An Operator Method.

Let x be an n-times differentiable function of the independent variable t. We denote the operation of differentiation with respect to t by the symbol D and call D a differential operator. So

$$Dx \equiv \frac{dx}{dt}, \quad D^2x \equiv \frac{d^2x}{dt^2} \quad \dots \quad D^nx \equiv \frac{d^nx}{dt^n}$$

Further extending this operator notation,

$$(D+c)x \equiv \frac{dx}{dt} + cx \qquad \text{and} \qquad (aD^n + bD^m)x = a\frac{d^nx}{dt^n} + b\frac{d^mx}{dt^m}.$$

In this notation

$$a_0 \frac{d^nx}{dt^n} + a_1 \frac{d^{n-1}x}{dt^{n-1}} + \dots + a_{n-1}\frac{dx}{dt} + a_n x$$

can be written

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1}D + a_n)x.$$

Example Consider the linear differential operator

$$3D^2 + 5D - 2$$

and $x = t^3$. We have, since $\frac{dx}{dt} = 3t^2 \quad \frac{d^2x}{dt^2} = 6t$

$$(3D^2 + 5D - 2)x = 18t + 15t^2 - t^3$$

We denote the operator $a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1}D + a_n$ by L, that is $L \equiv a_0 D^n + \dots + a_n$.

Now suppose that f_1 and f_2 are both n-times differentiable functions of t and c_i are constants. It is easy to show that

$$L(c_1 f_1 + c_2 f_2) = c_1 Lf_1 + c_2 Lf_2.$$

Now let $L_1 \equiv \sum_{i=0}^{m} a_i D^{m-i}$ and $L_2 \equiv \sum_{j=0}^{n} b_j D^{n-j}$

be two linear differential operators. Let $L_1(r) = \sum_{i=0}^{m} a_i r^{m-i}$

and $L_2(r) = \sum_{j=0}^{n} b_j r^{n-j}$ be the two polynomial in r. Let us denote

the product of the polynomial $L_1(r)$ and $L_2(r)$ by $L(r)$; that is

$$L(r) = L_1(r) \cdot L_2(r).$$

Thm, if f is a function possessing $n+m$ derivatives, it can be shown that

$$L_1 L_2 f = L_2 L_1 f = L f.$$

We now consider a linear system of the form.

$$L_1 x + L_2 y = f_1(t) \qquad (I).$$
$$L_3 x + L_4 y = f_2(t)$$

where $L_1 \equiv \sum_{i=0}^{m} a_i D^{m-i}$, $L_2 \equiv \sum_{\dot{\delta}=0}^{n} b_{\delta} D^{n-\dot{\delta}}$, $L_3 \equiv \sum_{k=0}^{p} d_k D^{p-k}$

and $L_4 \equiv \sum_{l=0}^{q} \beta_l D^{q-l}$

A simple example of a system which may be expressed in the form (I).

For example

$$2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t \implies 2\frac{dv}{dt} - 3x - 2\frac{dy}{dt} = t.$$

$$2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y = 2 \implies 2\frac{dx}{dt} + 3x + 2\frac{dy}{dt} + 8y = 2.$$

$$\implies (2D - 3)x - 2Dy = t$$
$$(2D + 3)x + (2D + 8)y = 2.$$

Returning the system (1), we apply the operator L_4 to the first equation of (1) and the operator L_2 to the second equation of (1), obtaining

$$L_4 L_1 x + L_4 L_2 y = L_4 f_1$$
$$L_2 L_3 x + L_2 L_4 y = L_2 f_2$$
$$\overline{\qquad\qquad\qquad\qquad\qquad}$$
$$L_4 L_1 x - L_2 L_3 x = L_4 f_1 - L_2 f_2$$

Since $L_4 L_2 = L_2 L_4$

$$(L_4 L_1 - L_2 L_3)x = L_4 f_1 - L_2 f_2.$$

or

Let $L_5 = L_4 L_1 - L_2 L_3$ and $g(t) = L_4 f_1 - L_2 f_2$. Then we have

$$L_5 x = g_1$$

②

which is a linear differential equation with constant coefficients.

We can solve the equation $L_5 x = \delta_1$.

Similarly we first obtain the equation $L_5 y = \delta_2$ and then solve this equation.

Example: Solve the system

$$2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t$$

$$2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y = 2.$$

Solution:
$$(2D-3)x - 2Dy = t \qquad \backslash(2D+8)$$
$$(2D+3)x + (2D+8)y = 2 \quad .\backslash 2D$$

$$\Rightarrow \quad (2D+8)(2D-3)x - 2D(2D+8)y = (2D+8)t$$
$$+ \quad 2D(2D+3)x + 2D(2D+8)y = 2D.2$$
$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$
$$(4D^2 - 6D + 16D - 24 + 4D^2 + 6D)x = 2 + 8t$$
$$(8D^2 + 16D - 24)x = 8t + 2$$
$$(D^2 + 2D - 3)x = t + \tfrac{1}{4}.$$

$$m^2 + 2m - 3 = (m+3)(m-1) = 0 \Rightarrow m = -3 \quad m = 1$$

$$y_c = c_1 e^{-3t} + c_2 e^{t}$$

$$y_p = At + B \Rightarrow y_p' = A \qquad y_p'' = 0$$
$$2A - 3(At + B) = t + \tfrac{1}{4}. \quad \Rightarrow \quad -3A = 1 \qquad 2A - 3B = \tfrac{1}{4}$$
$$A = -\tfrac{1}{3} \qquad B = -\tfrac{11}{36}$$

$$\cancel{x} = c_1 e^{t} + c_2 e^{-3t} - \tfrac{1}{3}t - \tfrac{11}{36}.$$

$$- (2D+3)(2D-3)x + 2D(2D+3)y = -(2D+3)t$$
$$+ \quad (2D+3)(2D-3)x + (2D+8)(2D-3)y = (2D-3)2$$
$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$
$$(4D^2 + 6D + 4D^2 - 6D + 16D - 24)y = -2 + 3t - 6$$
$$(8D^2 + 16D - 24)y = -3t - 8$$

$$y_p = At + B \Rightarrow 16A - 24At - 24B = -3t - 8 \qquad A = \tfrac{-3}{-24} = \tfrac{1}{8}$$
$$-24B = -10 \Rightarrow B = \tfrac{5}{12}.$$

$$y = k_1 e^t + k_2 e^{-3t} + \frac{1}{8}t + \frac{5}{12}.$$

$$\begin{vmatrix} 2D-3 & -2D \\ 2D+3 & 2D+8 \end{vmatrix} = 8D^2 + 16D - 24.$$ Since this is of order two, the number of independent constants in the general solution of the system must also be two.

By substituting in the first equation,

$$\left(2C_1 e^t + 2C_2 e^{-3t} - \frac{2}{3}\right) + \left(-2k_1 e^t - 2k_2 e^{-3t} - \frac{1}{4}\right)^{-3C_1 e^t - 3C_2 e^{-3t} + t + \frac{11}{12}t}$$

$$(+C_1 - 2k_1)e^t + (-C_2 - 2k_2)e^{-3t} + t - \frac{2}{3} - \frac{1}{4} + \frac{11}{12} = t$$

$$(-C_1 - 2k_1)(e^t + e^{-3t}) + t = t$$

$$-C_1 - 2k_1 = 0 \qquad C_1 = -2k_1. \quad \text{or} \quad k_1 = -\frac{1}{2}C_1.$$
$$-C_2 - 2k_2 = 0 \qquad C_2 = -2k_2 \quad \text{or} \quad k_2 = -\frac{1}{2}C_2.$$

The general solution

$$x = C_1 e^t + C_2 e^{-3t} - \frac{1}{3}t - \frac{11}{36}$$

$$y = -\frac{1}{2}C_1 e^t - \frac{1}{2}C_2 e^{-3t} + \frac{1}{8}t + \frac{5}{12}. \qquad \boxed{3}$$

④

Exercises

1) $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2x - 4y = e^t$ $\qquad$ $\dfrac{dx}{dt} + \dfrac{dy}{dt} - y = e^{4t}$

$\left. \begin{array}{l} (D-2)x + (D-4)y = e^t \\ Dx + (D-1)y = e^{4t} \end{array} \right\} \Rightarrow$ $\begin{array}{l} (D^2 - 3D + 2)x = (D-1)e^t \\ (-D^2 + 4D)x = (-D+4)e^{4t} \end{array}$

$\underline{\qquad\qquad\qquad}$

$(D+2)x = 0$

$m + 2 = 0 \Rightarrow m = -2$ $\quad$ $\underline{x = ce^{-2t}}$

$-D(D-2) + (-D^2 + 4D)y = -D \cdot e^t \qquad = -e^t$

$(D-2)Dx + (D^2 - 3D + 2)y = (D-2)e^{4t} = 4e^{4t} - 2e^{4t} = 2e^{4t}$

$\underline{\qquad\qquad}$

$(D+2)y = 2e^{4t} - e^t \qquad \Rightarrow \qquad \dfrac{dy}{dt} + 2y = 2e^{4t} - e^t$

$D + 2 \neq 0 \Rightarrow m = -2$ $\quad$ $\underline{y_c = e^{-2t}}$ $\qquad$ $4Ae^{4t} + Be^t + 2Ae^{4t} + 2Be^t$

$y_p = Ae^{4t} + Be^t \Rightarrow \dfrac{dy_p}{dt} = 4Ae^{4t} + Be^t$ $\nearrow$ $= 6Ae^{4t} + 3Be^t = 2e^{4t} - e^t$

$\qquad\qquad\qquad\qquad$ $A = \frac{1}{3}$ $B = -\frac{1}{3}$

$\qquad \underline{y = ke^{-2t} + \frac{1}{3}e^{4t} - \frac{1}{3}e^t}$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2x - 4y = e^t \Rightarrow$ $-2ce^{-2t} + 2ke^{-2t} + \frac{4}{3}e^{4t} - \frac{1}{3}e^t - 2ce^{-2t}$

$-4ke^{-2t} - \frac{4}{3}e^{4t} + \frac{4}{3}e^t = (-4c - 6k)e^{-2t} + e^t = e^t$

$(-4c - 6k) = 0 \Rightarrow k = -\frac{2}{3}c.$

$\qquad \left(ce^{-2t}, \; -\frac{2}{3}ce^{-2t} + \frac{1}{3}e^{4t} - \frac{1}{3}e^t \right)$

2) $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x = -2t$ $\qquad$ $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 3x - y = t^2$

3) $\dfrac{dx}{dt} + \dfrac{dy}{dt} + x = e^{3t}$ $\qquad$ $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - 3y = e^t$

$(D+1)x + Dy = e^{3t}$ $\qquad$ $\big| \begin{smallmatrix} D & -3 \\ 1 & -D \end{smallmatrix}$ $\qquad$ $(D+1)(D-3)x = (D-3)e^{3t}$

$(D-1)x + (D-3)y = e^t$ $\qquad \big|$ $\qquad$ $-D(D-1)x = (-D)e^t$

$(D^2-2D-3)x=0$

$(-2D^2+D)x=-e^t$ $\Rightarrow$ $(-D-3)x=-e^t$

$$x_c=ce^{-3t}$$

$x_p=Ae^t$ $\Rightarrow$ $\quad -Ae^t-3Ae^t=-4Ae^t=-e^t$ $\Rightarrow$ $A=\frac{1}{4}$.

$$\boxed{x=ce^{-3t}+\frac{1}{4}e^t}$$

$-(D-1)(D+1)x-D(D-1)y=-(D-1)e^{3t}$ $\quad = -3e^{3t}+e^{3t}=-2e^{3t}$

$(D-1)(D+1)x+(D+1)(D-3)y=(D+1)(e^t) = 2e^t$

$(-D-3)y=-2e^{3t}+2e^t$ $\Rightarrow$ $\frac{dy}{dt}+3y=2e^{3t}-2e^t$

$y_p=Ae^{3t}+Be^t$ $\Rightarrow$ $y_p'=3Ae^{3t}+Be^t$ $\Rightarrow$ $3Ae^{3t}+Be^t+3Ae^{3t}+3Be^t$

$\quad = 6Ae^{3t}+4Be^t=2e^{3t}-2e^t$ $\Rightarrow$ $A=\frac{1}{3}$ $\quad B=-\frac{1}{2}$

$$\boxed{y=ke^{-3t}+\frac{1}{3}e^{3t}-\frac{1}{2}e^t}$$

$-3ce^{-3t}+\frac{1}{4}e^t-3ke^{-3t}+e^{3t}-\frac{1}{2}e^t+ce^{-3t}+\frac{1}{4}e^t=e^{3t}$

$(-2c-3k)e^{-3t}=0$ $\Rightarrow$ $k=-\frac{2}{3}$

$(-2D+1)(D+1)$

$-2D^2-2D+D+1$

$(ce^{-3t}+\frac{1}{4}e^t, -\frac{2}{3}e^{-3t}+\frac{1}{3}e^{3t}-\frac{1}{2}e^t)$

4) $\frac{dx}{dt}+\frac{dy}{dt}-x-2y=2e^t$ $\qquad\qquad \frac{dx}{dt}+\frac{dy}{dt}-3x-4y=e^{2t}$

5) $2\frac{dx}{dt}+\frac{dy}{dt}-x-y=e^{-t}$ $\qquad\qquad \frac{dx}{dt}+\frac{dy}{dt}+2x+y=e^t$

$(2D-1)x+(D-1)y=e^{-t}$ $\quad |_{D+1}$ $\qquad (2D^2+D-1)x=-e^{-t}+e^t=0$

$(D+2)x+(D+1)y=e^t$ $\quad |_{-D+1}$ $\qquad (-D^2-D+2)x=-e^t+e^t=0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{+}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (D^2+1)x=0$

$m^2+1=0$ $\quad m=\pm i$ $\qquad x=\cancel{e^{\pm it}}$ $\quad \underline{x=c_1\sin t+c_2\cos t}$

$(D^2+D-2)y=(D+2)e^{-t}=-e^{-t}+2e^{-t}=e^{-t}$

$(-2D^2-D+1)y=(-2D+1)e^t=-2e^t+e^t=-e^t$

(6)

$(-D^2-1)y = -e^t + e^{-t} \Rightarrow (D^2+1)y = e^t - e^{-t}$

$y_c = k_1 \sin t + k_2 \cos t$

$y_p = Ae^t + Be^{-t} \Rightarrow y_p'' = Ae^t + Be^{-t} \Rightarrow 2Ae^t + 2Be^{-t} = e^t - e^{-t}$

$A = \frac{1}{2} \quad B = -\frac{1}{2} \quad y = k_1 \sin t + k_2 \cos t + \frac{1}{2}e^t - \frac{1}{2}e^{-t}$

$(2) \Rightarrow \underline{c_1 \cos t - c_2 \sin t} + \underline{k_1 \cos t - k_2 \sin t} + \frac{1}{2}e^t + \frac{1}{2}e^{-t} + 2c_1 \sin t + 2c_2 \cos t$

$k_1 \sin t + k_2 \cos t + \frac{1}{2}e^t - \frac{1}{2}e^{-t} = e^t$

$(c_1 + k_1 + 2c_2 + k_2)\cos t + (-c_2 - k_2 + 2c_1 + k_1)\sin t = 0$

$k_1 + k_2 = -c_1 - 2c_2 \qquad k_1 = \frac{-3c_1 - c_2}{2} \qquad k_2 = \frac{c_1 - 3c_2}{2}$

$k_1 - k_2 = -2c_1 + c_2$

6) $\quad 2\frac{dx}{dt} + \frac{dy}{dt} - 3x - y = t \qquad 1) - \frac{dx}{dt} + \frac{dy}{dt} - 4x - y = e^t$

$(-D+6)(D-2)$

$-D^2 + 2D + 6D - 12)$

?) $\quad \frac{dx}{dt} + \frac{dy}{dt} - x - 6y = e^{3t} \qquad \frac{dx}{dt} + 2\frac{dy}{dt} - 2x - 6y = t$

$-D \cdot 1$

$(D-1)x + (D-6)y = e^{3t} \qquad (2D^2 - 8D + 6)x = (2D - 6)e^{3t}$

$(D-2)x + (2D-6)y = t \qquad \underline{(-D^2 + 8D - 12)x = (-D+6)t}$

$(D^2 - 6)x = 6e^{3t} - 6e^{3t} - 1 + 6t = 6t - 1$

$m^2 - 6 = 0 \Rightarrow m = \pm\sqrt{6} \Rightarrow x_c = c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t}$

$x_p = At + B \Rightarrow x_p'' = 0 \Rightarrow \frac{d^2x}{dt^2} - 6x = -6At - 6B = 6t - 1 \Rightarrow A = +1 \quad B = -\frac{1}{6}$

$x = c_1 e^{\sqrt{6}t} + c_2 e^{-\sqrt{6}t} + t + \frac{-1}{6}$

$(D^2 - 8D + 12)y = (D-2)e^{3t} = 3e^{3t} - 2e^{3t} = e^{3t}$

$(-2D^2 + 8D - 6)y = (-D+1)t = -1 + t$

$\Rightarrow (-D^2 + 6)y = e^{3t} + t + 1$

$y_p = Ae^{3t} + Bt + c \Rightarrow y_p'' = 9Ae^{3t}$

$-9Ae^{3t} + 6Ae^{3t} + 6Bt + 6c = e^{3t} + t - 1$

$A = -\frac{1}{3} \quad B = \frac{1}{6} \quad c = -\frac{1}{6}$

$y = k_1 e^{\sqrt{6}t} + k_2 e^{-\sqrt{6}t} - \frac{e^{3t}}{3} + \frac{t}{6} - \frac{1}{6}$

8) $\frac{dx}{dt} + \frac{dy}{dt} - x - 3y = \partial t$ $\frac{dx}{\partial t} + 2\frac{dy}{\partial t} - 2x - 3y = 1$

9) $\frac{dx}{\partial t} + \frac{dy}{dt} + 2y = \sin t$ $\frac{dx}{\partial t} + \frac{dy}{\partial t} - x - y = 0$

10) $\frac{dx}{\partial t} - \frac{dy}{\partial t} + 2x + 4y = t$ $\frac{dx}{\partial t} + \frac{dy}{\partial t} - x - y = 1$

18) $\frac{d^2 x}{\partial t^2} + \frac{dy}{\partial t} = e^{2t}$ $\frac{dx}{\partial t} + \frac{dy}{\partial t} - x - y = 0$

19) $\frac{d^2 x}{dt^2} + \frac{dy}{dt} - x + y = 1$ $\frac{d^2 y}{dt^2} + \frac{dx}{\partial t} - x + y = 0$

21) $\frac{d^2 x}{dt^2} + \frac{dy}{\partial t} = t + 1$ $\frac{dx}{\partial t} + \frac{dy}{\partial t} - 3x + y = 2t - 1$

→ Solution of 19 $(D^2 - 1)x + (D+1)y = 1$ $(D^4 - 1)x = (D^2+1)1 = 1$
                     $(D-1)x + (D^2+1)y = 0$ $-(D^2-1)x = -(D+1)0 = 0$

$\left((D^4-1) - (D^2-1)\right)x = (D^2-1)(D^2+1) - 1)x = D^2(D^2-1)x = 1$

$m^2(m^2-1) = 0 \Rightarrow m = 0, 0, -1, 1$ $X_c = C_1 + C_2 t + C_3 e^t + C_4 e^{-t}$

$\frac{d^4 x}{dt^4} - \frac{d^2 x}{dt^2} = 1$ $X_p = At^2 \Rightarrow$ $0 - 2A = 1 \Rightarrow A = -\frac{1}{2}$

$X_p = \sum_{n=0}^{\infty} A_n t^n$ $X_p'' = \sum_{n=2} n(n-1)A_n t^{n-2}$ $X_p^{IV} = \sum_{n=4} n(n-1)(n-2)(n-3)A_n t^{n-4}$

$X_p^{IV} - X_p'' = 1 \Leftrightarrow$

$$x = C_1 + C_2 t + C_3 e^t + C_4 e^{-t} - \frac{t^2}{2}$$

$(D^2-1)x + (D+1)y = 1$
$-(D^2-1)x - (D+1)(D^2+1)y = +(D+1)0 = 0$

$(D+1)(D-D^2-1)y = -D^2(D+1)y = 1 \Rightarrow y_c = k_1 + k_2 t + k_3 e^{-t}$

$y_p = At^2 \Rightarrow$ $y_p'' = 2A$ $y_p''' = 0 \Rightarrow$ $-2A = 1 \Rightarrow y = -\frac{1}{2}$

$$y = k_1 + k_2 t + k_3 e^t - \frac{t^2}{2}$$

$k_1 = 1 + C_1 - C_2$ $k_2 = (1 + C_2)$ $k_3 = C_4$

⑧

Using these values for k_1 and k_2 in (7.59), the resulting pair (7.50) and (7.59) constitute the general solution of the system (7.51). That is, the general solution of (7.51) is given by

$$x = c_1 e^t + c_2 e^{-3t} - \frac{1}{3}t - \frac{11}{36}$$

$$y = -\frac{1}{2}c_1 e^t + \frac{3}{2}c_2 e^{-3t} + \frac{1}{8}t + \frac{5}{12}$$

where c_1 and c_2 are arbitrary constants. If we had chosen k_1 and k_2 as the independent constants in (7.57), then the general solution of the system (7.51) would have been written

$$x = -2k_1 e^t + \frac{2}{3}k_2 e^{-3t} - \frac{1}{3}t - \frac{11}{36}$$

$$y = k_1 e^t + k_2 e^{-3t} + \frac{1}{8}t + \frac{5}{12}$$

Exercises

Use the operator method described in this section to find the general solution of each of the following linear systems.

1. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2x - 4y = e^t$

   $\dfrac{dx}{dt} + \dfrac{dy}{dt} - y = e^{4t}$

2. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x = -2t$

   $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 3x - y = t^2$

3. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - 3y = e^t$

   $\dfrac{dx}{dt} + \dfrac{dy}{dt} + x = e^{3t}$

4. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - 2y = 2e^t$

   $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 3x - 4y = e^{4t}$

5. $2\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - y = e^{-t}$

   $\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = e^t$

6. $2\dfrac{dx}{dt} + \dfrac{dy}{dt} - 3x - y = t$

   $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 4x - y = e^t$

7. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - 6y = e^{3t}$

   $\dfrac{dx}{dt} + 2\dfrac{dy}{dt} - 2x - 6y = t$

8. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - 3y = 3t$

   $\dfrac{dx}{dt} + 2\dfrac{dy}{dt} - 2x - 3y = 1$

9. $\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x = \sin t$

   $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - y = 0$

10. $\dfrac{dx}{dt} - \dfrac{dy}{dt} - 2x + 4y = t$

    $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - y = 1$

11. $2\dfrac{dx}{dt} + \dfrac{dy}{dt} + x + 5y = 4t$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + 2y = 2$

12. $\dfrac{dx}{dt} + \dfrac{dy}{dt} - x + 5y = t^2$

$\dfrac{dx}{dt} + 2\dfrac{dy}{dt} - 2x + 4y = 2t + 1$

13. $2\dfrac{dx}{dt} + \dfrac{dy}{dt} + x + y = t^2 + 4t$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + 2y = 2t^2 - 2t$

14. $3\dfrac{dx}{dt} + 2\dfrac{dy}{dt} - x + y = t - 1$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} - x = t + 2$

15. $2\dfrac{dx}{dt} + 4\dfrac{dy}{dt} + x - y = 3e^t$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + 2y = e^t$

16. $2\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - y = -2t$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} + x - y = t^2$

17. $2\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - y = 1$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x - y = t$

18. $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} = e^{2t}$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} - x - y = 0$

19. $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} - x + y = 1$

$\dfrac{d^2y}{dt^2} + \dfrac{dx}{dt} - x + y = 0$

20. $\dfrac{d^2x}{dt^2} - \dfrac{dy}{dt} = e^t$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} - 4x - y = 2e^t$

21. $\dfrac{d^2x}{dt^2} - \dfrac{dy}{dt} = t + 1$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} - 3x + y = 2t - 1$

7.4 Applications

A. Applications to Mechanics

Systems of linear differential equations originate in the mathematical formulation of numerous problems in mechanics. We consider one such problem in the following example. Another mechanics problem leading to a linear system is given in the exercises at the end of this section.

Example 7.16. On a smooth horizontal plane BC (for example, a smooth table top) an object A_1 is connected to a fixed point P by a massless spring S_1 of natural length L_1. An object A_2 is then connected to A_1 by a massless spring S_2 of natural length L_2 in such a way that the fixed point P and the centers of gravity A_1 and A_2 all lie in a straight line (see Figure 7.1).

The object A_1 is then displaced a distance a_1 to the right or left of its equilibrium position O_1, the object A_2 is displaced a distance a_2 to the right or left of its equilibrium position O_2, and at time $t = 0$ the twoobjects are released (see Figure 7.2). What are the positions of the two objects at any time $t > 0$?

THE LAPLACE TRASFORM.

Definition: let F be a real-valued function of the real variable t, defined for $t > 0$. let s be a variable which we shall assume to be real, and consider the function f defined by

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt \qquad (2)$$

for all values of s for which this integral exists. The function f defined by the integral (2) is called the Laplace Trasform of the function F. We shall denote the Laplace Trasform f of F by $\mathcal{L}(F)$ and shall denote $f(s)$ by $\mathcal{L}\{F(t)\}$.

Examples 1) Consider the function F defined by $F(t) = 1$, for $t > 0$.

$$\mathcal{L}(1) = \int_0^\infty e^{-st} \cdot 1\, dt = \lim_{R \to \infty} \int_0^R e^{-st}\, dt = \lim_{R \to \infty} -\frac{e^{-st}}{s}\Big|_0^R$$

$$= \left(\lim_{R \to \infty} \frac{-e^{-sR}}{s}\right) - \frac{-e^0}{s} = \frac{1}{s} \quad \text{for all } s > 0. \quad \text{Thus we have}$$

$$\mathcal{L}(1) = \frac{1}{s} \quad (s > 0).$$

2). $F(t) = t$, for $t > 0$.

$$\mathcal{L}(t) = \int_0^\infty e^{-st} \cdot t\, dt = \lim_{R \to \infty} \int_0^R e^{-st} \cdot t\, dt = \lim_{R \to \infty} \left(\frac{-(st+1)}{s^2} \cdot e^{-st}\right)\Big|_0^R$$

$$= \frac{1}{s^2} - \lim_{R \to \infty}\left(\frac{(sR+1) \cdot e^{-sR}}{s^2}\right) = \frac{1}{s^2} \cdot \text{Thus we have}$$

$$\mathcal{L}(t) = \frac{1}{s^2} \quad (s > 0).$$