TPG4155 Computer Methods in Engineering Exercise 8

Deadline: October 30 2023
Information: The exercise must be handed in electronically through Blackboard. You may typeset the exercises with LaTeX, Word, Jupyter etc., or scan your handwritten sheets. It is strongly recommended to take this opportunity to learn how to use LaTeX(which you can also use inside Jupyter). In problems where plots or source code are required, include printed/exported versions in the file you send in. The file format must be PDF or Jupyter. Exercises delivered too late will be rejected unless you have made an agreement with the teaching assistant *in advance*.

In this exercise we are going to solve the Helmholtz equation in one dimension. The equation describes harmonic waves and reads:

$$\frac{d^2 p(x)}{d^2 x} + k^2(x)p(x) = 0 \tag{1}$$

Here x is the position, $k = \omega/c(x)$ is a wavenumber. ω is the angular frequency related to the frequency f by $f = 2\pi f$. $c(x)$ is the wave speed and p is the pressure.

We will use the finite difference method to solve the Helmholtz equation on an interval $x = 0$ to L, where L is the length of the interval. We use a numerical grid defined by setting $x = n\Delta x$, where Δx is the size of each grid cell, and $n = 0, \cdots, N$ and $N = L/\Delta x$. The discretized pressure is written $p_n = p(n\Delta x)$ and $s_n = s(n\Delta x)$. Also $k_n = \omega/c(n\Delta x)$. We will assume constant boundary conditions so that $p(0) = p(L) = 1.0$ These boundary conditions corresponds to a standing wave solution, so we will assume that the wavenumber fulfills the condition

$$kL = 2l\pi, \tag{2}$$

where $l = 1, 2, \cdots$.

Problem 1

Prove that the harmonic wave

$$p(x) = \exp(ikx) \tag{3}$$

where i is the imaginary unit, is a solution of equation (1).

Solution:
Differentiating equation (3) two times with respect to x

$$\frac{d^2p(x)}{d^2x} = -k^2p(x) \tag{4}$$

Inserting the equation above into the Helmholtz equation gives

$$-k^2p(x) + k^2p(x) = 0, \tag{5}$$

which proves that the harmonic wave given by (3) is a solution.

Problem 2

Use the following approximation for second order derivatives

$$p''(x) = \frac{p(x+\Delta x) - 2p(x) + p(x-\Delta x)}{\Delta x^2} \tag{6}$$

to show that equation (1) can be written in discretized form as

$$p_{n+1} - p_n r_n^2 + p_{n-1} = 0 \tag{7}$$

for the internal points where $n = 1, \cdots, N-1$. and where $r_n = 2 - \Delta x^2 k_n^2$

Solution:
Using equation (6) on equation (1) we get:

$$\frac{p_{n+1} - 2p_n + p_{n-1}}{\Delta x^2} + k_n^2 p_n = 0 \tag{8}$$

Which can be immediately rearranged to

$$p_{n+1} - p_n(2 - \Delta x^2 k_n^2) + p_{n-1} = 0 \tag{9}$$

Which gives

$$p_{n+1} - p_n r_n^2 + p_{n-1} = 0 \tag{10}$$

for the internal grid points $n = 1, \cdots, N-1$.

Problem 3

Show that by using the boundary conditions we can obtain the following matrix equation for p_n

$$\mathbf{A}\vec{p} = \vec{s}, \tag{11}$$

where $\mathbf{A}$ is an $N-1 \times N-1$ matrix,

$$\mathbf{A} = \begin{bmatrix} -r_1^2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -r^2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -r^2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -r_{N-1}^2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -r_{N-1}^2 \end{bmatrix}$$

, The unknowns p_n are given by the vector

$$\vec{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{N-1} \end{bmatrix}$$

Finally, the right hand side vector s with length $N-1$ is given by

$$\vec{s} = \begin{bmatrix} -p_0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -p_N \end{bmatrix}$$

Solution:
The finite-difference solution given in equation (10) gives for internal points

$$\begin{aligned} p_2 - p_1 r^2 + p_0 &= 0 \\ p_3 - p_1 r_2^2 + p_1 &= 0 \\ \vdots &= 0 \\ p_N - p_{N-1} r^2 + p_{N-2} &= 0 \end{aligned}$$

Rearranging the above equations give

$$\begin{aligned} p_0 - p_1 r_n^2 + p_2 &= 0 \\ p_1 - p_1 r_2^2 + p_3 &= 0 \\ \vdots &= 0 \\ p_{N-2} - p_{N-1} r^2 + p_N &= 0 \end{aligned}$$

Moving the boundary value terms to the right hand side

$$\begin{aligned} -p_1 r_n^2 + p_2 &= -p_0 \\ p_1 - p_1 r_2^2 + p_3 &= 0 \\ \vdots &= 0 \\ p_{N-2} - p_{N-1} r^2 &= -p_N \end{aligned}$$

Recognizing that the first row is a dot product between the vector $(1, -r^2, 1, 0, \cdots, 0)$ of length $N-1$ and the vector $(p_1, \cdots, p_{N-1})$ we can write down the first row of the the matrix given in (3). Proceeding in a similar way with the other equations we get all rows of the A matrix.

Problem 4

Implement the finite difference solution for the Helmholtz equation given above by using the matrix A and solving for the unknown $\vec{p}$. Assume that the velocity is constant $c = 2000m/s$, $\Delta x = 1m$ and that $L = 1000m$. Also use a frequency which fulfills one of the conditions in equation (2). Different values of l in equation (2) corresponds to a standing wave. Use both the conjugate gradient method and the LU dcomposition method to solve the equation system. Compute the error of the solution by computing the norm of the difference and divide by $\sqrt{N}$:

$$err = \frac{1}{\sqrt{N}}||\mathbf{A}\mathbf{p} - \mathbf{s}||, \tag{12}$$

where $\mathbf{p}$ is the solution vector for either the conjugate-gradient or the LU decomposition method. Also try to compute the solution by decreasing Δx, increasing N but keeping the length L fixed. How does the error change with the number of points N?

Solution:
See attached Jupyter notebook.