed subsets.

There is a homogeneous Nullstellensatz. We will state it now for completeness and its proof is usually given side by side with the affine Nullstellensatz.

Theorem 2.17 (Projective Nullstellensatz)**.** *Let k be algebraically closed and let $\mathfrak{a}$ be a homogeneous ideal of S which is missing at least one of the x_i. If $f \in I(Z(\mathfrak{a}))$ then there exists a $r \geq 1$ such that $f^r \in \mathfrak{a}$.*

Proof. See Atiyah-Macdonald. $\square$

Corollary 2.18 (Fundamental Theorem of Projective Space)**.** *Let k be algebraically closed. The operations $Z(-)$ and $I(-)$ satisfy the following:*

1. *Let X be a closed set in $\mathbb{P}^r_k$. Then $Z(I(X)) = X$.*
2. *Let J be a homogeneous ideal of S. Then $IZ(J) = \sqrt{J}$.*
3. *$Z(-)$ is a bijection from the set of homogeneous prime ideals of $k[x_1, ..., x_n]$ which are missing at least one x_i to the set of irreducible closed subsets of $\mathbb{P}^r_k$. Its inverse function is $I(-)$.*
4. *The bijection in (3) restricts to a bijection from $\mathbb{P}^r_k$ to the set of homogeneous maximal ideals of $k[x_1, ..., x_n]$ which are missing at least one x_i.*

The parity of the two fundamental theorems allows for an open cover of any PAV by a finite collection of AAV's.

Theorem 2.19. *Let U_i denote the set of points of $\mathbb{P}^r_k$ for which the i^{th} coordinate is nonzero. Then the U_i form an open cover of $\mathbb{P}^r_k$ and there are homeomorphisms*

$$U_i \cong \mathbb{A}^r_k$$

which restrict to homeomorphisms between AAV's and QPAV's. In particular, every PAV has an open cover of QPAV's which are homeomorphic to various AAV's.

Proof. Define

$$\varphi_i : U_i \to \mathbb{A}^r_k$$

by

$$\varphi_i\left(\left[\frac{a_0}{a_i} : ... : \frac{a_{i-1}}{a_i} : 1 : \frac{a_{i+1}}{a_i} : ... : \frac{a_r}{a_i}\right]\right) = \left(\frac{a_0}{a_i}, ..., \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, ..., \frac{a_r}{a_i}\right)$$

Checking that the U_i form an open cover of $\mathbb{P}^r_k$ and that the φ_i are homeomorphisms is left to the following exercise. Let X be a PAV in $\mathbb{P}^r_k$. Then any $Y_i = X \cap U_i$ is a QAAV in X. Let ψ_i be restriction of φ_i to Y_i. Since Y_i is closed in U_i, we see ψ_i is a homeomorphism from a QAAV to a AAV. $\square$

Exercise 2.20 (★★☆☆☆)**.** Show that the U_i in the proof of theorem 2.19 are open in $\mathbb{P}^r_k$. Show that the φ_i are homeomorphisms.

Example 2.21. The PAV $\mathbb{P}^1$ is covered by $U_i \cong \mathbb{A}^1$, $i = 0, 1$. More specifically

$$U_0 = \mathbb{P}^1 - \{[0:1]\} \quad U_1 = \mathbb{P}^1 - \{[1:0]\}$$

The point $[0 : 1]$ (resp. $[1 : 0]$) is called "a point at infinity" with respect to U_0 (resp. U_1). Its absence is due to the fact that it can't ever be reached, in some sense, via traveling a finite distance in $\mathbb{A}^1$. Taking an affine chart U_i on $\mathbb{P}^r$ gives "a hyperplane at infinity", i.e. after moving an infinite distance away from the origin in $U_i \cong \mathbb{A}^r$ one would find a copy of $\mathbb{A}^{r-1}$ missing from U_i.

Exercise 2.22 (★★★☆☆)**.** Let X be a PAV in $\mathbb{P}^r_k$ with its usuall affine algebraic cover U_i and charts $\varphi_i : U_i \to \mathbb{A}^r$.

1. Explain why the φ_i send irreducible closed subsets of X to irreducible closed subsets of $\mathbb{A}^r$.
2. Show that if Y is an irreducible closed subset of X then $Y_i = U_i \cap X$ are AAV's of the same dimension as Y.

3. Conclude that $\dim X = \dim A[X]$.

Exercise 2.23 (★★★★☆). Let k be algebraically closed. Let $f \in S = k[x_0, x_1, x_2]$ be a homogeneous and irreducible polynomial of degree d.

1. Explain why $X = Z(f)$ is a PAV. Use the previous exercise to conclude that X is a projective algebraic curve, i.e. a PAV of dimension one.
2. Show that any copy of $\mathbb{P}^1$ in $\mathbb{P}^2$ intersects $Z(f)$ in at most d distinct points.
3. A line $\mathbb{P}^1$ in $\mathbb{P}^2$ is said to be general if it intersects X in exactly d points. Let $f = x_2 x_1^2 - x_0^3$ and assume this polynomial is irreducible. Give an example of a general line and a non-general line. The examples of lines you give will need to be defined via charts.
4. Let f be a linear polynomial. Remark on why $Z(f)$ is homeomorphic to $\mathbb{P}^1$. Show that the only non-general line relative to this f is $Z(f)$.

3 The Category of Varieties over a field k

Up to this point, we have only been considering homeomorphisms between algebraic varieties. Unfortunately these maps are not specific enough to compare intricacies of algebraic varieties. For example the projective line $\mathbb{P}^1$ and the curve

$$x_2 x_1^2 = x_0^3$$

in $\mathbb{P}^2$ have charts into $\mathbb{A}^1$ which are homeomorphic to $\mathbb{A}^1$ but exercise 2.23 suggests that these algebraic varieties differ. However, exercise 2.23 can't very well be trusted because of the misleading nature of embeddings. That said, these curves are indeed different and one can prove this by checking that the coordinate rings of these algebraic varieties are not isomorphic. Unfortunately, the word "different" doesn't have a meaning yet because we haven't defined what it means for varieties to be "isomorphic". In this section, we will do just that but the discussion must start with a reformulation of our intuitive ideas about what a variety should be.

Hopefully someone has wondered why we use the word "algebraic" next to every definition involving the word variety. The reason is that all of these objects were originally defined *as subsets* of either $\mathbb{A}^r_k$ or $\mathbb{P}^r_k$. We do not have any examples of algebraic varieties which appear *outside* of $\mathbb{A}^r_k$ or $\mathbb{P}^r_k$ because the definitions of AAV and PAV do not allow for this. Without allowing for varieties to exist outside of these cases, we would be starving ourselves of intricate underlying relationships between AG and other areas of math. The first major example of this will be encountered in the section on rational maps which, among other results, will prove that structure preserving maps between curves are in correspondence to field extensions. The proof of this requires a discussion of curves whose points are in bijection with a certain class of discrete valuation rings. This connection wouldn't be possible if we only considered varieties as objects that are born of $\mathbb{A}^r_k$ or $\mathbb{P}^r_k$. Therefore, we must define a precursor of a variety independently of affine or projective space. This is accomplished by introducing locally ringed spaces. The road to these objects requires a deep understanding of sheaves and their morphisms. We have abbreviated the terminology as much as possible so that we may transition to varieties more quickly. To avoid terseness, we pad with examples throughout.

Definition 3.1. Let X be a topological space. A pre-sheaf of rings on X is an assignment

$$U \mapsto \mathcal{F}(U)$$

for all open $U \subset X$ where the various $F(U)$ are rings satisfying:

1. The empty set is sent to the zero ring
2. Inclusions of open sets $V \subseteq U$ are sent to ring homomorphisms

$$\mathcal{F}(U) \to \mathcal{F}(V)$$

which are called restriction maps with notation $s|_V$ denoting the image of s in $\mathcal{F}(V)$

3. The identity map on open sets is sent to the identity map on rings
4. If $V \subseteq U \subseteq W$ then for $s \in \mathcal{F}(W)$
$$(s|_U)|_V = s|_V$$

Elements of $\mathcal{F}(U)$ are called sections over U. The key to understanding pre-sheaves is to understand what "restriction" means in the context of how the sheaf is defined. We will practice with a few simple but important examples.

Example 3.2. Let $X = \mathbb{R}^2$ in its usual topology. Define $\mathcal{F}(U)$ to be the set of continuous functions from U into $\mathbb{R}$. Define the maps assigned to inclusions $V \subseteq U$ as literal function restriction. We must check that $\mathcal{F}(U)$ is a ring. This follows from the fact that continuity behaves well under addition and multiplication. Furthermore, we may define $\mathcal{F}(U) = 0$ when U is the empty set. The last axiom of a pre-sheaf, which is sometimes called the composition rule, is automatically satisfied because the needed ring homomorphisms are defined by literal function restriction and this always satisfies the composition rule. Thus $\mathcal{F}$ is a presheaf and restriction in the context of $\mathcal{F}$ is good ol' function restriction.

Example 3.3. Let $X = \mathbb{R}^2$ again. Let $\mathbb{Z}$ denote the presheaf of rings on X which sends all nonempty open sets to $\mathbb{Z}$ and sends the empty set to $0 \in \mathbb{Z}$. Define restriction maps to be the identity map when possible and let all other restriction maps be the trivial map. This is routinely checked to be a presheaf of rings on X. Let P be the origin in X. Define a presheaf of rings $\mathcal{G}$ on X which agrees with the presheaf $\mathbb{Z}$ whenever $P \in U$ and is trivial otherwise. Restriction maps follow the same convention as they did for the presheaf $\mathbb{Z}$. The first presheaf in this example is called the constant presheaf of rings at $\mathbb{Z}$ and the other presheaf is called the skyscraper sheaf of $\mathbb{Z}$ over P. Over the constant sheaf, restriction means "do nothing". Whereas in the skyscraper presheaf, restriction means "report $\mathbb{Z}$ if P is present and report zero otherwise".

Definition 3.4. Let X be a topological space. A sheaf of rings on X is a pre-sheaf of rings $\mathcal{F}$ satisfying the following: if (U_i, s_i) is an open cover of X with $s_i \in \mathcal{F}(U_i)$ which agree on intersections of the U_i, then there is a unique section $s \in \mathcal{F}(X)$ satisfying
$$s|_{U_i} = s_i$$
for all i.

This last axiom is called the gluing axiom and for good reason. When $\mathcal{F}$ is a sheaf of rings, we may "glue" sections together whenever the sections agree on an overlap, i.e. when
$$s|_{U \cap V} = t|_{U \cap V}$$
for $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$.

Exercise 3.5 (★★☆☆☆)**.** There are two examples of presheaves of rings in example 3.3 and one example in 3.4. Which, if any, of these three presheaves are also sheaves of rings?

Exercise 3.6 (★★★☆☆)**.** Let $X = \{P, Q\}$ in the discrete topology. Construct a presheaf of rings which is not a sheaf of rings.

The goal now is to construct a sheaf of rings for an arbitrary algebraic variety which gives us the ability to utilize the most fundamental commutative algebra technique: localization of rings. We will now briefly review localization. The exercises for the rest of this section are exceptionally punishing if avoided now. That said, most consist of routine fact checking from definitions.

Definition 3.7. Let R be a ring. Let S be a multiplicatively closed subset of R which includes 1. Define the set
$$S^{-1}R = \left\{ \frac{r}{s} \mid r \in R \text{ and } s \in S \right\} / \sim$$
wherein $\frac{r}{s} = \frac{a}{b}$ if and only if $0 = u(rb - sa)$ for some $u \in S$

Exercise 3.8 (★★☆☆☆)**.** The set $S^{-1}R$ is a ring under the usual notions of addition and multiplication of fractions.

Let P be a prime ideal of R. Define
$$R_P = (R - P)^{-1}R$$
which we call the localization of R at P (check that $R - P$ is multiplicatively closed with 1 if P is prime).

Definition 3.9. A local ring is a ring A with precisely one maximal ideal $\mathfrak{m}$. We refer to local rings as the pair $(A, \mathfrak{m})$.

Exercise 3.10 (★★☆☆☆)**.** Let P be a prime ideal of the ring R. Let $\mathfrak{m}$ denote the ideal of R_P generated by the image of P in R_P. Then I is not the unit ideal of R_P if and only if I is contained in $\mathfrak{m}$. Conclude that $(R_P, \mathfrak{m})$ is a local ring.

Let X be an AAV in $\mathbb{A}^r_k$. Let U be open in X. Let
$$s : U \to k$$
be a continuous function when k is given the topology of $\mathbb{A}^1_k$. Then s is regular at $P \in U$ if s agrees with a rational function $\frac{f}{g}$ for $f, g \in k[x_1, ..., x_r]$ in some neighborhood V of P such that g never vanishes over V. The function s is regular over U if s is regular at every point of U. Define a presheaf of rings $\mathcal{O}_X$ on X which assigns a nonempty U to the set of functions which are regular over U. Let the empty set be assigned the zero ring. Restriction is defined to be literal function restriction. Once its checked that the $\mathcal{O}_X(U)$ are indeed rings, the other axioms of a presheaf are a quick consequence of the fact that restriction is defined to be literal function restriction.

Exercise 3.11 (★★☆☆☆)**.** Check that $\mathcal{O}_X(U)$ is a ring.

Proposition 3.12. *Define $\mathcal{O}_X$ on X as above. Then $\mathcal{O}_X$ is a sheaf of rings on X.*

Proof. Let $s \in \mathcal{O}_X(U)$ and $t \in \mathcal{O}_X(V)$ agree over $U \cap V$. Let q be the result of gluing s and t. Then q is continuous. Suppose $P \in U \cup V$. Let W be a neighborhood of P in $U \cup V$ for which
$$s(x) = \frac{f}{g} \quad t(x) = \frac{a}{b}$$
for polynomials f, g, a, b with g and b never vanishing over U. We may also assume s and t agree over U or just take more intersections as necessary. These functions agree with q over W by construction. Since P was arbitrary, we obtain an open over of $U \cup V$ for which q restricts to a rational function over any set in this cover. For unicity, we need only use the well known fact that if r_1 and r_2 are rational functions without singularities over an open set U in X which agree on U, then they are equal on U. □

We must also define a sheaf of rings on PAV's. Let X be a PAV in $\mathbb{P}^r_k$. Let U be open in X. Let
$$s : U \to k$$
be a continuous function when k is given the structure of $\mathbb{A}^1_k$. Then s is regular at $P \in U$ if s agrees with a rational function $\frac{f}{g}$ for homogeneous $f, g \in k[x_1, ..., x_r]$ of the same degree in some neighborhood V of P such that g never vanishes over V. All other conventions for $\mathcal{O}_X$ are the same as the affine case. The proof of the previous proposition goes through just fine for the projective case of $\mathcal{O}_X$ despite the additional conditions on homogeneity and degree of polynomials. Without these conditions, regular functions on a PAV wouldn't be well defined: When s is regular at $P = [a_0, ..., a_r]$ there is some neighborhood U of P such that for any $Q \in U$ and $\lambda \neq 0$
$$s(\lambda Q) = \frac{f(\lambda Q)}{g(\lambda Q)} = \lambda^{\deg f - \deg g} \cdot \frac{f(Q)}{g(Q)} = 1 \cdot s(Q)$$

This precisely says in some neighborhood of P, the regular function s takes values independently of the choice of homogeneous coordinates.

We extend the affine (resp. projective) definition of $\mathcal{O}_X$ to QAAV's (resp. QPAV's). The following result is absolutely crucial. The proofs of which are found in chapter I.3 of Hartshorne.

Theorem 3.13. *Let X be an AAV. Then*

1. $A[X] = \mathcal{O}_X(X)$
2. *Let $P \in X$. Define $\mathcal{O}_{X,P}$ to be the set of regular functions at P modulo agreement in a neighborhood of P. Then $\mathcal{O}_{X,P}$ is a local ring whose maximal ideal $\mathfrak{m}_P$ consists of equivalence classes which can be represented by a regular function which vanishes at P.*
3. *Let $\mathfrak{m}$ be the maximal ideal of $A[X]$ which corresponds to P. There is an isomorphism of rings $\mathcal{O}_{X,P} \cong A[X]_{\mathfrak{m}_P}$*

Theorem 3.14. *Let X be a PAV. Then*

1. $A[X] = \mathcal{O}_X(X)$
2. *Let $P \in X$. Define $\mathcal{O}_{X,P}$ to be the set of regular functions at P modulo agreement in a neighborhood of P. Then $\mathcal{O}_{X,P}$ is a graded local ring whose homogeneous maximal ideal $\mathfrak{m}_P$ consists of equivalence classes which are represented by regular functions which vanish at P.*
3. *Let $\mathfrak{m}$ be the homogeneous maximal ideal of $A[X]$ which corresponds to P. There is an isomorphism of graded rings $\mathcal{O}_{X,P} \cong A[X]_{\mathfrak{m}_P}$.*

Example 3.15. We will now distinguish the affine line from the vanishing of $y^2 - x^3$ in $\mathbb{A}^2$ which we denote by X. Let P be a point on the affine line. Then

$$\mathcal{O}_{\mathbb{A},P} \cong A[X]_{\mathfrak{m}_P} \cong \mathbb{C}[x]_{(x-p)}$$

which has Krull dimension one and if $Q = (0,0)$ is the cusp of X

$$\mathcal{O}_{X,Q} \cong A[X]_{\mathfrak{m}_Q} \cong (\mathbb{C}[x,y]/(y^2 - x^3))_{(x)} \cong (\mathbb{C}[x,y]/(y^2 - x^3))_{(x,y)} \cong \mathbb{C}[t^2,t^3]_{(t^2,t^3)}$$

which has Krull dimension two. Thus the local rings on these two curves are distinct because every local ring on the affine line is dimension one since P was arbitrary. Later on, we will learn that these dimensions are actually tracking dimensions of tangent spaces to corresponding points.

It's a natural construction to obtain the underlying space of a variety from its corresponding prime ideal. However, this does little to explain how to obtain a sheaf of rings from this construction. Given a prime ideal $I \subseteq S = k[x_1, ..., x_n]$ let $X = Z(I)$. We know $\mathcal{O}_X(X) \cong S/I$. Every open subset of X is of the form $U \cap X$ for some open U of $\mathbb{A}^n_k$. Let $\mathcal{O}$ denote the structure sheaf of $\mathbb{A}^n_k$. Let $r : \mathcal{O}(\mathbb{A}^n_k) \to \mathcal{O}(U)$ denote restriction. Define

$$\mathcal{O}_X(U \cap X) = \mathcal{O}(U)/r(I)$$

Note that $r(I)$ is just the set of polynomials of S which vanish on X and are restricted to U.

Example 3.16. Let $f(x,y) = x$. Let $I = (f)$ in $S = \mathbb{C}[x,y]$. Let $X = Z(I)$ which is indeed a variety, hence a line in the plane. Let P denote the origin and let $U = \mathbb{A}^2 - P$. Then

$$\mathcal{O}_X(U \cap X) = \mathcal{O}(U)/r(I) \cong \mathcal{O}(U)/(f_U)$$

The ring $\mathcal{O}(U)$ is the set of all regular function on U, i.e. those functions which are locally rational. Thus

$$\mathcal{O}(U) \cong \mathbb{C}\left[x,y,u,v\right]/(xu-1, yv-1)$$

The open set U is sometimes called a distinguished set and, in some sense, the coordinate ring of any distinguished set is obtained by letting one or more variables become units. Hence we may write

$$\mathcal{O}(U) \cong \mathbb{C}\left[x, y, \frac{1}{x}, \frac{1}{y}\right]$$

so that x and y are units but, for example, $x-1$ and $y-1$ are not units.

Recall that the original goal was to define a variety without ever mentioning affine or projective space. For this to be possible, we need a “holding tank” of sorts for these more abstract varieties. This is where the definition of locally ringed space comes into play. We briefly mention that the process of taking equivalence classes to form $\mathcal{O}_{X,P}$ on an algebraic variety X works for arbitrary presheaves of rings on any space. The ring $\mathcal{O}_{X,P}$ is called the stalk at P. However, stalks aren't always local rings, as the next definition implies.

Definition 3.17. A locally ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ whose associated stalks $\mathcal{O}_{X,P}$ are local rings.

The previous two theorems tell us that algebraic varieties are indeed locally ringed spaces. Let $(X, \mathcal{O}_X)$ be a locally ringed space. What conditions should we impose on $(X, \mathcal{O}_X)$ in order to obtain a variety? First, X should be homeomorphic to an algebraic variety Y. Second, the local rings of both spaces should all line up via that homeomorphism. Turns out, the correct definition of isomorphisms of varieties is a bit special because its a homeomorphism f which actually induces an isomorphism of local rings via the pullback map $s \mapsto s \circ f$. There is no analogous situation for arbitrary locally ringed spaces. Hence we must decouple the homeomorphism from the induced maps on local rings when defining structure preserving maps between locally ringed spaces.

Definition 3.18. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be locally ringed spaces. Let $f : X \to Y$ be a continuous function. The pushdown of f is the sheaf of rings $f_*\mathcal{O}_X$ on Y defined by

$$f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$$

Its easy to check that the pushdown is indeed a sheaf of rings on Y. Note that this definition wouldn't make sense without the continuity assumption on f. Furthermore, $f_*\mathcal{O}_X$ has nothing to do with $\mathcal{O}_Y$. The point of using the pushdown is to give two sheaves of rings on the same space, namely Y in our case. This allows us to compare sheaves.

Definition 3.19. A morphism of locally ringed spaces is a pair $(f, f^\sharp)$ where $f : X \to Y$ is continuous and $f^\sharp$ is an indexing of homomorphisms of rings,

$$f^\sharp(U) : \mathcal{O}_Y(U) \to f_*\mathcal{O}_X(U)$$

for U open in Y such that for all open $V \subseteq U$ the diagram

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{f^\sharp(U)} & f_*\mathcal{O}_X(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_Y(V) & \xrightarrow{f^\sharp(V)} & f_*\mathcal{O}_X(V) \end{array}$$

commutes (the vertical maps are restriction). In addition, we require the limit of every $f^\sharp(U)$ to be a local homomorphism, i.e. the induced map

$$f^\sharp_P : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$$

must satisfy

$$(f^\sharp_P)^{-1}(\mathfrak{m}_P) = \mathfrak{m}_{f(P)}$$

An isomorphism of locally ringed spaces is a morphism $(f, f^\sharp)$ such that f is a homeomorphism and every $f^\sharp(U)$ is an isomorphism.

The induced map in the previous definition comes from a construction which we have not seen yet. Given a point in a locally ringed space $P \in X$ there are two ways to obtain the local ring $\mathcal{O}_{X,P}$. The first way is via taking the set of all functions which are regular at P modulo agreement over a neighborhood of P. The other way comes from taking the direct limit. We will introduce a special case of the direct limit now. Given a sequence of ring homomorphisms

$$A_0 \to A_1 \to ...$$

$$d_i : A_i \to A_{i+1}$$

define $A = \lim_i A_i$ to be the unique ring satisfying the following universal mapping property: if $\varphi : B \to A_i$ then there is a unique $\psi : B \to A$ such that

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & A_j \arrow[d, "d_j"] \\ & A \end{tikzcd}$$

commutes. Existence is not obvious but the construction can be found on wikipedia. Uniqueness comes from a simple but clever use of the universal property. Given a different linear sequence (Bi, d_i) with maps $A_i \to B_i$ there is a unique map $A \to B$ where B is the direct limit of the B_i.

Fix P in a locally ringed space X. Let

$$U_0 \supseteq U_1 \supseteq ...$$

be a sequence of inclusions of open sets with $P \in U_i$ for all $i \geq 0$. Taking restriction maps gives

$$\mathcal{O}_X(U_0) \to \mathcal{O}_X(U_1) \to ...$$

meaning there is a direct limit to associate to P provided this sequence contains all other linear sequences of inclusion of open sets. Turns out, $\mathcal{O}_{X,P}$ is this direct limit. Let

$$(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

be a morphism of locally ringed spaces. Fix $P \in X$. Let U_i denote a sequence of inclusions of open sets in Y such that $f(P) \in U_i$ for all $i \geq 0$. Then there are ring homs

$$f^\sharp(U_i) : \mathcal{O}_Y(U_i) \to f_*\mathcal{O}_X(U_i)$$

which give rise to a unique ring hom $f^\sharp_P$ via taking the direct limit, i.e.

$$f^\sharp_P : \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$$

This is a direct consequence of the fact that

$$f^{-1}(U_0) \supseteq f^{-1}(U_1) \supseteq ...$$

and all of these open sets contain P. With this, the definition of a morphism is simply asking for two things:

1. taking $f^\sharp$ of a section must commute with restriction
2. the unique ring homs $f^\sharp_P$ must satisfy $(f^\sharp_P)^{-1}(\mathfrak{m}_P) = \mathfrak{m}_{f(P)}$

The second condition is what was meant by "we require the limit of every $f^\sharp(U)$ to be a local homomorphism". This condition is awkward to include but necessary because we needed a canonical way of constructing ring homs between stalks and the direct limit is the industry standard.

After 3 pages of definitions, we can finally define the category of varieties.

Definition 3.20. A variety over k is a locally ringed space which is isomorphic to an AAV, PAV, QAAV, or QPAV over k. An affine variety (resp. projective variety) is any locally ringed space which is isomorphic to an AAV (resp. a PAV). A morphism of varieties is a morphism of locally ringed spaces.

Exercise 3.21 (★★★☆☆). Prove that compositions of morphisms of locally ringed spaces are morphisms of locally ringed spaces.

Exercise 3.22 (★★☆☆☆). Let X and Y affine varieties over k. Let $\varphi : A[Y] \to A[X]$ be a k-algebra hom. Define

$$f : X \to Y \quad f(P) = \varphi^{-1}(I(P))$$

and define

$$f^\sharp(U) : \mathcal{O}_Y(U) \to f_*\mathcal{O}_X(U) \quad f^\sharp(U)(s) = s \circ f$$

Show that $(f, f^\sharp)$ is a morphism of varieties. Conclude that giving a k-algebra hom is equivalent to giving a morphism of varieties.

Remark 3.23. In the previous exercise, $f^\sharp$ is defined by the classic pullback map from topology, i.e. $s \mapsto s \circ f$. This is not the only possible $f^\sharp$ that one can define from a continuous f (recall that f and $f^\sharp$ were decoupled in the definition of a morphism). However, this particular $f^\sharp$ is special because it gives the needed equivalence between algebra homs and morphisms of varieties in exer. 3.21. For an arbitrary morphism $(g, g^\sharp)$ of locally ringed spaces $g^\sharp$ does not have to be $s \mapsto s \circ g$. This is quite common in examples outside of the category of varieties.

4 Rational Maps

At this point we've encountered different varieties, and we have a certain kind of map between two varieties, called a morphism, which you can recall is a continuous map $X \to Y$ where pulling back a regular function on Y gives you a regular function on X.

In the following sections we're going to introduce a different kind of map, called a rational map, between varieties, which will allow us a little bit more nuance. The basic idea is that a rational map will only be a partial function – we won't map all of X to Y, in the same way that the "function" $1/x : \mathbf{R} \to \mathbf{R}$ doesn't actually have domain $\mathbf{R}$. But because of the rigidness of varieties, rational maps are still going to give us a lot of information even if they're only defined on part of a variety. This nuance is important for later results we might see, like the classification of varieties or for resolutions of singularities, just to name-drop two important ideas so that you believe we have some reason to care. Otherwise, why worry about anything other than honest morphisms between varieties, right?

Before we get started, let's consider two salient ideas:

1. The nonempty open subsets of a(n irreducible)[1] variety are dense (*big*!).
2. If you have a morphism, then it's determined by a nonempty open subset. In other words:

Lemma 4.1. *If $\varphi : X \to Y$ and $\psi : X \to Y$ are two morphisms of varieties X and Y such that there is a nonempty open subset $U \subseteq X$ such that $\varphi|_U = \psi|_U$, then $\varphi = \psi$.*

Exercise 4.2 (★★☆☆☆). Prove **Lemma 4.1**. Hint: one such proof could take the following steps:

0. Since Y is a variety, it lives inside of $\mathbf{P}^n$ for some n. Without loss of generality, we can compose φ and ψ with the inclusion map $Y \hookrightarrow \mathbf{P}^n$, so it suffices to prove **Lemma 4.1** in the case where $Y = \mathbf{P}^n$.
1. Show the map $\varphi \times \psi : X \to \mathbf{P}^n \times \mathbf{P}^n$, $x \mapsto (\varphi(x), \psi(x))$ is continuous.
2. Show that the diagonal $\Delta = \{(p, p)\} \subseteq \mathbf{P}^n \times \mathbf{P}^n$ is closed.
3. Using the hypotheses of the lemma, where does $\varphi \times \psi(U)$ live inside of $\mathbf{P}^n \times \mathbf{P}^n$?
4. Use the fact that $U \subseteq X$ is dense and Δ is closed to state where $\varphi \times \psi(X)$ lives in $\mathbf{P}^n \times \mathbf{P}^n$.
5. Interpret step 4 as the conclusion to the lemma.

One additional way to motivate rational maps is that they're trying to ask and answer the converse to number 2 above. If we do have a function that's only defined on a dense open subset, can it extend to a map on the whole space? The answer is actually "no, not always," but even if not, rational maps suffice to say a lot.

Now let's begin by actually defining a rational map.

[1] Our varieties are, by definition, irreducible, but as you may know that's not a universal convention. So sometimes I'll specifically point out the irreducibility condition whenever it's relevant to the claim I'm making. Not always though because the parentheticals get cumbersome.

Definition 4.3. Let X and Y be two varieties. A **rational map** from X to Y is a pair of an open subset $U \subseteq X$ and a morphism $\varphi_U : U \to Y$. We write $\langle U, \varphi_U \rangle$ for this pair. We also put an equivalence relation on the set of pairs $\langle U, \varphi_U \rangle$, where $\langle U, \varphi_U \rangle$ is equivalent to $\langle V, \varphi_V \rangle$ if φ_U and φ_V agree in the only place they could, $U \cap V$. Sometimes you'll see the notation $\varphi : X \dashrightarrow Y$ to denote a rational map.

Right off the bat there's some work to be done with this definition.

> **Exercise 4.4** (★☆☆☆☆)**.** Check that the relation we just described on the pairs $\langle U, \varphi_U \rangle$ is indeed an equivalence relation; that is, it is symmetric, reflexive, and transitive. Hint: use **Lemma 4.1**.

Okay, even once you've done **Exercise 4.4**, this is still a pretty esoteric definition right now. A rational map $X \dashrightarrow Y$ is a class of an open subset $U \subseteq X$ and a morphism $\varphi_U : U \to Y$, which means we're not dealing with functions on the nose but equivalence classes of both subsets and morphisms. Yuck! Fortunately though, we can still say a lot. First let's just cook up some examples, without worrying about the equivalence relation yet.

Example 4.5. Any morphism $\varphi : X \to Y$ is a rational map. One representative of the equivalence class is simply $\langle X, \varphi \rangle$.

Example 4.6. Let's see a more interesting example. Let $\mathbf{A}^2$ have coordinates (x, y) and we'll cook up a rational map $\mathbf{A}^2 \dashrightarrow \mathbf{A}^1$ that is not a morphism. It'll be the map $(x, y) \mapsto y/x$, with open subset $D(x) = \{(x, y) \in \mathbf{A}^2 \mid x \neq 0\}$.

Remark 4.7. Let's turn our attention to the equivalence relation now, and see how it actually helps simplify the picture, rather than being a cumbersome piece of data to carry around.

It actually is the case that I could be a bit sloppy in **Example 4.6** and get away with simply saying $\mathbf{A}^2 \dashrightarrow \mathbf{A}^1$ is $(x, y) \mapsto y/x$. In other words, I could omit the open subset, even though the definition requires us to carry it around. The reason for that is because $D(x)$ is the largest open subset that permits $(x, y) \mapsto y/x$ to be a morphism. Extending it any further introduces 0s in the denominator!

But now you might ask, what about a smaller open subset? And this is where the equivalence relation kicks in, and simplifies the picture. If we did something rather silly and chose a different open subset like $D(xy) = \{(x, y) \in \mathbf{A}^2 \mid xy \neq 0\}$, then $D(x)$ and $D(xy)$ are two open subsets where $(x, y) \mapsto y/x$ is defined – and must agree with itself! – on their intersection, which is just $D(xy)$ since $D(xy) \subseteq D(x)$.

And really $D(xy)$ was an arbitrary example; the same argument works for any open subset where the morphism $(x, y) \mapsto y/x$ is defined. So in other words, if we fix the morphism part of a rational map, then we may permit ourselves to occasionally be sloppy and omit the open subset because it's implied to be maximal.

Definition 4.8. Given a rational map $\varphi : X \dashrightarrow Y$, the largest open subset on which φ is defined is called the **domain of** φ, and the complement of the domain is called the **locus of indeterminancy**.

Here's some more examples.

Example 4.9. Consider the rational map $\mathbf{P}^2 \dashrightarrow \mathbf{P}^1$ defined by $[x : y : z] \mapsto [x : y]$.

> **Exercise 4.10** (★☆☆☆☆)**.** What is the locus of indeterminancy of the rational map in **Example 4.9**? Can that rational map extend to a honest morphism $\mathbf{P}^2 \to \mathbf{P}^1$?

Example 4.11. Here's an example where we can see a bit of a complication re-arise with regards to the equivalence classes. What happens if we don't a priori fix one defining equation for the morphism? It turns out that the equivalence class doesn't just carry the data of the domain of the rational map, but also the data of it as a morphism too. Consider the variety $V(xw - yz) \subseteq \mathbf{A}^4 = \{(x, y, z, w)\}$. Let's define a rational map $V(xw - yz) \dashrightarrow \mathbf{A}^1$. It'll be defined to be

$$(x, y, z, w) \mapsto \begin{cases} x/y & \text{if } y \neq 0 \\ z/w & \text{if } w \neq 0. \end{cases}$$

Now notice that when both $y \neq 0$ and $w \neq 0$, we get $x/y = z/w$, because on our variety, $xw - yz = 0$ and therefore $x/y = z/w$ via cross multiplication. So this is indeed a well-defined rational map with domain $\{(x, y, z, w) \mid y \neq 0 \text{ or } w \neq 0\}$. This means that our $V(xw - yz) \dashrightarrow \mathbf{A}^1$ is a rational map that can't be described as a single morphism – it must be defined piecewisely, and the equivalence relation allows us to see that if you define it as $\langle D(y), x/y\rangle$ and I define it as $\langle D(w), z/w\rangle$, we are each defining the same rational map (whence the same equivalence class).

5 The Category of Varieties with Dominant Rational Maps

☡ **Warning! 5.1.** Without the soon-to-be-defined notion of "dominant" rational maps, we won't be able to get off the ground. The attempt to define a category given by "the objects are varieties and the arrows are rational maps" *doesn't* actually form a category! That's because we can't necessarily compose rational maps, since they're not defined everywhere. It could be the case that the image of the first rational map might lie in the locus of indeterminancy of the second.

Exercise 5.2 (★☆☆☆☆)**.** Consider the morphism $\varphi : \mathbf{A}^2 \to \mathbf{A}^3$ defined by $(x, y) \mapsto (x^2, xy, y^2)$ thought of as a rational map. Consider the rational map $\psi : \mathbf{A}^3 \dashrightarrow \mathbf{A}^1$ defined by $(a, b, c) \mapsto \frac{1}{ac-b^2}$. Explain why the composition $\psi \circ \varphi : \mathbf{A}^2 \dashrightarrow \mathbf{A}^1$ is not defined.

Fortunately all hope is not lost, as long as we impose one more condition on our rational maps.

Definition 5.3. We say that a rational map $\varphi : X \dashrightarrow Y$ is **dominant** if for some (and hence every) pair $\langle U, \varphi_U\rangle$, the image of φ_U is dense in Y.

Exercise 5.4 (★★☆☆☆)**.** Explain why you can compose *dominant* rational maps.

Now our title has been justified. The category we get contains (irreducible) varieties as objects and dominant rational maps as arrows. And,

Definition 5.5. When we have an isomorphism in this category, which is simply to say we have a rational map $\varphi : X \dashrightarrow Y$ with an inverse $\psi : Y \dashrightarrow X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$, we call φ (and ψ) a **birational map** and call X and Y **birationally equivalent** (or sometimes just **birational** to each other).

Exercise 5.6 (★★★★☆)**.** In the last section we listed a lot of examples of rational maps.

- Any morphism $X \to Y$ (**Example 4.5**)
- $\mathbf{A}^2 \dashrightarrow \mathbf{A}^1$, $(x, y) \mapsto y/x$ (**Example 4.6**)
- $\mathbf{P}^2 \dashrightarrow \mathbf{P}^1$, $[x : y : z] \mapsto [x : y]$ (**Example 4.9**)
- $V(xw - yz) \dashrightarrow \mathbf{A}^1$, $(x, y, z, w) \mapsto x/y$ or z/w (**Example 4.11**)

Of those that were dominant (which ones were those?), which of them are birational maps? What are their inverses?

Example 5.7. Depending on how much you do of **Exercise 5.6**, you may (or may not!) now have several examples of birational maps. Here's at least one more for your benefit. Consider the variety $V(y^2 - x^3) \subseteq \mathbf{A}^2$. We'll show that $V(y^2 - x^3)$ is birationally equivalent to $\mathbf{A}^1$ (which we'll give coordinate t). In one direction, we define $\varphi : \mathbf{A}^1 \dashrightarrow V(y^2 - x^3)$ by $t \mapsto (t^2, t^3)$, and in the other direction, define $\psi : V(y^2 - x^3) \dashrightarrow \mathbf{A}^1$ by $(x, y) \mapsto y/x$. Indeed, see that

$$\psi \circ \varphi(t) = \psi(t^2, t^3) = \frac{t^3}{t^2} = t, \text{ and}$$

$$\varphi \circ \psi(x, y) = \varphi\left(\frac{y}{x}\right) = \left(\left(\frac{y}{x}\right)^2, \left(\frac{y}{x}\right)^3\right) = \left(\frac{y^2}{x^2}, \frac{y^3}{x^3}\right) = \left(\frac{x^3}{x^2}, \frac{y^3}{y^2}\right) = (x, y).$$

Therefore $V(y^2 - x^3)$ and $\mathbf{A}^1$ are birationally equivalent, as desired.

Exercise 5.8 (★★★☆☆)**.** If you're avoiding **Exercise 5.6** (don't!), but at the very least, you should at least do one example. Show that $V(x^3 + y^3 - xy)$ and $\mathbf{A}^1$ are birationally equivalent.

Remark 5.9. We're knee deep in the mathematics currently, but let me just point out that heuristically, **Exercise 5.8** is believable because when we draw $V(x^3+y^3-xy)$, it looks like an $\mathbf{A}^1$ with a knot in it! This is the intuition that you want to have kinda half-stewing in the back of your mind, and it's something that textbooks tend to omit unless they draw lots of pictures. Hopefully the speaker for these notes (whether that's me or someone else) is self-aware enough to draw these pictures for you. If not, ask me about them! In fact, this intuition gets to be made precise by the time we get to **Corollary 5.20**; for now, let me give you the slogan to have in mind that jives with the story the pictures tell: two varieties are birational if and only if they have isomorphic (dense) open subsets.

Definition 5.10. We call a variety **rational** if it is birationally equivalent to $\mathbf{P}^n$ for some n. I'm not super fond of this; it may take more words to say "birationally equivalent to some $\mathbf{P}^n$" but it's explicit. Ho hum.

Exercise 5.11 (★★☆☆☆)**.** Show that $\mathbf{P}^n \times \mathbf{P}^m$ is rational by constructing an explicit birational map $\mathbf{P}^n \times \mathbf{P}^m \dashrightarrow \mathbf{P}^{n+m}$. As a corollary, show that if X and Y are rational, then $X \times Y$ is rational. Hint: this is an exercise in combinatorics.

We've seen some specific examples of rational maps and how they may not be morphisms, but now we have a notion of birational equivalence, just like we have a notion of isomorphism of varieties. Do these coincide; i.e., if two varieties are birationally equivalent, need they be isomorphic?

Exercise 5.12 (☆☆☆☆☆)**.** *Certainly* the converse is true; if X and Y are isomorphic then they are birationally equivalent. Why?

It turns out that the answer to our question is "no." (– which is good! We're not retreading old ground.) There are varieties that are birationally equivalent but not isomorphic.

Exercise 5.13 (★★☆☆☆)**.** Prove that $\mathbf{P}^1$ and $\mathbf{A}^1$ are birationally equivalent, but not isomorphic.

Also, now that we have a category, another thing you might have on your mind is: have we seen this category before? In the same way that, for example, finite dimensional $\mathbf{R}$-vector spaces and matrices over $\mathbf{R}$ describe the same thing in different languages, maybe this category of varieties and dominant rational maps is equivalent to a different category we can describe? The answer is yes! (In fact, this question and its affirmative answer will pop up many times throughout algebraic geometry; see the remark after **Theorem 5.14** for just how big a deal this is.)

Theorem 5.14. *There is an equivalence of categories*

$$\{\textit{varieties and domainant rational maps}\} \leftrightarrow \{\textit{finitely generated field extensions of } \mathbf{C}\}.$$

Remark 5.15. First, a small comment: We know how to define varieties over algebraically closed fields other than $\mathbf{C}$. In this setting, **Theorem 5.14** remains true, replacing $\mathbf{C}$ by your field.

Next, a big comment: **Theorem 5.14** is remarkable! An equivalence of categories is an incredibly strong thing; it's basically the magic that makes algebraic geometry happen. What it means is that any question you would want to ask or answer in one category can be asked and answered in the equivalent category. A question that is hard in one category might be easy in the other, and just the ability to translate itself can motivate questions. This is exactly the same magic that we saw on day 1 when we took a $\mathbf{C}$-algebra like $\mathbf{C}[x,y]/(y-x^2)$ and turned its maximal ideals into points on the parabola $y = x^2$!

*Proof of **Theorem 5.14**.* To show this, we need to produce:

- a field extension, if we're given a variety,

- a map of fields, if we're given a dominant rational map, and
- a way to do both of these things in reverse.

Let's start by producing a field extension, given a variety. That part is easy because we've already done it in a past talk. Let X be a variety; we get a field by taking the function field $K(X)$. Recall

$$K(X) = \{\langle U, f\rangle \text{ a rational function on } X \mid U \subseteq X \text{ and } f \text{ is regular on } U\}$$

modulo the relation $\langle U, f\rangle \sim \langle V, g\rangle$ if $f = g$ on $U \cap V$.

☡ **Warning! 5.16.** This is the first time we've reintroduced rational *functions* in these notes. Keep them distinct from rational *maps*! Their definitions are dangerously similar! In fact, that's not a coincidence:

Exercise 5.17 (★★☆☆☆). Definition/sanity check: show that a rational function in $K(X)$ is the same as a rational map $X \dashrightarrow \mathbf{A}^1$.

The function field $K(X)$ is a field extension of $\mathbf{C}$, so we are done with this part. In fact, let us note here that we're really showing an equivalence of categories

$$\{\text{varieties and domainant rational maps}\} \leftrightarrow \{\text{function fields } K(-)\},$$

and that the latter is in fact the finitely generated field extensions of $\mathbf{C}$.

Moving on, given a dominant rational map $\varphi : X \dashrightarrow Y$ which we'll write $\langle U, \varphi_U\rangle$, we need to produce a map of fields. We'll actually do so "contravariently," which means that our output will be a map which goes $K(Y) \to K(X)$ – the direction has been reversed. Here's how we get it. Let $\langle V, f\rangle \in K(Y)$ be a rational function, where $V \subseteq Y$ and f is regular on V. Since $\langle U, \varphi_U\rangle$ is dominant, $\varphi_U(U)$ is dense in Y, so $\varphi_U^{-1}(V)$ is a nonempty open subset of X. Since φ_U is a morphism, $f \circ \varphi_U$ is regular on $\varphi_U^{-1}(V)$, and hence it defines an equivalence class $\langle \varphi_U^{-1}(V), f \circ \varphi_U\rangle \in K(X)$. So that's our map; given a dominant rational map $\langle U, \varphi_U\rangle : X \dashrightarrow Y$, we get a map $K(Y) \to K(X)$ defined by

$$\langle V, f\rangle \mapsto \langle \varphi_U^{-1}(V), f \circ \varphi_U\rangle.$$

Exercise 5.18 (★★★☆☆). Do the due diligence of checking this is well-defined on equivalence classes. It's not particularly enlightening in my opinion though.

Okay, now we need to do each of these constructions in reverse. Again, if we have a finitely generated field extension of $\mathbf{C}$, then it is a function field $K(X)$ for some variety X.

What about a map of $\mathbf{C}$-algebras $\eta : K(Y) \to K(X)$; how do we get a rational map $X \dashrightarrow Y$? The variety Y has an affine cover, so without loss of generality, let Y be affine, since we just need to define this map on a local open subset and we can shrink to an affine one. Y has a coordinate ring $A(Y) = \mathbf{C}[y_1, \ldots, y_n]/I(Y)$. Using η, we see that $\eta(y_1), \ldots, \eta(y_n) \in K(X)$, so there exists $U \subseteq X$ such that $\eta(y_i)$ is regular on U. Thus η induces an injective $\mathbf{C}$-algebra homomorphism $A(Y) \hookrightarrow \mathcal{O}(U)$. This corresponds to an injective morphism of varieties $U \hookrightarrow Y$, hence a dominant rational map $X \dashrightarrow Y$. □

Remark 5.19. There are a couple of things we black boxed in the proof of **Theorem 5.14** above. Feel free to check them if you like.

Corollary 5.20 (Corollary to **Theorem 5.14**.)**.** *The following are equivalent.*

1. *X and Y are birationally equivalent.*
2. *There exists open subsets $U \subseteq X$ and $V \subseteq Y$ such that $U \cong V$ as varieties.*
3. *$K(X) \cong K(Y)$ as field extensions over $\mathbf{C}$.*

Exercise 5.21 (★★☆☆☆). Prove **Corollary 5.20** by showing (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Note that only (1) $\Rightarrow$ (2) needs any work since the other two implications are essentially the theorem (confirm this for yourself).

Exercise 5.22 (★★★☆☆)**.** Let's allow **Theorem 5.14** to suggest some mathematics to us. Consider $X = \mathbf{P}^1$ with homogeneous coordinates $[x : y]$, but you may reduce to the affine cover where $y = 1$, so we have local coordinate x. Consider $Y = \mathbf{A}^1$ with coordinate t. Find a dominant rational map $\mathbf{P}^1 \dashrightarrow \mathbf{A}^1$ corresponding to the morphism of function fields $K(\mathbf{A}^1) \to K(\mathbf{P}^1)$ given by $t \mapsto x^2$.

Example 5.23. Here's another application of **Theorem 5.14**. We'll show that $\mathbf{P}^2$ is birationally equivalent to $X = V(xy - zw) \subseteq \mathbf{P}^3 = \{[x : y : z : w]\}$. But we won't actually produce a birational map $\mathbf{P}^2 \dashrightarrow X$ with birational inverse! Instead, we'll compute the function fields $K(X)$ and $K(\mathbf{P}^2)$, then apply **Theorem 5.14**.

To compute $K(X)$, first reduce to the affine subset of X where $w \neq 0$. This is valid since rational functions are equivalence classes defined on open subsets. So in coordinate language we may set $w = 1$ and identify this affine subset as sitting inside $\mathbf{A}^3 = \{(x, y, z)\}$. Here, the coordinate ring of the variety X is

$$A(X) = \mathbf{C}[x, y, z]/I(X) = \mathbf{C}[x, y, z]/(xy - z).$$

This ring is isomorphic to $\mathbf{C}[x, y]$ via the map $f(x, y) \mapsto f(x, y, xy)$. The field of fractions of $\mathbf{C}[x, y]$ is $\mathbf{C}(x, y)$, and thus $K(X) = \mathbf{C}(x, y)$. And also we know $K(\mathbf{P}^2)$ is $\mathbf{C}(x, y)$ as well, so since $K(X) \cong K(\mathbf{P}^2)$, X and $\mathbf{P}^2$ are birationally equivalent (even though we never described a birational map!).

Remark 5.24. It requires a few commutative algebra facts to properly prove, so we won't do so here unless you twist my arm later, but it's a neat fact that any variety X with $\dim(X) = r$ is birational to a hyperplane in $\mathbf{P}^{r+1}$. Recall that a hyperplane is defined by the vanishing of a single equation.

6 Blow Ups and Singularities: The Reason for Birational Maps

Now we're getting into (one of) the big kahunas. This section of the notes might be skipped and returned to at a later date because it's technical and we might not need it anytime soon. But if we're now at the stage where we're needing/wanting it, or if you're reading it of your own volition, be not afraid! (Although those three words should carry all the connotations you'd imagine – if a biblical wheel-of-eyes old testament angel screams discordantly at you not to fear blow ups, would you?)

https://twitter.com/reactjpg/status/1330804563796926464/photo/1

So, what is a blow up, and why should we care? The true answer, which we'll cycle back to towards the end, is that blow ups give us a way to "fix" singularities. What does this mean? The idea is: say we have a variety which is singular at some places. As an example to keep in the back of your mind, consider the cuspidal cubic $V(y^2 - x^3)$, which is singular at the origin. We'd like a way to understand "basically all" of our variety by comparing it to something similar which isn't singular and is therefore nicer. Isomorphisms of varieties can't do this for us – any singular variety *must* be isomorphic to another singular variety – but birational equivalence is *exactly* the tool we need! We already know our slogan: varieties are birational when they have isomorphic open subsets, and this is exactly the way in which we can compare "basically all" of our singular variety to one that is smooth! A birational equivalence between, for example, $V(y^2 - x^3)$ and $\mathbf{A}^1$, is the precise way to see that the cuspidal cubic is basically the affine line, up to the cusp point, which gets unkinked. Blow ups are a systematic way to fix your singularities, no matter what horrendous variety you might start with!

Let's ease our way into this topic not by jumping right in with a definition, but instead a motivating example.

Example 6.1. Let $\mathbf{A}^2 \dashrightarrow \mathbf{A}^1$ be a rational map defined by $(x, y) \mapsto y/x$. We have seen back in **Example 4.6** that this is a rational map with domain $D(x) = \{(x, y) \in \mathbf{A}^2 \mid x \neq 0\}$. Suppose we're a bit miffed about that and want to extend this rational map to a larger domain. One thing we can do to extend this map is to replace $\mathbf{A}^1$ with $\mathbf{P}^1$; that is, we get a map $(x, y) \mapsto [x : y]$. This is, honest to goodness, how you might naturally want to extend the map $(x, y) \mapsto y/x$, since its outputs are slopes of lines through the origin, and that's what determines points in $\mathbf{P}^1$. In fact, the extension $(x, y) \mapsto [x : y]$ is now clearly defined everywhere except $(0, 0) \in \mathbf{A}^2$. Naturally, we'd ask: how does this construction compare to our original rational map?

We answer this by considering the graph of our extension. We'll denote it by Γ and recall that

$$\Gamma = \{(x, y) \times [x : y]\} \subseteq \mathbf{A}^2 \times \mathbf{P}^1.$$

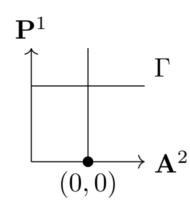
(I'm writing $(x, y) \times [x : y]$ because it's a lot more tractable than $((x, y), [x : y])$.) We'll visualize this graph as we do with graphs in other contexts in math: draw an axis for the input (in this case $\mathbf{A}^2$), an axis for the output ($\mathbf{P}^1$), and Γ is a picture sitting in the $\mathbf{A}^2 \times \mathbf{P}^1$-plane, just like a graph $(x, f(x))$ would in the xy-plane. I'd love to actually draw this for you, but I don't have a great way to do so that's helpful until we do the work to unpack Γ by hand.

Fortunately, we don't have to actually draw Γ for what I'm about to argue (though I'll give you a really crappy picture at the end and you can tell me if it helps). We're just going to think about what happens when we take Γ and project it to the $\mathbf{A}^2$-factor. When we're away from $(0, 0) \in \mathbf{A}^2$, the projection $\pi : \Gamma \to \mathbf{A}^2$ is an isomorphism, because like I said, (x, y) determines a slope y/x (which is ∞ if $x = 0$ and $y \neq 0$) and hence a point in $\mathbf{P}^1$, so π has an inverse $(x, y) \mapsto (x, y) \times [x : y]$, where $[x : y]$ represents a single well-defined unique slope, the slope of the line from the origin to (x, y).

But what happens above $(0, 0) \in \mathbf{A}^2$? It turns out that we get a whole copy of $\mathbf{P}^1$ inside of Γ above $(0, 0)$! How can we verify that claim? Consider a line $y = mx$ which passes through the origin in $\mathbf{A}^2$. Every point (x, y) on this line, minus the origin, gets sent to $(x, y) \times m$, to abuse notation and use m to denote $[x : y]$, since it's supposed to represent our slope. If we take the closure of this line, meaning to now include the origin, all those points will still be sent to $(x, y) \times m$. But yet m was arbitrary, so the origin gets sent to $(0, 0) \times [x : y]$ for any $[x; y] \in \mathbf{P}^1$.

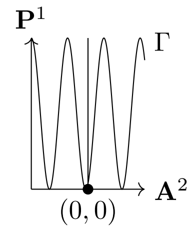
So what's going on here? We understand the extension by understanding Γ, and we understand Γ by understanding its projection $\pi : \Gamma \to \mathbf{A}^2$. The projection π is an isomorphism outside of $(0, 0)$ and contracts a whole copy of $\mathbf{P}^1$ to $(0, 0)$. This gives us the idea of "blowing up" the origin in $\mathbf{A}^2$.

Here's that picture. It's "honest" but because of that not particularly enlightening without the accompanying explanation.



See that away from $(0, 0)$, Γ is a horizontal "line," which is supposed to look like the same "line" given by the $\mathbf{A}^2$-axis, hence an isomorphism of varieties there. And at $(0, 0)$, Γ has a vertical "line," which is supposed to look like the entire $\mathbf{P}^1$-axis. So we've taken $\mathbf{A}^2$, left it completely undisturbed away from $(0, 0)$, and taken $(0, 0)$ and stretched it into a whole copy of $\mathbf{P}^1$ (i.e., blown it up).

Remark 6.2. As a quick aside, there is one lie in the picture of Γ above. Writing the preimage of $\mathbf{A}^2 \setminus \{(0,0)\}$ as a horizontal line helps highlight the fact that it's isomorphic to $\mathbf{A}^2 \setminus \{(0,0)\}$, but it does suggest that the $\mathbf{P}^1$ factor is always constant, which is definitely not true. Different $(x,y) \in \mathbf{A}^2$ are going to have different slopes $[x:y]$. A slightly more honest picture would be something like



which highlights the fact that as you vary the element $(x,y) \in \mathbf{A}^2$, you'll get different slopes from the origin to (x,y) and hence different values $[x:y]$ in the $\mathbf{P}^1$-factor. But adding this honesty doesn't really contribute much to our understanding of how a blow up actually works, so we'll go back to the picture that lies a little bit from now on.

Believe it or not (well, believe it; it's true!), the construction we've just described in **Example 6.1** is actually general!

Definition 6.3. The **blow up of $\mathbf{A}^n$ at the origin** is the closed subset $\Gamma \subseteq \mathbf{A}^n \times \mathbf{P}^{n-1}$ defined by

$$\Gamma = \left\{ x_i y_j = x_j y_i \mid i,j \in \{1,\dots,n\}, \mathbf{A}^n = \{(x_1,\dots,x_n)\}, \mathbf{P}^{n-1} = \{[y_1 : \cdots : y_n]\} \right\}.$$

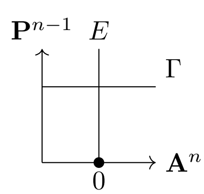
Exercise 6.4 (★☆☆☆☆)**.** Do the typechecking necessary to verify that **Example 6.1** satisfies the definition of a blow up in **Definition 6.3**. It's just a matter of relabeling a couple of things in our example to match the notation of the definition.

Remark 6.5. There is a natural map $\varphi : \Gamma \to \mathbf{A}^n$ defined by the composition $\Gamma \hookrightarrow \mathbf{A}^n \times \mathbf{P}^{n-1} \xrightarrow{\pi} \mathbf{A}^n$. The results we saw in **Example 6.1** hold:

1. If $P \in \mathbf{A}^n$ is not the origin, then $\varphi^{-1}(P)$ is a single point.
2. The preimage of the origin, $\varphi^{-1}(0)$, which is called the exceptional divisor, is all of $\mathbf{P}^{n-1}$.
3. The points of $\varphi^{-1}(0)$ are in bijection with the set of lines through 0 in $\mathbf{A}^n$.
4. Γ is irreducible.

To check (4), since we didn't in **Example 6.1**, observe that Γ is the union of $\Gamma \setminus \varphi^{-1}(0)$ and $\varphi^{-1}(0)$. The first piece is isomorphic to $\mathbf{A}^n \setminus 0$, hence irreducible. But observe that $\varphi^{-1}(0)$ is in the closure of a subset $L \subseteq \Gamma \setminus \varphi^{-1}(0)$ (this subset is $L = \{x_i = a_i t, y_i = a_i \mid t \in \mathbf{A}^1 \setminus 0, a_i \in \mathbf{C} \text{ not all } 0\}$). Hence $\Gamma \setminus \varphi^{-1}(0)$ is dense in Γ, and thus Γ is irreducible.

It's the same picture as before! We'll write E for the exceptional divisor $\varphi^{-1}(0)$. Also note that Γ is still the entire graph, while E is only the "vertical line" $\varphi^{-1}(0)$.



Okay, fantastic; all blow ups of $0 \in \mathbf{A}^n$ are captured by our toy example! But there are two wrinkles, one not important and one important. The first wrinkle is: what if we want to blow up at a different point $P \in \mathbf{A}^n$? The reason this one is not important is because you just have to first make a linear change of coordinates sending P to 0, and then do a blow up at 0, which we already know how to do. The second wrinkle, which isn't quite so trivial, is: what if we want to blow up a different variety other than $\mathbf{A}^n$?

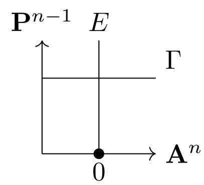
Definition 6.6. Let X be a closed subvariety of $\mathbf{A}^n$ passing through 0 (or not, remember we can do a change of coordinates). The **blow up of** X **at the origin** is $\widetilde{X} = \overline{\varphi^{-1}(X \setminus 0)} \subseteq \Gamma$, where $\varphi : \Gamma \to \mathbf{A}^n$ is defined in **Remark 6.5**. By abuse of notation, we'll write $\varphi : \widetilde{X} \to X$ for $\varphi|_{\widetilde{X}}$.

Remark 6.7. Just like we've been seeing for $\mathbf{A}^n$, if we blow up a variety X, then $\varphi : \widetilde{X} \to X$ induces an isomorphism $\widetilde{X} \setminus \varphi^{-1}(0) \cong X \setminus 0$. By **Corollary 5.20**, $\widetilde{X}$ and X are birationally equivalent. One other comment to make is that right now, the definition of the blow up $\widetilde{X}$ appears to depend on how X has been embedded in $\mathbf{A}^n$, but this is not the case; blowing up is actually intrinsic to the variety X and doesn't depend on an embedding.

In addition to the exceptional divisor, we have a few more names for the parts of a blow up:

Definition 6.8. Given any birational map $\varphi : X \dashrightarrow Y$, the locus where it is not an isomorphism is called the **exceptional locus**, E. If $V \subseteq Y$, then $\varphi^{-1}(V)$ is called the **total transform of** V. If we write $Z = \varphi(E)$, then for any $V \nsubseteq Z$, we call $\overline{\varphi^{-1}(V \setminus Z)}$ the **strict transform of** V.

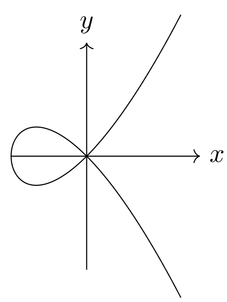
So in our ongoing picture:



given our birational map $\Gamma \dashrightarrow \mathbf{A}^n$, we have that E is the exceptional locus, and letting $V = \mathbf{A}^n$, the total transform of V is Γ and the strict transform of V is $\overline{\Gamma \setminus E}$.

Example 6.9. Let's do one explicit example of the blow up of a variety together. Just as was the case for $\mathbf{A}^n$, you should imagine a blow up as stretching out or pulling apart your variety near the origin according to the different directions of lines through the origin. Let's try to mesh that intuition with the mathematics as we work through this example.

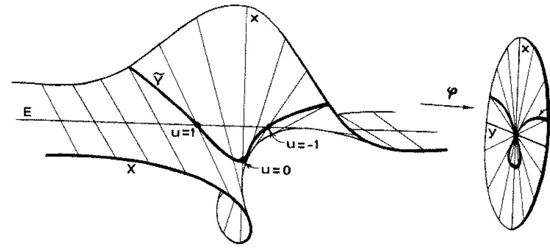
Let $X = V(y^2 - x^2(x+1))$. In $\mathbf{A}^2$, we can picture X as the curve



Let's write $[t : u]$ for the coordinates of $\mathbf{P}^1$. We've already seen that $\Gamma \subseteq \mathbf{A}^2 \times \mathbf{P}^1$, the blow up of $\mathbf{A}^2$ at $(0,0)$, is given by $xu = ty$, either through **Definition 6.3** directly, or through your translation in **Exercise 6.4**.

The total transform of X is determined by the simultaneous system $y^2 = x^2(x+1)$ and $xu = ty$ in $\mathbf{A}^2 \times \mathbf{P}^1$. We'll describe this space using two charts on $\mathbf{P}^1$: the canonical ones where $t \neq 0$ and where $u \neq 0$. In the chart $t \neq 0$, let $t = 1$ and treat u as an affine parameter. Our system becomes $y^2 = x^2(x+1)$ and $y = xu$, so we can substitute xu for y to see that $x^2u^2 - x^2(x+1) = 0$ in $\mathbf{A}^3 = \{(x,y,u)\}$. This equation factors, giving us two irreducible components. The first is defined by $x = y = 0$ and u free. This is the exceptional curve E. The second is $u^2 = x+1$ and $y = xu$. This is the blow up $\widetilde{X}$. One way to conceptualize $\widetilde{X}$ is that we've stretched out $\mathbf{A}^2$ at the origin, which has the effect of parametrizing the curve so that it "sort of" no longer self intersects (if by "sort of" we mean on the strict transform), at the cost of introducing that exceptional curve E. You can see this a bit more precisely by realizing that $\widetilde{X} \cap E = \{u = 1, u = -1\}$, which correspond to the slopes of the two branches of X at the origin.

Here's a picture from Hartshorne to help you visualize what's going on here.



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Exercise 6.10 (★★☆☆☆)**.** What happens on the chart $u \neq 0$? Can you draw a picture?

Exercise 6.11 (★★★☆☆)**.** Ready to do a blow up yourself? Let Y be the cuspidal cubic defined by $V(y^2 - x^3) \subseteq \mathbf{A}^2$. Blow up $(0,0)$. Let E be the exceptional curve and let $\widetilde{Y}$ be the strict transform of Y. Show that E meets $\widetilde{Y}$ in one point and that $\widetilde{Y} \cong \mathbf{A}^1$. Draw plenty of pictures!

Exercise 6.12 (★★★☆☆)**.** Here's another blow up. Let Y be the cone $V(x^2 + y^2 - z^2) \subseteq \mathbf{A}^3$. Blow up $(0,0,0)$. Describe it. Draw pictures!

Exercise 6.13 (★★★☆☆)**.** What about blowing up $(0,0)$ in $V(x^3 + y^3 - xy)$?

Exercise 6.14 (★★★★☆)**.** Can you generalize the blow up of a variety in $\mathbf{A}^2$ at the origin? Let's say that the variety is defined as the vanishing of a single irreducible polynomial $f \in \mathbf{C}[x,y]$. What can you say about the blow up in relation to f? The exceptional divisor? The strict transform? Okay, now what about $V(f_1, \dots, f_s)$?

Now we finally (modulo the fact that I *will not* prove this for you) reach one of the name-dropped reasons to care that I gave in the introduction.

Theorem 6.15 (Hironaka 1964)**.** *Let X be a variety over $\mathbf{C}$, possibly singular at some points. There exists a nonsingular variety Y and a proper, birational map $Y \dashrightarrow X$. This is called a resolution of the singularities of X. You can construct Y explicitly by repeatedly blowing up along nonsingular subvarieties.*

Remark 6.16. The proof of this is hard[2], which is again why I'm not proving this for you. But look at how incredible this theorem is! You hand me any variety you like, with the grossest defining equations or the nastiest singularities you can muster, and no matter what, we can just blow up a finite number of times to get a birationally equivalent nonsingular variety. Wow! Also, for what it's worth, the problem is still open[3] in characteristic p; the proof requires the fact that $\operatorname{char} \mathbf{C} = 0$.

I also didn't define proper here; it's a bit technical but we may see it eventually. If you're just looking for a reason why that word is there, it's essentially so that the resolution of singularities isn't something dumb like just taking Y to be the subvariety of nonsingular points. If you've seen proper maps in topology, it's got the same vibes in algebraic geometry, but the definition is more technical (as are all topological notions) since our spaces aren't Hausdorff.

[2] at least the one Hironaka gave; wikipedia says there's modern proofs which are 1/10 the size of Hironaka's and approachable in an introductory graduate course. I guess I trust that but I didn't personally check and regardless, it's still a digression we don't need at this time

[3] we'll discuss in person

7 Examples and Motivations for Scheme Theory

This section focuses heavily on fundamental examples that help one understand how to think about varieties in the neo-classical way. Some examples also serve to explain the limitations of using varieties to solve problems from other areas of math. This will foreshadow another enlargement of the category of varieties into the category of schemes over a base field.

Varieties, as we have currently defined them, have three "blind-spots". These blind-spots are as follows:

1. Cases where we would like to study a locally ringed space whose global sections do not form an integral domain
2. The ability to calculate tangent lines from a purely algebraic setting
3. Solving number theory problems

The reader has likely encountered the first blind-spot by accident when attempting to use the Nullstellensatz on an ideal which wasn't prime! The result is a reducible algebraic set. If we let such a set have global sections, the resulting ring couldn't possibly be an integral domain. The second blind-spot is rather subtle. The idea is to formalize the correspondence between powers of non-radical ideals which vanish over a variety and the orders of their non-vanishing derivatives.

This last blind-spot is primarily motivated by the Weil conjectures which are, for the most part, unsolved. These conjectures are highly related to the Riemann-zeta hypothesis and likely lie at the heart of all modern algebraic number theory. The most famous accomplishment of AG related to number theory is the proof of Fermat's last theorem. The theorem states there are no nontrivial integer solutions to $x^n + y^n = z^n$ when $n \geq 3$. The proof was first given by contradiction by Andrew Wiles in 1994. First, Wiles proved a special case of the Modularity Conjecture which roughly states that smooth curves called elliptic curves are in one to one correspondence with functions on the upper half complex plane called modular forms. The special case concerned semistable elliptic curves. If any of the equations of Fermat's last theorem had a nontrivial solution, one could construct a semistable elliptic curve over a finite field of characteristic p with integer solutions modulo p. This curve should correspond to a modular form but the integer solutions make this impossible so one reaches a contradiction. The construction of the needed elliptic curve requires more technology than locally ringed spaces. Hence varieties aren't general enough to handle problems of this caliber.

Recall that a variety is a pair $(X, \mathcal{O}_X)$ consisting of a locally ringed space X and its structure sheaf of rings $\mathcal{O}_X$ which is isomorphic (as a locally ringed space) to a pair $(Y, \mathcal{O}_Y)$ where Y is a QAAV or a QPAV and $\mathcal{O}_Y$ is its sheaf of rings of regular functions.

Let X denote the origin in $\mathbb{A}^3$. Consider the vanishing set of $I = (x, y, z)^2$ which is X. However, I is not prime. We call $Z(I)$ a non-reduced point. In general, when a closed subset Y is given by an ideal which is not a prime ideal, we say Y is non-reduced. Otherwise, we say Y is reduced. Any non-reduced algebraic set Y has a canonical reduced algebraic set associated to it. In our current setting, if Y is irreducible then $Y = Z(I^n)$ for some prime ideal I and some $n \geq 2$. Thus $Y = Z(I)$ as sets. If Y is not irreducible then Y is the union of two or more distinct varieties $Y_i = Z(I_i)$ which are also closed in Y. Hence there are n_i for which

$$S/I(Y) \cong S/(I_1^{n_1} ... I_m^{n_m})$$

It's sometimes helpful to mentally picture non-reduced varieties as being "thick" but still occupying the same exact amount of space that one would expect the corresponding reduced variety to occupy. Note that the global sections of an affine (resp. projective) algebraic set Y are an integral domain iff Y is an affine (resp. projective) variety iff Y is an irreducible space which is also reduced.

There is more subtle information we can obtain by studying the geometry of non-radical ideals.

Example 7.1. Let I be an ideal of $S = k[x_1, ..., x_n]$. Then $Z(I)$ determines a closed subset of $\mathbb{A}^n_k$. We know that $Z(I)$ must be a union of AAV's of the form $Z(P)$ where P is a prime ideal of S. We wish to calculate a maximal list of distinct prime ideals P_i for which

$$Z(I) = Z(P_1) \cup ... \cup Z(P_m)$$

If I is a radical ideal, the Nullstellensatz guarantees that this occurs precisely when

$$I = P_1 \cap ... \cap P_m$$

where the P_i all contain I but are minimal in the following sense: if $I \subseteq Q \subseteq P_i$ for a prime ideal Q then $Q = P_i$. Provided all of the P_i are pairwise distinct primes, this list of prime ideals is unique. Let $I = (x^2, y)$ in $S = k[x, y]$. This ideal is not a radical ideal since its vanishing set is the origin which is already married to the radical ideal (x, y). However, this ignores some important geometric data. Let P denote the origin. As sets, $Z(I)$ is equal to P. As locally ringed spaces, we can distinguish $Z(I)$ from P by stating that $Z(I)$ has $\{P\}$ as its underlying set but comes equipped with a horizontal tangent line. The tangent line represents the surviving first derivative of x^2 in the horizontal direction. To elaborate: let $f = 3+14x-6y+5x^2+2xy+x^3$. Say I write down f on a piece of paper that I keep hidden from you. Then, I write down f modulo I

$$f \equiv 3 + 14x$$

on a new piece of paper which I then show you. What information about f can you deduce from only the second paper? You know the constant term is $f(0, 0)$ which isn't killed off by I so you deduce that the constant term is 3. The term $14x$ also survived giving

$$14 = \partial f/\partial x(0, 0)$$

However, any data concerning the partial with respect to y has been lost. Therefore, I killed off everything except position and velocity in the horizontal direction. Thus we may think of $Z(I)$ as the origin along with a horizontal tangent line. Given a polynomial f, $Z(I)$ scans f and returns its constant term (position) and the x coefficient (velocity in the horizontal direction). This object is by no means a variety but it still conveys fundamental geometric properties.

Example 7.2. Let $I = (x^2, xy, y^2)$. Again, I is contained in (x, y) and $Z(I)$ is just the origin. We see constant terms have survived along with the first x-partial. Yet y is not included in I so the first y-partial also survived. Hence $Z(I)$ may be thought of as the origin along with all possible tangent lines through the origin. Given a polynomial f, $Z(I)$ scans f and returns its constant term and a vector in the plane with coordinates given by the coefficients of x and y.

Example 7.3. Let $I = (x) \cap (x^2, xy, y^2)$. Looking at the vanishing sets of these ideals, we see $Z(I)$ is the y-axis in the plane together with an embedded component at the origin. This embedded component comes equipped with the ability to read off all first order derivatives evaluated at the origin.

By now, the reader should be getting the feeling that the Nullstellensatz doesn't give us the whole picture. If you re-read the fundamental theorems which follow from it, Hilbert's Nullstellensatz is a statement about set inclusion. We need fundamental theorems which are deep enough to pick up the subtle geometric information which survives quotients by non-radical ideals as demonstrated in the previous three examples.

Exercise 7.4 (★☆☆☆☆). Give an ideal I of $S = \mathbb{C}[x_1, ..., x_n]$ such that $Z(I)$ has one embedded component of dimension r for each $1 \leq r \leq n - 1$.

Exercise 7.5 (★★☆☆☆). Let $I = (x) \cap (x^2, xy, y^2)$. Describe the geometric object in the plane associated to this ideal.

8 Affine Schemes