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Welch's t test has a gnarly formula for df.

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The formula for degrees of freedom is annoying to evaluate for mere mortals. So, unless otherwise instructed, we will use a conservative estimate (conservative w.r.t. type I error).

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Don't be surprised if other texts (or people) tell you to use $df = n_1 + n_2 - 2$. We only use this if we have a strong argument for why we believe $\sigma_1 = \sigma_2$.

Hypotheses under paired and unpaired

- With paired data, the statistic is a mean of differences. Usually we are wondering whether the population mean of differences is 0.

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Hypotheses under paired and unpaired (other notation)

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Example problem

An experiment has $n_1 = 4$ plants in the treatment group and $n_2 = 6$ plants in the control group. After some time, the plants' heights (in cm) are measured, resulting in the following data:

	value1	value2	value3	value4	value5	value6
sample 1:	16.4	14.2	19.4	17.3		
sample 2:	10.3	9.9	9.4	11	10.4	10.7

1. Determine degrees of freedom.
2. Determine $t^\star$ for a 98% confidence interval.
3. Determine SE.
4. Determine a lower bound of the 98% confidence interval of $\mu_2 - \mu_1$.
5. Determine an upper bound of the 98% confidence interval of $\mu_2 - \mu_1$.
6. Determine $|t_{\text{obs}}|$ under the null hypothesis $\mu_2 - \mu_1 = 0$.
7. Determine a lower bound of the two-tail p-value.
8. Determine an upper bound of two-tail p-value.
9. Do you reject the null hypothesis with a two-tail test using a significance level $\alpha = 0.02$? (yes or no)

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$$SE = \sqrt{\frac{(s_1)^2}{n_1} + \frac{(s_2)^2}{n_2}} = \sqrt{\frac{(2.15)^2}{4} + \frac{(0.571)^2}{6}} = 1.1$$

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$$CI = (-11.494, -1.506)$$

We find t_{obs}.

$$t_{\text{obs}} = \frac{(\overline{x_2} - \overline{x_1}) - (\mu_2 - \mu_1)_0}{SE} = \frac{(10.3 - 16.8) - 0}{1.1} = -5.91$$

We find $|t_{\text{obs}}|$.

$$|t_{\text{obs}}| = 5.91$$

We find the bounds of the confidence interval.

$$CI = (\overline{x_2} - \overline{x_1}) \pm t^{\star} SE$$

$$CI = (-11.494,\ -1.506)$$

We find t_{obs}.

$$t_{\text{obs}} = \frac{(\overline{x_2} - \overline{x_1}) - (\mu_2 - \mu_1)_0}{SE} = \frac{(10.3 - 16.8) - 0}{1.1} = -5.91$$

We find $|t_{\text{obs}}|$.

$$|t_{\text{obs}}| = 5.91$$

We use the table to determine bounds on p-value. Remember, $df = 3$ and p-value $= P(|T| > |t_{\text{obs}}|)$.

We find the bounds of the confidence interval.

$$CI = (\overline{x_2} - \overline{x_1}) \pm t^\star SE$$

$$CI = (-11.494, -1.506)$$

We find t_{obs}.

$$t_{\text{obs}} = \frac{(\overline{x_2} - \overline{x_1}) - (\mu_2 - \mu_1)_0}{SE} = \frac{(10.3 - 16.8) - 0}{1.1} = -5.91$$

We find $|t_{\text{obs}}|$.

$$|t_{\text{obs}}| = 5.91$$

We use the table to determine bounds on p-value. Remember, $df = 3$ and p-value $= P(|T| > |t_{\text{obs}}|)$.

$$0.005 < p\text{-value} < 0.01$$

We find the bounds of the confidence interval.

$$CI = (\overline{x_2} - \overline{x_1}) \pm t^\star SE$$

$$CI = (-11.494, -1.506)$$

We find t_{obs}.

$$t_{\text{obs}} = \frac{(\overline{x_2} - \overline{x_1}) - (\mu_2 - \mu_1)_0}{SE} = \frac{(10.3 - 16.8) - 0}{1.1} = -5.91$$

We find $|t_{\text{obs}}|$.

$$|t_{\text{obs}}| = 5.91$$

We use the table to determine bounds on p-value. Remember, $df = 3$ and p-value $= P(|T| > |t_{\text{obs}}|)$.

$$0.005 < p\text{-value} < 0.01$$

We should consider both comparisons to make our decision.

We find the bounds of the confidence interval.

$$CI = (\overline{x_2} - \overline{x_1}) \pm t^\star SE$$

$$CI = (-11.494, -1.506)$$

We find t_{obs}.

$$t_{\text{obs}} = \frac{(\overline{x_2} - \overline{x_1}) - (\mu_2 - \mu_1)_0}{SE} = \frac{(10.3 - 16.8) - 0}{1.1} = -5.91$$

We find $|t_{\text{obs}}|$.

$$|t_{\text{obs}}| = 5.91$$

We use the table to determine bounds on p-value. Remember, $df = 3$ and p-value $= P(|T| > |t_{\text{obs}}|)$.

$$0.005 < p\text{-value} < 0.01$$

We should consider both comparisons to make our decision.

$$|t_{\text{obs}}| > t^\star$$

$$p\text{-value} < \alpha$$

We find the bounds of the confidence interval.

$$CI = (\overline{x_2} - \overline{x_1}) \pm t^{\star} SE$$

$$CI = (-11.494,\ -1.506)$$

We find t_{obs}.

$$t_{\text{obs}} = \frac{(\overline{x_2} - \overline{x_1}) - (\mu_2 - \mu_1)_0}{SE} = \frac{(10.3 - 16.8) - 0}{1.1} = -5.91$$

We find $|t_{\text{obs}}|$.

$$|t_{\text{obs}}| = 5.91$$

We use the table to determine bounds on p-value. Remember, $df = 3$ and p-value $= P(|T| > |t_{\text{obs}}|)$.

$$0.005 < p\text{-value} < 0.01$$

We should consider both comparisons to make our decision.

$$|t_{\text{obs}}| > t^{\star}$$

$$p\text{-value} < \alpha$$

Thus, we reject the null hypothesis. Also notice the confidence interval does not contain 0.

Answer list

1. 3
2. 4.54
3. 1.1
4. -11.494
5. -1.506
6. 5.909
7. 0.005
8. 0.01
9. yes

Example problem 2

An experiment has $n_1 = 6$ plants in the treatment group and $n_2 = 8$ plants in the control group. After some time, the plants' heights (in cm) are measured, resulting in the following data:

	value1	value2	value3	value4	value5	value6	value7	value8
sample 1:	0.81	0.98	1.39	1.34	0.78	1.11		
sample 2:	1.31	1.3	1.45	1.42	1.22	1.37	1.34	1.31

1. Determine degrees of freedom.
2. Determine $t^{\star}$ for a 98% confidence interval.
3. Determine SE.
4. Determine a lower bound of the 98% confidence interval of $\mu_2 - \mu_1$.
5. Determine an upper bound of the 98% confidence interval of $\mu_2 - \mu_1$.
6. Determine $|t_{\text{obs}}|$ under the null hypothesis $\mu_2 - \mu_1 = 0$.
7. Determine a lower bound of the two-tail p-value.
8. Determine an upper bound of two-tail p-value.
9. Do you reject the null hypothesis with a two-tail test using a significance level $\alpha = 0.02$? (yes or no)

Solution 2

These data are unpaired. We might as well find the sample means and sample standard deviations (use a calculator's built-in function for standard deviation).

$$\overline{x_1} = 1.07$$
$$\overline{x_2} = 1.34$$
$$s_1 = 0.259$$
$$s_2 = 0.0729$$

We make a conservative estimate of the degrees of freedom using the appropriate formula.

$$df \;=\; \min(n_1, n_2) - 1 \;=\; \min(6, 8) - 1 \;=\; 5$$

We use the t table to find $t^\star$ such that $P(|T| < t^\star) = 0.98$

$$t^\star = 3.36$$

We use the SE formula for unpaired data.

$$SE = \sqrt{\frac{(s_1)^2}{n_1} + \frac{(s_2)^2}{n_2}} = \sqrt{\frac{(0.259)^2}{6} + \frac{(0.0729)^2}{8}} = 0.109$$

We find the bounds of the confidence interval.

$$CI = (\overline{x_2} - \overline{x_1}) \pm t^{\star} SE$$
$$CI = (-0.096, 0.636)$$

We find t_{obs}.

$$t_{\text{obs}} = \frac{(\overline{x_2} - \overline{x_1}) - (\mu_2 - \mu_1)_0}{SE} = \frac{(1.34 - 1.07) - 0}{0.109} = 2.48$$

We find $|t_{\text{obs}}|$.

$$|t_{\text{obs}}| = 2.48$$

We use the table to determine bounds on p-value. Remember, $df = 5$ and p-value $= P(|T| > |t_{\text{obs}}|)$.

$$0.05 < p\text{-value} < 0.1$$

We should consider both comparisons to make our decision.

$$|t_{\text{obs}}| < t^{\star}$$

$$p\text{-value} > \alpha$$

Thus, we retain the null hypothesis. Also notice the confidence interval does contain 0.

Answer list

1. 5
2. 3.36
3. 0.109
4. -0.096
5. 0.636
6. 2.481
7. 0.05
8. 0.1
9. no