Symplectic Manifold Obstructions

Exploring topics discussed in passing during the course

Motivation: In understanding the course and the theorems we proved, I like drawing lots of examples and seeing how to utilize the results proven. That's predicated on having a rich collection of examples of symplectic manifolds, so the most natural question is "when can a manifold be given a symplectic structure?" Also, in understanding the course, I wanted to verify for myself some of the results we mentioned in passing but didn't have time to explore in the class. So this talk is merging those two goals: what are some easy-to-see obstructions to having a symplectic structure, especially from some of the theorems we only mentioned in passing?

Setting: we have a smooth $2n$-dimensional manifold M and a closed, nondegenerate 2-form ω. Since ω is nondegenerate, $\omega \wedge \cdots \wedge \omega = \omega^n \neq 0$ and hence is a volume form.

Symplectomorphisms, defined to be diffeomorphisms that preserve the symplectic structure, i.e., φ such that $\varphi^*\omega = \omega$.

So first natural question: what's the difference between a symplectomorphism and a volume-preserving diffeomorphism? Certainly, every symplectomorphism preserves volume since preserving ω implies preserving ω^n.

In $\mathbf{R}^2$, $\omega_0 = dx_1 \wedge dy_1$ is a volume form, so a symplectomorphism is an area/volume preserving diffeomorphism. And

Theorem (Moser) If $U \subseteq \mathbf{R}^2$ is diffeomorphic to $B^2(1)$ and $A(U) = A(B^2(1))$, then there exists an area preserving diffeomorphism $U \to B^2(1)$. Effectively, there's nothing to know that can't be known by studying the topology only.

So we have to look at higher dimensions to try and find obstructions. And we do reach our first one, due to Gromov:

Theorem (Gromov, Nonsqueezing) Given $B^{2n}(R)$ and $\{(x_1, y_1, ..., x_n, y_n) \in \mathbf{R}^{2n} \mid |{x_1}^2 + {y_1}^2| < r^2\} = Cyl(r)$, if $B^{2n}(R) \hookrightarrow Cyl(r)$ symplectically, then $R \leq r$.

So here's our first obstruction; there's plenty of volume preserving embeddings from the ball to any cylinder (the cylinder has infinite volume), but there's an obstruction if we want to do this symplectically.

If we have time, we may return to this statement and give a proof sketch, but I want to turn to other rudimentary obstructions, like the fundamental group. We know from alg top due to Van Kampen's theorem that any group has some topological space as its fundamental group. It's an easy construction: given a presentation $\langle g_i \mid r_j \rangle$, build a CW complex with 1 0-cell, i 1-cells glued to make a rose, and realize each relation r_j by gluing 2 cells to make a loop along the relation trivial.

Example $\mathbf{Z} \times \mathbf{Z} = {F(a,b)}/{\langle aba^{-1}b^{-1} \rangle}$ is realized as the fundamental group of the torus, because we glue a 2-cell inside

$$\begin{tikzcd} * \arrow[r, "b"] & * \\ * \arrow[r, "b"] \arrow[u, "a"] & * \arrow[u, "a"'] \end{tikzcd}$$

which is homotopy equivalent to

$$\begin{tikzcd} * \arrow[loop left, "a"] \arrow[loop right, "b"] \end{tikzcd}$$

It's a quick fact that this generalizes to manifolds, assuming your group is finitely generated and your manifold is of dimension at least 4. (Build your rose with disks attached and embed it into $\mathbf{R}^5$; the disks do not intersect in their interiors since G is fin gen. Take a tubular neighborhood of this complex in $\mathbf{R}^5$ and then take its boundary, call it M. One can show it's a 4-manifold, and it clearly still has $\pi_1(M) = G$.)

So that leads to our main question: does the same hold true if we have to additionally impose a symplectic structure on M? If not, we have an obstruction to having a symplectic manifold: if we encounter a manifold with a fundamental group not realizable as the fundamental group of a symplectic manifold, then we know our manifold isn't symplectic. Unfortunately:

Theorem (Gompf) Any finitely generated group G is the fundamental group of some compact 4-dimensional symplectic manifold.

The proof begins by building a sort of "connect sum" that respects symplectic structure. Here is the idea:

Suppose we have $\iota_1 : N \to (M_1, \omega_1)$ and $\iota_2 : N \to (M_2, \omega_2)$ where N is a compact symplectic submanifold of each of codimension 2. Suppose both have trivial normal bundles. We will (*) be able to find neighborhoods U_1 and U_2 of $\iota_1(N)$ and $\iota_2(N)$ in M_1 and M_2 which are symplectomorphic to the trivial disk bundle $N \times B(\varepsilon)$ ($B(\varepsilon)$ has standard symplectic form $dx \wedge dy$). Then remove the center of each $B(\varepsilon)$ to get an annulus A. Since A admits a self symplectomorphism that reverses the boundary components (to see this:)

$$dx \wedge dy = rdr \wedge d\theta = d\frac{r^2}{2} \wedge d\theta$$

means my self symplectomorphism is

$$(r, \theta) \mapsto (\sqrt{\varepsilon^2 - r^2}, -\theta)$$

this implies we can glue M_1 and M_2 along U_1 and U_2 using the symplectomorphism so that the symplectic forms agree on the gluing. Denote the construction $M_1 \# M_2$.

Let's first prove the claim that we can find such neighborhoods U_1 and U_2:

Lemma Let $N \subseteq (M, \omega)$ be a compact symplectic submanifold with trivial normal bundle. Then we can find a neighborhood $U = U(N)$ and a symplectomorphism $\varphi : (N \times B(\varepsilon), \omega \times \omega_0) \to (U, \omega)$ from trivial disk bundle to U which sends $N \times \{0\}$ to N in the obvious way.

Proof. Since $N \subseteq M$ is symplectic, we can specify its normal bundle by orthogonality with respect to ω, and since the bundle is trivial, we can fix a diffeomorphism $f : N \times B(\varepsilon) \to U$ sending $N \times \{0\}$ to N such that f_* sends the fibers $\{x\} \times B(\varepsilon)$ to fibers $T_x N$ of the normal bundle of N. f need not be a symplectomorphism, though.

On $N \times B(\varepsilon)$, we have two forms: $\omega_\alpha := f^*\omega$ and $\omega_\beta := \omega \times \omega_0$. These agree on N.

Let $\omega_t = (1-t)\omega_\alpha + t\omega_\beta$ be the convex combination between ω_α and ω_β. We use Moser's trick to get an isotopy φ_t from $\varphi_0 = \mathrm{id}$ to φ_1 where φ_t is such that ${\varphi_t}^*\omega_t = \omega_\alpha$ and $\varphi_t|_N = \mathrm{id}$.

To do this, let X_t denote the vector field generating the flow φ_t. Differentiating ${\varphi_t}^*\omega_t = \omega_\alpha$, we get

$${\varphi_t}^* \left(\frac{d\omega_t}{dt} + d\iota_{X_t}\omega_t \right) = 0,$$

since $d\omega_t = 0$ as ω_t is closed. (Is this by Cartan's formula?) We claim $\frac{d\omega_t}{dt} = \omega_\beta - \omega_\alpha = d\eta$ for some 1-form η that vanishes on N.

To see this, let $\tau = \omega_\beta - \omega_\alpha$. Let $\pi_s : N \times B(\varepsilon) \to N \times B(\varepsilon)$ be fiberwise multiplication by s. Then,

$$\frac{d}{ds}{\pi_s}^*\tau = {\pi_s}^*\mathcal{L}_{r\frac{\partial}{\partial r}}\tau = d{\pi_s}^*\iota_{r\frac{\partial}{\partial r}}\tau,$$

(ERIC: $\mathcal{L}$ is the Lie derivative, commutes with ${\pi_s}^*$) and

$$\tau = {\pi_1}^*\tau - {\pi_0}^*t = \int_0^1 \frac{d}{ds}{\pi_s}^*\tau ds = d\int_0^1 {\pi_s}^*\iota_{r\frac{\partial}{\partial r}}\tau ds$$

(that was just FTC) so let

$$\eta = \int_0^1 {\pi_s}^*\iota_{r\frac{\partial}{\partial r}}\tau ds,$$

which vanishes on N since τ vanishes on N.

Now since $\frac{d\omega_t}{dt} = d\eta$, there exists some X_t such that $\iota_{X_t}\omega_t = -\eta$, since every ω_t is nondegenerate on $N \times B(\varepsilon)$. We then have an ODE with initial condition $\varphi_0 = \mathrm{id}$, so we can solve for the isotopy φ_t on $N \times B(\varepsilon')$ (shrinking ε as needed for existence of ODE solutions), as we needed to show.

Having done Moser's trick, the composition $\varphi_1 f$ is the desired symplectomorphism. $\square$

Note that above is just a special case of the Weinstein symplectic neighborhood theorem.

So now we have a way to symplectically connect sum two symplectic manifolds. This tool is used in the proof of Gompf's theorem.

Proof. Let $G = \langle g_1, ..., g_k \mid r_1, ..., r_\ell \rangle$. Let F be a compact Riemann surface of genus k. Let $\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_k$ be a standard basis of oriented simple curves for the first homology of F.

Draw this picture.

Let $\gamma_1, ..., \gamma_\ell$ be smoothly immersed circles representing the words $r_1, ..., r_\ell$: substitute every g_i in $r_1, ..., r_\ell$ by α_i. Then let $\gamma_{\ell+i} = \beta_i$ for $i = 1, .., k$.

This means $G = {}^{\pi_1(F)}\!\big/_{\langle \gamma_1, ..., \gamma_{\ell+k} \rangle}$.

Now suppose ρ is a closed 1-form on F that restricts to a volume form on each γ_i. Such a ρ may not exist as of yet, so let us later modify what we have so far to guarantee such a ρ. For now we assume it does, and put a (*) here to return to this fact.

Suppose we have ρ. Let $\mathbf{T}^2$ be the torus. Let $\alpha = S^1 \times \{x\}$ and $\beta = \{x\} \times S^1$ be oriented circles in $\mathbf{T}^2$. Let ω be the product symplectic structure on $F \times \mathbf{T}^2$.

Then $T_i := \gamma_i \times \beta \subseteq F \times \mathbf{T}^2$ are Lagrangian immersed tori (i.e., half-dimensional and the restriction of the symplectic form is 0).

Let θ be the pullback of the volume form on β under projection from $\mathbf{T}^2$. Let $\eta = {\pi_1}^*\rho \times {\pi_2}^*\theta$. Then η is a closed 2-form that restricts to a symplectic form on each T_i.

Let t be small. Then $\widehat{\omega} := \omega + t\eta$ is a symplectic form on $F \times \mathbf{T}^2$, and $(T_i, \widehat{\omega})$ are symplectic immersed manifolds.

Perturb γ_i inside the 3-manifold $F \times \alpha$ to ensure that each T_i is disjointly embedded and disjoint from $\{z\} \times \mathbf{T}^2$ for some point $z \in F$. If the perturbation is sufficiently small and smooth, the T_i will remain symplectic with regard to $\widehat{\omega}$.

Since the normal bundles of perturbed γ_i in $F \times \alpha$ are trivial, so are the normal bundles of T_i in $F \times \mathbf{T}^2$.

Let W denote a complex surface such that N is a symplectic torus in W with trivial normal bundle, and such that $W \setminus N$ is simply connected. Such a N, W exists, because we can take $\mathbf{CP}^2$, take two nondegenerate cubics that intersect in 9 points, and blow up along those intersection points. N can be chosen as a generic fiber of the elliptic "fibration" over $\mathbf{CP}^1$.

We now have all the tools we need. Taking $F \# W$ along a torus $\iota : \mathbf{T}^2 \to F$ has the effect of killing $\iota_* \pi_1(\mathbf{T}^2)$ in $\pi_1(F)$. Therefore, let

$$M = F \# W \# W \# \cdots \# W$$

summing $k + \ell + 1$ times along the tori T_i and $\{z\} \times \mathbf{T}^2$. Then

$$\pi_1(M) = {}^{\pi_1(F)}\!\big/_{\langle \iota_* \pi_1(\mathbf{T}^2)_i \rangle} = {}^{\pi_1(F)}\!\big/_{\langle \gamma_i \rangle} = G.$$

Let's return to our starred claim. We need to show that there exists ρ a 1-form on F with $d\rho = 0$ and $\rho|_{\gamma_i}$ a volume form for all i.

Without loss of generality, we can assume all γ_i are transverse so that their union forms a graph in the torus. Call it Γ.

Let $\gamma = \{y\} \times S^1$ be a circle in $\mathbf{T}^2$ oriented parallel to β. Let D be a small disk, disjoint from α and β and intersecting γ in an arc.

For each edge of Γ e, choose a small disk that intersects the interior of the edge in an arc. Attach via symplectic connect sum a copy of $\mathbf{T}^2$ that glues this disk with the D above, so that γ will be connect summed with e, matching orientations of γ and e.

Write $\gamma_1, ..., \gamma_m$ to replace the old γ_is now with copies of $\gamma = \{y\} \times S^1$ attached, and include copies of α and β for each attached torus.

We still have $G = {}^{\pi_1(F)}\!\big/_{\langle \gamma_1, ..., \gamma_m \rangle}$. We also now know that each edge of Γ formed by the (new) γ_is has a segment which lies in a copy of α, β, or γ.

Now we can begin to construct the desired ρ. Let ρ_0 be a closed 1-form on $\mathbf{T}^2$ that vanishes near D and suppose ρ_0 has positive integrals over α, β, and $\gamma \setminus D$. Such a ρ_0 exists, since we can construct it by collapsing a neighborhood of D to a point, projecting to the diagonal of $\mathbf{T}^2$ so that the projections from α, β, and γ have degree 1, and pulling back the volume form from the diagonal circle.

So put such a ρ_0 form on each copy of $\mathbf{T}^2 \setminus D$, and extend it by zero to all of F. Call this 1-form ρ^*.

Then ρ^* is a closed 1-form that has a positive integral over every edge of Γ.

This implies there exists a volume form θ_i on each γ_i such that

$$\int_e \theta_i = \int_e \rho^*$$

for each edge e in γ_i.

Thus, $\theta_i = \rho^* + df$ for some function f on e. Without loss of generality, f vanishes at the vertices of Γ. Thus, extend f to a smooth function on all of F, and set $\rho = \rho^* + df$. Then ρ is a 1-form on all of F such that

$$d\rho = d\rho^* + ddf = 0 + 0$$

and by construction

$$\rho|_{\gamma_i} = \rho^* + df|_{\gamma_i} = \theta_i,$$

a volume form. The starred claim is proven, and the proof of Gompf completed. □

So the fundamental group is no obstruction to having a symplectic manifold. We now switch gears and return to the Gromov Nonsqueezing Theorem, which does provide an obstruction. Due to time, we're going to give a proof sketch. Recall the theorem:

Gromov Nonsqueezing Theorem If $B^{2n}(R) \hookrightarrow Cyl(r)$ symplectically, then $R \leq r$.

Proof. By contradiction. Assume $\varphi : B^{2n}(R) \hookrightarrow Cyl(r)$ and that $r < R$. There exists $R' \in (r, R)$ such that $B^{2n}(R') \subseteq \mathrm{Int}(Cyl(r))$.

There exists $\varepsilon > 0$ such that φ extends to a symplectic embedding (still denoted φ)

$$\varphi : B^{2n}(R' + \varepsilon) \to Cyl(r - \varepsilon) \subseteq Cyl(r)$$

Let J_0 be the standard almost complex structure, compatible with ω given symplectic form. We push forward J_0 from $B^{2n}(R' + \varepsilon)$ by φ to $\varphi(B^{2n}(R' + \varepsilon)) \subseteq Cyl(r - \varepsilon)$. Then extend to J_1 over all of $\mathbf{C}^n = \mathbf{R}^{2n}$ via

$$J_1 = \begin{cases} \varphi_*(J_0) & \text{on } \varphi\left(B^{2n}\left(R' + \frac{\varepsilon}{2}\right)\right) \\ J_0 & \text{on } \mathbf{C}^n \setminus D(r - \varepsilon) \times [-k+1, k-1]^{2n-2} \\ & \text{smooth extension elsewhere, so that } J_1 \text{ is still compatible to } \omega. \end{cases}$$

The k is such that

$$\varphi(B^{2n}(R' + \varepsilon)) \subseteq D(r - \varepsilon) \times [-k+1, k-1]^{2n-2}.$$

The proof of the theorem follows three more steps:

Step 1 Prove there exists a J_1 holomorphic map $f : (D, \partial D) \to (\mathbf{C}^n, \mathbf{C}^n \setminus \varphi(B^{2n}(R' + \varepsilon)))$ such that

$$\int_{\mathrm{im}\, f} \omega_0 \leq A(Cyl(r) \cap \mathbf{R}^2)(\text{area of a flat disk}) = \pi r^2$$

and $\varphi(0) \in \mathrm{im}\, f$.

Step 2 Consider the preimage of $\mathrm{im}\, f$; i.e., $\varphi^{-1}(\mathrm{im}\, f) \cap B^{2n}(R)$.

Since $\mathrm{im}\,(f|_{\partial D}) \subseteq \mathbf{C}^n \setminus \varphi(B^{2n}(R' + \varepsilon))$, $\varphi^{-1}(\mathrm{im}\, f) \cap B^{2n}(R)$ is a proper surface in $B^{2n}(R)$ that passes through the origin.

Also

$$J_1|_{\varphi(B^{2n}(R'+\varepsilon))} = \varphi_* J_0$$

and $\operatorname{im} f$ is a J_1 holomorphic curve. So $\varphi^{-1}(\operatorname{im} f)$ is a J_0 holomorphic curve in $\mathbf{C}^n$, and hence a minimal surface w.r.t. the standard metric on $\mathbf{C}^n$.

Then apply the following monotonicity formula:

Monotonicity Formula A proper minimal surface S passing through the origin in $B^{2n}(R) \to \mathbf{C}^n$ has $A(S) \geq \pi R^2$. $A(S) = \pi R^2$ if and only if S is a flat disk.

With this result, $A(\varphi^{-1}(\operatorname{im} f) \cap B^{2n}(R)) \geq \pi R^2$.

Step 3 The contradiction:

Lemma 1 If $f : (S, j) \to (M, J)$ is J holomorphic and S is closed without boundary, then $A(f) = \int f^*\omega$.

Lemma 2 If f is J holomorphic, $f^*\omega = |\partial_J f|^2 dA \geq 0$ where dA is the area form on (S, j).

Now

$$\begin{aligned}
\pi R^2 &\leq A(\varphi^{-1}(\operatorname{im} f) \cap B^{2n}(R)) \\
&= \int_{\varphi^{-1}(\operatorname{im} f) \cap B^{2n}(R)} \omega_0 \text{ (lemma 1)} \\
&= \int_{\varphi^{-1}(\operatorname{im} f) \cap B^{2n}(R)} \varphi^*\omega_0 \text{ (φ is symplectic)} \\
&= \int_{\operatorname{im} f \cap \varphi(B^{2n}(R))} \omega_0 \text{ (change of variables)} \\
&\leq \int_{\operatorname{im} f} \omega_0 \text{ (lemma 2)} \\
&\leq \pi r^2,
\end{aligned}$$

contradicting $r < R$.

It only remains to justify **Step 1**. To do this, recall that we have J_1 an almost complex structure on $Cyl(r) = D(r) \times \mathbf{C}^{n-1}$ with J_1 standard on $\mathbf{C}^n \setminus D(r-\varepsilon) \times [-k+1.k-1]^{2n-2}$. So J_1 is standard near the boundary of $D\left(r - \frac{\varepsilon}{2}\right) \times [-k, k]^{2n-2}$ which we will denote $D(r, \varepsilon, k)$.

So we push forward J_1 on $D(r, \varepsilon, k)$ to $D\left(r - \frac{\varepsilon}{2}\right) \times \mathbf{T}^{2n-2}(k)$ (still denoting it J_1), where $\mathbf{T}^{2n-2}(K)$ is the torus gluing $-k$ to k in $[-k, k]$; i.e., $\mathbf{T}^{2n-2}(k) = S^1(k) \times \cdots \times S^1(k)$ $2n-2$ times.

Since $\varphi(B^{2n}(r+\varepsilon)) \subseteq D(r, \varepsilon, k)$, $\varphi(B^{2n}(r+\varepsilon))$ can be thought of as a subset of $D\left(r - \frac{\varepsilon}{2}\right) \times \mathbf{T}^{2n-2}(k)$, and J_1 is still the standard almost complex structure near the boundary of this space.

Embed $D\left(r - \frac{\varepsilon}{2}\right)$ into $S^2\left(\frac{r}{2}\right)$ via an area preserving map $\psi : D\left(r - \frac{\varepsilon}{2}\right) \to S^2\left(\frac{\varepsilon}{2}\right)$. Embed $D\left(r - \frac{\varepsilon}{2}\right) \times \mathbf{T}^{2n-2}(k)$ into $S^2\left(\frac{r}{2}\right) \times \mathbf{T}^{2n-2}(k)$ via the symplectic map $\psi \times \mathrm{id} : D\left(r - \frac{\varepsilon}{2}\right) \times \mathbf{T}^{2n-2}(k) \to S^2\left(\frac{r}{2}\right) \times \mathbf{T}^{2n-2}(k)$.

Write $\omega_{r,k} = \omega_r \oplus \omega_k$ to be the product of standard symplectic structures on $S^2\left(\frac{r}{2}\right) \times \mathbf{T}^{2n-2}(k)$.

Extend the structure $(\psi \times \mathrm{id})_* J_1$ on $(\psi \times id)(D\left(r - \frac{\varepsilon}{2}\right) \times \mathbf{T}^{2n-2}(k))$ to all of $S^2\left(\frac{r}{2}\right) \times \mathbf{T}^{2n-2}(k)$. Denote it $\widetilde{J_1}$. Extend so that $\widetilde{J_1}$ is compatible to $\omega_{r,k}$.

Note that $\pi_2(S^2\left(\frac{r}{2}\right) \times \mathbf{T}^{2n-2}(k)) \cong \mathbf{Z} = \langle [S^2\left(\frac{r}{2}\right) \times \{x\}] \rangle = \langle A \rangle$.

Let $p_0 \in S^2\left(\frac{r}{2}\right) \times \mathbf{T}^{2n-2}(k)$ be the point corresponding to $\varphi(0) \in Cyl(r)$.

Now, we claim **Step 1** is proven when we find a $\widetilde{J_1}$ holomorphic sphere $\widetilde{C}$ with $[\widetilde{C}] = A \in \pi_2(S^2\left(\frac{r}{2}\right) \times \mathbf{T}^{2n-2}(k))$ with $p_0 \in \widetilde{C}$.

To see this claim, suppose such a $\widetilde{C}$ exists. Let $u : S^2 \to S^2\left(\frac{r}{2}\right) \times \mathbf{T}^{2n-2}(k)$ be the map representing $\widetilde{C}$; i.e., $\operatorname{im} u = \widetilde{C}$. Since $[\widetilde{C}] = A$, by Lemma 1 above,

$$\int_{\operatorname{im} u} \omega_{r,k} = \pi r^2.$$

Then since $[\widetilde{C}] = [S^2\left(\frac{r}{2}\right) \times \{x\}]$, $\pi u : S^2 \to S^2\left(\frac{r}{2}\right)$ is surjective. Now we can choose the curve $\operatorname{im} f$ needed in **Step 1** to be

$$C := (\psi \times \mathrm{id})^{-1}(\widetilde{C}) \subseteq D\left(r - \frac{\varepsilon}{2}\right) \times \mathbf{T}^{2n-2}(k),$$

regarded as a subset of $Cyl\left(r - \frac{\varepsilon}{2}\right) \subseteq Cyl(r)$. C is a proper surface in $Cyl\left(r - \frac{\varepsilon}{2}\right)$, so $C \cap \varphi(B^{2n}(r+\varepsilon))$ is a proper surface in $\varphi(B^{2n}(r+\varepsilon))$, since $\varphi(B^{2n}(r+\varepsilon)) \subseteq Cyl\left(r - \frac{\varepsilon}{2}\right)$.

And to check that **Step 1** is complete, see that

$$
\begin{aligned}
\int_C \omega_0 &= \int_C (\psi \times \mathrm{id})^* \omega_{r,k} \text{ (symplectic)} \\
&= \int_{(\psi \times \mathrm{id})^{-1}(\widetilde{C})} (\psi \times \mathrm{id})^* \omega_{r,k} \text{ (definition)} \\
&= \int_{\widetilde{C} \cap (\psi \times \mathrm{id})(D\left(r - \frac{\varepsilon}{2}\right) \times \mathbf{T}^{2n-2}(k))} \omega_{r,k} \text{ (change of variables)} \\
&\leq \int_{\widetilde{C}} \omega_{r,k} \text{ (lemma 2)} \\
&= \pi r^2,
\end{aligned}
$$

and see that since $p_0 \in \widetilde{C}$, $\varphi(0) \in C$.

$\square$