We will use the notation $X \sim \mathcal{B}(n,\ p)$ to say X is binomially distributed with n trials and chance of success p (on each trial).

3.25: (a): Yes. Each trial is independent and has the same probability of success (and each trial will either be a success or failure).

(b): Let $X \sim \mathcal{B}(10,\ 0.697)$. We are asked to find the probability $P(X = 6)$.

$$\begin{aligned} P(X = 6) &= \binom{10}{6}(0.697)^6(1 - 0.697)^4 \\ &= 210 \times 0.697^6 \times 0.303^4 \\ &= \boxed{0.203} \end{aligned}$$

(c): This is the same thing as part (b), just worded differently. Each child either did or didn't. If 6 did, then 4 didn't.

$$\boxed{0.203}$$

(d): Let $X \sim \mathcal{B}(n = \mathbf{5},\ p = 0.697)$. We can calculate the entire probability mass function.

k	$P(X = k)$ before simplification	$P(X = k)$
0	$\binom{5}{0}0.697^0 \times 0.303^5$	0.00255395
1	$\binom{5}{1}0.697^0 \times 0.303^5$	0.02937469
2	$\binom{5}{2}0.697^2 \times 0.303^3$	0.13514297
3	$\binom{5}{3}0.697^3 \times 0.303^2$	0.31087342
4	$\binom{5}{4}0.697^4 \times 0.303^1$	0.35755573
5	$\binom{5}{5}0.697^5 \times 0.303^0$	0.16449924

We are asked to determine the probability $P(X \leq 2)$.

$$\begin{aligned} P(X \leq 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= 0.00255 + 0.02937 + 0.13514 \\ &\approx \boxed{0.167} \end{aligned}$$

Also, if you are interested... R will evaluate this with a single command:

```
pbinom(2, 5, 0.697)
```

Also, we can use `dbinom(k,n,p)` to get the simple probabilities.

```
> dbinom(seq(0,5), 5, 0.697)
[1] 0.00255395 0.02937469 0.13514297 0.31087342 0.35755573 0.16449924
> pbinom(2, 5, 0.697)
[1] 0.1670716
```

(e): (Same distribution as (d).) It is easiest to recognize the complement of "at least 1" is "none".

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - 0.00255 \\ &= \boxed{0.99745} \end{aligned}$$

3.26: **(a):** Yes. Each trial will be "success" or "fail", and each trial has the same probability of "success".

(b): Let $X \sim \mathcal{B}(100,\ 0.90)$. We are asked to calculate the probability $P(X = 97)$.

$$\begin{aligned} P(X = 97) &= \binom{100}{97}(0.9)^{97}(0.1)^3 \\ &= 161700 \times 0.9^{97} \times 0.1^3 \\ &= \boxed{0.005891602} \end{aligned}$$

We could also use R.

```
> dbinom(97,100,0.9)
[1] 0.005891602
```

(c): This is the same question as (b).

$$\boxed{0.00589}$$

(d): Let $Y \sim \mathcal{B}(10,\ 0.9)$, where Y is the number of people who had chickenpox from a random sample of 10 Americans. We recognize "at least one" as the complement of "none".

$$\begin{aligned} P(Y = 0) &= \binom{10}{0}(0.9)^0(0.1)^{10} \\ &= 0.000000001 \end{aligned}$$

We use the complement rule.

$$\begin{aligned} P(Y \geq 1) &= 1 - P(Y = 0) \\ &= 1 - 0.000000001 \\ &= 0.999999999 \\ &\approx \boxed{1} \end{aligned}$$

(e): We continue to let $Y \sim \mathcal{B}(10,\ 0.9)$. We rephrase the question in terms of number of people who *had* chickenpox. We want to find the probability that *at least* 7 Americans had chickenpox.

$$\begin{aligned} P(Y \geq 7) &= P(Y = 7) + P(Y = 8) + P(Y = 9) + P(Y = 10) \\ &= \binom{10}{7}(0.9)^7(0.1)^3 + \binom{10}{8}(0.9)^8(0.1)^2 + \binom{10}{9}(0.9)^9(0.1)^1 + \binom{10}{10}(0.9)^{10}(0.1)^0 \\ &\approx \boxed{0.9872} \end{aligned}$$

3.27: Let $X \sim \mathcal{B}(50,\ 0.70)$. As a reminder, that means $n = 50$ and $p = 0.7$.

(a): We can calculate the mean and standard deviation of X using the formulas for binomial distributions.

$$\mu = np = 50 \times 0.70 = \boxed{35}$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{(50)(0.7)(0.3)} = \boxed{3.24}$$

(b): Well, 45 would have a pretty high z score. Remember, we can often approximate a binomial distribution with a normal distribution.

$$z = \frac{45 - 35}{3.24} = 3.09$$

Yeah. If $|z| > 2$, we consider it unusual, as it happens less than 5% of the time due to chance. If $|z| > 3$, we consider it rare, as it happens less than 0.3% of the time due to chance.

(c): Hmm... we can do this a few ways. The first is to use R.

```
> 1 - pbinom(44, 50, 0.7)
[1] 0.0007228617
> sum(dbinom(seq(45,50), 50, 0.7))
[1] 0.0007228617
```

We can do this by hand...

k	$P(X = k)$ before simplification	$P(X = k)$
$\vdots$	$\vdots$	$\vdots$
45	$\binom{50}{45}(0.7)^{45}(0.3)^5$	0.000551
46	$\binom{50}{46}(0.7)^{46}(0.3)^4$	0.000140
47	$\binom{50}{47}(0.7)^{47}(0.3)^3$	0.000028
48	$\binom{50}{48}(0.7)^{48}(0.3)^2$	0.000004
49	$\binom{50}{49}(0.7)^{49}(0.3)^1$	0.000000
50	$\binom{50}{50}(0.7)^{50}(0.3)^0$	0.000000

We sum the probabilities to get $P(X \geq 45) = 0.00072$.

As a third technique, we can use the **normal approximation** with **continuity correction**. Let $Y \sim \mathcal{N}(35,\ 3.24)$.

$$\begin{aligned} P(X \geq 45) &\approx P(Y \geq 44.5) \\ &= 1 - \Phi\left(\frac{44.5 - 35}{3.24}\right) \\ &= 0.00168 \end{aligned}$$

But, notice the normal approximation did not work out great... it was off by a factor of more than 2. **The normal approximation does not work well in the tails**. However, all three techniques agree that the probability is quite low.

3.28: **(a):** Let $X \sim \mathcal{B}(120,\ 0.9)$. To determine the mean and standard deviation, we use the formulas for binomial distributions.

$$\mu = np = (120)(0.9) = 108$$
$$\sigma = \sqrt{np(1-p)} = \sqrt{(120)(0.9)(0.1)} = 3.286$$

We expect about 108 with a standard deviation of 3.28 people to have had chickenpox.

(b): 105 is within 1 standard deviation of 108.

$$z = \frac{105 - 108}{3.28} = -0.914$$

So, this does not seem unusual.

(c): We are asked to calculate $P(X \leq 105)$. This is easy with a computer.

```
> pbinom(105, 120, 0.9)
[1] 0.2181634
```

To do it by hand, we will use a normal approximation with the continuity correction. Let $Y \sim \mathcal{N}(108,\ 3.286)$.

$$\begin{aligned} P(X \leq 105) &\approx P(Y \leq 105.5) \\ &= \Phi\left(\frac{105.5 - 108}{3.286}\right) \\ &= 0.2233872 \end{aligned}$$

So, either way we get about 22% chance, which agrees with us not being surprised in part (b).

3.29: Let $X \sim \mathcal{B}(2500,\ 0.7)$, where X represents the number of students who accept. We will definitely use a normal approximation here. In fact, we are dealing with such a large n we don't even need to make a continuity correction.

We first calculate the mean and standard deviation using the formulas for binomial distributions.

$$\mu = np = 2500 \times 0.7 = 1750$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{2500(0.7)(0.3)} = 22.9$$

Now, let $Y \sim \mathcal{N}(1750,\ 22.9)$. The normal approximation (with continuity correction) can be used.

$$\begin{aligned} P(X > 1786) &\approx P(Y > 1786.5) \\ &= 1 - \Phi\left(\frac{1786.5 - 1750}{22.9}\right) \\ &= 0.056 \end{aligned}$$

Let's use R to calculate it more exactly.

```
> 1-pbinom(1786,2500,0.7)
[1] 0.05506358
```

Wow, in this case the normal approximation did a great job!

3.30: Let $X \sim \mathcal{B}(15000,\ 0.09)$. Remember, that means $n = 15000$ and $p = 0.09$. We want to calculate the probability $P(X \geq 1500)$. Using a computer, we can do that easily.

```
> 1-pbinom(1499,15000,0.09)
[1] 1.326331e-05
```

Let's also use a normal approximation. We first calculate the mean and standard deviation.

$$\mu = np = (15000)(0.09) = 1350$$

$$\sigma = \sqrt{np(1-p)} = \sqrt{(15000)(0.09)(0.91)} = 35.04996$$

Now we define a continuous random variable $Y \sim \mathcal{N}(1350,\ 35)$, and we use the normal approximation.

$$\begin{aligned} P(X \geq 1500) &\approx P(Y > 1499.5) \\ &= 1 - \Phi\left(\frac{1499.5 - 1350}{35}\right) \\ &= 0.0000097 \end{aligned}$$

We don't expect the normal approximation to work in the tails. But we know the chance is low!

3.31: In all cases, the sides are all equally likely, so we can just say 1 in 4 sides corresponds to "success", but which side is "success" depends on the case. So, we can just define random variable $X \sim \mathcal{B}(3,\ 0.25)$, where X is the number of successes. We make a table, where we use k to represent specific (possible) outcomes.

k	$P(X = k)$ before simplification	$P(X = k)$
0	$\binom{3}{0}(0.25)^0(0.75)^3$	0.421875
1	$\binom{3}{1}(0.25)^1(0.75)^2$	0.421875
2	$\binom{3}{2}(0.25)^2(0.75)^1$	0.140625
3	$\binom{3}{3}(0.25)^3(0.75)^0$	0.015625

(a): Reread as "at least one success". This is the complement of "no successes".

$$P(X \geq 1) = 1 - P(X = 0) = 1 - 0.422 = \boxed{0.578}$$

(b): Reread as "exactly two successes", and look at the table.

$$P(X = 2) = \boxed{0.141}$$

(c): Reread as "exactly one success", and look at the table.

$$P(X = 1) = \boxed{0.422}$$

(d): Reread as "at most two success". This is the complement of "exactly three successes".

$$P(X \leq 2) = 1 - P(X = 3) = 1 - 0.0156 = \boxed{0.9844}$$

3.32: Let $X \sim \mathcal{N}(10,\ 0.07)$, where X represents the number of teenagers suffering from arachnophobia.

(a): We want to calculate $P(X \geq 1)$. This is the complement of $P(X = 0)$.

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - \binom{10}{0}(0.07)^0(0.93)^{10} \\ &= \boxed{0.516} \end{aligned}$$

(b): We want to calculate $P(X = 2)$.

$$\begin{aligned} P(X = 2) &= \binom{10}{2}(0.07)^2(0.93)^8 \\ &= \boxed{0.123} \end{aligned}$$

(c): We want to calculate $P(X \leq 1)$.

$$\begin{aligned} P(X \leq 1) &= \binom{10}{0}(0.07)^0(0.93)^{10} + \binom{10}{1}(0.07)^1(0.93)^9 \\ &= \boxed{0.848} \end{aligned}$$

(d): No. There is a 15% chance that, in the tent, more than 1 teenager is afraid of spiders.

3.33: **(a):** $0.125 \times (1 - 0.125) = \boxed{0.109}$

(b): $\binom{2}{1}(0.125)^1(1 - 0.125)^1 = \boxed{0.219}$

(c): $\binom{6}{2}(0.125)^2(1 - 0.125)^4 = \boxed{0.137}$

(d): Complement. $1 - \binom{6}{0}(0.125)^0(1 - 0.125)^6 = \boxed{0.551}$

(e): Geometric. $(1 - 0.125)^3(0.125) = \boxed{0.0837}$

(f): We can calculate a z score. First we need μ and σ of the binomial distribution $\mathcal{B}(6,\ 0.75)$.

$$\mu = (6)(0.75) = 4.5$$

$$\sigma = \sqrt{(6)(0.75)(0.25)} = 1.06$$

$$z = \frac{2 - 4.5}{1.06} = -2.36$$

This z score is considered unusual because $|-2.36| > 2$.

We could also calculate the probability of having 2 **or fewer** children with brown eyes.

$$\begin{aligned} P(X \leq 2) &= \binom{6}{0}(0.75)^0(0.25)^6 + \binom{6}{1}(0.75)^1(0.25)^5 + \binom{6}{2}(0.75)^2(0.25)^4 \\ &= 0.0376 \end{aligned}$$

So, having 2 **or fewer** kids with brown eyes only happens about 4% of the time. This is low enough to be unusual.

3.34: (a): Let $X_{\text{a}} \sim \mathcal{B}(3,\ 0.25)$. We are asked for $P(X_{\text{a}} = 2)$.

$$P(X_{\text{a}} = 2) \quad = \quad \binom{3}{2}(0.25)^2(0.75)^1 \quad = \quad \boxed{0.14}$$

(b): Let $X_{\text{b}} \sim \mathcal{B}(3,\ 0.25)$. We are asked for $P(X_{\text{b}} = 0)$.

$$P(X_{\text{b}} = 0) \quad = \quad \binom{3}{0}(0.25)^0(0.75)^3 \quad = \quad \boxed{0.42}$$

(c): $X_{\text{c}} \sim \mathcal{B}(3,\ 0.25)$.

$$\begin{aligned} P(X_{\text{c}} \geq 1) &= 1 - P(X_{\text{c}} = 0) \\ &= 1 - \binom{3}{0}(0.25)^0(0.75)^3 \\ &= \boxed{0.578} \end{aligned}$$

(d): Geometric.

$$(1 - 0.25)^2(0.25) = \boxed{0.14}$$