DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS

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BASIC DEFINITIONS

Definition: An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a differential equation

Examples

(1) $\dfrac{d^2y}{dx^2} + xy\left(\dfrac{dy}{dx}\right)^2 = 0$

(2) $\dfrac{d^4x}{dt^4} + 5\dfrac{d^2x}{dt^2} + 3x = \sin t$

(3) $\dfrac{\partial v}{\partial s} + \dfrac{\partial v}{\partial t} = v$

(4) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

* Bir veya daha fazla bağımlı değişkenin bir veya daha fazla bağımsız değişkene göre türevlerini içeren denklemler d.d. denir.

* Sadece bir tek bağımsız değişken içeren adi-diferansiyelli denklemlere a.d.d. denir.

* En az bir bağımsız değişken içeren kısmi-türevli denklemlere k.d.d. denir.

Definitions: A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation

A differential equation involving partial derivatives of one or more dependent variables with respect to one or more independent variables is called a partial differential equation

Examples

(1) The equation $\dfrac{d^2y}{dx^2} + xy\left(\dfrac{dy}{dx}\right)^2 = 0$ is an ordinary differential equation with the dependent variable y and with the independent variable x.

(2) The equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ is a partial

differential equation with the dependent variable u and the independent variables x, y and z.

<u>Definition:</u> The order of highest ordered derivative involved in a differential equation is called the order of the differential equation.

Bir d. denklemdeki en yüksek dereceden türevin derecesine denklemin derecesi denir.

<u>Examples:</u>

(1) The equation $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^3 = 0$ is of the second order.

(2) The equation $\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x^6 = \sin t$ is an ordinary differential equation of the fourth order.

(3) The partial differentia equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ of the second order.

<u>Definition:</u> A differential equation is called linear if

(a) every dependent variable and every derivative involved occurs to the first degree only.

(b) no products of dependent variables and/or derivatives occur, and

(c) no transcendental functions of the dependent variable or its derivatives occur.

A differential equation which is not linear is called a nonlinear differential equation.

Therefore, a linear differential equation of order n, in the dependent variable y and independent variable x, is an equation that can be expressed in the form

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x)$$

where $a_0(x)$ is not identically zero.

Examples:

(1) The following ordinary differential equation are both linear:

(a) $$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

(b) $$\frac{d^4y}{dx^4} + x^2\frac{d^3y}{dx^3} + \sin x.\frac{dy}{dx} = xe^x$$

(2) The following ordinary differential equation are all nonlinear:

(a) $$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0$$

(b) $$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^2 + 6y = 0$$

(c) $$\frac{d^2y}{dx^2} + 5y\frac{dy}{dx} + 6y = 0$$

(d) $$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + \sin y = 0.$$

(3) The partial differential equation

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$$

is linear, but the differential equation

$$\left(\frac{\partial y}{\partial t}\right)^2 = \frac{\partial^2 y}{\partial x^2}$$

is not.

Exercises:

Classify each of the following differential equasion as ordinary or partial differential equations; state the order of each equation; and determine whether the equation under consideration is linear or non-linear.

1) $\frac{dy}{dx}+x^2y=xe^x$

2) $\frac{d^3y}{dx^3}+4\frac{d^2y}{dx^2}-5\frac{dy}{dx}+3y=\sin x$

3) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

4) $x^2dy+y^2dx=0$

5) $\frac{d^4y}{dx^4}+3\left(\frac{d^2y}{dx^2}\right)^5+5y=0$

6) $\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+u=0$

7) $\frac{d^2y}{dx^2}+y\sin x=0$

8) $\frac{d^2y}{dx^2}+x\sin y=0$

9) $\frac{d^6x}{dt^6}+\left(\frac{d^4x}{dt^4}\right)\left(\frac{d^3x}{dt^3}\right)+tx=t$

10) $\left(\frac{dr}{ds}\right)^3=\sqrt{\frac{d^2r}{ds^2}+1}$

Solutions:

Definitions: Consider the n^{th}-order ordinary differential equation

$$F\left[x,y,\frac{dy}{dx},\dots,\frac{d^ny}{dx^n}\right]=0 \qquad (I)$$

where F is a real function of its $(n+2)$ arguments $x,y,\frac{dy}{dx},\dots,\frac{d^ny}{dx^n}$.

(1) Let f be a real function defined for all x in a real interval I and having n^{th} derivative (and hence also all lower ordered derivatives) for all $x\in I$. The function f is called an explicit solution of the differential equation (I) on I if it fullfills the following the requirements:

(A) $F[x,f(x),f'(x),\dots,f^{(n)}(x)]$ is defined for all $x\in I$, and

(B) $F[x, f(x), f'(x), \ldots, f^{(n)}(x)] = 0$

for all $x \in I$.

(2) A relation $g(x,y)=0$ is called an implicit solution of (I) if this relation defines at least one real function f of the variable x on an interval I such that this function is an explicit solution of (I) on this interval.

Examples:

(1) $f(x) = 2\sin x + 3\cos x$ is an explicit solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0 \quad (\text{II})$$

for all $x \in \mathbb{R}$. First note that f is defined and has a second derivative for all $x \in \mathbb{R}$. Next observe that

$$f'(x) = 2\cos x - 3\sin x$$
$$f''(x) = -2\sin x - 3\cos x$$

By substituting $f''(x)$ for $\frac{d^2y}{dx^2}$ and $f(x)$ for y in the differential equation (II), it reduces to the identity

$$(-2\sin x - 3\cos x) + (2\sin x + 3\cos x) = 0,$$

which holds for all real x.

(2) The relation $x^2+y^2 = 25$ is an implicit solution of the differential equation

$$x + y\frac{dy}{dx} = 0 \quad (\text{III})$$

on the interval defined by $-5 < x < 5$. From the relation we have two real functions f_1 and f_2 given by

$$f_1(x) = \sqrt{25 - x^2} \quad \text{and} \quad f_2(x) = -\sqrt{25 - x^2}$$

for all $x \in I = (-5, 5)$, and both of these function are explicit solutions of the equation (III) on I. Let us illustrate this for the function f_1.

$$f_1(x) = \sqrt{25-x^2} \Rightarrow f_1'(x) = \frac{-x}{\sqrt{25-x^2}}$$

for all $x \in I$. By substituting

$$x + y\frac{dy}{dx} = x + \left(\cancel{\sqrt{25-x^2}}\right)\cdot\left(\frac{-x}{\cancel{\sqrt{25-x^2}}}\right) = x + (-x) = 0$$

which holds for all real $x \in I$.

(3) Let us consider the first order differential equation

$$\frac{dy}{dx} = 2x$$

The function f defined for all real x by

$$f(x) = x^2 + c$$

is a solution of the equation for any constant c. This solution f is called a general solution of the equation.

(4) Consider the second-order differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

and consider the function f defined for all real x by

$$f(x) = c_1 \sin x + c_2 \cos x$$

We can easily verify that this function f is a solution of the equation for any real values of the constants c_1 and c_2.

<u>Definition:</u> Let

$$F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^ny}{dx^n}\right) = 0 \qquad \text{(IV)}$$

be an n^{th} order-ordinary differential equation.

(A) A solution of (IV) containing n essentially arbitrary constant will be called a "general solution of" (III).

(B) A solution of (IV) obtained from a general solution of (IV) by giving particular values to one or more of the n essentially arbitrary constant will be called a particular solution of (IV).

(C) A solution of (IV) which can not be obtained from any general solution of (IV) by any choice of the n-th essentially arbitrary constant will be called a singular solution of (IV).

Example: (1) Consider the second-order differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0.$$

$f(x)=c_1e^x+c_2e^{2x}$ is a general solution of the equation.

$$\left.\begin{array}{l} f_1(x)=5e^x+e^{2x} \\ f_2(x)=\frac{1}{2}e^x+c_2e^{2x} \\ f_3(x)=ce^x+\sqrt{2}e^{2x} \end{array}\right\} \text{are all particular solutions of the equation.}$$

$f_4(x)=e^{2x}$

(2) Consider the first-order differential equation

$$\left(\frac{dy}{dx}\right)^2-4y=0.$$

$f(x)=(x+c)^2$ is a general solution of the equation

$f_1(x)=x^2$

$f_2(x)=(x+\pi)^2$ are particular solutions of the equation.

$f_3(x)=x^2+2x+1$

$g(x)=0$ is a singular solution of the equation.

Exercises: (1) Show that each of the functions defined in column I below is a solution of the corresponding differential equation in column II on every interval $a<x<b$ of the x axis.

	I	II
(a)	$f(x)=x+3e^{-x}$	$\frac{dy}{dx}+y=x+1$
(b)	$f(x)=2e^{3x}-5e^{4x}$	$\frac{d^2y}{dx^2}-7\frac{dy}{dx}+12y=0$
(c)	$f(x)=e^{x}+2x^2+6x+7$	$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=4x^2$
(d)	$f(x)=\frac{1}{1+x^2}$	$(1+x^2)\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=0.$

(2) Show that $x^3+3xy^2=1$ is an implicit solution of $2xy\frac{dy}{dx}+x^2+y^2=0$ on $(0,1)$.

(3) Show that $5x^2y^2-2x^3y^2=1$ is an implicit solution of $x\frac{dy}{dx}+y=x^3y^3$ on $(0,\frac{5}{2})$

(4) Show that $f(x)=c_1e^{4x}+c_2e^{-2x}$ is a general solution of $y''-2y'-8y=0$.

(5) Show that $f(x)=c_1e^{2x}+c_2xe^{2x}+c_3e^{-2x}$ is a general solution of $y'''-2y''-4y'+8y=0$.

(6) For certain values of the constant m the function f defined by $f(x)=e^{mx}$ is a solution of $y'''-3y''-4y'+12y=0$. Determine all such values of m.

(7) For certain values of the constant n the function f defined by $g(x)=x^n$ is a solution of $x^3y'''+2x^2y''-10xy'-8y=0$. Determine all such values of n.

Initial Value Problems:

Problem: Find a solution f of the differential equation

$$\frac{dy}{dx} = 2x$$

such that at $x=1$ this solution f has the value 4.

Solution: We know that $f(x) = x^2 + c$ is a solution of the equation. If we take $c = 3$ then $f(x) = x^2 + 3$ satisfy the condition $f(1) = 4$.

This entire problem may be written in the following form:

$$\frac{dy}{dx} = 2x \quad \text{and} \quad y(1) = 4.$$

The problem is called an initial-value problem.

Problem: Find a solution f of the initial-value problem:

$$\frac{d^2y}{dx^2} + y = 0 \quad \text{and} \quad y(0) = 1 \quad y'(0) = 5.$$

Solution: We already know that $f(x) = c_1 \sin x + c_2 \cos x$ is a general solution of the equation.

$$f(0) = 1 \Rightarrow \quad c_1 \sin 0 + c_2 \cos 0 = 1 \Rightarrow c_2 = 1$$

$$f'(0) = 5 \Rightarrow \quad c_1 \cos 0 - c_2 \sin 0 = 5 \Rightarrow c_1 = 5$$

A solution is $f(x) = 5\sin x + \cos x$.

Definition: Consider the first-order differential equation $\frac{dy}{dx} = f(x, y)$ where f is a continuous function of x and y in some domain D of the xy-plane; and let (x_0, y_0) be a point of D. The initial-value problem associated with the equation is to find a solution f of the equation, defined on some interval containg x_0, and satisfying the initial condition $f(x_0)=y_0$.

In the customary abbreviated notation, this initial-value problem may be written

$$\frac{dy}{dx}=f(x,y) \quad \text{and} \quad y(x_0)=y_0 .$$

Theorem (Existence and Uniqueness Theorem): Consider the differential equation $\frac{dy}{dx}=f(x,y)$. where

(a) f is a continuous function of x and y in some domain D of the xy-plane, and

(b) the partial derivative $\frac{\partial f}{\partial y}$ is also a continuous function of x and y in D; and let $(x_0,y_0)\in D$. Then there exists a unique solution ϕ of the equation defined on some interval $|x-x_0|\leq h$, where h is sufficiently small, which satisfies the condition $\phi(x_0)=y_0$.

Examples: (1) Consider the initial-value problem

$$\frac{dy}{dx}=x^2+y^2 \quad \text{and} \quad y(1)=3$$

Let us the above theorem:

$$f(x,y)=x^2+y^2 \quad \text{and} \quad \frac{\partial f}{\partial y}=2y$$

Both of the functions f and $\frac{\partial f}{\partial y}$ are continuous in every domain $D\subset\mathbb{R}^2$. The point $(1,3)$ certainly lies in some such domain D. Thus, there is a unique solution ϕ of the differential equation $\frac{dy}{dx} = x^2+y^2$, defined on some interval $|x-1|<h$ about $x_0 = 1$, which satisfies the initial condition, that is which such that $\phi(1) = 3$.

(2) Consider the two problems

$$(A) \quad \frac{dy}{dx} = \frac{y}{\sqrt{x}} \quad \text{and} \quad y(1) = 2$$

$$(B) \quad \frac{dy}{dx} = \frac{y}{\sqrt{x}} \quad \text{and} \quad y(0) = 2$$

Here $f(x,y) = \frac{y}{\sqrt{x}}$ and $\frac{\partial f}{\partial y} = \frac{1}{\sqrt{x}}$. These functions are both continuous except for $x=0$. In Problem (A), $x_0 = 1$, $y_0 = 2$. The square of side 1 centered about $(1,2)$ does not contain the y axis, and so f and $\frac{\partial f}{\partial y}$ satisfy the required hypotheses in this square. Problem (A) has a unique solution.

At $(0,2)$ neither f nor $\frac{\partial f}{\partial y}$ are continuous. The point $(0,2)$ can not be included in a domain D where the required hypothesis are satisfied. Thus we can not conclude from the theorem that Problem (B) has a solution. (