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The Method of Undetermined Coefficients.

We now consider the nonhomogeneous differential equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = b(x) \qquad (1)$$

where the coefficients $a_0, a_1, \dots, a_n$ are constants but where the nonhomogeneous term b is a nonconstant function of x. Recall that the general solution of (1) may be written $y = y_c + y_p$ where y_c is the complementary function and y_p is a particular integral.

Definition: We call a function a UC function if it is either a function defined by one of the following:

(i) x^n $(n \in \mathbb{Z}^+ \cup \{0\})$ (ii) e^{ax} $(a \in \mathbb{R} \setminus \{0\})$

(iii) $\sin(bx+c)$ $(b, c \in \mathbb{R}, b \neq 0)$ (iv) $\cos(bx+c)$ $(b, c \in \mathbb{R}, b \neq 0)$

or a function defined as a finite product of two or more functions of these four type.

Consider a UC function f. The set of functions consisting of itself and all linearly independent UC functions of which the successive derivatives of f are either constant multiples or linear combinations will be called the UC set of f.

Examples (1) $f(x) = x^3$ is a UC function. Computing derivatives of f, we find

$f'(x) = 3x^2$ $f''(x) = 6x$ $f'''(x) = 6$ $f^{(n)}(x) = 0$ for $n > 3$.

Thus the UC set of x^3 is the set $S = \{x^3, x^2, x, 1\}$.

(2) $f(x) = \sin 2x$ is a UC function.

$f'(x) = 2\cos 2x$, $f''(x) = -4\sin 2x$, $f''' = -8\cos 4x, \dots$

UC set of $\sin 2x$ is the set $S = \{\sin 2x, \cos 2x\}$.

(3) $f(x) = x^2 \sin x$ is a UC function.

$f'(x) = 2x\sin x + x^2 \cos x$ $f''(x) = 2\sin x + 4x\cos x - x^2 \sin x$

$f'''(x) = 6\cos x - 6x\sin x - x^2\cos x$ ....

Thus the UC set of $x^2\sin x$ is the set

$$S = \{x^2\sin x, x^2\cos x, x\sin x, x\cos x, \sin x, \cos x\}.$$

Let b be a finite linear combination $b = A_1u_1 + A_2u_2 + \dots + A_mu_m$ of UC functions $u_1, u_2, \dots, u_m$, the A_i being known constants. Assuming the complementary function y_c has already been obtained.

(1) For each of the UC functions $u_1, \dots, u_n$ of which b is a linear combination, form the corresponding UC set, thus obtaining the respective sets $S_1, S_2, \dots, S_m$.

(2) Suppose that one of the UC sets so formed, say S_j, is identical with or completely included in another, say b_k, In this case, we omit the 'identical or smaller set S_j from further consideration.

(3) Suppose now one of these US sets, say S_l, includes one or more members which are solutions of the corresponding homogeneous differential equation. If this is the case, we multiply each member of S_l by the lowest positive integral power of x so that the resulting revised set will contain no members which are solutions of the corresponding homogeneous equation.

(4) Now form a linear combination of all of the elements of all of the sets with unknown constant coefficients.

(5) determine these unknown coefficients by substituting the linear combination formed in step (4) into the dif. equation.

Examples (1) $y'' - 2y' - 3y = 2e^x - 10\sin x$.

$m^2 - 2m - 3 = (m+1)(m-3) = 0$ $m = -1, 3 \Rightarrow y_c = c_1e^{-x} + c_2e^{3x}$.

UC functions are $u_1(x) = e^x$ $u_2(x) = \sin x$.

UC sets respectively $S_1 = \{e^x\}$ $S_2 = \{\sin x, \cos x\}$.

Let $y = Ae^x + B\sin x + C\cos x$.

$y' = Ae^x + B\cos x - C\sin x$ $\quad y'' = Ae^x - B\sin x - C\cos x$

$(Ae^x - B\sin x - C\cos x) - 2(Ae^x + B\cos x - C\sin x) - 3(Ae^x + B\sin x + C\cos x)$

$= -4Ae^x + (-4B + 2C)\sin x + (-4C - 2B)\cos x = 2e^x - 10\sin x$

$$\begin{array}{l} -4A = 2 \\ -4B + 2C = -10 \\ -4C - 2B = 0 \end{array} \Rightarrow \begin{array}{l} A = -\frac{1}{2} \\ B = 2 \\ C = -1 \end{array} \Rightarrow y_p = -\frac{1}{2}e^x + 2\sin x - \cos x$$

$$y = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{2}e^x + 2\sin x - \cos x.$$

(2) $y'' - 3y' + 2y = 2x^2 + e^x + 2xe^x + 4e^{3x}$

$m^2 - 3m + 2 = (m-1)(m-2) = 0 \Rightarrow m = 1, 2$ $\quad y_c = c_1 e^x + c_2 e^{2x}$

$u_1 = x^2,\ u_2 = e^x,\ u_3 = xe^x \quad u_4 = e^{3x}$

$S_1 = \{x^2, x, 1\},\ S_2 = \{e^x\},\ S_3 = \{xe^x, e^x\},\ S_4 = \{e^{3x}\}$

Since $S_2 \subset S_3$, delete S_2. multiply $\bar{S}_3$ by x, then

Since $e^x \in S_3$ appears in y_c.

$S_1 = \{x^2, x, 1\},\ \bar{S}_3 = \{x^2e^x, xe^x\},\ S_4 = \{e^{3x}\}.$

$$y = Ax^2 + Bx + C + De^{3x} + Ex^2e^x + Fxe^x$$
$$y' = 2Ax + B + 3De^{3x} + 2Exe^x + Ex^2e^x + Fe^x + Fxe^x$$
$$y'' = 2A + 9De^{3x} + 2Ee^x + 4Exe^x + Ex^2e^x + 2Fe^x + Fxe^x.$$

Substituting

$(2A - 3B + 2C) + (2B - 6A)x + 2Ax^2 + 2De^{3x} + (-2Exe^x + (2E - F)e^x = 2x^2 + e^x + 2xe^x + 4e^{3x}$

$$\left.\begin{array}{l} 2A - 3B + 2C = 0 \\ 2B - 6A = 0 \\ 2A = 2 \\ 2D = 4 \\ -2E = 2 \\ 2E - F = 1 \end{array}\right\} \Rightarrow \begin{array}{l} A = 1 \\ B = 3 \\ C = \frac{7}{2} \\ D = 2 \\ E = -1 \\ F = -3 \end{array} \Rightarrow$$

$y_p = x^2 + 3x + \frac{7}{2} + 2e^{3x} - x^2e^x - 3xe^x$

$\underline{y = y_c + y_p}$

(3) $y^{IV} + y'' = 3x^2 + 4\sin x - 2\cos x.$

$m^4 + m^2 = m^2(m^2 + 1) = 0 \Rightarrow m = 0, 0, -i, i \Rightarrow y_c = c_1 + c_2 x + c_3\sin x + c_4\cos x.$

$u_1 = 3x^2$ $\quad u_2 = \sin x$ $\quad u_3 = \cos x$

$S_1 = \{x^2, x, 1\}$, $\quad S_2 = \{\sin x, \cos x\}$, $\quad S_3 = \{\cos x, \sin x\}$.

$S_3 = S_2 \Rightarrow$ Delete S_3.

$x, 1 \in S_1$ appear in $y_c \Rightarrow$ $\quad S_1' = \{x^4, x^3, x^2\}$

$\sin x, \cos x \in S_2$ " " " $\Rightarrow$ $\quad S_2' = \{x\sin x, x\cos x\}$.

$$y = Ax^4 + Bx^3 + Cx^2 + Dx\sin x + Ex\cos x$$

$$y' = 4Ax^3 + 3Bx^2 + 2Cx + D\sin x + Dx\cos x + E\cos x - Ex\sin x$$

$$\rightarrow y'' = \underline{12Ax^2} + \underline{6Bx} + \underline{2C} + \underline{2D\cos x} - \underline{Dx\sin x} - \underline{2E\sin x} - \underline{Ex\cos x}.$$

$$y''' = 24Ax + 6B - 3D\sin x - Dx\cos x - 3E\cos x + Ex\sin x$$

$$\rightarrow y^{iv} = \underline{24A} - \underline{4D}\cos x + \underline{Dx\sin x} + \underline{4E\sin x} + \underline{Ex\cos x}$$

$$12Ax^2 + (24A + 2C) + 6Bx + 2E\sin x - 2D\cos x = 3x^2 + 4\sin x - 2\cos y$$

$$\left.\begin{aligned} 12A &= 3 \\ 6B &= 0 \\ 24A + 2C &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 1/4 \\ B &= 0 \\ C &= -3 \end{aligned} \qquad \left.\begin{aligned} 2E &= 4 \\ -2D &= -2 \end{aligned}\right\} \Rightarrow D = 1 \quad E = 2$$

$$y_p = \frac{x^4}{4} - 3x^2 + x\sin x + 2x\cos x.$$

(4) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 2e^x - 10\sin x$, $\quad y(0) = 2 \quad y'(0) = 4$.

From the first example $\quad y = c_1e^{3x} + c_2e^{-x} - \frac{1}{2}e^x + 2\sin x - \cos x$.

$$y' = 3c_1e^{3x} - c_2e^{-x} - \frac{1}{2}e^x + 2\cos x + \sin x$$

$y(0) = 2 \Rightarrow 2 = c_1 + c_2 - \frac{1}{2} - 1 \Rightarrow c_1 + c_2 = \frac{7}{2}$ $\qquad c_1 = \frac{3}{2} \quad c_2 = 2$

$y'(0) = 4 \Rightarrow 4 = 3c_1 - c_2 - \frac{1}{2} + 2 \Rightarrow 3c_1 - c_2 = \frac{5}{2}$

$$y = \frac{3}{2}e^{3x} + 2e^{-x} - \frac{1}{2}e^x + 2\sin x - \cos x.$$

Variation of Parameters

Exercises · find the general solutions of each of the followings

1. $y'' - 3y' + 2y = 4x^2$ 2. $y'' - 2y' - 8y = 4e^{2x} - 21e^{-3x}$

3. $y'' + 2y' + 5y = 6\sin 2x + 7\cos 2x.$

$m^2 + 2m + 5 = (m+1)^2 + 4 = 0 \Rightarrow m = -1 \mp 2i$

$y_c = (c_1 \sin 2x + c_2 \cos 2x)e^{-x}$

$u_1 = \sin 2x \quad u_2 = \cos 2x \Rightarrow S_1 = \{\sin 2x, \cos 2x\} = S_2.$

$S_1 \cap \{\sin 2x \cdot e^{-x}, \cos 2x \cdot e^{-x}\} = \emptyset$ so

$y_p = A\sin 2x + B\cos 2x$

$y_p' = 2A\cos 2x - 2B\sin 2x \Rightarrow y_p'' = -4A\sin 2x - 4B\cos 2x.$

$\underline{-4A\sin 2x} \underline{-4B\cos 2x} + \underline{4A\cos 2x} \underline{-4B\sin 2x} + \underline{5A\sin 2x} + \underline{5B\cos 2x}$

$= (-4A - 4B + 5A)\sin 2x + (-4B + 4A + 5B)\cos 2x$

$= (A - 4B)\sin 2x + (B + 4A)\cos 2x = 6\sin 2x + 7\cos 2x$

$$\left.\begin{array}{l} A - 4B = 6 \\ B + 4A = 7 \end{array}\right\} \Rightarrow \begin{array}{r} A - \cancel{4B} = 6 \\ + \underline{\cancel{4B} + 16A = 28} \end{array} \quad 17A = 34 \quad A = 2 \quad B = -1 \quad y_p = 2\sin 2x - \cos 2x$$

$y = (c_1 \sin 2x + c_2 \cos 2x)e^{-x} + 2\sin 2x - \cos 2x.$

4) $y'' + 2y' + 2y = 10\sin 4x$ 5) $y'' + 2y' + 4y = \cos 4x$

Solution: $m^2 + 2m + 4 = (m+1)^2 + 3 = 0 \quad m = -1 \mp \sqrt{3}i$ ~~$y_c = (c_1$~~

$y_c = (c_1 \sin\sqrt{3}x + c_2 \cos\sqrt{3}x)e^{-x}$

$u = \sin 4x \Rightarrow S = \{\sin 4x, \cos 4x\}.$

$y_p = A\sin 4x + B\cos 4x$

$y_p' = 4A\cos 4x - 4B\sin 4x. \quad y_p^{4} = -16A\sin 4x - 16B\cos 4x$

~~$-16A\sin 4x - 16B\cos 4x + 8A\cos 4x - 8B\sin 4x + 16A\cos 4x + 16B\sin 4x$~~

$(-16A + 8B)\sin 4x + (-16B$

$\underline{-16A\sin 4x} \underline{-16B\cos 4x} + \underline{8A\cos 4x} \underline{-8B\sin 4x} + \underline{4A\sin 4x} + \underline{4B\cos 4x}$

$= (-12A - 8B)\sin 4x + (-12B + 8A)\cos 4x = \cos 4x$

$$\begin{array}{l} -12A - 8B = 0 \\ -12B + 8A = 1 \end{array} \quad A = -\frac{2B}{3} \quad -12B - \frac{16B}{3} = \frac{-52B}{3} = 1 \Rightarrow B = \frac{-3}{52}$$

$A = \frac{-2\left(\frac{-3}{52}\right)}{3} = \frac{2}{52} \neq \frac{1}{26}$

$y = (c_1 \sin\sqrt{3} + c_2 \cos\sqrt{3}x)e^{-x} + \frac{1}{52}(2\sin 4x - 3\cos 4x)$.

6) $y''' + 2y'' + 3y' - 10y = 8xe^{-2x}$ 7) $y''' + y'' + 3y' - 5y = 5\sin 2x + 10x^2 - 3x + 7$.

Solution: $m^3 + m^2 + 3m - 5 = 0$ $m^2 + 2m + 5 = (m+1)^2 + 4 = 0$ $m = -1 \mp 2i$

$$\begin{array}{l|l} m^3 + m^2 + 3m - 5 & m-1 \\ \cline{2-2} -\ m^3 - m^2 & m^2 + 2m + 5 \\ \hline 2m^2 + 3m & \\ -\ 2m^2 - 2m & \\ \hline 5m - 5 & \end{array}$$

$y_c = c_1 e^{x} + (c_2 \sin 2x + c_3 \cos 2x)e^{-x}$

$u_1 = \sin 2x$ $u_2 = x^2$ (No need the others).

$S_1 = \{\sin 2x, \cos 2x\}$ $S_2 = \{x^2, x, 1\}$.

$y = A\sin 2x + B\cos 2x + Cx^2 + Dx + E$

$y' = 2A\cos 2x - 2B\sin 2x + 2Cx + D$

$y'' = -4A\sin 2x - 4B\cos 2x + 2C$

$y''' = -8A\cos 2x + 8B\sin 2x$.

$(8B - 4A - 6B - 5A)\sin 2x + (-8A - 4B + 6A - 5B)\cos 2x + Cx^2 + (2C + D)x + E + D + 2C$

$(2B - 9A)\sin 2x + (-2A - 9B)\cos 2x + Cx^2 + (2C + D)x + E = 5\sin 2x + 10x^2 - 3x + 7$.

$2B - 9A = 5$

$-2A - 9B = 0$ $B = \frac{-2A}{9}$

$\frac{-4A}{9} - 9A = \frac{\overset{-17}{\cancel{-85}}A}{9} = \cancel{5}$ $A = -\frac{9}{17}$ $B = \frac{2}{17}$

$-5C = 10$ $-5D + 6C = -3$ $C = -2$

~~$10C + D = -3$~~ $-5E + 3D + 2C = 7$

~~$2C + D + E = 7$~~

$D = \frac{-9}{\cancel{-23}\,5}$ $E = \frac{-82}{\cancel{+13}\,25}$

$y = c_1 e^{x} + (c_2 \sin 2x + c_3 \cos 2x)e^{-x} - \frac{9}{17}\sin 2x + \frac{2}{17}\cos 2x - 2x^2 - \frac{9x}{5} - \frac{82}{25}$

8. $4y''' - 4y'' - 5y' + 3y = 3x^3 - 8$ ✗ 9) $y'' + y' - 6y = 10e^{2x} - 18e^{3x} - 6x - 11$.

10. $y''' - 3y'' + 4y = 4e^{x} - 18e^{-x}$ 11) $y''' + y' = 2x^2 + 4\sin x$.

Solution $m^3 + m = m(m^2 + 1) = 0$ $m = 0, \mp i$

$y_c = c_1 + (c_2 \sin x + c_3 \cos x)$

$u_1 = x^2$ $u_2 = \sin x$ $S_1 = \{x^2, x, 1\}$ $S_2 = \{\sin x, \cos x\}$.

$S_1' = \{x^3, x^2, x\}$ $S_2' = \{x\sin x, x\cos x\}$.

$y = Ax^3 + Bx^2 + Cx + Dx\sin x + Ex\cos x$

$y' = 3Ax^2 + 2Bx + C + D\sin x + Dx\cos x + E\cos x - Ex\sin x$.

$y'' = 6Ax + 2B + 2D\cos x - Dx\sin x - 2E\sin x - Ex\cos x$

$y''' = 6A + 3D\sin x - Dx\cos x - 3E\cos x + Ex\sin x$

$y''' + y' = (6A + C) + 2Bx + 3Ax^2 + (-3D + D)\sin x + (-3E + E)\cos x = 2x^2 + 4\sin x$

$6A + C = 0 \Rightarrow C = -4$ $\quad -2D = 4 \quad D = -2$ $\quad y_p = \frac{2}{5}x^3 - 4x - 2x\sin x$

$2B = 0$ $\quad E = 0$

$3A = 2 \Rightarrow A = \frac{2}{3}$

12) $y^{IV} - 3y''' + 2y'' = 3e^{-x} + 6e^{2x} - 6x$ $\quad$ 13) $y''' - 6y'' + 11y' - 6y = xe^x - 4e^{2x} + 6e^{4x}$

Solution: $m^3 - 6m^2 + 11m - 6 = 0 = (m-1)(m-2)(m-3) = 0 \quad m = 1, 2, 3$

$$\begin{array}{l|l} m^3 - 6m^2 + 11m - 6 & m-1 \\ \hline -\ m^3 - m^2 & m^2 - 5m + 6 \\ \hline -5m^2 + 11m & \\ -\ -5m^2 + 5m & \\ \hline 6m - 6 & \end{array}$$

$y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$

$u_1 = xe^x \quad u_2 = e^{2x} \quad u_3 = e^{4x}$

$S_1 = \{xe^x, e^x\} \quad S_2 = \{e^{2x}\} \quad S_3 = \{e^{4x}\}$.

$S_1' = \{x^2e^x, xe^x\}$. $\quad S_2' = \{xe^{2x}\}$.

~~$y_p = Axe^x + Be^x + Cxe^{2x} + De^{4x}$~~

~~$y' = Ae^x + Axe^x + Be^x + Ce^{2x} + 2Cxe^{2x} + 4De^{4x}$~~

~~$y'' = 2Ae^x + Axe^x + Be^x + 4Ce^{2x} + 4Cxe^{2x} + 16De^{4x}$~~

~~$y''' = 3Ae^x + Axe^x + Be^x + 12Ce^{2x} + 8Cxe^{2x} + 64De^{4x}$.~~

~~$(3A - 12A + 11A)e^x + (12C - 24C + 11C)e^{2x} + (8C - 24C + 22C - 6C)xe^{2x}$.~~

~~$(64D - 96D + 44D - 6D)e^{4x} = xe^x - 4e^{2x} + 6e^{4x}$~~

~~$2A = 0$~~ $\quad y_p = \frac{x^2e^x}{4} + \frac{3xe^x}{4} + 4xe^{2x} + \text{[illegible]}e^{4x}$

14) $y'' + y = x\sin x$ $\quad$ 15) $y'' + 4y = 12x^2 - 16x\cos 2x$

These equations simplify at once to the following:

$$\begin{cases} c_1 + c_2 = \dfrac{7}{2} \\ 3c_1 - c_2 = \dfrac{5}{2}. \end{cases}$$

From these two equations we obtain

$$\begin{cases} c_1 = \dfrac{3}{2} \\ c_2 = 2. \end{cases}$$

Substituting these values for c_1 and c_2 into Equation (4.27) we obtain the unique solutio of the given initial-value problem in the form

$$y = \frac{3}{2}e^{3x} + 2e^{-x} - \frac{1}{2}e^{x} + 2\sin x - \cos x.$$

Exercises

Find the general solution of each of the differential equations in Exercises 1 through 17.

1. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4x^2.$

2. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 8y = 4e^{2x} - 21e^{-3x}.$

3. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = 6\sin 2x + 7\cos 2x.$

4. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 10\sin 4x.$

5. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 4y = \cos 4x.$

6. $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 10y = 8xe^{-2x}.$

7. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 5y = 5\sin 2x + 10x^2 - 3x + 7.$

8. $4\dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 3y = 3x^3 - 8x.$

9. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 10e^{2x} - 18e^{3x} - 6x - 11.$

10. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4y = 4e^{x} - 18e^{-x}.$

11. $\dfrac{d^3y}{dx^3} + \dfrac{dy}{dx} = 2x^2 + 4\sin x.$

12. $\dfrac{d^4y}{dx^4} - 3\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} = 3e^{-x} + 6e^{2x} - 6x.$

13. $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = xe^x - 4e^{2x} + 6e^{4x}.$

14. $\dfrac{d^2y}{dx^2} + y = x\sin x.$

15. $\dfrac{d^2y}{dx^2} + 4y = 12x^2 - 16x\cos 2x.$

16. $\dfrac{d^4y}{dx^4} - 5\dfrac{d^3y}{dx^3} + 7\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 5\sin x - 12\sin 2x.$

17. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} = 18x^2 + 16xe^x + 4e^{3x} - 9.$

Solve the initial-value problems in Exercises 18 through 21.

18. $\begin{cases} \dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 9x^2 + 4. \\ y(0) = 6 \\ y'(0) = 8. \end{cases}$

19. $\begin{cases} \dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 13y = 5\sin 2x \\ y(0) = 1 \\ y'(0) = -2. \end{cases}$

20. $\begin{cases} \dfrac{d^2y}{dx^2} + y = 3x^2 - 4\sin x \\ y(0) = 0 \\ y'(0) = 1. \end{cases}$

21. $\begin{cases} \dfrac{d^3y}{dx^3} - 4\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 6y = 3xe^x + 2e^x - \sin x \\ y(0) = \dfrac{33}{40} \\ y'(0) = 0 \\ y''(0) = 0. \end{cases}$

For each of the differential equations in Exercises 22 through 30 *set up* the correct linear combination of functions with undetermined literal coefficients to use in finding a particular integral by the method of undetermined coefficients. (Do not actually find the particular integrals.)

22. $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = x^3 + x + e^{-2x}.$

23. $\frac{d^2y}{dx^2} + 9y = e^{3x} + e^{-3x} + e^{3x}\sin 3x.$

24. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2e^x + 3xe^{2x} + 5x^2.$

25. $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} - 8y = xe^{2x} + x^2e^{3x}.$

26. $\frac{d^4y}{dx^4} + 3\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = x^2e^{-x} + 3e^{-1/2x}\cos\frac{\sqrt{3}}{2}x.$

27. $\frac{d^4y}{dx^4} - 16y = x^2\sin 2x + x^4e^{2x}.$

28. $\frac{d^6y}{dx^6} + 2\frac{d^5y}{dx^5} + 5\frac{d^4y}{dx^4} = x^3 + x^2e^{-x} + e^{-x}\sin 2x.$

29. $\frac{d^4y}{dx^4} + 3\frac{d^2y}{dx^2} - 4y = \cos^2 x - \cosh x.$

30. $\frac{d^4y}{dx^4} + 10\frac{d^2y}{dx^2} + 9y = \sin x \sin 2x.$

4.4 Variation of Parameters

A. The Method

While the process of carrying out the method of undetermined coefficients is actual quite straightforward (involving only techniques of college algebra and differentiation the method applies in general to a rather small class of problems. For example, it woul not apply to the apparently simple equation

$$\frac{d^2y}{dx^2} + y = \tan x.$$

We thus seek a method of finding a particular integral which applies in all cases (includin variable coefficients) in which the complementary function is known. Such a method is th method of *variation of parameters*, which we now consider.

We shall develop this method in connection with the general second order line differential equation with variable coefficients

(4.29)
$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = b(x).$$

Suppose that y_1 and y_2 are linearly independent solutions of the corresponding homo geneous equation

(4.30)
$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0.$$

Then the complementary function of equation (4.29) is

$$c_1y_1 + c_2y_2,$$