Computer Methods in Engineering
Exercise on LU decomposition

Problem 1

Consider a one-dimensional rod of length $L = 8$, divided into N equally spaced nodes. The thermal conductivity $k(x)$ varies along the rod as follows:

- $k(x) = k_1 = 1$ for $0 \leq x < L/4$
- $k(x) = k_2 = 2$ for $L/4 \leq x < L/2$
- $k(x) = k_3 = 3$ for $L/2 \leq x < 3L/4$
- $k(x) = k_4 = 1$ for $3L/4 \leq x \leq L$

The steady-state temperature distribution $T(x)$ in the rod is governed by the equation:

$$\frac{d}{dx}\left(k(x)\frac{dT}{dx}\right) = 0$$

with boundary conditions $T(0) = T_0 = 30$ and $T(L) = T_L = 20$.

a) Discretize the equation using the finite difference method for $N = 9$, of which we have 7 interior nodes, while the first and last nodes are the boundary conditions at T_0 and T_L. Write down the resulting linear system $\mathbf{AT} = \mathbf{b}$.

Solution:

For node i, the finite difference approximation is:

$$\frac{1}{\Delta x^2}\left[k_{i+\frac{1}{2}}(T_{i+1} - T_i) - k_{i-\frac{1}{2}}(T_i - T_{i-1})\right] = 0$$

where $k_{i+\frac{1}{2}}$ is the conductivity between nodes i and $i+1$. This leads to a tridiagonal system of equations for the interior nodes $T_1, T_2, \ldots, T_7$. The matrix $\mathbf{A}$ will have the following structure:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & 0 \\ 0 & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & a_{N,N-1} & a_{NN} \end{bmatrix}$$

The right-hand side vector $\mathbf{b}$ will incorporate the boundary conditions:

$$\mathbf{b} = \begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \\ b_N \end{bmatrix}$$

where b_1 and b_N will include contributions from T_0 and T_L respectively.

The equations for each interior node can be explicitly written out, leading to a system of 7 equations with 7 unknowns. For the first interior node, the equation is given as:

$$\frac{k_1}{\Delta x}(T_2 - T_1) - \frac{k_1}{\Delta x}(T_1 - T_0) = 0$$

Since $\Delta x = L/(N-1) = 1$, we can substitute the values of k and rearrange to get the equation

$$-2T_1 + T_2 = -T_0 = -30$$

For the second interior node, the equation is:

$$k_2(T_3 - T_2) - k_1(T_2 - T_1) = 0$$

which simplifies to

$$k_1T_1 - (k_1 + k_2)T_2 + k_2T_3 = 0$$
$$T_1 - 3T_2 + 2T_3 = 0$$

Continuing this process for all interior nodes, we can fill in the matrix **A** *and vector* **b** *as follows:*

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & -4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & -5 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & -6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} -30 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -20 \end{bmatrix}$$

b) Use LU decomposition to solve the linear system for the temperature at the interior nodes.

Solution:

To solve the system $\mathbf{AT} = \mathbf{b}$ using LU decomposition, we first factor **A** *as $\mathbf{A} = \mathbf{LU}$, where* **L** *is lower triangular and* **U** *is upper triangular. The solution proceeds in two steps:*

1. Solve $\mathbf{Ly} = \mathbf{b}$ for $\mathbf{y}$ using forward substitution. 2. Solve $\mathbf{UT} = \mathbf{y}$ for $\mathbf{T}$ using backward substitution.

For our matrix, the LU decomposition yields:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.8 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.8333 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.9 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.9091 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.7857 & 1 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2.5 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2.4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.3333 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.2727 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1.2143 \end{bmatrix}$$

First, solve $\mathbf{Ly} = \mathbf{b}$ *for* $\mathbf{y}$*, then solve* $\mathbf{UT} = \mathbf{y}$ *for the unknown temperatures at the interior nodes. This gives the complete solution for the temperature distribution.*

This gives a vector $\mathbf{y}$ *as:*

$$\mathbf{y} = \begin{bmatrix} -30.00 \\ -15.00 \\ -12.00 \\ -10.00 \\ -9.00 \\ -8.18 \\ -26.43 \end{bmatrix}$$

Finally, solving $\mathbf{UT} = \mathbf{y}$ *gives:*

$$\mathbf{T} = \begin{bmatrix} 28.24 \\ 26.47 \\ 25.59 \\ 24.71 \\ 24.12 \\ 23.53 \\ 21.76 \end{bmatrix}$$

c) Plot the temperature distribution along the rod.

Solution:

The temperature distribution along the rod is shown in the figure below.