

Conditional probability

Relapse

Researchers randomly assigned 72 chronic users of cocaine into three groups: desipramine (antidepressant), lithium (standard treatment for cocaine) and placebo. Results of the study are summarized below.

		no	
	relapse	relapse	total
desipramine	10	14	24
lithium	18	6	24
placebo	20	4	24
total	48	24	72

http://www.oswego.edu/~srp/stats/2_way_tbl_1.htm

Marginal probability

What is the probability that a patient relapsed?

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$$P(\text{relapsed}) = \frac{48}{72} \approx 0.67$$

Joint probability

What is the probability that a patient received the antidepressant (desipramine) and relapsed?

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$$P(\text{relapsed and desipramine}) = \frac{10}{72} \approx 0.14$$

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Conditional probability (cont.)

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$$P(\text{relapse} \mid \text{desipramine}) = \frac{10}{24} \approx 0.42$$

$$P(\text{relapse} \mid \text{lithium}) = \frac{18}{24} \approx 0.75$$

$$P(\text{relapse} \mid \text{placebo}) = \frac{20}{24} \approx 0.83$$

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$$P(\text{desipramine} \mid \text{relapse}) = \frac{10}{48} \approx 0.21$$

$$P(\text{lithium} \mid \text{relapse}) = \frac{18}{48} \approx 0.375$$

$$P(\text{placebo} \mid \text{relapse}) = \frac{20}{48} \approx 0.42$$

General multiplication rule

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- It is useful to think of B as the outcome of interest and A as the condition.

Independence and conditional probabilities

Consider the following (hypothetical) distribution of gender and major of students in an introductory statistics class:

	social science	non-social science	total
female	30	20	50
male	30	20	50
total	60	40	100

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- The probability that a randomly selected student is a social science major given that they are female is $\frac{30}{50} = 0.6$.
- Since $P(SS|M)$ also equals 0.6, major of students in this class does not depend on their gender: $P(SS | F) = P(SS)$.

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- Conceptually: Giving B doesn't tell us anything about A .
- Mathematically: We know that if events A and B are independent, $P(A \text{ AND } B) = P(A) \times P(B)$. Then,

$$P(A|B) = \frac{P(A \text{ AND } B)}{P(B)} = \frac{P(A) \times P(B)}{P(B)} = P(A)$$

Bayes' Theorem

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Example of using Bayes' Theorem

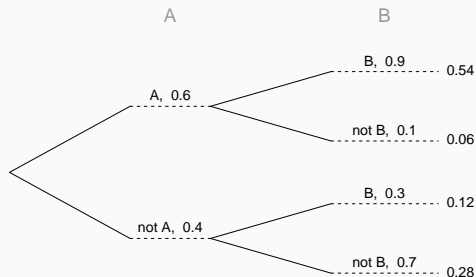
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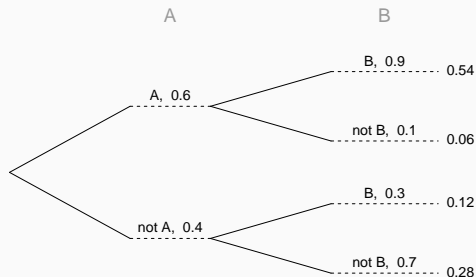
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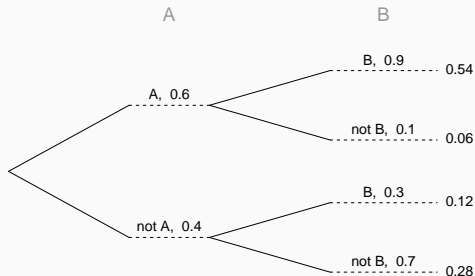
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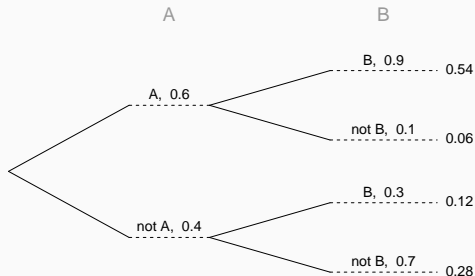


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$$P(A|B) = \frac{0.54}{0.54+0.12} = \frac{0.54}{0.66} \\ \approx 0.818$$

Breast cancer screening

- American Cancer Society estimates that about 1.7% of women have breast cancer.

<http://www.cancer.org/cancer/cancerbasics/cancer-prevalence>

- Susan G. Komen For The Cure Foundation states that mammography correctly identifies about 78% of women who truly have breast cancer.

<http://ww5.komen.org/BreastCancer/AccuracyofMammograms.html>

- An article published in 2003 suggests that up to 10% of all mammograms result in false positives for patients who do not have cancer.

<http://www.ncbi.nlm.nih.gov/pmc/articles/PMC1360940>

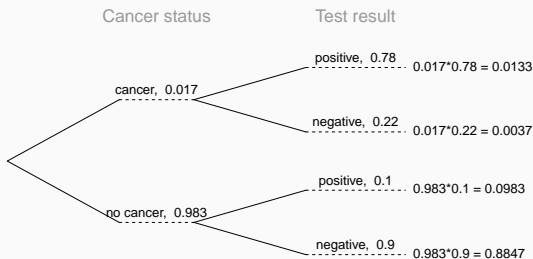
Note: *These percentages are approximate, and very difficult to estimate.*

Inverting probabilities

When a patient goes through breast cancer screening there are two competing claims: patient **has cancer** and patient **doesn't have cancer**. If a mammogram yields a positive result, what is the probability that patient actually has cancer?

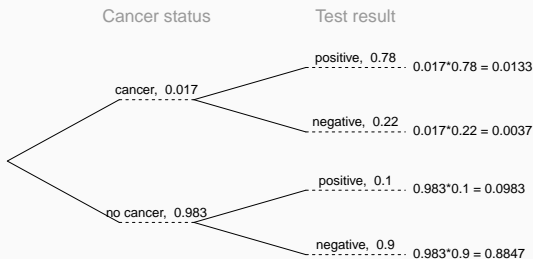
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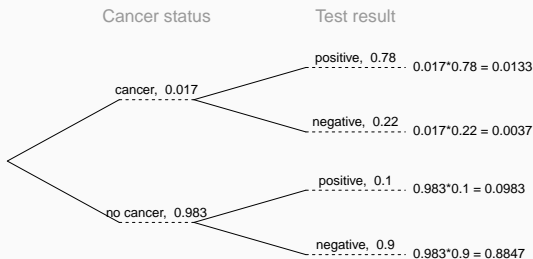
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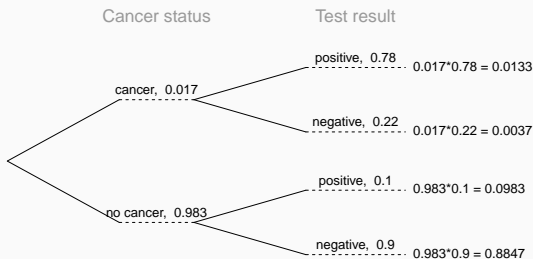
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$$P(C|+) = \frac{P(C \text{ AND } +)}{P(+)}$$

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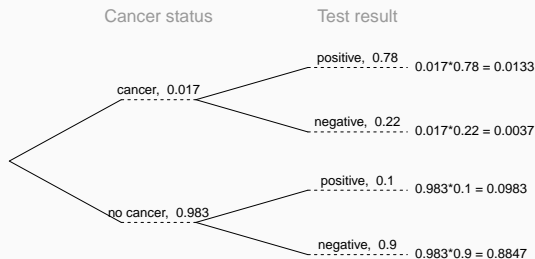
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$$\begin{aligned} P(C|+) &= \frac{P(C \text{ AND } +)}{P(+)} \\ &= \frac{0.0133}{0.0133 + 0.0983} \end{aligned}$$

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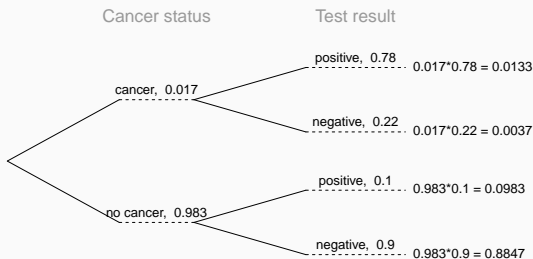
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Note: Tree diagrams are useful for inverting probabilities: we are given $P(+|C)$ and asked for $P(C|+)$.

Practice

Suppose a woman who gets tested once and obtains a positive result wants to get tested again. In the second test, what should we assume to be the probability of this specific woman having cancer?

- (a) 0.017
- (b) 0.12
- (c) 0.0133
- (d) 0.88

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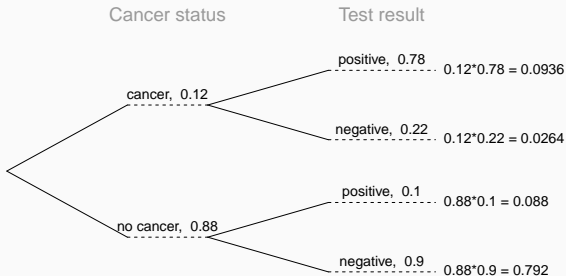
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What is the probability that this woman has cancer if this second mammogram also yielded a positive result?

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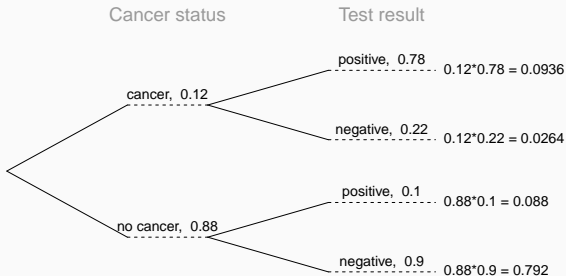
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$$P(C|+) = \frac{P(C \text{ and } +)}{P(+)} = \frac{0.0936}{0.0936 + 0.088} = 0.52$$

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- *Bayes' Theorem:*

$$P(\text{outcome } A_1 \text{ of variable 1} \mid \text{outcome } B \text{ of variable 2}) \\ = \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_k)P(A_k)}$$

where A_2, \dots, A_k represent all other possible outcomes of variable 1.

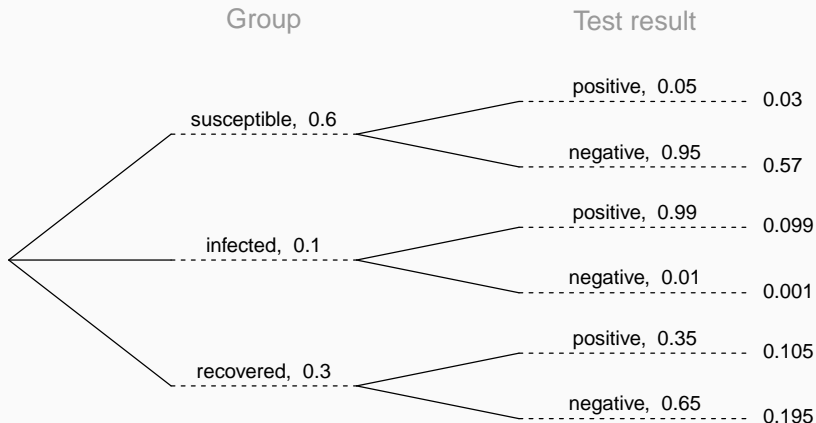
Application activity: Inverting probabilities

A common epidemiological model for the spread of diseases is the SIR model, where the population is partitioned into three groups: Susceptible, Infected, and Recovered. This is a reasonable model for diseases like chickenpox where a single infection usually provides immunity to subsequent infections. Sometimes these diseases can also be difficult to detect.

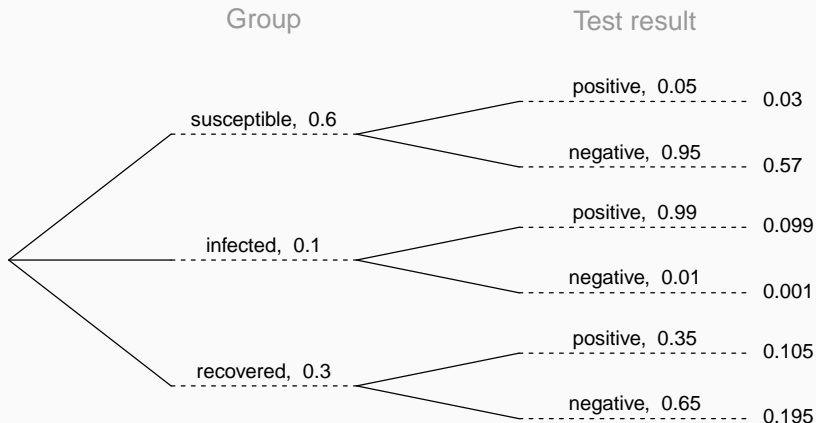
Imagine a population in the midst of an epidemic where 60% of the population is considered susceptible, 10% is infected, and 30% is recovered. The only test for the disease is accurate 95% of the time for susceptible individuals, 99% for infected individuals, but 65% for recovered individuals. (Note: In this case accurate means returning a negative result for susceptible and recovered individuals and a positive result for infected individuals).

Draw a probability tree to reflect the information given above. If the individual has tested positive, what is the probability that they are actually infected?

Application activity: Inverting probabilities (cont.)



Application activity: Inverting probabilities (cont.)



$$P(\text{inf}|+) = \frac{P(\text{inf and } +)}{P(+)} = \frac{0.099}{0.03 + 0.099 + 0.105} \approx 0.423$$