(7)

Variation of Parameters

This method only for the general second order linear differential equation with variable coefficients

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = b(x). \quad (1)$$

Suppose that y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0 \quad (2).$$

We assume a solution of the form

$$\underline{y_p = v_1 y_1 + v_2 y_2}$$

where v_1 and v_2 are functions on x. Then

$$y_p' = v_1 y_1' + v_2 y_2' + v_1' y_1 + v_2' y_2.$$

If we assume $v_1' y_1 + v_2' y_2 = 0$, then (3)

$$\underline{y_p' = v_1 y_1' + v_2 y_2'}$$

Again differentiating (3), we obtain

$$\underline{y_p'' = v_1 y_1'' + v_2 y_2'' + v_1' y_1' + v_2' y_2}. \quad (4).$$

We impose the basic condition that y_p be a solution of (1).
Thus we substitute (4), (3) and y_p respectively.

$$v_1\underbrace{[a_0 y_1'' + a_1 y_1' + a_2 y_1]}_{=0} + v_2\underbrace{[a_0 y_2'' + a_1 y_2' + a_2 y_2]}_{=0} + a_0[v_1' y_1' + v_2' y_2'] = b.$$

(which are solutions of (2))

Thus $v_1' y_1' + v_2' y_2' = \frac{b}{a_0}$. Thus we have.

$$y_1 v_1' + y_2 v_2' = 0 \quad \text{(by assumption)}.$$

$$y_1' v_1' + y_2' v_2' = \frac{b}{a_0}.$$

Then $v_1' = \dfrac{\begin{vmatrix} 0 & y_2 \\ \frac{b}{a_0} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \dfrac{-b y_2}{a_0 W(y_1, y_2)}$ similarly $v_2' = \dfrac{b y_1}{a_0 W(y_1, y_2)}$.

Examples $y'' + y = \tan x$

$m^2 + 1 = 0 \qquad m = \mp i \qquad y_c = c_1 \sin x + c_2 \cos x \qquad (y_1 = \sin x,\ y_2 = \cos x)$

$$W(y_1, y_2) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1$$

$$V_1' = \frac{-b\, y_2}{a_0\, W(y_1,y_2)} = \frac{-\tan x \cdot \cos x}{1 \cdot (-1)} = +\sin x \Rightarrow V_1(x) = -\cos x + c_3.$$

$$V_2' = \frac{b\, y_1}{a_0\, W(y_1,y_2)} = \frac{+\tan x \cdot \sin x}{-1} = \frac{-\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$

$\Rightarrow V_2 = \sin x - \ln|\sec x + \tan x| + c_4.$

$$y_p = c_3 \sin x + c_4 \cos x - (\cos x)\left(\ln|\sec x + \tan x|\right)$$

The general solution is

$$y = C_1 \sin x + C_2 \cos x - (\cos x)\left(\ln|\sec x + \tan x|\right)$$

$(C_1 = c_1 + c_3,\ C_2 = c_2 + c_4)$.

(2) $y''' - 6y'' + 11y' - 6y = e^x$

$m^3 - 6m^2 + 11m - 6 = (m-1)(m-2)(m-3) \qquad y_1 = e^x \quad y_2 = e^{2x} \quad y_3 = e^{3x}$

$$y_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

$$y_p = V_1 e^x + V_2 e^{2x} + V_3 e^{3x} \qquad \text{(condition first)}.$$

Since we have three conditions (because we have three functions).

$$y_p' = V_1 e^x + 2V_2 e^{2x} + 3V_3 e^{3x} + \underbrace{V_1' e^x + V_2' e^{2x} + V_3' e^{3x}}_{=0} \quad \text{(condition second)}.$$

$$y_p' = V_1 e^x + 2V_2 e^{2x} + 3V_3 e^{3x}$$

$$y_p'' = V_1 e^x + 4V_2 e^{2x} + 9V_3 e^{3x} + \underbrace{V_1' e^x + 2V_2' e^{2x} + 3V_3' e^{3x}}_{=0} \quad \text{(condition third)}.$$

$$y_p'' = V_1 e^x + 4V_2 e^{2x} + 9V_3 e^{3x}$$

$$y_p''' = V_1 e^x + 8V_2 e^{2x} + 27V_3 e^{3x} + V_1' e^x + 4V_2' e^{2x} + 9V_3' e^{3x}$$

Substituting $v'e^x + 4v_2'e^{2x} + 9v_3'e^{3x} = e^x$

Thus we have
$$\begin{cases} v_1'e^x + v_2'e^{2x} + v_3'e^{3x} = 0 \\ v_1'e^x + 2v_2'e^{2x} + 3v_3'e^{3x} = 0 \\ v_1'e^x + 4v_2'e^{2x} + 9v_3'e^{3x} = e^x \end{cases}$$

$$w(y_1,y_2,y_3) = e^{6x}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2e^{6x}$$

$$v_1' = \frac{\begin{vmatrix} 0 & e^{2x} & e^{3x} \\ 0 & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}}{2e^{6x}} = \frac{1}{2}$$
Similarly $v_2' = -e^{-x}$ $\quad v_3' = \frac{1}{2}e^{-2x}$

$v_1 = \frac{x}{2}$ $\quad v_2 = e^{-x}$ $\quad v_3 = -\frac{1}{4}e^{-2x}$.

$$y_p = \frac{x}{2}e^x + e^{-x}.e^{2x} - \frac{1}{4}e^{-2x}e^{3x} = \frac{1}{2}xe^x + \frac{3}{4}e^x$$

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x} + \frac{1}{2}xe^x + \frac{3}{4}e^x. \quad \text{or}$$
$$= c_1e^x + c_2e^{2x} + c_3e^{3x} + \frac{1}{2}xe^x$$

(3) $(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 6(x^2+1)^2$

We solved that $y_c = c_1x + c_2(x^2-1)$ is the general solution of the corresponding homogeneous equation.

$$y_p = v_1x + v_2(x^2-1)$$

$$y_p' = v_1 + v_2 2x + v_1'x + v_2'(x^2-1)$$

By the assumption $\quad v_1'x + v_2'(x^2-1) = 0,$

$$y_p' = v_1 + 2xv_2$$
$$y_p'' = v_1' + 2v_2 + 2xv_2'$$

By substituting.

$$(x^2+1)(v_1' + 2v_2 + 2xv_2') - 2x(v_1 + 2xv_2) + 2(v_1x + v_2(x^2-1))$$
$$(x^2+1)v_1' + 2x(x^2+1)v_2' + (-2x+2x)v_1 + (2x^2+2-4x^2+2x^2-2)v_2$$
$$= (x^2+1)v' + 2x(x^2+1)v_2' = 6(x^2+1)^2.$$

$x v_1' + (x^2-1) v_2' = 0$ or $x v_1' + (x^2-1) v_2' = 0$

$(x^2+1) v_1' + 2x(x^2+1) v_2' = 6(x^2+1)^2$ $v_1' + 2x v_2' = 6(x^2+1)$

$$W = \begin{vmatrix} x & x^2-1 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 + 1 = \underline{x^2+1}$$

$$v_1' = \frac{\begin{vmatrix} 0 & x^2-1 \\ 6(x^2+1) & 2x \end{vmatrix}}{x^2+1} = -6(x^2-1) = -6x^2+6 \Rightarrow \underline{v_1 = -2x^3+6x}$$

$$v_2' = \frac{\begin{vmatrix} x & 0 \\ 1 & 6(x^2+1) \end{vmatrix}}{(x^2+1)} = 6x \Rightarrow \underline{v_2' = 3x^2}$$

$$y = c_1 x + c_2(x^2-1) + x^4 + 3x^2$$

Theorem Let f be a particular integral of

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x) y = f(x)$$

Let g be a particular integral of

$$a_0(x)\frac{d^n y}{dx^n} + \dots\dots + a_n(x) y = G(x).$$

Then $k_1 f + k_2 g$ is a particular integral of

$$a_0(x)\frac{d^n y}{dx^n} + \dots\dots + a_n(x) y = k_1 F + k_2 G$$

where k_1 and k_2 are constants.

Example Find a particular integral of $y'' + y = 3e^x + 5\tan x$.

We found $y_{p_1} = \frac{1}{2}e^x$ is a particular solution of $y'' + y = 3e^x$ and $y_{p_2} = -(\cos x)\ln|\sec x + \tan x|$ is a particular solution of $y'' + y = \tan x$.

Thus $y_p = \frac{3}{2}e^x - 5(\cos x)\ln|\sec x + \tan x|$ is ~~the~~ particular integral of $y'' + y = 3e^x + 5\tan x$.

Exercises

Find the general solution of each of the followings

1) $y''+y=\cot x$ $(y_p=\sin x.\ln|\csc x-\cot x|$ 2) $y''+y=\tan 2x$

3) $y''+y=\sec x$ $(y_p=\cos x.\ln|\cos x|+x\sin x)$ 4) $y''+y=\sec^3 x$

5) $y''+4y=\sec^2 2x$

Solution: $m^2+4=0 \Rightarrow$ $y_c=c_1\sin 2x+c_2\cos 2x$.

$$y_p=V_1.\sin 2x+V_2.\cos 2x$$

$$y_p'=2V_1\cos 2x-2V_2\sin 2x+\underbrace{\sin 2x.V_1'+\cos 2x.V_2'}_{=0.}$$

$$y_p''=-4V_1\sin 2x-4V_2\cos 2x+2\cos 2x.V_1'-2\sin 2x.V_2'$$

$$\cancel{-4V_1\sin 2x}-\cancel{4V_2\cos 2x}+2\cos 2x.V_1'-2\sin 2x.V_2'+\cancel{4V_1\sin 2x}+\cancel{4V_2\cos 2x}$$

$$=2\cos 2x.V_1'-2\sin 2x.V_2'=\sec^2 2x.$$

$$\sin 2x.V_1'+\cos 2x.V_2'=0 \qquad \Rightarrow W=\begin{vmatrix}\sin 2x & \cos 2x\\ 2\cos 2x & -2\sin 2x\end{vmatrix}=-2.$$

$$2\cos 2x.V_1'-2\sin 2x.V_2'=\sec^2 2x$$

$$V_1'=\frac{\begin{vmatrix}0 & \cos 2x\\ \sec^2 2x & -2\sin 2x\end{vmatrix}}{-2}=\frac{\sec 2x}{-2}\Rightarrow V_1=+\frac{1}{4}\ln|\sec 2x+\tan 2x|$$

$$V_2'=\frac{\begin{vmatrix}\sin 2x & 0\\ 2\cos 2x & \sec^2 2x\end{vmatrix}}{-2}=\frac{\sin 2x}{-2\cos^2 2x}\Rightarrow V_2=-\frac{1}{4}\frac{1}{\cos 2x}$$

$$y_p=\frac{\sin 2x}{4}\ln|\sec 2x+\tan 2x|-\frac{1}{4}$$

6) $y''-2y'+y=e^x\ln x$ $(x>0)$ 7) $y''-2y'+y=xe^x\ln x$.

Solution $m^2-2m+1=(m-1)^2=0$ $m=1,1$ $\underline{y_c=c_1e^x+c_2xe^x}$

$$y_p=V_1e^x+V_2xe^x$$

$$y_p'=V_1e^x+V_2(e^x+xe^x)+\underbrace{V_1'e^x+V_2'xe^x}_{=0.}$$

$$y_p'=e^xV_1+(x+1)e^xV_2$$

$$y_p''=e^xV_1+(x+2)e^xV_2+e^xV_1'+(x+1)e^xV_2'$$

$$\cancel{e^x v_1 + (x+2)e^x v_2} + e^x v_1' + (x+1)e^x v_2' \cancel{-2e^x v_1 - 2(x+1)e^x v_2} \cancel{+ v_1 e^x + v_2 x e^x}$$

$$= e^x v_1' + (x+1)e^x v_2' = x e^x \ln x \qquad w = \begin{vmatrix} e^x & (x+1)e^x \\ e^x & x e^x \end{vmatrix} = -e^{2x}$$

$$e^x v_1' + x e^x v_2' = 0$$

$$v_1' = \frac{\begin{vmatrix} x e^x \ln x & (x+1)e^x \\ 0 & x e^x \end{vmatrix}}{-e^{2x}} = -x^2 \ln x \Rightarrow v_1 = \frac{-1}{3}(x^3 \ln x - x^3)$$

$$v_2' = \frac{\begin{vmatrix} e^x & x e^x \ln x \\ e^x & 0 \end{vmatrix}}{-e^{2x}} = x \ln x \Rightarrow v_2 = \frac{1}{2}(x^2 \ln x - x^2).$$

$$y_c = \frac{-e^x}{3}(x^3 \ln x - x^3) + \frac{e^x}{2}(x^3 \ln x - x^3)$$

$$= -x^3 e^x \left(\frac{\ln x - 1}{3} - \frac{\ln x - 1}{2} \right) = \frac{x^3 e^x}{6}(\ln x - 1) \quad \text{Yanlış.}$$

$$y_c = \frac{-5x^3 e^x}{36} + \frac{x^3 e^x \ln x}{6}. \quad \text{(Doğrusu).}$$

9) $y'' - 2y + y = (\ln x)^2$

8) $y'' - 2y + x = e^x \sin^{-1} x$

10) $y'' + 3y' + 2y = \dfrac{e^{-x}}{x}$

11) $y'' + 3y' + 2y = \dfrac{1}{1+e^x}$

12) $y'' + 3y' + 2y = \dfrac{1}{1+e^{2x}}$

13) $y'' + y = \dfrac{1}{1+\sin x}.$

Let us apply this theorem to an equation of the form (4.64), where the coefficients $a_0, \ldots, a_n$ are constants. Suppose that F is a simple UC function but that G is not. Then we could *not* apply the method of undetermined coefficients to find a particular integral of Equation (4.64). However, we could find a particular integral of equation (4.62) by the method of undetermined coefficients, and then use variation of parameters to find a particular integral of Equation (4.63). Then applying Theorem 4.12, the appropriate linear combination of these two particular integrals, found by different methods, is a particular integral of Equation (4.64).

Example 4.37. Find a particular integral of

(4.65) $$\frac{d^2y}{dx^2} + y = 3e^x + 5\tan x.$$

We consider the two equations

(4.66) $$\frac{d^2y}{dx^2} + y = e^x$$

and

(4.67) $$\frac{d^2y}{dx^2} + y = \tan x.$$

Since the function defined by e^x is a UC function, a particular integral of Equation (4.66) may be found by the method of undetermined coefficients. Letting $y_p = Ae^x$, we find at once that $A = \frac{1}{2}$; hence a particular integral of (4.66) is

$$y_p = \frac{1}{2}e^x.$$

Since the function defined by $\tan x$ is *not* a UC function, we turn to variation of parameters to find a particular integral of Equation (4.67). We have already solved this problem in Example 4.34; we found there that a particular integral of (4.67) is

$$y_p = -(\cos x)\ln|\sec x + \tan x|.$$

Thus, applying Theorem 4.12, a particular integral of Equation (4.65) is

$$y_p = \frac{3}{2}e^x - 5(\cos x)\ln|\sec x + \tan x|.$$

Exercises

Find the general solution of each of the differential equations in Exercises 1 through 13.

1. $\dfrac{d^2y}{dx^2} + y = \cot x.$

2. $\dfrac{d^2y}{dx^2} + y = \tan^2 x.$

3. $\dfrac{d^2y}{dx^2} + y = \sec x.$

4. $\dfrac{d^2y}{dx^2} + y = \sec^3 x.$

√5. $\dfrac{d^2y}{dx^2} + 4y = \sec^2 2x.$

6. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \ln x, \quad (x > 0).$

7. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \ln x, \quad (x > 0).$

8. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = (\ln x)^2, \quad (x > 0).$

9. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^x \sin^{-1} x.$

10. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \dfrac{e^{-x}}{x}.$

11. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \dfrac{1}{1 + e^x}.$

12. $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = \dfrac{1}{1 + e^{2x}}.$

13. $\dfrac{d^2y}{dx^2} + y = \dfrac{1}{1 + \sin x}.$

14. Find the general solution by two methods:

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3y = x^2 e^x.$$

15. Given that $y = x$ and $y = \dfrac{1}{x+1}$ are linearly independent solutions of

$$(2x + 1)(x + 1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0,$$

find the general solution of

$$(2x + 1)(x + 1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = (2x + 1)^2.$$

16. Given that $y = \sin x$ and $y = x\sin x$ are linearly independent solutions of

$$(\sin^2 x)\frac{d^2y}{dx^2} - 2\sin x\cos x\frac{dy}{dx} + (\cos^2 x + 1)y = 0,$$

find the general solution of

$$(\sin^2 x)\frac{d^2y}{dx^2} - 2\sin x\cos x\frac{dy}{dx} + (\cos^2 x + 1)y = \sin^3 x.$$