1. **Problem:**
   Your boss wants to know what proportion of a very large population is special. You already know the proportion approximately 0.36. But, your boss wants to guarantee that the margin of error of a 95% confidence interval will be less than 0.004 (which is 0.4 percentage points). How large of a sample is needed?

Solution: We are given the confidence level and the margin of error.

$$\gamma = 0.95$$
$$ME = 0.004$$

Determine $z^\star$ such that $P(|Z| < z^\star) = 0.95$. It is helpful to get the percentile of $z^\star$ by using $\ell = \frac{1+\gamma}{2} = \frac{1+0.95}{2} = 0.975$. This lets you find $z^\star$ such that $P(Z < z^\star) = 0.975$.

$$z^\star = 1.96$$

Use the appropriate formula.

$$n = \hat{p}(1-\hat{p})\left(\frac{z^\star}{ME}\right)^2$$

$$= (0.36)(0.64)\left(\frac{1.96}{0.004}\right)^2$$

$$= 55319.04$$

When determining a necessary sample size, always round up (ceiling).

$$n = 55320$$

2. **Problem:**
A fair 8-sided die has a discrete uniform distribution with an expected value of $\mu = 4.5$ and a standard deviation $\sigma = 2.29$.

You are told to check if a 8-sided die has an expected value different than 4.5. You are told to roll the die 151 times and do a significance test with a significance level of 0.01.

You then roll the die 151 times and get a mean of 4.011. Should we conclude the die is unfair?

(a) What kind of hypothesis test is appropriate?

(b) State the hypotheses

(c) Determine the p-value.

(d) Decide whether we reject or retain the null hypothesis.

(e) Do we conclude the die is unfair?

Solution: We should use a two-tail test of population mean.

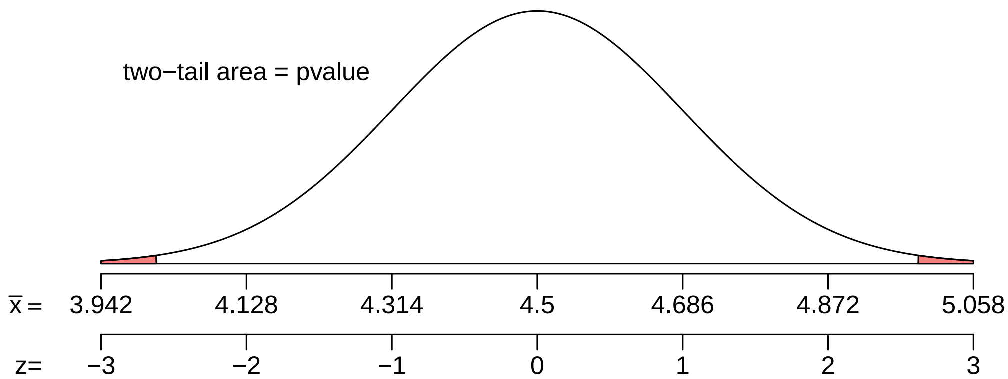
State the hypotheses:

$$H_0 \text{ claims } \mu = 4.5$$
$$H_A \text{ claims } \mu \neq 4.5$$

Find the standard error.

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{2.29}{\sqrt{151}} = 0.186$$

Make a sketch of the null's sampling distribution.



Find the z score.

$$z = \frac{\bar{x} - \mu_0}{\sigma_{\bar{x}}} = \frac{4.011 - 4.5}{0.186} = -2.62$$

Find the p-value (using formula for left-tail test of mean).

$$\begin{aligned} p\text{-value} &= P(|Z| > 2.62) \\ &= 2 \cdot P(Z < -2.62) \\ &= \boxed{0.0088} \end{aligned}$$

Compare the p-value and the significance level ($\alpha = 0.01$).

$$p\text{-value} < \alpha$$

Yes, we reject the null hypothesis.

We conclude the die is unfair.

(a) Right-tail single mean test
(b) Hypotheses: H_0 claims $\mu = 4.5$ and H_A claims $\mu \neq 4.5$.
(c) p-value = 0.0088
(d) Yes, we reject the null hypothesis.
(e) We conclude the die is unfair.

3. **Problem:**
   A traffic engineer wants to determine the average speed of cars on the Longfellow Bridge. She knows the population standard deviation of speeds is 18 mph. To determine a 86% confidence interval with a margin of error of 1 mph, what sample size is needed?

Solution: We are given the population standard deviation, confidence level, and margin of error.

$$\sigma = 18$$
$$\gamma = 0.86$$
$$ME = 1$$

Determine the critical z value, $z^\star$, such that $P(|Z| < z^\star) = 0.86$. Remember, $\ell = \frac{\gamma+1}{2} = \frac{0.86+1}{2} = 0.93$

$$z^\star = 1.48$$

Use the formula for sample size.

$$n = \left(\frac{z^\star \sigma}{ME}\right)^2$$

$$= \left(\frac{(1.48)(18)}{1}\right)^2$$

$$= 709.6896$$

Round up.

$$n = 710$$

4. **Problem:**
   Brahim wants to estimate the average mass of the beans in a large bag. Somehow, Brahim is certain that the standard deviation of the beans in the bag is 58 milligrams. He takes a sample of size 132 and finds the sample mean to be 491 milligrams. What would be the 98% confidence interval?

Solution: We are given the sample size, sample mean, population standard deviation, and confidence level.

$$n = 132$$
$$\bar{x} = 491$$
$$\sigma = 58$$
$$\gamma = 0.98$$

Determine the critical z value, $z^{\star}$, such that $P(|Z| < z^{\star}) = 0.98$. Remember, $\ell = \frac{\gamma+1}{2} = \frac{0.98+1}{2} = 0.99$

$$z^{\star} = 2.33$$

Use the formula for bounds (mean, σ known).

$$\begin{aligned} LB &= \bar{x} - z^{\star}\frac{\sigma}{\sqrt{n}} & \qquad UB &= \bar{x} + z^{\star}\frac{\sigma}{\sqrt{n}} \\ &= 491 - 2.33 \times \frac{58}{\sqrt{132}} & &= 491 + 2.33 \times \frac{58}{\sqrt{132}} \\ &= 479.24 & &= 502.76 \end{aligned}$$

We are 98% confident that the population mean is between 479.24 and 502.76 milligrams.

$$CI = (479.24,\ 502.76)$$

5. **Problem:**
   A student is taking a multiple choice test with 700 questions. Each question has 5 choices. You want to detect whether the student does better than random guessing, so you decide to run a hypothesis test with a significance level of 0.02.

   Then, the student takes the test and gets 159 questions correct.

   (a) What kind of hypothesis test is appropriate?
   (b) State the hypotheses
   (c) Determine the p-value.
   (d) Decide whether we reject or retain the null hypothesis.
   (e) Do we think the student did significantly better than random guessing?

Solution: This is a right-tail (one-tail) proportion test because we only care whether the student does better than random.

Determine the null population proportion.

$$p_0 = \frac{1}{5} = 0.2$$

State the hypotheses.

$$H_0 \quad \text{claims} \quad p = 0.2$$
$$H_A \quad \text{claims} \quad p > 0.2$$

Determine the standard error.

$$\sigma_{\hat{p}} = \sqrt{\frac{p_0(1-p_0)}{n}} = \sqrt{\frac{0.2(1-0.2)}{700}} = 0.0151$$

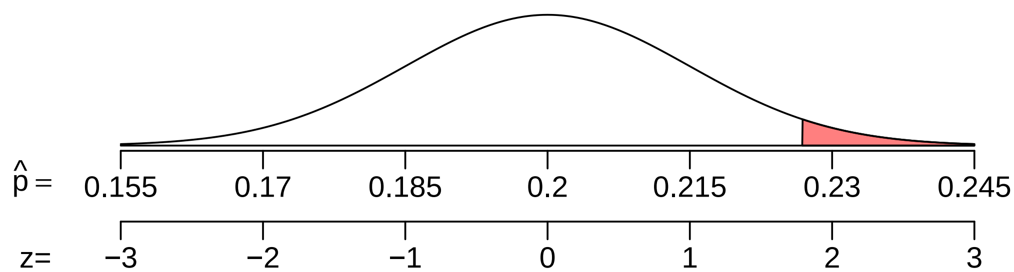
Determine the sample proportion.

$$\hat{p} = \frac{159}{700} = 0.227$$

Determine a z score. For simplicity, we ignore the continuity correction.

$$z = \frac{\hat{p} - p_0}{\sigma_{\hat{p}}} = \frac{0.227 - 0.2}{0.0151} = 1.79$$

Make a sketch of the null's sampling distribution. The p-value is a right area.



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To determine that right area, we use the z table.

$$\begin{aligned} p\text{-value} &= P(\hat{p} > 0.227) \\ &= P(Z > 1.79) \\ &= 1 - P(Z < 1.79) \\ &= 0.0367 \end{aligned}$$

Compare p-value to α (which is 0.02).

$$p\text{-value} > \alpha$$

Make the conclusion: we retain the null hypothesis.

We think the student might just be guessing.

(a) Right tail (one-tail) proportion test
(b) Hypotheses: H_0 claims $p = 0.2$ and H_A claims $p > 0.2$.
(c) The p-value is 0.0367
(d) We retain the null hypothesis.
(e) We think the student might just be guessing.

6. **Problem:**
A null hypothesis claims a population has a mean $\mu = 100$ and a standard deviation $\sigma = 22$. You decide to run one-tail test on a sample of size $n = 286$ using a significance level $\alpha = 0.1$ to detect if the actual population mean is more than 100. You then collect the sample and find it has mean $\bar{x} = 101.27$.

   (a) What kind of hypothesis test is appropriate?
   (b) State the hypotheses
   (c) Determine the p-value.
   (d) Decide whether we reject or retain the null hypothesis.

Solution: We should use a right-tail test of population mean.

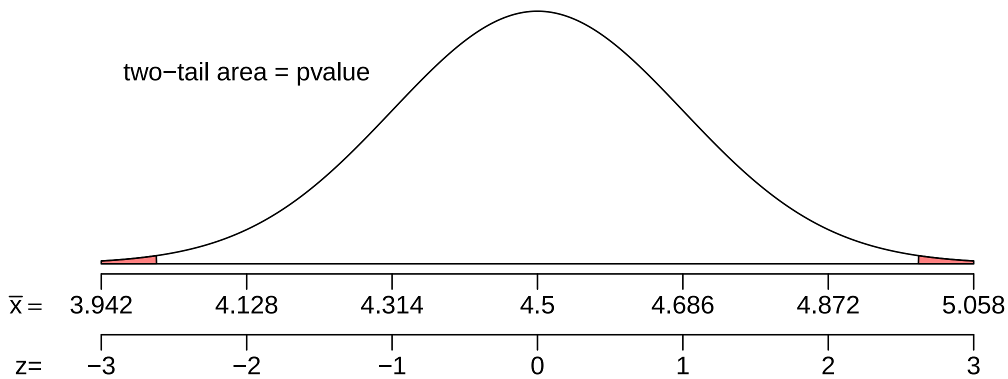
State the hypotheses:

$$H_0 \text{ claims } \mu = 100$$
$$H_A \text{ claims } \mu > 100$$

Find the standard error.

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{22}{\sqrt{286}} = 1.301$$

Make a sketch of the null's sampling distribution.



Find the z score.

$$z = \frac{\bar{x} - \mu_0}{\sigma_{\bar{x}}} = \frac{101.27 - 100}{1.301} = 0.98$$

Find the p-value (using formula for left-tail test of mean).

$$\begin{aligned} p\text{-value} &= P(Z > 0.98) \\ &= 1 - P(Z < 0.98) \\ &= 1 - 0.8365 \\ &= \boxed{0.1635} \end{aligned}$$

Compare the p-value and the significance level ($\alpha = 0.1$).

$$p\text{-value} > \alpha$$

No, we do not reject the null hypothesis.

(a) Right-tail single mean test
(b) Hypotheses: H_0 claims $\mu = 100$ and H_A claims $\mu < 100$.
(c) p-value = 0.1635
(d) No, we do not reject the null hypothesis.

7. **Problem:**
   A new virus has been devastating corn production. When exposed, 44.7% of common seedlings die within a week. We are trying to develop a resistant strain of corn.

   When we expose 700 seedlings of our strain to the virus, 41.1% die within a week. Using a significance level of 0.05, can we conclude that our strain is significantly more resistant?

   (a) What kind of hypothesis test is appropriate?
   (b) State the hypotheses
   (c) Determine the p-value.
   (d) Decide whether we reject or retain the null hypothesis.
   (e) Do we think our strain is significantly more resistant?

Solution: This is a left-tail (one-tail) proportion test because we only care whether a lower percentage of seedlings will die.

State the hypotheses.

$$H_0 \quad \text{claims} \quad p = 0.447$$
$$H_A \quad \text{claims} \quad p < 0.447$$

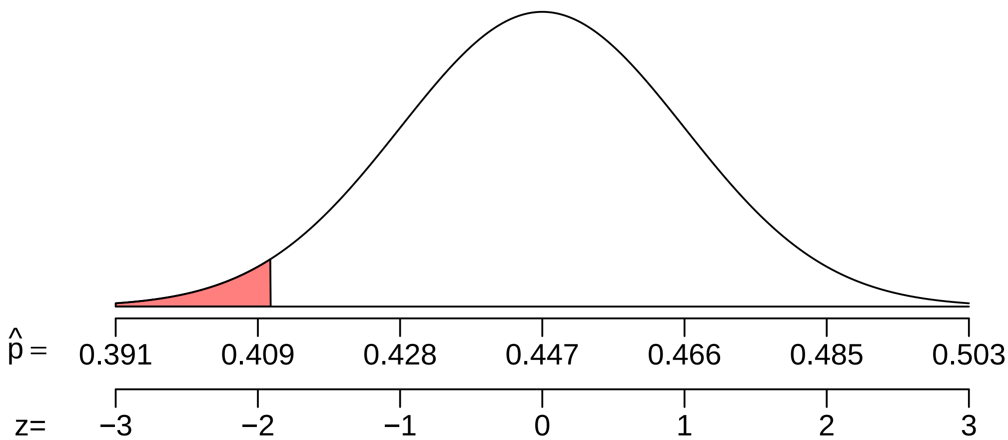
Determine the standard error.

$$\sigma_{\hat{p}} = \sqrt{\frac{p_0(1-p_0)}{n}} = \sqrt{\frac{0.447(1-0.447)}{700}} = 0.0188$$

Determine a z score. For simplicity, we ignore the continuity correction.

$$z = \frac{\hat{p} - p_0}{\sigma_{\hat{p}}} = \frac{0.411 - 0.447}{0.0188} = -1.91$$

Make a sketch of the null's sampling distribution. The p-value is a left area.



To determine that left area, we use the z table.

$$\begin{aligned} p\text{-value} &= P(\hat{p} < 0.411) \\ &= P(Z < -1.91) \\ &= 0.0281 \end{aligned}$$

Compare p-value to α (which is 0.05).

$$p\text{-value} < \alpha$$

Make the conclusion: we reject the null hypothesis.

We think our strain is more resistant than common corn.

(a) Left-tail (one-tail) proportion test
(b) Hypotheses: H_0 claims $p = 0.447$ and H_A claims $p < 0.447$.
(c) The p-value is 0.0281
(d) We reject the null hypothesis.
(e) We think our strain is more resistant than common corn.

8. **Problem:**
   Some snails have clockwise shells, and others have counter-clockwise shells. You want to construct a 80% confidence interval of the population proportion with clockwise shells. You hope the margin of error will be about 0.01. How many snails do you need?

Solution: We are given the confidence level and the margin of error.

$$\gamma = 0.8$$
$$ME = 0.01$$

Determine $z^\star$ such that $P(|Z| < z^\star) = 0.8$. It is helpful to get the percentile of $z^\star$ by using $\ell = \frac{1+\gamma}{2} = \frac{1+0.8}{2} = 0.9$. This lets you find $z^\star$ such that $P(Z < z^\star) = 0.9$.

$$z^\star = 1.28$$

Use the appropriate formula.

$$n = \frac{1}{4}\left(\frac{z^\star}{ME}\right)^2$$

$$= \frac{1}{4}\left(\frac{1.28}{0.01}\right)^2$$

$$= 4096$$

When determining a necessary sample size, always round up (ceiling).

$$n = 4096$$

9. **Problem:**
As an ornithologist, you wish to determine the average body mass of *Catharus fuscescens*. You randomly sample 16 adults of *Catharus fuscescens*, resulting in a sample mean of 41.52 grams and a sample standard deviation of 3.32 grams. Determine a 98% confidence interval of the true population mean.

Solution: We are given the sample size, sample mean, sample standard deviation, and confidence level.

$$\begin{aligned} n &= 16 \\ \bar{x} &= 41.52 \\ s &= 3.32 \\ \gamma &= 0.98 \end{aligned}$$

Find the degrees of freedom.

$$\begin{aligned} df &= n - 1 \\ &= 16 - 1 \\ &= 15 \end{aligned}$$

Determine the critical t value, $t^\star$, such that $P(|T| < t^\star) = 0.98$ and $df = 15$.

$$t^\star = 2.6$$

Use the formula for bounds (mean, σ unknown).

$$\begin{aligned} LB &= \bar{x} - t^\star \frac{s}{\sqrt{n}} \\ &= 41.52 - 2.6 \times \frac{3.32}{\sqrt{16}} \\ &= 39.4 \end{aligned} \qquad \begin{aligned} UB &= \bar{x} + t^\star \frac{s}{\sqrt{n}} \\ &= 41.52 + 2.6 \times \frac{3.32}{\sqrt{16}} \\ &= 43.7 \end{aligned}$$

We are 98% confident that the population mean is between 39.4 and 43.7.

$$CI = (39.4,\ 43.7)$$

10. **Problem:**
A null hypothesis claims a population has a mean $\mu = 120$. You decide to run two-tail test on a sample of size $n = 12$ using a significance level $\alpha = 0.1$.

You then collect the sample:

180.4	152.7	128.5	163.7	72
143.7	101.2	144.5	117.8	142.7
167.9	120.3			

(a) Determine the p-value.

(b) Do you reject the null hypothesis?

Solution: State the hypotheses.

$$H_0 \text{ claims } \mu = 120$$
$$H_A \text{ claims } \mu \neq 120$$

Find the mean and standard deviation of the sample.

$$\bar{x} = 136.283$$

$$s = 30.379$$

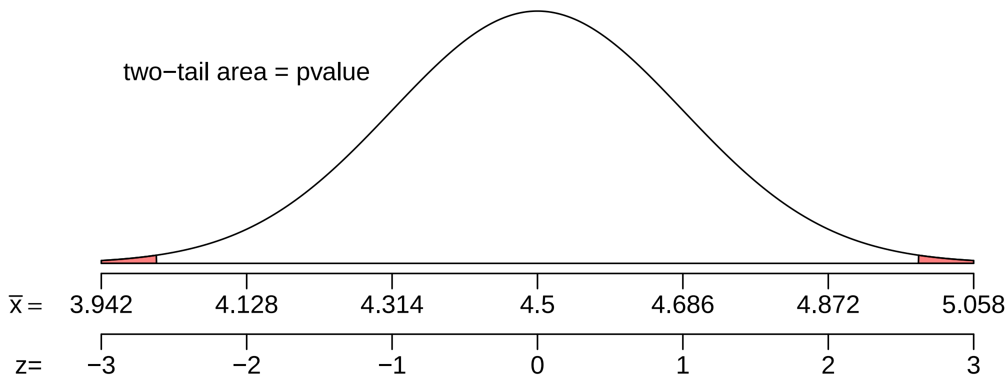
Determine the degrees of freedom.

$$df = 12 - 1 = 11$$

Find the standard error.

$$\sigma_{\bar{x}} = \frac{s}{\sqrt{n}} = \frac{30.379}{\sqrt{12}} = 8.77$$

Make a sketch of the null's sampling distribution.



Find the t score.

$$t = \frac{\bar{x} - \mu_0}{\sigma_{\bar{x}}} = \frac{136.283 - 120}{8.77} = 1.86$$

Find the p-value.

$$p\text{-value} = P(|T| > 1.86)$$

We can't get an exact value with our table, but we can determine an interval that contains the p-value. (Look at row with $df = 11$.)

$$P(|T| > 2.2) = 0.05$$

$$P(|T| > 1.8) = 0.1$$

Basically, because t is between 2.2 and 1.8, we know the p-value is between 0.05 and 0.1.

$$0.05 < p\text{-value} < 0.1$$

Compare the p-value and the significance level ($\alpha = 0.1$).

$$p\text{-value} < \alpha$$

Yes, we reject the null hypothesis.

(a) $0.05 < p\text{-value} < 0.1$

(b) Yes, we reject the null hypothesis.

11. **Problem:**
A random sample of size 1900 was found to have a sample proportion of 80% (because there were 1520 successes). Determine a 92% confidence interval of the population proportion.

(a) Find the lower bound of the confidence interval.

(b) Find the upper bound of the condifence interval.

Solution: Identify the givens.

$$n = 1900$$
$$\hat{p} = 0.8$$
$$\gamma = 0.92$$

Determine $z^\star$ such that $P(|Z| < z^\star) = 0.92$. It is helpful to know that $\ell = \frac{\gamma+1}{2} = \frac{0.92+1}{2} = 0.96$. (Use the z-table to find $z^\star$.)

$$z^\star = 1.75$$

Use the formula (proportion) for the bounds.

$$LB = \hat{p} - z^\star\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \qquad UB = \hat{p} + z^\star\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$= 0.8 - 1.75\sqrt{\frac{(0.8)(0.2)}{1900}} \qquad = 0.8 + 1.75\sqrt{\frac{(0.8)(0.2)}{1900}}$$

$$= 0.784 \qquad = 0.816$$

Determine the interval.

$$CI = (0.784\,,\ 0.816)$$

We are 92% confident that the true population proportion is between 78.4% and 81.6%.

(a) The lower bound = 0.784, which can also be expressed as 78.4%.

(b) The upper bound = 0.816, which can also be expressed as 81.6%.

12. **Problem:**
A fair coin should have a 50% chance of landing on either side. Someone is mildly suspicious that a coin is unfair.

You are asked to judge the fairness of the coin by flipping it 6000 times and counting how many heads are flipped. You are told to use a significance level of 0.05.

Then, you actually flip the coin 6000 times and get 2934 heads. Should we conclude this coin is unfair?

(a) What kind of hypothesis test is appropriate?
(b) State the hypotheses
(c) Determine the *p*-value.
(d) Decide whether we reject or retain the null hypothesis.
(e) Do you think the coin is unfair?

Solution: We should use a two-tail proportion test.

State the hypotheses.

$$H_0 \text{ claims } p = 0.5$$
$$H_A \text{ claims } p \neq 0.5$$

Determine the standard error.

$$\sigma_{\hat{p}} = \sqrt{\frac{p_0(1-p_0)}{n}} = \sqrt{\frac{0.5(1-0.5)}{6000}} = 0.00645$$

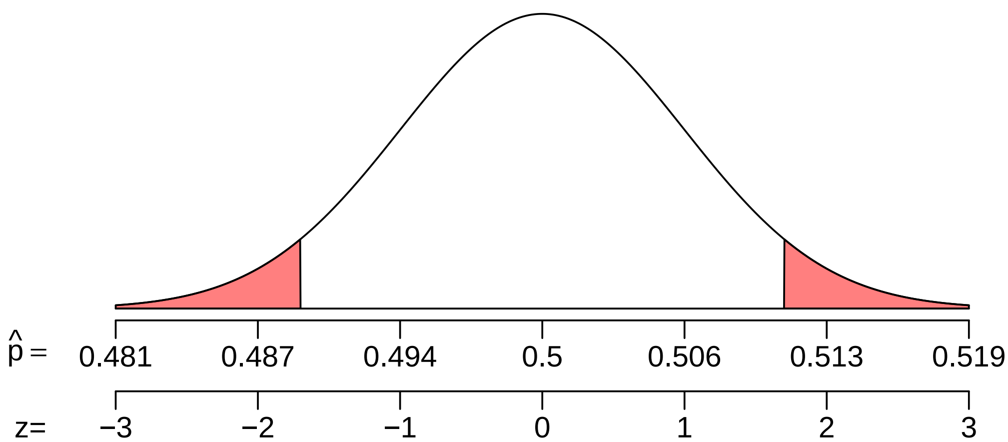
Determine the sample proportion.

$$\hat{p} = 0.489$$

Determine a z score. For simplicity, we ignore the continuity correction.

$$z = \frac{\hat{p} - p_0}{\sigma_{\hat{p}}} = \frac{0.489 - 0.5}{0.00645} = -1.7$$

Make a sketch of the null's sampling distribution. The p-value is a two-tail area.



To determine that two-tail area, we use the z table.

$$\begin{aligned} p\text{-value} &= P(|Z| > 1.7) \\ &= 2 \cdot P(Z < -1.7) \\ &= 0.0892 \end{aligned}$$

Compare p-value to α (which is 0.05).

$$p\text{-value} > \alpha$$

Make the conclusion: we don't reject the null hypothesis.

We conclude the coin could be fair.

(a) Two-tail proportion test
(b) Hypotheses: H_0 claims $p = 0.5$ and H_A claims $p \neq 0.5$.
(c) The p-value is 0.0892
(d) We don't reject the null hypothesis.
(e) We conclude the coin could be fair.