Computer Methods in Enigineering
Exercise on parabolic equations

In this exercise we will continue the work from the earlier exercise on solutions to Elliptic equations. Then we solved the Laplace equation $\nabla^2 f = 0$. Now, we will solve the diffusivity equation $\eta \nabla^2 p = \partial p / \partial t$.

The solution of the diffusivity equations can be obtained by different methods. In this exercise we will use the two methods we learned in the lecture, namely explicit and implicit solution. Both are based on the finite difference approximation. The centered difference for the second order derivative is given as

$$\frac{\partial^2 p}{\partial x^2} \simeq \frac{p(x + \Delta x, y, t) - 2p(x, y, t) + p(x - \Delta x, y, t)}{(\Delta x)^2}$$

Equivalently, in the y-direction, we have

$$\frac{\partial^2 p}{\partial y^2} \simeq \frac{p(x, y + \Delta y, t) - 2p(x, y, t) + p(x, y - \Delta y, t)}{(\Delta y)^2}$$

For the explicit solution, we use a forward difference for the time derivative

$$\frac{\partial p}{\partial t} \simeq \frac{p(x, y, t + \Delta t) - p(x, y, t)}{\Delta t}$$

For the two-dimensional system, our diffusivity equation $\eta \nabla^2 p = \partial p / \partial t$ is then given as

$$\eta \nabla^2 p = 0 \tag{1}$$

$$\eta \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) p = \frac{\partial p}{\partial t} \tag{2}$$

$$\eta \frac{\partial^2 p}{\partial x^2} + \eta \frac{\partial^2 p}{\partial y^2} = \frac{\partial p}{\partial t} \tag{3}$$

Assuming the same spatial step-length $\Delta x = \Delta y$, these second order partial derivatives can then be approximated using finite differences as

$$P^t_{i+1,j} - 2P^t_{i,j} + P^t_{i-1,j} + P^t_{i,j+1} - 2P^t_{i,j} + P^t_{i,j-1} = \frac{(\Delta x)^2}{\eta \Delta t} \left(P^{t+\Delta t}_{i,j} - P^t_{i,j} \right) \tag{4}$$

$$P^t_{i+1,j} + P^t_{i,j+1} - 4P^t_{i,j} + P^t_{i-1,j} + P^t_{i,j-1} = \frac{(\Delta x)^2}{\eta \Delta t} \left(P^{t+\Delta t}_{i,j} - P^t_{i,j} \right) \tag{5}$$

where we have used the index notation $P^t_{i,j} = P(i\Delta x, j\Delta y, t)$.

Just like in the previous exercise, assume a sand-body connecting two fluid reservoirs at different pressure. The left reservoir has a pressure $p_l = 1 \times 10^5$ Pa, while the

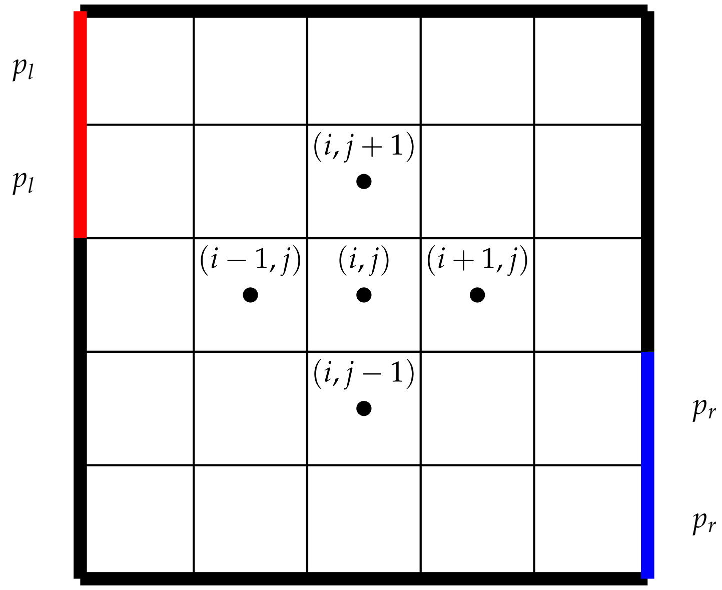

Figure 1: Grid representation of the sand-body connecting the two reservoirs. Thick black lines indicate no-flow boundaries, while red lines indicate boundaries connected to the left reservoir, and blue lines indicate boundaries connected to the right reservoir.

right reservoir has a pressure $p_r = 2 \times 10^5$ Pa. The sand-body has a shape between the two reservoirs as outlined in Fig. 1, where the grid cell size is 100 m × 100 m. Further, assume a viscosity of 1×10^{-3} Pa s, a permeability of 1×10^{-10} m^2, and assume a sand body thickness of 10 m.

Problem 1

Assume the porous medium is filled with water at an initial pressure of $p = p_l = 1 \times 10^5$ Pa, and fully connected to the left and right boundary at the indicated boundaries. To simplify, we can assume that the porous medium stretches a half grid block into the two reservoirs where the connections are indicated.

a) Write out the explicit equation for this system i two dimensions.

b) Create a code to solve the pressure field using an explicit scheme, when we define $\eta = 1$, $\Delta x = \Delta y = 1$.

c) Solve the explicit scheme at time $t = 0.1, 1.0, 10.0$. How does the longer times compare with the elliptic solution in the previous exercise?

Problem 2

We will now solve the same system using an implicit solution.

a) Create a code to solve the pressure field using an implicit scheme, using the same input as in the previous explicit scheme.

b) Solve the explicit scheme at time $t = 0.1, 1.0, 10.0$. How does this compare with the previous explicit solution, and with the elliptic solution in the previous exercise?