Higher Order Linear Differential Equations

Definition: A linear differential equation of order n is an equation of the form

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x), \quad (1)$$

where a_0 is not identically zero. We assume that $a_0, a_1, \dots, a_n$ and b are continuous real functions on a real interval $a \le x \le b$ and that $a_0(x) \neq 0$ for any x on $a \le x \le b$. The right hand member is called the nonhomogeneous term. If b is identically zero the equation reduces to

$$a_0(x)\frac{d^n y}{dx^n} + \dots + a_n(x)y = 0$$

and is then called homogeneous.

Examples (1): $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + x^3 y = e^x$ is a linear dif. eq. of 2nd order.

(2) $\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + 3x^2\frac{dy}{dx} - 5y = \sin x$ is a linear dif. eq. of 3rd order.

Theorem Consider nth order linear differential equation (1). Let x_0 be any point of the interval $[a,b]$ and c_i $(0 \le i \le n-1)$ be n arbitrary real constants. If $a_0(x) \neq 0$ for ~~any~~ $x \in [a,b]$, then there exists a unique solution f of (1) such that

$$f(x_0) = c_0, \quad f'(x_0) = c_1, \dots, \quad f^{n-1}(x_0) = c_{n-1}$$

and this solution is defined over the entire interval $[a,b]$.

Example: Consider the initial-value problem

$$y'' + 3xy' + x^3 y = e^x, \quad y(1) = 2 \quad y'(1) = -5.$$

The coefficients $1, 3x$ and x^3, as well as the nonhomogeneous term e^x are all continuous for all value of x, $-\infty < x < \infty$.

The point x_0 here is the point 1.

The real numbers c_0, c_1 are 2 and -5 respectively.

Thus there exists a solution of the given problem which is unique, and is defined for all $x \in \mathbb{R}$.

Corollary: Let f be a solution of the nth order homogeneous linear differential equation such that

$$g(x_0) = g'(x_0) = \cdots = g^{n-1}(x_0) = 0 \qquad x_0 \in [a,b].$$

Then $f(x) = 0$ for all $x \in [a,b]$.

This corollary tells us that this solution is the "trivial" solution f such that $f(x) = 0$ for all $x \in [a,b]$.

Example: The solution f of $y''' + 2y'' + 4xy' + x^2 y = 0$ which is such that $f(2) = f'(2) = f''(2) = 0$ is the trivial solution f such that $f(x) = 0$ for all x.

Definition If f_i $(1 \le i \le m)$ are m given functions, and c_i $(1 \le i \le m)$ are m constants, then the expression

$$c_1 f_1 + c_2 f_2 + \cdots + c_m f_m$$

is called a linear combination of $f_1, \ldots, f_m$.

Theorem: Any linear combination of solutions of the homogeneous linear differential equation is also a solution.

Examples (1): $f_1(x) = \sin x$ and $f_2(x) = \cos x$ are solutions of $y'' + y = 0$. It is easy to verify that $c_1 \sin x + c_2 \cos x$ is also a solution for any constants c_1 and c_2.

(2) It is easy to verify that e^x, e^{-x} and e^{2x} are solutions of $y''' - 2y'' - y' + 2y = 0$. Then it is clear that $c_1 e^x + c_2 e^{-x} + c_3 e^{2x}$ is also a solution for any $c_1, c_2, c_3 \in \mathbb{R}$.

Definition: The n function $f_1, \ldots, f_n$ are called linearly dependent on $a \le x \le b$ if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$$

for all $x \in [a,b]$.

For example, $f_1(x)=x$ and $f_2(x)=2x$ are linearly dependent since $-2f_1+f_2=0$.

Definition: The n function $f_1,\dots,f_n$ are called linearly independent on the interval $a\le x\le b$ if they are not linearly dependent there. That is, The function $f_1,\dots,f_n$ are linearly independent on $a\le x\le b$ if the relation

$$c_1f_1(x)+\cdots+c_nf_n(x)=0$$

for all $x\in[a,b]$ implies that $c_1=\cdots=c_n=0$.

For example x and x^2 are linearly independent on $0\le x\le 1$, since $c_1x+c_2x^2=0$ for all $x\in[0,1]$ implies that $c_1=c_2=0$.

Theorem: The nth order homogeneous linear differential equation always possesses n solutions which are linearly independent. Further if $f_1,\dots,f_n$ are n linearly independent solutions of its, then every solution f of its can be expressed as a linear combination $c_1f_1+\dots+c_nf_n$ of these n linear independent solutions by proper choice of the constante $c_1,\dots,c_n$.

Thus the solution $c_1f_1+\dots+c_nf_n$ is a "general" solution.

Definition: If $f_1,\dots,f_n$ are n linearly independent solutions of the nth order homogenous linear differential equation on $a\le x\le b$, then the function f defined by $f(x)=c_1f_1(x)+\dots+c_nf_n(x)$, where $c_i\in\mathbb{R}$, is called a general solution of the equation on $a\le x\le b$.

Examples (1) We have observed that $\sin x$ and $\cos x$ are solutions of $y''+y=0$ for all $x\in\mathbb{R}$. Thus, the general solution of $y''+y=0$ is

$$c_1\sin x+c_2\cos x$$

since $\sin x$ and $\cos x$ linearly independent.

(2) The solutions e^x, e^{-x}, and e^{2x} of $\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}-\frac{dy}{dx}+2y=0$ may be shown to be linearly independent for all $x\in\mathbb{R}$. Thus the general solution may be expressed as the linear combination

$$f(x)=c_1e^x+c_2e^{-x}+c_3e^{2x}$$

Definition: Let $f_1, \dots, f_n$ be n real functions each of which has an $(n-1)$st derivative on real interval. The determinant

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of these n functions.

We observe that $W(f_1, \dots, f_n)$ is itself a real function defined on $a \le x \le b$. Its value at x is denoted by $W(f_1, \dots, f_n)(x)$.

Theorem: (a) The n solutions $f_1, \dots, f_n$ of the nth order homogeneous linear differential equation are linearly independent on $a \le x \le b$ if and only if the Wronskian of $f_1, \dots, f_n$ is different from zero for some $x \in [a,b]$.

(b) The Wronskian of n solutions $f_1, \dots, f_n$ is either identically zero on $[a,b]$ or else is never zero on $a \le x \le b$.

Examples (1) The solutions $\sin x$ and $\cos x$ of $y'' + y = 0$ are linearly independent. Indeed

$$W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0.$$

(2) The solution e^x, e^{-x}, e^{2x} of $y''' - 2y'' - y' + 2y = 0$ are linearly independent. Indeed

$$W(e^x, e^{-x}, e^{2x}) = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix} = e^x \cdot e^{-x} \cdot e^{2x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = -6e^{2x} \neq 0.$$

Theorem: Let f be a nontrivial solution of the nth order homogeneous linear differential equation. The transformation $y = fu$ reduces the equation to an $(n-1)$st order homogeneous linear differential equation in the dependent variable $w = \frac{du}{dx}$.

Suppose f is a known nontrivial solution of

$$a_0(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x) y = 0 \qquad (2).$$

let us make the transformation $y = fu$ where u is a function of x.

Then $$\frac{dy}{dx} = f\frac{du}{dx} + u\frac{df}{dx}$$

$$\frac{d^2y}{dx^2} = f\frac{d^2u}{dx^2} + 2\frac{df}{dx}\cdot\frac{du}{dx} + \frac{d^2f}{dx^2}\cdot u.$$

Thus $$a_0\left(f\frac{d^2u}{dx^2} + 2\frac{df}{dx}\cdot\frac{du}{dx} + \frac{d^2f}{dx^2}u\right) + a_1\left(f\frac{du}{dx} + u\frac{df}{dx}\right) + a_2 fu = 0$$

or $$a_0 f\frac{d^2u}{dx^2} + \left[2a_0\frac{df}{dx} + a_1 f\right]\frac{du}{dx} + \left[a_0\frac{d^2f}{dx^2} + a_1\frac{df}{dx} + a_2 f\right]u = 0.$$

Since f is a solution of (2), the coefficient of u is zero, and so

$$a_0 f\frac{d^2u}{dx^2} + \left[2a_0\frac{df}{dx} + a_1 f\right]\frac{du}{dx} = 0$$

Letting $w = \frac{du}{dx}$, this becomes

$$a_0 f\frac{dw}{dx} + \left[2a_0\frac{df}{dx} + a_1 f\right]w = 0$$

which is first-order homogeneous linear differential. This equation is separable; thus assuming $f \neq 0$, and $a_0 \neq 0$, we may write

$$\frac{dw}{w} = -\left[2\frac{f'}{f} + \frac{a_1}{a_0}\right]dx$$

Thus integrating

$$\ln|w| = -\ln f^2 - \int\frac{a_1}{a_0}dx + \ln|k|$$

or $$w = \frac{c\,e^{-\int\frac{a_1}{a_0}dx}}{f^2}$$

Choosing $c=1$, recalling that $\frac{du}{dx} = w$, and integrating again, we have

$$u = \int\frac{e^{-\int\frac{a_1}{a_0}dx}}{f^2}dx$$

Finally, we obtain

$$y = f\int\frac{e^{-\int\frac{a_1}{a_0}dx}}{f^2}dx. = g(x).$$

Since $W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} f & fu \\ f' & fu' + f'u \end{vmatrix} = f^2 u' = e^{-\int \frac{a_1}{a_0} dx} \neq 0$.

f and g are independent. Thus the linear combination

$$c_1 f + c_2 f \int \frac{e^{-\int \frac{a_1}{a_0} dx}}{f^2} dx$$

is the general solution of (2).

Theorem: let f be a nontrivial solution of $a_0(x) y'' + a_1(x) y' + a_2(x) y = 0$

Then $c_1 f + c_2 f \int \frac{e^{-\int \frac{a_1}{a_0} dx}}{f^2} dx$ is the general solution.

Examples: Given that $y = x$ is a solution of $(x^2+1) y'' - 2x y' + 2y = 0$.

Notice that $y = x$ satisfy the equation.

Let $y = xu \Rightarrow \frac{dy}{dx} = x \frac{du}{dx} + u \qquad \frac{d^2y}{dx^2} = x \frac{d^2u}{dx^2} + 2 \frac{du}{dx}$.

$$(x^2+1)\left(x \frac{d^2u}{dx^2} + 2 \frac{du}{dx}\right) - 2x\left(x \frac{du}{dx} + u\right) + 2xu = 0$$

$$x(x^2+1) \frac{d^2u}{dx^2} + \left(2(x^2+1) - 2x^2\right) \frac{du}{dx} = 0$$

$w = \frac{du}{dx} \Rightarrow \quad x(x^2+1) \frac{dw}{dx} + 2w = 0 \Rightarrow \frac{dw}{2w} = -\frac{dx}{x(x^2+1)}$

or $\frac{dw}{w} = \left(-\frac{2}{x} + \frac{2x}{x^2+1}\right) dx \Rightarrow \ln|w| = -2\ln|x| + \ln|x^2+1| \Rightarrow w = \frac{x^2+1}{x^2}$

$\frac{du}{dx} = w \Rightarrow \quad du = (1 + x^{-2})\, dx \Rightarrow u = x - x^{-1} = x - \frac{1}{x}$

$g(x) = xu = x\left(x - \frac{1}{x}\right) = x^2 - 1 \Rightarrow \underline{\underline{y = c_1 x + c_2(x^2-1)}}$

Nonhomogeneous Equation

Theorem: Let v be any solution of n^{th} order linear differential equation

$$a_0(x) \frac{d^n y}{dx^n} + \dots + a_n(x) y = b(x) \qquad (3)$$

Let u be any solution of the corresponding homogeneous equation

$$a_0(x)\frac{d^n y}{dx^n}+\cdots+a_n(x)y=0 \qquad (4)$$

Then $u+v$ is also a solution of the given nonhomogeneous equation

<u>Example</u>: Observe that $y=x$ is a solution of $y''+y=x$ and that $y=\sin x$ is a solution of the corresponding homogeneous equation $y''+y=0$. Then it is easy to verify that $\sin x+x$ is also a solution of $y''+y=x$.

<u>Definition</u>: Consider the nth-order nonhomogeneous equation (3) and the corresponding homogeneous equation (4).

(i) The general solution of (4) is called the complementary function of the equation (3). We shall denote by y_c

(ii) Any particular solution of (3) is called a particular integral of (3). We shall denote by y_p.

(iii) The solution y_c+y_p of (3) is called the general solution of (3).

Thus to find the general solution of (3), we need merely find (a) The complementary function y_c and (b) a particular integral.

<u>Example</u>: Consider the differential equation $y''+y=x$. The complementary function $y_c=c_1\sin x+c_2\cos x$. A particular integral $y_p=x$. Thus the general solution $y=c_1\sin x+c_2\cos x+x$.

Exercises

(1) Consider the dif. eq. $y'' - 5y' + 6y = 0$.

(a) Show that e^{2x}, e^{3x} are linearly independent solution.

(b) Write the general solution.

(c) Find the solution satisfies $y(0) = 2$, $y'(0) = 3$.

Solution $f_1 = e^{2x} \Rightarrow f_1' = 2e^{2x} \Rightarrow f_1'' = 4e^{2x}$

$f_2 = e^{3x} \Rightarrow f_2' = 3e^{3x} \Rightarrow f_2'' = 9e^{2x}$.

Since $4e^{2x} - 10e^{2x} + 6e^{2x} = 0$ and $9e^{2x} - 15e^{2x} + 6e^{2x} = 0$, e^{2x} and e^{3x} are solutions.

$$\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} = e^{5x} \neq 0, \text{ they are independent.}$$

The general solution $\underline{y = c_1 e^{2x} + c_2 e^{3x}}$.

$2 = c_1 + c_2$ and $y' = 2c_1 e^{2x} + 3c_2 e^{3x}$ $3 = 2c_1 + 3c_2$

$3 = 2(c_1 + c_2) + c_2 = 4 + c_2 \Rightarrow$ $c_2 = -1$ $c_1 = 3$

$$\underline{y = 3e^{2x} - e^{3x}}$$

2. Consider the dif. eq. $x^2 y'' + xy' - 4y = 0$

(a) Show that x^2 and $\frac{1}{x^2}$ are linearly independent solution. $(0 < x < \infty)$.

(b) Write the general solution

(c) Find the solution which satisfies $y(2) = 3$, $y'(2) = -1$ $\left(y = \frac{x^2}{4} + \frac{8}{x^2}\right)$

3. The functions e^x and e^{4x} are both solutions of the differential equation

$$y'' - 5y' + 4y = 0$$

Show that these solutions are linearly independent. $(-\infty < x < \infty)$

4. Given that e^{-x}, e^{3x}, and e^{4x} are all solutions of $y''' - 6y'' + 5y' + 12y = 0$. Show that they are linearly independent and write the general solution.

Solution

$$\begin{vmatrix} e^{-x} & e^{3x} & e^{4x} \\ -e^{-x} & 3e^{3x} & 4e^{4x} \\ e^{-x} & 9e^{3x} & 16e^{4x} \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ -1 & 3 & 4 \\ 1 & 9 & 16 \end{vmatrix} = 20e^{6x} \neq 0. \quad y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{4x}$$

5) Given that $y=x^2$ is a solution of $(x^3-x^2)y''-(x^3+2x^2-2x)y'+(2x^2+2x-2)y=0$

Write the general solution.

Solution: $y=x^2u \Rightarrow y'=x^2\frac{du}{dx}+2xu \Rightarrow y''=x^2\frac{d^2u}{dx^2}+4x\frac{du}{dx}+2u$

$$(x^3-x^2)\left(x^2\frac{d^2u}{dx^2}+4x\frac{du}{dx}+2u\right)+(-x^3-2x^2+2x)\left(x^2\frac{du}{dx}+2xu\right)+(2x^2+2x-2)x^2u=0$$

$$(x^5-x^4)\frac{d^2u}{dx^2}+(4x^4-4x^3-x^5-2x^4+2x^3)\frac{du}{dx}+(2x^5-2x^2+2x^2-4x^3-2x^4+2x^4+2x^3-2x^2)u$$

$$(x^5-x^4)\frac{d^2u}{dx^2}+(-x^5+2x^4-2x^3)\frac{du}{dx}=0$$

$w=\frac{du}{dx} \Rightarrow (x^5-x^4)\frac{dw}{dx}+(-x^5+2x^4-2x^3)w=0$

$$\Rightarrow \frac{dw}{w}=\left(\frac{x^5-2x^4+2x^3}{x^5-x^4}\right)dx=\left(1+\frac{-x^4+2x^3}{x^5-x^4}\right)dx$$

$$=\left(1+\frac{2-x}{x-x^2}\right)dx=\left(1+\frac{2}{x}+\frac{3}{1-x}\right)dx$$

$$\Rightarrow \ln|w|=x+2\ln|x|-3\ln|x-1|=x+\ln\left|\frac{x^2}{(x-1)^3}\right| \Rightarrow w=\frac{x^2}{(x-1)^3}e^x$$

$$\frac{du}{dx}=\frac{x^2}{(x-1)^3}e^x \qquad u=\int\frac{x^2e^x}{(x-1)^3} \qquad ?$$

$\boxed{y=c_1x^2+c_2xe^x}$

6) Given that $y=e^{2x}$ is a solution of $(2x+1)y''-4(x+1)y'+4y=0$.

Write down the general solution.

Solution: $y=e^{2x}u \Rightarrow y'=e^{2x}\frac{du}{dx}+2e^{2x}u \Rightarrow y''=e^{2x}\frac{d^2u}{dx^2}+4e^{2x}\frac{du}{dx}+4e^{2x}u$

$$(2x+1)\left(e^{2x}\frac{d^2u}{dx^2}+4e^{2x}\frac{du}{dx}+4e^{2x}u\right)-4(x+1)\left(e^{2x}\frac{du}{dx}+2e^{2x}u\right)+4e^{2x}u$$

$$(2x+1)e^{2x}\frac{d^2u}{dx^2}+\left((2x+1)4e^{2x}-4(x+1)e^{2x}\right)\frac{du}{dx}+\left((2x+1)4e^{2x}-4(x+1)2e^{2x}+4e^{2x}\right)u$$

$$(2x+1)e^{2x}\frac{d^2u}{dx^2}+(4xe^{2x})\frac{du}{dx}=0.$$

$w=\frac{du}{dx} \Rightarrow (2x+1)e^{2x}\frac{dw}{dx}+4xe^{2x}w=0$

$$\frac{dw}{w}=-\frac{4xe^{2x}}{(2x+1)e^{2x}}dx=-\frac{4x}{2x+1}dx=\left(-2+\frac{2}{2x+1}\right)dx$$

(8)

$\ln|w| = -2x + \ln|2x+1| \Rightarrow w = (2x+1)e^{-2x}$.

$\frac{du}{dx} = (2x+1)e^{-2x} \Rightarrow u = \int (2x+1)e^{-2x}dx$

$u = 2x+1 \Rightarrow du = 2dx$

$dv = e^{-2x}dx \quad v = -\frac{1}{2}e^{-2x}$

$\Rightarrow u = \frac{-(2x+1)}{2}e^{-2x} + \frac{1}{2}\int 2e^{-2x}dx = \frac{(-2x-1)}{2}e^{-2x} - \frac{1}{2}e^{-2x}$

$= -(x+1)e^{-2x} \Rightarrow f = e^{2x}u = -(x+1)$

$\underline{y = c_1 e^{2x} + c_2(x+1)}$

7) Consider the nonhomogeneous dif. eq. $y'' - 3y' + 2y = 4x^2$.

(a) Show that e^x and e^{2x} are linearly independent solution of $y'' - 3y' + 2y = 0$

(b) What is the complementary function

(c) Show that $2x^2 + 6x + 7$ is a particular integral

(d) Write the general solution.

Solution a) $f_1 = e^x = f_1' = f_1'' \Rightarrow e^x - 3e^x + 2e^x = 0$

$f_2 = e^{2x} \Rightarrow f_2' = 2e^{2x} \Rightarrow f_2'' = 4e^{2x} \quad 4e^{2x} - 6e^{2x} + 2e^{2x} = 0$.

$$\begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x} \neq 0 \quad \checkmark.$$

(b) $y_c = c_1 e^x + c_2 e^{2x}$

(c) $y = 2x^2 + 6x + 7 \Rightarrow y' = 4x + 6 \quad y'' = 4$

$4 - 3(4x+6) + 2(2x^2 + 6x + 7) = \cancel{4} - \cancel{12x} - \cancel{18} + 4x^2 + \cancel{12x} + \cancel{14} = 4x^2 \checkmark$.

(d) $y = c_1 e^x + c_2 e^{2x} + 2x^2 + 6x + 7$. $\checkmark$.