Redo Calculation of Expanded Potential

TAKING EXPANDED POTENTIAL

$$H = \frac{p_x^2}{2} - \frac{p_y^2}{2} + \frac{\omega_x^2}{2}x^2 - \frac{\omega_y^2}{2}y^2 + \lambda\left(x^4 + 4y^2x^2 + y^4\right) + \dots$$

$$\frac{d}{dt}\hat{x}(t) = \frac{-i}{\hbar}\left[\hat{x}, H\right].$$

Using $\left[\hat{q}_i, \hat{p}_j\right] = i\hbar\delta_{ij}$

$$i\hbar\frac{d}{dt}\hat{x}(t) = \tfrac{1}{2}\left[\hat{x}, p_x^2\right] - \tfrac{1}{2}\cancelto{0}{\left[\hat{x}, \hat{p}_y^2\right]} + \frac{\omega_x^2}{2}\cancel{\left[\hat{x}, \hat{x}^2\right]} - \frac{\omega_y^2}{2}\cancel{\left[\hat{x}, \hat{y}^2\right]} + \lambda\cancel{\left[\hat{x}, \hat{x}^4\right]} + \lambda\cancel{\left[\hat{x}, \hat{y}^2\hat{x}^2\right]} + \lambda\cancel{\left[\hat{x}, \hat{y}^4\right]}$$

$$i\hbar\frac{d}{dt}\hat{x} = \frac{1}{2}\left[\hat{x}, \hat{p}_x^2\right]$$

$$= \frac{1}{2}\left(\left[\hat{x}, \hat{p}_x\right]\hat{p}_x + \hat{p}_x\left[\hat{x}, \hat{p}_x\right]\right)$$

$$\cancel{i\hbar}\frac{d}{dt}\hat{x} = \frac{1}{2}\left(2\hat{p}_x\right)\cancel{i\hbar}$$

$$\boxed{\frac{d}{dt}\hat{x} = \hat{p}_x} \quad (1)$$

CAN SHOW FOR $\hat{Y}$ OPERATOR THAT

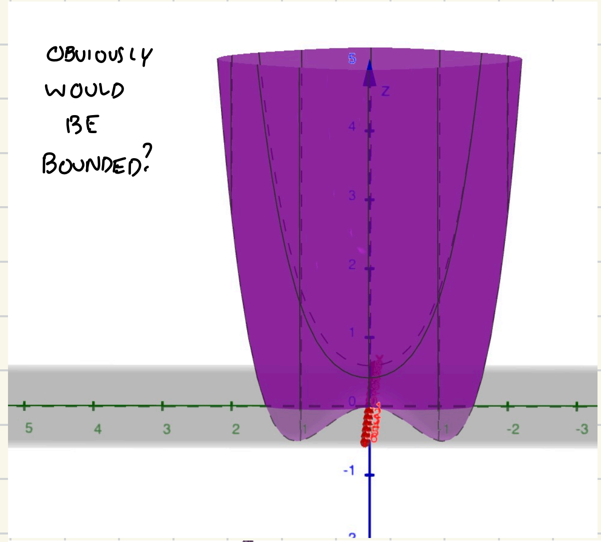
$$\boxed{\frac{d}{dt}\hat{y} = -\hat{p}_y} \quad (2)$$

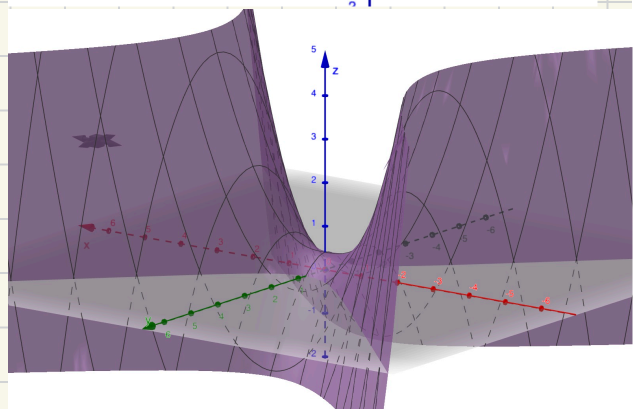
FIRST ORDER POTENTIAL EXPANSION

$$V_{\text{tot}} = \frac{\omega_x^2}{2}x^2 - \frac{\omega_y^2}{2}y^2 + \lambda(x^4 + 4y^2x^2 + y^4) + \cdots,$$

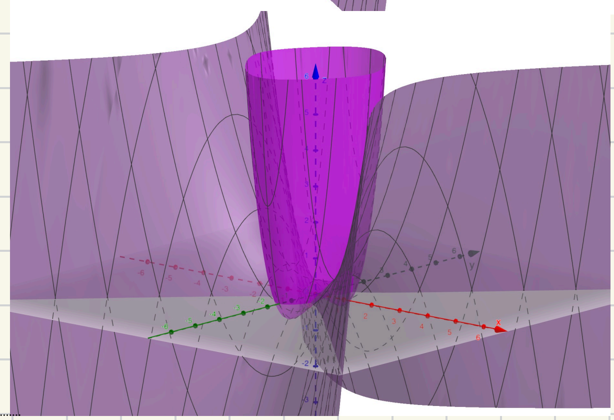
TRUE POTENTIAL

$$V = \frac{x^2}{2} - \frac{y^2}{2} + \lambda\left[(x^2 - y^2 - 1)^2 - 4x^2\right]^{-1/2}$$





BOTH ON SAME PLOT



Now looking at $\hat{p}_x$

$$-i\hbar\frac{d}{dt}\hat{p}_x = \left[\hat{H}, \hat{p}_x\right] = \frac{1}{2}\cancelto{0}{\left[\hat{p}_x^2, \hat{p}_x\right]} - \cancelto{0}{\left[\hat{p}_y^2, \hat{p}_x\right]} + \frac{\omega_x^2}{2}\left[\hat{x}^2, \hat{p}_x\right] - \frac{\omega_y^2}{2}\cancelto{0}{\left[\hat{y}^2, \hat{p}_x\right]} + \lambda\left[\hat{x}^4, \hat{p}_x\right] + 4\lambda\left[\hat{y}^2\hat{x}^2, \hat{p}_x\right] + \lambda\cancelto{0}{\left[\hat{y}^4, \hat{p}_x\right]}$$

$$= \frac{\omega_x^2}{2}\left[\hat{x}^2, \hat{p}_x\right] + \lambda\left[\hat{x}^4, \hat{p}_x\right] + 4\left[\hat{y}^2\hat{x}^2, \hat{p}_x\right] = \frac{\omega_x^2}{2}\left(\left[\hat{x}, \hat{p}_x\right]\hat{x} + \hat{x}\left[\hat{x}, \hat{p}_x\right]\right) + \lambda\left(\left[x^2, p_x\right]\hat{x}^2 + \hat{x}^2\left[x^2, p_x\right]\right) + 4\lambda\left(\left[\hat{x}^2, p_x\right]\hat{y}^2 + \hat{x}^2\cancelto{0}{\left[y^2, p_x\right]}\right)$$

$$= \frac{\omega_x^2}{\cancel{2}}\left(\cancel{2}i\hbar\hat{x}\right) + \lambda\left(\left(\left[x, p_x\right]\hat{x} + \hat{x}\left[x, p_x\right]\right)\hat{x} + \hat{x}\left(\left[x, p_x\right]\hat{x} + \hat{x}\left[x, p_x\right]\right)\right) + 4\lambda\left(\left[x, p_x\right]\hat{x} + \hat{x}\left[x, p_x\right]\right)\hat{y}^2$$

$$= i\hbar\omega_x^2\hat{x} + \lambda\left(\left(i\hbar\hat{x} + i\hbar\hat{x}\right)\hat{x} + \hat{x}\left(i\hbar\hat{x} + i\hbar\hat{x}\right)\right) + 4\lambda\left(2i\hbar\hat{x}\right)\hat{y}^2$$

$$= i\hbar\omega_x^2\hat{x} + \lambda\left(4i\hbar\hat{x}^2\right) + 8\lambda i\hbar\hat{x}\hat{y}^2$$

$$\cancel{-i\hbar}\frac{d}{dt}p_x = \cancel{i\hbar}\omega_x^2\hat{x} + 4\lambda\cancel{i\hbar}\hat{x}^2 + 8\lambda\cancel{i\hbar}\hat{x}\hat{y}^2$$

Likely incorrect as ghosts don't have regular commutator relations.

$$\boxed{\frac{d}{dt}\hat{p}_x = -\omega_x^2\hat{x} - 4\lambda\hat{x}^2 - 8\lambda\hat{x}\hat{y}^2 \qquad (3)}$$

Can argue from inspection that

$$\boxed{\frac{d}{dt}p_y = +\frac{\omega_y^2}{2}\hat{y} - 4\lambda\hat{y}^2 - 8\lambda\hat{x}^2\hat{y} \qquad (4)}$$

(Note sign flip for oscillator term)

Numerical Integration

$$y_1 = y_0 + \frac{dy}{dt}\bigg|_{t_0}(t_1 - t_0)$$

$$y_{i+1} = y_i + \frac{dy}{dt}\bigg|_{t_i}(t_{i+1} - t_i) \longrightarrow$$

For our formula

$$x_{i+1} = x_i + p_{x_i} = x_i + \int_{t_i}^{t_{i+1}}\left(\omega_x^2 x + 4\lambda\left(\hat{x}^3 + 2\hat{y}^2\hat{x} + 2\hat{x}\hat{y}^2\right)\right)dt$$

EULER'S NUMERICAL INTEGRATION

$$y'(t) = f(t, y(t)),\quad y(0) = y_0$$

$$\boxed{y_{i+1} = y_i + \frac{d}{dt}\hat{y}\bigg|_{t_i}\cdot\Delta t_{i,i+1}}$$

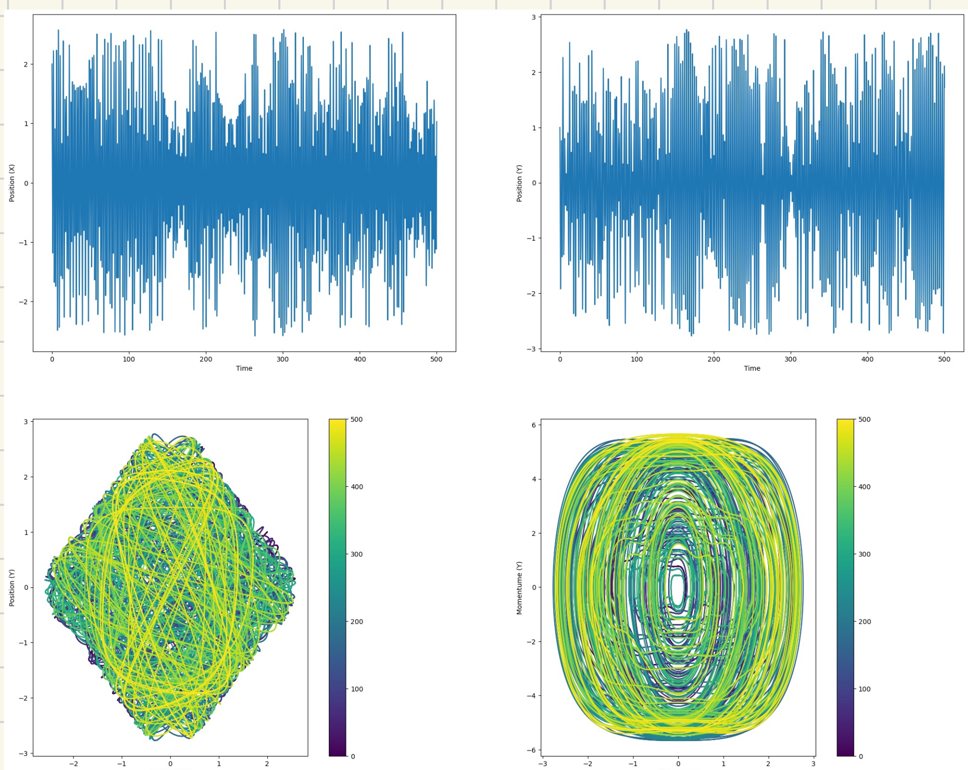
Euler Method

$$\int_{t_n}^{t_{n+1}} f(\tau, y(\tau))\,d\tau \approx \Delta t\, f(t_n, y(t_n))$$

PLOT OF THIS RELATION

$p_x(0) = \emptyset$ $\quad x_0 = 2$

$p_y(0) = \emptyset$ $\quad y_0 = 1$



Full Interaction Potential

$$H = \frac{p_x^2}{2} - \frac{p_y^2}{2} + \frac{x^2}{2} - \frac{y^2}{2} + \lambda\left[(x^2-y^2-1)^2 - 4x^2\right]^{-1/2}$$

$$\boxed{\frac{d\hat{x}}{dt} = p_x} \quad (1) \qquad \boxed{\frac{d\hat{y}}{dt} = -p_y} \quad (2)$$

$$-i\hbar\frac{d\hat{p}_x}{dt} = [\hat{H}, \hat{p}_x] = \frac{1}{2}\cancelto{0}{[\hat{p}_x^2, \hat{p}_x]} - \frac{1}{2}\cancel{[p_y^2, p_x]} + \frac{1}{2}[\hat{x}^2, \hat{p}_x] - \frac{1}{2}\cancelto{0}{[\hat{y}^2, \hat{p}_x]}$$

$$+ \lambda\left[\left((x^2-y^2-1)^2 - 4x^2\right)^{-1/2}, \hat{p}_x\right] \qquad [F(\hat{x}), \hat{p}_x] = i\hbar\frac{\partial}{\partial\hat{x}}F(x)$$

$$= i\hbar\hat{x} + \lambda\, i\hbar\left(\frac{\partial}{\partial x}\left((x^2-y^2-1)^2 - 4x^2\right)^{-1/2}\right)$$

$$= i\hbar\hat{x} + \lambda\, i\hbar\left(\frac{-2\hat{x}^3 + 2\hat{x}\hat{y}^2 + 6\hat{x}}{\left((x^2-y^2-1)^2 - 4x^2\right)^{3/2}}\right)$$

$$\boxed{\frac{d\hat{p}_x}{dt} = -\hat{x} - \lambda\left(\frac{-2\hat{x}^3 + 2\hat{x}\hat{y}^2 + 6\hat{x}}{\left((x^2-y^2-1)^2 - 4x^2\right)^{3/2}}\right)} \quad (3)$$

Now similarly for p_y

$$-i\hbar\frac{d}{dt}p_y = -i\hbar\hat{y} + i\hbar\lambda\left(\frac{2yx^2 - 2y^3 - 2y}{\left((-x^2+y^2+1)^2 - 4y^2\right)^{3/2}}\right)$$

$$\boxed{\frac{dp_y}{dt} = \hat{y} - \lambda\left(\frac{2yx^2 - 2y^3 - 2y}{\left((-x^2+y^2+1)^2 - 4y^2\right)^{3/2}}\right)} \quad (4)$$