The Homogeneous Linear Equation With Constant Coefficients

We shall be concerned with the equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = 0 \qquad (1)$$

where $a_0, a_1, \dots, a_n$ are real constants. Here we need a function such that its derivatives are constant multiples of itself. We know exponential function f such that $f(x) = e^{mx}$ $(m \in \mathbb{R})$ have this property, that is

$$\frac{d^k [e^{mx}]}{dx^k} = m^k e^{mx}.$$

Thus we shall seek solutions of (1) of the form $y = e^{mx}$, $(m \in \mathbb{R})$, where the constant m will be chosen such that e^{mx} does satisfy the equation. Assuming that $y = e^{mx}$ is a solution for certain m, we have.

$$\frac{dy}{dx} = m e^{mx}, \quad \frac{d^2 y}{dx^2} = m^2 e^{mx}, \dots, \quad \frac{d^n y}{dx^n} = m^n e^{mx}.$$

Substituting in (1), we obtain

$$a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \dots + a_{n-1} m e^{mx} + a_n e^{mx} = 0$$

or since $e^{mx} \neq 0$

$$a_0 m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0 \qquad (2)$$

This equation is called the auxiliary equation or the characteristic equation of (1). Here, to solve (1), we write the characteristic equation (2) and solve it for m. Three cases arise, according as the roots of (2) are real and distinct, real and repeated, or complex.

Case 1 : Distinct Real Roots

Suppose the roots of (2) are n distinct real numbers $m_1, \dots, m_n$. Then $e^{m_1 x}, e^{m_2 x}, \dots, e^{m_n x}$ are n distinct solution of (1). Further, using the Wronskian determinant one may show that these n solutions are linearly independent.

Theorem: With the notation above $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$ is the general solution of (1).

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Examples (1) $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$

The auxiliary equation is $m^2-3m+2=0$. Here $m=1,2$ the roots. Thus $y=c_1e^{x}+c_2e^{2x}$ is the general solution.

Examples (2) $\frac{d^3y}{dx^3}-4\frac{d^2y}{dx^2}+\frac{dy}{dx}+6y=0$

The auxiliary equation is $m^3-4m^2+m+6=0$. We observe that $m=-1$ is a root of this equation. By synthetic division,

$(m+1)(m^2-5m+6)=(m+1)(m-2)(m-3)=0 \Rightarrow m=-1,2,3$

$y=c_1e^{-x}+c_2e^{2x}+c_3e^{3x}$ is the general solution.

Case II Repeated Real Roots

Consider the differential equation $y''-6y'+9y=0$. The auxiliary equation is $m^2-6m+9=(m-3)^2=0$. The roots of this equation are $m_1=3, m_2=3$. Since e^{3x} and e^{3x} are linearly dependent, we must find a linearly independent solution. We already know the one of the solution e^{3x}, so we reduce the order of the dif eqution.

Let $y=e^{3x}u$. Then

$$\frac{dy}{dx}=e^{3x}\frac{du}{dx}+3e^{3x}u \qquad \frac{d^2y}{dx^2}=e^{3x}\frac{d^2u}{dx^2}+6e^{3x}\frac{du}{dx}+9e^{3x}u.$$

Supstiting into the equation we have

$$\left(e^{3x}\frac{d^2u}{dx^2}+6e^{3x}\frac{du}{dx}+9e^{3x}u\right)-6\left(e^{3x}\frac{du}{dx}+3e^{3x}u\right)+9e^{3x}u=0$$

or $$e^{3x}\frac{d^2u}{dx^2}=0$$

Let $w=\frac{du}{dx}$, we have the first-order equation $e^{3x}\frac{dw}{dx}=0$

or simply $\frac{dw}{dx}=0$, and so $w=c$. Choose $c=1=w$, then

$\frac{du}{dx}=1 \Rightarrow du=dx \Rightarrow u=x$

Thus the second solution is xe^{3x}. Therefore $y=c_1e^{3x}+c_2xe^{3x}$ the general solution of $y''-6y'+9y=0$.

Theorem: (i) Consider the nth order homogeneous linear differential equation (1) with constant coefficients. If the auxiliary equation (2) has the real root m occuring k times, then the part of the general solution of (1) corresponding to this k-fold repeated root is

$$(c_1+c_2x+\cdots+c_kx^{k-1})e^{mx}.$$

(ii) If, further, the remaining roots are the distinct real numbers $m_{k+1},\ldots,m_n$, then the general solution of (1) is

$$y=(c_1+c_2x+\cdots+c_kx^{k-1})e^{mx}+c_{k+1}e^{m_{k+1}x}+\cdots+c_ne^{m_nx}.$$

(iii) If, however, any of the remaining roots are also repeated, then the parts of the general solution of (1) corresponding to each of these other repeated roots are expressions similar to that the corresponding to m in part (i).

Examples (1) Find the general solution of $y'''-4y''-3y'+18y=0$

The auxiliary equation $m^3-4m^2-3m+18=0$ has the roots $3,3,-2$.

The general solution is

$$y=c_1e^{3x}+c_2xe^{3x}+c_3e^{-2x}.$$

(2) Find the general solution of $y^{iv}-5y'''+6y''+4y'-8y=0$

The auxiliary equation is $m^4-5m^3+6m^2+4m-8=0$, with roots $2,2,2,-1$.

The general solution is

$$y=c_1e^{2x}+c_2xe^{2x}+c_3x^2e^{2x}+c_4e^{-x}.$$

Case III: Conjugate Complex Roots.

Now suppose that the auxiliary equation has the complex number $a+bi$ as a non repeted root. Since the coefficients are real, $a-bi$

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is also a nonrepeated root. The corresponding part of the general solution is $k_1 e^{(a+bi)x} + k_2 e^{(a-bi)x}$ $(k_1, k_2 \in \mathbb{R})$. The solutions defined by $e^{(a+bi)x}$ and $e^{(a-bi)x}$ are complex function. By using Euler's formula $(e^{i\theta} = \cos\theta + i\sin\theta)$ we have

$$k_1 e^{(a+bi)x} + k_2 e^{(a-bi)x} = k_1 e^{ax}(\cos bx + i\sin bx) + k_2(\cos bx - i\sin bx)$$

$$= e^{ax}[(k_1+k_2)\cos bx + i(k_1-k_2)\sin bx]$$

$$= [c_1 \sin bx + c_2 \cos bx] e^{ax}$$

where $c_1 = i(k_1-k_2)$, $c_2 = k_1+k_2$. Thus the part of the general solution corresponding to the nonrepeated conjugate complex roots $a \mp bi$ is

$$e^{ax}[c_1 \sin bx + c_2 \cos bx].$$

Theorem (i) Consider the nth order homogeneous linear differential equation with constant coefficients. If the auxiliary equation has the conjugate complex roots $a+bi$ and $a-bi$, neither repeated, then corresponding part of the general solution $y = e^{ax}(c_1 \sin bx + c_2 \cos bx)$.

(ii) If, however $a+bi$ and $a-bi$ are each k-fold roots of the auxiliary equation then the corresponding part of the general solution

$$y = e^{ax}[(c_1 + c_2 x + \cdots + c_k x^{k-1}] \sin bx + [c_{k+1} + \cdots + c_{2k} x^{k-1}] \cos bx.$$

Examples: Find the general solution of $y'' + y = 0$.

The auxiliary equation $m^2 + 1 = 0$ has the roots $m = \mp i$. The general solution

$$y = c_1 \sin x + c_2 \cos x.$$

(2) Find the general solution of $y'' - 6y' + 25y = 0$

The auxiliary equation $m^2 - 6m + 25 = (m-3)^2 + 16 = 0 \Rightarrow (m-3)^2 = -16$

$\Rightarrow m - 3 = \mp 4i \Rightarrow m = 3 \mp 4i$

The general solution

$$y = e^{3x}(c_1 \sin 4x + c_2 \cos 4x).$$

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(3) Find the general solution of $y^{iv}-4y'''+14y''-20y'+25y=0$.

The auxiliary equation is

$$m^4-4m^3+14m^2-20m+25=(m^2-2m+5)(m^2-2m+5)$$

$$(m^2-2m+5)=0 \Rightarrow (m-1)^2+4=0 \Rightarrow m=1\mp 2i.$$

The solution is

$$y=[(c_1+c_2x)\sin 2x+(c_3+c_4x)\cos 2x]e^x.$$

(4) Solve the initial-value problem $y''-6y'+25y=0$ $y(0)=-3$ $y'(0)=-1$.

$$m^2-6m+25=(m-3)^2+16=0 \Rightarrow m=3\mp 4i \Rightarrow$$

$$y=(c_1\sin 4x+c_2\cos 4x)e^{3x}.$$

$$y'=(4c_1\cos 4x-4c_2\sin 4x).e^{3x}+(c_1\sin 4x+c_2\cos 4x)3e^{3x}.$$

$y(0)=-3 \Rightarrow -3=c_2$ $\quad y'(0)=-1 \Rightarrow -1=4c_1+3c_2 \Rightarrow c_1=2.$

$$y=e^{3x}(2\sin 4x-3\cos 4x).$$

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Exercises

Find the general solution of the following:

1) $y''-5y+6y=0$

2) $y''-2y-3y=0$

3) $4y''-12y+5y=0$

Solution: $4m^2-12m+5=(2m-3)^2-4=0$ $\qquad 2m-3=\mp 2$

$\Rightarrow m=\frac{3\mp 2}{2}=\frac{1}{2},\frac{5}{2}.$

$y=c_1e^{x/2}+c_2e^{5x/2}.$

5) $y'''-3y''-y'+3y=0$

Solution: $m^3-3m^2-m+3=0$ $\quad m=1$ is a root. $m=-1$ is also enough.

$$\begin{array}{l|l} m^3-3m^2-m+3 & m^2-1 \\ \hline \underline{m^3-m} & m-3 \\ -3m^2+3 & \end{array}$$

$\Rightarrow m=-1,1,3.$

$y=c_1e^{-x}+c_2e^{x}+c_3e^{3x}.$

6) $y'''-6y''+5y+12y=0$

7) $y''-8y'+16y=0$

8) $4y''+4y'+y=0$

9) $y''-4y+13y=0$

10) $y''+9y=0$

11) $y'''-5y''+7y-3y=0$

Solution: $m^3-5m^2+7m-3=0$ $\quad m=1$

$$\begin{array}{l|l} m^3-5m^2+7m-3 & m-1 \\ \hline \underline{m^3-m^2} & m^2-4m+3 \\ -4m^2+7m & \\ \underline{-4m^2+4m} & \\ 3m-3 & \end{array}$$

$\Rightarrow (m-1)(m^2-4m+3)=(m-1)(m-1)(m-3)$

$m^2-4m+3=(m-2)^2-1$ $\qquad m=1,1,3.$

$y=(c_1+c_2x)e^{x}+c_3e^{3x}.$

12) $4y'''+4y''-7y'+2y=0$

13) $y'''-6y''+12y-8y=0$

16) $y'''+4y''+5y'+6y=0.$

Solution $m^3+4m^2+5m+6=0$ $\quad \underline{m=-3}$

$$\begin{array}{l|l} m^3+4m^2+5m+6 & m+3 \\ \hline \underline{m^3+3m^2} & m^2+m+2 \\ m^2+5m & \\ \underline{m^2+3m} & \\ 2m+6 & \end{array}$$

$(m+3)(m^2+m+2)$ $\qquad (m+\frac{1}{2})^2+\frac{7}{4}=0$

$m=\frac{-1\mp\sqrt{7}i}{2}$

$y=c_1e^{-3x}+\left(c_2\sin\frac{\sqrt{7}x}{2}+c_3\cos\frac{\sqrt{7}x}{2}\right)e^{-x/2}$

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15) $y''' - y'' + y' - y = 0$ 16) $y^{V} - 2y^{IV} + y''' = 0$

<u>Solution:</u> $m^5 - 2m^4 + m^3 = m^3(m^2 - 2m + 1) = m^3(m-1)^2 = 0 \Rightarrow m = 0, 0, 0, 1, 1.$

$$y = (c_1 + c_2 x + c_3 x^2) + c_4 e^x + c_5 x e^x$$

17) $y^{IV} - 3y^3 - 2y^2 + 2y' + 12y = 0$

$m^4 - 3m^3 - 2m^2 + 2m + 12 = 0$ $m = 2.$

$(m-2)(m^3 - m^2 - 4m - 6) = 0$

$(m-2)(m-3)(m^2 + 2m + 2)$

$(m+1)^2 + 1 = 0$

$m + 1 = \mp i$

$m = -1 \mp i$

$$\begin{array}{l|l} m^4 - 3m^3 - 2m^2 + 2m + 12 & m-2 \\ \hline -\ m^4 - 2m^3 & m^3 - m^2 - 4m - 6 \\ \quad -m^3 - 2m^2 & \\ -\ -m^3 + 2m^2 & \\ \quad -4m^2 + 2m & \\ -\ -4m^2 + 8m & \\ \quad -6m + 12 & \end{array}$$

$$\begin{array}{l|l} m^3 - m^2 - 4m - 6 & m-3 \\ \hline -\ m^3 - 3m^2 & m^2 + 2m + 2 \\ \quad 2m^2 - 4m - 6 & \\ -\ 2m^2 - 6m & \\ \quad 2m - 6 & \end{array}$$

$$y = c_1 e^{2x} + c_2 e^{3x} + (c_3 \sin x + c_4 \cos x) e^{-x}.$$

18) $y^{IV} + 6y''' + 15y'' + 20y' + 12y = 0$

19) $y^{IV} + y = 0$

<u>Solution</u> ~~$(m^4+1) = 0 \Rightarrow (m^2-1)(m^2+1) = 0$ $m = \mp 1, \mp i$.~~

~~$y = c_1 e^{-x} + c_2 e^{x} + (c_3 \sin x + c_4 \cos x)$.~~

$m^4 + 1 = 0 \Rightarrow m^4 = -1 \Rightarrow m = \sqrt[4]{-1} = \pm \frac{\sqrt{2}}{2} \mp \frac{\sqrt{2}}{2} i$

$$y = e^{\frac{\sqrt{2}x}{2}}\left(c_1 \sin \tfrac{\sqrt{2}x}{2} + c_2 \cos x \tfrac{\sqrt{2}x}{2}\right) + e^{-\frac{\sqrt{2}x}{2}}\left(c_3 \sin \tfrac{\sqrt{2}x}{2} + c_4 \sin \tfrac{\sqrt{2}x}{2}\right).$$

20) $y^{V} = 0$

<u>Solution</u> $m^5 = 0 \Rightarrow m = 0, 0, 0, 0, 0 \Rightarrow y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4.$

21) Solve the initial value problems:

21) $y'' - y - 12y = 0$ $y(0) = 3$ $y'(0) = 5$

22) $9y'' - 6y + y = 0$ $y(0) = 3$ $y'(0) = -1$

23) $y'' - 4y' + 29y = 0$ $y(0) = 0$ $y'(0) = 5.$

<u>Solution</u> $m^2 - 4m + 29 = (m-2)^2 + 5^2 = 0$ $m = 2 \mp 5i$

$$y = (c_1 \sin 5x + c_2 \cos 5x) e^{2x}$$

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$$y' = 5c_1\cos 5x - 5c_2\sin 5x)e^{2x} + 2(c_1\sin 5x + c_2\cos 5x)e^{2x}.$$

$y(0)=0 \Rightarrow \quad 0 = c_2 \quad y'(0)=5 \Rightarrow 5 = 5c_1 + 2c_2 \Rightarrow c_1 = 1$

$$y = (\sin 5x)e^{2x}$$

24) $4y'' + 4y + 37y = 0 \quad y(0)=2 \quad y'(0) = -4.$

$4m^2 + 4m + 37 = (2m+1) + 36 = 0 \qquad m = \dfrac{-1 \mp 6i}{2}$

$$y = (c_1\sin 3x + c_2\cos 3x)e^{-x/2}$$

$$y' = (3c_1\cos 3x - 3c_2\sin 3x)e^{-x/2} - \frac{1}{2}(c_1\sin 3x + c_2\cos 3x)e^{-x/2}$$

$y(0)=2 \Rightarrow \quad 2 = c_2 \quad y'(0) = -4 = 3c_1 - \frac{1}{2}c_2 = 3c_1 - 1 \qquad c_1 = -1$

$$y = (-\sin 3x + 2\cos 3x)e^{-x/2}.$$

25) $y''' - 6y'' + 11y - 6y = 0 \quad y(0)=0 \quad y'(0)=0 \quad y''(0)=2.$

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A particular integral is given by

$$y_p = x.$$

Thus the general solution of the given equation may be written

$$y = y_c + y_p = c_1 \sin x + c_2 \cos x + x.$$

In the remaining sections of this chapter we shall proceed to study methods of obtaining the two constituent parts of the general solution.

Exercises

1. Theorem 4.1 applies to one of the following problems but not to the other. Determine to which of the problems the theorem applies and state precisely the conclusion which can be drawn in this case. Explain why the theorem does not apply to the remaining problem.

(a) $\begin{cases} \dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = e^x \\ y(0) = 5 \\ y'(0) = 7. \end{cases}$ (b) $\begin{cases} \dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = e^x \\ y(0) = 5 \\ y'(1) = 7. \end{cases}$

2. Answer orally: What is the solution of the following initial-value problem? Why?

$$\begin{cases} \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + x^2y = 0 \\ y(1) = 0 \\ y'(1) = 0. \end{cases}$$

3. Prove Theorem 4.2 for the case $m = n = 2$. That is, prove that if $f_1(x)$ and $f_2(x)$ are two solutions of

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0,$$

then $c_1f_1(x) + c_2f_2(x)$ is also a solution of this equation, where c_1 and c_2 are arbitrary constants.

4. Consider the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$$

(a) Show that e^{2x} and e^{3x} are linearly independent solutions of this equation on the interval $-\infty < x < \infty$.

(b) Write the general solution of the given equation.

(c) Find the solution which satisfies the conditions $y(0) = 2$, $y'(0) = 3$. Explain why this solution is unique.

5. Consider the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0.$$

(a) Show that x^2 and $\dfrac{1}{x^2}$ are linearly independent solutions of this equation on the interval $0 < x < \infty$.

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(b) Write the general solution of the given equation.

(c) Find the solution which satisfies the conditions $y(2) = 3$, $y'(2) = -1$. Over what interval is this solution defined?

6. The functions e^x and e^{4x} are both solutions of the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0.$$

(a) Show that these solutions are linearly independent on the interval $-\infty < x < \infty$.

(b) What theorem enables us to conclude at once that $2e^x - 3e^{4x}$ is also a solution of the given differential equation?

(c) Show that the solution of part (b) and the solution e^x are also linearly independent on $-\infty < x < \infty$.

7. Given that e^{-x}, e^{3x}, and e^{4x} are all solutions of

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 12y = 0,$$

show that they are linearly independent on the interval $-\infty < x < \infty$ and write the general solution.

8. Verify the truth of Theorem 4.6 for the equation

$$(x^2 - 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0,$$

given that $f(x) = x$ is a solution.

9. Given that $y = e^{2x}$ is a solution of

$$(2x + 1)\frac{d^2y}{dx^2} - 4(x + 1)\frac{dy}{dx} + 4y = 0,$$

find a linearly independent solution by reducing the order. Write the general solution.

10. Given that $y = x^2$ is a solution of

$$(x^3 - x^2)\frac{d^2y}{dx^2} - (x^3 + 2x^2 - 2x)\frac{dy}{dx} + (2x^2 + 2x - 2)y = 0,$$

find a linearly independent solution by reducing the order. Write the general solution.

11. Prove Theorem 4.8 for the case $n = 2$. That is, prove that if u is any solution of

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$$

and v is any solution of

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = b(x),$$

then $u + v$ is also a solution of this latter nonhomogeneous equation.

12. Consider the nonhomogeneous differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x^2.$$

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(a) Show that e^x and e^{2x} are linearly independent solutions of the corresponding homogeneous equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$$

(b) What is the complementary function of the given nonhomogeneous equation?
(c) Show that $2x^2 + 6x + 7$ is a particular integral of the given equation.
(d) What is the general solution of the given equation?

4.2 The Homogeneous Linear Equation With Constant Coefficients

A. Introduction

In this section we consider the special case of the nth order homogeneous linear differential equation in which all of the coefficients are real constants. That is, we shall be concerned with the equation

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = 0 \tag{4.11}$$

where $a_0, a_1, \ldots, a_{n-1}\ a_n$ are real constants. We shall show that the general solution of this equation can be found explicitly.

In an attempt to find solutions of a differential equation we would naturally inquire whether or not any familiar type of function might possibly have the properties which would enable it to be a solution. The differential equation (4.11) requires a function f having the property such that if it and its various derivatives are each multiplied by certain constants, the a_i, and the resulting products, $a_i\frac{d^{n-i}f}{dx^{n-i}}$, are then added, the result will equal zero. For this to be the case we would need a function such that its derivatives were constant multiples of itself. Do we know of functions f having this property that $\frac{d^k}{dx^k}[f(x)] = cf(x)$ for all x? The answer is "Yes," for the exponential function f such that $f(x) = e^{mx}$, where m is a constant, is such that

$$\frac{d^k}{dx^k}[e^{mx}] = m^ke^{mx}.$$

Thus we shall seek solutions of (4.11) of the form $y = e^{mx}$, where the constant m will be chosen such that e^{mx} *does* satisfy the equation. Assuming then that $y = e^{mx}$ is a solution for certain m, we have:

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

$$\vdots$$

$$\frac{d^ny}{dx^n} = m^ne^{mx}.$$

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which reduces at once to

$$c_2 = -3. \tag{4.21}$$

Applying condition (4.18), $y'(0) = -1$, to Equation (4.20), we obtain

$$-1 = e^0[(3c_1 - 4c_2)\sin 0 + (4c_1 + 3c_2)\cos 0]$$

which reduces to

$$4c_1 + 3c_2 = -1. \tag{4.22}$$

Solving Equations (4.21) and (4.22) for the unknowns c_1 and c_2, we find

$$\begin{cases} c_1 = 2 \\ c_2 = -3. \end{cases}$$

Replacing c_1 and c_2 in Equation (4.19) by these values, we obtain the unique solution of the given initial-value problem in the form

$$y = e^{3x}(2\sin 4x - 3\cos 4x).$$

We may write this in an alternate form by first multiplying and dividing by $\sqrt{(2^2) + (-3)^2} = \sqrt{13}$ to obtain

$$y = \sqrt{13}e^{3x}\left[\frac{2}{\sqrt{13}}\sin 4x - \frac{3}{\sqrt{13}}\cos 4x\right].$$

From this we may express the solution in the alternate form

$$y = \sqrt{13}e^{3x}\sin(4x + \phi),$$

where the angle ϕ is defined by the equations

$$\begin{cases} \sin\phi = -\dfrac{3}{\sqrt{13}} \\ \cos\phi = \dfrac{2}{\sqrt{13}}. \end{cases}$$

Exercises

Find the general solution of each of the differential equations in Exercises 1 through 24.

1. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0.$

2. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} - 3y = 0.$

3. $4\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 5y = 0.$

4. $3\dfrac{d^2y}{dx^2} - 14\dfrac{dy}{dx} - 5y = 0.$

5. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 3y = 0.$

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6. $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 12y = 0.$

7. $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0.$

8. $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0.$

9. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0.$

10. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 0.$

11. $\frac{d^2y}{dx^2} + 9y = 0.$

12. $4\frac{d^2y}{dx^2} + y = 0.$

13. $\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 3y = 0.$

14. $4\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 2y = 0.$

15. $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} - 8y = 0.$

16. $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0.$

17. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0.$

18. $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0.$

19. $\frac{d^5y}{dx^5} - 2\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} = 0.$

20. $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = 0.$

21. $\frac{d^4y}{dx^4} - 3\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 12y = 0.$

22. $\frac{d^4y}{dx^4} + 6\frac{d^3y}{dx^3} + 15\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 12y = 0.$

23. $\frac{d^4y}{dx^4} + y = 0.$

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24. $\dfrac{d^5y}{dx^5} = 0.$

Solve the initial-value problems in Exercises 25 through 30.

25. $\begin{cases} \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 12y = 0 \\ y(0) = 3 \\ y'(0) = 5. \end{cases}$

26. $\begin{cases} 9\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + y = 0 \\ y(0) = 3 \\ y'(0) = -1. \end{cases}$

27. $\begin{cases} \dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 29y = 0 \\ y(0) = 0 \\ y'(0) = 5. \end{cases}$

28. $\begin{cases} 4\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 37y = 0 \\ y(0) = 2 \\ y'(0) = -4. \end{cases}$

29. $\begin{cases} \dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0 \\ y(0) = 0 \\ y'(0) = 0 \\ y''(0) = 2. \end{cases}$

30. $\begin{cases} \dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 8y = 0 \\ y(0) = 2 \\ y'(0) = 0 \\ y''(0) = 0. \end{cases}$

31. The roots of the auxiliary equation, corresponding to a certain 10th-order homogeneous linear differential equation with constant coefficients, are

$$4, 4, 4, 4, 2 + 3i, 2 - 3i, 2 + 3i, 2 - 3i, 2 + 3i, 2 - 3i.$$

Write the general solution.

32. Given that $\sin x$ is a solution of

$$\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0,$$

find the general solution.

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