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Convex Programming

Definition 1.1.1

Given $x, y \in R^n$, the point $z := \lambda x + (1 - \lambda)y$ is said to be a convex combination of x, y for all $\lambda \in [0, 1]$. The combination is said to be strict if $0 < \lambda < 1$.

More generally, the convex combination of k points $x^1, \dots, x^k \in R^n$ is defined as $\sum_{i=1}^k \lambda_i x^i$, with $\lambda_i, \dots, \lambda_k \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$.

Definition 1.1.2

A set $X \subseteq R^n$ is said to be convex if $\forall x, y \in X$ we have that X contains all the convex combinations of x and y , i.e. $z := [\lambda x + (1 - \lambda)y], \forall \lambda \in [0, 1]$.

Proposition 1.1.1

The intersection of two convex set $A, B \in R^n$ is still a convex set.

Proof:

Given $x, y \in A \cap B$, for all $\lambda \in [0, 1]$ we have $z := \lambda x + (1 - \lambda)y \in A$ by convexity of A , $z \in B$ for convexity of B , hence $z \in A \cap B$, as requested.

Definition 1.1.3

A function $f : X \rightarrow R$ defined on a convex set $X \subseteq R^n$ is said to be convex if $\forall x, y \in X$ and $\forall \lambda \in [0, 1]$ we have that $f(z) \leq \lambda f(x) + (1 - \lambda)f(y)$, where $z = \lambda x + (1 - \lambda)y$.

Theorem 1.1.1

Let $X = \{x \in R^n : g_i(x) \leq 0, i = 1, \dots, m\}$. If for all $i \in \{1, \dots, m\}$ the functions $g_i : R^n \rightarrow R$ are convex, then the set X is convex.

Proof:

Clearly $X = \cap_{i=1}^m X_i$, where $X_i := \{x \in R^n : g_i(x) \leq 0\}$. By proposition 1.1.1, it is then sufficient to prove that each set X_i is convex. Indeed, given any two elements x and y of X_i and a generic point $z = \lambda x + (1 - \lambda)y, \lambda \in [0, 1]$, by the convexity hypothesis of the function g_i we can write $g_i(z) = g_i(\lambda x + (1 - \lambda)y) \leq \lambda g_i(x) + (1 - \lambda)g_i(y) \leq 0$, where the latter inequality is valid since $g_i(x) \leq 0, g_i(y) \leq 0$, and $0 \leq \lambda \leq 1$. It follows that $g_i(z) \leq 0$, hence $z \in X_i$. Given the arbitrariness of x, y and z , one thus has that X_i is convex, as requested.

Theorem 1.1.2

Consider a convex programming problem, i.e., a problem $\min\{f(x) : x \in X\}$ where $X \subseteq R^n$ is a convex set and $f : X \rightarrow R$ is a convex function. Every locally optimal solution is also a globally optimal solution.

Proof:

Let \tilde{x} be any locally optimal solution. By the local optimum definition, there exists then $\epsilon > 0$ such that $f(\tilde{x}) \leq f(z)$

for all $z \in I_\epsilon(\tilde{x}) := \{x \in X : |x - \tilde{x}| \leq \epsilon\}$. We have to prove that $f(\tilde{x}) \leq f(y)$ for all $y \in X$. Given any $y \in X$, consider the point z belonging to the segment that connects \tilde{x} to y and define as $z := \lambda\tilde{x} + (1 - \lambda)y$, where $\lambda < 1$ is chose very close to the value 1 so that $z \in I_\epsilon(\tilde{x})$ and hence $f(\tilde{x}) \leq f(z)$. By the convexity hypothesis of f it follows that $f(\tilde{x}) \leq f(z) = f(\lambda\tilde{x} + (1 - \lambda)y) \leq \lambda f(\tilde{x}) + (1 - \lambda)f(y)$, from which, dividing by $(1 - \lambda) > 0$, we obtain $f(\tilde{x}) \leq f(y)$, as requested.

Simplex Algorithm

Definition 4.1.1

The sets $\{x \in R^n : \alpha^T x \leq \alpha_0\}$ and $\{x \in R^n : \alpha^T x = \alpha_0\}$ are called affine half-space and hyperplane, respectively, induced by (α, α_0) .

Definition 4.1.2

A (convex) polyhedron is defined as the intersection of a finite number of affine half-spaces and hyperplanes. The sets of feasible solutions of Linear Programming problems are hence polyhedra.

Definition 4.1.3

A bounded (i.e. there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in P$) polyhedron P is called polytope.

Definition 4.1.4

A point x of a polyhedron P is said to be an extreme point or a vertex of P if it cannot be expressed as a strict convex combination of other two points of the polyhedron, i.e, if there exist no $y, z \in P, y \neq z$ and $\lambda \in (0, 1)$ such that $x = \lambda y + (1 - \lambda)z$.

Theorem 4.1.1 Minkowski-Weyl theorem

Every point of a polytope can be obtained as the convex combination of its vertices.

Theorem 4.1.2

If the set P of the feasible solutions of the linear programming problem $\min\{c^T x : x \in P\}$ is bounded, then there exists at least one optimal vertex of P .

Proof:

Let x^1, \dots, x^k be the vertices of P and $z^* := \min\{c^T x^i : i = 1, \dots, k\}$. Given any $y \in P$, we need to prove that $c^T y \geq z^*$. Indeed, $y \in P$ implies the existence of multipliers $\lambda_1, \dots, \lambda_k \geq 0, \sum_{i=1}^k \lambda_i = 1$, such that $y = \sum_{i=1}^k \lambda_i x^i$. Hence we have $c^T y = c^T \sum_{i=1}^k \lambda_i x^i = \sum_{i=1}^k \lambda_i (c^T x^i) \geq \sum_{i=1}^k \lambda_i z^* = z^*$.

Definition 4.1.5

A collection of m linearly independent columns of A is said to be a basis of A . The x_j variables associated with the basic columns are called basic variables; the remaining variables are called non-basic variables.

Definition 4.1.6

The solution obtained imposing $x_F = 0$ and $x_B = B^{-1}b$ is said to be the basic solution associated with basis B. The basic solution (and by extension, basis B itself) is said to be feasible if $x_B = B^{-1}b \geq 0$.

Definition 4.1.7

A basis B is said to be degenerate if $B^{-1}b$ has one or more zero components.

Theorem 4.1.3

A point $x \in P$ is a vertex of the not empty polyhedron $P := \{x \geq 0 : Ax = b\}$ if and only if x is a basic feasible solution of the system $Ax = b$.

Proof: