MATH 181 2ND EXAM SOLUTIONS

SPRING 2019

Name: _____

- Write your **full name** on the line above.

- Show your work. Incorrect answers with work can receive partial credit.

- Attempt every question; showing you understand the question earns some credit.

- If you run out of room for an answer, continue on the back of the page. Before doing so, write "see back" with a circle around it.

- You can use 1 page (front and back) of notes in addition to the formula sheet (on page 2) and z-table (last page).

- You can use (and probably need) a calculator.

- You can use the Geogebra Scientific Calculator instead of a calculator. You need to put your phone on **airplane mode** and then within the application, start **exam mode**; you should see a green bar with a timer counting up.

- If a question is confusing or ambiguous, please ask for clarification; however, you will not be told how to answer the question.

- **Box your final answer**.

- You can rip off the z-table, but please keep the rest of the test intact.

Do not write in this grade table.

Question:	**Q1**	**Q2**	**Q3**	**Q4**	**Q5**	**Q6**	Total
Points:	10	10	10	10	10	10	60
Score:							

Normal Distribution:

$X \sim \mathcal{N}(\mu, \sigma)$

$\mu =$ population mean

$\sigma =$ population standard deviation

$x =$ possible value of X

$\ell =$ percentile of x (left area)

$\Phi(z) =$ standard normal cumulative function

$$z = \frac{x - \mu}{\sigma}$$

$$P(X < x) = \Phi(z)$$

$$\ell = \Phi(z)$$

$$z = \Phi^{-1}(\ell)$$

Bernoulli Distribution:

$X \sim \text{Bern}(p)$

$X = 0$ for fail or 1 for success

$p =$ probability of success

$$P(X = 0) = 1 - p$$

$$P(X = 1) = p$$

$$\mu = p$$

$$\sigma = \sqrt{p(1 - p)}$$

Geometric Distribution:

$X \sim \text{Geo}(p)$

$X =$ number of trials until first success

$p =$ probability of success on each trial

$n =$ a possible number of trials

$$P(X = n) = (1 - p)^{n-1}(p)$$

$$\mu = \frac{1}{p}$$

$$\sigma = \sqrt{\frac{1 - p}{p^2}}$$

Mean-Sampling Distribution:

$\bar{X} =$ sample mean

$s =$ sample standard deviation

$n =$ sample size

$\mu =$ population mean

$\sigma =$ population standard deviation

$SE =$ standard error

$$SE = \frac{\sigma}{\sqrt{n}}$$

If $n \geq 30$ (or if population is normal) then:

$$\bar{X} \sim \mathcal{N}(\mu, SE)$$

Binomial Distribution:

$X \sim \mathcal{B}(n, p)$

$X =$ number of successes from n trials

$p =$ probability of success on each trial

$n =$ number of trials

$k =$ a possible number of successes

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$\mu = np$$

$$\sigma = \sqrt{np(1 - p)}$$

If $np \geq 10$ and $n(1 - p) \geq 10$, then

$$X \sim \mathcal{N}(\mu, \sigma)$$

Continuity correction:

$$P(X \leq k) \approx \Phi\left(\frac{k + 0.5 - \mu}{\sigma}\right)$$

Confidence Interval:

$CI =$ confidence interval

$\gamma =$ confidence level

$\bar{x} =$ sample mean

$s =$ sample standard deviation

$$z^{\star} = \Phi^{-1}\left(\frac{\gamma + 1}{2}\right)$$

$$SE \approx \frac{s}{\sqrt{n}}$$

$$CI = \bar{x} \pm z^{\star} SE$$

Hypothesis testing:

$H_0 : \quad \mu = \mu_0$

$H_A : \quad \mu \neq \mu_0$

$\bar{x} =$ a possible/specific/observed sample mean

$s =$ sample standard deviation

$\alpha =$ significance level

$$\sigma \approx s$$

$$z = \frac{\bar{x} - \mu_0}{SE}$$

$$p\text{-value} = P(|Z| > |z|)$$
$$= 2 \cdot \Phi(-|z|)$$

If p-value $< \alpha$, then reject H_0, else retain H_0.

Q1. (10 points) Let random variable X be normally distributed with mean $\mu = 33$ and standard deviation $\sigma = 4$.

(a) Evaluate $P(X < 37)$.

Solution: We find a z-score.
$$z = \frac{37 - 33}{4} = 1$$

We use the z table to evaluate the probability.

$$P(X < 37) \; = \; P(Z < 1) \; = \; \Phi(1) \; = \; \boxed{0.8413}$$

(b) Determine x such that $P(X < x) = 0.33$.

Solution: We find the z-score from the z-table.

$$z \; = \; \Phi^{-1}(0.33) \; = \; -0.44$$

We convert this z-score into an x-score.

$$
\begin{aligned}
x \; &= \; \mu + z\sigma \\
&= \; 33 + (-0.44)(4) \\
&= \; \boxed{31.24}
\end{aligned}
$$

Q2. (10 points) Imagine a scratch-off lottery has a chance of success $p = 0.01$.

(a) What is the mean number of trials until the first success?

> **Solution:** This situaton is described by a geometric distribution.
>
> $$\mu = \frac{1}{p} = \frac{1}{0.01} = \boxed{100}$$
>
> The mean number of trials is 100. The expected number of trials is 100.

(b) What is the probability of getting the first success on the twelfth trial?

> **Solution:** This situation is still described by a geometric distribution. So, we use the geometric probability formula: $P(X = n) = (1 - p)^{n-1}p$.
>
> $$\begin{aligned} P(X = 12) &= (1 - 0.01)^{12-1}(0.01) \\ &= (0.99)^{11}(0.01) \\ &= \boxed{0.00895} \end{aligned}$$

Q3. (10 points) Let each trial have a chance of success $p = 0.25$. We will predict what happens when we have 100 trials.

(a) What is the probability of getting exactly 26 successes?

> **Solution:** This situation is described by a binomial distribution.
>
> $$P(X = 26) = \binom{100}{26}(0.25)^{26}(0.75)^{74}$$
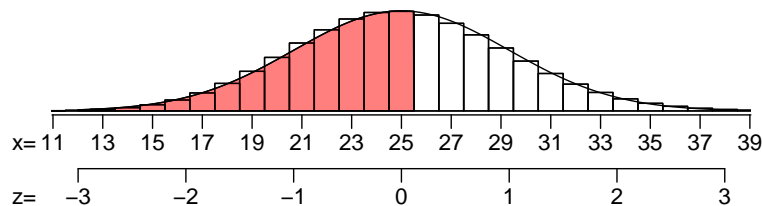> $$= \boxed{0.09}$$

(b) What is the probability of getting fewer than 26 successes? (Use a normal approximation and continuity correction.)

> **Solution:** We find the parameters needed to construct a normal approximation.
>
> $$\mu = np = (100)(0.25) = 25$$
>
> $$\sigma = \sqrt{np(1-p)}$$
> $$= \sqrt{(100)(0.25)(0.75)}$$
> $$= 4.33$$
>
> Here is a sketch:
>
>
>
> We find the z score of the boundary.
>
> $$z = \frac{25.5 - 25}{4.33} = 0.12$$
>
> We calculate the probability.
>
> $$P(X < 26) = P(X < 25.5) = P(Z < 0.12) = \boxed{0.55}$$

(c) What is the probability of getting more than 26 successes?

> **Solution:** The easiest way to do this is by recognizing these three probabilities are mutually exclusive and exhaustive.
> $$1 - 0.09 - 0.55 = \boxed{0.36}$$

Q4. (10 points) A uniformly distributed population has a mean $\mu = 250$ and a standard deviation $\sigma = 18$. When a sample of size $n = 144$ is taken, what is the probability of getting a sample mean between 247 and 253?

Solution: This situation is described by a mean-sampling distribution. We find the standard error.

$$SE \;=\; \frac{\sigma}{\sqrt{n}} \;=\; \frac{18}{\sqrt{144}} \;=\; 1.5$$

Even though the population is uniformly distributed, the sampling distribution is approximately normal.
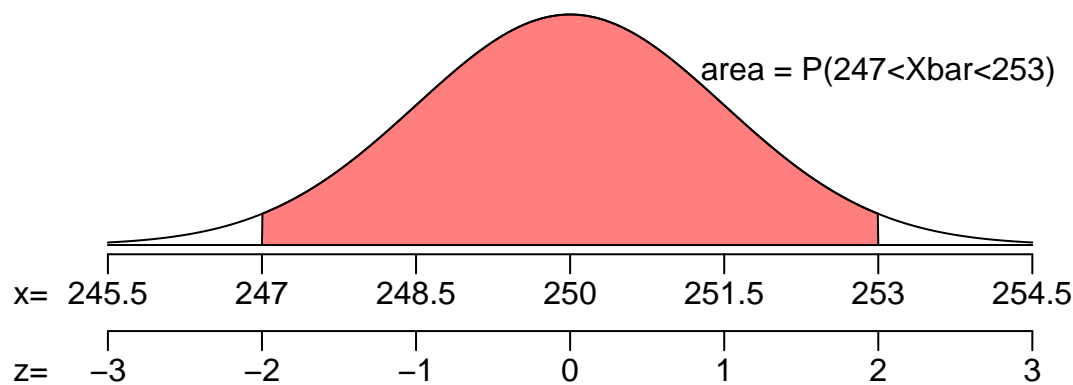
$$\bar{X} \sim \mathcal{N}(250,\; 1.5)$$

We find the z-scores.

$$z_{\text{LOWER}} \;=\; \frac{247 - 250}{1.5} \;=\; -2$$

$$z_{\text{UPPER}} \;=\; \frac{253 - 250}{1.5} \;=\; 2$$

Here is a sketch:

area = P(247<Xbar<253)

x= 245.5	247	248.5	250	251.5	253	254.5
z= −3	−2	−1	0	1	2	3

We find the probability.

$$
\begin{aligned}
P(247 < \bar{X} < 253) \;&=\; P(-2 < Z < 2) \\
&=\; \Phi(2) - \Phi(-2) \\
&=\; \boxed{0.9545}
\end{aligned}
$$

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Q5. (10 points) A sample of size $n = 144$ has a mean $\bar{x} = 253.92$ and a standard deviation $s = 17.54$. Construct a confidence interval of the population's mean based on the sample using a confidence level $\gamma = 0.99$.

Solution: We find $z^\star$.

$$
\begin{aligned}
z^\star &= \Phi^{-1}\left(\frac{0.99 + 1}{2}\right) \\
&= \Phi^{-1}(0.995) \\
&= 2.58
\end{aligned}
$$

We find the standard error.
$$
SE = \frac{17.54}{\sqrt{144}} = 1.4617
$$

We determine the confidence interval.

$$
\begin{aligned}
CI &= \bar{x} \pm z^\star SE \\
&= 253.92 \pm (2.58)(1.4617) \\
&= \boxed{(250.1,\ 257.7)}
\end{aligned}
$$

Q6. (10 points) Imagine you thought a uniformly distributed population had a mean $\mu = 250$ and standard deviation $\sigma = 18$. However, you are skeptical, so you decide to run a two-tailed hypothesis test with a significance level $\alpha = 0.01$.

Your sample of size $n = 144$ results in a mean $\bar{x} = 253.92$.

(a) State the hypotheses.

Solution:
$$H_0: \quad \mu = 250$$
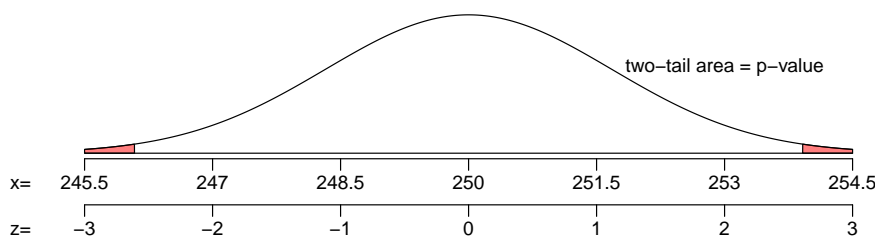$$H_A: \quad \mu \neq 250$$

(b) Describe and sketch the null's sampling distribution.

Solution: We need the standard error for a sampling distribution.

$$SE = \frac{18}{\sqrt{144}} = 1.5$$

We think the null's sampling distribution is normal with mean $\mu = 250$ and standard deviation $\sigma = 1.5$.

$$\bar{X} \sim \mathcal{N}(250, \ 1.5)$$

(c) Find a z score of the sample mean.

Solution:
$$z = \frac{253.92 - 250}{1.5} = 2.61$$

(d) Calculate the p-value.

Solution:
$$p\text{-value} = 2 \cdot \Phi(-2.61) = (2)(0.0045) = 0.009$$

(e) Make your conclusion.

Solution: We reject the null hypothesis because $0.009 < 0.01$.