4

Linear Equations and Bernoulli Equations.

__Definition:__ A first-order ordinary differential equation is called linear if it can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x). \quad (1)$$

Let us write the equation (1) in the form

$$(P(x)y - Q(x))\,dx + dy = 0 \quad (2)$$

It is clear that the equation (2) is not exact unless $P(x)=0$ and not separable in general. Let us multiply equation (2) by $\mathcal{g}(x)$, obtaining

$$(\mathcal{g}(x)P(x)y - \mathcal{g}(x)Q(x))\,dx + \mathcal{g}(x)\,dy = 0 \quad (3)$$

$\mathcal{g}(x)$ is an integrating factor of (3) iff Equation (3) is exact, that is

$$\frac{\partial(\mathcal{g}(x)P(x)y - \mathcal{g}(x)Q(x))}{\partial y} = \frac{\partial(\mathcal{g}(x))}{\partial x}$$

or

$$\mathcal{g}(x)P(x) = \frac{d\mathcal{g}}{dx} \quad (4)$$

It is clear that the equation (4) is separable, and so

$$\frac{d\mathcal{g}}{\mathcal{g}} = P\,dx$$

Integrating, we obtain the particular solution

$$\ln|\mathcal{g}| = \int P\,dx$$

or

$$\mathcal{g} = e^{\int P\,dx} \quad (5).$$

Thus the linear equation (1) possesses an integrating factor of the form (5). Multiplying (2.26) by (2.30) gives

$$e^{\int P\,dx}\frac{dy}{dx} + e^{\int P\,dx}Py = Qe^{\int P\,dx}$$

or

$$\frac{d\left(e^{\int P\,dx}y\right)}{dx} = Qe^{\int P\,dx}$$

Integrating this

$$e^{\int P\,dx}y = \int e^{\int P\,dx}Q\,dx + c$$

Theorem: The linear differential equation $\frac{dy}{dx} + P(x)y = Q(x)$ has an integrating of the form $e^{\int P dx}$. The general solution of this equation is

$$y = e^{-\int P dx}\left(\int e^{\int P dx} Q\, dx + c\right)$$

Examples: (1) $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$

Here $P(x) = \frac{2x+1}{x}$, and hence

$$\mu(x) = e^{\int \frac{2x+1}{x} dx} = e^{\int (2+\frac{1}{x}) dx} = e^{2x+\ln x} = x e^{2x}$$

Multiplying the equation by μ, we obtain

$$x e^{2x} \frac{dy}{dx} + e^{2x}(2x+1)y = x$$

or

$$\frac{d(xe^{2x}y)}{dx} = x \Rightarrow xe^{2x}y = \frac{x^2}{2} + c$$

$$\Rightarrow y = \frac{1}{2} x e^{-2x} + \frac{c}{x} e^{-2x}$$

(2) Solve the initial-value problem

$$(x^2+1)\frac{dy}{dx} + 4xy = x \qquad y(2) = 1.$$

Dividing (x^2+1), $\frac{dy}{dx} + \frac{4x}{(x^2+1)} y = \frac{x}{x^2+1}$

$P(x) = \frac{4x}{(x^2+1)} \Rightarrow \mu(x) = e^{\int \frac{4x}{x^2+1} dx} = e^{2\ln(x^2+1)} = (x^2+1)^2$

Multiplyin $\mu(x)$, $(x^2+1)^2 \frac{dy}{dx} + (x^2+1)4x\, y = x(x^2+1)$

$$\frac{d[(x^2+1)^2 y]}{dx} = x^3 + x$$

$(x^2+1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + c \Rightarrow y = 1$ and $(x) = 2$ $\quad 25 = 4 + 2 + c$, $c = \frac{19}{6}$

$$\Rightarrow (x^2+1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + 19 \quad //.$$

(3) Solve the differential equation $y^2 dx + (3xy-1)\,dy = 0$

Solving for $\frac{dy}{dx}$, $\frac{dy}{dx} = \frac{y^2}{1-3xy}$ which is not linear in y.

Solving for $\frac{dx}{dy}$, $\frac{dx}{dy} = \frac{1-3xy}{y^2} = \frac{1}{y^2} - \frac{3x}{y}$. or

$$\frac{dx}{dy} + \left(\frac{3}{y}\right)x = \frac{1}{y^2}$$

which is linear in x.

$P(y) = \frac{3}{y} \Rightarrow \mu(y) = e^{\int \frac{3}{y}dy} = e^{3\ln|y|} = y^3$.

$$y^3 \frac{dx}{dy} + 3y^2 x = y \Rightarrow \frac{d(y^3 x)}{dy} = y$$

$$y^3 x = \frac{y^2}{2} + c \Rightarrow x = \frac{1}{2y} + \frac{c}{y^3}$$

Bernoulli Equations

Definition: An equation of the form
$$\frac{dy}{dx} + P(x)y = Q(x).y^n \qquad (6)$$
is called a Bernoulli Dif. Equation.

Notice that if $n=1$, then the equation (6) is separable.

Theorem: Suppose that $n \neq 1$. Then the transformation $v = y^{1-n}$ reduces the Bernoulli equation (6) to a linear equation in v.

Proof: We first multiply (6) by y^{-n}.
$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x) \qquad (7)$$

Let $v = y^{1-n}$, then $\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$ and Equation (2.38) transforms into
$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x) \quad \text{or}$$
$$\frac{dv}{dx} + (1-n)P(x)\,v = (1-n)\,Q(x)$$
which is linear in v.

Examples: (1) $\frac{dy}{dx} + y = xy^3$

multiplying by y^{-3}, $y^{-3}\frac{dy}{dx} + y^{-2} = x$

let $v = y^{-2} \Rightarrow \frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$. Then, transforming

$$\frac{dv}{dx} - 2v = -2x$$

$P(x) = -2 \Rightarrow \mathcal{g}(x) = e^{\int -2dx} = e^{-2x}$. Multiplying e^{-2x}

$$e^{-2x}\frac{dv}{dx} - 2e^{-2x}v = -2xe^{-2x}$$

$$\frac{d(e^{-2x}v)}{dx} = -2xe^{-2x} \Rightarrow v = x + \frac{1}{2} + ce^{2x}$$

Thus $\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}$.

(2) $\frac{dy}{dx} - \frac{1}{3x}y = -2x^4y^4$

Multiplying y^{-4}, $y^{-4}\frac{dy}{dx} - \frac{1}{3x}y^{-3} = -2x^4$.

If $v = y^{-3} \Rightarrow \frac{dv}{dx} = -3y^{-4}\frac{dy}{dx}$. Transforming $v = y^{-3}$,

$$\frac{1}{-3}\frac{dv}{dx} - \frac{1}{3x}v = -2x^4 \quad \text{or} \quad \frac{dv}{dx} + \frac{1}{x}v = 6x^4$$

$\mathcal{g}(x) = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$. multiplying $\mathcal{g}(x) = x$

$$x\frac{dv}{dx} + v = 6x^5 \Rightarrow \frac{d(xv)}{dx} = 6x^5 \Rightarrow xv = x^6 + c \Rightarrow v = x^5 + \frac{c}{x}$$

$$\Rightarrow \frac{1}{y^3} = x^5 + \frac{c}{x} //.$$

Multiplying (2.41) by e^{-2x}, we find

$$e^{-2x}\frac{dv}{dx} - 2e^{-2x}v = -2xe^{-2x}$$

or

$$\frac{d}{dx}[e^{-2x}v] = -2xe^{-2x}.$$

Integrating, we find

$$e^{-2x}v = \frac{1}{2}e^{-2x}(2x + 1) + c$$

or

$$v = x + \frac{1}{2} + ce^{2x}.$$

But

$$v = \frac{1}{y^2}.$$

Thus we obtain the solution of (2.39) in the form

$$\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}.$$

Exercises

Solve the given differential equations in Exercises 1 through 15.

1. $\dfrac{dy}{dx} + \dfrac{3y}{x} = 6x^2.$

2. $x^4\dfrac{dy}{dx} + 2x^3y = 1.$

3. $\dfrac{dx}{dt} + \dfrac{x}{t^2} = \dfrac{1}{t^2}.$

4. $(u^2 + 1)\dfrac{dv}{du} + 4uv = 3u.$

5. $x\dfrac{dy}{dx} + \dfrac{2x + 1}{x + 1}y = x - 1.$

6. $(x^2 + x - 2)\dfrac{dy}{dx} + 3(x + 1)y = x - 1.$

7. $xdy + (xy + y - 1)dx = 0.$

8. $ydx + (xy^2 + x - y)dy = 0.$

9. $\dfrac{dr}{d\theta} + r\tan\theta = \cos\theta.$

10. $\cos\theta\, dr + (r\sin\theta - \cos^4\theta)d\theta = 0.$

11. $(\cos^2 x - y\cos x)dx - (1 + \sin x)dy = 0.$

12. $\dfrac{dy}{dx} - \dfrac{y}{x} = -\dfrac{y^2}{x}.$

13. $x\dfrac{dy}{dx} + y = -2x^6y^4.$

14. $dy + (4y - 8y^{-3})x\,dx = 0.$

15. $\dfrac{dx}{dt} + \dfrac{t+1}{2t}x = \dfrac{t+1}{xt}.$

Solve the initial-value problems in Exercises 16 though 22.

16. $\begin{cases} x\dfrac{dy}{dx} - 2y = 2x^4 \\ y(2) = 8. \end{cases}$

17. $\begin{cases} \dfrac{dy}{dx} + 3x^2y = x^2 \\ y(0) = 2. \end{cases}$

18. $\begin{cases} 2x\,(y+1)dx - (x^2+1)dy = 0 \\ y(1) = -5. \end{cases}$

19. $\begin{cases} \dfrac{dr}{d\theta} + r\tan\theta = \cos^2\theta \\ r\left(\dfrac{\pi}{4}\right) = 1. \end{cases}$

20. $\begin{cases} \dfrac{dx}{dt} - x = \sin 2t \\ x(0) = 0. \end{cases}$

21. $\begin{cases} \dfrac{dy}{dx} + \dfrac{y}{2x} = \dfrac{x}{y^3} \\ y(1) = 2. \end{cases}$

22. $\begin{cases} x\dfrac{dy}{dx} + y = (xy)^{3/2} \\ y(1) = 4. \end{cases}$

23. Consider the equation $a\dfrac{dy}{dx} + by = ke^{-\lambda x}$, where a, b, and k are positive constants and λ is a nonnegative constant.

(a) Solve this equation.

(b) Show that if $\lambda = 0$ every solution approaches k/b as $x \to \infty$ but if $\lambda > 0$ every solution approaches 0 as $x \to \infty$.

24. Solve the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} + y = f(x), \quad \text{where } f(x) = \begin{cases} 2, & 0 \leq x < 1, \\ 0, & x \geq 1, \end{cases} \\ y(0) = 0. \end{cases}$$

25. Solve the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} + y = f(x), \quad \text{where } f(x) = \begin{cases} 5, & 0 \leq x < 10, \\ 1, & x \geq 10, \end{cases} \\ y(0) = 6. \end{cases}$$

26. Consider the differential equation $\dfrac{dy}{dx} + P(x)y = 0.$

(a) Show that if f and g are two solutions of this equation and c_1 and c_2 are arbitrary constants, then $c_1 f + c_2 g$ is also a solution of this equation.

(b) Extending the result of (a), show that if $f_1, f_2, \cdots, f_n$ are n solutions of this equation and $c_1, c_2, \cdots, c_n$ are n arbitrary constants, then

$$\sum_{k=1}^{n} c_k f_k$$

is also a solution of this equation.

27. Consider the differential equation

$$\text{(A)} \qquad \frac{dy}{dx} + P(x)y = 0,$$

where P is continuous on a real interval I.

(a) Show that the function f such that $f(x) = 0$ for all $x \in I$ is a solution of this equation.

(b) Show that if f is a solution of (A) such that $f(x_0) = 0$ for some $x_0 \in I$, then $f(x) = 0$ for all $x \in I$.

(c) Show that if f and g are two solutions of (A) such that $f(x_0) = g(x_0)$ for some $x_0 \in I$, then $f(x) = g(x)$ for all $x \in I$.

28. (a) Prove that if f and g are two different solutions of

$$\text{(A)} \qquad \frac{dy}{dx} + P(x)y = Q(x),$$

then $f - g$ is a solution of the equation

$$\frac{dy}{dx} + P(x)y = 0.$$

(b) Thus show that if f and g are two different solutions of Equation (A) and c is an arbitrary constant, then

$$c(f - g) + f$$

is a general solution of (A).

29. (a) Let f_1 be a solution of

$$\frac{dy}{dx} + P(x)y = Q_1(x)$$

and f_2 be a solution of

$$\frac{dy}{dx} + P(x)y = Q_2(x),$$

where P, Q_1, and Q_2 are all defined on the same real interval I. Prove that $f_1 + f_2$ is a solution of

$$\frac{dy}{dx} + P(x)y = Q_1(x) + Q_2(x)$$

on I.

(b) Use the result of (a) to solve the equation

$$\frac{dy}{dx} + y = 2\sin x + 5\sin 2x.$$

30. (a) Extend the result of Exercise 29(a) to cover the case of the equation

$$\frac{dy}{dx} + P(x)y = \sum_{k=1}^{n} Q_k(x),$$

where P, $Q_k(k = 1,2, \cdots, n)$ are all defined on the same real interval I.

(b) Use the result obtained in (a) to solve the equation

$$\frac{dy}{dx} + y = \sum_{k=1}^{5} \sin kx.$$

31. (a) Show that the transformation $v = f(y)$ reduces the equation

$$\frac{df(y)}{dy}\frac{dy}{dx} + P(x)f(y) = Q(x)$$

to a linear equation in v.

(b) Use the result of (a) to solve the equation

$$(y + 1)\frac{dy}{dx} + x(y^2 + 2y) = x.$$

32. The equation

$$\text{(A)} \qquad \frac{dy}{dx} = A(x)y^2 + B(x)y + C(x)$$

is called Riccati's Equation.

(a) Show that if $A(x) = 0$ for all x, then Equation (A) is a linear equation, whereas if $C(x) = 0$ for all x, then equation (A) is a Bernoulli equation.

(b) Show that if f is any solution of the equation (A), then the transformation

$$y = f + \frac{1}{v}$$

reduces (A) to a linear equation in v.

(c) Consider the Riccati equation

$$\text{(B)} \qquad \frac{dy}{dx} = (1 - x)y^2 + (2x - 1)y - x.$$

Exercises

Solve the given differential equations below.

1) $\frac{dy}{dx} + \frac{dy}{\cancel{x}} = 6x^2$

2) $x^4 \frac{dy}{dx} + 2x^3 y = 1$

3) $\frac{dx}{dt} + \frac{x}{t^2} = \frac{1}{t^2}$

Solution $P(t) = \frac{1}{t^2} \Rightarrow \mu(t) = e^{\int \frac{1}{t^2} dt} = e^{-1/t}$

$\Rightarrow e^{-1/t} \frac{dx}{dt} + \frac{e^{-1/t}}{t^2} x = \frac{e^{-1/t}}{t^2} \Rightarrow \frac{d(e^{-1/t} x)}{dt} = \frac{e^{-1/t}}{t^2}$

$\Rightarrow d(e^{-1/t} x) = \int \frac{e^{-1/t}}{t^2} dt \Rightarrow e^{-1/t} x = e^{-1/t} + c \Rightarrow \underline{x = 1 + ce^{1/t}}$

$u = e^{-1/t} \Rightarrow du = \frac{e^{-1/t}}{t^2} dt$.

4) $(u^2+1) \frac{dv}{du} + 4uv = 3u$.

5) $x \frac{dy}{dx} + \frac{2x+1}{x+1} y = x - 1$

6) $x dy + (xy + y - 1) dx = 0$.

Solution: $\frac{dy}{dx} + \frac{xy + y - 1}{x} = 0 \Rightarrow \frac{dy}{dx} + (1 + \frac{1}{x}) y = \frac{1}{x}$

$\mu(x) = e^{\int (1 + \frac{1}{x}) dx} = e^{x + \ln|x|} = xe^x$

$xe^x \frac{dy}{dx} + (1 + \frac{1}{x}) xe^x y = e^x \Rightarrow xe^x \frac{dy}{dx} + (x+1) e^x y = e^x$

$\Rightarrow \frac{d(xe^x y)}{dx} = e^x \Rightarrow xe^x y = e^x + c \Rightarrow y = \frac{1}{x} + \frac{c}{xe^x}$

7) $y dx + (xy^2 + x - y) dy = 0$

8) $\frac{dr}{d\theta} + r \tan\theta = \cos\theta$

9) $(\cos^2 x - y\cos x)dx - (1+\sin x)dy = 0$

$$\frac{dy}{dx} - \frac{\cos^2 x - y\cos x}{1+\sin x} = 0 \Rightarrow \frac{dy}{dx} + \frac{\cos x}{1+\sin x} y = \frac{\cos^2 x}{1+\sin x}.$$

$$I = \int \frac{\cos x}{1+\sin x}dx = \int \frac{du}{u} = \ln|u| = \ln|1+\sin x| \Rightarrow \mathcal{I}(x) = 1+\sin x$$

$u = 1+\sin x \Rightarrow du = \cos x\, dx$

$$\Rightarrow (1+\sin x)\frac{dy}{dx} + (\cos x)y = \cos^2 x \Rightarrow \frac{d((1+\sin x)y)}{dx} = \cos^2 x.$$

$$J = \int \cos^2 x\, dx = \int \frac{\cos 2x + 1}{2}dx = \frac{1}{2}\left(\frac{1}{2}\sin 2x + x\right) + c$$

$$\Rightarrow (1+\sin x)y = \frac{1}{4}\sin 2x + \frac{x}{2} + c$$

10) $\frac{dy}{dx} - \frac{y}{x} = -\frac{y^2}{x}$

11) $x\frac{dy}{dx} + y = -2x^6 y^4.$

Solution $\frac{dy}{dx} + \frac{1}{x}y = -2x^5 y^4 \Rightarrow y^{-4}\frac{dy}{dx} + \frac{1}{x}y^{-3} = -2x^5$

$v = y^{-3} \Rightarrow \frac{dv}{dx} = -3y^{-4}\frac{dy}{dx} \Rightarrow \frac{dv}{dx} - \frac{3}{x}v = 6x^5$

$\mathcal{I}(x) = e^{\int -\frac{3}{x}dx} = e^{-3\ln|x|} = x^{-3}$ $\qquad x^{-3}\frac{dv}{dx} - 3x^{-4}v = 6x^2$

$\frac{d(x^{-3}v)}{dx} = 6x^2 \Rightarrow x^{-3}v = 2x^3 + c$ $\qquad v = 2 + cx^3 \Rightarrow \frac{1}{y^3} = 2 + cx^3$?

$(x^3y^3 = (2x^3 + c))$

12) $dy + (4y - 8y^{-3})x\,dx = 0$

Solution: $\frac{dy}{dx} + 4y - 8y^{-3}x = 0 \Rightarrow \frac{dy}{dx} + 4yx = 8y^{-3}x$

$y^3\frac{dy}{dx} + 4y^4 x = 8x$ $\qquad v = y^4 \Rightarrow \frac{dv}{dx} = 4y^3\frac{dy}{dx}$

$\frac{dv}{dx} + 16xv = 32x$ $\qquad \mathcal{I}(x) = e^{\int 16x\,dx} = e^{8x^2}$

$e^{8x^2}\frac{dv}{dx} + 16xe^{8x^2}v = 32x.e^{8x^2} \Rightarrow \frac{d(e^{8x^2}v)}{dx} = 32xe^{8x^2}$

$e^{8x^2}v = 2e^{8x^2} + c \Rightarrow V = 2 + \frac{c}{e^{8x^2}}$ $\qquad \underline{y^4 = 2 + ce^{-8x^2}}$

6

CamScanner ile tarandı

13) $\frac{dx}{dt} + \frac{t+1}{2t}x = \frac{t+1}{xt}$ $\quad (x^2 = 2 + ct^{-1}e^{-t})$

Solve the initial value problems

14) $x\frac{dy}{dx} - 2y = 2x^4$ $\quad y(2) = 8$

15) $\frac{dy}{dx} + 3x^2 y = x^2$ $\quad y(0) = 2$

16) $2x(y+1)\,dx - (x^2+1)\,dy = 0$ $\quad y(1) = -5$

Solution $-\frac{dy}{dx} + \frac{2x(y+1)}{x^2+1} = 0 \Rightarrow \frac{dy}{dx} - \frac{2x}{x^2+1}y - \frac{2x}{x^2+1} = 0$

$\frac{dy}{dx} - \frac{2x}{x^2+1}y = \frac{2x}{x^2+1}$

$\mathcal{J}(x) = e^{\int \frac{-2x}{x^2+1}dx} = e^{-\ln|x^2+1|} = \frac{1}{x^2+1}$

$\frac{1}{x^2+1}\frac{dy}{dx} - \frac{2x}{(x^2+1)}y = \frac{2x}{(x^2+1)^2} \Rightarrow \frac{d\left(\frac{1}{x^2+1}y\right)}{dx} = \frac{2x}{(x^2+1)^2}$

$\int \frac{2x}{(x^2+1)}dx = \int \frac{du}{u^2} = -u^{-1} = \frac{-1}{x^2+1}$ $\quad u = x^2+1$

$\frac{1}{x^2+1}y = \frac{-1}{x^2+1} + c \Rightarrow y = -1 + c(x^2+1)$

$-5 = -1 + c(1^2+1) \Rightarrow 2c = -4 \quad \underline{c = -2}$

$y = -1 - 2(x^2+1)$

17) $\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3}$ $\quad y(1) = 2$

Solution: $\frac{dy}{dx} + \frac{1}{2x}y = xy^{-3} \Rightarrow y^3\frac{dy}{dx} + \frac{1}{2x}y^4 = x$

$u = y^4 \Rightarrow 4y^3\frac{dy}{dx} = \frac{du}{dx}$ $\quad \frac{du}{dx} + \frac{2}{x}u = 4x$

$\mathcal{J}(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = x^2$

$x^2\frac{du}{dx} + 2xu = 4x^3 \Rightarrow \frac{d(x^2u)}{dx} = 4x^3 \Rightarrow x^2u = x^4 + c$

$u = x^2 + cx^{-2} \Rightarrow y^4 = x^2 + cx^{-2}$ $\quad 16 = 1 + c \Rightarrow c = 15$

$y^4 = x^2 + \frac{15}{x^2}$ //

7