Instructions

This test is to be taken as an individual without outside assistance or notes. If you are suspected of cheating, you will fail this exam and your transgression will be reported.

You can either use a scientific calculator or (with a smartphone) GeoGebra Scientific Calculator.

To use GeoGebra, you must first put your smartphone on **Airplane Mode**. Then, in GeoGebra, turn on **Exam Mode**. You must leave exam mode on for the entire exam. You won't be able to use your smartphone for anything else. After you are done, you will show me how long exam mode has been running, and if that time is not how long you've been sitting, you will fail this exam.

Read each question carefully and show your work. If the answer is wrong, partial credit is awarded for coherent work.

Grade Table

(do not write in this table)

question	available points	earned points
1	20	
2	20	
3	10	
4	10	
5	10	
6	10	
7	20	
EC1	10	
Total	100	

1. **Problem**
A farm produces 4 types of fruit: *A*, *B*, *C*, and *D*. The fruits' masses follow normal distributions, with parameters dependent on the type of fruit.

Type of fruit	Mean mass (g)	Standard deviation of mass (g)
A	133	6
B	102	10
C	113	13
D	135	9

One specimen of each type is weighed. The results are shown below.

Type of fruit	Mass of specimen (g)
A	127.8
B	114.6
C	122.6
D	148.4

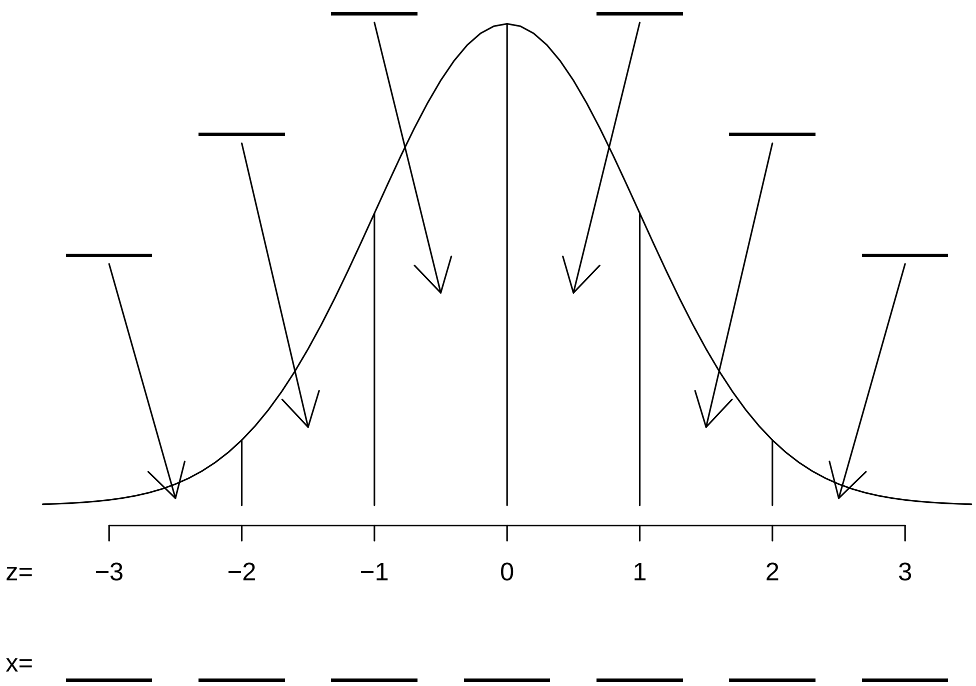
(a) Calculate a z score for each specimen.

(b) Which specimen is the most unusually large (relative to others of its type)?

(c) Which specimen is the most unusually small (relative to others of its type)?

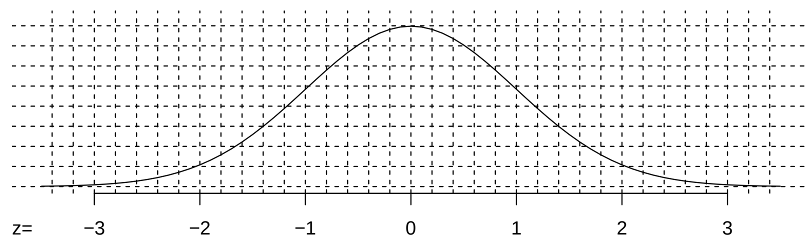
2. **Problem**
A normal random variable X has a mean $\mu = 44.2$ and standard deviation $\sigma = 1.1$. Please label the density curve with:

(a) The appropriate values of x.
(b) The areas of the sections.

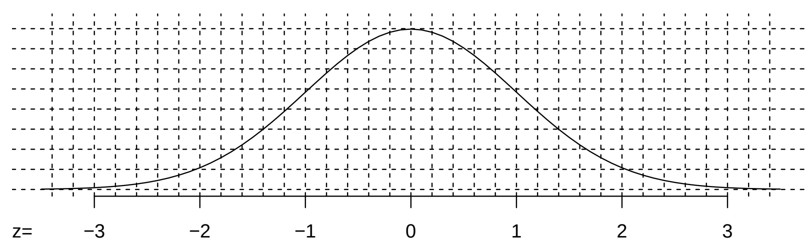


3. **Problem**
Let X be normally distributed with mean 67 and standard deviation 14. Please calculate the probabilities shown below and also shade a corresponding region under the density curve.

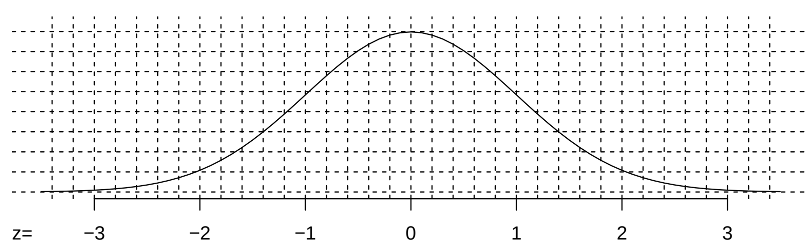
(a) $P(X < 46)$



(b) $P(X > 60)$



(c) $P(57.2 < X < 76.8)$



4. **Problem**
A very large population has a mean of 49 and a standard deviation of 9.24. When a random sample of size 144 is taken, what is the probability that the **sample mean** ($\bar{x}$) is between 48.8 and 49.4?

(a) Determine the mean and standard deviation of the sampling distribution.

(b) Determine $P(48.8 < \overline{X} < 49.4)$. **Draw a sketch**

5. **Problem**
A very large population has a population proportion $p = 0.65$. When a random sample of size 100 is taken, what is the probability that the **sample proportion** ($\hat{p}$) is more than 0.62?

Do NOT use a continuity correction.

(a) Determine the mean and standard deviation of the sampling distribution.

(b) Determine $P(\hat{p} > 0.62)$. **Draw a sketch**

6. **Problem**
Let random variable W have mean $\mu_W = 18$ and standard deviation $\sigma_W = 6$. Let random variable X represent the **sum** of $n = 64$ instances of W.

   (a) Determine the expected value of X. $\mu_X = ?$
   (b) Determine the standard deviation of X. $\sigma_X = ?$
   (c) Using normal approximation, determine $P(X < 1198.56)$.
   (d) Using normal approximation, determine $P(X > 1172.64)$.

7. **Problem**

   Let each trial have a chance of success $p = 0.84$. If 169 trials occur, what is the probability of getting at least 131 but at most 152 successes?

   In other words, let $X \sim \text{Bin}(n = 169,\ p = 0.84)$ and find $P(131 \leq X \leq 152)$.

   Use a normal approximation along with the continuity correction.

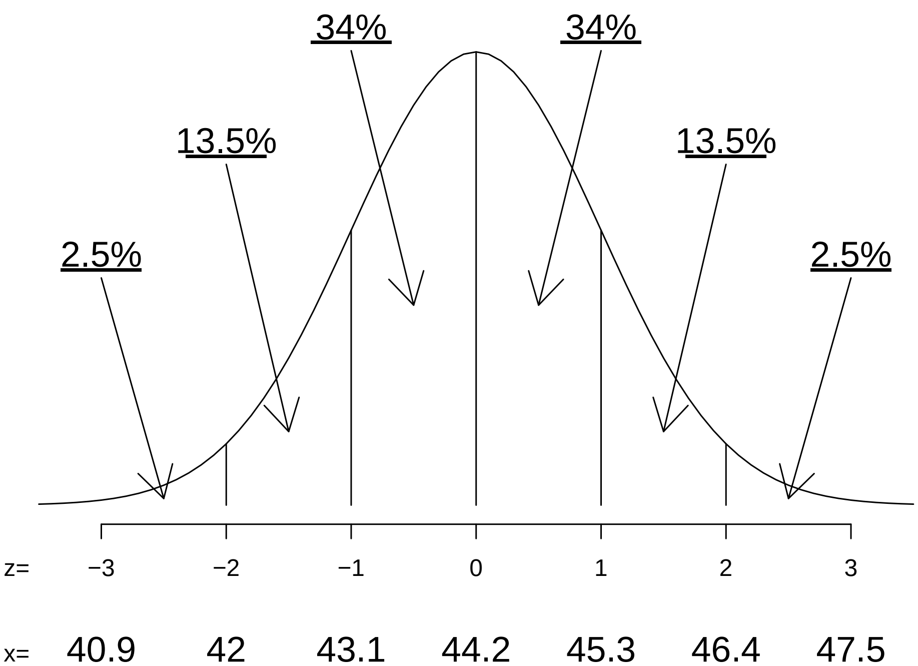
1. (a) We calculate the z-scores.

Type of fruit	formula	z-score
A	$z = \frac{127.8-133}{6}$	-0.87
B	$z = \frac{114.6-102}{10}$	1.26
C	$z = \frac{122.6-113}{13}$	0.74
D	$z = \frac{148.4-135}{9}$	1.49

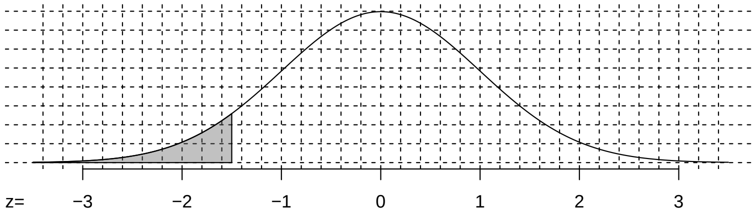
(b) The largest z-score corresponds to the most unusually large, so the specimen of type D is the most unusually large.

(c) The smallest z-score corresponds to the most unusually small, so the specimen of type A is the most unusually large.

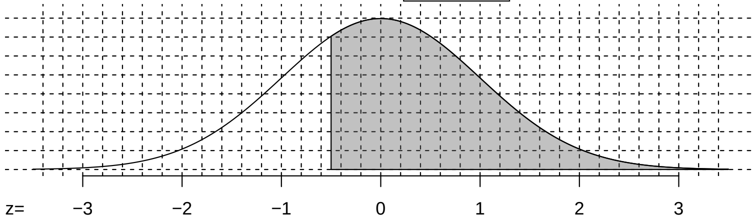
2. The filled in areas and x values are shown below.



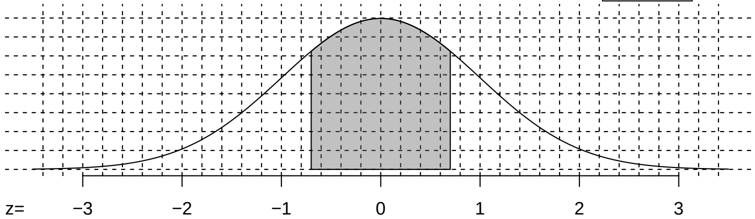
3. (a) $P(X < 46) = P(Z < -1.5) = \boxed{0.0668}$



(b) $P(X > 60) = P(Z > -0.5) = \boxed{0.6915}$

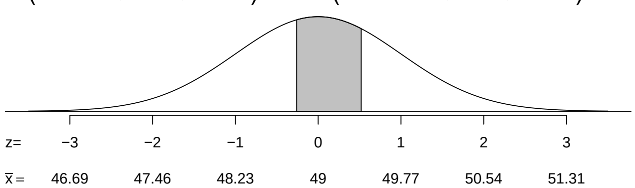


(c) $P(57.2 < X < 76.8) = P(-0.7 < Z < 0.7) = \boxed{0.516}$



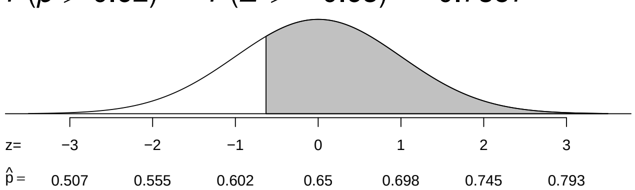
4. (a) Central limit of average formulas: $\mu_{\bar{x}} = 49$ and $\sigma_{\bar{x}} = \frac{9.24}{\sqrt{144}} = 0.77$.

(b) $P(48.8 < \overline{X} < 49.4) = P(-0.26 < Z < 0.52) = 0.3011$



5. (a) Use $\hat{p}$ sampling formulas: $\mu_{\hat{p}} = 0.65$ and $\sigma_{\hat{p}} = \frac{\sqrt{(0.65)(0.35)}}{\sqrt{100}} = 0.047697$.

(b) $P(\hat{p} > 0.62) = P(Z > -0.63) = 0.7357$



6. (a) $\mu_X = n\mu_W = (64)(18) = \boxed{1152}$

(b) $\sigma_X = \sigma_W\sqrt{n} = (6)(\sqrt{64}) = \boxed{48}$

(c) $P(X < 1198.56) = P(Z < 0.97) = \boxed{0.834}$

(d) $P(X > 1172.64) = P(Z > 0.43) = 1 - 0.6664 = \boxed{0.3336}$

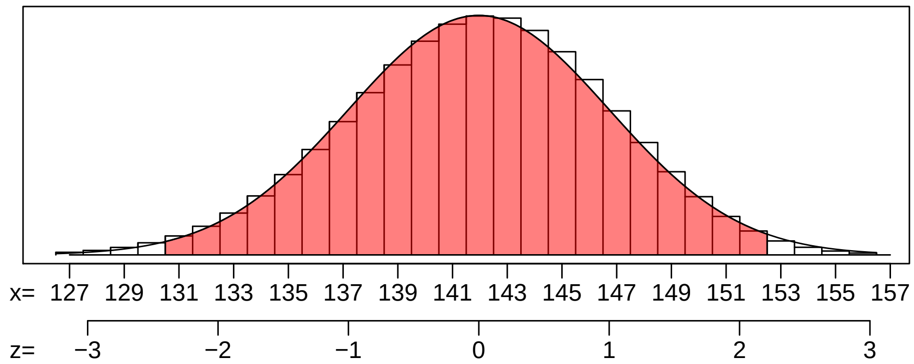
7. Find the mean.

$$\mu = np = (169)(0.84) = 141.96$$

Find the standard deviation.

$$\sigma = \sqrt{np(1-p)} = \sqrt{(169)(0.84)(1-0.84)} = 4.7659$$

Make a sketch, specifically try to picture whether you need to add or subtract 0.5 for the continuity correction.



Find the z scores.

$$z_1 = \frac{130.5 - 141.96}{4.7659} = -2.4$$

$$z_2 = \frac{152.5 - 141.96}{4.7659} = 2.21$$

Find the percentiles (from z-table).

$$\ell_1 = 0.0082$$

$$\ell_2 = 0.9864$$

Calculate the probability.

$$P(131 \leq X \leq 152) = 0.9864 - 0.0082 = 0.9782$$

Normal Distributions

$$z = \frac{x - \mu}{\sigma}$$

$$x = \mu + z\sigma$$

Central Limit Theorem

- **If:**
  - Random variable W has mean μ_w and standard deviation σ_w.
  - Random variable X represents the **sum** of n instances of W.

    $$X = W_1 + W_2 + W_3 + \cdots + W_n$$

  - Random variable Y represents the **mean** of n instances of W.

    $$Y = \frac{W_1 + W_2 + W_3 + \cdots + W_n}{n}$$

- **Then**:
  - The following formulas are exactly true:

    $$\mu_x = n\mu_w \qquad \mu_y = \mu_w$$

    $$\sigma_x = \sigma_w\sqrt{n} \qquad \sigma_y = \frac{\sigma_w}{\sqrt{n}}$$

  - X and Y are both approximately normal (if $n \geq 30$).
  - X and Y are exactly normal if W is normal.

Special case of CLT: Bernoulli, Binomial, and Proportion Sampling

- **If:**
  - Random variable W is Bernoulli:

w	$P(w)$
0	q
1	p

  - Random variable X represents the sum of n instances of W. (Binomial)
  - Random variable $\hat{p}$ represents the mean of n instances of W. (Proportion sampling)

- **Then**:
  - The following formulas are exactly true:

    $$\mu_w = p \qquad \mu_x = np \qquad \mu_{\hat{p}} = p$$

    $$\sigma_w = \sqrt{pq} \qquad \sigma_x = \sqrt{pq}\sqrt{n} \qquad \sigma_{\hat{p}} = \frac{\sqrt{pq}}{\sqrt{n}}$$

  - X and $\hat{p}$ are both approximately normal (if $np \geq 10$ and $nq \geq 10$).