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Individual	Weight in 2010	Weight in 2020	Diff
Marion	140	135	-5
Sylvester	190	249	59
Florence	183	183	0
David	90	134	44
Gertrude	208	180	-28
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Two sets of observations are paired if each observation in one set has a special correspondence or connection with exactly one observation in the other data set.

Unpaired Data

Two separate random samples would produce unpaired data.

year=2010	
Individual	Weight
Lonzo	140
Rosalia	190
Leora	183
Otis	90
Edward	208
⋮	⋮

year=2020	
Individual	Weight
Henry	310
Harvey	250
Phoebe	210
Donna	150
John	110
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What would an implied question be?

> Did humans (as a species) tend to gain weight over time?

We will discuss unpaired analysis in Chapter 5.3 (next class).
With paired data, we consider a **mean of differences**.
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Derivation of paired formulas

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Basically, we can treat these differences just like any other independent and identically distributed random variables.

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- I would at least prefer using $\overline{X_{\text{diff}}}$ and $\overline{x_{\text{diff}}}$ to emphasize we are finding a mean of differences.
- The book's notation of μ_{diff} (for the population's true difference) is useful. We could also use $E(D)$ or μ_D.
- In order to match the book as much as possible, I will now use $x_{\text{diff},i}$ and $\overline{X_{\text{diff}}}$ and $\overline{x_{\text{diff}}}$ and μ_{diff}.

Example problem

A teacher wonders if, on average, a random student will perform about the same on two exams. She decides to run a two-tail t test on a random sample of size $n = 5$ with a signficance level $\alpha = 0.05$.

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Here are the results of her study:

Student	Exam 1	Exam 2
Norma	98	96
Elliot	15	10
Walton	61	61
Mable	80	79
Loretta	10	8

Perform the t test.

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Find the differences.

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$$s = \sqrt{\frac{\sum_{i=1}^{n} (x_{\mathsf{diff},i} - \overline{x_{\mathsf{diff}}})^2}{n-1}} = \sqrt{\frac{(0)^2 + (3)^2 + (2)^2 + (1)^2 + (0)^2}{5-1}} = 1.87$$

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We maintain that maybe students do equally well on both tests.

Practice

The following table has paired data. Test the hypotheses of whether or not the differences have a population average of 0. Use $\alpha = 0.1$.

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