Misuse of the equals sign: An entrenched practice from early primary years to tertiary mathematics

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Introduction

In mathematics we frequently need to express equality between expressions. Robert Recorde, born in Wales in about 1510 (Figure 1), is credited with inventing the equals sign that we use today. Up until this time, equality was expressed in words. Recorde's first use of the equals sign was in 1557 in *The Whetstone of Witte* (Figure 2). Translated into modern English, Recorde's explanation of his equals sign reads: "And to avoid the tedious repetition of these words 'is equal to' I will set as I do often in work, use a pair of parallel lines of one length, thus: ═══, because no 2 things can be more equal" (https://archive.org/details/TheWhetstoneOfWitte).

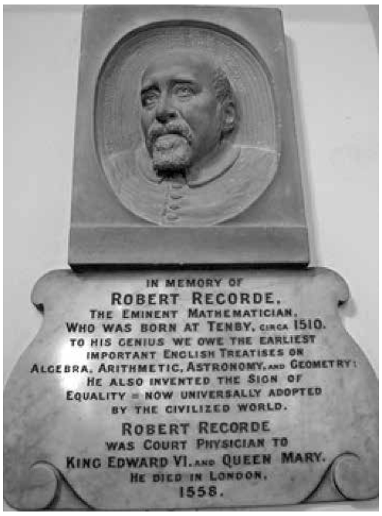

Figure 1. Memorial to Robert Recorde in St Mary's Church, Tenby, Wales.

And to a-
uoide the tediouſe repetition of theſe woordes : is e-
qualle to : I will ſette as I doe often in woorke uſe, a
paire of paralleles, or Gemowe lines of one lengthe,
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equalle. And now marke theſe nombers.

Figure 2. Robert Recorde's introduction of the equals sign in The Whetstone of Witte.

Symbolic literacy

Before looking at the origins of misuse of the equals sign, we consider briefly the broader aspect of symbolic literacy in mathematics. Symbolic literacy implies the notion of 'symbol sense' described by Arcavi (1994, 2005), which includes among other components the ability to manipulate, 'read through' symbolic expressions, realise that symbols can play different roles in different contexts and develop an intuitive feel for those differences. Skemp (1982) identified two levels of mathematical language, distinguishing between the surface structures (syntax) of mathematical symbol systems and the deep structures which embody the meaning of a mathematical communication—the mathematical ideas themselves, and their relationships. Serfati (2005) also provides a framework for thinking about mathematical symbols. Simplifying Serfati's framework, a symbol such as the equals sign can be considered in terms of three different aspects:

- its 'physical' attributes (what it looks like), including the category the symbol belongs to (for example, a letter, a numeral, a specific shape);
- the syntax, that is, the rules it must obey in the symbolic writing—this includes the number of operands for symbols standing for operators but also the 'legitimacy' of a symbol being juxtaposed to adjacent symbols;
- the meaning of the symbol as commonly agreed by the community of mathematicians.

To work with a mathematical symbol, then, one not only has to recognise it in the text, but must select the right meaning and appropriate syntax in that context, which sometimes has to be interpreted very locally (for example, the symbol "–" in front of a number, or between matrices). In a study involving first year university physics students, Torigoe and Gladding (2011) found that students' performance was highly correlated to their understanding of symbols. We anticipate that similar outcomes apply to other mathematical sciences at university, with the consequence that students may struggle with the mathematical content and be discouraged from continuing with mathematics and other tertiary subjects that involve advanced mathematics.

'Do something' operational view of the equals sign

Misuse of the equals sign by primary and junior secondary students, where "=" has taken on an operational meaning, has been the subject of much research and discussion over many decades (for example, Renwick, 1932). In school mathematics, students encounter equivalences such as $8 + 3 = 11$, $3x - 5 = 23 - x$. It seems, though, that instead of serving a relational role between two equivalent expressions, the equals sign has been misconstrued as a cue that an answer is required, that is, an operation must be performed. Behr, Erlwanger and Nichols (1980) described students' view of the equals sign as a 'do something signal'. Research suggests that this operational view of the equals sign stems from the early years where there is a strong focus on completing addition and subtraction calculations. When practising addition and subtraction in exercises such as $6 + 7 =$, and $8 – 3 =$, the equals sign is serving an operational role and students have learned to recognise that they must 'write the answer'. McNeil et al. (2006) note that it is not necessary for students to interpret the equals sign as a symbol of equivalence in order to correctly answer standard arithmetic 'operations equals answer' equations. Hence students see the equals sign as a signal to perform the operation preceding it. MacGregor and Stacey (1999) point out that "the language of arithmetic focuses on answers. The language of algebra focuses on relationships. For example, compare the arithmetic statement $287 + 146 = 433$ with a typical algebraic statement, $2(x + 1) = 2x + 2$. The arithmetic statement gives an answer, and the '=' sign indicates that this answer has been found" (p. 79). By contrast, the algebraic statement represents a relationship of equivalence.

Misinterpretation of the meaning of the equals sign is reinforced in several situations encountered by primary and junior secondary students. First, many calculators use the button labelled = for 'calculate the answer'. For example, entering 23×12 and pressing = gives the answer 276. (By contrast, some scientific calculators use the button labelled EXE (execute) for 'calculate the answer', retaining the equality meaning for =). Second, in a spreadsheet, entering, for example, the formula =A1*2+B1 in cell C1 is an instruction for the spreadsheet to perform a series of operations on the numbers in cells A1 and B1, so in this sense it is an operational use of the equals sign. However, the equals sign also has a relational role in expressing the equality relation C1=A1*2+B1, that is, the number in cell C1 is always equal to twice the number in cell A1 plus the number in cell B1: changing the values of the numbers in cells A1 and/or B1 leads to a new calculation of the value of the number in C1 so that the equality relationship is maintained. Third, in slogans such as "Effort = Success", the equals sign is a shorthand way of saying "leads to", mimicking the operational interpretation in mathematics.

Kieran (1981) notes that many students find difficulty with the equality (relational) interpretation of the equals sign:

> …the concept of equivalence is an elusive one not only for elementary school students but for high schoolers as well. That the equals sign is a 'do something signal' is a thread which seems to run through the interpretation of equality sentences throughout elementary school, high school, and even college. Early elementary school children, despite efforts to teach them otherwise, view the equals sign as a symbol which separates a problem and its answer. This thinking remains as children get older and advance to the upper elementary grades (p. 324).

Baroody and Ginsburg (1983) cite Collis (1974), who suggests that young children (6–10 years) require closure of an operation on two numbers, so, for example, $4 + 5 = ?$ is meaningful only when the child sees 9 written on the right-hand side of the equation.

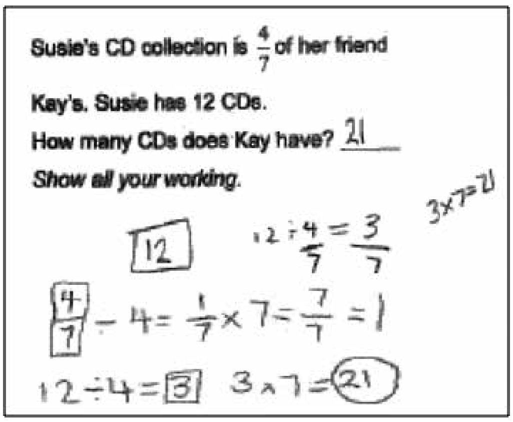
While it is acceptable for the equals sign to have an operational meaning, a consequence of students using it to mean 'then I did this' is that they write strings of false equalities, with each equals sign representing a step in a multi-step calculation. For example, when asked to perform calculations such as $5 \times (13 + 27)$, many students write $13 + 27 = 40 \times 5 = 200$, as they would perform the calculation on their calculator. Students need to be trained to set out their working for multi-step problems in a logical sequence of number sentences where the equals sign shows that the number expressions on each side are equal. Kieran (1992, p. 393) notes the methods of Year 6 students when solving the following problem:

> Daniel went to visit his grandmother, who gave him $1.50. Then he bought a book costing $3.20. If he has $2.30 left, how much money did he have before visiting his grandmother?

The students commonly wrote "2.30 + 3.20 = 5.50 – 1.50 = 4" rather than writing "2.30 + 3.20 = 5.50", followed by "5.50 – 1.50 = 4". The students are viewing the equals sign as a 'gives' sign, ignoring the symmetry property associated with the equals sign.

Figure 3 shows the written responses of two Year 6 students who were solving fraction problems (observed by one of the authors of this article). Both students obtained the correct answer and their calculations show that they understood the solution process. However, they have both used the '=' sign in an operational way: "I got this then I did this".

Baroody and Ginsburg (1983) report that young students have difficulty with non-standard number sentences such as $13 = 7 + 6$, $6 + 4 = 3 + 7$, and $8 = 8$. Falkner, Levi and Carpenter (1999) found that when students from grades 1 to 6 encountered different forms of number sentences, they tried to adapt their belief that an operation was on the left and the answer on the right, rather



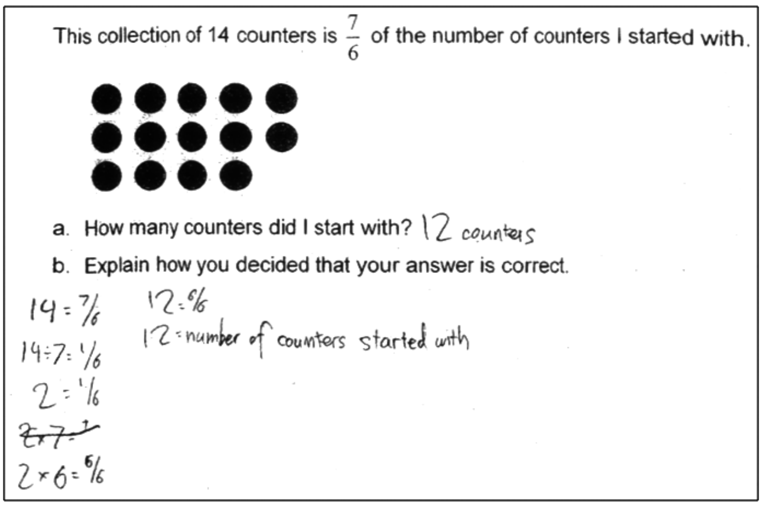


Figure 3. Written calculations of two Year 6 students showing an operational use of the equals sign.

than looking for a relation. When asked what number should be written in the box to make the number sentence $8 + 4 = \square + 5$ true, no more than 10% of students in any grade gave the correct answer, 7, and performance did not improve with age. Follow-up discussion of true and false number sentences showed that students readily accepted that $4 + 5 = 9$ was true and $12 - 5 = 9$ was false. However, when number sentences were expressed in non-standard forms, many students were confused. Instead of always using the standard 'operations equals answer' equation context, for example, $3 + 4 = 7$, it is important to use non-standard contexts such as $3 + 4 = 5 + 2$ to highlight the equality relationship between the quantities on each side of an equation.

Capraro et al. (2007) investigated the approach to the equals sign in Chinese textbooks, noting that multiple representations for equality in textbooks and teacher books assisted students to correctly interpret the equals sign:

> …all of the textbooks introduce the equals sign in conjunction with '>' and '<' before introducing the concepts of addition and subtraction. Correspondingly, teachers are encouraged to teach the equals sign within various comparison contexts. Thus, Chinese students encounter the concept of the equals sign as a relational symbol from the very beginning (p. 87).

In algebra, students need both the operational and relational meanings of the equals sign. When substituting numbers for pronumerals in an algebraic expression, students are able to rely on their operational understanding to evaluate the expression. Similarly, it is possible for students to solve algebraic equations such as $2x + 3 = 11$ without recourse to the relational meaning, for example, by a guess and check approach. However, if they fail to understand that the expressions on each side of an equation are equal, then they have difficulty, for example, understanding why we can subtract x from both sides when solving the equation $2x + 3 = x + 11$. Researchers have found that students who understand that the equals sign is a relational symbol of equality are more successful in solving algebraic equations. Carpenter, Franke and Levi (2003, p. 22) contend that a "limited conception of what the equal sign means is one of the major stumbling blocks in learning algebra".

First year undergraduate mathematics students' misuse of the equals sign

Our observations of the written work of first year university mathematics students show that inappropriate use of the equals sign is not confined to school mathematics, supporting the findings of Godfrey and Thomas (2008). We look now at a sample of the written solutions of first semester undergraduate students enrolled in Calculus 1 in a major Australian university. During their weekly tutorials the students completed worksheet exercises and problems based on their current lecture topics. It was the normal practice in these tutorials for students to write their solutions on whiteboards. The tutor moved around the tutorial room, checking students' progress, pointing out errors in the students' solutions and suggesting appropriate methods when students were unsure how to proceed. We take solutions to the question shown in Figure 5, based on the students' recent lectures on complex numbers, as an example.

Find an argument θ, $-\pi < \theta \le \pi$, for the following complex numbers
(a) $-5)1 - i)$ (b) $-1-\sqrt{3}i$ (c) $\dfrac{-2+2i}{-1-\sqrt{3}i}$

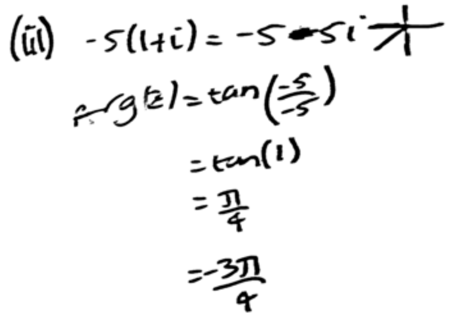
Figure 4. Complex numbers question from tutorial exercise worksheet.

Considering a generic complex number, $a + bi$, the appropriate symbolic form for the argument θ is

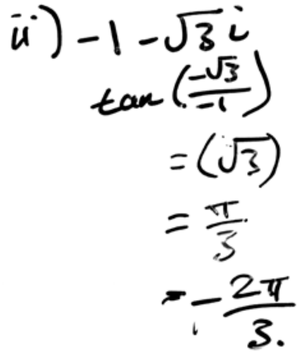
$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \text{ or } \theta = \arctan\left(\frac{b}{a}\right)$$

taking into account, of course, the signs of a and b to determine the appropriate angle. The students whose solutions are shown in Figures 5a, 5b and 5c have each obtained the correct values for the arguments but all three demonstrate inappropriate use of the equals sign: 'and then I did this'. In Figure 5c we see that the student has used a further, but legitimate, meaning of the equals sign to assign names to the complex numbers in the numerator and denominator. However, instead of including two additional lines for this assigning of names ($-2 + 2i = w$ and $-1 - 3i = x$), the student has merely inserted "$= w$" and "$= x$" beside the numerator and denominator respectively.

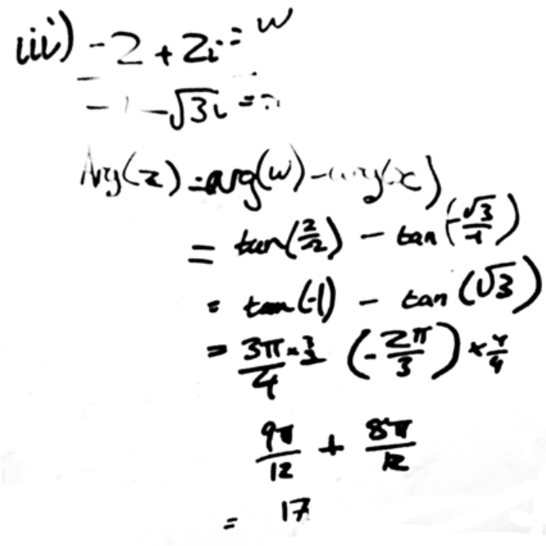
In addition to a consistent misuse of the equals sign, the students are also confused between the tangent and the inverse operation, using "tan" instead of "$\tan^{-1}$" or "arctan". The responses in Figure 5 suggest that the students do not consider the syntax of expressions, not recognising that 'tan' prompts for its argument to be an angle. It would seem that students should be encouraged to verbalise their symbolic expressions, stating orally that the argument is equal to 'the angle whose tangent is'. The students' syntax, if read aloud, does not make sense. They seem to be working out the answer without expecting that the symbols they are writing convey a meaning to the reader. Their responses suggest they are using '=' to say: "and then 'I did something', and the result is…". These meanings of the '=' sign are deeply set in students' thinking. The work shown in Figures 5a, 5b and 5c suggests that students have thought about the meaning of the symbols, indicating the size and position of the angle, locating the complex number on the Argand plane, but have taken this into consideration only once they had finished their calculations. It is clear that the students' focus is on the new aspects of working with complex numbers.



$$-5(1+i)=-5-5i$$
$$\begin{aligned}\operatorname{Arg}(z)&=\tan\left(\frac{-5}{-5}\right)\\&=\tan(1)\\&=\frac{\pi}{4}\\&=\frac{-3\pi}{4}\end{aligned}$$



$$-1-\sqrt{3}i$$
$$\tan\left(\frac{-\sqrt{3}}{-1}\right)$$
$$=\sqrt{3}$$
$$=\frac{\pi}{3}$$
$$=-\frac{2\pi}{3}$$



$$\begin{aligned}-2+2i&=w\\-1-\sqrt{3}i&=x\end{aligned}$$
$$\begin{aligned}\arg(z)&=\arg(w)-\arg(x)\\&=\tan\left(\frac{2}{-2}\right)-\tan\left(\frac{-\sqrt{3}}{-1}\right)\\&=\tan(-1)-\tan\left(\sqrt{3}\right)\\&=\frac{3\pi\times\frac{3}{3}}{4}-\left(-\frac{2\pi}{3}\right)\times\frac{4}{4}\\&\frac{9\pi}{12}+\frac{8\pi}{12}\\&=\frac{17\pi}{12}\end{aligned}$$

Figure 5. Inappropriate use of the equals sign.

Implications for teaching

Research (for example, Baroody & Ginsburg, 1983; Stacey & MacGregor, 1997) has shown that appropriate instruction enables students to develop an understanding of the equivalence relationship when two expressions are linked by the equals sign. In the early years where students become familiar with 'operation equals answer', for example, $4 + 3 = 7$, they should be exposed also to non-standard equations such as $7 = 4 + 3$, $4 + 2 = 3 \times 2$.

Usiskin (2012, p. 4) asserts that "mathematics is both a written language and a spoken language, for—particularly in school mathematics—we have words for virtually all the symbols. Familiarity with this language is a precursor to all understanding". It is the spoken aspect of mathematics that needs to receive greater emphasis. Teaching students to read aloud symbolic mathematical statements is an important part of developing their symbolic, and hence, mathematical literacy. When students write $2 + 5 = 7 - 3 = 4$, they are using a shortcut way of saying '2 + 5 makes 7 and then take away 3 makes 4' but by using the equals sign in $2 + 5 = 7 - 3 = 4$ they are in fact writing a nonsense statement: $2 + 5 = 4$. They should be encouraged to recognise that their written calculations must make logical sense:

$$2 + 5 = 7$$
$$7 - 3 = 4$$

The *Senior Secondary Curriculum: Mathematics* (ACARA, 2014) in its *Representation of General capabilities* emphasises the importance of communication:

> In the senior years these literacy skills and strategies enable students to express, interpret, and communicate complex mathematical information, ideas and processes. Mathematics provides a specific and rich context for students to develop their ability to read, write, visualise and talk about complex situations involving a range of mathematical ideas. Students can apply and further develop their literacy skills and strategies by shifting between verbal, graphic, numerical and symbolic forms of representing problems in order to formulate, understand and solve problems and communicate results. This process of translation across different systems of representation is essential for complex mathematical reasoning and expression. Students learn to communicate their findings in different ways, using multiple systems of representation and data displays to illustrate the relationships they have observed or constructed.

The illustrations in Figures 3 and 5 highlight what happens when students do not expect mathematics to be read with logical meaning. The notion of expecting symbols to have meaning and a habit of checking the meaning of the symbols used is an aspect of working mathematically that needs to be cultivated at all levels: primary, secondary and tertiary. In particular, ensuring that students understand the relational role of the equals sign is an important step in developing symbol sense.

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