What's on the mind of air traffic controllers and algebraic geometers? #2

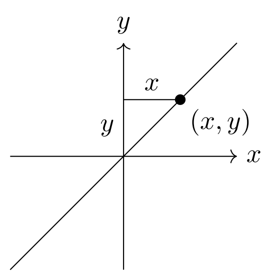
Abstract

(Subtitled: blowing up points on the plane.) A mathematician's job is always easier when they're working with smooth objects. Topologists and differential geometers have access to smoothing tools like homeomorphisms, bump functions, and partitions of unity. But algebraic geometers are restricted to working with polynomials, so these tools aren't accessible. Instead we get access to a clever tool called a blow up, which we attain by changing the codomain of a function. In this talk, we'll describe how to blow up a point on the plane, and then use that construction to resolve singularities of curves. The talk will be generally accessible to an audience that's familiar with polynomials.

1 Motivation

Consider the function $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $f(x, y) = y/x$, for which we clearly can't plug in $x = 0$. What does this function describe?

One way to conceptualize this function is that each point (x, y) on the plane maps to $(y - 0)/(x - 0)$; i.e., this is the slope of the line passing through (x, y) and the origin. Phrased in this way, we have a natural way to extend this function so that we could plug in $x = 0$. A line through the origin and $(0, y)$ is a vertical line, so it should have slope ∞.



So we add a point at ∞ to our codomain. So that ∞ plays nice with the topology we already have, we need to describe the space $\mathbf{R} \cup \{\infty\}$. There are a few ways to do this:

1. analytically this is allowing sequences to converge to ∞,
2. topologically this is the one-point compactification, or
3. our way:

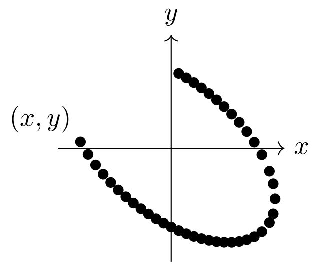
Define a new set $\mathbf{P}$. Its elements are written $[x : y]$ where $x, y \in \mathbf{R}$ and not both of x and y are zero. We also give this set an equivalence relation where $[x : y]$ is equal to $[\lambda x : \lambda y]$ as long as $\lambda \neq 0$. This is called projective space and parametrizes lines through the origin. Indeed, any point $(x, y) \neq (0, 0)$ determines a line through the origin, and $(\lambda x, \lambda y)$ is on that same line.

Thanks to this equivalence relation, if you hand me a point $[x : y]$ and $x \neq 0$, I can let $\lambda = 1/x$ and rescale to the point $[1 : y/x]$, which you should think of as the point $y/x \in \mathbf{R}$. And if $x = 0$, the point $[0 : y] \sim [0 : 1]$ is the point ∞.

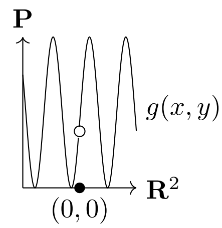
So we can now plug in any $(x, y) \neq (0, 0)$ into the function $g : \mathbf{R} \times \mathbf{R} \to \mathbf{P}$, $g(x, y) = [x : y]$. Of course, there is a natural question: by how much does this new construction differ from f? Let's graph $((x, y), [x : y]) \subseteq \mathbf{R}^2 \times \mathbf{P}$.

Certainly our construction behaves well away from $(0,0) \in \mathbf{R}^2$. It's either y/x or our new point ∞, since the slope of the line passing through the origin and (x, y) is either y/x or ∞. At the very least, since g is a function on $\mathbf{R}^2 \setminus \{(0,0)\}$, there is only a single value.

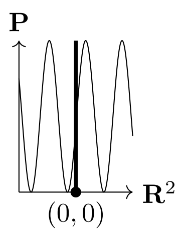
As I spiral around the plane:



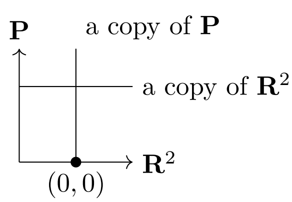
I get varying slopes $[x : y] \in \mathbf{P}$:



But what happens to $((x, y), [x : y])$ above $(0,0)$? Another way to ask this: what is the slope of a line passing through the origin and $(0,0)$? Well, clearly that's not enough to determine a unique line. In fact, any slope $m \in \mathbf{R} \cup \{\infty\}$ is attainable. So above $(0,0)$ we get an entire copy of $\mathbf{P}$.



This is the general idea of what a blow up of $(0,0)$ in $\mathbf{R}^2$ is. We leave all the other points $(x, y) \in \mathbf{R}^2$ alone (by which we mean each $(x, y) \neq (0,0)$ only has a singleton in its preimage), but we stretch out the origin into an entire copy of $\mathbf{P}$. To simplify our picture, it's perhaps easier to think of a blow up in the following way.



This space can be expressed as a polynomial inside of $\mathbf{R}^2 \times \mathbf{P}$. If $\mathbf{R}^2 = \{(x, y)\}$ and $\mathbf{P} = \{[s : t]\}$, then this graph is

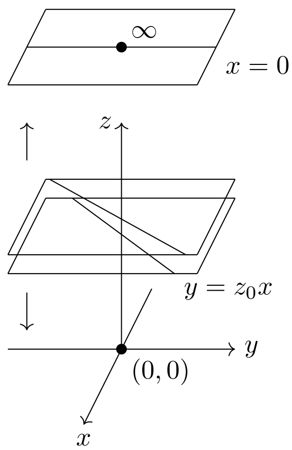
$$xt = ys.$$

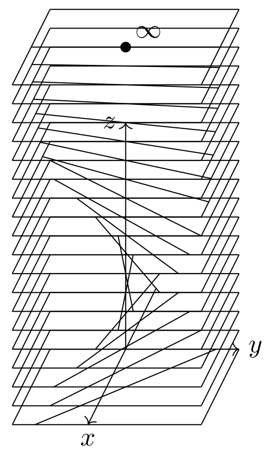
We can actually draw this three dimensionally. Break $\mathbf{P}$ into $[s : t] = [1 : t/s] = z$ if $s \neq 0$ and ∞ if $s = 0$. We have two cases:

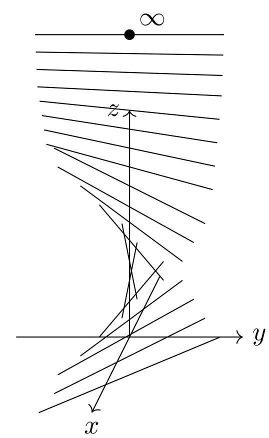
1. If $s \neq 0$, then $xt = ys$ is the same as $xt/s = y$, i.e., $y = zx$.

2. If $s = 0$, then $xt = 0$, but since t cannot also be 0, $x = 0$.

This means our three dimensional picture should be thought of as a stack of lines in horizontal plane cross-sections, where the height of the stack z determines the slope of the line $y = zx$. Once we reach height ∞, we get a vertical line on our plane.





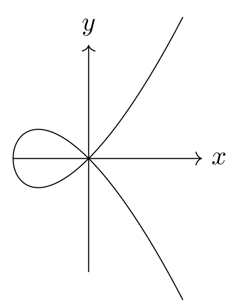


The blow up is a tower of twisting lines, where preimage over $(x, y) \neq (0, 0)$ is a single line but the preimage of $(0, 0)$ is the center of the whole stack. The visualization is that you're untwisting all the lines through the origin upwards into a big stack.

2 How Does a Blow Up Smooth Things Out?

The key idea is that by untwisting lines through the origin, you're allowing a curve to pass through the origin multiple times, as long as its entries/exits occur at different directions. What a blow up essentially does is it takes a singular curve and pulls it apart according to all its tangent lines at $(0, 0)$.

Consider the polynomial curve $y^2 = x^2(x + 1)$. Its graph looks like this.



This curve is singular at the origin; we call it a nodal cubic. Its tangent space at the origin is dimension 2, corresponding to the fact that we've got two tangent lines: one has a slope of 1 and the other -1. Intuitively, a blow up should smooth out this curve so that it will pass through the preimage of the origin twice: once at $z = 1$ and once at $z = -1$. Can we make this intuition precise with a computation?

Sure! Computing the blow up of this polynomial is solving the simultaneous system

$$\begin{cases} y^2 = x^2(x+1) \\ y = zx \end{cases}$$

which we can ask our college algebra students to compute. By substitution, we get

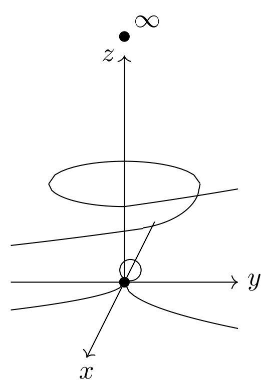
$$\begin{aligned} (zx)^2 &= x^2(x+1) \\ z^2x^2 - x^2(x+1) &= 0 \\ x^2(z^2 - x - 1) &= 0, \end{aligned}$$

so $x^2 = 0$ or $z^2 = x + 1$. If $x^2 = 0$, then $x = 0$, and so too does y; z is left free. This factor is the preimage of $(0, 0)$. If $z^2 = x + 1$, then $z^2 = x + 1$ and $y = zx$ gives us our smoothing. Notice that this smoothing intersects the preimage of $(0, 0)$ in two places. Plug in 0 for x and y:

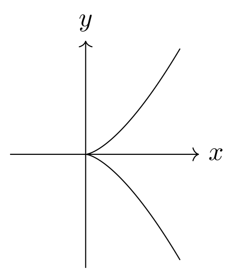
$$\begin{aligned} z^2 &= 0 + 1 \qquad\qquad & 0 &= z \cdot 0 \\ z &= \pm 1, \end{aligned}$$

just like we wanted!

Here's the three dimensional picture.



We can do another example: the polynomial curve $y^2 = x^3$ is the cuspidal cubic.



Geometrically, we already know what the blow up will look like. (Exercise for the reader of these notes: draw it! I did during the talk.) Algebraically, we have

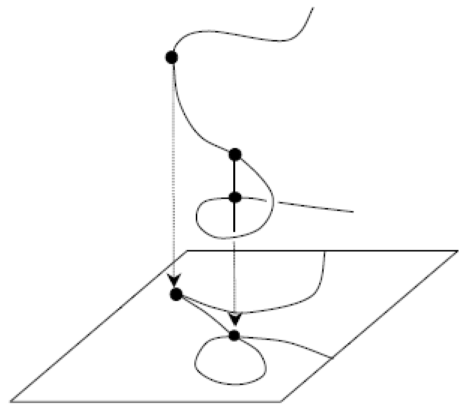
$$\begin{cases} y^2 = x^3 \\ y = zx, \end{cases}$$

so

$$\begin{aligned} (zx)^2 &= x^3 \\ z^2x^2 - x^3 &= 0 \\ x^2(z^2 - x) &= 0. \end{aligned}$$

If $x^2 = 0$, then $x = y = 0$ and z is free; once again we get a copy of the preimage of 0. In fact, we always will. Otherwise $z^2 = x$ and $y = zx$ is the smoothing. Indeed, we get the tilted parabola I just drew (or you just did)!

This process is a systematic algorithm to smoothing out any singular curve you like. By blowing up, we spread out self-intersection into different branches. We can draw the messiest curve possible, and a sequence of blow ups works to smooth it out.



The main idea is that: every singular curve is the shadow of a smooth curve, given by projecting its parametrization.

3 There's More to the Story

Of course, the sandbox we're playing in now is not the end. Blow ups aren't restricted to $\mathbf{R}^2$, by which we mean we can blow up points in $\mathbf{R}^n$, or we can blow up points in F^n for your favorite field F. Our favorite field today is $\mathbf{C}$, because it's algebraically closed and its characteristic is 0. In all of these settings, a blow up is just the polynomial equations

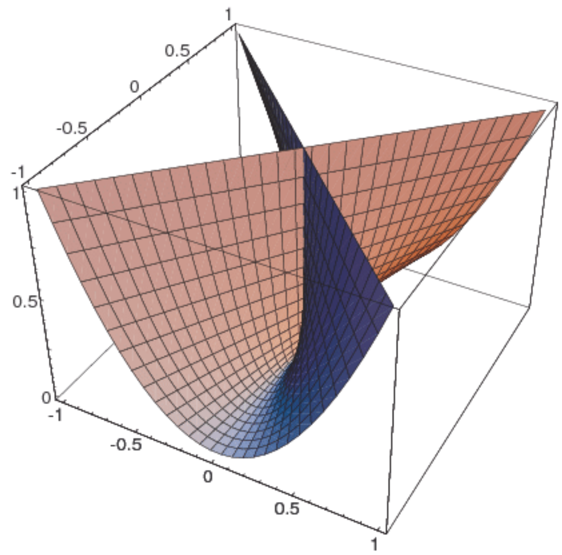
$$\left\{x_i y_j = x_j y_i \mid F^n = \{(x_1, \dots, x_n)\}, \mathbf{P}_F^{n-1} = \{[y_1 : \cdots : y_n]\}\right\} \subseteq F^n \times \mathbf{P}_F^{n-1}.$$

For example, to blow up the origin in $\mathbf{C}^3 = \{(x, y, z)\}$, let $\mathbf{P}_{\mathbf{C}}^2 = \{[s : t : u]\}$ and we have the equations

$$\begin{cases} xt = ys \\ xu = zs \\ yu = zt \end{cases}$$

inside of $\mathbf{C}^3 \times \mathbf{P}_{\mathbf{C}}^2$. While we can't draw it, we can compute the blow up of a polynomial passing through $(0, 0, 0) \in \mathbf{C}^3$.

You can also blow up subspaces, not just single points. Sometimes this is actually necessary; a construction called the Whitney umbrella is an example of a singularity called a pinch point. Its defining equation is $x^2 - y^2 z = 0$ in $\mathbf{R}^3$. You can draw it by fixing a parabola P and a line L parallel to P's axis, lying on its perpendicular bisecting plane. Then for each point on P, draw the line through that point which is perpendicular to L. However, I prefer to picture it as a stack of crossed lines $z_0 y^2 = x^2$ parameterized by height z, where as you descend the lines pivot closer together.



If you try to blow up the pinch point, the surface will still be pinched, since every neighborhood of the pinch point intersects itself. Instead, you have to blow up the entire z-axis.

If we acknowledge these technical difficulties, there is a marvelous theorem beloved by birational geometers.

Theorem (Hironaka 1964)**.** *Let f be a polynomial in $\mathbf{C}^n$. Consider the graph of points where f is equal to* 0*; we call such a thing a variety and we will give this one the name X. If X is singular at some collection of points, then there exists a nonsingular variety Y and a birational morphism $Y \dashrightarrow X$, which is to say that Y and X are homeomorphic on dense subsets. Y is called a resolution of the singularities of X. You build Y and the map $Y \dashrightarrow X$ via repeated iterations of blowing up.*

The hypotheses in the theorem are necessary: Hironaka's theorem only works in characteristic 0, hence our choice of favorite field. When you have a field of characteristic $p > 0$, you don't have resolutions of singularities (instead you have something different called alterations due to de Jong). You can still compute blow ups, but the idea of the proof is that you can attach a number to a variety which tracks its distance from being smooth, and blowing up in characteristic 0 decreases that number, showing that you reach a smooth variety after finitely many blow ups. But in characteristic p, this number may increase under blowing up.

4 Why Would You Want to Resolve Singularities?

Of course Hironaka's theorem begs the question: what do you gain by looking at varieties up to birational equivalence? What invariants do you keep; i.e., what properties don't change under blow up? Let X be a variety over $\mathbf{C}$. Here is a short list of technical invariants and objects of study:

- If X is a curve, its genus.
- If $X \subseteq \mathbf{P}^n$ is smooth, its fundamental group.
- Let Ω_X be the cotangent bundle of X. Let $K_X = \wedge^{\dim X}\Omega_X$. The plurigenera $P_d = \dim_{\mathbf{C}} H^0(X, K_X^d)$ and the Kodaira dimension $\kappa(X) = \min\{k \in \mathbf{Z}_{\geq 0} \mid \text{the set } \{P_d/d^k\} \text{ is bounded}\}$ or $\kappa(X) = -\infty$ if $P_d = 0$ for all $d > 0$.
- The Hodge numbers $h^{p,0} = \dim_{\mathbf{C}} H^0(X, \wedge^p\Omega_X)$. (But the Hodge numbers $h^{p,q} = \dim_{\mathbf{C}} H^q(X, \wedge^p\Omega_X)$ for $q \neq 0$ may change under blow up.)
- The number $K_X^2 + \rho$, where K_X^2 is the self intersection of K_X and ρ is the rank of Néron-Severi group $\operatorname{Pic}(X)/\operatorname{Pic}^0(X)$, which is also Hodge theoretic.
- The minimal model program asks us to find a variety X' birational to X where X' is "as simple as possible" depending on $\kappa(X)$. It suffices to work with smooth X thanks to Hironaka, but X' may not be smooth - it may have what are called "terminal singularities." Partial results are known.