You and your friends are sledding down a snowy hill. In order to maximize sledding excitement, you are curious whether starting from a greater height will increase or decrease the amount of sled-time. This is easily testable, but, being huge nerds, you decide to instead spend the afternoon inside calculating the answer.

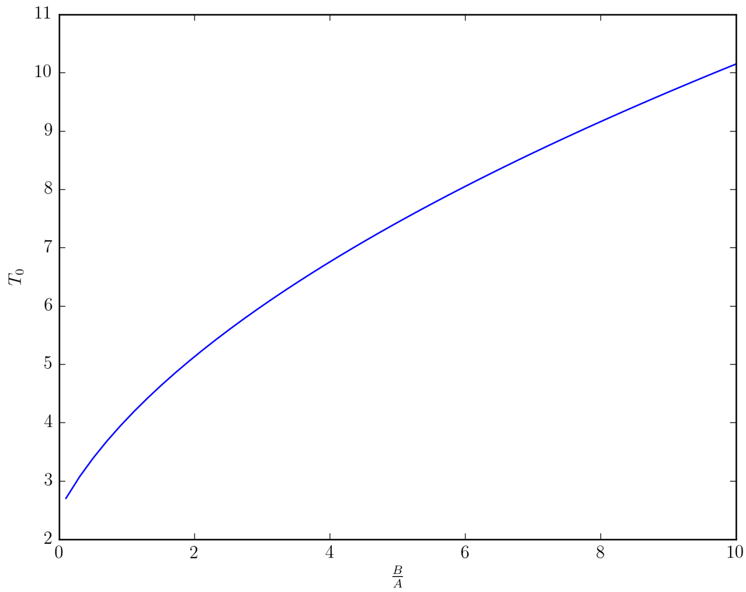
The height of the hill can be described by the function $y(x) = \frac{x^2}{A}$, $x > 0$. Find the time it takes to reach $x = 0$ if you start from rest at $y = B$.

1. First, using conservation of energy, write down the velocity of the sled as a function of the height.
2. Now in the previous result, replace v^2 with $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2$, and solve for dt.
3. Now you should have a term that looks like $dx^2 + dy^2$. Factor out a dx^2 and replace the remaining $(\frac{dy}{dx})^2$ with the actual derivative squared which you can calculate from $y(x)$
4. Now that you know $dt = (\cdots)dx$ and you also know the bounds on x are $(0, \sqrt{BA})$, you can integrate the function to find the time.
5. Does starting from a greater height increase or decrease the time to hit the bottom?

Note: You can eventually end up with an integral that looks like the following:

$$T_0 = \int_0^1 \sqrt{\frac{1 + 4\frac{B}{A}\gamma}{\gamma - \gamma^2}} d\gamma$$

This integral has, as far as I can tell, no closed form solution, However, the following plot shows the value of the integral as a function of $\frac{B}{A}$.(calculated numerically)



Solution

1. Conservation of energy states that the initial energy is equal to final energy, or any energy in between. The initial energy is:

$$E_i = mgB$$

   The energy at some height y is:

$$E_f = mgy + \frac{1}{2}mv^2$$

   Equating these and solving for v^2 yields:

$$v^2 = 2g(B - y)$$

2. Substituting v^2 with $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2$ gives:

$$\begin{aligned}
(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 &= 2g(B - y) \\
\frac{dx^2 + dy^2}{dt^2} &= 2g(B - y) \\
\frac{dx^2 + dy^2}{2g(B - y)} &= dt^2 \\
dt &= \sqrt{\frac{dx^2 + dy^2}{2g(B - y)}}
\end{aligned}$$

3. Factoring out a dx gives:

$$dt = \sqrt{\frac{1 + (\frac{dy}{dx})^2}{2g(B - y)}}dx$$

   And replacing $\frac{dy}{dx}$ with the actual derivative gives

$$dt = \sqrt{\frac{1 + 4\frac{x^2}{A^2}}{2g(B - \frac{x^2}{A})}}dx$$

4. Integrating both sides gives:

$$t = \int_0^{\sqrt{AB}} \sqrt{\frac{1 + 4\frac{x^2}{A^2}}{2g(B - \frac{x^2}{A}}}dx$$

   Make integration substitution $\alpha = \frac{x^2}{AB}$ to simplify the integral gives:

$$t = \sqrt{\frac{A}{8g}} \int_0^1 \sqrt{\frac{1 + 4\frac{B}{A}\alpha}{\alpha - \alpha^2}}dx$$

   Call the remaining integral $T_0(\frac{B}{A})$.

$$t = \sqrt{\frac{A}{8g}} T_0 \left(\frac{B}{A}\right)$$

5. The figure shows that T_0 is an increasing function of $\frac{B}{A}$, higher B yields a larger time with A held constant. Therefore, starting at a larger height results in a longer sled ride! ☺