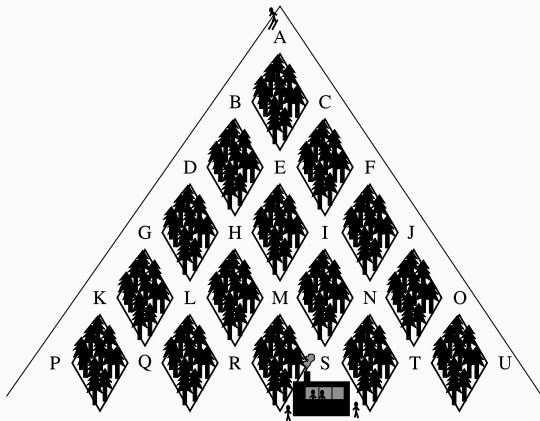


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Pascal's Triangle

[illegible]

Laney has 5 toes on her right foot. She wants to choose three of these nails to paint green. How many different ways can Laney do this?

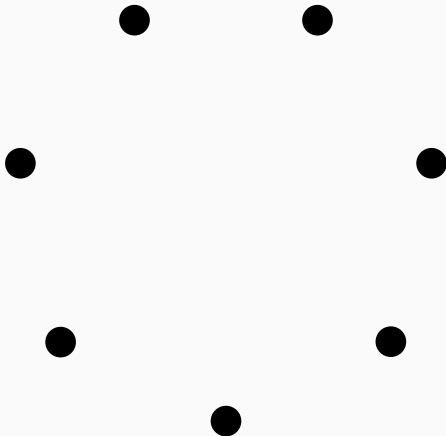


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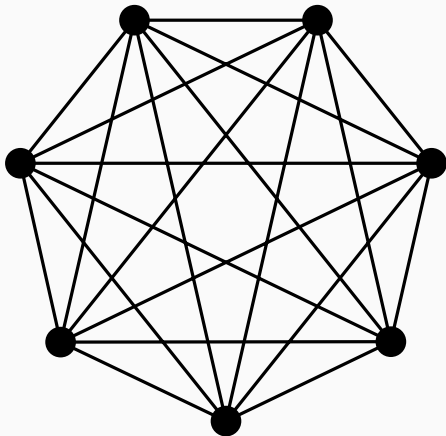


GGGxx	GGxGx	GGxxG	GxGGx	GxGxG
GxxGG	xGGGx	xGGxG	xGxGG	xxGGG

When given 7 dots, how many distinct line segments connect 2 of those dots? In other words, with 7 nodes, how many edges can be drawn?



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CCxCxxx	CxCxxxC	xCCCxxx	xCxxCCx	xxCxCxC
CCxxCxx	CxxCCxx	xCCxCxx	xCxxCxC	xxCxxCC
CCxxxCx	CxxCxCx	xCCxxCx	xCxxxCC	xxxCCCx
CCxxxxC	CxxCxxC	xCCxxxC	xxCCCxx	xxxCCxC
CxCCxxx	CxxxCCx	xCxCCxx	xxCCxCx	xxxGxCC
CxCxCxx	CxxxCx C	xCx CxCx	xxCCxxC	xxxxCCC

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CCxxxxC	CxxCxxC	xCCxxxC	xxCCCxx	xxxCCxC
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$$\binom{7}{3} = \frac{7!}{4! \cdot 3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} =$$

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Notice, these rearrangements are like anagrams.

Combinatorics: combinations

Combinations: list of all anagrams of a “word” which contains only 2 letters. Often we use 1 for “yes” or “success” and use 0 for “no” or “failure”.

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We define:

n = word length

r = how many 1s

The typical problem: We have n objects and we will choose r of them as “yes” (and the rest as “no”). How many possibilities exist?

$$n \text{ choose } r = {}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}$$

Evaluating n choose r with technology

If we wanted to evaluate $\binom{40}{27}$...

Geogebra Scientific Calculator:

```
nCr(40, 27)
```

R:

```
> choose(40,27)  
[1] 12033222880
```

TI Calculator:

```
40 nCr 27
```


Binomial distribution

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What is the probability of rolling 5 dice and getting 3 successes?

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Well... first let's do something easier...

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What is the probability of rolling 5 dice and getting (in this order) success, fail, success, success, and fail.

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Thus,

$$P(3 \text{ successes}) = 10 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 \approx \mathbf{0.032}$$

Binomial mass function

Let X represent the number of successes when n trials are performed and each trial has p chance of success. We use a formula to calculate the probability that X is k .

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

For example, if $n = 4$ and $p = 0.1$, then:

k	$P(X = k)$ unsimped	$P(X = k)$
0	$(1)(0.1)^0(0.9)^4$	0.6561
1	$(4)(0.1)^1(0.9)^3$	0.2916
2	$(6)(0.1)^2(0.9)^2$	0.0486
3	$(4)(0.1)^3(0.9)^1$	0.0036
4	$(1)(0.1)^4(0.9)^0$	0.0001

Practice

Find the probabilities of $X \sim \text{Binomial}(n = 2, p = 0.4)$.

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k	$P(X = k)$ unsimplified	$P(X = k)$ simplified
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1	$(2)(0.4)^1(0.6)^1$	0.48
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$$\mu = np$$

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$$\mu = (0)(0.4) + (1)(0.6) = 0.6$$

$$\sigma = \sqrt{(0 - 0.6)^2(0.4) + (1 - 0.6)^2(0.6)} = 0.4899$$

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$$\begin{aligned}\sigma &= \sqrt{(0 - p)^2(1 - p) + (1 - p)^2p} \\ &= \sqrt{p^2(1 - p) + (1 - p)^2p} \\ &= \sqrt{p^2 - p^3 + p - 2p^2 + p^3} \\ &= \sqrt{p - p^2} \\ &= \sqrt{p(1 - p)}\end{aligned}$$

A binomial is a sum of Bernoulli trials

In chapter 2.4 we learned the following rules.

$$E(W_1 + W_2 + \cdots + W_n) = E(W_1) + E(W_2) + \cdots + E(W_n)$$

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For a specific p , for all i between 1 and n , let $W_i \sim \text{Bernoulli}(p)$. Let X represent the sum of those variables, making $X \sim \text{Binomial}(n, p)$.

$$X = \sum_{i=1}^n W_i$$

If so, then we know (by using those rules):

$$E(X) = np$$

$$\text{Var}(X) = np(1 - p)$$

$$\text{SD}(X) = \sqrt{np(1 - p)}$$

Binomial mean and standard deviation

Let $X \sim \text{Binomial}(n, p)$. The mean (expected value) of a binomial distribution:

$$\mu = np$$

The standard deviation of a binomial distribution:

$$\sigma = \sqrt{np(1 - p)}$$

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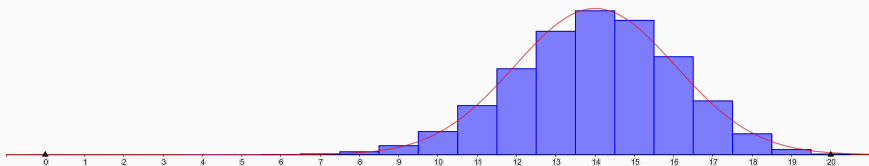
Let's overlay two density functions: the discrete binomial function and the continuous normal function.

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Let's overlay two density functions: the discrete binomial function and the continuous normal function.



Rule of thumb:

If $np \geq 10$ and $n(1 - p) \geq 10$, then the normal approximation will work well (except in the tails).

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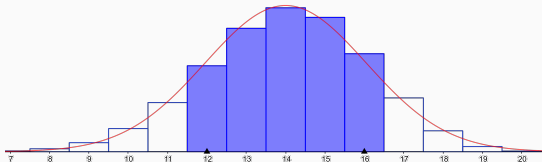
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Estimate $P(12 \leq X \leq 14)$ using the normal approximation.

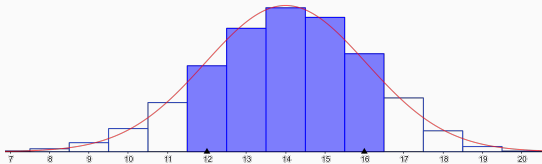


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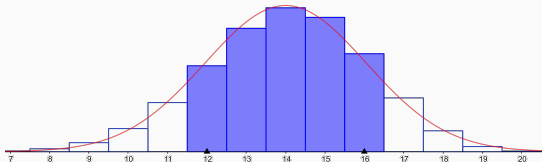
$$P(12 \leq X \leq 16) \approx P(11.5 < Y < 16.5)$$

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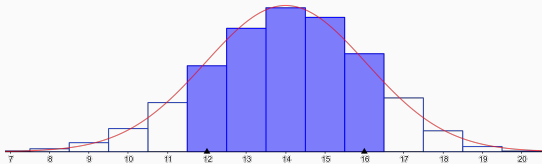
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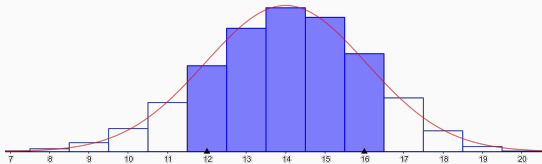
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$$P(12 \leq X \leq 16) \approx P(11.5 < Y < 16.5)$$

$$z_1 = \frac{11.5 - 14}{2.05} = -1.22 \quad z_2 = \frac{16.5 - 14}{2.05} = 1.22$$

$$P(12 \leq X \leq 16) \approx \Phi(1.22) - \Phi(-1.22) = \boxed{0.78}$$