Gravitational Waves and Analysis through Fourier Techniques

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The Fourier transform is a useful tool in signal analysis that allows functions dependent on time to be broken into their fundamental frequencies. When applied to gravitational waves, this gives us a method to separate the background noises picked up by our apparatus from our desired data. An example is given that describes its application to the 'spin down' of neutron stars. This also includes a method of accounting for Doppler shifted frequencies due to the Earth's rotation and revolution about the sun. We can then apply the methods outline within this paper to allow for gravitational wave astronomy to become a viable method for gaining information about some of the most massive objects in the universe.

I. INTRODUCTION

The Fourier transform allows signals that are functions of time to be decomposed into their fundamental frequencies. This means that a superposition of signals can be broken down into their specific frequencies for later spectral analysis. The detection of gravitational waves requires the analysis of continuous, month long data series. In order to analyze a signal, scientists need to convert the time based nature of the wave into a frequency based function. When looking at gravitational waves emitted by pulsars or merging black holes, a Doppler shift needs to be considered due to the unique position of a spinning neutron star and the motion of the Earth. Therefore all possible frequencies need to be tested. Each segment of data is tested by looking at the frequency spectrum of the data and removing the background noise, largely through Fourier methods, from candidate frequencies that match a specific criteria associated with a specific pulsar.

Pulsars, in accordance with the general theory of relativity, are believed to be sources of continuous gravitational waves. They are believed to be slowly losing energy due to a partial mass imbalance on their surface. Changes in physical distances and slight disturbances in space time can be detected through the use of large scale laser interferometer observatories. An interferometer is a system composed of two intersecting light beams of equal length being directed into an extremely small detector, as shown in Figure (1).

These beams happen to be in phase, so upon reaching a detector the power associated with the beam is a constant value. If there is a slight disturbance in the environment through a shift in spacetime, the beams will be slightly out of phase resulting in a noticeable disturbance picked up by the detector.

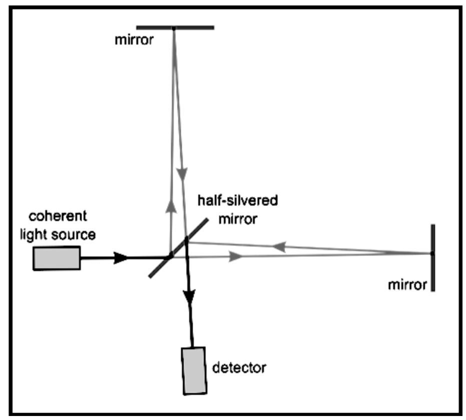


FIG. 1. Diagram of a large scale interferometer. An interferometer is used to measure variations in physical space due to the existence of gravitational waves.

II. THE FOURIER TRANSFORM

A Fourier transform can transform between a time to frequency domain. The method of Fourier transforms can also be applied to position spaces and momentum spaces, where the corresponding transform transposes the function between the two spaces. However we will be looking exclusively at transforms involving time and frequency within this paper. The transform takes a function on the interval $0 < t < \infty$ and transforms it from its time domain to its respective frequency domain. Linear operations performed in one domain have a corresponding operation in the other domain. The transform from time to frequency is given by the function:

$$\tilde{X}(\omega) = \int_0^\infty x(t)e^{-i\omega t}dt \tag{1}$$

where

$$\tilde{X}(\omega) : -\infty < \omega < \infty$$

This transform will then give us a frequency spectrum of our original function, which has numerous applications in mathematics. If we desire to preform linear operations within our frequency domain in order to see its effects within the time domain, the reverse Fourier transform is given by the equation:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{X}(\omega)e^{i\omega t}dt \tag{2}$$

where

$$x(t) : 0 < t < \infty$$

This gives us a complete correspondence between a function and the frequency and time domains.

A. Simple Fourier Transform Example: Top Hat Function

A fairly straightforward example of the mapping properties of the Fourier transform can be illustrated by transforming a top hat function. The example described below was given by Professor Jorge Peñarrubia at Edinburgh University [6]. If we define a function Π_x as

$$\Pi_x = \begin{cases} h & d-a \leq x \leq d+a \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Our definition of the Fourier transform described by equation (1) allows us to describe $\tilde{F}(k)$ as

$$\tilde{F}(k) = \int_{-inf}^{inf} \Pi_x e^{-ikx}dx \tag{4}$$

which simplifies due to the limitations associated with Π_x to

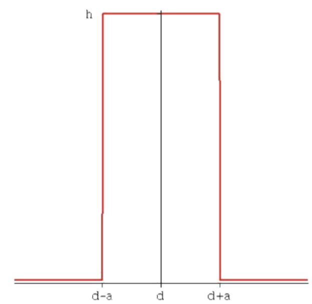
$$\tilde{F}(k) = h\int_{d-a}^{d+a} e^{-ikx}dx \tag{5}$$

Making a substitution $u = x - d$ simplifies the calculations. From here we can evaluate the integral as described below

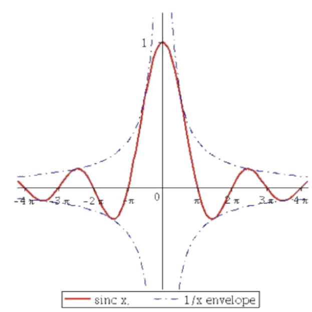
$$\tilde{F}(k) = he^{ikd}\int_{-a}^{a} e^{-iku}du = he^{ikd}\left[\frac{e^{-iku}}{-ik}\right]_{-a}^{a}$$

$$\tilde{F}(k) = \frac{2he^{ikd}\sin(\text{ka})}{k} = 2he^{ikd}\text{sinc}(\text{ka}) \tag{6}$$

From this we have deduced that a Fourier transform of a piecewise function centered at the point $x = d$ maps to a sinc function in k-space. This example is not particularly revealing of the usefulness of the Fourier Transform. Still this example illustrates the ability to change the nature of a function or even differential equation, which can often be solved much easier in an equivalent space. An illustration of the associated transform can be seen on the following column.



Sketch of top-hat function defined in Eqn. (3)



Sketch of $\text{sinc}\, x \equiv \frac{\sin x}{x}$

FIG. 2. An example of the application of the Fourier transform converting a function within the time domain to its frequency domain. The transform shown is of the piecewise function described in the section prior.

III. METHODS AND APPLICATIONS

The use of the Fast Fourier transform (FFT) can be utilized in the analysis of data obtained from gravitational wave interferometers. The Fast Fourier Transform is an algorithm that can compute and greatly reduce the complexity of a series of Discrete Fourier transforms (DFT's), or its inverses, rapidly through factorizing a matrix composed of DFT's. As a result, the FFT can take data streams obtained over month to year long periods and decompose the signal into a manageable frequency spectrum, where background noise and sources of disturbances can then be managed more easily. The main problem associated with this method is that there are numerous sources of background disturbances. These disturbances include the rotation of the Earth and the Earth's orbit around the sun, which can cause a doppler effect that shifts the frequencies of incoming waves. The signals picked up by the a ground based interferometer therefore need to be broken down into their fundamental frequencies in order to remove the frequencies that do not meet a specified criteria. The following method was first described by D.C. Srivastava and S.K. Sahay in their paper "Data Analysis of Continuous Gravitational Wave Signal: Fourier Transform I" [1]. In general, a ground based interferometer detector has a response R(t) composed of the polarization states of a signal are given by the function

$$R(t) = F_+(t)h(t)_+ + F_x(t)h_x(t) \tag{7}$$

where the functions $h(t)_+$ and $h(t)_x$ represent the polarization states of the system, which are dependent on the initial phase of the incoming beam. $F(t)_+$ and $F(t)_x$ are beam pattern functions and correspond to the variation in amplitude. The subscripts "x" and "+" specify which arm of the interferometer either beam is associated with. We have also made the assumption that the beam arms are completely orthogonal.

The response of our detector as a function of frequency can be described by

$$\tilde{R}(\omega) = \tilde{F_+}(\omega) + \tilde{F_x}(\omega) \tag{8}$$

where

$$\tilde{F_+}(\omega) = \int h_+(t)F_+(t)e^{-i\omega t}dt \tag{9}$$

and

$$\tilde{F_x}(\omega) = \int h_x(t)F_x(t)e^{-i\omega t}dt \tag{10}$$

Difficulty may arise when evaluating these integrals, as some may not be able to be solved analytically. Additionally, data points extracted at a rate of one point per fifteen seconds integrated over a period of a year results in approximately two million data points to sort through. This results in a monumental effort to sort through unwanted signals. However, through performing the Fast Fourier Transform on the response $R(t)$, we can observe the complete response of a pulsar that is emitting gravitational waves at a specific frequency and associated background noise.

A. Example: Spin Down Pulsars

A specific application of the Fourier approach described above is the case of a pulsar experiencing a 'spin down.' Pulsars are thought to lose energy through gravitational waves and electromagnetic braking, resulting in an unstable rotational frequency that varies according to the age and size of the pulsar. Professor Srivastav and Doctor Sahay have developed a method to account for this varying rotational frequency in their paper "Data Analysis of Continuous Gravitational Wave: Fourier transform II." [2]. Within the paper they state that over an observation time period T_0, we can divide the total interval into M different parts, each with a time interval of Δt. The total period of integration is given by $T_0 = M\Delta t$. The time interval is small so that the frequency associated can be considered constant for a given interval. For this example we define our functions $h_+(t)$ and $h_x(t)$, with $h_+(t) = h_{o+}\cos(\phi t)$ and $h_x(t) = h_{ox}\sin(\phi t)$ and ϕ being taken as the phase of gravitational wave signal and h_0 being an arbitrary amplitude. For this method, we will be integrating over the nth interval, and sum the corresponding results. Let

$$\tilde{h}(t)_i = \int_{t_0+n\Delta t}^{t_0+(n+1)\Delta t} h(t')_i e^{-i2\pi\omega t'}dt' \tag{11}$$

with

$$t' = t + t_o + n\Delta t \tag{12}$$

The Fourier transform for the interval under consideration (for $h_+(t)$) is given by

$$\tilde{h}_+(\omega) = \int_0^{\Delta t} h_{o+}\cos(\phi(t+t_o))e^{-i2\pi\omega(t+t_o+n\Delta t)}dt + \int_0^{\Delta t} h_{o+}\cos(\phi(n\Delta t))e^{-i2\pi\omega(t+t_o+n\Delta t)}dt \tag{13}$$

The specific steps involved in evaluating the integral described in equation (13) are extremely arduous and consider detailed initial conditions of the system. A full description of the steps involved can be found in reference [2]. After integration, we obtain a solution of the form

$$\tilde{h}_+(\omega) = \frac{f_o + f}{2\omega_{\text{orb}} f_{\text{orb}}} \sum_{k=-\infty}^{k=\infty} \sum_{m=-\infty}^{m=\infty} e^{iA} B(C - iD) \tag{14}$$

where the terms A,B,C,D are functions dependent on the phase ϕ of the gravitational wave. The term ω_{orb} is the orbital angular velocity of the earth around the sun. When performing data analysis of the obtained frequency spectrum a criteria has to be imposed in order to remove the majority of undesired frequencies. This criteria is obtained through the application of expected frequencies from numerical

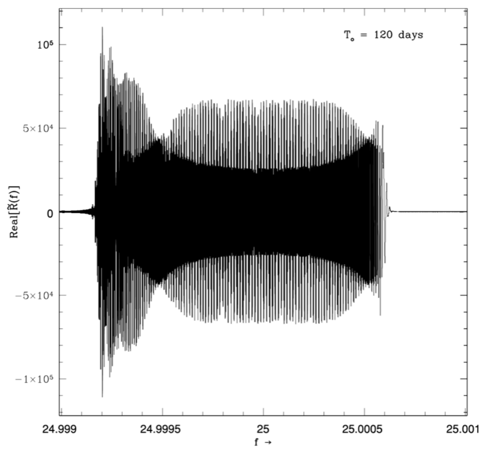


FIG. 3. An example of the noise associated with a Fourier transformed frequency spectrum centered around $f = 25$ Hz. Each spike corresponds to a specific signal picked up by the detector on the interferometer.

models to the obtained data. Due to the possibility of Doppler shifted data due to the frequencies associated with the earths rotation and orbit around the sun, all frequencies require testing across all sky positions of the tested pulsar. In order for a true gravitational wave to be confirmed a candidate signal must fit the parameters for multiple segments divided into two day periods, as a gravitational wave should persist from one segment to another.

IV. RESULTS

The application of the mathematical methods to the detection of gravitational waves will greatly cut down time spent on data analysis. Furthermore, a frequency based approach towards detecting gravitational waves would ultimately be impossible without Fourier methods, as it would likely be impossible to separate the background noise from the desired data. The detecting capabilities of the large scale interferometers are constantly being upgraded, allowing for more sensitive measurements. The efficiency of the method described above will depend entirely upon the frequency spectrum detected. Since gravitational waves exist on an extremely miniature scale, the usefulness of the Fourier transform is ultimately limited to the detector itself. The application of the FFT to gravitational wave detection allowed for an extremely important testing of Einstein's general theory of relativity. As of September 14, 2015, the first gravitational wave has been confirmed at two separate laser interferometer gravitational wave observatories across the United states. Prior to this discovery, this was the last remaining unconfirmed prediction from the general theory of relativity. Additionally, this allows for a new era of large scale observation through the application of gravitational wave astronomy. There are limitations associated with astronomy based upon electromagnetic radiation, as nebulae and massive objects can obscure view. Through Fourier techniques the structure of the universe can now be seen through a new lens.

V. DISCUSSION

The Fourier transform is an extremely useful tool for decomposing a signal into its superimposed frequencies. Problems can often be solved in an equivalent domain much easier due to the unique properties of the Fourier Transform. We can then transform the solution back to our original domain and have a valid solution to our original problem. Signals appear in the frequency domain as Dirac delta functions centered upon its fundamental frequency and, by imposing boundary conditions upon the frequency spectrum, we can eliminate signals that fail to meet a specific criteria. The specific analysis of the Fast Fourier Transform allows for its application towards the detection of gravitational waves and gravitation wave astronomy. It usefulness comes from its ability to separate background interference from incoming signals, which is extremely invaluable due to the scale of the waves being observed. The applications of the FFT are not limited to the topics within this paper, and can be further explored in the field of gravitational wave spectral analysis. This includes varying time intervals of integration and an N-component signal, which is predicted to emit quadrupole gravitational waves at two separate frequencies. This process is also described by David Thompson in his paper "Gravitational Wave Signal Identification and Transformations in Time-Frequency Domain" [4]. Additionally, we can modify our current equations to allow for a non zero initial time and phase. The methods obtained by Professor Srivasta and Dr. Sahay [1] greatly reduce the time consumed by data analysis. These methods described in the sections above also account for non-ideal situations, such as Doppler shifted waves and incoming frequencies that vary with time.

VI. ACKNOWLEDGEMENTS

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