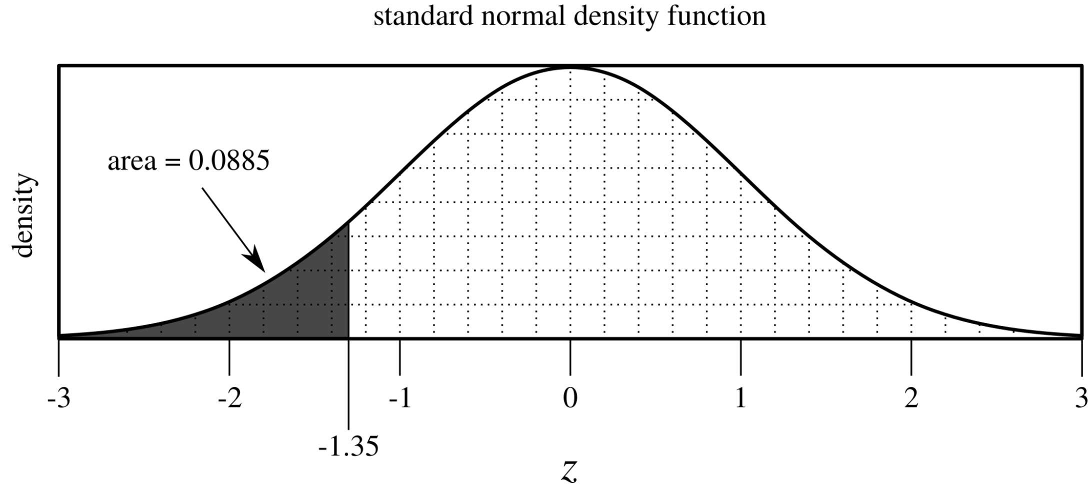
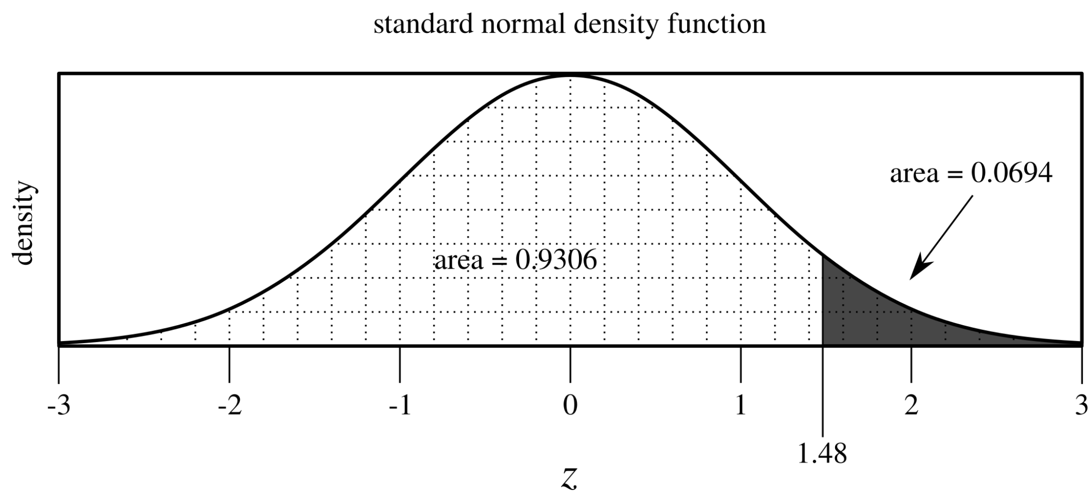
3.1: (a): Below is a detailed sketch.



We are finding a **left** area. This is the easiest. We can use the standard normal table directly.

$$P(Z < -1.35) \quad = \quad \boxed{0.0885}$$

(b): Below is a detailed sketch.



We are finding a **right** area. We can find the complementary left area from the table.

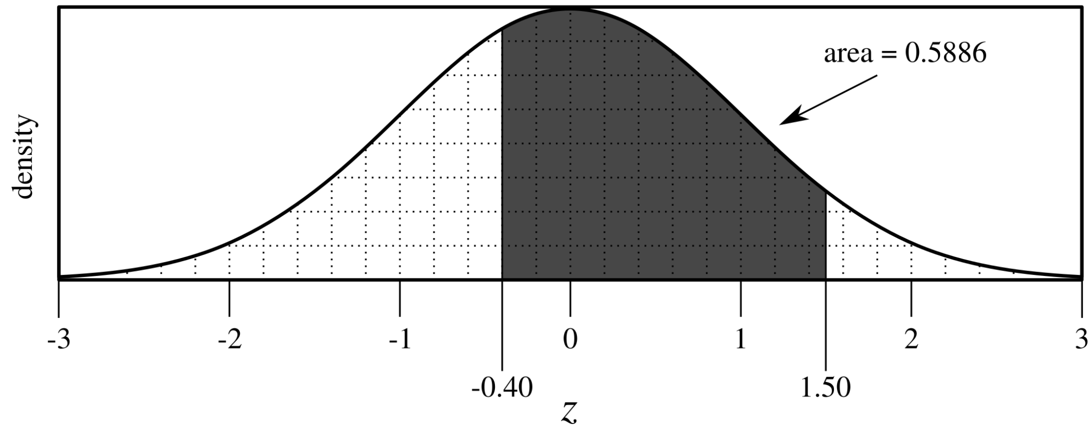
$$P(Z < 1.48) \quad = \quad 0.9306$$

Then, we can calculate the desired probability.

$$P(Z > 1.48) \quad = \quad 1 - 0.9306 \quad = \quad \boxed{0.0694}$$

(c): Below is a sketch of the area we wish to determine (along with the answer).

standard normal density function



I'll call this doubly bounded area a **sectional** area. In order to use the table, we will need to subtract two left areas.

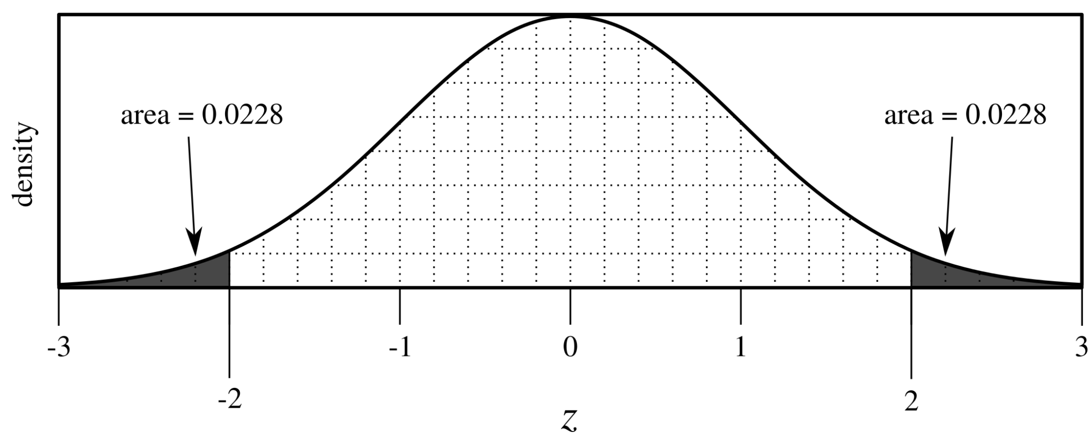
$$P(Z < 1.5) = 0.9332$$

$$P(Z < -0.4) = 0.3446$$

$$P(-0.4 < Z < 1.5) = 0.9332 - 0.3446 = \boxed{0.5886}$$

(d): Below is a sketch of the (symmetric) **two-tailed** area we wish to determine.

standard normal density function



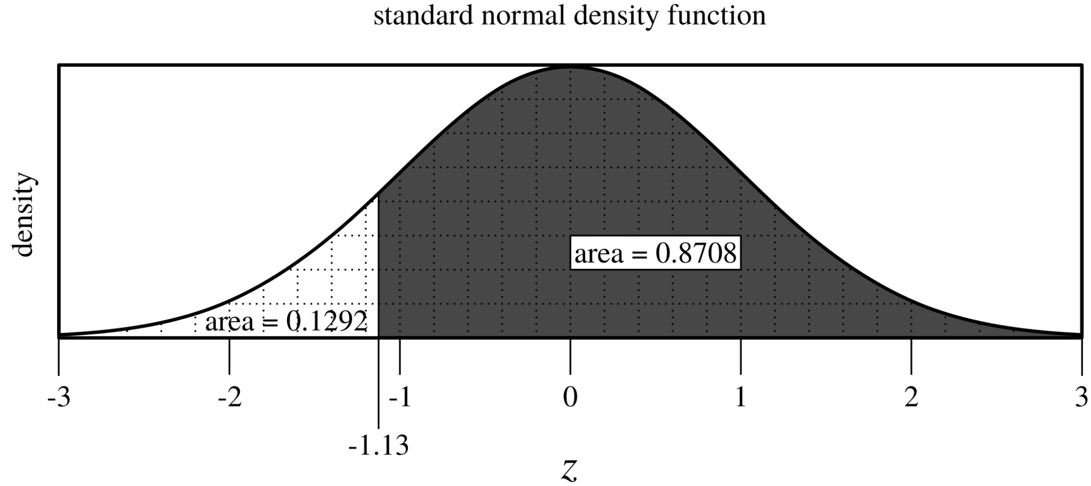
There are a few ways to find this two-tailed area. Either way we need the left area.

$$P(Z < -2) = 0.0228$$

In this case, we can use symmetry to find the two-tailed area.

$$P(|Z| > 2) = 0.0228 + 0.0228 = \boxed{0.0456}$$

3.2: **(a):** Below is a detailed sketch.

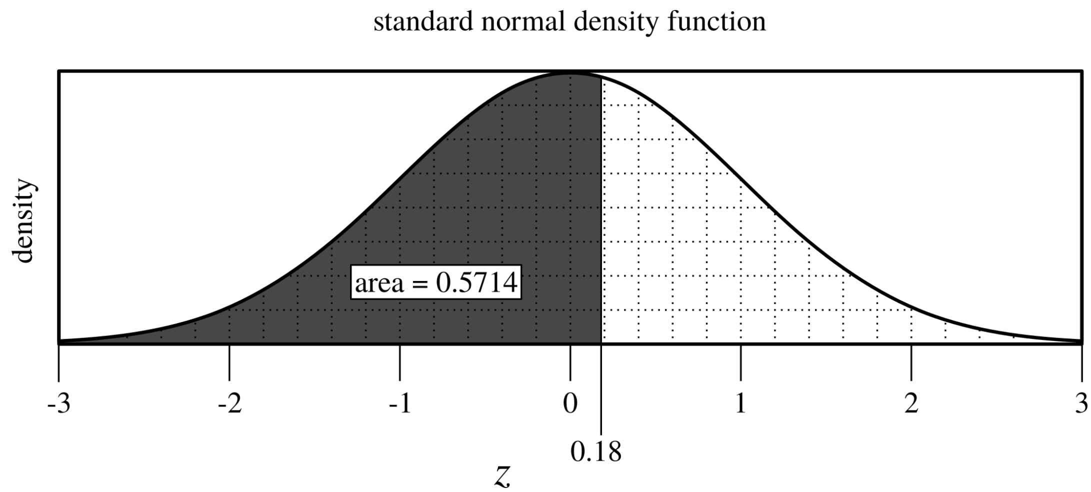


We are finding a **right** area.

$$P(Z < -1.13) = 0.1292$$

$$P(Z > -1.13) = 1 - 0.1292 = \boxed{0.8708}$$

(b): Below is a detailed sketch.



We are finding a **left** area.

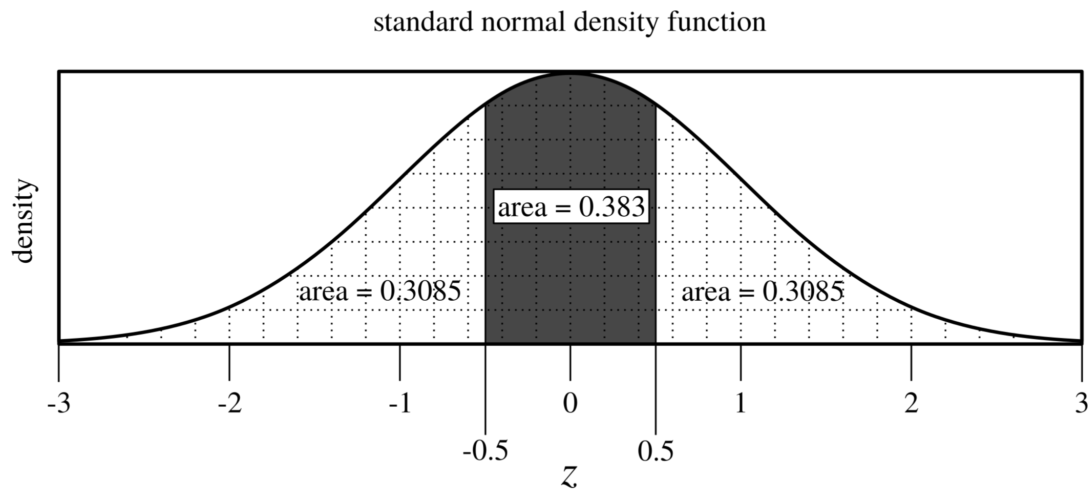
$$P(Z < 0.18) = \boxed{0.5714}$$

(c): The area above $z = 8$ is so tiny that we can call it 0. If you have a more advanced way to look this up than the table, you might find a more precise answer:

$$P(X > 8) = 0.000000000000000622 \quad = \quad 6.22 \times 10^{-16}$$

However, from the table we can conclude the area is **less than 0.0002**.

(d): Below is a sketch of the (symmetric) **central** area we wish to determine.



There are a few ways to find this central area. I like to find the left area of the left boundary.

$$P(Z < -0.5) = 0.3085$$

We can use symmetry to find the central area.

$$P(|Z| < 0.5) \quad = \quad 1 - 0.3085 - 0.3085 \quad = \quad \boxed{0.383}$$

I will point out that if you do this with precise software, you will get 0.3829 instead...

3.3: **(a):** Let random variable V represent the Verbal Reasoning score of a random person. Let random variable Q represent the Quantitative Reasoning score of a random person.

$$V \sim \mathcal{N}\left(151,\ 7\right)$$

$$Q \sim \mathcal{N}\left(153,\ 7.67\right)$$

(b): We calculate the z-scores.

$$Z_{\text{V, Sophia}} = \frac{160 - 151}{7} = 1.29$$

$$Z_{\text{Q, Sophia}} = \frac{157 - 153}{7.67} = 0.5215$$

(c): Sophia's score on Verbal was 1.29 standard deviations above the mean. Sophia's score on Quantitative was 0.52 standard deviations above the mean.

(d): Relative to other people, Sophia did better on Verbal Reasoning.

(e): To find her percentile, we need to use the Z table. At this point I want to introduce a new syntax.

$$\Phi(k) \equiv P(Z < k)$$

Basically, $\Phi(z)$ means use the standard normal table with the given value of z.

$$\Phi(1.29) = 0.90$$

$$\Phi(0.52) = 0.70$$

Sophia was 90th percentile in Verbal and 70th percentile in Quantitative.

(f): 10% did better than Sophia on Verbal. 30% did better than Sophia on Quantitative.

(g): We are more interested in which test Sophia does unusually well on compared to others, not just the raw score.
Analogy: just because a specific watermelon might be bigger than a specific orange does not mean the watermelon is more unusually large.

(h): The answer to (b) is the same. I guess (c) is still valid too. The other answers would need to change because they used a normal assumption.

3.4: **(a):** Let random variable M represent the finishing time of a random man from ages 30 to 34. Let random variable W represent the finishing time of a random woman from ages 25 to 29.

$$M \sim \mathcal{N}\left(4313,\ 583\right)$$

$$W \sim \mathcal{N}\left(5261,\ 807\right)$$

(b): We calculate the standard scores. Let Z_{Leo} represent Leo's standard score. Let Z_{Mary} represent Mary's standard score.

$$Z_{\text{Leo}} = \frac{4948 - 4313}{583} = 1.09$$

$$Z_{\text{Mary}} = \frac{5513 - 5261}{807} = 0.31$$

Leo finished 1.09 standard deviations later than average. Mary finished 0.31 standard deviation later than average.

(c): Mary ranked better in her group because her standard score is lower.

(d): We first find the percent who were faster than Leo by using the table.

$$\Phi(1.09) = 0.8621$$

Then, we can determine the percent who were slower (had longer times).

$$1 - 0.8621 = \boxed{0.1379}$$

(e): We first find the percent who were faster than Mary by using the table.

$$\Phi(0.31) = 0.6217$$

Then, we can determine the percent who were slower (had longer times).

$$1 - 0.6217 = \boxed{0.3783}$$

(f): Our answers to (b) and (c) would still be valid. The others need reconsideration as they assumed a normal distribution.

3.5: (a): We now need to use the standard normal table **backwards**. We scan for a percentile nearest 0.8, then determine the value of Z associated with that percentile.

$$\Phi(0.84) = 0.8$$

I want to introduce another notation, for using the standard normal table in reverse.

$$\Phi(z) = a \quad \Longleftrightarrow \quad \Phi^{-1}(a) = z$$

Thus, I could write:

$$Z = \Phi^{-1}(0.8) = 0.84$$

Now, we convert this Z score into a Quantitative Reasoning score.

$$\begin{aligned} Q &= Z\sigma + \mu \\ &= (0.84)(7.67) + 153 \\ &= \boxed{159.4} \end{aligned}$$

(b): We find the Z score from the percentile, which is 30th percentile (worse than 70%).

$$Z = \Phi^{-1}(0.3) = -0.52$$

We calculate the Verbal Reasoning score.

$$\begin{aligned} V &= Z\sigma + \mu \\ &= (-0.52)(7) + 151 \\ &= \boxed{147.4} \end{aligned}$$

3.6: (a): We find Z.

$$Z = \Phi^{-1}(0.05) = -1.64$$

We find the cutoff time for 5th percentile in Men's.

$$\begin{aligned} M_{5\%} &= Z\sigma + \mu \\ &= (-1.64)(583) + 4313 \\ &\approx \boxed{3357 \text{ seconds}} \end{aligned}$$

(b): We find Z. Remember, this is 90th percentile.

$$Z = \Phi^{-1}(0.90) = 1.28$$

We find the cutoff time for 90th percentile in Women's.

$$\begin{aligned} W_{90\%} &= Z\sigma + \mu \\ &= (1.28)(807) + 5261 \\ &\approx \boxed{6294 \text{ seconds}} \end{aligned}$$

3.7: (a): Let random variable T follow the temperature distribution. We wish to evaluate $P(T \geq 83)$. We first find Z.

$$Z = \frac{83 - 77}{5} = 1.2$$

We use the table (and a sketch).

$$\begin{aligned} P(T \geq 83) &= P(Z \geq 1.2) \\ &= 1 - \Phi(1.2) \\ &= 1 - 0.8849 \\ &= \boxed{0.1151} \end{aligned}$$

(b): We are told to find x such that $P(T \leq x) = 0.10$. In other words we are looking for the cutoff of the 10th percentile. We first find the z score.

$$z = \Phi^{-1}(0.1) = -1.28$$

We convert this z into x.

$$\begin{aligned} x &= z\sigma + \mu \\ &= (-1.28)(5) + 77 \\ &= \boxed{70.6} \end{aligned}$$

The coldest 10% of days are below 70.6°F.

3.8: **(a):** Let random variable $R \sim \mathcal{N}\left(0.147,\ 0.33\right)$ represent the annual return. We wish to evaluate $P(R < 0)$. We calculate the z score.

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{0 - 14.7}{33}$$

$$= -0.45$$

We calculate a left area.

$$\begin{aligned} P(R < 0) &= P(Z < -0.45) \\ &= \Phi(-0.45) \\ &= \boxed{0.3264} \end{aligned}$$

(b): We want to find x such that $P(R > x) = 0.15$. We find z by using the standard normal table in reverse. Also, we are looking for the 85th percentile.

$$\begin{aligned} z &= \Phi^{-1}(0.85) \\ &= 1.04 \end{aligned}$$

We convert z into x.

$$\begin{aligned} x &= z\sigma + \mu \\ &= (1.04)(33) + 14.7 \\ &= \boxed{49.02} \end{aligned}$$

So, the cutoff is a return of 49 percent. Only in 15% of years is the return higher than 49%.