More Exercises for Linear Dif Equation

1) If f and g are two solution of $\frac{dy}{dx}+P(x)y=0$, then, for $c_1, c_2 \in \mathbb{R}$, $c_1 f + c_2 g$ is also a solution of this equation.

<u>Solution:</u> Since f is a solution, then $\frac{df}{dx}+P(x).f(x)=0$ and
Since g is a solution, then $\frac{dg}{dx}+P(x)g(x)=0$.

Thus, for $c_1, c_2 \in \mathbb{R}$,

$$\frac{d(c_1 f + c_2 g)}{dx} + P(x)(c_1 f + c_2 g) = \cdots = 0$$

2. Prove that if f and g are two different solutions of $\frac{dy}{dx}+P(x)y=Q(x)$ then, for $c \in \mathbb{R}$, $c(f-g)+f$ is also a solution of $\frac{dy}{dx}+P(x)y=Q(x)$.

<u>Solution:</u> easy.

3. Let f_1 be a solution of $\frac{dy}{dx}+P(x)y=Q_1(x)$ and f_2 be a solution of $\frac{dy}{dx}+P(x)y=Q_2(x)$. on I Prove that f_1+f_2 is a solution of

$$\frac{dy}{dx}+P(x)y=Q_1(x)+Q_2(x), \quad \text{on } I.$$

4: Show that the transformation $v=f(y)$ reduces the equation

$$\frac{d(f(y))}{dy} \cdot \frac{dy}{dx} + P(x).f(y) = Q(x).$$

to a linear equation in v.

<u>Solution:</u> Let $v=f(y)$. Then $\frac{dv}{dx}=\frac{dv}{dy}\cdot\frac{dy}{dx}=\frac{d(f(y))}{dy}\cdot\frac{dy}{dx}$.

It follows that $\frac{dv}{dx}+P(x)v=Q(x)$.

5) Solve the equation

$$(y+1)\frac{dy}{dx} + x(y^2+2y) = x^3$$

Let $v = y^2+2y \Rightarrow \frac{dv}{dx} = \frac{dv}{dy}\cdot\frac{dy}{dx} = 2y+2 \cdot \frac{dy}{dx} = 2(y+1)\cdot\frac{dy}{dx}$.

$\Rightarrow \frac{1}{2}\frac{dv}{dx} + xv = x \Rightarrow \frac{dv}{dx} + 2xv = 2x \Rightarrow dv + 2xv\,dx = 2x\,dx$

$\Rightarrow dv + (2xv - 2x)dx = 0 \qquad \mu = \frac{1}{v-1} \Rightarrow \int \frac{dv}{v-1} + \int 2x\,dx = 0$

$\Rightarrow \ln|v-1| + x^2 = \ln|c| \Rightarrow e^{x^2}\cdot(v-1) = c \Rightarrow \underline{e^{x^2}(y^2+2y-1) = c}$

The equation $\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x)$ is called Riccati's Equation. It is clear that

if $A(x) = 0 \Rightarrow$ the equation is linear equation
If $C(x) = 0 \Rightarrow$ the equation is Bernoulli equation.

Lemma: If f is any solution of $\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x)$ (A), then the transformation $y = f + \frac{1}{v}$ reduces (A) to a linear equation in v.

Proof: Let $y = f + \frac{1}{v} \Rightarrow \frac{dy}{dx} = \frac{df}{dx} - \frac{1}{v^2}\cdot\frac{dv}{dx}$. $\Rightarrow$

$$\frac{df}{dx} - \frac{1}{v^2}\frac{dv}{dx} = A(x)\cdot\left(f+\frac{1}{v}\right)^2 + B(x)\left(f+\frac{1}{v}\right) + C(x)$$

$$\Rightarrow \underbrace{\left(\frac{df}{dx} - A(x)f^2 - B(x)f - C(x)\right)}_{=0} - \frac{1}{v^2}\frac{dv}{dx} = A(x)\left(\frac{2f}{v} + \frac{1}{v^2}\right) + \frac{B(x)}{v}$$

$$\Rightarrow \frac{dv}{dx} = -2f\cdot v\cdot A(x) - A(x) - B(x)\cdot v$$

$$\Rightarrow \frac{dv}{dx} + (2f\,A(x) + B(x))\,v = -A(x).$$

Examples

(1) Consider the Riccati equation $\frac{dy}{dx} = (1-x)y^2 + (2x-1)y - x$.

and observe that $f(x) = 1$ is a solution.

$$y = 1 + \frac{1}{v} \Rightarrow \frac{dy}{dx} = -\frac{1}{v^2}\frac{dv}{dx} \Rightarrow -\frac{1}{v^2}\frac{dv}{dx} = (1-x)(1+\frac{1}{v})^2 + (2x-1)(1+\frac{1}{v}) - x$$

$$\Rightarrow -\frac{1}{\cancel{v^2}}\frac{dv}{dx} = \frac{\cancel{v^2} - \cancel{xv^2} + \cancel{2v} - \cancel{2xv} + 1 - x + \cancel{2xv^2} - \cancel{v^2} + \cancel{2xv} - \cancel{v} - x^2 v}{\cancel{v^2}}$$

$$-\frac{dv}{dx} = \cancel{xv^2} + v - x^2 \cancel{- xv} + 1 \Rightarrow \frac{dv}{dx} = -v + x^2 - 1 \Rightarrow \frac{dv}{dx} + v = x^2 - 1$$

$$\mu = e^{\int dx} = e^x \Rightarrow \quad e^x \frac{dv}{dx} + e^x v = (x^2-1)e^x$$

$$\Rightarrow \frac{d(e^x v)}{dx} = (x^2-1)e^x \Rightarrow e^x v = -e^x + \int x^2 e^x dx$$

$$\Rightarrow e^x v = -e^x + x^2 e^x - 2xe^x + 2e^x + c$$

$$v = (x^2 e^x - 2xe^x + e^x + c)e^{-x} \Rightarrow \underline{(x^2 - 2x + 1) + ce^{-x}}$$

Finding Integral Factors

Suppose that the equation $Mdx + Ndy = 0$ is not exact and that μ is an integrating factor of it. Then the equation

$$\mu M dx + \mu N dy = 0$$

is exact. So

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

or
$$N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu.$$

which is a partial differential equation. Since we are in no position to attempt to solve such an equation, we assume that μ depends upon x alone (or y alone). So $\frac{\partial \mu}{\partial y} = 0$. Thus, we have.

$$N\frac{\partial \mu}{\partial x} = \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu$$

It follows that $\frac{\partial \mu}{\partial x} = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu$ and so

$$\mu = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx}.$$

If μ depend on y alone then similarly $\mu = e^{\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dy}$

Teorem: Consider the differential equation $Mdx + Ndy = 0$ (i)

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ depends on x only, then. $e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx}$

is an integrating factor of the equation.

Example: $(2x^2+y)dx + (x^2y - x)dy = 0$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 - (2xy - 1) = -2xy + 2$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{2-2xy}{x^2y-x}=\frac{2(1-xy)}{-x(1-xy)}=-\frac{2}{x}.$$

$$\mu(x)=e^{\int -\frac{2}{x}dx}=e^{-2\ln|x|}=\underline{x^{-2}}$$

$$\left(2+\frac{y}{x^2}\right)dx+\left(y-\frac{1}{x}\right)dy=0$$

$$u=\int(2+yx^{-2})\,dx+\phi(y)=2x-yx^{-1}+\phi(y)$$

$$\cancel{\frac{\partial u}{\partial x}=2+2yx}\qquad \frac{\partial u}{\partial y}=-x^{-1}+\frac{\partial \phi}{\partial y}=y-\cancel{\frac{1}{x}}\Rightarrow \phi=\frac{y^2}{2}.$$

$$2x-\frac{y}{x}+\frac{y^2}{2}=c.$$

<u>Teorem 2.7</u> Consider the equation

$$(a_1x+b_1y+c_1)\,dx+(a_2x+b_2y+c_2)\,dy=0$$

(i) If $\frac{a_2}{a_1}\neq\frac{b_2}{b_1}$ and (h,k) is a solution of $\begin{array}{l}a_1x+b_1y=-c_1\\ a_2x+b_2y=c_2\end{array}$

then the transformation $\begin{array}{l}x=X+h\\ y=Y+h\end{array}$ reduces the equation to the homogeneous equation $(a_1X+b_1Y)\,dX+(a_2X+b_2Y)\,dY=0$.

(i) If $\frac{a_2}{a_1}=\frac{b_2}{b_1}=k$, then the transformation $z=a_1x+b_1y$ reduces the equation to a separable equation in the variables x and z.

<u>Example:</u> $(x-2y+1)\,dx+(4x-3y-6)\,dy=0.$

<u>Solution:</u> Clearly $\frac{a_2}{a_1}=4\neq\frac{b_2}{b_1}=\frac{3}{2}$.

$$\left.\begin{array}{l}x-2y=-1\\ 4x-3y=6\end{array}\right\}\Rightarrow \begin{array}{r}4x-3y=6\\ +\ -4x+8y=+4\\ \hline 5y=10\Rightarrow \underline{y=2}\end{array}\qquad x-4=-1\Rightarrow \underline{x=3}$$

$$\left.\begin{aligned} x &= X+3 \\ y &= Y+2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} dx &= dX \\ dy &= dY \end{aligned}\right] \Rightarrow (X+3-2(Y+2)+1)\,dX$$

$$+(4(X+3)-3(Y+2)-6)\,dY=0 \Rightarrow (X+2Y)\,dX+(4X-3Y)\,dY=0$$

which homogeneous (of degree 1).

$$Y=vX \Rightarrow \frac{dY}{dX}=v+X\frac{dv}{dX} \qquad \frac{dY}{dX}=\frac{2Y-X}{4X-3Y}$$

$$\Rightarrow v+X\frac{dv}{dX}=\frac{2vX-X}{4X-3vX}=\frac{2v-1}{4-3v} \Rightarrow X\frac{dv}{dX}=\frac{2v-1-4v+3v^2}{4-3v}$$

$$X\frac{dv}{dX}=\frac{3v^2-2v-1}{4-3v} \Rightarrow \frac{3v-4}{3v^2-2v-1}dv+\frac{dX}{X}=0$$

$$\int\frac{\frac{15}{4}}{3v+1}dv+\int\frac{-\frac{1}{4}dv}{v-1}+\ln|X|=\frac{\ln|c|}{4}$$

$$\frac{5}{4}\ln|3v+1|-\frac{1}{4}\ln|v-1|+\ln|X|=\frac{\ln|c|}{4}$$

$$\frac{(3v+1)^5}{v-1}\cdot X^4=c$$

$$\frac{\frac{(3Y+X)^5}{X^5}}{\frac{Y-X}{X}}\cdot X^4=c \Rightarrow \frac{(3Y+X)^5}{(Y-X)}=c \Rightarrow (x+3y-9)^5=c(y-x+1)$$

(2) $(x+2y+3)\,dx+(2x+4y-1)\,dy=0.$

$$\frac{a_2}{a_1}=2=\frac{b_2}{b_1}. \qquad z=x+2y. \qquad \cancel{\frac{dy}{dx}=\frac{d\left(\frac{z-x}{2}\right)}{dx}=\frac{1}{2}(dz-dx)}$$

$$\Rightarrow dz=dx+2dy \Rightarrow dy=\frac{dz-dx}{2}. \Rightarrow (z+3)\,dx+(2z-1)\cdot\frac{dz-dx}{2}=0$$

$$\Rightarrow \underline{(2z+6)}\,dx+(2z-1)\,dz-\underline{(2z-1)}\,dx=0 \Rightarrow \int 7\,dx+\int(2z-1)\,dz=0$$

$$7x + z^2 - z = c \Rightarrow 7x + (x+2y)^2 - (x+2y) = c$$

$$\boxed{x^2 + 4xy + 4y^2 + 6x - 2y = c}$$

Alıştırmalar

1) $(5xy+4y^2+1)\,dx + (x^2+2xy)\,dy = 0$

2) $(2x+\tan y)\,dx + (x-x^2\tan y)\,dy = 0$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = (\tan^2 y \cancel{+1}) - (\cancel{1} - 2x\tan y) = \tan y(\tan y + 2x)$$

$$\mathcal{J} = e^{\int -\tan y\,dy} = e^{\int \frac{-\sin y\,dy}{\cos y}} = e^{+\ln|\cos y|} = \left(\frac{1}{\cos y}\right)^{-1} = \cos y.$$

$\Rightarrow (2x\cos y + \sin y)\,dx + (x\cos y - x^2\sin y)\,dy = 0$ is exact.

$$u = \int (2x\cos y + \sin y)\,dx + \phi(y) = x^2\cos y + x.\sin y + \phi(y)$$

$$\frac{\partial u}{\partial y} = -x^2\sin y + x\cos y + \frac{d\phi}{dy} = N \Rightarrow \underline{\phi = 0}$$

$$\boxed{x^2\cos y + x\sin y = c}$$

3) $[y^2(x+1)+y]\,dx + (2xy+1)\,dy = 0$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = (2y(x+1)+1) - 2y = 2xy+1$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{-(2x+1)}{2x+1} = -1$$

$$\Rightarrow \mathcal{J} = e^{\int -dx} = e^{-x} \Rightarrow (y^2(x+1)+y)e^{-x}\,dx + (2xy+1)e^{-x}\,dy = 0$$

$$u = \int (2xy+1)e^{-x}\,dy + \phi(x) = xy^2e^{-x} + ye^{-x} + \phi$$

$$\frac{\partial u}{\partial x} = \cancel{y^2e^{-x}} - xy^2e^{-x} - ye^{-x} + \frac{d\phi}{dx} = xy^2e^{-x} + \cancel{y^2e^{-x}} + ye^{-x}$$

(Cevap: $\underline{xy^2e^{x} + ye^{x}} = c$)

4) $(4xy^2+6y)\,dx + (5x^2y+8x)\,dy = 0.$

5) $(5x+2y+1)dx+(2x+y+1)dy=0$ $(5x^2+4xy+y^2+2x+2y=c)$

6) $(3x-y+1)dx+(-6x+2y+3)dy=0$

7) $(x-2y-3)dx+(2x+y-1)dy=0$ $\left(\ln(c(x^2+y^2-2x+2y+2))+4\arctan\left(\frac{y+1}{x-1}\right)=0\right)$

8) $(6x+4y+1)dx+(4x+2y+2)dy=0.$

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3. $[y^2(x + 1) + y]dx + (2xy + 1)dy = 0.$

4. $(4xy^2 + 6y)dx + (5x^2y + 8x)dy = 0.$

[*Hint.* This differential equation has an integrating factor of the form x^py^q.]

Solve each differential equation in Exercises 5 through 7 by making a suitable transformation.

5. $(5x + 2y + 1)dx + (2x + y + 1)dy = 0.$

6. $(3x - y + 1)dx - (6x - 2y - 3)dy = 0,$

7. $(x - 2y - 3)dx + (2x + y - 1)dy = 0.$

Solve the initial-value problems in Exercises 8 through 10.

8. $\begin{cases}(6x + 4y + 1)dx + (4x + 2y + 2)dy = 0\\ y(\frac{1}{2}) = 3.\end{cases}$

9. $\begin{cases}(3x - y - 6)dx + (x + y + 2)dy = 0\\ y(2) = -2.\end{cases}$

10. $\begin{cases}(2x + 3y + 1)dx + (4x + 6y + 1)dy = 0\\ y(-2) = 2.\end{cases}$

11. Prove Theorem 2.6.

12. Prove Theorem 2.7.

13. Show that if μ and ν are integrating factors of

(A) $$Mdx + Ndy = 0$$

such that μ/ν is not constant, then

$$\mu = c\nu$$

is a solution of Equation (A) for every constant c.

14. Show that if the equation

(A) $$Mdx + Ndy = 0$$

is homogeneous and $Mx + Ny \neq 0$, then $1/(Mx + Ny)$ is an integrating factor of (A).

15. Show that if the equation $Mdx + Ndy = 0$ is both homogeneous and exact and if $Mx + Ny$ is not a constant, then the solution of this equation is $Mx + Ny = c$, where c is an arbitrary constant.

SUGGESTED READING

I. Basic Methods:

Agnew (1)
Ford (17)
Kaplan (30)
Martin and Reissner (38)
Rainville (45)

II. Further Methods:

Ince (26)
Kamke (28)