Probability - definitions, conditional probability, tree diagrams, Bayes' theorem, contingency tables, probability distributions, and binomial distribution

Independence

$$\text{Events } A,\ B \text{ are independent}$$
$$\Updownarrow$$
$$P(A \text{ and } B) = P(A) \cdot P(B)$$
$$\Updownarrow$$
$$P(A \text{ given } B) = P(A)$$
$$\Updownarrow$$
$$P(B \text{ given } A) = P(B)$$

(If any of the 4 statements is true, all are true)

Mutual exclusivity (disjointness)

Events A, B are mutually exclusive (disjoint)

$$\Updownarrow$$

$$P(A \text{ and } B) = 0$$

$$\Updownarrow$$

$$P(A \text{ or } B) = P(A) + P(B)$$

Exhaustivity

Events A, B are exhaustive

$$\updownarrow$$

$$P(A \text{ or } B) = 1$$

Sometimes it takes more than two events for exhaustion.

Events A, B, C are exhaustive

$$\updownarrow$$

$$P(A \text{ or } B \text{ or } C) = 1$$

Complementarity

Events A, B are complementary

$$\updownarrow$$

Events A, B are disjoint and exhaustive

$$\updownarrow$$

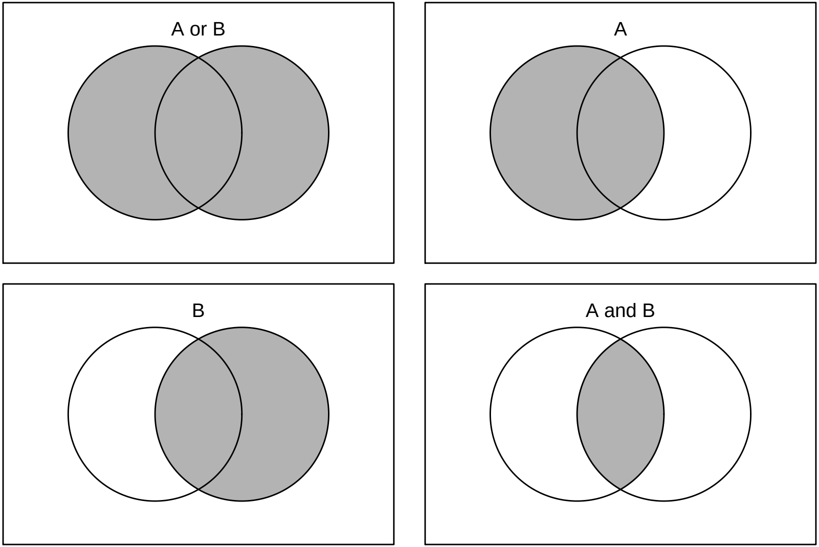
$$P(A \text{ and } B) = 0 \quad \text{and} \quad P(A \text{ or } B) = 1$$

$$\updownarrow$$

A is "not B". B is "not A".

General addition rule

$$P(A \text{ or } B) \;=\; P(A) + P(B) - P(A \text{ and } B)$$



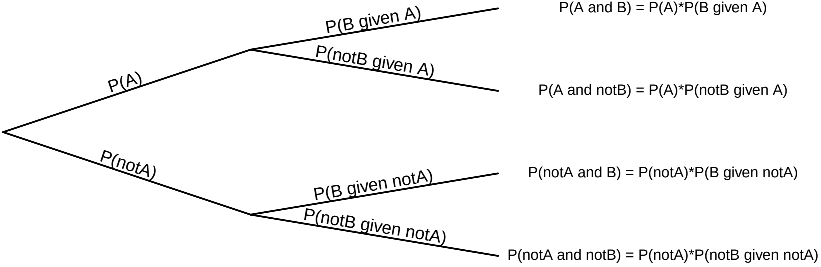
Definition of conditional probability and general product rule

$$P(B \text{ given } A) = \frac{P(A \text{ and } B)}{P(A)}$$

$$P(A \text{ and } B) = P(A) \cdot P(B \text{ given } A)$$

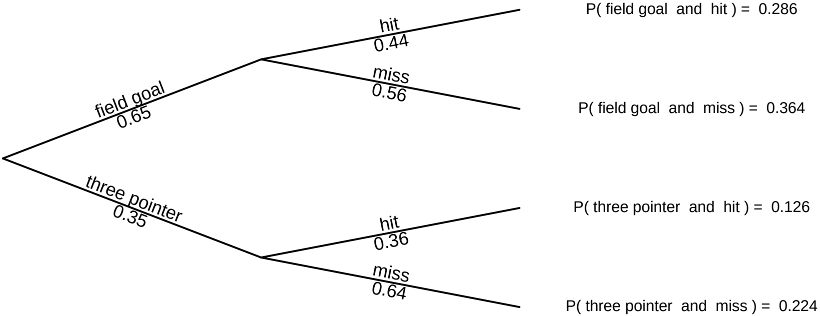
Tree diagrams

A basic, general tree diagram is shown below.

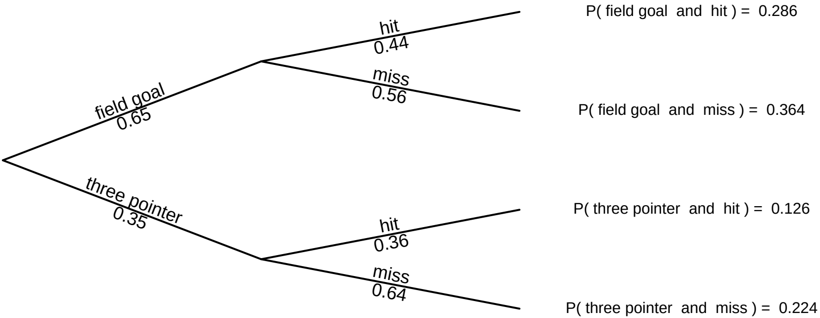


Example tree

Let's imagine a basketball player has the following probabilities (based on James Harden 2019-2020). When James shoots during play, 65% of attempts are field goals and 35% of attempts are three-pointers. Of the field goals, 44% are hits and 56% are misses. Of the three-pointers, 36% are hits and 64% are misses.



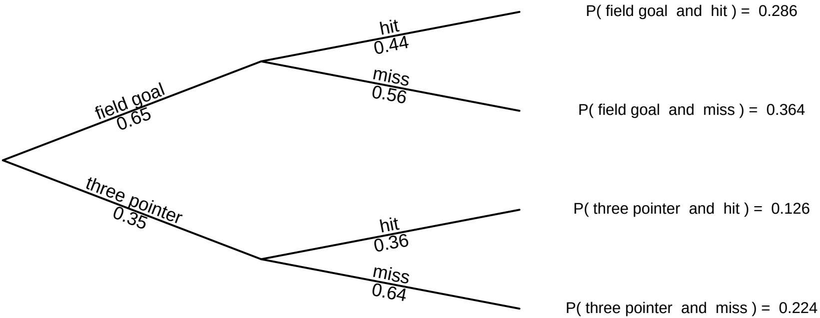
Example problems



- What is the probability that James hits a shot?

$$P(\text{hit}) =$$

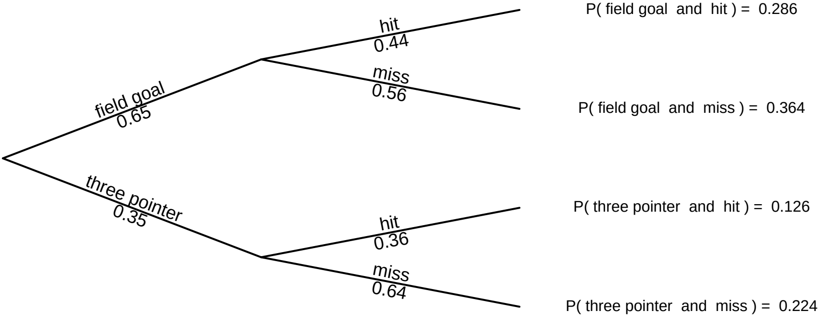
Example problems



- What is the probability that James misses a three-pointer?

$$P(\text{miss given threepointer}) =$$

Example problems



- What is the probability that James attempted a three-pointer given he missed the shot?

$$P(\text{threepointer given miss}) =$$

Two-way Contingency Table

	hit	miss	total
FG	0.286	0.364	0.65
3P	0.126	0.224	0.35
total	0.412	0.588	1

$$P(\text{hit}) = 0.412$$

$$P(\text{miss given 3P}) = \frac{0.224}{0.35} = 0.64$$

$$P(\text{3P given miss}) = \frac{0.224}{0.588} = 0.381$$

Bayes' Theorem

$$P(A \text{ given } B) \;=\; \frac{P(B \text{ given } A) \cdot P(A)}{P(B)}$$

Probability distribution

A probability distribution is a list of events that are pairwise disjoint and collectively exhaustive. A probability distribution unambiguously describes a random variable.

- Example - 5 card poker hands

Event	Probability
Royal Flush	0.00000154
Straight flush (excluding royal flush)	0.000015
Four of a kind	0.000240
Full house	0.001441
Flush (not royal or straight)	0.003925
Three of a kind	0.021128
Two pair	0.047539
One pair	0.422569
No pair/High card	0.501177

Discrete Probability Distribution

The events are discrete numbers.

- Example: straight up bet of a dollar on an single number in French Roulette

Event	Probability
-1	36/37
35	1/37

Mean and Standard Deviation of Discrete Probability Distribution

x	$P(x)$
-1	36/37
35	1/37

$$\mu = \sum x \cdot P(x)$$

The mean of a probability distribution (μ) is also called the expected value, denoted with an upper-case E, with the variable denoted in parentheses.

$$E(X) = \mu = \sum x \cdot P(x)$$

The standard deviation is determined with the equation below.

$$\sigma = \sqrt{\sum (x - \mu)^2 \cdot P(x)}$$

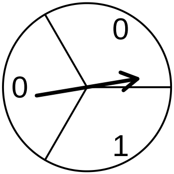
Expected value and standard deviation of roulette

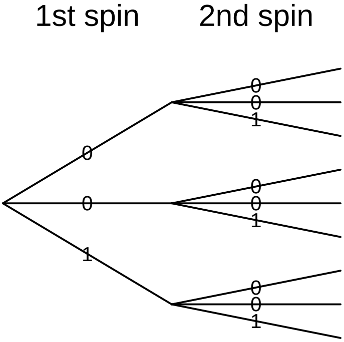
$$\mu = \sum x \cdot P(x)$$

$$\sigma = \sqrt{\sum (x-\mu)^2 \cdot P(x)}$$

x	$P(x)$	$x \cdot P(x)$	$x - \mu$	$(x-\mu)^2$	$(x-\mu)^2 \cdot P(x)$
-1	36/37	-36/37	-36/37	0.946676	0.9211
35	1/37	35/37	1296/37	1226.89	33.16
		$\mu = -1/37$			$\sigma^2 = 34.08$ $\sigma = \sqrt{34.08} = 5.8$

Consider summing (or averaging) 2 spins of a Bernoulli



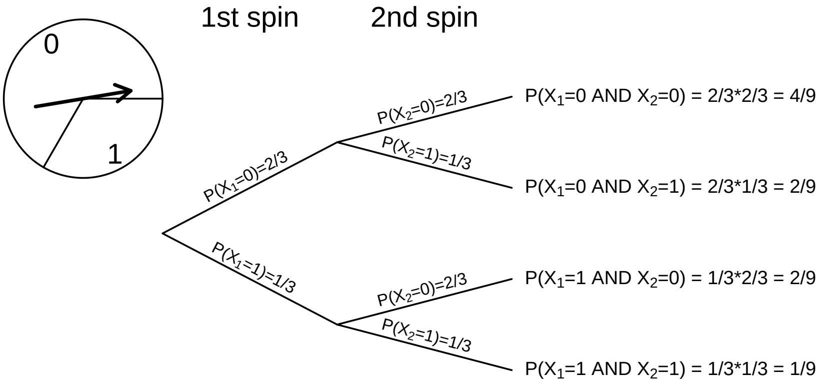


Total	Average
0	0
0	0
1	0.5
0	0
0	0
1	0.5
1	0.5
1	0.5
2	1

$$P(\text{Total} = 1) \;=\; \frac{4}{9}$$

$$\mu_{\text{total}} = \frac{0+0+1+0+0+1+1+1+2}{9} = \frac{6}{9} = \frac{2}{3}$$

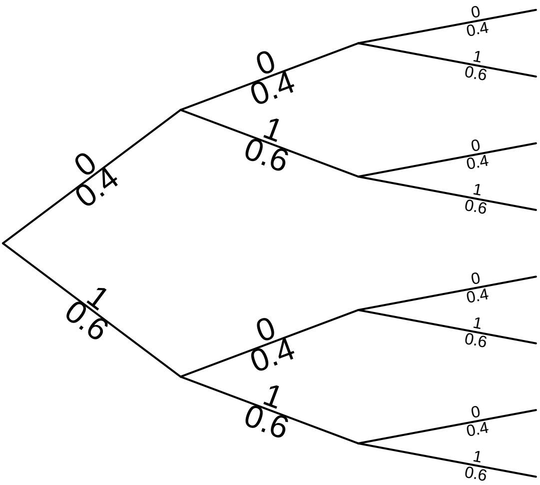
Same thing, different representation



$$P(\text{total} = 1) \;=\; P(0,1) + P(1,0) \;=\; \frac{2}{9} + \frac{2}{9} \;=\; \frac{4}{9}$$

$$\mu_{\text{total}} \;=\; \left(0 \cdot \frac{4}{9} + 1 \cdot \frac{2}{9} + 1 \cdot \frac{2}{9} + 2 \cdot \frac{1}{9}\right) \;=\; \frac{6}{9} \;=\; \frac{2}{3}$$

Consider 3 spins of a Bernoulli spinner with $p = 0.6$



P(0,0,0) = 0.4*0.4*0.4

P(0,0,1) = 0.4*0.4*0.6

P(0,1,0) = 0.4*0.6*0.4

P(0,1,1) = 0.4*0.6*0.6

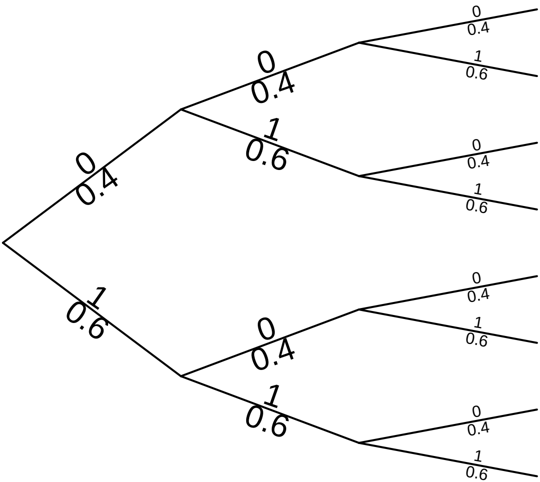
P(1,0,0) = 0.6*0.4*0.4

P(1,0,1) = 0.6*0.4*0.6

P(1,1,0) = 0.6*0.6*0.4

P(1,1,1) = 0.6*0.6*0.6

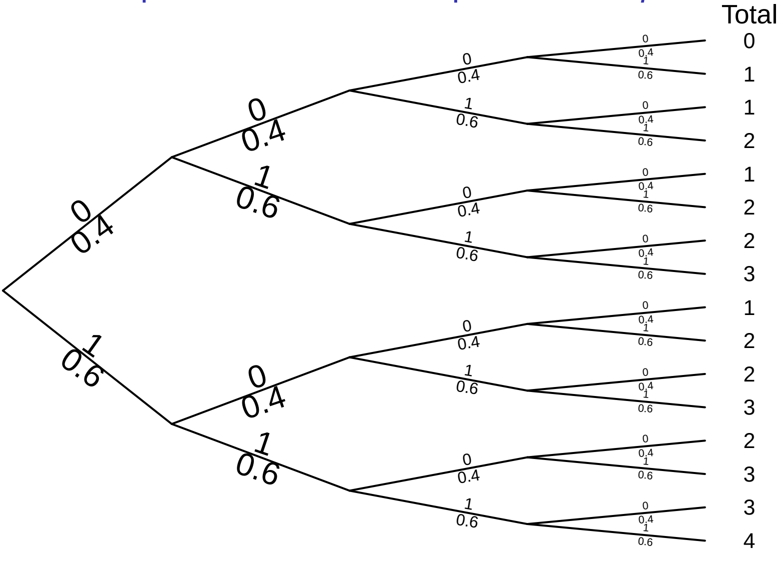
Consider 3 spins of a Bernoulli spinner with $p = 0.6$



Total	Average	Prob
0	0	0.064
1	0.3333	0.096
1	0.3333	0.096
2	0.6667	0.144
1	0.3333	0.096
2	0.6667	0.144
2	0.6667	0.144
3	1	0.216

$$P(\text{Total=2}) \quad = \quad 0.144 + 0.144 + 0.144 \quad = \quad 0.432$$

Consider 4 spins of a Bernoulli spinner with $p = 0.6$



$$P(\text{Total}=2) \;=\; (6)(0.6)^2(0.4)^2 \;=\; 0.3456$$

Representing total of 4 spins with probability distribution

Event	Probability
total=0	$(1)(0.6)^0(0.4)^4$
total=1	$(4)(0.6)^1(0.4)^3$
total=2	$(6)(0.6)^2(0.4)^2$
total=3	$(4)(0.6)^3(0.4)^1$
total=4	$(1)(0.6)^4(0.4)^0$

Pascal's Triangle

										1										
									1		1									
								1		2		1								
							1		3		3		1							
						1		4		6		4		1						
					1		5		10		10		5		1					
				1		6		15		20		15		6		1				
			1		7		21		35		35		21		7		1			
		1		8		28		56		70		56		28		8		1		
	1		9		36		84		126		126		84		36		9		1	
1		10		45		120		210		252		210		120		45		10		1

$$_7C_3 = \frac{7!}{3!4!} = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 4\cdot 3\cdot 2\cdot 1} = 35$$

Binomial Distribution

- Let W represent a Bernoulli distribution with p as the chance of success.
- Let n represent the number of spins.
- Let X represent the total of n spins of W.
- We say X follows a binomial distribution.

$$P(X = x) \quad = \quad {}_nC_x p^x (1 - p)^{n-x}$$