Separable Equations

Definition: An equation of the form

$$F(x)\,G(y)\,dx + f(x)\,g(y)\,dy = 0 \qquad (1)$$

is called an equation with variables separable or simply a separable equation.

In general the separable equation (1) is not exact, but it posses an obvious integrating factor, namely $\frac{1}{f(x)\,G(y)}$. For if we multiply (1) by this expression, we separate the variables, reducing (1) to the equivalent equation

$$\frac{F(x)}{f(x)}\,dx + \frac{g(y)}{G(y)}\,dy = 0 \qquad (2)$$

This equation is exact, since $\frac{\partial(F/f)}{\partial y} = 0 = \frac{\partial(g/G)}{\partial x}$.

Denoting $F(x)/f(x)$ by $M(x)$ and $g(y)/G(y)$ by $N(y)$, equation (2) takes the form $M(x)\,dx + N(y)\,dy = 0$. Since M is a function of x only and N is a function of y only, we see at once that the solution is $\int M(x)\,dx + \int N(y)\,dy = c$ when c is an arbitrary constant.

Examples (1) $(x^3+x^2)y\,dx + x^2(y^3+2y)\,dy = 0$. Separating the variables by dividing x^2y we obtain

$$\frac{x^3+x^2}{x^2}\,dx + \frac{y^3+2y}{y}\,dy = 0 \quad \text{or}$$

$$(x+1)\,dx + (y^2+2)\,dy = 0$$

From this we have

$$\int (x+1)\,dx + \int (y^2+2)\,dy = c \quad \text{or} \quad \frac{x^2}{2} + x + \frac{y^3}{3} + 2y = c.$$

(2) Solve the initial-value problem

$$2x\sin y\,dx + (x^2+1)\cos y\,dy = 0 \quad y(1) = \frac{\pi}{2}$$

Dividing by $\sin y \cdot (x^2+1)$ we obtain $\frac{x}{x^2+1}dx + \frac{\cos y}{\sin y}dy = 0$

Thus $\int \frac{x\,dx}{x^2+1} + \int \frac{\cos y\,dy}{\sin y} = c_0 \Rightarrow \frac{1}{2}\ln(x^2+1) + \ln|\sin y| \mp c_0$

Choosing $2c_0 = \ln|c|$, we have $\ln(x^2+1) + 2\ln|\sin y| = \ln|c|$

or $\ln((x^2+1)\sin^2 y) = \ln c \Rightarrow (x^2+1)\sin^2 y = c$

$x=1$ and $y = \frac{\pi}{2} \Rightarrow 2 \cdot \sin^2\frac{\pi}{2} = c \Rightarrow c = 2$

$$\underline{(x^2+1)\sin^2 y = 2}$$

(3) Solve the initial value problem

$$2(y-1)dy - (3x^2+4x+2)dx = 0 \quad y(0) = -1.$$

$\int 2(y-1)dy - \int (3x^2+4x+2)dx = 0 \Rightarrow y^2 - 2y - x^3 - 2x^2 - 2x = c$

$x=0$ and $y=-1 \Rightarrow 1+2 = c \Rightarrow c = 3 \Rightarrow (y-1)^2 = x^3+2x^2+2x+4$

$\Rightarrow y = 1 \mp \sqrt{x^3+2x^2+2x+4} \underset{y(0)=-1}{\Longrightarrow} \underline{y = 1 - \sqrt{x^3+2x^2+2x+4}}.$

Homogeneous Equations

<u>Definition:</u> The first-order differential equation $M\,dx + N\,dy = 0$ is said to be homogeneous if, when we write in the form $\frac{dy}{dx} = f(x,y)$, there exists g such that $f(x,y)$ can be expressed in the form $g(\frac{y}{x})$.

<u>Examples:</u> The differential equation $(x^2-3y^2)dx + 2xy\,dy = 0$ is homogeneous. To see this

$$\frac{dy}{dx} = \frac{3y^2-x^2}{2xy} = \frac{3y}{2x} - \frac{x}{2y} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{1}{\frac{y}{x}}\right)$$

Take $g(t) = \frac{3}{2}t - \frac{1}{2}\cdot\frac{1}{t}$ so that $\frac{dy}{dx} = g(\frac{y}{x})$.

(2) The equation $(y+\sqrt{x^2+y^2})\,dx - x\,dy = 0$ is homogeneous.

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} = \frac{y}{x} \mp \frac{\sqrt{x^2+y^2}}{\sqrt{x^2}} = \frac{y}{x} \mp \sqrt{1+(\frac{y}{x})^2}$$

Take $g(t) = t \mp \sqrt{1+t^2}$, depending on the sign of x.

(3) The differential equation $\frac{dy}{dx} = \ln x - \ln y + \frac{x+y}{x-y}$ is homogeneous.

$$\frac{dy}{dx} = \ln(x/y) + \frac{1+y/x}{1-y/x}.$$ Take $g(t) = \ln(t) + \frac{1+t}{1-t}$.

A function F is called homogeneous of degree n if $F(tx,ty) = t^n F(x,y)$.

For example $F(x,y) = x^2+y^2$ is homogeneous of degree 2, since

$$F(tx,ty) = (tx)^2 + (ty)^2 = t^2 F(x,y).$$

Now suppose that the function M and N in the dif. equation $M\,dx + N\,dy = 0$ are both homogeneous of the same degree n.

Since $M(tx,ty) = t^n M(x,y)$, we have

$$M(1,\frac{y}{x}) = M(\frac{1}{x}\cdot x, \frac{1}{x}\cdot y) = (\frac{1}{x})^n M(x,y).$$

Similarly $N(1,\frac{y}{x}) = (\frac{1}{x})^n N(x,y)$.

Now we find

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)} = -\frac{(\frac{1}{x})^{-n} M(1,\frac{y}{x})}{(\frac{1}{x})^{-n} N(1,\frac{y}{x})} = -\frac{M(1,\frac{y}{x})}{N(1,\frac{y}{x})}$$

Thus we conclude that if M and N in $M\,dx + N\,dy = 0$ are both homogeneous functions of the same degree, then the differential equation is a homogeneous differential equation.

Example: Consider $(x^2-3y^2)dx + 2xy\,dy = 0$. It is obvious that both M and N are homogeneous of degree 2.

Theorem: If $M dx + N dy = 0$ is a homogeneous equation, then the change of variables $y = vx$ transforms the equation into a separable equation in the variables v and x.

Proof: Since $M dx + N dy = 0$ is homogeneous, it may be written in the form $\frac{dy}{dx} = g(\frac{y}{x})$.

Let $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and $\frac{dy}{dx} = g(v)$.

Thus $v + x\frac{dv}{dx} = g(v)$ or $(v - g(v))\,dx + x\,dv = 0$. or

$\frac{dx}{x} + \frac{dv}{v-g(v)} = 0$ which separable. ∎

Thus to solve a homogeneous differential equation $M dx + N dy = 0$, we let $y = vx$. Then we have

$$\int \frac{dx}{x} + \int \frac{dv}{v-g(v)} = c$$

Let $F(v) = \int \frac{dv}{v-g(v)}$. Then we have $F(\frac{y}{x}) + \ln|x| = c$ which is the solution of $M dx + N dy = 0$.

Examples: (1). Solve the equation $(x^2-3y^2)dx + 2xy\,dy = 0$

$\frac{dy}{dx} = -\frac{x^2-3y^2}{2xy} = \frac{3y}{2x} - \frac{x}{2y}$. Let $y = vx$, we obtain

$v + x\frac{dv}{dx} = \frac{3v}{2} - \frac{1}{2v} \Rightarrow x\frac{dv}{dx} = \frac{v}{2} - \frac{1}{2v} = \frac{v^2-1}{2v}$

$\Rightarrow 2vx\,dv = (v^2-1)dx \Rightarrow \frac{2v\,dv}{v^2-1} = \frac{dx}{x}$

$\ln|v^2-1| = \ln|x| + \ln|c| \Rightarrow v^2-1 = cx \Rightarrow \frac{y^2}{x^2} - 1 = cx$

$\Rightarrow \underline{y^2 - x^2 - cx^3 = 0}$

(2) Solve the initial-value problem

$(y+\sqrt{x^2+y^2})\,dx - x\,dy = 0 \qquad y(1)=0.$

It is clear that it is homogeneous of degree 1. As before

$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} = \frac{y}{x} + \sqrt{1+(y/x)^2}$. let $y = vx$

$v + x\frac{dv}{dx} = v + \sqrt{1+v^2} \Rightarrow x\frac{dv}{dx} = \sqrt{1+v^2} \Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}.$

$\ln|v+\sqrt{v^2+1}| = \ln|x| + \ln|c| \Rightarrow v + \sqrt{v^2+1} = cx$

$\Rightarrow \frac{y}{x} + \sqrt{\frac{y^2}{x^2}+1} = cx$ or $y + \sqrt{x^2+y^2} = cx^2$

$y=0$ at $x=1 \Rightarrow 1 = c \Rightarrow y + \sqrt{x^2+y^2} = x^2.$

~~(3) Solve the initial-value problem~~

~~$(x^2+3y^2)\,dx - 2xy\,dy = 0, \quad y(2)=6.$~~

$\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$

$$x\frac{dv}{dx} = \sqrt{1 + v^2}.$$

or

Separating variables, we find

$$\frac{dv}{\sqrt{v^2 + 1}} = \frac{dx}{x}.$$

Using tables, we perform the required integrations to obtain

$$\ln |v + \sqrt{v^2 + 1}| = \ln |x| + \ln |c|,$$

or

$$v + \sqrt{v^2 + 1} = cx.$$

Now replacing v by $\dfrac{y}{x}$, we obtain the general solution of the differential equation in the form

$$\frac{y}{x} + \sqrt{\frac{y^2}{x^2} + 1} = cx$$

or

$$y + \sqrt{x^2 + y^2} = cx^2.$$

The initial condition requires that $y = 0$ when $x = 1$. This gives $c = 1$ and hence

$$y + \sqrt{x^2 + y^2} = x^2,$$

from which it follows that

$$y = \tfrac{1}{2}(x^2 - 1).$$

Exercises

Solve each of the differential equations in Exercises 1 through 14.

1. $4xydx + (x^2 + 1)dy = 0.$
2. $(xy + 2x + y + 2)dx + (x^2 + 2x)dy = 0.$
3. $2r(s^2 + 1)dr + (r^4 + 1)ds = 0.$
4. $\csc y\,dx + \sec x\,dy = 0.$
5. $\tan\theta\,dr + 2r\,d\theta = 0.$
6. $(e^v + 1)\cos u\,du + e^v(\sin u + 1)dv = 0.$
7. $(x + 4)(y^2 + 1)dx + y(x^2 + 3x + 2)dy = 0.$
8. $(x + y)dx - xdy = 0.$
9. $(2xy + 3y^2)dx - (2xy + x^2)dy = 0.$
10. $v^3du + (u^3 - uv^2)dv = 0.$
11. $\left(x\tan\frac{y}{x} + y\right)dx - xdy = 0.$
12. $(2s^2 + 2st + t^2)ds + (s^2 + 2st - t^2)dt = 0.$

13. $(x^3 + y^2\sqrt{x^2 + y^2})dx - xy\sqrt{x^2 + y^2}dy = 0$

14. $(\sqrt{x + y} + \sqrt{x - y})dx + (\sqrt{x - y} - \sqrt{x + y})dy = 0$

Solve the initial-value problems in Exercises 15 through 20.

15. $(y + 2)dx + y(x + 4)dy = 0$

$y(-3) = -1$

16. $8\cos^2 y\,dx + \csc^2 x\,dy = 0$

$y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$

17. $(3x + 8)(y^2 + 4)dx - 4y(x^2 + 5x + 6)dy = 0$

$y(1) = 2$

18. $(x^2 + 3y^2)dx - 2xy\,dy = 0$

$y(2) = 6$

19. $(2x - 5y)dx + (4x - y)dy = 0$

$y(1) = 4$

20. $(3x^2 + 9xy + 5y^2)dx - (6x^2 + 4xy)dy = 0$

$y(2) = -6$

21. Show that the homogeneous equation

$$(Ax^2 + By^2)dx + (Cxy + Dy^2)dy = 0$$

is exact if and only if $2B = C$.

22. Solve by two methods:

$$(x^2 + 2y^2)dx + (4xy - y^2)dy = 0.$$

23. (a) Prove that if $M\,dx + N\,dy = 0$ is a homogeneous equation, then the change of variables $x = uy$ transforms this equation into a separable equation in the variables y and u.

(b) Use the result of (a) to solve the equation of Example 2.12 of the text.

(c) Use the result of (a) to solve the equation of Example 2.13 of the text.

24. Suppose the equation $M\,dx + N\,dy = 0$ is homogeneous. Show that the transformation $x = r\cos\theta$, $y = r\sin\theta$ reduces this equation to a separable equation in the variables r and θ.

25. Suppose the equation

(A) $$M\,dx + N\,dy = 0$$

is homogeneous.

(a) Show that equation (A) is invariant under the transformation

$$x = k\xi$$

(B) $$y = k\eta,$$

where k is a constant.

Exercises

Solve each of the differential equations below:

(1) $4xy\,dx + (x^2+1)\,dy = 0$

(2) $(xy+2x+y+2)\,dx + (x^2+2x)\,dy = 0$

Solution: $(xy+2x+y+2) = x(y+2) + (y+2) = (x+1)(y+2)$

Dividing by $(y+2)(x^2+2x)$,

$$\frac{(x+1)}{x^2+2x}dx + \frac{dy}{y+2} = \int\frac{(x+1)\,dx}{(x+1)^2-1} + \int\frac{dy}{y+2} = 0$$

$$\frac{1}{2}\ln|x^2+2x| + \ln|y+2| = \frac{1}{2}\ln c \Rightarrow \ln(x^2+2x)(y+2)^2 = \ln c$$

$$(x^2+2x)(y+2)^2 = c.$$

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(3) $2r(s^2+1)\,dr + (r^4+1)\,ds = 0$

Dividing by $(s^2+1)(r^4+1)$

$$\int\frac{2r\,dr}{r^4+1} + \int\frac{ds}{s^2+1} = 0 \quad \int\frac{du}{u^2+1} + \int\frac{ds}{s^2+1} = 0$$

$r^2 = u$

arctan

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(5) $\csc y\,dx + \sec x\,dy = 0.$

(6) $\tan\theta\,dr + 2r\,d\theta = 0$

Dividing $2r\tan\theta \Rightarrow$ $\dfrac{dr}{2r} + \dfrac{d\theta}{\tan\theta} = 0 \Rightarrow \dfrac{dr}{2r} + \dfrac{\cos\theta\,d\theta}{\sin\theta} = 0$

$\Rightarrow \frac{1}{2}\ln|r| + \ln|\sin\theta| = \frac{1}{2}\ln|c| \Rightarrow \ln|r.\sin^2\theta| = \ln c$

$$r\sin^2\theta = c.$$

5. $(e^v+1)\cos u\,du + e^v(\sin u+1)\,dv = 0$

Solution: Dividing $(e^v+1)(\sin u+1)$

$\int \frac{\cos u\,du}{\sin u+1} + \int \frac{e^v}{e^v+1}\,dv = 0 \Rightarrow \ln|\sin u+1| + \ln|e^v+1| = \ln c$

$\Rightarrow (\sin u+1).(e^v+1) = c$

6. $(x+y)\,dx - x\,dy = 0$

7. $(2xy+3y^2)\,dx - (2xy+x^2)\,dy = 0$

Solution Homogeneous of degree 2.

$$\frac{dy}{dx} = \frac{2xy+3y^2}{2xy+x^2} = \frac{y}{x}\cdot\frac{(2x+3y)}{(2y+x)} = \frac{y}{x}\cdot\frac{(2+3\frac{y}{x})}{(2\frac{y}{x}+1)}$$

Let $y = vx \Rightarrow$ $\cancel{v + x\frac{dv}{dx} = v\cdot\frac{2+3v}{2v+1}}$

$$v + x\frac{dv}{dx} = \frac{2x^2v+3x^2v^2}{2x^2v+x^2} = \frac{2v+3v^2}{2v+1}$$

$$x\frac{dv}{dx} = \frac{2v+3v^2-2v^2-v}{2v+1} = \frac{v^2+v}{2v+1} \Rightarrow \int\frac{2v+1}{v^2+v}\,dv = \int\frac{dx}{x}$$

$\ln|v^2+v| = \ln|x| + \ln|c| \Rightarrow v^2+v = cx \qquad y^2+xy = cx^3$

8. $v^3\,du + (u^3-uv^2)\,dv = 0$

9. $(x\tan\frac{y}{x}+y)\,dx - x\,dy = 0$

Dividing by x $\quad (\tan\frac{y}{x}+\frac{y}{x})\,dx - dy = 0 \Rightarrow \frac{dy}{dx} = \tan\frac{y}{x}+\frac{y}{x}$

Let $y = xv \Rightarrow v + x\frac{dv}{dx} = \tan v + v \Rightarrow \frac{dv}{\tan v} = \frac{dx}{x}$

$\Rightarrow \int\frac{\cos v\,dv}{\sin v} = \int\frac{dx}{x} \Rightarrow \ln|\sin v| = \ln|x| + \ln|c| \Rightarrow \sin v = cx$

$\sin\frac{y}{x} = cx.$

10) $(2s^2+2st+t^2)\,ds + (s^2+2st-t^2)\,dt = 0$

11) $(x^3+y^2\sqrt{x^2+y^2})\,dx - xy\sqrt{x^2+y^2}\,dy = 0$

Solution: Homogeneous of degree 3.

Let $y = vx$ $\quad \dfrac{dy}{dx} = \dfrac{x^3+y^2\sqrt{x^2+y^2}}{xy\sqrt{x^2+y^2}} = \dfrac{x^3+x^2v^2\sqrt{x^2(1+v^2)}}{x^2v\sqrt{x^2(1+v^2)}}$

$= \dfrac{1+v^2\sqrt{1+v^2}}{v\sqrt{1+v^2}} = \dfrac{1}{v\sqrt{1+v^2}} + v$

$v + x\dfrac{dv}{dx} \Rightarrow \int v\sqrt{1+v^2}\,dv = \int \dfrac{dx}{x}$ +

$1+v^2 = u \Rightarrow 2v\,dv = du$ $\qquad \frac{1}{2}\int \sqrt{u}\,du = \ln x + \ln c.$

$\Rightarrow \frac{1}{3}u\sqrt{u} = \ln|cx| \Rightarrow \dfrac{(1+v^2)\sqrt{1+v^2}}{3} = \ln|cx|.$

$\dfrac{(x^2+y^2)\sqrt{x^2+y^2}}{3x^3} = \ln|cx|$ $\qquad (x^2+y^2)\sqrt{x^2+y^2} = 3x^3\ln|cx|.$

Solve the initial-value problems below:

12) $8\cos^2 y\,dx + \csc^2 x\,dy = 0$ $\qquad y(\frac{\pi}{12}) = \frac{\pi}{4}$

13) $(3x+8)(y^2+4)\,dx - 4y(x^2+5x+6)\,dy = 0$ $\qquad y(1) = 2$

14) $(x^2+3y^2)\,dx - 2xy\,dy = 0$ $\qquad y(2) = 6$

15) $(2x-5y)\,dx + (4x-y)\,dy = 0$ $\qquad y(1) = 4$

16) $(3x^2+9xy+5y^2)\,dx - (6x^2+4xy)\,dy = 0$ $\qquad y(2) = -6.$

Solution: Homogeneous of degree 2.

$y = vx$ $\qquad \dfrac{dy}{dx} = \dfrac{3x^2+9xy+5y^2}{6x^2+4xy} = \dfrac{3x^2+9x^2v+5x^2v^2}{6x^2+4x^2v} = \dfrac{5v^2+9v+3}{4v+6}$

$$\cancel{\frac{(4v+6)\,dv}{5v^2+8v+3} = \frac{dx}{x}}$$

$$\cancel{\int \frac{2}{5}\cdot\frac{10v+15}{5v^2+8v+3}\,dv = \frac{2}{5}\int\frac{10v+8}{5v^2+8v+3}\,dv\int\frac{6\,dv}{5v^2+8v+3}}$$

$$\therefore\; v + x\frac{dv}{dx} = \frac{5v^2+8v+3}{4v+6} \Rightarrow x\frac{dv}{dx} = \frac{5v^2+8v+3-4v^2-6v}{4v+6}$$

$$x\frac{dv}{dx} = \frac{v^2+3v+3}{4v+6} \Rightarrow \int\frac{(4v+6)\,dv}{(v^2+3v+3)} = \int\frac{dx}{x}$$

$$\Rightarrow 2\int\frac{(2v+3)\,dv}{v^2+3v+3} = \int\frac{dx}{x} \Rightarrow 2\ln|v^2+3v+3| = \ln|x| + \ln|c|$$

$$(v^2+3v+3)^2 = cx \Rightarrow (y^2-3yx+3x^2)^2 = cx^5$$

$$x=2 \quad y=-6 \qquad (36+36+12)^2 = c2^5 \Rightarrow (18.9)^2 = c2^5$$

$$8^6\,9 = c2^5$$

$$c = 162 \Rightarrow \underline{(y^2-3yx+3x^2)^2 = \frac{9}{2}x^5}$$