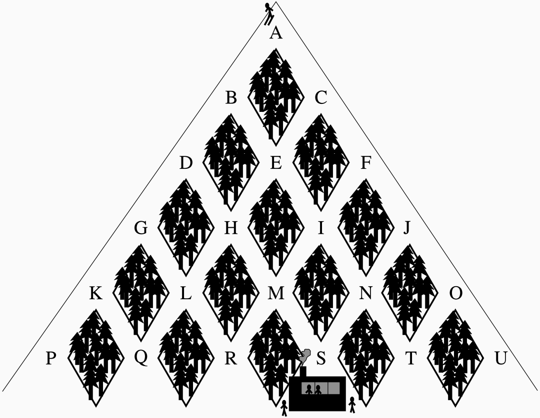
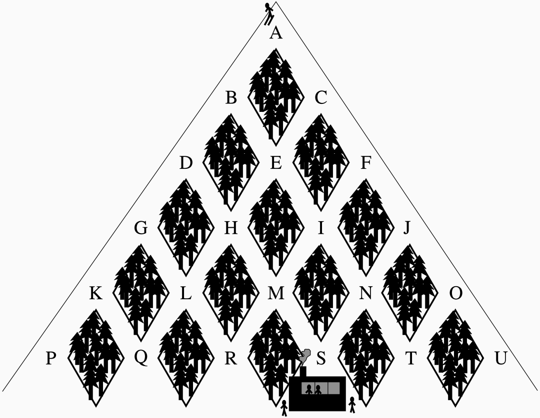


Laney is at the top of the mountain (point A). She hopes to ski **down** to point S (without going through trees or up hill). How many routes are possible?



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10

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Laney has 5 toes on her right foot. She wants to choose three of these nails to paint green. How many different ways can Laney do this?

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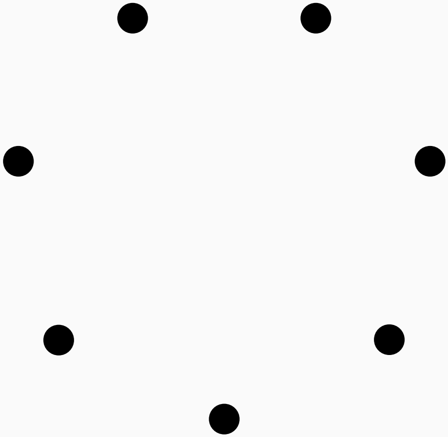
GGGxx	*GGxGx*	*GGxxG*	*GxGGx*	*GxGxG*
GxxGG	*xGGGx*	*xGGxG*	*xGxGG*	*xxGGG*

$\boxed{10}$

When given 7 dots, how many distinct line segments connect 2 of those dots? In other words, with 7 nodes, how many edges can be drawn?



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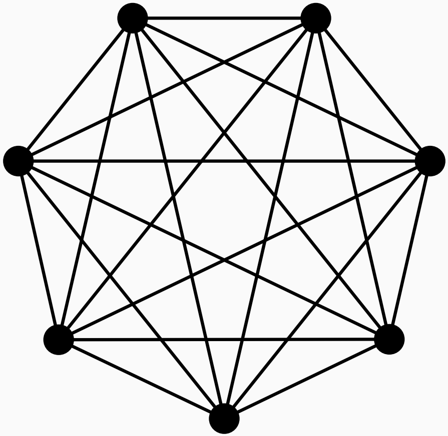

$\boxed{21}$

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CCxCxxx	CxCxxxC	xCCCxxx	xCxxCCx	xxCxCxC
CCxxCxx	CxxCCxx	xCCxCxx	xCxxCxC	xxCxxCC
CCxxxCx	CxxCxCx	xCCxxCx	xCxxxCC	xxxCCCx
CCxxxxC	CxxCxxC	xCCxxxC	xxCCCxx	xxxCCxC
CxCCxxx	CxxxCCx	xCxCCxx	xxCCxCx	xxxCxCC
CxCxCxx	CxxxCxC	xCxCxCx	xxCCxxC	xxxxCCC

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CCxxxxC	CxxCxxC	xCCxxxC	xxCCCxx	xxxCCxC
CxCCxxx	CxxxCCx	xCxCCxx	xxCCxCx	xxxCxCC
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$$\binom{7}{3} = \frac{7!}{4! \cdot 3!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} =$$

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CCxxxCx	CxxCxCx	xCCxxCx	xCxxxCC	xxxCCCx
CCxxxxC	CxxCxxC	xCCxxxC	xxCCCxx	xxxCCxC
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Notice, these rearrangements are like anagrams.

Combinatorics: combinations

Combinations: list of all anagrams of a “word” which contains only 2 letters. Often we use 1 for “yes” or “success” and use 0 for “no” or “failure”.

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We define:

$$n = \text{word length}$$

$$r = \text{how many 1s}$$

The typical problem: We have n objects and we will choose r of them as “yes” (and the rest as “no”). How many possibilities exist?

$$n \text{ choose } r = {}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}$$

Evaluating n choose r with technology

If we wanted to evaluate $\binom{40}{27}$...

Geogebra Scientific Calculator:

```
nCr(40, 27)
```

R:

```
> choose(40,27)
[1] 12033222880
```

TI Calculator:

```
40 nCr 27
```

Binomial distribution

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What is the probability of rolling 5 dice and getting 3 successes?

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Well... first let’s do something easier...

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What is the probability of rolling 5 dice and getting (in this order) success, fail, success, success, and fail.

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Thus,

$$P(\textit{3 successes}) = \mathbf{10}\left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 \approx \mathbf{0.032}$$

Binomial mass function

Let X represent the number of successes when n trials are performed and each trial has p chance of success. We use a formula to calculate the probability that X is k.

$$P(X = k) \quad = \quad \binom{n}{k} p^k (1-p)^{n-k}$$

For example, if $n = 4$ and $p = 0.1$, then:

k	$P(X = k)$ unsimped	$P(X = k)$
0	$(1)(0.1)^0(0.9)^4$	0.6561
1	$(4)(0.1)^1(0.9)^3$	0.2916
2	$(6)(0.1)^2(0.9)^2$	0.0486
3	$(4)(0.1)^3(0.9)^1$	0.0036
4	$(1)(0.1)^4(0.9)^0$	0.0001

Practice

Find the probabilities of $X \sim Binomial(n = 2, p = 0.4)$.

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$$\mu = (0)(0.36) + (1)(0.48) + (2)(0.16) = 0.8$$

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A Binomial is a sum of Bernoulli trials

A Bernoulli trial is a random variable that can take on two possible values, 0 or 1, and has a p chance of being 1.

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$$\mu = (0)(0.4) + (1)(0.6) = 0.6$$

$$\sigma = \sqrt{(0 - 0.6)^2(0.4) + (1 - 0.6)^2(0.6)} = 0.4899$$

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$$\begin{aligned}\sigma &= \sqrt{(0-p)^2(1-p) + (1-p)^2 p} \\ &= \sqrt{p^2(1-p) + (1-p)^2 p} \\ &= \sqrt{p^2 - p^3 + p - 2p^2 + p^3} \\ &= \sqrt{p - p^2} \\ &= \sqrt{p(1-p)}\end{aligned}$$

A binomial is a sum of Bernoulli trials

In chapter 2.4 we learned the following rules.

$$E(W_1 + W_2 + \cdots + W_n) = E(W_1) + E(W_2) + \cdots + E(W_n)$$

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For a specific p, for all i between 1 and n, let $W_i \sim Bernoulli(p)$. Let X represent the sum of those variables, making $X \sim Binomial(n, p)$.

$$X = \sum_{i=1}^{n} W_i$$

If so, then we know (by using those rules):

$$E(X) = np$$

$$Var(X) = np(1-p)$$

$$SD(X) = \sqrt{np(1-p)}$$

Binomial mean and standard deviation

Let $X \sim Binomial(n, p)$. The mean (expected value) of a binomial distribution:

$$\mu = np$$

The standard deviation of a binomial distribution:

$$\sigma = \sqrt{np(1-p)}$$

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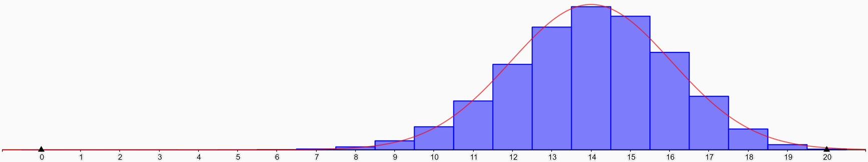

Rule of thumb:

If $np \geq 10$ and $n(1-p) \geq 10$, then the normal approximation will work well (except in the tails).

Practice

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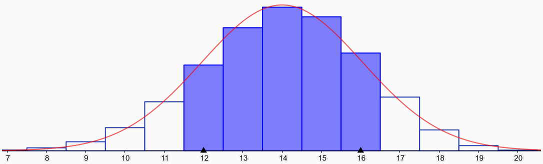
Estimate $P(12 \leq X \leq 16)$ using the normal approximation.



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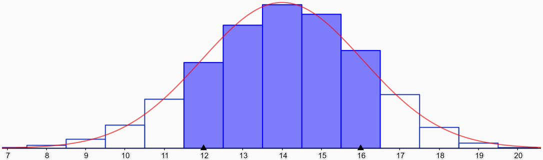

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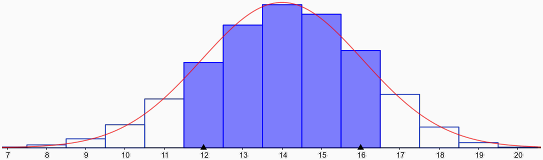

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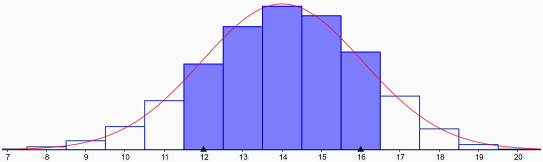

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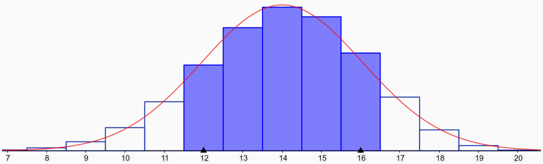

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$z_1 = \frac{11.5-14}{2.05} = -1.22 \qquad z_2 = \frac{16.5-14}{2.05} = 1.22$

$$P(12 \leq X \leq 16) \approx \Phi(1.22) - \Phi(-1.22) = \boxed{0.78}$$