FIRST ORER EQUATIONS

A first order equation can either be given in the derivatitive form

$$\frac{dy}{dx} = f(x,y) \quad \text{or in the form} \quad M(x,y)\,dx + N(x,y)\,dy = 0$$

An equation in one of these forms may readily be written in the other form. For example, the equation

$$\frac{dy}{dx} = \frac{x^2+y^2}{x-y} \Rightarrow (x^2+y^2)\,dx + (y-x)\,dy = 0$$

The equation

$$(\sin x + y)\,dx + (x+3y)\,dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{\sin x + y}{x+3y}$$

Exact Differential Equations

__Definition:__ Suppose $u = F(x,y)$, when F has continuous first partial derivatives. The total differential du is defined by the formula

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

__Example:__ Let $u = xy^2 + 2x^3y$. Then $\frac{\partial u}{\partial x} = y^2 + 6x^2y$ and $\frac{\partial u}{\partial y} = 2xy + 2x^3$

and so $du = (y^2+6x^2y)\,dx + (2xy+2x^3)\,dy$.

__Definition:__ The expression

$$M\,dx + N\,dy \qquad (1)$$

is called an exact differential if there exists some u for which this expression is the total differential du. In other words, the expression (1) is an exact dif. if there exists some u such that

$$\frac{\partial u}{\partial x} = M \quad \text{and} \quad \frac{\partial u}{\partial y} = N$$

If $M\,dx + N\,dy$ is an exact differential, then the differential equation

$$M\,dx + N\,dy = 0$$

is called an exact dif. equation.

Example : The differential equation $y^2dx+2xydy=0$ is an exact differential equation. Indeed, it is the total differential of $u=xy^2$.

Theorem : The differential equation $Mdx+Ndy=0$, where M and N have continuous first partial derivatives, is exact if and only if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

Proof: If the differential equation is exact, then $Mdx+Ndy$ is an exact differential. By the definition of an exact differential, there exist a u such that $\frac{\partial u}{\partial x}=M$ and $\frac{\partial u}{\partial y}=N$.

Then $\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial M}{\partial y}$ and $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial N}{\partial x}$.

Since $\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 u}{\partial x\partial y}$ (by the continuousness of ...), $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

$(\Leftarrow)$ Suppose that $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. We must prove that there exists a u such that $\frac{\partial u}{\partial x}=M$ and $\frac{\partial u}{\partial y}=N$.

We can certainly find some u satisfying either $\frac{\partial u}{\partial x}=M$ or $\frac{\partial u}{\partial y}=N$ ($u=\int Mdx$ or $u=\int Ndy$). Assume that u satisfies $\frac{\partial u}{\partial x}=M$ and proceed.

$$u=\int M\partial x+\phi(y) \qquad (2)$$

Differentiating (2.10) partially with respect to y, we obtain

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\int M\partial x+\frac{\partial \phi}{\partial y}$$

and so, we must have $N = \frac{\partial}{\partial y}\int M\,\partial x + \frac{\partial \phi}{\partial y}$

and hence $\frac{\partial \phi}{\partial y} = N - \frac{\partial}{\partial y}\int M\,\partial x$. Since ϕ is a function of y only, the derivative $\frac{d\phi}{\partial y}$ must also be independent of x. That is $N - \frac{\partial}{\partial y}\int M\,\partial x$ must be independent of x. Since

$$\frac{\partial}{\partial x}\left[N - \frac{\partial}{\partial y}\int M\,\partial x\right] = \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial x\partial y}\int M\,dx = \frac{\partial N}{\partial x} - \frac{\partial^2}{\partial y\partial x}\int M\,\partial x$$

$$= \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0.$$

Thus we see that

$$\phi(y) = \int\left[N - \int \frac{\partial M}{\partial y}\,\partial x\right]\partial y.$$

Substituting this into (2), we have

$$u = \int M\,\partial x + \int\left[N - \int\frac{\partial M}{\partial y}\,\partial x\right]dy. \quad (*)$$

This u is clearly satisfy $\frac{\partial u}{\partial x} = M$ and $\frac{\partial u}{\partial y} = N$, and so $M\,dx + N\,dy = 0$ is exact. $\blacksquare$

Examples: (1) Consider the equation $y^2dx + 2xy\,dy = 0$. Here $M = y^2$, $N = 2xy$. Since $\frac{\partial M}{\partial y} = 2y = \frac{\partial N}{\partial x}$, the equation is exact.

(2) Consider the equation $y\,dx + 2x\,dy = 0$. We have $M = y$ and $N = 2x$. Since $\frac{\partial M}{\partial y} = 1 \neq 2 = \frac{\partial N}{\partial x}$, the equation is not exact.

(3) Consider the equation $(2x\sin y + y^3e^x)dx + (x^2\cos y + 3y^2e^x)dx$

Here $M = 2x\sin y + y^3e^x$ and $N = x^2\cos y + 3y^2e^x$

Since $\frac{\partial M}{\partial y} = 2x\cos y + 3y^2e^x = \frac{\partial N}{\partial x}$,

the equation is exact.

If the equation $M\,dx + N\,dy = 0$ is exact, then there exists some u such that $\frac{\partial u}{\partial x} = M$ and $\frac{\partial u}{\partial y} = N$. Then the equation may be written

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \quad \text{or} \quad du = 0$$

The relation $u = c$ is obviously a solution of this. However we use ⊛ to find a general solution.

<u>Examples</u> (1) $(3x^2+4xy)\,dx + (2x^2+2y)\,dy = 0$.

Here $N = 2x^2+2y$ and $M = 3x^2+4xy$. Since $\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x}$ the equation is exact

$$u = \int M\,\partial x + \phi(y) = \int (3x^2+4xy)\,\partial x + \phi(y)$$
$$= x^3 + 2x^2y + \phi(y).$$

Then $\frac{\partial u}{\partial y} = 2x^2 + \frac{d\phi}{dy}$ which is equal to N and so

$\frac{d\phi}{dy} = 2y \Rightarrow d\phi = 2y\,dy \Rightarrow \phi = y^2 + c_0$. Therefore

$$u = x^3+3x^2y+y^2+c_0.$$

The general solution is $u = c_1$ or $x^3+2x^2y+y^2+c_0 = c_1$. or equivalently

$$x^3+2x^2y+y^2 = c$$

(2) Solve the initial-value problem

$$(2x\cos y+3x^2y)\,dx + (x^3-x^2\sin y-y)\,dy = 0$$
$$y(0) = 2.$$

Since $\frac{\partial M}{\partial y} = -2x\sin y + 3x^2 = \frac{\partial N}{\partial x}$, the equation is exact.

$$u = \int M\,\partial x + \phi(y) = \int (2x\cos y + 3x^2y)\,\partial x + \phi(y)$$
$$= x^2\cos y + x^3y + \phi(y).$$

$$N = \frac{\partial u}{\partial y} = -x^2 \sin y + x^3 + \frac{d\phi}{dy} \Rightarrow \frac{d\phi}{dy} = -y \Rightarrow \phi = -\frac{y^2}{2},$$

and so $\quad x^2\cos y + x^3 y - \frac{y^2}{2} = C$

is a general solution.

From $y(0) = 2$, we find $C = -2$. Thus the solution is

$$x^2\cos y + x^3 y - \frac{y^2}{2} + 2 = 0.$$

Integrating Factors

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Integrating Factors

Given the differential equation $Mdx + Ndy = 0$. If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ the the equation is not exact. We can multiply the equation by some expression which will transform it into an equivalent exact equation.

Let us consider the equation $ydx + 2xdy = 0$. which is not exact. However, if we multiply it by y it is transformed into equivalent equation $y^2dx + 2xydy = 0$

which is exact.

$$u = \int M\partial x + \phi(y) = \int y^2 \partial x + \phi(y) = xy^2 + \phi(y)$$

$$\frac{\partial u}{\partial y} = 2xy + \frac{d\phi}{dy} = N = 2xy \Rightarrow \quad \frac{d\phi}{dy} = 0 \Rightarrow \phi = c$$

$$u = xy^2$$

Thus $xy^2 = c$ is a solution of $y^2dx + 2xydy = 0$.

<u>Definition</u>: Suppose that the differential equation $Mdx + Ndy = 0$ is not exact but the differential equation $\gamma Mdx + \gamma Ndy = 0$ is exact where $\gamma = F(x,y)$ for a suitably chosen function F. Then γ is called integrating factor of the first equation.

<u>Example</u>: Consider the differential equation $(3y + 4xy^2)dx + (2x + 3x^2y)dy = 0$

$\frac{\partial M}{\partial y} = 3 + 8xy \quad \frac{\partial N}{\partial x} = 2 + 6xy$, and so it is not exact.

Let $\gamma = x^2y$. Then $(3x^2y^2 + 4x^3y^3)dx + (2x^3y + 3x^4y^2)dy = 0$ is exact.

Since $\frac{\partial(\gamma M)}{\partial y} = 6x^2y + 12x^3y^2 = \frac{\partial(\gamma N)}{\partial x}$. Hence $\gamma = x^2y$ is an integrating factor.

Exercises

In Exercises 1 through 10 determine whether or not each of the given equations is exact; solve those which are exact.

1. $(3x + 2y)dx + (2x + y)dy = 0.$
2. $(y^2 + 3)dx + (2xy - 4)dy = 0.$
3. $(2xy + 1)dx + (x^2 + 4y)dy = 0.$
4. $(3x^2y + 2)dx - (x^3 + y)dy = 0.$
5. $(6xy + 2y^2 - 5)dx + (3x^2 + 4xy - 6)dy = 0.$
6. $(\theta^2 + 1)\cos r dr + 2\theta \sin r d\theta = 0.$
7. $(y\sec^2 x + \sec x \tan x)dx + (\tan x + 2y)dy = 0.$
8. $\left(\dfrac{x}{y^2} + x\right)dx + \left(\dfrac{x^2}{y^3} + y\right)dy = 0.$
9. $\left(\dfrac{2s - 1}{t}\right)ds + \left(\dfrac{s - s^2}{t^2}\right)dt = 0.$
10. $\dfrac{2y^{3/2} + 1}{x^{1/2}}dx + (3x^{1/2}y^{1/2} - 1)dy = 0.$

Solve the initial-value problems in Exercises 11 through 14.

11. $\begin{cases} (2xy - 3)dx + (x^2 + 4y)dy = 0, \\ y(1) = 2. \end{cases}$

12. $\begin{cases} (2y\sin x\cos x + y^2\sin x)dx + (\sin^2 x - 2y\cos x)dy = 0, \\ y(0) = 3. \end{cases}$

13. $\begin{cases} \left(\dfrac{3 - y}{x^2}\right)dx + \left(\dfrac{y^2 - 2x}{xy^2}\right)dy = 0, \\ y(-1) = 2. \end{cases}$

14. $\begin{cases} \dfrac{1 + 8xy^{2/3}}{x^{2/3}y^{1/3}}dx + \dfrac{2x^{4/3}y^{2/3} - x^{1/3}}{y^{4/3}}dy = 0, \\ y(1) = 8. \end{cases}$

15. In each of the following equations determine the constant A such that the equation is exact, and solve the resulting exact equation.

(a) $(x^2 + 3xy)dx + (Ax^2 + 4y)dy = 0.$

(b) $\left(\dfrac{1}{x^2} + \dfrac{1}{y^2}\right)dx + \left(\dfrac{Ax + 1}{y^3}\right)dy = 0.$

16. In each of the following equations determine the most general function $N(x,y)$ such that the equation is exact.

(a) $(x^3 + xy^2)dx + N(x,y)dy = 0.$

(b) $(x^{-2}y^{-2} + xy^{-3})dx + N(x,y)dy = 0$.

17. Consider the differential equation*

$$(4x + 3y^2)dx + 2xydy = 0.$$

(a) Show that this equation is not exact.
(b) Find an integrating factor of the form x^n, where n is a positive integer.
(c) Multiply the given equation through by the integrating factor found in (b) and solve the resulting exact equation.

18. Consider a differential equation of the form

$$[y + xf(x^2 + y^2)]dx + [yf(x^2 + y^2) - x]dy = 0.$$

(a) Show that an equation of this form is not exact.
(b) Show that $1/(x^2 + y^2)$ is an integrating factor of an equation of this form.

19. Use the result of Exercise 18(b) to solve the equation

$$[y + x(x^2 + y^2)^2]dx + [y(x^2 + y^2)^2 - x]dy = 0.$$

2.2 Separable Equations and Equations Reducible to This Form

A. *Separable Equations*

DEFINITION. An equation of the form

(2.17) $$F(x)\,G(y)dx + f(x)g(y)dy = 0$$

is called an *equation with variables separable* or simply a *separable equation.*

For example, the equation $(x^3 + x^2)ydx + x^2(y^3 + 2y)dy = 0$ is a separable equation. In general the separable equation (2.17) is not exact, but it possesses an obvious integrating factor, namely $1/f(x)G(y)$. For if we multiply Equation (2.17) by this expression, we separate the variables, reducing (2.17) to the equivalent equation

(2.18) $$\frac{F(x)}{f(x)}dx + \frac{g(y)}{G(y)}dy = 0.$$

This equation is exact, since

$$\frac{\partial}{\partial y}\left[\frac{F(x)}{f(x)}\right] = 0 = \frac{\partial}{\partial x}\left[\frac{g(y)}{G(y)}\right].$$

Denoting $F(x)/f(x)$ by $M(x)$ and $g(y)/G(y)$ by $N(y)$, Equation (2.18) takes the form $M(x)dx + N(y)dy = 0$. Since M is a function of x only and N is a function of y only, we see at once that the solution is

(2.19) $$\int M(x)dx + \int N(y)dy = c,$$

where c is an arbitrary constant. Thus the problem of solving the separable equation (2.17) has reduced to that of performing the integrations indicated in Equation (2.19). It is in this sense that separable equations are the simplest first-order differential equations.

Exercises

Determine whether or not each of the given equation is exact; solve the exact ones.

1) $(3x+2y)\,dx + (2x+y)\,dy = 0$

$$\frac{\partial M}{\partial y} = 2 = \frac{\partial N}{\partial x} \Rightarrow \text{exact}$$

$$u = \int (3x+2y)\,dx + \phi(y) = \frac{3x^2}{2} + 2yx + \phi(y)$$

$$2x+y = 2x + \frac{d\phi}{dy} \Rightarrow \quad d\phi = y\,dy \Rightarrow \phi = \frac{y^2}{2}$$

$$\frac{3x^2}{2} + 2yx + \frac{y^2}{2} = c \quad \text{or} \quad 3x^2 + 4yx + y^2 = c.$$

2) $(y^3+3)\,dx + (2xy-4)\,dy = 0$ (not exact)

3) $(2xy+1)\,dx + (x^2+4y)\,dy = 0$ $(x + x^2y + 2y^2 = c)$

4) $(3x^2y+2)\,dx - (x^3+y)\,dy = 0$

$$\frac{\partial M}{\partial y} = 3x^2 = \frac{\partial N}{\partial x} \Rightarrow \text{exact}$$

$$u = \int (3x^2y+2)\,dx + \phi(y)$$

$$u = x^3y + 2x + \phi(y)$$

$$x^3 + y = x^3 + \frac{d\phi}{dy} \Rightarrow \quad d\phi = y\,dy \Rightarrow \phi = \frac{y^2}{2}$$

$$u = x^3y + 2x + \frac{y^2}{2}$$

$$2x^3y + 4x + y^2 = c.$$

5) $(6xy+2y^2-5)\,dx + (3x^2+4xy-6)\,dy = 0$ $(3x^2y + 2y^2x - 5x - 6y = c)$

6) $(\theta^2+1)\cos r\,dr + 2\theta \sin r\,d\theta = 0$

$$\frac{\partial M}{\partial \theta} = 2\theta\cos r = \frac{\partial N}{\partial r} \Rightarrow \text{exact}$$

$$u = \int (\theta^2+1)\cos r\,dr + \phi(\theta) = (\theta^2+1)\sin r + \phi(\theta)$$

$$2\theta \sin r = 2\theta \sin r + \frac{d\phi}{d\theta}$$

$$\phi = c.$$

$(\theta^2+1)\sin r = c$ is a solution.

7) $(y\sec^2 x + \sec x \tan x)\,dx + (\tan x + 2y)\,dy = 0$

$$\frac{\partial M}{\partial y} = \sec^2 x = \frac{\partial N}{\partial x} \Rightarrow \text{exact}$$

$$u = \int (y\sec^2 x + \sec x \tan x\,dx + \phi(y)$$

$$= \int y\sec^2 x\,dx + \int \sec x \tan x\,dx + \phi(y).$$

$$= y\tan x + \sec x + \phi(y)$$

$$\tan x + 2y = \frac{\partial u}{\partial y} = \tan x + \frac{d\phi}{dy} \Rightarrow d\phi = 2y\,dy \quad \phi = y^2$$

$$y\tan x + \sec x + y^2 = c.$$

8) $\left(\frac{x}{y^2} + x\right)dx + \left(\frac{x^2}{y^3} + y\right)dy = 0$

9) $\left(\frac{2s-1}{t}\right)ds + \left(\frac{s-s^2}{t^2}\right)dt = 0$ $\quad (s^2 - s = ct)$

Solve the initial-value problems.

10) $(2xy-3)\,dx + (x^2+4y)\,dy = 0$ $\quad y(1) = 2$ $\quad (x^2y - 3x + 2y^2 = 7)$

11) $(2y\sin x\cos x + y^2 \sin x)\,dx + (\sin^2 x - 2y\cos x)\,dy = 0$ $\quad y(0) = 3$

12) $\left(\frac{3-y}{x^2}\right)dx + \left(\frac{y^2-2x}{xy^2}\right)dy = 0$ $\quad y(-1) = 2.$

$$\frac{\partial M}{\partial y} = \frac{-1}{x^2} \stackrel{?}{=} \frac{\partial N}{\partial x} = \frac{-2xy^2 - y^2(y^2-2x)}{(xy^2)^2} = \frac{-y^4}{x^2y^4} = \frac{-1}{x^2} \quad \checkmark.\ \text{exact}.$$

$$u = \int \left(\frac{3-y}{x^2}\right)dx + \phi(y) = \frac{y-3}{x} + \phi(y)$$

$$\frac{y^2-2x}{xy^2} = \frac{\partial u}{\partial y} = \frac{1}{x} + \frac{d\phi}{dy} \Rightarrow \frac{d\phi}{dy} = \cancel{\frac{1}{x}} - \frac{2}{y^2} - \cancel{\frac{1}{x}} \quad d\phi = \frac{-2dy}{y^2}$$

$$\Rightarrow \phi = \frac{+2}{y} \quad \Rightarrow u = \frac{y-3}{x} + \frac{2}{y} = c \Rightarrow y^2 - 3y + 2x = cxy$$

$$4 - 6 - 2 = -2c \Rightarrow c = 2$$

$$\Rightarrow y^2 - 3y + 2x = cxy$$

$$y^2 - 3y + 2x = 2xy.$$

13) $\quad y$