SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

We consider a basic type of system of two linear differential equations in two unknown function.

$$\frac{dx}{dt} = a_{11}(t)\,x + a_{12}(t)\,y + b_1(t) \qquad (I)$$

$$\frac{dy}{dt} = a_{21}(t)\,x + a_{22}(t)\,y + b_2(t)$$

Assuming $a_{11}, \dots, a_{22}$, b_1 and b_2 are all continuous on a real interval $a \le t \le b$. If $b_1(t) = b_2(t) = 0$ for all t, then the system (I) is called homogeneous; otherwise the system is said to be nonhomogeneous.

By a solution of the system (I) we mean an ordered pair of real functions (f, g) (II) each having continuous derivative on the real interval $a \le t \le b$, such that

$$\frac{df(t)}{dt} = a_{11}(t)\,f(t) + a_{12}(t)\,g(t) + b_1(t)$$

$$\frac{dg(t)}{dt} = a_{21}(t)\,f(t) + a_{22}(t)\,g(t) + b_2(t).$$

for all $t \in [a, b]$.

__Example__ The ordered pair of functions defined for all t by $(e^{5t}, -3e^{5t})$ is a solution of $\frac{dx}{dt} = 2x - y$ $\quad \frac{dy}{dt} = 3x + 6y$.

__Theorem:__ Let $x = f_1(t)$ $y = g_1(t)$ and $x = f_2(t)$ $y = g_2(t)$ be two solutions of the homogeneous linear system

$$\frac{dx}{dt} = a_{11}(t)\,x + a_{12}(t)\,y \qquad (III)$$

$$\frac{dy}{dt} = a_{21}(t)\,x + a_{22}(t)\,y$$

Let c_1 and c_2 be two arbitrary constants. Then $\underline{x = c_1 f_1 + c_2 f_2 \quad y = c_1 g_1 + c_2 g_2}$ (*) is also a solution of the system (III).

__Definition:__ The solution (*) is called a linear combination of the solutions.

Example: $\begin{array}{l} x = e^{5t} \\ y = -3e^{5t} \end{array}$ and $\begin{array}{l} x = e^{3t} \\ y = -e^{3t} \end{array}$ are two solutions of

$$\begin{array}{l} \dfrac{dx}{dt} = 2x - y \\ \dfrac{dy}{dt} = 3x + 6y \end{array}$$

. Thus $\begin{array}{l} x = c_1 e^{5t} + c_2 e^{3t} \\ y = -3c_1 e^{5t} - c_2 e^{3t} \end{array}$ is also a solution.

Definition: Let $\begin{array}{l} x = f_1(t) \\ y = g_1(t) \end{array}$ and $\begin{array}{l} x = f_2(t) \\ y = g_2(t) \end{array}$ be two solutions of (III).

These two solutions are linearly dependent on the interval $[a,b]$ if there exists constant c_1 and c_2, but not both zero, such that

$$c_1 f_1 + c_2 f_2 = c_1 g_1 + c_2 g_2 = 0 \quad \text{for all } t \in [a,b].$$

Otherwise these two solutions are linearly independent on $[a,b]$.

It is clear that $\begin{array}{l} x = e^{5t} \\ y = -3e^{5t} \end{array}$ and $\begin{array}{l} x = 2e^{5t} \\ y = -6e^{5t} \end{array}$ are linearly dependent by choosing $c_1 = -2c_2$ $(c_2 \in \mathbb{R} \setminus \{0\})$. However $\begin{array}{l} x = e^{5t} \\ y = -3e^{5t} \end{array}$ and $\begin{array}{l} x = e^{3t} \\ y = -e^{3t} \end{array}$ are linearly independent on $[a,b]$.

Theorem: There exist sets of two linearly independent solutions of the system (III). Every solution of the system can be written as a linear combination of a set of two linearly independent solutions.

Definition: Let $\begin{array}{l} x = f_1(t) \\ y = g_1(t) \end{array}$ and $\begin{array}{l} x = f_2(t) \\ y = g_2(t) \end{array}$ be two linearly independent solutions of the system (III). Let c_1 and c_2 be two arbitrary constants. Then the solution $\begin{array}{l} x = c_1 f_1 + c_2 f_2 \\ y = c_1 g_1 + c_2 g_2 \end{array}$ is called a general solution of the system (III).

Theorem: Let $\begin{array}{l} x = f_1(t) \\ y = g_1(t) \end{array}$ and $\begin{array}{l} x = f_2(t) \\ y = g_2(t) \end{array}$ be two solutions of the system (III). A necessary and sufficient condition that these two solutions be linearly independent on $[a,b]$ is that the determinant

$$\Delta(t) = \begin{vmatrix} f_1(t) & f_2(t) \\ g_1(t) & g_2(t) \end{vmatrix} \neq 0 \quad \text{for all } t \in [a,b].$$

Note that the determinant $\Delta(t)$ either is identically zero or vanishes for no value of t on the interval $[a,b]$.

For example $\begin{vmatrix} e^{5t} & e^{3t} \\ -3e^{5t} & -e^{3t} \end{vmatrix} = 2e^{8t} \neq 0$ for every closed interval $[a,b]$.

<u>Theorem</u>: Let $\begin{array}{l} x=F(t) \\ y=G(t) \end{array}$ be any solution of the nonhomogeneous system (I), and let $\begin{array}{l} x=f(t) \\ y=g(t) \end{array}$ be any solution of the corresponding homogeneous system (II). Then $\begin{array}{l} x=f(t)+F(t) \\ y=g(t)+G(t) \end{array}$ is also a solution of the nonhomogeneous system (I).

Then the solution $\begin{array}{l} x=c_1f_1(t)+c_2f_2(t)+F(t) \\ y=c_1g_1(t)+c_2g_2(t)+G(t) \end{array}$ is called a general solution of the system (I).

<u>Example</u> It is easy to verify that $\begin{array}{l} x=2t+1 \\ y=-t \end{array}$ is a solution of

$\frac{dx}{dt}=2x-y-5t$ $\quad \frac{dy}{dt}=3x+6y-4$. Thus the general solution of this system $(x=c_1e^{5t}+c_2e^{3t}+2t+1,\ y=-3c_1e^{5t}-c_2e^{3t}-t)$.

Homogeneous Linear Systems with Constant Coefficients.

Now we are concerned with the homogeneous linear system

$$\frac{dx}{dt}=a_1x+b_1y \qquad \frac{dy}{dt}=a_2x+b_2y. \qquad (II).$$

where $a_1,..,b_2 \in \mathbb{R}$.

Let $x=Ae^{\lambda t}$ and $y=Be^{\lambda t}$ where A, B and λ are constants be solutions of the system (II). Then

$$A\lambda e^{\lambda t}=a_1Ae^{\lambda t}+b_1Be^{\lambda t}$$
$$B\lambda e^{\lambda t}=a_2Ae^{\lambda t}+b_2Be^{\lambda t}$$

Thus
$$(a_1-\lambda)A+b_1B=0 \qquad (V)$$
$$a_2A+(b_2-\lambda)B=0.$$

This system obviously has the trivial solution $A=B=0$.

A necessary and sufficient condition that this system have a nontrivial solution is that the determinant

$$\begin{vmatrix} a_1-\lambda & b_1 \\ a_2 & b_2-\lambda \end{vmatrix} = 0.$$

or $\lambda^2 - (a_1+b_2)\lambda + (a_1b_2 - a_2b_1) = 0$. This equation is called the characteristic equation associated with the system (III). Its roots λ_1 and λ_2 are called characteristic roots.

Suppose $\lambda=\lambda_1$. Then substituting $\lambda=\lambda_1$ into the system (V), we may obtain nontrivial solution A_1, B_1. With these values, we obtain the nontrivial solution

$$f_1(t) = A_1 e^{\lambda_1 t} \qquad g_1(t) = B_1 e^{\lambda_1 t}$$

of the system (III).

(1) The roots λ_1 and λ_2 are Real and Distinct:

If so, we find $x=f_2(t)$ and $y=g_2(t)$ similarly.

Example: $\dfrac{dx}{dt} = 6x-3y$ and $\dfrac{dy}{dt} = 2x+y$.

$$\begin{vmatrix} 6-\lambda & -3 \\ 2 & 1-\lambda \end{vmatrix} = +\lambda^2 - 7\lambda + 6 + 6 = (\lambda^2 - 7\lambda + 12) = 0 \Rightarrow \lambda = 3, 4.$$

$$\left.\begin{array}{l}(6-\lambda)A_1 - 3B_1 = 0 \\ 2A_1 + (1-\lambda)B_1\end{array}\right\} \Rightarrow A_1 - B_1 = 0 \Rightarrow A_1 = B_1. \quad \text{Choose } A_1 = B_1 = 1$$

$$f_1(t) = e^{3t} \quad g_1(t) = e^{3t}$$

$$\lambda = 4 \Rightarrow \left.\begin{array}{l} 2A_2 - 3B_2 = 0 \\ 2A_1 - 3B_2 = 0\end{array}\right\} \Rightarrow 2A_2 = 3B_2 \qquad \text{Choose } A_2 = 3 \quad B_2 = 2.$$

$$f_2(t) = 3e^{4t} \quad g_2(t) = 4e^{4t}.$$

The general solution $(c_1e^{3t} + 3c_2e^{4t},\ c_1e^{3t} + 2c_2e^{4t})$.

(2) The roots λ_1 and λ_2 are Real and Equal

For the second solution we may use $x = Ate^{\lambda t}$ and $y = Bte^{\lambda t}$.

We must actually seek a second solution of the form
$x=(A_1t+A_2)e^{\lambda t}$ and $y=(B_1t+B_2)e^{\lambda t}$.

Example: $\frac{dx}{dt}=4x-y$ $\qquad$ $\frac{dy}{dt}=x+2y$

$$\begin{vmatrix} 4-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix}=\lambda^2-6\lambda+8+1=(\lambda-3)^2=0 \Rightarrow \lambda=\underline{\underline{3,3}}$$

$(a_1-\lambda)A+b_1B=0$, $\lambda=3 \Rightarrow A+(-B)=0 \Rightarrow A=B=1$

$f_1(t)=e^{3t}$ and $g_1(t)=e^{3t}$

For the second solution $x=(A_1t+A_2)e^{3t}$ and $y=(B_1t+B_2)e^{3t}$

$$\left.\begin{aligned} 3A_1t+3A_2+A_1 &= 4A_1t+4A_2-B_1t-B_2 \\ 3B_1t+3B_2+B_1 &= A_1t+A_2+2B_1t+2B_2 \end{aligned}\right\} \Rightarrow \begin{aligned} (A_1-B_1)t+(A_2-A_1-B_2)&=0 \\ (A_1-B_1)t+(A_2-B_1-B_2)&=0 \end{aligned}$$

$A_1-B_1=0 \Rightarrow \underline{A_1=B_1}$ and $A_2-B_2=A_1$ $\quad$ Choosing $A_1=1$. $A_2=1$

$\Rightarrow B_1=1$ $\quad B_2=0$.

$$\begin{aligned} x&=(t+1)e^{3t} \\ y&=te^{3t} \end{aligned} \Rightarrow \begin{aligned} x&=c_1e^{3t}+c_2(t+1)e^{3t} \\ y&=c_1e^{3t}+c_2te^{3t}. \end{aligned}$$

(3) The roots ~~λ_1 and~~ λ_2 are Conjugate Complex.

The roots λ_1 and λ_2 are the conjugate complex number $a\pm bi$.
Then the system (ii) has two real linearly independent solutions of the form.

$$\begin{aligned} x&=e^{at}(A_1\cos bt-A_2\sin bt) \\ y&=e^{at}(B_1\cos bt-B_2\sin bt) \end{aligned} \quad\text{and}\quad \begin{aligned} x&=e^{at}(A_2\cos bt+A_1\sin bt) \\ y&=e^{at}(B_2\cos bt+B_1\sin bt) \end{aligned}$$

Example: $\frac{dx}{dt}=3x+2y$ $\qquad$ $\frac{dy}{dt}=-5x+y$.

$$\begin{vmatrix} 3-\lambda & 2 \\ -5 & 1-\lambda \end{vmatrix}=\lambda^2-4\lambda+3+10=\lambda^2-4\lambda+13=(\lambda-2)^2+9=0 \Rightarrow$$

$\lambda=2\mp3i$.

Setting $\lambda = 2+3i$

$(1-3i)A + 2B = 0$ $\Rightarrow$ Choosing $A=2$ $B=-1+3i$
$-5A + (-1-3i)B = 0$

$x = 2e^{(2+3i)t} = e^{2t}\left(2\cos 3t + i(2\sin 3t)\right)$

$y = (-1+3i)e^{(2+3i)t} = e^{2t}\left[(-\cos 3t - 3\sin 3t) + i(3\cos 3t - \sin 3t)\right]$.

Since both real and imaginary parts of this solution of the system are themselves solutions of the system, thus two solutions are.

$x_1 = 2e^{2t}\cos t$ $\qquad$ $x_2 = 2e^{2t}\sin 3t$
$y_1 = -e^{2t}(\cos 3t + 3\sin 3t)$ $\qquad$ $y_2 = e^{2t}(3\cos 3t - \sin 3t)$..

The general solution

$x = 2e^{2t}(c_1\cos 3t + c_2\sin 3t)$
$y = e^{2t}\left(-c_1(\cos 3t + 3\sin 3t) + c_2(3\cos 3t - \sin 3t)\right)$.

Exercises

Find the general solution of each of the following.

1) Find the general solution of each of the following.

1) $\frac{dx}{dt} = 5x - 2y$, $\frac{dy}{dt} = 4x - y$ $\qquad$ 2) $\frac{dx}{dt} = x + 2y$, $\frac{dy}{dt} = 3x + 2y$ $\qquad$ 3) $\frac{dx}{dt} = 3x + y$, $\frac{dy}{dt} = 4x + 3y$.

<u>Solution:</u> $\begin{vmatrix} 3-\lambda & 1 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 - 4 = \lambda^2 + 6\lambda + 5 = (\lambda-1)(\lambda-5) = 0 \Rightarrow \lambda = 1, 5$

$\lambda = 5 \Rightarrow$ $\begin{matrix} -2A + B = 0 \\ 4A - 2B = 0 \end{matrix}$ $\Rightarrow B = 2A \Rightarrow A = 1$ $B = 2$ $\Rightarrow$ $x_1 = e^{5t}$, $y_1 = 2e^{5t}$

$\lambda = 1 \Rightarrow$ $\begin{matrix} 2A + B = 0 \\ 4A + 2B = 0 \end{matrix}$ $\Rightarrow B = -2A \Rightarrow A = 1$ $B = 2$ $\Rightarrow$ $x_2 = e^{t}$, $y_2 = -2e^{t}$

$x = c_1 e^{5t} + c_2 e^{t}$
$y = 2c_1 e^{5t} - 2c_2 e^{t}$.

4) $\dfrac{dx}{dt} = x+3y$
$\dfrac{dy}{dt} = 3x+y$

5) $\dfrac{dx}{dt} = 3x-y$
$\dfrac{dy}{dt} = 4x-y$

6) $\dfrac{dx}{dt} = 5x+4y$
$\dfrac{dy}{dt} = -x+y.$

Solution: $\begin{vmatrix} 5-\lambda & +4 \\ -1 & 1-\lambda \end{vmatrix} = \lambda^2-6\lambda+5+4 = (\lambda-3)^2=0 \Rightarrow \lambda=3,3$

$\lambda=3 \Rightarrow \begin{array}{l} 2A+4B=0 \\ -A-2B=0 \end{array} \Rightarrow A=-2B \Rightarrow B=1,\ A=-2$ $\quad \begin{array}{l} x_1=-2e^{3t} \\ y_1=e^{3t}. \end{array}$

$x=(A_1t+A_2)e^{3t} \Rightarrow \dfrac{dx}{dt} = A_1e^{3t} + 3(A_1t+A_2)e^{3t}$

$y=(B_1t+B_2)e^{3t} \Rightarrow \dfrac{dy}{dt} = B_1e^{3t} + 3(B_1t+B_2)e^{3t}$

By substituting.

$\left.\begin{array}{l} (3A_1t+A_1+3A_2)e^{3t} = (5A_1t+5A_2+4B_1t+4B_2)e^{3t} \\ (3B_1t+B_1+3B_2)e^{3t} = (-A_1t-A_2+B_1t+B_2)e^{3t} \end{array}\right\} \Rightarrow$

$\left.\begin{array}{l} (2A_1+4B_1)t-A_1+2A_2+4B_2=0 \\ (-A_1-2B_1)t+(B_1-2B_2-A_2)=0 \end{array}\right\} \Rightarrow \left.\begin{array}{ll} 2A_1+4B_1=0 & -A_1+2A_2+4B_2=0 \\ -A_1-2B_1=0 & -B_1-A_2-2B_2=0 \end{array}\right\}$

$\Rightarrow A_1=-2B_1,\quad -B_1=A_2-2B_2$ $\quad$ By choosing. $B_1=B_2=1$

$A_1=-2\quad A_2=1.$ $\quad \begin{array}{l} x_2=(-2t+1)e^{3t} \\ y_2=(t+1)e^{3t}. \end{array}$

$\Rightarrow \begin{array}{l} x=-2c_1e^{3t}+c_2(-2t+1)e^{3t} \\ y=c_1e^{3t}+c_2(t+1)e^{3t}. \end{array}$

7) $\dfrac{dx}{dt} = x-2y$ $\qquad \dfrac{dy}{dt} = 2x-3y.$

$\begin{vmatrix} 1-\lambda & -2 \\ 2 & -3-\lambda \end{vmatrix} = \lambda^2+2\lambda-3+4 = (\lambda+1)^2=0 \Rightarrow \lambda=-1,-1$

$\begin{array}{l} 2A-2B=0 \\ 2A-2B=0 \end{array} \Rightarrow B=A \quad A=1 \Rightarrow B=1$ $\quad \boxed{\begin{array}{l} x_1=e^{-t} \\ y_1=e^{-t}. \end{array}}$

$x = (A_1 t + A_2) e^{-t} \Rightarrow \frac{dx}{dt} = A_1 e^{-t} - (A_1 t + A_2) e^{-t}$

$y = (B_1 t + B_2) e^{-t} \qquad \frac{dy}{dt} = B_1 e^{-t} - (B_1 t + B_2) e^{-t}$

$A_1 - (A_1 t + A_2) = (A_1 t + A_2) - 2(B_1 t + B_2) \Rightarrow (2A_1 - 2B_1)t - A_1 + 2A_2 - 2B_2 = 0$

$B_1 - (B_1 t + B_2) = 2(A_1 t + A_2) - 3(B_1 t + B_2) \Rightarrow (2A_1 - 2B_1)t - B_1 - 2B_2 + 2A_2 = 0$

$\Rightarrow 2A_1 - 2B_1 = 0 \quad -A_1 + 2A_2 - 2B_2 = 0 \Rightarrow A_1 = B_1 \quad 2A_2 - 2B_2 = A_1$

$A_1 = 2$ al. $B_2 = 0 \Rightarrow B_1 = 2 \quad A_2 = 1$

$x_2 = (2t+1)e^{-t} \qquad y_2 = 2te^{-t}$

$x = c_1 e^{-t} + c_2 (2t+1) e^{-t}$

$y = c_1 e^{-t} + 2c_2 t e^{-t}$

8) $\frac{dx}{dt} = x - 4y$

$\frac{dy}{dt} = x + y$

9) $\frac{dx}{dt} = x - 3y$

$\frac{dy}{dt} = 3x + y$

10) $\frac{dx}{dt} = 4x - 2y$

$\frac{dy}{dt} = 5x + 2y$.

Solution. $\begin{vmatrix} 4-\lambda & -2 \\ 5 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 8 + 10 = (\lambda-3)^2 + 9 = 0 \quad \lambda = 3 \mp 3i$

$\lambda = 3+3i \Rightarrow (4-3-3i)A - 2B = 0 \qquad (1-3i)A - 2B = 0$

$5A + (2-3-3i)B = 0 \qquad 5A - (1+3i)B = 0$

Choosing $A = 2 \Rightarrow B = 1-3i$

$x = 2e^{(3+3i)t} = 2e^{3t}[\cos 3t + i\sin 3t] = e^{3t}(2\cos 3t + 2i\sin 3t)$

$y = (1-3i)e^{(3+3i)t} = (1-3i)e^{3t}[\cos 3t + i\sin 3t] = e^{3t}((\cos 3t + 3\sin 3t) + i(\sin 3t - 3\cos 3t))$

$\Rightarrow x_1 = e^{3t} 2\cos 3t \qquad x_2 = e^{3t} 2\sin 3t$

$y_1 = e^{3t}(\cos 3t + 3\sin 3t) \qquad y_2 = e^{3t}(\sin 3t - 3\cos 3t)$.

$\Rightarrow x = 2e^{3t}[c_1 \cos 3t + c_2 \sin 3t]$

$y = e^{3t}[c_1(\cos 3t + 3\sin 3t) + c_2(\sin 3t - 3\cos 3t)]$

11) $\frac{dx}{dt} = 3x - 2y$, $\frac{dy}{dt} = 2x + 3y$

Solution: $\begin{vmatrix} 3-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 + 4 = (\lambda-3)^2 + 4 = 0 \quad \lambda = 3 \mp 2i$

$\lambda = 3+2i$

$$-2iA - 2B = 0 \Rightarrow B = -Ai \qquad A = 1 \Rightarrow B = -i.$$
$$2A - 2iB = 0$$

$x = e^{(3+2i)t} = e^{3t}(\cos 2t + i\sin 2t)$

$y = i\,e^{(3+2i)t} = -i\,e^{3t}(\cos 2t + i\sin 2t) = e^{3t}(+\sin 2t - i\cos 2t).$

$$x = e^{3t}(c_1\cos 2t + c_2\sin 2t)$$
$$y = e^{3t}(+c_1\sin 2t - c_2\cos 2t).$$

12) $t\frac{dx}{dt} = x + y$ 13) $t\frac{dx}{dt} = 2x + 3y$

$t\frac{dy}{dt} = -3x + 5y$ $t\frac{dy}{dt} = 2x + y.$

Solution: $t = e^u \Rightarrow \frac{dt}{du} = e^u$ $\frac{dx}{dt} = \frac{dx}{du}\cdot\frac{du}{dt} = \frac{1}{t}\cdot\frac{dx}{du}.$

$u = \ln t \Rightarrow \frac{du}{dt} = \frac{1}{t}$

Substating. $\frac{dx}{du} = 2x + 3y$ and $\frac{dy}{du} = 2x + y$

$\begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 - 6 = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4) \Rightarrow \lambda = -1, \lambda = 4.$

$\lambda = -1 \Rightarrow$ $\begin{array}{l} 3A + 3B = 0 \\ 2A + 2B = 0 \end{array} \Rightarrow B = -A$ $\quad x_1 = e^{-u}$

$A = 1 \Rightarrow B = -1$ $\quad y_1 = -e^{-u}$

$\lambda = 4 \Rightarrow$ $\begin{array}{l} -2A + 3B = 0 \\ 2A - 3B = 0 \end{array} \Rightarrow 3B = 2A$ $\quad x_2 = 3e^{4u}$

$A = 3 \Rightarrow B = 2$ $\quad y_2 = 2e^{4u}.$

$$x = c_1 e^{-u} + 3c_2 e^{4u} = c_1\cdot\frac{1}{t} + 3c_2 t^4$$
$$y = -c_1 e^{-u} + 2c_2 e^{4u}. = -c_1\cdot\frac{1}{t} + 2c_2 t^4.$$

(17) $\frac{dx}{dt} = x+y-z$

$\frac{dy}{dt} = 2x+3y-4z$

$\frac{dz}{dt} = 4x+y-4z$

15) $\frac{dx}{dt} = x-y-z$

$\frac{dy}{dt} = x+3y+z$

$\frac{dz}{dt} = -3x-6y+6z$

16) $\frac{dx}{dt} = x-y-z$

$\frac{dy}{dt} = x+3y+z$.

$\frac{dz}{dt} = -3x+y-z$.

Solution: $\begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & 3-\lambda & 1 \\ -3 & 1 & -1-\lambda \end{vmatrix} = [(1-\lambda)(3-\lambda)(-1-\lambda)-1+3]-[3(3-\lambda)+(1-\lambda)$

$\begin{matrix} 1-\lambda & -1 & -1 \\ +1 & 3-\lambda & 1 \end{matrix}$

$+1+\lambda] = (-\lambda^3+3\lambda^2+\lambda-\cancel{3}-1+\cancel{3})-(11-3\lambda)$

$= -\lambda^3+3\lambda^2+4\lambda-12=0$

$\Rightarrow \lambda^3-3\lambda^2-4\lambda+12=0$ $\quad \underline{\lambda=2}$ $\quad \underline{\lambda=-2,2,3}$.

$\lambda^3-3\lambda^2-4\lambda+12 \,|\, \lambda-2$

$\lambda^2-\lambda-6 = (\lambda-3)(\lambda+2)$

$-\ \lambda^3-2\lambda^2$

$-\lambda^2-4\lambda+12$

$-\ -\lambda^2+2\lambda$

$-6\lambda+12$

$\left.\begin{matrix} 3A-B-C=0 \\ A+5B+C=0 \\ -3A+B+C=0 \end{matrix}\right\} \Rightarrow \underline{A=1}$ $\quad \underline{C=4}$

$B+C=3$

$-\ 5B+C=-1$

$-4B=4$

$\underline{B=-1}$

$x_1 = e^{-2t}$

$y_1 = -e^{-2t}$

$z_1 = 4e^{-2t}$.

$\lambda=2 \Rightarrow \begin{matrix} -A-B-C=0 \\ A+B+C=0 \\ -3A+B-3C=0 \end{matrix}$ $\quad \underline{A=1}$

$B+C=-1$

$-\ B-3C=3$

$4C=-4$

$\underline{C=-1}$

$B=0$.

$x_2 = e^{2t}$

$y_2 = 0$

$z_2 = -e^{2t}$.

$\lambda=3 \Rightarrow \begin{matrix} -2A-B-C=0 \\ A+C=0 \\ -3A+B-4C=0 \end{matrix}$ $\quad A=1$

$B+C=-2 \Rightarrow$ $B=-1$

$C=-1$

$B-4C=+3$

$x_3 = e^{3t}$

$y_3 = -e^{3t}$

$y_4 = -e^{3t}$.

$x = c_1 e^{-2t} + c_2 e^{2t} + c_3 e^{3t}$

$y = -c_1 e^{-2t} - c_3 e^{3t}$

$z = 4c_1 e^{-2t} - c_2 e^{2t} - c_3 e^{3t}$.

and

$$(7.43) \qquad \begin{cases} x = 2e^{2t}\sin 3t \\ y = e^{2t}(3\cos 3t - \sin 3t). \end{cases}$$

Finally, since the two solutions (7.42) and (7.43) are linearly independent we may write the general solution of the system (7.39) in the form

$$\begin{cases} x = 2e^{2t}(c_1\cos 3t + c_2\sin 3t) \\ y = e^{2t}[c_1(-\cos 3t - 3\sin 3t) + c_2(3\cos 3t - \sin 3t)], \end{cases}$$

where c_1 and c_2 are arbitrary constants.

Exercises

Find the general solution of each of the linear systems in Exercises 1 through 22.

1. $\begin{cases} \dfrac{dx}{dt} = 5x - 2y \\ \dfrac{dy}{dt} = 4x - y \end{cases}$

2. $\begin{cases} \dfrac{dx}{dt} = 5x - y \\ \dfrac{dy}{dt} = 3x + y \end{cases}$

3. $\begin{cases} \dfrac{dx}{dt} = x + 2y \\ \dfrac{dy}{dt} = 3x + 2y \end{cases}$

4. $\begin{cases} \dfrac{dx}{dt} = 2x + 3y \\ \dfrac{dy}{dt} = -x - 2y \end{cases}$

5. $\begin{cases} \dfrac{dx}{dt} = 3x + y \\ \dfrac{dy}{dt} = 4x + 3y \end{cases}$

6. $\begin{cases} \dfrac{dx}{dt} = 6x - y \\ \dfrac{dy}{dt} = 3x + 2y \end{cases}$

7. $\begin{cases} \dfrac{dx}{dt} = 3x - 4y \\ \dfrac{dy}{dt} = 2x - 3y \end{cases}$

8. $\begin{cases} \dfrac{dx}{dt} = 2x - y \\ \dfrac{dy}{dt} = 9x + 2y \end{cases}$

9. $\begin{cases} \dfrac{dx}{dt} = x + 3y \\ \dfrac{dy}{dt} = 3x + y \end{cases}$

10. $\begin{cases} \dfrac{dx}{dt} = 3x + 2y \\ \dfrac{dy}{dt} = 6x - y \end{cases}$

11. $\begin{cases} \dfrac{dx}{dt} = 3x - y \\ \dfrac{dy}{dt} = 4x - y \end{cases}$

12. $\begin{cases} \dfrac{dx}{dt} = 4x - y \\ \dfrac{dy}{dt} = x + 2y \end{cases}$

13. $\begin{cases} \dfrac{dx}{dt} = 5x + 4y \\ \dfrac{dy}{dt} = -x + y \end{cases}$

14. $\begin{cases} \dfrac{dx}{dt} = x - 2y \\ \dfrac{dy}{dt} = 2x - 3y \end{cases}$

15. $\begin{cases} \dfrac{dx}{dt} = x - 4y \\ \dfrac{dy}{dt} = x + y \end{cases}$

16. $\begin{cases} \dfrac{dx}{dt} = 2x - 3y \\ \dfrac{dy}{dt} = 3x + 2y \end{cases}$

17. $\begin{cases} \dfrac{dx}{dt} = x - 3y \\ \dfrac{dy}{dt} = 3x + y \end{cases}$

18. $\begin{cases} \dfrac{dx}{dt} = 5x - 4y \\ \dfrac{dy}{dt} = 2x + y \end{cases}$

19. $\begin{cases} \dfrac{dx}{dt} = 4x - 2y \\ \dfrac{dy}{dt} = 5x + 2y \end{cases}$

20. $\begin{cases} \dfrac{dx}{dt} = x - 5y \\ \dfrac{dy}{dt} = 2x - y \end{cases}$

21. $\begin{cases} \dfrac{dx}{dt} = 3x - 2y \\ \dfrac{dy}{dt} = 2x + 3y \end{cases}$

22. $\begin{cases} \dfrac{dx}{dt} = 6x - 5y \\ \dfrac{dy}{dt} = x + 2y \end{cases}$

23. Consider the linear system

$$\begin{cases} t\dfrac{dx}{dt} = a_1x + b_1y \\ t\dfrac{dy}{dt} = a_2x + b_2y, \end{cases}$$

where a_1, b_1, a_2, and b_2 are real constants. Show that the transformation $t = e^w$ transforms this system into a linear system with constant coefficients.

24. Use the result of Exercise 23 to solve the system

$$\begin{cases} t\dfrac{dx}{dt} = x + y \\ t\dfrac{dy}{dt} = -3x + 5y. \end{cases}$$

25. Use the result of Exercise 23 to solve the system

$$\begin{cases} t\dfrac{dx}{dt} = 2x + 3y \\ t\dfrac{dy}{dt} = 2x + y. \end{cases}$$

26. Consider the linear system

$$\begin{cases} \dfrac{dx}{dt} = a_1x + b_1y \\ \dfrac{dy}{dt} = a_2x + b_2y \end{cases}$$

where a_1, b_1, a_2, and b_2 are real constants. Show that the condition $a_2b_1 > 0$ is sufficient, but not necessary, for the system to have two real linearly independent solutions of the form

$$\begin{cases} x = Ae^{\lambda t} \\ y = Be^{\lambda t}. \end{cases}$$

Extend the methods of this section to solve the systems of three differential equations in Exercises 27 through 29.

27. $$\begin{cases} \dfrac{dx}{dt} = x + y - z \\ \dfrac{dy}{dt} = 2x + 3y - 4z \\ \dfrac{dz}{dt} = 4x + y - 4z \end{cases}$$

28. $$\begin{cases} \dfrac{dx}{dt} = x - y - z \\ \dfrac{dy}{dt} = x + 3y + z \\ \dfrac{dz}{dt} = -3x - 6y + 6z \end{cases}$$

29. $$\begin{cases} \dfrac{dx}{dt} = x - y - z \\ \dfrac{dy}{dt} = x + 3y + z, \\ \dfrac{dz}{dt} = -3x + y - z \end{cases}$$

7.3 An Operator Method

A. Differential Operators

In this section we shall present a symbolic operator method for solving linear systems with constant coefficients. This method depends upon the use of so-called *differential operators*, which we now introduce.

Let x be an n-times differentiable function of the independent variable t. We denote the operation of differentiation with respect to t by the symbol D and call D a differential operator. In terms of this differential operator the derivative dx/dt is denoted by Dx. That is,

$$Dx \equiv dx/dt.$$

In like manner, we denote the second derivative of x with respect to t by D^2x. Extending this, we denote the nth derivative of x with respect to t by D^nx. That is,

$$D^nx = \frac{d^nx}{dt^n} \; (n = 1, 2, \cdots).$$

Further extending this operator notation, we write

$$(D + c)x \quad \text{to denote} \quad \frac{dx}{dt} + cx$$

and

$$(aD^n + bD^m)x \quad \text{to denote} \quad a\frac{d^nx}{dt^n} + b\frac{d^mx}{dt^m},$$

where a, b, and c are constants.