Confidence Interval of Proportion

How to determine a confidence interval of a proportion

Definitions

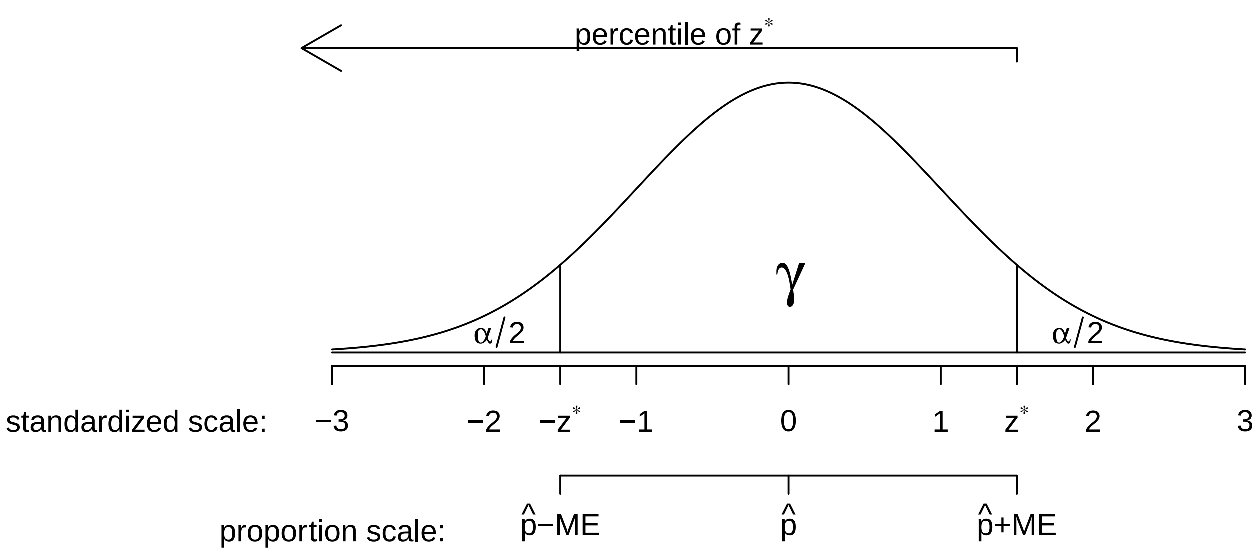
$$
\begin{aligned}
p &= \text{population proportion (to be estimated)}\\
\hat{p} &= \text{sample proportion (actually measured)}\\
n &= \text{sample size}\\
\gamma &= \text{confidence level} = \text{chance that a confidence interval will include } p\\
\alpha &= \text{error rate} = 1-\gamma\\
\sigma_{\hat{p}} &= \text{standard error (standard deviation of sampling distribution)}\\
z^\star &= \text{standardized radius of interval}\\
ME &= \text{margin of error (radius of interval)} = z^\star \cdot \sigma_{\hat{p}}\\
\ell &= \text{percentile associated with } z^\star\\
LB &= \text{lower bound of confidence interval} = \hat{p} - ME\\
UB &= \text{upper bound of confidence interval} = \hat{p} + ME\\
CI &= \text{confidence interval} = [LB,\ UB]
\end{aligned}
$$

General Problem

- **Given:** $\hat{p}$, n, and γ
- **Find:** The lower and upper bounds of the confidence interval.

General Procedure

We first determine $z^\star$ such that $P(|Z| < z^\star) = \gamma$. The graphic below suggests the strategy: determine α, find ℓ, and use the z-table to get $z^\star$.



Determine error rate. This represents how often confidence intervals will miss the true population proportion. This error rate is a two-tail area.

$$\alpha = 1 - \gamma$$

We can determine the percentile (ℓ) of $z^\star$.

$$\ell = 1 - \frac{\alpha}{2}$$

It should be mentioned that you could have gotten the percentile more directly.

$$\ell = \frac{\gamma + 1}{2}$$

Use the z-table to get z^*.

We estimate the standard error. (Technically we should use p, not $\hat{p}$, but we only know $\hat{p}$. We assume $\hat{p}$ is close enough to p for this estimation to be accurate.)

$$\sigma_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Determine the margin of error.

$$ME = z^\star \cdot \sigma_{\hat{p}}$$

Get the lower bound.

$$LB = \hat{p} - ME$$

Get the upper bound.

$$UB = \hat{p} + ME$$

Write the confidence interval in interval notation.

$$\boxed{CI = [LB, UB]}$$

We can summarize the procedure in two steps:

1. Determine $z^\star$ such that $P(|Z| < z^\star) = \gamma$. It is helpful to know $\ell = \frac{\gamma+1}{2}$.
2. Use the following expression to find the bounds:

$$\hat{p} \pm z^\star \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$