

# MATH 181 2ND EXAM PRACTICE A SOLUTIONS

## SPRING 2019

Name: \_\_\_\_\_

- Write your **full name** on the line above.
- Show your work. Incorrect answers with work can receive partial credit.
- Attempt every question; showing you understand the question earns some credit.
- If you run out of room for an answer, continue on the back of the page. Before doing so, write “see back” with a circle around it.
- You can use 1 page (front and back) of notes.
- You can use (and probably need) a calculator.
- You can use the Geogebra Scientific Calculator instead of a calculator. You need to put your phone on **airplane mode** and then within the application, start **exam mode**; you should see a green bar with a timer counting up.
- If a question is confusing or ambiguous, please ask for clarification; however, you will not be told how to answer the question.
- **Box your final answer.**
- A formula sheet is attached to this test.

Do not write in this grade table.

Question:	Q1	Q2	Q3	Q4	Total
Points:	10	10	10	10	40
Score:					

**Normal Distribution:**

$$X \sim \mathcal{N}(\mu, \sigma)$$

$\mu$  = population mean

$\sigma$  = population standard deviation

$x$  = possible value of  $X$

$\ell$  = percentile of  $x$  (left area)

$\Phi(z)$  = standard normal cumulative function

$$z = \frac{x - \mu}{\sigma}$$

$$P(X < x) = \Phi(z)$$

$$\ell = \Phi(z)$$

$$z = \Phi^{-1}(\ell)$$

**Bernoulli Distribution:**

$$X \sim \text{Bern}(p)$$

$X = 0$  for fail or 1 for success

$p$  = probability of success

$$P(X = 0) = 1 - p$$

$$P(X = 1) = p$$

$$\mu = p$$

$$\sigma = \sqrt{p(1 - p)}$$

**Geometric Distribution:**

$$X \sim \text{Geo}(p)$$

$X$  = number of trials until first success

$p$  = probability of success on each trial

$n$  = a possible number of trials

$$P(X = n) = (1 - p)^{n-1}(p)$$

$$\mu = \frac{1}{p}$$

$$\sigma = \sqrt{\frac{1 - p}{p^2}}$$

**Mean-Sampling Distribution:**

$\bar{X}$  = sample mean

$s$  = sample standard deviation

$n$  = sample size

$\mu$  = population mean

$\sigma$  = population standard deviation

$SE$  = standard error

$$SE = \frac{\sigma}{\sqrt{n}}$$

If  $n \geq 30$  (or if population is normal) then:

$$\bar{X} \sim \mathcal{N}(\mu, SE)$$

**Binomial Distribution:**

$$X \sim \mathcal{B}(n, p)$$

$X$  = number of successes from  $n$  trials

$p$  = probability of success on each trial

$n$  = number of trials

$k$  = a possible number of successes

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$\mu = np$$

$$\sigma = \sqrt{np(1 - p)}$$

If  $np \geq 10$  and  $n(1 - p) \geq 10$ , then

$$X \sim \mathcal{N}(\mu, \sigma)$$

Continuity correction:

$$P(X \leq k) \approx \Phi\left(\frac{k + 0.5 - \mu}{\sigma}\right)$$

**Confidence Interval:**

$CI$  = confidence interval

$\gamma$  = confidence level

$\bar{x}$  = sample mean

$s$  = sample standard deviation

$$z^* = \Phi^{-1}\left(\frac{\gamma + 1}{2}\right)$$

$$SE \approx \frac{s}{\sqrt{n}}$$

$$CI = \bar{x} \pm z^* SE$$

**Hypothesis testing:**

$$H_0 : \mu = \mu_0$$

$$H_A : \mu \neq \mu_0$$

$\bar{x}$  = a possible/specific/observed sample mean

$s$  = sample standard deviation

$\alpha$  = significance level

$$\sigma \approx s$$

$$z = \frac{\bar{x} - \mu_0}{SE}$$

$$\begin{aligned} p\text{-value} &= P(|Z| > |z|) \\ &= 2 \cdot \Phi(-|z|) \end{aligned}$$

If  $p\text{-value} < \alpha$ , then reject  $H_0$ , else retain  $H_0$ .

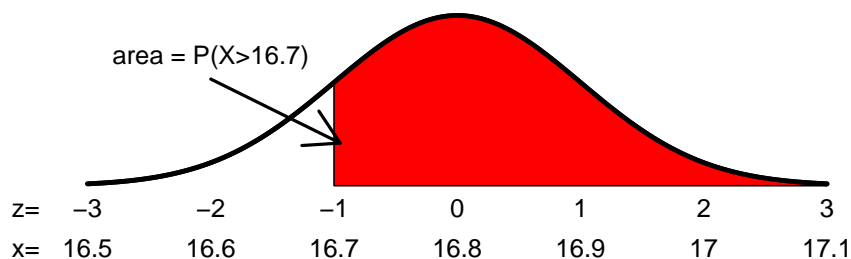
**Q1.** (10 points) Brood XIV is a population of 17-year cicadas in eastern United States, including Massachusetts. The juvenile lifespan is normally distributed with mean of 16.8 years and standard deviation of 0.1 years.

- (a) What is the probability of a random juvenile's lifespan being more than 16.7 years?  
In other words, let  $X \sim \mathcal{N}(16.8, 0.1)$  and find  $P(X > 16.7)$ .

**Solution:** Find the  $z$ -score.

$$z = \frac{16.7 - 16.8}{0.1} = -1$$

Draw a sketch.



Find the area.

$$\begin{aligned} P(X > 16.7) &= P(Z > -1) \\ &= 1 - P(Z < -1) \\ &= 1 - \Phi(-1) \\ &= 1 - 0.1587 \\ &= \boxed{0.8413} \end{aligned}$$

- (b) What is the IQR of juvenile lifespans?

**Solution:** We find the  $z$ -scores of 25th percentile and 75th percentile. So, let's find  $z_{\text{LOW}}$  such that  $P(Z < z_{\text{LOW}}) = 0.25$ .

$$z_{\text{LOW}} = \Phi^{-1}(0.25) = -0.67$$

Let's find  $z_{\text{HIGH}}$  such that  $P(Z < z_{\text{HIGH}}) = 0.75$ .

$$z_{\text{HIGH}} = \Phi^{-1}(0.75) = 0.67$$

We find the associated  $x$  scores.

$$x_{\text{LOW}} = 16.8 + (-0.67)(0.1) = 16.733$$

$$x_{\text{HIGH}} = 16.8 + (0.67)(0.1) = 16.867$$

To find IQR, we find the difference.

$$IQR = 16.867 - 16.733 = \boxed{0.134}$$

**Q2.** (10 points) A 20-sided die (icosahedron) has a 5% chance of landing on each side. Imagine that only one side is a success and the rest are fails.

(a) What is the chance the first success happens on the third roll?

**Solution:** We use a geometric model.  $p = 0.05$  and  $n = 3$ .

$$P(\text{Fail, Fail, Success}) = (0.95)^2(0.05) = \boxed{0.045}$$

(b) What is the chance of getting exactly 5 successes in 100 rolls?

**Solution:** We use a binomial model.  $p = 0.05$  and  $n = 100$  and  $k = 5$ . Let  $X$  represent the number of successes when 100 of these dice are thrown.

$$\begin{aligned} P(X = 5) &= \binom{100}{5}(0.05)^5(0.95)^{95} \\ &= \boxed{0.18} \end{aligned}$$

(c) What is the chance of getting between at least 10 and less than 30 successes in 300 rolls?

**Solution:** We hope to use the normal approximation to the binomial distribution. We first check to make sure we can use the normal approximation.

$$np = (300)(0.05) = 15 > 10$$

$$n(1 - p) = (300)(0.95) = 285 > 10$$

Great, we can. We determine the mean and standard deviation of the binomial distribution.

$$\mu = np = (300)(0.05) = 15$$

$$\sigma = \sqrt{np(1 - p)} = \sqrt{(300)(0.05)(0.95)} = 3.7749$$

We find the  $z$ -scores. REMEMBER THE CONTINUITY CORRECTIONS!

$$z_{\text{LOWER}} = \frac{10 - 0.5 - 15}{3.77} = -1.46$$

$$z_{\text{UPPER}} = \frac{30 - 0.5 - 15}{3.77} = 3.84$$

Because  $z_{\text{UPPER}}$  is larger than 3.5, we can ignore that upper bound (and just find a right area instead.)

$$P(10 \leq X < 30) \approx P(Z > -1.46) = 1 - P(Z < -1.46) = \boxed{0.93}$$

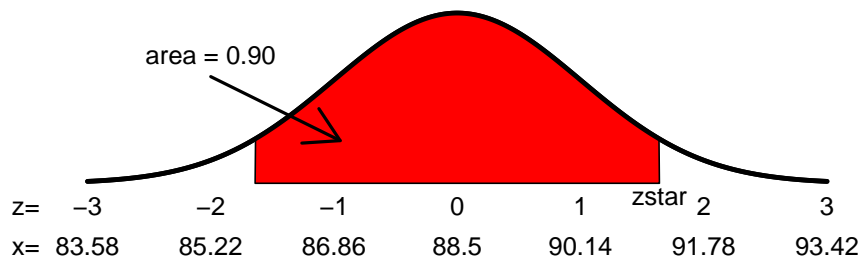
**Q3.** (10 points) You collect 45 measurements with a mean of 88.5 mm and a standard deviation of 11.0 mm.

(a) Determine a 90% confidence interval.

**Solution:** We calculate the standard error.

$$SE = \frac{11}{\sqrt{45}} = 1.63978$$

We determine  $z^*$  such that  $P(|Z| < z^*) = 0.90$ .



Using symmetry, we recognize how to find  $z^*$ .

$$z^* = \Phi^{-1}(0.95) = 1.64$$

We find the confidence interval.

$$\begin{aligned} CI &= \bar{x} \pm z^* SE \\ &= 88.5 \pm (1.64)(1.63978) \\ &= (85.81, 91.19) \end{aligned}$$

(b) Determine a 99% confidence interval.

**Solution:** The standard error is the same as above. We calculate a new  $z^*$ .

$$z^* = \Phi^{-1}(0.995) = 2.58$$

We find the confidence interval.

$$\begin{aligned} CI &= \bar{x} \pm z^* SE \\ &= 88.5 \pm (2.58)(1.63978) \\ &= (84.27, 92.73) \end{aligned}$$

- (c) If a normally distributed population has a mean of 90 and a standard deviation of 11, what is the chance that 45 measurements will have a mean lower than 88.5?

**Solution:** Let  $X$  represent a single measurement.

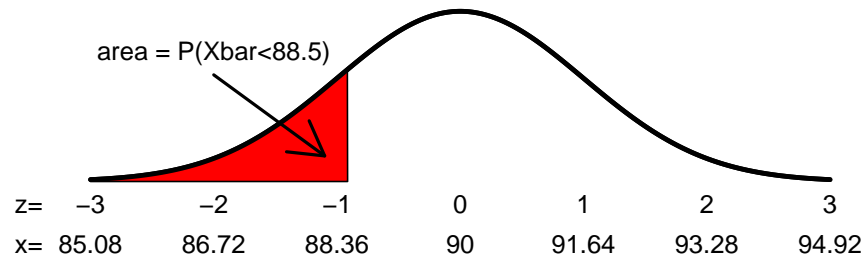
$$X \sim \mathcal{N}(90, 11)$$

Let  $\bar{X}$  represent the mean of 45 measurements.

$$\bar{X} \sim \mathcal{N}\left(90, \frac{11}{\sqrt{45}}\right)$$

$$\bar{X} \sim \mathcal{N}(90, 1.64)$$

We hope to calculate  $P(\bar{X} < 88.5)$ . We draw a sketch.



We calculate the  $z$ -score.

$$z = \frac{88.5 - 90}{1.64} = -0.91$$

We calculate the probability.

$$P(\bar{X} < 88.5) = P(Z < -0.91) = \boxed{0.1814}$$

**Q4.** (10 points) You had been told that adult elephants have a mean weight of 255 kg. You decided to measure the weights of 50 random elephants and run a hypothesis test with a significance level of 0.05.

Your sample has a mean of 249.8 kg and a standard deviation of 12.34 kg. What is your conclusion and why? Show your work for full credit.

**Solution:** We state the hypotheses.

$$H_0 : \mu = 255$$

$$H_A : \mu \neq 255$$

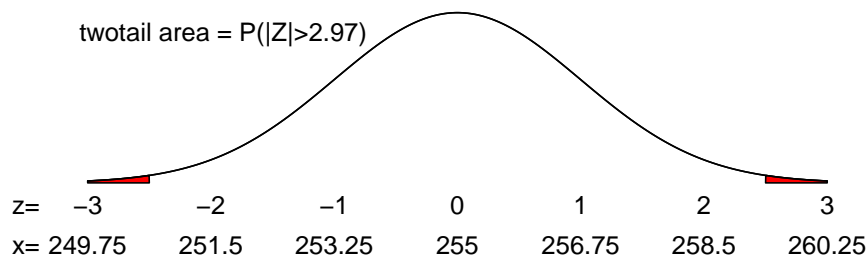
We describe the null's sampling distribution by assuming  $\sigma \approx 12.34$ . We calculate the standard error:  $SE = 12.34/\sqrt{50} = 1.75$

$$\bar{X}_0 \sim \mathcal{N}(255, 1.75)$$

We find the  $z$ -score of the actual sample's mean (249.8) under the null's sampling distribution.

$$z = \frac{249.8 - 255}{1.75} = -2.97$$

We sketch the null's sampling distribution, along with a two-tailed area using 249.8 (the actual sample's mean) as a boundary.



We determine the probability.

$$P(|Z| > 2.97) = 2P(Z < -2.97) = (2)(0.0015) = 0.003$$

$$p\text{-value} = 0.003$$

We compare the  $p$ -value to the significance level.

$$0.003 < 0.05$$

$$p\text{-value} < \alpha$$

We reject the null hypothesis. We conclude the true mean weight of elephants is not 255 kg.