(9)

The Cauchy-Euler Equation.

The equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = b(x)$$

when $a_0, a_1, \dots, a_n$ are constants is called Cauch-Euler Equation.

<u>Theorem:</u> The transformation $x = e^t$ reduces the equation

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = b x.$$

to a linear differential equation with constant coefficients.

We shall prove the theorem for the case of the second-order Cauchy-Euler differential equation

$$a_0 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = b(x).$$

Let $x = e^t$, assuming $x > 0$, we have $t = \ln x$. Then $\frac{dt}{dx} = \frac{1}{x}$ and

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x} \frac{dy}{dt}.$$

$$\frac{d^2 y}{dx^2} = -\frac{1}{x^2} \cdot \frac{dy}{dt} + \frac{1}{x} \cdot \frac{d^2 y}{dt^2} \cdot \frac{dt}{dx} = \frac{1}{x^2}\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right).$$

Thus $x \frac{dy}{dx} = \frac{dy}{dt}$ and $x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt}$

Substiting $a_0\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right) + a_1 \frac{dy}{dt} + a_2 y = b(e^t)$

$$a_0 \frac{d^2 y}{dt^2} + (a_1 - a_0) \frac{dy}{dt} + a_2 y = b(e^t).$$

<u>Examples</u> (1) $x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = 2x^3$

$x = e^t \Rightarrow t = \ln x$, $\frac{dt}{dx} = \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{1}{x} \frac{dy}{dt}$

and $\frac{d^2 y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right)$

$$\left(\frac{d^2y}{\partial t^2}-\frac{dy}{\partial t}\right)-2\frac{dy}{\partial t}+2y=\frac{d^2y}{\partial t^2}-3\frac{dy}{\partial t^2}+2y=(e^t)^3=e^{3t}$$

$m^2-3m+2=(m-1)(m-2)=0 \quad m=1,2 \Rightarrow y_c=c_1e^t+c_2e^{2t}.$

$u=e^{3t} \Rightarrow S=\{e^{3t}\} \Rightarrow \quad y_p=Ae^{3t} \Rightarrow y_p'=3Ae^{3t} \quad y_p''=9Ae^{3t}$

$9Ae^{3t}-9Ae^{3t}+2Ae^{3t}=e^{3t} \Rightarrow \quad A=1/2 \quad y_p=\frac{1}{2}e^{3t}$

$$y=c_1e^t+c_2e^{2t}+\frac{1}{2}e^{3t}$$

$t=\ln x \Rightarrow \quad y=c_1x+c_2x^2+\frac{x^3}{2}$

(2) $x^3\frac{d^3y}{\partial x^3}-4x^2\frac{d^2y}{\partial x^2}+8x\frac{dy}{\partial x}-8y=4\ln x.$

$x=e^t \Rightarrow t=\ln x \quad \frac{\partial t}{\partial x}=\frac{1}{x} \Rightarrow \frac{dy}{\partial x}=\frac{1}{x}\frac{\partial y}{\partial t}$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2}=\frac{1}{x^2}\left(\frac{d^2y}{\partial x^2}-\frac{dy}{\partial t}\right)$$

$$\Rightarrow \frac{d^3y}{\partial x^3}=-\frac{2}{x^3}\left(\frac{d^2y}{\partial x^2}-\frac{dy}{\partial t}\right)+\frac{1}{x^2}\left(\frac{d^3y}{\partial t^3}\cdot\frac{dt}{\partial x}-\frac{d^2y}{\partial x^2}\cdot\frac{dt}{\partial x}\right)$$

$$=\underline{\frac{-2}{x^3}\frac{d^2y}{\partial x^2}}+\underline{\frac{2}{x^3}\frac{dy}{\partial t}}+\frac{1}{x^3}\underline{\frac{d^3y}{\partial t^3}}-\underline{\frac{1}{x^3}\frac{d^2y}{\partial x^2}}$$

$$=\frac{1}{x^3}\left(\frac{d^3y}{dt^3}-3\frac{d^2y}{\partial t^2}+2\frac{dy}{\partial t}\right)$$

Substating

$$\underline{\frac{d^3y}{\partial t^3}}-\underline{3\frac{d^2y}{\partial t^2}}+\underline{2\frac{dy}{\partial t}}-\underline{4\frac{d^2y}{\partial x^2}}+\underline{4\frac{dy}{\partial t}}+\underline{8\frac{dy}{\partial t}}-8y=4\ln x$$

$$\frac{d^3y}{\partial t^3}-7\frac{d^2y}{\partial x^2}+14\frac{dy}{\partial t}-8y=4\ln e^t=4t.$$

$m^3-7m^2+14m-8=0=(m-1)(m-2)(m-4)=0 \Rightarrow m=1,2,4.$

$y_c=c_1e^t+c_2e^{2t}+c_3e^{4t}.$

$y_p=At+B. \Rightarrow y_p'=A \quad y_p''=y_p'''=0$

$14A-8\cdot(At+B)=4$

$14A - 8B = 0 \Rightarrow B = -\frac{7}{8}$ $\qquad y_p = -\frac{1}{2}t - \frac{7}{8}t$

$-8A = 4 \Rightarrow A = -\frac{1}{2}$

$$y = c_1 e^{t} + c_2 e^{2t} + c_3 e^{4t} - \frac{1}{2}t - \frac{7}{8}$$

$$= c_1 x + c_2 x^2 + c_3 x^4 - \frac{1}{2}\ln x - \frac{7}{8}$$

<u>Exercises</u>

Find the general solution of each of the followings.

1) $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 3y = 0$ $\qquad$ 2) $4x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 3y = 0$

3) $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 4y = 0.$

<u>Solution:</u> $x = e^t \Rightarrow t = \ln x \quad \dfrac{dt}{dx} = \dfrac{1}{x} \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dy}{dt}\cdot\dfrac{1}{x}$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + \frac{dy}{dt} + 4y = \frac{d^2y}{dt^2} - 4\cancel{\frac{dy}{dt}} = 0$$

$m^2 + 4 = 0 \Rightarrow m = \mp 2i \Rightarrow y_c = c_1 \sin 2t + c_2 \cos 2x$

$$+ c_2 \cos(2\ln x)$$

$$y = c_1 \sin(2\ln x) + c_2 \cos(2\ln x)$$

4) $x^3 \dfrac{d^3y}{dx^3} - 3x^2\dfrac{d^2y}{dx^2} + 6x\dfrac{dy}{dx} - 6 = 6x\ln x$

5) $x^3 \dfrac{d^3y}{dx^3} - x^2\dfrac{d^2y}{dx^2} - 6x\dfrac{dy}{dx} + 18y = 0.$

<u>Solution</u> $x = e^t \Rightarrow t = \ln x \quad \dfrac{dt}{dx} = \dfrac{1}{x} \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{x}\dfrac{dy}{dt}$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) \quad \frac{d^3y}{dx^3} = \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right).$$

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - \frac{d^2y}{dt^2} + \frac{dy}{dt} - 6\frac{dy}{dt} + 18y = 0$$

$$\frac{d^3y}{dt^3} - 4\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 18y = 0.$$

$$m^3-4m^2-3m+18=(m+2)(m^2-6m+9)=(m+2)(m-3)^2=0$$

$$m=-2,3,3$$

$$y=c_1e^{-2t}+c_2e^{3t}+c_3te^{3t}$$

$$=c_1x^{-2}+c_2x^3+c_3x^3.\ln|x|.$$

6) $x^2\frac{d^2y}{dx^2}-5x\frac{dy}{dx}+8y=2x^2$

7) $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+4y=2x\ln x.$

Solution: $x=e^t \Rightarrow t=\ln x \quad \frac{dt}{dx}=\frac{1}{x} \Rightarrow \frac{dy}{dx}=\frac{1}{x}\frac{dy}{dt}$

$$\frac{d^2y}{dx^2}=\frac{1}{x^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right).$$

$$\frac{d^2y}{dt^2}-\frac{dy}{dt}+\frac{dy}{dt}+4y=\frac{d^2y}{dt^2}+4y=2te^t$$

$m^2+4=0 \Rightarrow \quad m=\mp 2i \quad y_c=c_1\sin 2t+c_2\cos 2t.$

$u=te^t \quad S=\{te^t, e^t\} \quad y_p=Ate^t+Be^t$

$y_p'=(A+B)e^t+Ate^t. \quad y_p''=(2A+B)e^t+Ate^t$

$(2A+B)e^t+Ate^t+4Ate^t+4Be^t=2te^t$

$5Ate^t+(2A+2B)e^t=2te^t$

$5A=2 \quad A=\frac{2}{5} \quad B=-\frac{4}{25} \quad y_c=\frac{2}{5}te^t-\frac{4}{25}e^t$

$2A+5B=0$

$$=\frac{2}{5}\ln x.e^{\ln x}-\frac{4}{25}e^{\ln x}=\frac{2x}{5}\ln x-\frac{4}{5}x.$$

$$y=c_1\sin(\ln x^2)+c_2\cos(\ln x^2)+\frac{x}{5}\ln x^2-\frac{4x}{5}$$

8) $(x+2)^2\frac{d^2y}{dx^2}-(x+2)\frac{dy}{dx}-3y=0$

9) $(2x-3)^2 \frac{d^2y}{dx^2} - 6(2x-3)\frac{dy}{dx} + 12y = 0.$

Solution $u = 2x-3 \Rightarrow \frac{du}{dx} = 2$ $\quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2\frac{dy}{du}.$

$$\Rightarrow u^2 \frac{d^2y}{du^2} - 6u\frac{dy}{du} + 12y = 0.$$

$u = e^t \Rightarrow t = \ln u, \quad \frac{dt}{du} = \frac{1}{u} \Rightarrow \frac{dy}{du} = \frac{1}{u}\frac{dy}{dt}$

$$\frac{d^2y}{du^2} = \frac{1}{u^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6\frac{dy}{dt} + 12y = \frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 0$$

$$m^2 - 7m + 12 = (m-3)(m-4) = 0 \Rightarrow m = 3,4.$$

$$y = c_1 e^{3t} + c_2 e^{4t}$$
$$= c_1 u^3 + c_2 u^4$$
$$= c_1 (2x-3)^3 + c_2 (2x-3)^4.$$

2. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 4y = 0.$

3. $4x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 3y = 0.$

4. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 0.$

5. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 4y = 0.$

6. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 13y = 0.$

7. $x^3\dfrac{d^3y}{dx^3} - 3x^2\dfrac{d^2y}{dx^2} + 6x\dfrac{dy}{dx} - 6y = 0.$

8. $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} - 10x\dfrac{dy}{dx} - 8y = 0.$

9. $x^3\dfrac{d^3y}{dx^3} - x^2\dfrac{d^2y}{dx^2} - 6x\dfrac{dy}{dx} + 18y = 0.$

10. $x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = 4x - 6.$

11. $x^2\dfrac{d^2y}{dx^2} - 5x\dfrac{dy}{dx} + 8y = 2x^3.$

12. $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = 4\ln x, \quad (x > 0).$

13. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 4y = 2x\ln x, \quad (x > 0).$

14. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \dfrac{1}{1+x}.$

15. $x^3\dfrac{d^3y}{dx^3} - x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 2y = x^3.$

Solve the initial value problems in Exercises 16 through 20.

16. $$\begin{cases} x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 10y = 0 \\ y(1) = 5 \\ y'(1) = 4. \end{cases}$$

17. $$\begin{cases} x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = 0 \\ y(2) = 0 \\ y'(2) = 4. \end{cases}$$

18. $$\begin{cases} x^2\dfrac{d^2y}{dx^2} - 5x\dfrac{dy}{dx} + 8y = 2x^3 \\ y(-2) = 1 \\ y'(-2) = 7. \end{cases}$$

19. $$\begin{cases} x^2\dfrac{d^2y}{dx^2} - 6y = \ln x, \quad (x > 0) \\ y(1) = \frac{1}{6} \\ y'(1) = -\frac{1}{6}. \end{cases}$$

20. Solve:

$$(x + 2)^2\frac{d^2y}{dx^2} - (x + 2)\frac{dy}{dx} - 3y = 0.$$

21. Solve:

$$(2x - 3)^2\frac{d^2y}{dx^2} - 6(2x - 3)\frac{dy}{dx} + 12y = 0.$$

SUGGESTED READING

Agnew (1)
Coddington (12)
Ford (17)
Greenspan (20)
Kaplan (30)
Leighton (36)
Martin and Reissner (38)
Rainville (45)