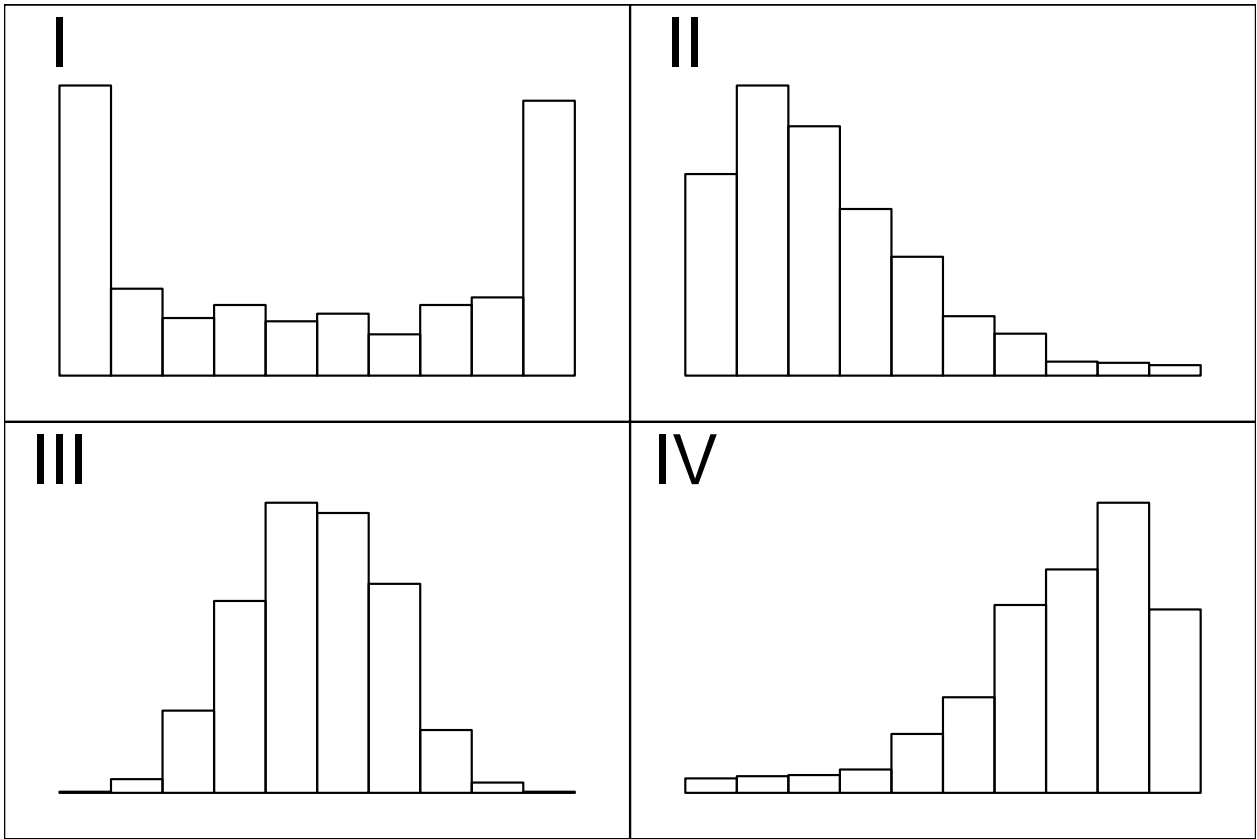


**MAT-181 FINAL TAKE-HOME EXAM**

This exam is to be taken without discussion or correspondence with any human. Please show work!

| question | available points | earned points |
|----------|------------------|---------------|
| 1        | 10               |               |
| 2        | 15               |               |
| 3        | 10               |               |
| 4        | 10               |               |
| 5        | 10               |               |
| 6        | 10               |               |
| 7        | 15               |               |
| 8        | 20               |               |
| EC       | 5                |               |
| EC       | 5                |               |
| Total    | 100              |               |

## 1. (10 Points)



For each description below, choose which histogram best fits (I, II, III, or IV). Each histogram should be used once.

- (a) The distribution of annual income for NBA basketball players where only a few are high-paid superstars.
- (b) The distribution of heights of adult women
- (c) The distribution of hours spent per week reading by adults. In this distribution, many people do not read much, and a similar number of people read a lot.
- (d) The distribution of ages at a skilled nursing facility, where most of the patients are elderly but a few are quite young.

**Solution:**

- (a) II
- (b) III
- (c) I
- (d) IV

## 2. (15 Points)

In a deck of strange cards, there are 462 cards. Each card has an image and a color. The amounts are shown in the table below.

|       | black | indigo | orange | teal | Total |
|-------|-------|--------|--------|------|-------|
| gem   | 23    | 39     | 41     | 12   | 115   |
| kite  | 14    | 15     | 31     | 16   | 76    |
| mop   | 25    | 46     | 19     | 48   | 138   |
| quilt | 37    | 17     | 29     | 50   | 133   |
| Total | 99    | 117    | 120    | 126  | 462   |

- (a) Is a kite or a quilt more likely to be teal?
- (b) What is the probability a random card is a quilt?
- (c) What is the probability a random card is teal given it is a mop?
- (d) What is the probability a random card is teal?
- (e) What is the probability a random card is both a quilt and orange?
- (f) What is the probability a random card is a kite given it is black?
- (g) What is the probability a random card is either a kite or orange (or both)?

**Solution:**

- (a)  $P(\text{teal given kite}) = 0.211$  and  $P(\text{teal given quilt}) = 0.376$ , so a quilt is more likely to be teal than a kite is.
- (b)  $P(\text{quilt}) = 0.288$
- (c)  $P(\text{teal given mop}) = 0.348$
- (d)  $P(\text{teal}) = 0.273$
- (e)  $P(\text{quilt and orange}) = 0.0628$
- (f)  $P(\text{kite given black}) = 0.141$
- (g)  $P(\text{kite or orange}) = 0.357$

3. (10 points)

A farm produces 4 types of fruit: *A*, *B*, *C*, and *D*. The fruits' masses follow normal distributions, with parameters dependent on the type of fruit.

| Type of fruit | Mean mass (g) | Standard deviation of mass (g) |
|---------------|---------------|--------------------------------|
| <i>A</i>      | 110           | 15                             |
| <i>B</i>      | 126           | 5                              |
| <i>C</i>      | 80            | 12                             |
| <i>D</i>      | 83            | 6                              |

One specimen of each type is weighed. The results are shown below.

| Type of fruit | Mass of specimen (g) |
|---------------|----------------------|
| <i>A</i>      | 83                   |
| <i>B</i>      | 126.4                |
| <i>C</i>      | 66.08                |
| <i>D</i>      | 94.1                 |

Which specimen is the most unusually far (in either direction) from average (relative to others of its type)?

**Solution:** We compare the absolute z-scores. The largest absolute z-score corresponds to the specimen that is most unusually far from average.

| Type of fruit | formula                       | absolute z-score |
|---------------|-------------------------------|------------------|
| <i>A</i>      | $z = \frac{ 83 - 110 }{15}$   | 1.8              |
| <i>B</i>      | $z = \frac{ 126.4 - 126 }{5}$ | 0.08             |
| <i>C</i>      | $z = \frac{ 66.08 - 80 }{12}$ | 1.16             |
| <i>D</i>      | $z = \frac{ 94.1 - 83 }{6}$   | 1.85             |

Thus, the specimen of type *D* is the most unusually far from average.

4. (10 points)

A tree's leaves were found to be normally distributed with a mean of 53.3 millimeters and a standard deviation of 6.4 millimeters. If you pick a random leaf from that tree, what is the probability the length is between 44.8 and 53.7 millimeters?



**Solution:**

$$\mu = 53.3$$

$$\sigma = 6.4$$

$$x_1 = 44.8$$

$$x_2 = 53.7$$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{44.8 - 53.3}{6.4} = -1.33$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{53.7 - 53.3}{6.4} = 0.06$$

$$P(x_1 < X < x_2) = P(z_1 < Z < z_2) = 0.5239 - 0.0918 = 0.4321$$

5. (10 points)

A species of duck is known to have a mean weight of 178.7 grams and a standard deviation of 36 grams. A researcher plans to measure the weights of 225 of these ducks sampled randomly. What is the probability the **sample mean** will be between 175.7 and 184.7 grams?

**Solution:**

$$n = 225$$

$$\mu = 178.7$$

$$\sigma = 36$$

$$SE = \frac{36}{\sqrt{225}} = 2.4$$

$$x_1 = 175.7$$

$$x_2 = 184.7$$

$$z_1 = \frac{x_1 - \mu}{SE} = \frac{175.7 - 178.7}{2.4} = -1.25$$

$$z_2 = \frac{x_2 - \mu}{SE} = \frac{184.7 - 178.7}{2.4} = 2.5$$

$$P(x_1 < \bar{X} < x_2) = P(z_1 < Z < z_2) = 0.9938 - 0.1056 = 0.8882$$

6. (10 points)

An ornithologist wishes to characterize the average body mass of *Melospiza georgiana*. She randomly samples 32 adults of *Melospiza georgiana*, resulting in a sample mean of 13.6 grams and a sample standard deviation of 1.14 grams. Determine a 95% confidence interval of the true population mean.

**Solution:** We are given the sample size, sample mean, sample standard deviation, and confidence level.

$$n = 32$$

$$\bar{x} = 13.6$$

$$s = 1.14$$

$$\gamma = 0.95$$

Find the degrees of freedom.

$$df = n - 1$$

$$= 32 - 1$$

$$= 31$$

Determine the critical  $t$  value,  $t^*$ , such that  $P(|T| < t^*) = 0.95$  and  $df = 31$ .

$$t^* = 2.04$$

Use the formula for bounds (mean,  $\sigma$  unknown).

$$LB = \bar{x} - t^* \frac{s}{\sqrt{n}}$$

$$= 13.6 - 2.04 \times \frac{1.14}{\sqrt{32}}$$

$$= 13.2$$

$$UB = \bar{x} + t^* \frac{s}{\sqrt{n}}$$

$$= 13.6 + 2.04 \times \frac{1.14}{\sqrt{32}}$$

$$= 14$$

We are 95% confident that the population mean is between 13.2 and 14 grams.

$$CI = (13.2, 14)$$

7. (15 points)

A student is taking a multiple choice test with 500 questions. Each question has 2 choices. You want to detect whether the student does significantly better than random guessing, so you decide to run a hypothesis test with a significance level of 0.05.

Then, the student takes the test and gets 268 questions correct.

(a) What kind of hypothesis test is appropriate?

(b) State the hypotheses.

(c) Determine the test statistic ( $z$  or  $t$ ), draw a sketch, and determine the  $p$ -value.

(d) Decide whether we reject or retain the null hypothesis.

(e) Did the student do significantly better than random guessing?

**Solution:** This is a right-tail (one-tail) proportion test because we only care whether the student does better than random.

Determine the null population proportion.

$$p_0 = \frac{1}{2} = 0.5$$

State the hypotheses.

$$H_0 \text{ claims } p = 0.5$$

$$H_A \text{ claims } p > 0.5$$

Determine the standard error.

$$\sigma_{\hat{p}} = \sqrt{\frac{p_0(1-p_0)}{n}} = \sqrt{\frac{0.5(1-0.5)}{500}} = 0.0224$$

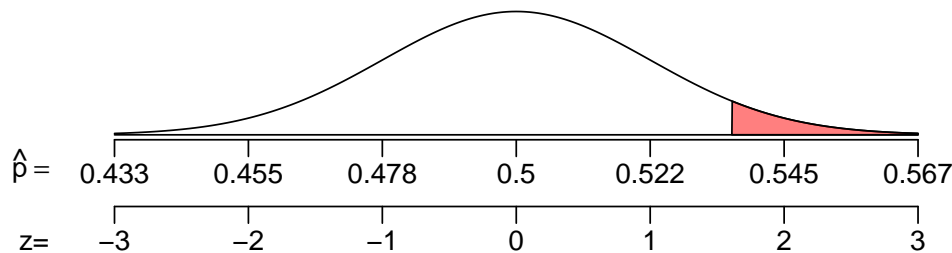
Determine the sample proportion.

$$\hat{p} = \frac{268}{500} = 0.536$$

Determine a z score. For simplicity, we ignore the continuity correction.

$$z = \frac{\hat{p} - p_0}{\sigma_{\hat{p}}} = \frac{0.536 - 0.5}{0.0224} = 1.61$$

Make a sketch of the null's sampling distribution. The  $p$ -value is a right area.



To determine that right area, we use the z table.

$$\begin{aligned} p\text{-value} &= P(\hat{p} > 0.536) \\ &= P(Z > 1.61) \\ &= 1 - P(Z < 1.61) \\ &= 0.0537 \end{aligned}$$

Compare  $p$ -value to  $\alpha$  (which is 0.05).

$$p\text{-value} > \alpha$$

Make the conclusion: we retain the null hypothesis.

We think the student might just be guessing.

- (a) Right tail (one-tail) proportion test
- (b) Hypotheses:  $H_0$  claims  $p = 0.5$  and  $H_A$  claims  $p > 0.5$ .
- (c) The  $p$ -value is 0.0537
- (d) We retain the null hypothesis.
- (e) We think the student might just be guessing.



8. (20 points) [Note: this question uses 2 pages.]

You have collected the following data:

| $x$         | $y$         | $xy$        |
|-------------|-------------|-------------|
| 62          | 2.2         |             |
| 46          | 3.1         |             |
| 73          | 3.4         |             |
| 85          | 3.7         |             |
| 65          | 4.8         |             |
| 99          | 4.4         |             |
| 71          | 6.5         |             |
| $\sum x =$  | $\sum y =$  | $\sum xy =$ |
| $\bar{x} =$ | $\bar{y} =$ |             |
| $s_x =$     | $s_y =$     |             |

(a) Complete the table.

(b) Calculate the correlation coefficient ( $r$ ) using the formula below.

$$r = \frac{\sum xy - n\bar{x}\bar{y}}{(n-1)s_x s_y}$$

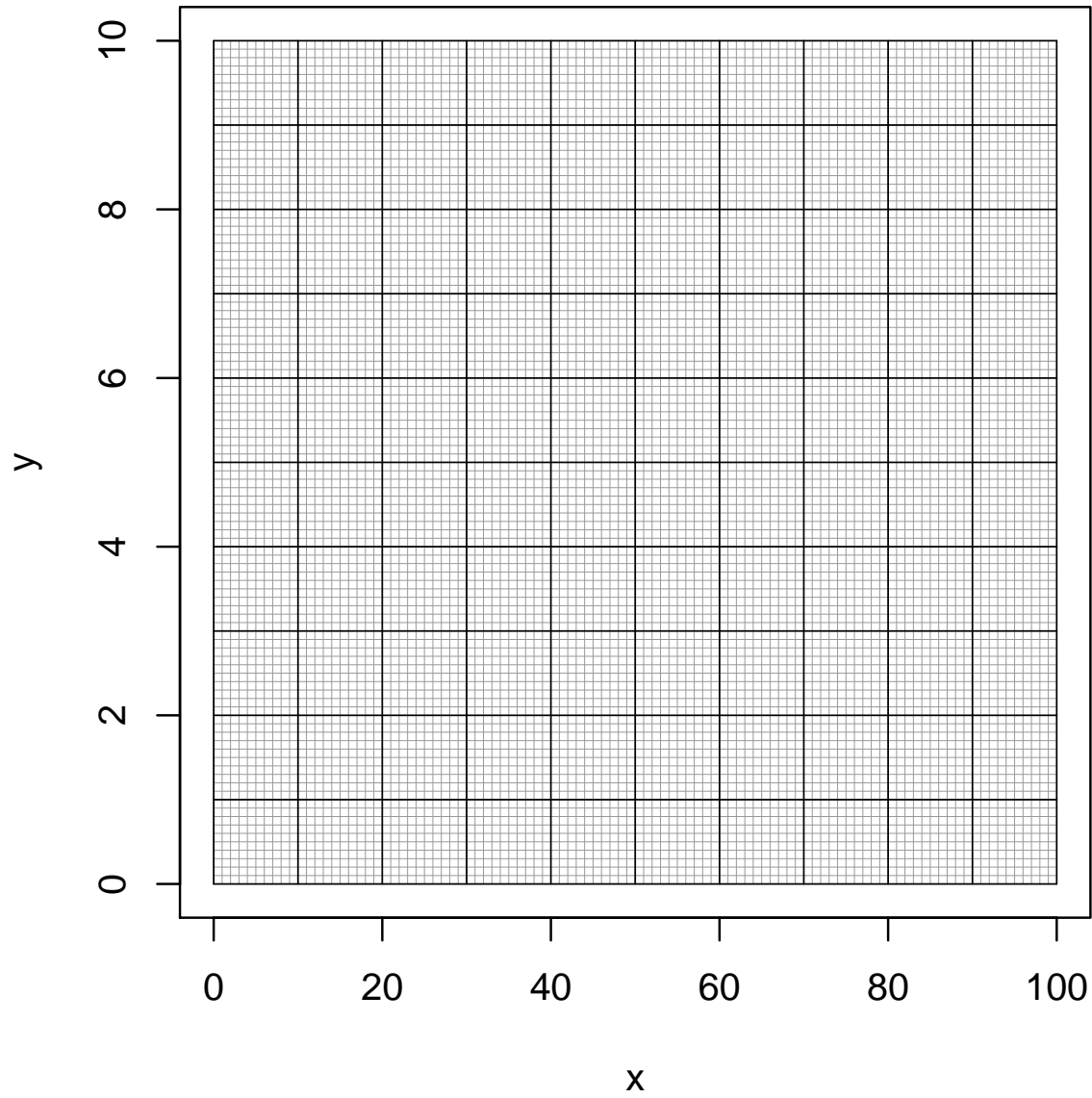
(c) The least-squares regression line will be represented as  $y = a + bx$ . Determine the parameters ( $b$  and  $a$ ) using the formulas below.

$$b = r \frac{s_y}{s_x}$$

$$a = \bar{y} - b\bar{x}$$

(d) Write the equation of the regression line (using the calculated values of  $a$  and  $b$ .)

(e) Please plot the data and a corresponding regression line.



**Solution:** Remember the formula for the correlation coefficient.

$$r = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{(n-1)s_x s_y}$$

We calculate the necessary values.

| $x$               | $y$               | $xy$                    |
|-------------------|-------------------|-------------------------|
| 62                | 2.2               | 136.4                   |
| 46                | 3.1               | 142.6                   |
| 73                | 3.4               | 248.2                   |
| 85                | 3.7               | 314.5                   |
| 65                | 4.8               | 312                     |
| 99                | 4.4               | 435.6                   |
| 71                | 6.5               | 461.5                   |
| $\sum x = 501$    | $\sum y = 28.1$   | $\sum x_i y_i = 2050.8$ |
| $\bar{x} = 71.57$ | $\bar{y} = 4.014$ |                         |
| $s_x = 16.95$     | $s_y = 1.387$     |                         |

$$r = \frac{2050.8 - (7)(71.57)(4.014)}{(7-1)(16.95)(1.387)} = 0.282$$

If you didn't round any of the steps up to here, you'd get an exact value which is pretty close to our value.

$$r_{\text{exact}} = 0.2809753$$

The regression line has the form

$$y = a + bx$$

So,  $a$  is the  $y$ -intercept and  $b$  is the slope. We have formulas to determine them:

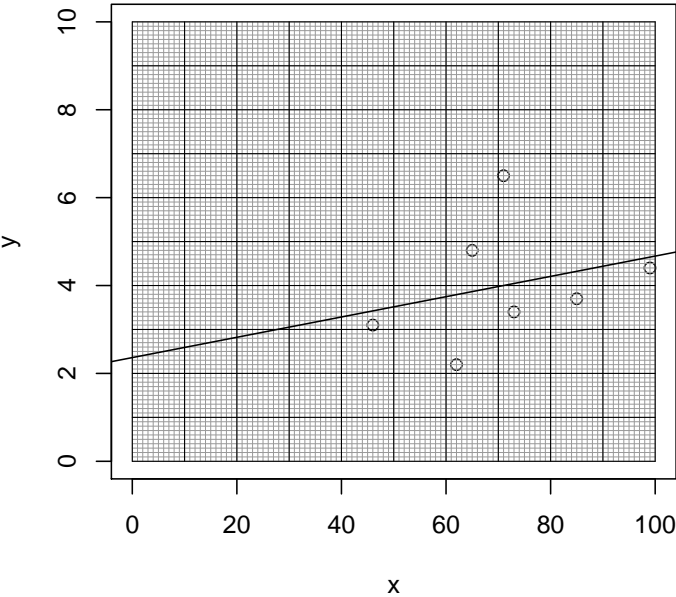
$$b = r \frac{s_y}{s_x} = 0.282 \cdot \frac{1.387}{16.95} = 0.0231$$

$$a = \bar{y} - b\bar{x} = 4.01 - (0.0231)(71.6) = 2.36$$

Our regression line:

$$y = 2.36 + (0.0231)x$$

Make a plot.



9. (Extra credit: 5 points)

Let each trial have a chance of success  $p = 0.92$ . If 238 trials occur, what is the probability of getting at least 212 but less than 225 successes?

In other words, let  $X \sim \text{Bin}(n = 238, p = 0.92)$  and find  $P(212 \leq X < 225)$ .

Use a normal approximation along with the continuity correction.

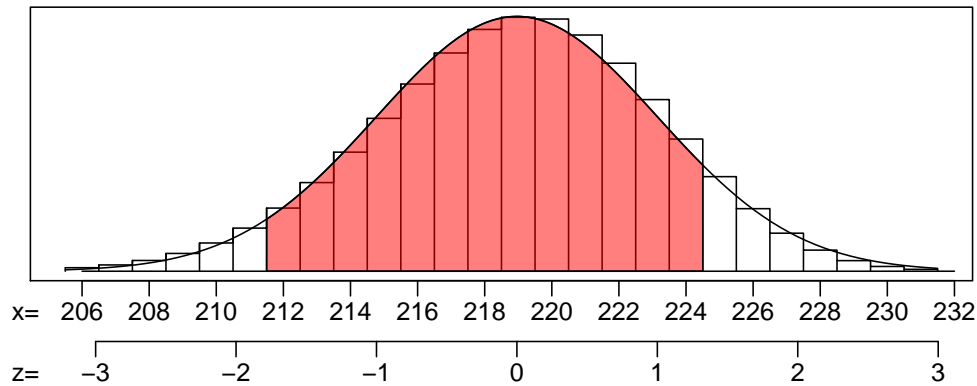
**Solution:** Find the mean.

$$\mu = np = (238)(0.92) = 218.96$$

Find the standard deviation.

$$\sigma = \sqrt{np(1-p)} = \sqrt{(238)(0.92)(1-0.92)} = 4.1853$$

Make a sketch, specifically try to picture whether you need to add or subtract 0.5 for the continuity correction.



Find the z scores.

$$z_1 = \frac{211.5 - 218.96}{4.1853} = -1.78$$

$$z_2 = \frac{224.5 - 218.96}{4.1853} = 1.32$$

Find the percentiles (from z-table).

$$\ell_1 = 0.0375$$

$$\ell_2 = 0.9066$$

Calculate the probability.

$$P(212 \leq X < 225) = 0.9066 - 0.0375 = 0.869$$

10. (Extra credit: 5 points)

A null hypothesis claims a population has a mean  $\mu = 150$ . You decide to run two-tail test on a sample of size  $n = 9$  using a significance level  $\alpha = 0.05$ .

You then collect the sample:

|       |       |       |       |       |
|-------|-------|-------|-------|-------|
| 224.8 | 167.6 | 122   | 201.3 | 171.3 |
| 186.1 | 183.4 | 146.1 | 156.6 |       |

- (a) Determine the  $p$ -value.
- (b) Do you reject the null hypothesis?

**Solution:** State the hypotheses.

$$H_0 \text{ claims } \mu = 150$$

$$H_A \text{ claims } \mu \neq 150$$

Find the mean and standard deviation of the sample.

$$\bar{x} = 173.244$$

$$s = 30.391$$

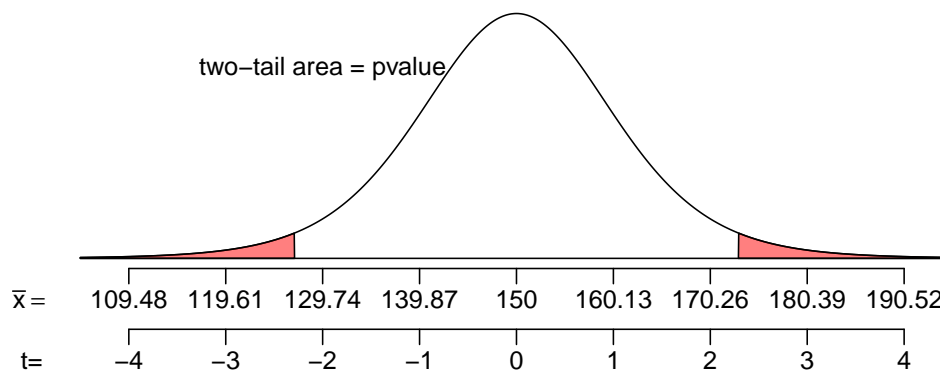
Determine the degrees of freedom.

$$df = 9 - 1 = 8$$

Find the standard error.

$$\sigma_{\bar{x}} = \frac{s}{\sqrt{n}} = \frac{30.391}{\sqrt{9}} = 10.13$$

Make a sketch of the null's sampling distribution.



Find the  $t$  score.

$$t = \frac{\bar{x} - \mu_0}{\sigma_{\bar{x}}} = \frac{173.244 - 150}{10.13} = 2.29$$

Find the  $p$ -value.

$$p\text{-value} = P(|T| > 2.29)$$

We can't get an exact value with our table, but we can determine an interval that contains the  $p$ -value. (Look at row with  $df = 8$ .)

$$P(|T| > 2.31) = 0.05$$

$$P(|T| > 1.86) = 0.1$$

Basically, because  $t$  is between 2.31 and 1.86, we know the  $p$ -value is between 0.05 and 0.1.

$$0.05 < p\text{-value} < 0.1$$

Compare the  $p$ -value and the significance level ( $\alpha = 0.05$ ).

$$p\text{-value} > \alpha$$

No, we do not reject the null hypothesis.



- (a)  $0.05 < p\text{-value} < 0.1$
- (b) No, we do not reject the null hypothesis.