Paired Data

Paired data often arise when measuring the same individuals twice (before and after a period).

Paired Data

Paired data often arise when measuring the same individuals twice (before and after a period).

Individual	Weight in 2010	Weight in 2020	Diff
Marion	140	135	-5
Sylvester	190	249	59
Florence	183	183	0
David	90	134	44
Gertrude	208	180	-28
$\vdots$	$\vdots$	$\vdots$	$\vdots$

What would an implied question be?

Paired Data

Paired data often arise when measuring the same individuals twice (before and after a period).

Individual	Weight in 2010	Weight in 2020	Diff
Marion	140	135	-5
Sylvester	190	249	59
Florence	183	183	0
David	90	134	44
Gertrude	208	180	-28
⋮	⋮	⋮	⋮

What would an implied question be?

Do individual humans tend to gain weight over time?

Paired Data

Paired data often arise when measuring the same individuals twice (before and after a period).

Individual	Weight in 2010	Weight in 2020	Diff
Marion	140	135	-5
Sylvester	190	249	59
Florence	183	183	0
David	90	134	44
Gertrude	208	180	-28
$\vdots$	$\vdots$	$\vdots$	$\vdots$

What would an implied question be?

Do individual humans tend to gain weight over time?

Two sets of observations are paired if each observation in one set has a special correspondence or connection with exactly one observation in the other data set.

Unpaired Data

Two separate random samples would produce unpaired data.

year=2010	
Individual	Weight
Lonzo	140
Rosalia	190
Leora	183
Otis	90
Edward	208
⋮	⋮

year=2020	
Individual	Weight
Henry	310
Harvey	250
Phoebe	210
Donna	150
John	110
⋮	⋮

What would an implied question be?

Unpaired Data

Two separate random samples would produce unpaired data.

year=2010	
Individual	Weight
Lonzo	140
Rosalia	190
Leora	183
Otis	90
Edward	208
⋮	⋮

year=2020	
Individual	Weight
Henry	310
Harvey	250
Phoebe	210
Donna	150
John	110
⋮	⋮

What would an implied question be?

Did humans (as a species) tend to gain weight over time?

Unpaired Data

Two separate random samples would produce unpaired data.

year=2010	
Individual	Weight
Lonzo	140
Rosalia	190
Leora	183
Otis	90
Edward	208
⋮	⋮

year=2020	
Individual	Weight
Henry	310
Harvey	250
Phoebe	210
Donna	150
John	110
⋮	⋮

What would an implied question be?

Did humans (as a species) tend to gain weight over time?

We will discuss unpaired analysis in Chapter 5.3 (next class).
With paired data, we consider a **mean of differences**.
With unpaired data, we consider a **difference of means**.

Derivation of paired formulas

Let random variable D_i represent the (unknown) difference from a (yet to be) randomly selected individual i.

Derivation of paired formulas

Let random variable D_i represent the (unknown) difference from a (yet to be) randomly selected individual i.
We want to predict what happens when we find a mean of differences.

$$\bar{D} = \frac{D_1 + D_2 + D_3 + ... + D_n}{n}$$

Derivation of paired formulas

Let random variable D_i represent the (unknown) difference from a (yet to be) randomly selected individual i.
We want to predict what happens when we find a mean of differences.

$$\bar{D} = \frac{D_1 + D_2 + D_3 + ... + D_n}{n}$$

The central limit theorem still applies!

Derivation of paired formulas

Let random variable D_i represent the (unknown) difference from a (yet to be) randomly selected individual i.
We want to predict what happens when we find a mean of differences.

$$\bar{D} = \frac{D_1 + D_2 + D_3 + ... + D_n}{n}$$

The central limit theorem still applies!

As $n \to \infty$, $\bar{D}$ becomes normally distributed.

Derivation of paired formulas

Let random variable D_i represent the (unknown) difference from a (yet to be) randomly selected individual i.
We want to predict what happens when we find a mean of differences.

$$\bar{D} = \frac{D_1 + D_2 + D_3 + ... + D_n}{n}$$

The central limit theorem still applies!

As $n \to \infty$, $\bar{D}$ becomes normally distributed.

Basically, we can treat these differences just like any other independent and identically distributed random variables.

Note about notation

- I used $\bar{D}$ for the random variable representing an unknown mean of differences.

Note about notation

- I used $\bar{D}$ for the random variable representing an unknown mean of differences.
- I would use $\bar{d}$ for a specific (observed, critical, etc) mean of difference.

Note about notation

- I used $\bar{D}$ for the random variable representing an unknown mean of differences.
- I would use $\bar{d}$ for a specific (observed, critical, etc) mean of difference.
- The book uses $\bar{x}_{\mathsf{diff}}$ for both of these concepts. This is misleading, as it looks like a difference of means, not a mean of differences.

Note about notation

- I used $\bar{D}$ for the random variable representing an unknown mean of differences.
- I would use $\bar{d}$ for a specific (observed, critical, etc) mean of difference.
- The book uses $\bar{x}_{\text{diff}}$ for both of these concepts. This is misleading, as it looks like a difference of means, not a mean of differences.
- I would at least prefer using $\overline{X_{\text{diff}}}$ and $\overline{x_{\text{diff}}}$ to emphasize we are finding a mean of differences.

Note about notation

- I used $\bar{D}$ for the random variable representing an unknown mean of differences.
- I would use $\bar{d}$ for a specific (observed, critical, etc) mean of difference.
- The book uses $\bar{x}_{\text{diff}}$ for both of these concepts. This is misleading, as it looks like a difference of means, not a mean of differences.
- I would at least prefer using $\overline{X_{\text{diff}}}$ and $\overline{x_{\text{diff}}}$ to emphasize we are finding a mean of differences.
- The book's notation of μ_{diff} (for the population's true difference) is useful. We could also use $E(D)$ or μ_D.
- In order to match the book as much as possible, I will now use $x_{\text{diff},i}$ and $\overline{X_{\text{diff}}}$ and $\overline{x_{\text{diff}}}$ and μ_{diff}.

Example problem

A teacher wonders if, on average, a random student will perform about the same on two exams. She decides to run a two-tail t test on a random sample of size $n = 5$ with a signficance level $\alpha = 0.05$.

Example problem

A teacher wonders if, on average, a random student will perform about the same on two exams. She decides to run a two-tail t test on a random sample of size $n = 5$ with a signficance level $\alpha = 0.05$.
Here are the results of her study:

Student	Exam 1	Exam 2
Norma	98	96
Elliot	15	10
Walton	61	61
Mable	80	79
Loretta	10	8

Perform the t test.

Example problem solution

Find the differences.

Example problem solution

Find the differences.

i	$x_{1,i}$	$x_{2,i}$	$x_{\text{diff},i}$
1	98	96	-2
2	15	10	-5
3	61	61	0
4	80	79	-1
5	10	8	-2

Example problem solution

Find the differences.

i	$x_{1,i}$	$x_{2,i}$	$x_{\text{diff},i}$
1	98	96	-2
2	15	10	-5
3	61	61	0
4	80	79	-1
5	10	8	-2

Find the (differences') sample mean.

Example problem solution

Find the differences.

i	$x_{1,i}$	$x_{2,i}$	$x_{\text{diff},i}$
1	98	96	-2
2	15	10	-5
3	61	61	0
4	80	79	-1
5	10	8	-2

Find the (differences') sample mean.

$$\overline{x_{\text{diff}}} = \frac{\sum_{i=1}^{n} x_{\text{diff},i}}{n} = \frac{-2-5+0-1-2}{5} = -2$$

Example problem solution

Find the differences.

i	$x_{1,i}$	$x_{2,i}$	$x_{\text{diff},i}$
1	98	96	-2
2	15	10	-5
3	61	61	0
4	80	79	-1
5	10	8	-2

Find the (differences') sample mean.

$$\overline{x_{\text{diff}}} = \frac{\sum_{i=1}^{n} x_{\text{diff},i}}{n} = \frac{-2-5+0-1-2}{5} = -2$$

Find the (differences') sample standard deviation.

Example problem solution

Find the differences.

i	$x_{1,i}$	$x_{2,i}$	$x_{\text{diff},i}$
1	98	96	-2
2	15	10	-5
3	61	61	0
4	80	79	-1
5	10	8	-2

Find the (differences') sample mean.

$$\overline{x_{\text{diff}}} = \frac{\sum_{i=1}^{n} x_{\text{diff},i}}{n} = \frac{-2-5+0-1-2}{5} = -2$$

Find the (differences') sample standard deviation.

$$s = \sqrt{\frac{\sum_{i=1}^{n} (x_{\text{diff},i} - \overline{x_{\text{diff}}})^2}{n-1}} = \sqrt{\frac{(0)^2 + (3)^2 + (2)^2 + (1)^2 + (0)^2}{5-1}} = 1.87$$

We are doing a two-tail test with the following:

$$n = 5 \qquad \overline{x_{\textsf{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$

We are doing a two-tail test with the following:

$$n = 5 \qquad \overline{x_{\mathsf{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$

State the hypotheses.

We are doing a two-tail test with the following:

$$n = 5 \qquad \overline{x_{\text{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$

State the hypotheses.

$$H_0: \quad \mu_{\text{diff}} = 0 \qquad\qquad H_A: \quad \mu_{\text{diff}} \neq 0$$

We are doing a two-tail test with the following:

$$n = 5 \qquad \overline{x_{\textsf{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$

State the hypotheses.

$$H_0: \quad \mu_{\textsf{diff}} = 0 \qquad\qquad H_A: \quad \mu_{\textsf{diff}} \neq 0$$

Determine the critical value, $t^\star$, such that $P(|T| > t^\star) = 0.05$.

We are doing a two-tail test with the following:

$$n = 5 \qquad \overline{x_{\text{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$

State the hypotheses.

$$H_0: \quad \mu_{\text{diff}} = 0 \qquad\qquad H_A: \quad \mu_{\text{diff}} \neq 0$$

Determine the critical value, $t^\star$, such that $P(|T| > t^\star) = 0.05$.

$$t^\star = 2.78$$

We are doing a two-tail test with the following:

$$n = 5 \qquad \overline{x_{\text{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$

State the hypotheses.

$$H_0: \quad \mu_{\text{diff}} = 0 \qquad\qquad H_A: \quad \mu_{\text{diff}} \neq 0$$

Determine the critical value, $t^\star$, such that $P(|T| > t^\star) = 0.05$.

$$t^\star = 2.78$$

Find the standard error (the standard deviation of the differences' sampling distribution).

We are doing a two-tail test with the following:

$$n = 5 \qquad \overline{x_{\text{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$

State the hypotheses.

$$H_0 : \quad \mu_{\text{diff}} = 0 \qquad\qquad H_A : \quad \mu_{\text{diff}} \neq 0$$

Determine the critical value, $t^\star$, such that $P(|T| > t^\star) = 0.05$.

$$t^\star = 2.78$$

Find the standard error (the standard deviation of the differences' sampling distribution).

$$SE = \frac{s}{\sqrt{n}} = \frac{1.87}{\sqrt{5}} = 0.837$$

We are doing a two-tail test with the following:

$$n = 5 \qquad \overline{x_{\text{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$

State the hypotheses.

$$H_0 : \quad \mu_{\text{diff}} = 0 \qquad\qquad H_A : \quad \mu_{\text{diff}} \neq 0$$

Determine the critical value, $t^\star$, such that $P(|T| > t^\star) = 0.05$.

$$t^\star = 2.78$$

Find the standard error (the standard deviation of the differences' sampling distribution).

$$SE = \frac{s}{\sqrt{n}} = \frac{1.87}{\sqrt{5}} = 0.837$$

Calculate an observed t score.

We are doing a two-tail test with the following:

$$n = 5 \qquad \overline{x_{\textsf{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$

State the hypotheses.

$$H_0 : \quad \mu_{\textsf{diff}} = 0 \qquad\qquad H_A : \quad \mu_{\textsf{diff}} \neq 0$$

Determine the critical value, $t^\star$, such that $P(|T| > t^\star) = 0.05$.

$$t^\star = 2.78$$

Find the standard error (the standard deviation of the differences' sampling distribution).

$$SE = \frac{s}{\sqrt{n}} = \frac{1.87}{\sqrt{5}} = 0.837$$

Calculate an observed t score.

$$t_{\textsf{obs}} = \frac{(-2) - 0}{0.837} = -2.39$$

From the previous slides:

$$n = 5 \qquad \overline{x_{\text{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$

$$t^\star = 2.78 \qquad SE = 0.837 \qquad t_{\text{obs}} = -2.39$$

From the previous slides:

$$\begin{array}{llll} n = 5 & \overline{x_{\text{diff}}} = -2 & s = 1.87 & \alpha = 0.05 \\ t^{\star} = 2.78 & SE = 0.837 & t_{\text{obs}} = -2.39 & \end{array}$$

We can determine a p-value. Remember we are doing a two-tail test, so p-value $= P(|T| > 2.39)$.

From the previous slides:

$$n = 5 \qquad \overline{x_{\text{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$
$$t^{\star} = 2.78 \qquad SE = 0.837 \qquad t_{\text{obs}} = -2.39$$

We can determine a p-value. Remember we are doing a two-tail test, so p-value $= P(|T| > 2.39)$.

$$0.05 \quad < \quad p\text{-value} \quad < \quad 0.1$$

From the previous slides:

$$n = 5 \qquad \overline{x_{\text{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$
$$t^\star = 2.78 \qquad SE = 0.837 \qquad t_{\text{obs}} = -2.39$$

We can determine a p-value. Remember we are doing a two-tail test, so p-value $= P(|T| > 2.39)$.

$$0.05 \quad < \quad p\text{-value} \quad < \quad 0.1$$

We can compare t_{obs} and $t^\star$. We can also compare p-value and α.

From the previous slides:

$$n = 5 \qquad \overline{x_{\text{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$
$$t^\star = 2.78 \qquad SE = 0.837 \qquad t_{\text{obs}} = -2.39$$

We can determine a p-value. Remember we are doing a two-tail test, so p-value $= P(|T| > 2.39)$.

$$0.05 \quad < \quad p\text{-value} \quad < \quad 0.1$$

We can compare t_{obs} and $t^\star$. We can also compare p-value and α.

$$|t_{\text{obs}}| < |t^\star|$$

From the previous slides:

$$n = 5 \qquad \overline{x_{\mathsf{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$
$$t^\star = 2.78 \qquad SE = 0.837 \qquad t_{\mathsf{obs}} = -2.39$$

We can determine a p-value. Remember we are doing a two-tail test, so p-value $= P(|T| > 2.39)$.

$$0.05 \quad < \quad p\text{-value} \quad < \quad 0.1$$

We can compare t_{obs} and $t^\star$. We can also compare p-value and α.

$$|t_{\mathsf{obs}}| < |t^\star|$$

$$p\text{-value} > \alpha$$

From the previous slides:

$$n = 5 \qquad \overline{x_{\text{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$

$$t^\star = 2.78 \qquad SE = 0.837 \qquad t_{\text{obs}} = -2.39$$

We can determine a p-value. Remember we are doing a two-tail test, so p-value $= P(|T| > 2.39)$.

$$0.05 \quad < \quad p\text{-value} \quad < \quad 0.1$$

We can compare t_{obs} and $t^\star$. We can also compare p-value and α.

$$|t_{\text{obs}}| < |t^\star|$$

$$p\text{-value} > \alpha$$

Thus, we retain the null hypothesis.

From the previous slides:

$$n = 5 \qquad \overline{x_{\mathsf{diff}}} = -2 \qquad s = 1.87 \qquad \alpha = 0.05$$
$$t^\star = 2.78 \qquad SE = 0.837 \qquad t_{\mathsf{obs}} = -2.39$$

We can determine a p-value. Remember we are doing a two-tail test, so p-value $= P(|T| > 2.39)$.

$$0.05 \quad < \quad p\text{-value} \quad < \quad 0.1$$

We can compare t_{obs} and $t^\star$. We can also compare p-value and α.

$$|t_{\mathsf{obs}}| < |t^\star|$$

$$p\text{-value} > \alpha$$

Thus, we retain the null hypothesis.

We maintain that maybe both tests are equally challenging.

Practice

The following table has paired data. Test the hypotheses of whether or not the differences have a population average of 0. Use $\alpha = 0.1$.

i	$x_{1,i}$	$x_{2,i}$
1	50	54
2	23	25
3	96	97
4	47	49
5	10	16

Practice

The following table has paired data. Test the hypotheses of whether or not the differences have a population average of 0. Use $\alpha = 0.1$.

i	$x_{1,i}$	$x_{2,i}$
1	50	54
2	23	25
3	96	97
4	47	49
5	10	16