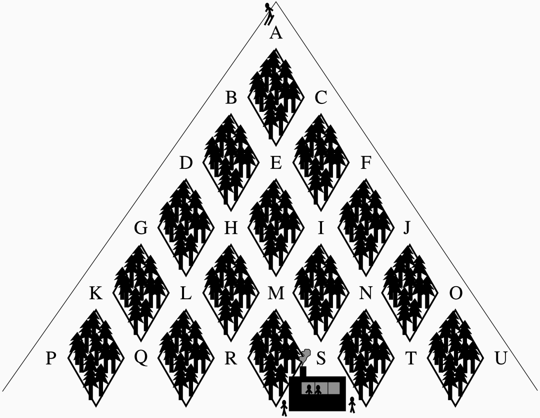
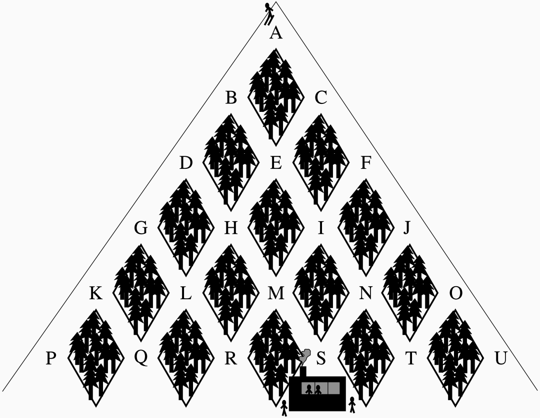


Laney is at the top of the mountain (point A). She hopes to ski **down** to point S (without going through trees or up hill). How many routes are possible?



Laney is at the top of the mountain (point A). She hopes to ski **down** to point S (without going through trees or up hill). How many routes are possible?

10

Pascal’s Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

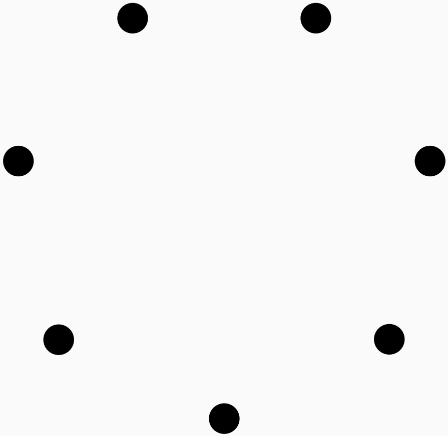
Laney has 5 toes on her right foot. She wants to choose three of these nails to paint green. How many different ways can Laney do this?

Laney has 5 toes on her right foot. She wants to choose three of these nails to paint green. How many different ways can Laney do this?

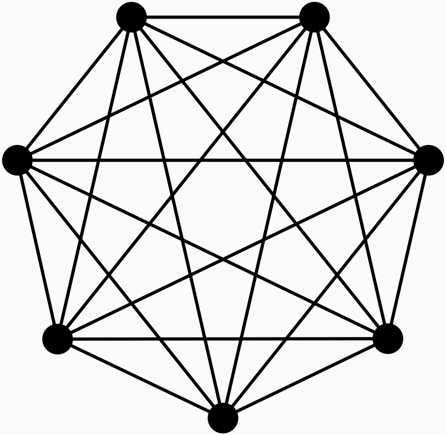
GGGxx *GGxGx* *GGxxG* *GxGGx* *GxGxG*
GxxGG *xGGGx* *xGGxG* *xGxGG* *xxGGG*

$\boxed{10}$

When given 7 dots, how many distinct line segments connect 2 of those dots? In other words, with 7 nodes, how many edges can be drawn?



When given 7 dots, how many distinct line segments connect 2 of those dots? In other words, with 7 nodes, how many edges can be drawn?



$\boxed{21}$

If there are 7 possible pizza toppings, and you will choose 3 of them, how many different pizzas are possible?

If there are 7 possible pizza toppings, and you will choose 3 of them, how many different pizzas are possible?

CCCxxxx	CxCxxCx	CxxxxCC	xCxCxxC	xxCxCCx
CCxCxxx	CxCxxxC	xCCCxxx	xCxxCCx	xxCxCxC
CCxxCxx	CxxCCxx	xCCxCxx	xCxxCxC	xxCxxCC
CCxxxCx	CxxCxCx	xCCxxCx	xCxxxCC	xxxCCCx
CCxxxxC	CxxCxxC	xCCxxxC	xxCCCxx	xxxCCxC
CxCCxxx	CxxxCCx	xCxCCxx	xxCCxCx	xxxCxCC
CxCxCxx	CxxxCxC	xCxCxCx	xxCCxxC	xxxxCCC

If there are 7 possible pizza toppings, and you will choose 3 of them, how many different pizzas are possible?

CCCxxxx	CxCxxCx	CxxxxCC	xCxCxxC	xxCxCCx
CCxCxxx	CxCxxxC	xCCCxxx	xCxxCCx	xxCxCxC
CCxxCxx	CxxCCxx	xCCxCxx	xCxxCxC	xxCxxCC
CCxxxCx	CxxCxCx	xCCxxCx	xCxxxCC	xxxCCCx
CCxxxxC	CxxCxxC	xCCxxxC	xxCCCxx	xxxCCxC
CxCCxxx	CxxxCCx	xCxCCxx	xxCCxCx	xxxCxCC
CxCxCxx	CxxxCxC	xCxCxCx	xxCCxxC	xxxxCCC

$$\binom{7}{3} = \frac{7!}{4! \cdot 3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} =$$

If there are 7 possible pizza toppings, and you will choose 3 of them, how many different pizzas are possible?

CCCxxxx	CxCxxCx	CxxxxCC	xCxCxxC	xxCxCCx
CCxCxxx	CxCxxxC	xCCCxxx	xCxxCCx	xxCxCxC
CCxxCxx	CxxCCxx	xCCxCxx	xCxxCxC	xxCxxCC
CCxxxCx	CxxCxCx	xCCxxCx	xCxxxCC	xxxCCCx
CCxxxxC	CxxCxxC	xCCxxxC	xxCCCxx	xxxCCxC
CxCCxxx	CxxxCCx	xCxCCxx	xxCCxCx	xxxCxCC
CxCxCxx	CxxxCxC	xCxCxCx	xxCCxxC	xxxxCCC

$$\binom{7}{3} = \frac{7!}{4! \cdot 3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = \boxed{35}$$

If there are 7 possible pizza toppings, and you will choose 3 of them, how many different pizzas are possible?

CCCxxxx	CxCxxCx	CxxxxCC	xCxCxxC	xxCxCCx
CCxCxxx	CxCxxxC	xCCCxxx	xCxxCCx	xxCxCxC
CCxxCxx	CxxCCxx	xCCxCxx	xCxxCxC	xxCxxCC
CCxxxCx	CxxCxCx	xCCxxCx	xCxxxCC	xxxCCCx
CCxxxxC	CxxCxxC	xCCxxxC	xxCCCxx	xxxCCxC
CxCCxxx	CxxxCCx	xCxCCxx	xxCCxCx	xxxCxCC
CxCxCxx	CxxxCxC	xCxCxCx	xxCCxxC	xxxxCCC

$$\binom{7}{3} = \frac{7!}{4! \cdot 3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = \boxed{35}$$

Notice, these rearrangements are like anagrams.

Combinatorics: combinations

Combinations: list of all anagrams of a “word” which contains only 2 letters. Often we use 1 for “yes” or “success” and use 0 for “no” or “failure”.

Combinatorics: combinations

Combinations: list of all anagrams of a “word” which contains only 2 letters. Often we use 1 for “yes” or “success” and use 0 for “no” or “failure”.

for example: 0011 0101 0110 1001 1010 1100

Combinatorics: combinations

Combinations: list of all anagrams of a “word” which contains only 2 letters. Often we use 1 for “yes” or “success” and use 0 for “no” or “failure”.

for example: 0011 0101 0110 1001 1010 1100

We define:

$$n = \text{word length}$$

$$r = \text{how many 1s}$$

The typical problem: We have n objects and we will choose r of them as “yes” (and the rest as “no”). How many possibilities exist?

$$n \text{ choose } r = {}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}$$

Evaluating n choose r with technology

If we wanted to evaluate $\binom{40}{27}$...

Geogebra Scientific Calculator:

```
nCr(40, 27)
```

R:

```
> choose(40,27)
[1] 12033222880
```

TI Calculator:

```
40 nCr 27
```

Binomial distribution

Imagine a dice game where a 6 is “success” and anything else is “failure”.

What is the probability of rolling 5 dice and getting 3 successes?

Imagine a dice game where a 6 is “success” and anything else is “failure”.

What is the probability of rolling 5 dice and getting 3 successes?

Well... first let’s do something easier...

Imagine a dice game where a 6 is “success” and anything else is “failure”.

What is the probability of rolling 5 dice and getting (in this order) success, fail, success, success, and fail.

$$P(10110) = ?$$

Imagine a dice game where a 6 is “success” and anything else is “failure”.

What is the probability of rolling 5 dice and getting (in this order) success, fail, success, success, and fail.

$$P(10110) = ?$$

$$P(10110) = \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \approx 0.0032$$

Imagine a dice game where a 6 is “success” and anything else is “failure”.

What is the probability of rolling 5 dice and getting (in this order) success, fail, success, success, and fail.

$$P(10110) = ?$$

$$P(10110) = \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \approx 0.0032$$

What is the probability of rolling 5 dice and getting (in this order) fail, fail, success, success, and success.

$$P(00111) = ?$$

Imagine a dice game where a 6 is “success” and anything else is “failure”.

What is the probability of rolling 5 dice and getting (in this order) success, fail, success, success, and fail.

$$P(10110) = ?$$

$$P(10110) = \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} \approx 0.0032$$

What is the probability of rolling 5 dice and getting (in this order) fail, fail, success, success, and success.

$$P(00111) = ?$$

$$P(00111) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \approx 0.0032$$

Imagine a dice game where a 6 is “success” and anything else is “failure”.

What is the probability of rolling 5 dice and getting 3 successes?

Imagine a dice game where a 6 is “success” and anything else is “failure”.

What is the probability of rolling 5 dice and getting 3 successes?

We need to consider how many (disjoint) ways we can accomplish 3 successes.

Imagine a dice game where a 6 is “success” and anything else is “failure”.

What is the probability of rolling 5 dice and getting 3 successes?

We need to consider how many (disjoint) ways we can accomplish 3 successes.

11100 11010 11001 10110 10101

10011 01110 01101 01011 00111

$$\binom{5}{3} = 10$$

There are ten ways to get 3 successes from 5 trials.

Imagine a dice game where a 6 is “success” and anything else is “failure”.

What is the probability of rolling 5 dice and getting 3 successes?

We need to consider how many (disjoint) ways we can accomplish 3 successes.

11100 11010 11001 10110 10101

10011 01110 01101 01011 00111

$$\binom{5}{3} = 10$$

There are ten ways to get 3 successes from 5 trials.

Each way has an equal probability.

$$P(\textit{a way}) = \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 \approx 0.0032$$

Imagine a dice game where a 6 is “success” and anything else is “failure”.

What is the probability of rolling 5 dice and getting 3 successes?

We need to consider how many (disjoint) ways we can accomplish 3 successes.

11100   11010   11001   10110   10101

10011   01110   01101   01011   00111

$$\binom{5}{3} = 10$$

There are ten ways to get 3 successes from 5 trials.

Each way has an equal probability.

$$P(\textit{a way}) = \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 \approx 0.0032$$

Thus,

$$P(\textit{3 successes}) = \mathbf{10}\left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 \approx \mathbf{0.032}$$

Binomial mass function

Let X represent the number of successes when n trials are performed and each trial has p chance of success. We use a formula to calculate the probability that X is k.

$$P(X = k) \quad = \quad \binom{n}{k} p^k (1-p)^{n-k}$$

For example, if $n = 4$ and $p = 0.1$, then:

k	$P(X = k)$ unsimped	$P(X = k)$
0	$(1)(0.1)^0(0.9)^4$	0.6561
1	$(4)(0.1)^1(0.9)^3$	0.2916
2	$(6)(0.1)^2(0.9)^2$	0.0486
3	$(4)(0.1)^3(0.9)^1$	0.0036
4	$(1)(0.1)^4(0.9)^0$	0.0001

Practice

Find the probabilities of $X \sim Binomial(n = 2, p = 0.4)$.

Practice

Find the probabilities of $X \sim Binomial(n = 2, p = 0.4)$.

k	$P(X = k)$ unsimplified	$P(X = k)$ simplified
0	$(1)(0.4)^0(0.6)^2$	*0.36*
1	$(2)(0.4)^1(0.6)^1$	*0.48*
2	$(1)(0.4)^2(0.6)^0$	*0.16*

Practice

Find the probabilities of $X \sim Binomial(n = 2, p = 0.4)$.

k	$P(X = k)$ unsimplified	$P(X = k)$ simplified
0	$(1)(0.4)^0(0.6)^2$	*0.36*
1	$(2)(0.4)^1(0.6)^1$	*0.48*
2	$(1)(0.4)^2(0.6)^0$	*0.16*

Determine $P(X \geq 1)$.

Practice

Find the probabilities of $X \sim Binomial(n = 2, p = 0.4)$.

k	$P(X = k)$ unsimplified	$P(X = k)$ simplified
0	$(1)(0.4)^0(0.6)^2$	*0.36*
1	$(2)(0.4)^1(0.6)^1$	*0.48*
2	$(1)(0.4)^2(0.6)^0$	*0.16*

Determine $P(X \geq 1)$.

$$P(X \geq 1) = 0.64$$

Determine the expected value.

Practice

Find the probabilities of $X \sim Binomial(n = 2, p = 0.4)$.

k	$P(X = k)$ unsimplified	$P(X = k)$ simplified
0	$(1)(0.4)^0(0.6)^2$	0.36
1	$(2)(0.4)^1(0.6)^1$	0.48
2	$(1)(0.4)^2(0.6)^0$	0.16

Determine $P(X \geq 1)$.

$$P(X \geq 1) = 0.64$$

Determine the expected value.

$$\mu = (0)(0.36) + (1)(0.48) + (2)(0.16) =$$

Practice

Find the probabilities of $X \sim Binomial(n = 2, p = 0.4)$.

k	$P(X = k)$ unsimplified	$P(X = k)$ simplified
0	$(1)(0.4)^0(0.6)^2$	0.36
1	$(2)(0.4)^1(0.6)^1$	0.48
2	$(1)(0.4)^2(0.6)^0$	0.16

Determine $P(X \geq 1)$.

$$P(X \geq 1) = 0.64$$

Determine the expected value.

$$\mu = (0)(0.36) + (1)(0.48) + (2)(0.16) = 0.8$$

Practice

Let $X \sim Binomial(20, 0.8)$. Calculate $P(X = 15)$.

Practice

Let $X \sim Binomial(20, 0.8)$. Calculate $P(X = 15)$.

$$\binom{20}{15}(0.8)^{15}(0.2)^5 =$$

Practice

Let $X \sim Binomial(20, 0.8)$. Calculate $P(X = 15)$.

$$\binom{20}{15}(0.8)^{15}(0.2)^5 = \boxed{0.1745595}$$

We are about to derive the following rules for binomials:

$$\mu = np$$

$$\sigma = \sqrt{np(1-p)}$$

Determine the expected value and standard deviation of X.

Practice

Let $X \sim Binomial(20, 0.8)$. Calculate $P(X = 15)$.

$$\binom{20}{15}(0.8)^{15}(0.2)^5 = \boxed{0.1745595}$$

We are about to derive the following rules for binomials:

$$\mu = np$$

$$\sigma = \sqrt{np(1-p)}$$

Determine the expected value and standard deviation of X.

$$\mu = (20)(0.8) = 16$$

Practice

Let $X \sim Binomial(20, 0.8)$. Calculate $P(X = 15)$.

$$\binom{20}{15}(0.8)^{15}(0.2)^{5} = \boxed{0.1745595}$$

We are about to derive the following rules for binomials:

$$\mu = np$$

$$\sigma = \sqrt{np(1-p)}$$

Determine the expected value and standard deviation of X.

$$\mu = (20)(0.8) = 16$$

$$\sigma = \sqrt{(20)(0.8)(0.2)} = 1.788854$$

A Binomial is a sum of Bernoulli trials

A Bernoulli trial is a random variable that can take on two possible values, 0 or 1, and has a p chance of being 1.

A Bernoulli trial is a random variable that can take on two possible values, 0 or 1, and has a p chance of being 1.

Let $W \sim Bernoulli(p = 0.6)$.

w	$P(W = w)$
0	0.4
1	0.6

Determine μ and σ.

A Bernoulli trial is a random variable that can take on two possible values, 0 or 1, and has a p chance of being 1.

Let $W \sim Bernoulli(p = 0.6)$.

w	$P(W = w)$
0	0.4
1	0.6

Determine μ and σ.

$$\mu = (0)(0.4) + (1)(0.6) = 0.6$$

$$\sigma = \sqrt{(0 - 0.6)^2(0.4) + (1 - 0.6)^2(0.6)} = 0.4899$$

Now, try this more generally. Let $W \sim Bernoulli(p)$.

Now, try this more generally. Let $W \sim Bernoulli(p)$.

w	$P(W = w)$
0	$(1 - p)$
1	p

Now, try this more generally. Let $W \sim Bernoulli(p)$.

w	$P(W = w)$
0	$(1 - p)$
1	p

Determine μ and σ.

Now, try this more generally. Let $W \sim Bernoulli(p)$.

w	$P(W = w)$
0	$(1 - p)$
1	p

Determine μ and σ.

$$\mu = (0)(1 - p) + (1)(p) = \boxed{\text{p}}$$

$$\begin{aligned}
\sigma &= \sqrt{(0 - p)^2(1 - p) + (1 - p)^2 p} \\
&= \sqrt{p^2(1 - p) + (1 - p)^2 p} \\
&= \sqrt{p^2 - p^3 + p - 2p^2 + p^3} \\
&= \sqrt{p - p^2} \\
&= \sqrt{p(1 - p)}
\end{aligned}$$

A binomial is a sum of Bernoulli trials

In chapter 2.4 we learned the following rules.

$$E(W_1 + W_2 + \cdots + W_n) = E(W_1) + E(W_2) + \cdots + E(W_n)$$

$$Var(W_1 + W_2 + \cdots + W_n) = Var(W_1) + Var(W_2) + \cdots + Var(W_n)$$

A binomial is a sum of Bernoulli trials

In chapter 2.4 we learned the following rules.

$$E(W_1 + W_2 + \cdots + W_n) = E(W_1) + E(W_2) + \cdots + E(W_n)$$

$$Var(W_1 + W_2 + \cdots + W_n) = Var(W_1) + Var(W_2) + \cdots + Var(W_n)$$

For a specific p, for all i between 1 and n, let $W_i \sim Bernoulli(p)$. Let X represent the sum of those variables, making $X \sim Binomial(n, p)$.

$$X = \sum_{i=1}^{n} W_i$$

If so, then we know (by using those rules):

$$E(X) = np$$

$$Var(X) = np(1-p)$$

$$SD(X) = \sqrt{np(1-p)}$$

Binomial mean and standard deviation

Let $X \sim Binomial(n, p)$. The mean (expected value) of a binomial distribution:

$$\mu = np$$

The standard deviation of a binomial distribution:

$$\sigma = \sqrt{np(1-p)}$$

Binomial Distributions are (often) approximately normal

Binomial Distributions are (often) approximately normal

Let $X \sim Binomial(n = 20, p = 0.7)$, which has $\mu = 14$ and $\sigma = 2.05$.

Binomial Distributions are (often) approximately normal

Let $X \sim Binomial(n = 20, p = 0.7)$, which has $\mu = 14$ and $\sigma = 2.05$.

Let $Y \sim N(\mu = 14, \sigma = 2.05)$.

Binomial Distributions are (often) approximately normal

Let $X \sim Binomial(n = 20, p = 0.7)$, which has $\mu = 14$ and $\sigma = 2.05$.

Let $Y \sim N(\mu = 14, \sigma = 2.05)$.

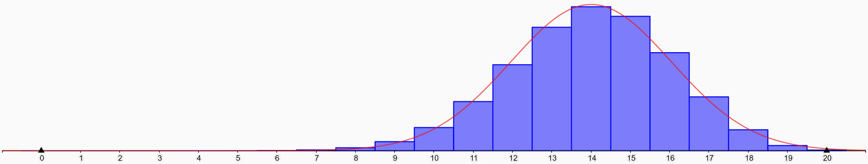
Let's overlay two density functions: the discrete binomial function and the continuous normal function.

Binomial Distributions are (often) approximately normal

Let $X \sim Binomial(n = 20,\ p = 0.7)$, which has $\mu = 14$ and $\sigma = 2.05$.

Let $Y \sim N(\mu = 14,\ \sigma = 2.05)$.

Let's overlay two density functions: the discrete binomial function and the continuous normal function.



Rule of thumb:

If $np \geq 10$ and $n(1-p) \geq 10$, then the normal approximation will work well (except in the tails).

Practice

Let $X \sim Binomial(n = 20, p = 0.7)$, which has $\mu = 14$ and $\sigma = 2.05$.

Practice

Let $X \sim Binomial(n = 20, p = 0.7)$, which has $\mu = 14$ and $\sigma = 2.05$.

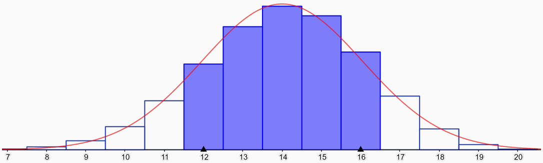
Let $Y \sim N(\mu = 14, \sigma = 1.79)$.

Practice

Let $X \sim Binomial(n = 20,\ p = 0.7)$, which has $\mu = 14$ and $\sigma = 2.05$.

Let $Y \sim N(\mu = 14,\ \sigma = 1.79)$.

Estimate $P(12 \leq X \leq 16)$ using the normal approximation.

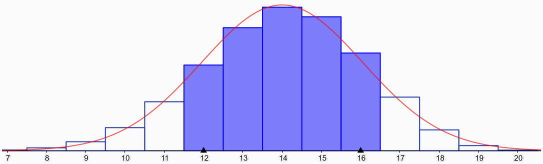


Practice

Let $X \sim Binomial(n = 20,\ p = 0.7)$, which has $\mu = 14$ and $\sigma = 2.05$.

Let $Y \sim N(\mu = 14,\ \sigma = 1.79)$.

Estimate $P(12 \leq X \leq 16)$ using the normal approximation.



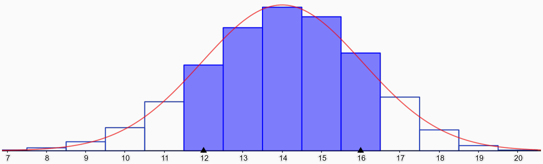
$$P(12 \leq X \leq 16) \approx P(11.5 < Y < 16.5)$$

Practice

Let $X \sim Binomial(n = 20, p = 0.7)$, which has $\mu = 14$ and $\sigma = 2.05$.

Let $Y \sim N(\mu = 14, \sigma = 1.79)$.

Estimate $P(12 \leq X \leq 16)$ using the normal approximation.



$$P(12 \leq X \leq 16) \approx P(11.5 < Y < 16.5)$$

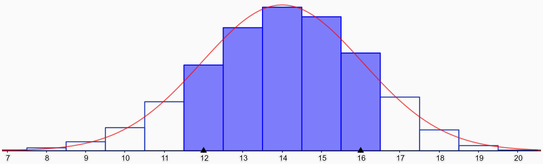
$z_1 = \frac{11.5-14}{2.05} = -1.22$

Practice

Let $X \sim Binomial(n = 20, p = 0.7)$, which has $\mu = 14$ and $\sigma = 2.05$.

Let $Y \sim N(\mu = 14, \sigma = 1.79)$.

Estimate $P(12 \leq X \leq 16)$ using the normal approximation.



$$P(12 \leq X \leq 16) \approx P(11.5 < Y < 16.5)$$

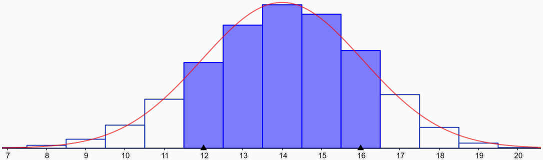
$z_1 = \frac{11.5-14}{2.05} = -1.22 \qquad z_2 = \frac{16.5-14}{2.05} = 1.22$

Practice

Let $X \sim Binomial(n = 20, p = 0.7)$, which has $\mu = 14$ and $\sigma = 2.05$.

Let $Y \sim N(\mu = 14, \sigma = 1.79)$.

Estimate $P(12 \leq X \leq 16)$ using the normal approximation.



$$P(12 \leq X \leq 16) \approx P(11.5 < Y < 16.5)$$

$$z_1 = \frac{11.5-14}{2.05} = -1.22 \qquad z_2 = \frac{16.5-14}{2.05} = 1.22$$

$$P(12 \leq X \leq 16) \approx \Phi(1.22) - \Phi(-1.22) = \boxed{0.78}$$