The module of Kähler differentials

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**CARES** 

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# Outline

#### Topics:

- Why?
- Derivations
- Definition of the Kähler differentials
- Construction of the Kähler differentials
- The first fundamental exact sequence
- The second fundamental exact sequence
- Where do you go from here?

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#### Conventions:

- k is a ring, and all rings are commutative and unital
- a k-algebra is a ring A with a structure map  $\varphi: k \to A$

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As algebraists, this is formal symbol moving, not  $\varepsilon$ -neighborhoods. (But will a geometric picture remain?)

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This is some of the cal 1 rules... sorta. Is it enough?

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In almost all contexts we will care about,  $\varphi: k \to A$  is injective, so we will typically write c for  $\varphi(c)$ , and then  $\delta(cf) = c\delta(f)$ .

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Not only that, but  $\operatorname{Der}_k(A; M)$  is an A-submodule via the action  $(f \cdot \delta)(g) = f\delta(g)$ . We can add, subtract, and scale derivations.

# Definition of the Kähler differentials Definition. The module of Kähler differentials of A over $k, \ \Omega_{A/k},$

 $\Omega_{A/k}$ 

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$$A \xrightarrow{d} \Omega_{A/k}$$

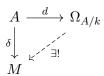
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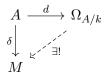
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That is, there is an isomorphism of A-modules

$$\operatorname{Hom}_A(\Omega_{A/k}, M) \cong \operatorname{Der}_k(A; M)$$

given by composition with the universal derivation  $d:A \to \Omega_{A/\underline{k}}$ .

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But what about in general?

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- $d\varphi(c) = 0$ .

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But this shouldn't necessarily sit well with us: where is the geometry?

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Let  $d: A \to K'$  be defined by  $d(f) = 1 \otimes f - f \otimes 1$ .

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You can think: take a Taylor series and truncate it to get the first order differentiation. We'll see more geometry later!

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Let  $M = Adx_1 \oplus \cdots \oplus Adx_n$ . The partial derivative  $\partial_i : A \to Adx_i$  is a derivation, so  $\delta = \sum \partial_i$  is a derivation  $A \to M$ .

$$A$$

$$\delta \downarrow$$

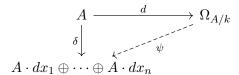
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Using the universal property, we get a unique A-module map  $\psi$  such that the diagram commutes.

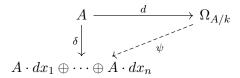


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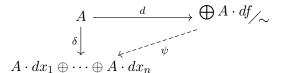
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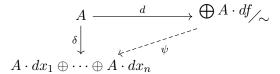
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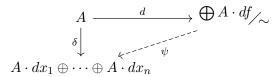


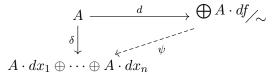
Let  $\Omega_{A/k} \cong \bigoplus Adf/\sim$ , which was our first construction.



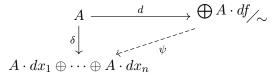


 $\psi$  is injective: If  $\psi(df) = 0$ , then  $\delta(f) = 0$ , so  $\partial_i(f) = 0 dx_i$  for all i. Thus f is  $x_i$ -free, i.e.,  $f \in k$ , so df = 0 in  $\bigoplus A df / \sim$ .

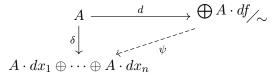




$$\psi\left(dx_{i}\right) = \delta\left(x_{i}\right)$$



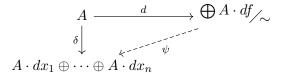
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 $\psi$  is injective: If  $\psi(df) = 0$ , then  $\delta(f) = 0$ , so  $\partial_i(f) = 0 dx_i$  for all i. Thus f is  $x_i$ -free, i.e.,  $f \in k$ , so df = 0 in  $\bigoplus A df / \sim$ .  $\psi$  is surjective:  $A dx_1 \oplus \cdots \oplus A dx_n$  has an A-basis  $\{1 dx_1, \ldots, 1 dx_n\}$ . The element  $dx_i \in \bigoplus A df / \sim$  satisfies

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so  $\{dx_1,\ldots,dx_n\}\subseteq\bigoplus Adf/\sim$  maps to the basis  $\{1dx_1,\ldots,1dx_n\}$  under  $\psi$ .

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### Construction of the Kähler differentials Another example (goal: to generalize)! Let $A = k[x, y]/(y - x^2)$ .

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Now the above sequence is exact.

$$(f)_{f^2} \to \Omega_{k[x,y]/k} \otimes_{k[x,y]} A \to \Omega_{A/k} \to 0$$

What happens when  $A \neq k[x,y]/(f)$ ? E.g.,  $k[x,y]/(f_1,\ldots,f_s)$ ? Or, more generally, ring maps  $k \to R \twoheadrightarrow S$ ? The FFES gives us

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Repeat the same argument as before. We get:

**Theorem.** Let R woheadrightarrow S be a map of k-algs. Let  $I = \ker(R woheadrightarrow S)$ . The following sequence of S-modules is exact.

$$I_{I^2} \xrightarrow{f \mapsto df \otimes 1} \Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to 0$$

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Proof.

[00RU].



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Corollary. If  $A \cong k[x_1, \dots, x_n]/(f_1, \dots, f_s)$ , then
$$\Omega_{A/k} \cong \operatorname{coker} \left[ \frac{\partial f_i}{\partial x_j} \right].$$

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#### Proof.

Let  $R = k[x_1, \ldots, x_n]$ , S = A, and observe that

$$df_i = \sum_{j=1}^s \frac{\partial f_i}{\partial x_j} dx_j.$$



# The second fundamental exact sequence Examples are now easy:

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2 If A' = k[x, y, z]/(xy, xz, yz), then

$$\Omega_{A'/k} \cong \frac{A'dx \oplus A'dy \oplus A'dz}{(xdy + ydx, xdz + zdx, ydz + zdy)}.$$

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3 If  $A'' = k[x_1, \dots, x_n]/(f_1, \dots, f_s)$ , then

$$\Omega_{A''/k} \cong \frac{A''dx_1 \oplus \cdots A''dx_n}{(df_1, \dots, df_s)}.$$

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Suppose  $(R, \mathfrak{m}, k)$  is a local ring, so it can be understood as in correspondence with a point x of some LRS X. Using the map of k-algs  $k \to R \twoheadrightarrow R/\mathfrak{m} = k$ , we get

$$\mathfrak{m}_{\mathfrak{m}^2} \xrightarrow{\varphi} \Omega_{R/k} \otimes_R k \to \Omega_{k/k} = 0$$

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But  $\varphi$  is injective too! To see this, we'll use the fact that  $\operatorname{Hom}(-,k)$  is left exact, and check that

$$\operatorname{Hom}_k(\Omega_{R/k} \otimes_R k, k) \xrightarrow{\varphi_*} \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$$

is surjective.

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Why  $\operatorname{Hom}(-, k)$ ?!?!

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Why  $\operatorname{Hom}(-,k)$ ?!?! Because  $\mathfrak{m}/\mathfrak{m}^2$  is the Zariski cotangent space at x, and its k-vector space dual  $\operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2,k)$  is the tangent space, so it's reasonable to look at.

$$\mathfrak{m}_{\mathfrak{m}^2} \xrightarrow{\varphi} \Omega_{R/k} \otimes_R k \to 0$$
$$0 \to \operatorname{Hom}_k(\Omega_{R/k} \otimes_R k, k) \xrightarrow{\varphi_*} \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$$

Why  $\operatorname{Hom}(-,k)$ ?!?! Because  $\mathfrak{m}/\mathfrak{m}^2$  is the Zariski cotangent space at x, and its k-vector space dual  $\operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2,k)$  is the tangent space, so it's reasonable to look at.

Idea: To show  $\varphi_*$  is surjective, we show any k-linear morphism  $\psi: \mathfrak{m}/\mathfrak{m}^2 \to k$  lifts to  $\Omega_{R/k} \otimes_R k \to k$ . Define a map  $R \to k$  by r = a + b for  $a \in k$  and  $b \in \mathfrak{m}$ ; check that  $r \mapsto \psi(b)$  is a derivation. Then show  $\varphi_*$  is surjective via universal property of  $\Omega_{R/k}$ .

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So Kähler differentials tell us geometry!  $\Omega_{R/k} \otimes_R k \cong \mathfrak{m}/\mathfrak{m}^2$  is the cotangent space.

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 $\Omega_{A/k}$  is differentiation in module form. From last year's CARES:  $J^1A$  is differentiation in k-algebra form. One might wonder: is there a connection? Yes! And it's exactly what you hope. The functor Sym:  $\mathbf{Mod}_k \to \mathbf{Alg}_k$  is the natural way to take a module to an algebra. And indeed,

$$J^1A \cong \operatorname{Sym} \Omega_{A/k}$$
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You can also build out differential forms à la Calculus 3 in the natural way. Let  $\Omega^p_{A/k}$  be the pth exterior power of  $\Omega_{A/k}$  in the category of A-modules.

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The differential  $d: \Omega^p_{A/k} \to \Omega^{p+1}_{A/k}$  satisfies  $d^2 = 0$  and there is a multiplicative map  $\Omega^p_{A/k} \otimes_A \Omega^q_{A/k} \to \Omega^{p+q}_{A/k}$ , so we get a differential graded algebra / cochain complex  $\Omega^{\bullet}_{A/k}$ .

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Connect this to Duncan's 15 Sept talk about the Koszul complex and Čech / sheaf cohomology!

Homological algebra and derived functors: we have two sequences which are exact on the right:

$$#1: k \to R \to S \Rightarrow \Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to \Omega_{S/R} \to 0.$$

$$#2: k \to R \xrightarrow{\psi} S \Rightarrow \ker \psi /_{\ker \psi^2} \to \Omega_{R/k} \otimes_R S \to \Omega_{S/k} \to 0.$$

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You might want to extend to long exact sequences. This is kinda funky since  $\mathbf{Alg}_k$  is not an abelian category. But it can be done homotopically. You get something called the cotangent complex  $\mathbf{L}_{A/k}$ .

# Thank you!

Exact sequences. The Stacks project https://stacks.math.columbia.edu Tags: [00RS] [00RU]

Jet spaces. Jet schemes and singularities, Lawrence Ein & Mircea Mustață  $\mathbf{Ex}\ 2.5$ 

Homotopy and  $\mathbf{L}_{A/k}$ . An introduction to homological algebra, Charles Weibel §8.8.

DAG IV: Deformation Theory, Jacob Lurie