TPG4155 Computer methods in Engineering
Exercise 4

Problem 1
The forward and backward numerical derivative of a function f can be expressed as

$$f'(k\Delta x + \Delta x/2) \approx d_x^+ f(k\Delta x) \quad = \quad d^+ f_k = \frac{f_{k+1} - f_k}{\Delta x}, \tag{1}$$

$$f'(k\Delta x - \Delta x/2) \approx d_x^- f(k\Delta x) \quad = \quad d^- f_k = \frac{f_k - f_{k-1}}{\Delta x}. \tag{2}$$

Here $f_k = f(k\Delta x)$ where $k = 0, 1, 2, \cdots, N-1$, Δx is the sampling interval and N is the number of samples.

a) Write two functions, Df and Db which performs backward and forward differentiation of a function. The input arguments to the functions are an array of length N with the function samples, an output array of length N and the sampling interval dx. The function Df should compute the derivatives of the input array at points $k\Delta x + \Delta x/2$ for $k = 0, 2, \cdots, N-2$ while Db should compute the derivative at points $k\Delta x - \Delta x/2$ for $k = 1, 2, \cdots, N-1$.

Solution:
See attached wave1d Jupyter notebook.

b) Check that the routines work by differentiating a known function.

Solution:

Problem 2
The one-dimensional acoustic equations of motion are given as

$$\ddot{u}(x,t) \quad = \quad \frac{1}{\rho(x)} \frac{d\sigma(x,t)}{dx}, \tag{3}$$

$$\ddot{\sigma}(x,t) \quad = \quad \kappa(x) \frac{d\ddot{u}(x,t)}{dx} + s(x,t). \tag{4}$$

An algorithm for solving the equations numerically is

1. Set $t = 0, \sigma(x, t = 0) = 0$ and $\sigma(x, t = -\Delta t) = 0$ for all $x = l\Delta x, l = 0, \ldots, N-1$
2. Compute the accelerations $\ddot{u}(x,t)$ at time t, for all x=$l\Delta x, l = 0, \ldots, N-2$ by using the equations

$$\ddot{u}(x + \Delta x/2, t) \quad = \quad \rho^{-1}(x + \Delta x/2) d_x^+ \sigma(x,t), \tag{5}$$

3. Compute the stress $\sigma(x, t + \Delta t)$ at the future time $t + \Delta t$ and for all $x = l\Delta x$, $l = 1, \ldots, N-1$ by using the equation

$$\begin{aligned} \sigma(x, t + \Delta t) &= 2\sigma(x, t) - \sigma(x, t - \Delta t) \\ &+ \Delta t^2 \kappa(x) [d_x^- [\ddot{u}(x + \Delta x/2, t)] \end{aligned} \tag{6}$$

4. Add the source term for a single grid position $x_s = m_s \Delta x$

$$\sigma(x_s, t + \Delta t) = \sigma(x_s, t + \Delta t) + \Delta t^2 s(x_s, t)/\Delta x. \tag{7}$$

5. set $t = t + \Delta t$ and go to 2. Stop if $t = (N_t - 1)\Delta t$ is reached.

The derivative operators d_x^+ and d_x^- are the forward and backward derivatives computed by th Df and Db functions.

a) Implement the above scheme for solution of the one-dimensional wave equation. Use three arrays, sigma2, sigma1 and sigma0, of length N for the stresses corresponding to times $t + \delta t$, t, and $t - \delta t$. Use an additional array u for the acceleration and if necessary additional arrays for holding intermediate results.

The source s should be implemented as a separate function with the time index k $t = k\Delta t$ as an argument. Use a Ricker wavelet with peak frequency(fp) of 30Hz, and delay (tp) of 0.1 seconds. See the attached ricker jupyter notebook for the python code of a function implementing the ricker wavelet. The boundary conditions are implicit and no extra code is needed. Plot the pulse.

Make separate arrays to hold the density and bulk modulus κ. The stability condition for the algorithm is

$$c\Delta t/\Delta x \leq 1, \tag{8}$$

where $c = \sqrt{(\kappa/\rho)}$ is the wave velocity.

Solution:

See attached Jupyter notebook. See figure 1.

b) Create a model with constant velocity and density with length of 3000m. The wave velocity should be equal to 2000 m/s and the density set to 1000kg/m^3. The source should have a peak frequency of 30Hz. Select suitable $\Delta x = 5.0$m and $\Delta t = 0.0001$ seconds. Place the source at a position of 1500m and simulate the stress for approximately one second. Plot the resulting stress as function of position and time as an image using f.ex a gray scale color table.

Solution:

See figure 2

Problem 3

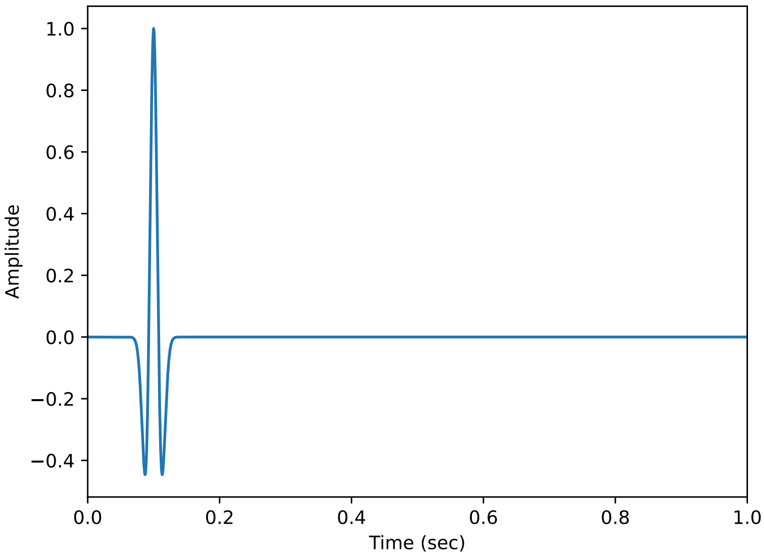


Figure 1: Ricker pulse with peak frequency 30Hz

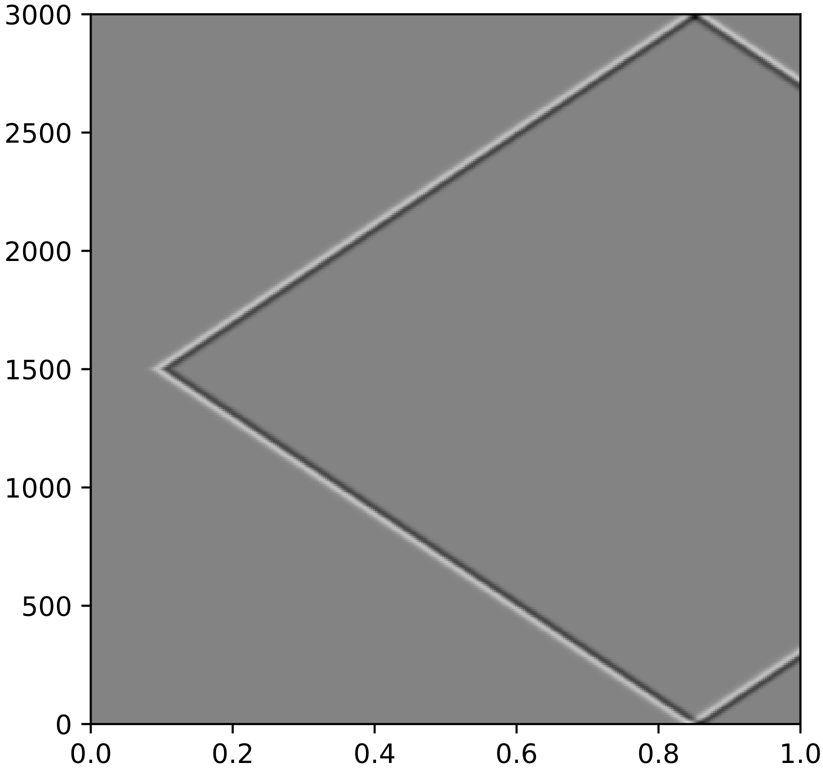


Figure 2: Stress for homogeneous model with $c = 2000$ m/s.

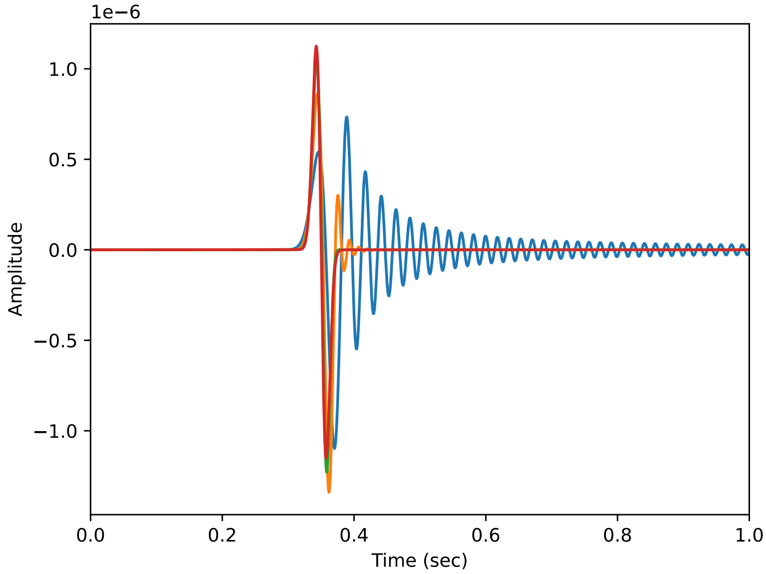


Figure 3: Pressure recorded at position of 2000m for $\Delta x = 10, 5, 2.5$ and 1m grid spacing.

a) Perform four simulations with $\Delta x = 10, 5, 2.5$ and 1m. Record the pressure at a depth of 2000m and compute the relative error between simulations for 10 and 5m, 5 and 2.5m and 2.5m and 1m. The error between two simulations can be computed using

$$\epsilon = \frac{\sqrt{\sum_{i=0}^{N}[p_1(i\Delta t) - p_2(i\Delta t)]^2}}{\sqrt{\sum_{i=0}^{N-1} p_1^2(i\Delta t)}} \tag{9}$$

Plot the resulting error as a function of the wavelength divided by Δx, i.e. the number of wavelengths per spatial grid point.

Solution:

See figures 3 and 4.

Problem 4

a) Make a model containing reflectors of your own device, compute and plot the solution. Also plot the velocity and density.

Solution:

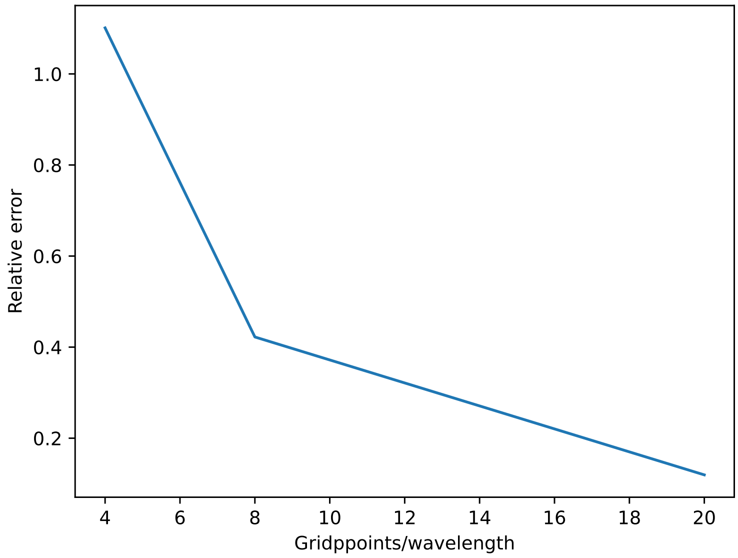


Figure 4: Relative between pressure at 2000m for $\Delta x = 10, 5, 2.5$ and 1m grid spacing.

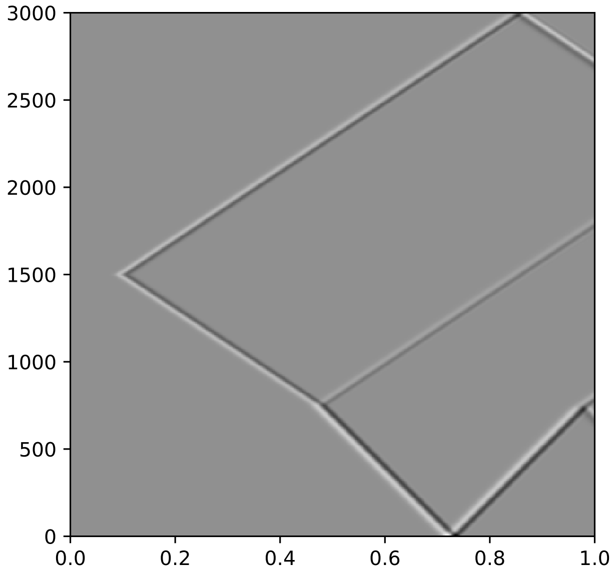
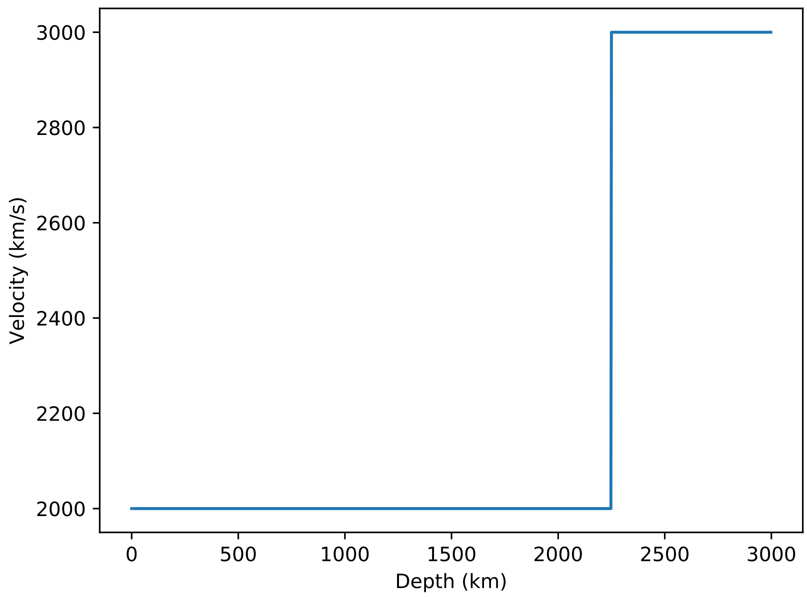
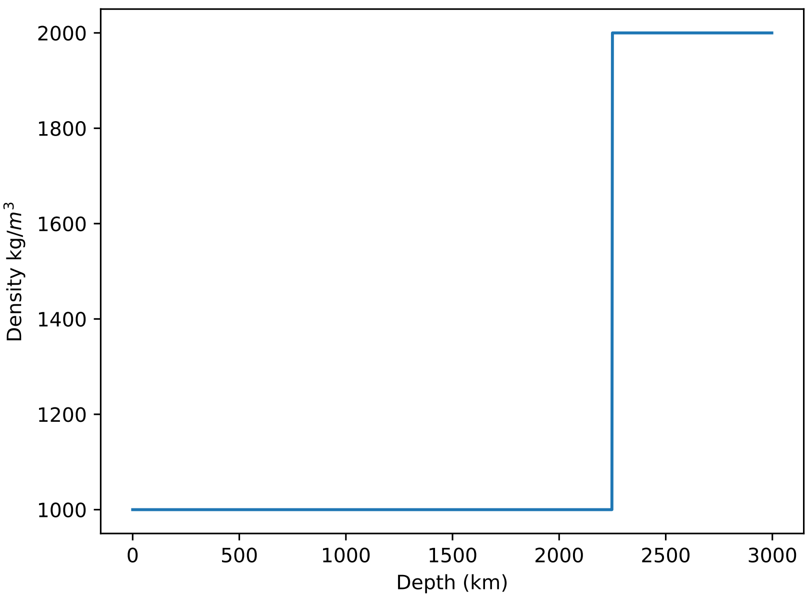


Figure 5: Stress as function of space and time



Velocity (km/s)
3000
2800
2600
2400
2200
2000
0
500
1000
1500
2000
2500
3000
Depth (km)



Density kg/m^3
2000
1800
1600
1400
1200
1000
0
500
1000
1500
2000
2500
3000
Depth (km)