TPG4155 Computer Methods in Enigineering Exercise 3

In this exercise we will work on solutions to Elliptic equations, more specifically, the Laplace equation $\nabla^2 f = 0$. In our case, we will consider flow in porous media. Such flow is governed by the Darcy equation:

$$\vec{q} = -\frac{k}{\mu}\nabla p$$

Here q is the volumetric flow rate, k is the permeability (a measure for how well the porous medium allows for transport of fluids), μ is the viscosity of the fluid, and p is the fluid pressure. As shown in the lecture, at steady state this gives the Laplace equation $\nabla^2 p = 0$. We will consider a two-dimensional model, thus

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

The solution of the Laplace equations can be obtained by different methods. In this exercise we will use two solution techniques. Both will be based on the finite difference approximation. The centered difference for the second order derivative is given as

$$\frac{\partial^2 p}{\partial x^2} \simeq \frac{f(x+\Delta x) - 2p(x) + p(x-\Delta x)}{(\Delta x)^2}$$

Equivalently, in the y-direction, we have

$$\frac{\partial^2 p}{\partial y^2} \simeq \frac{f(y+\Delta y) - 2p(y) + p(y-\Delta y)}{(\Delta y)^2}$$

For the two-dimensional system, our Laplace equation $\nabla^2 p = 0$ is given as,

$$\nabla^2 p = 0 \tag{1}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) p = 0 \tag{2}$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \tag{3}$$

These two second order derivatives can then be approximated using finite differences as

$$P_{i+1,j} - 2P_{i,j} + P_{i-1,j} + P_{i,j+1} - 2P_{i,j} + P_{i,j-1} = 0 \tag{4}$$

$$P_{i+1,j} + P_{i,j+1} - 4P_{i,j} + P_{i-1,j} + P_{i,j-1} = 0 \tag{5}$$

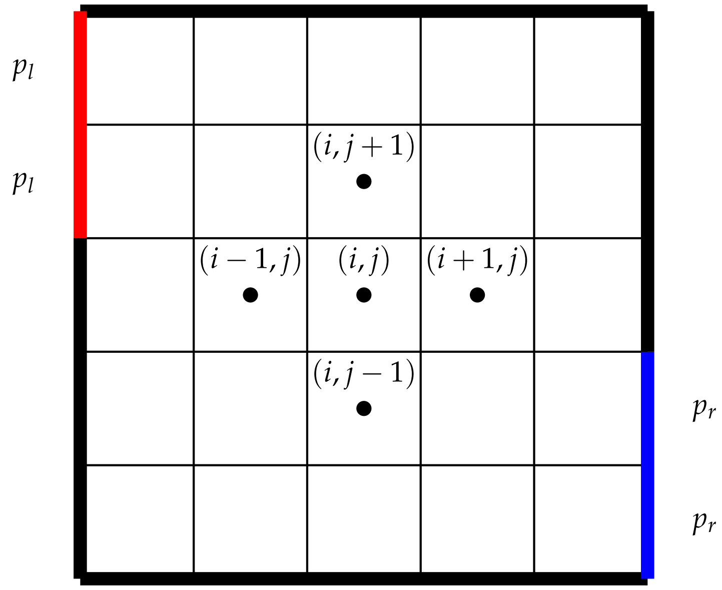

Figure 1: Grid representation of the sand-body connecting the two reservoirs. Thick black lines indicate no-flow boundaries, while red lines indicate boundaries connected to the left reservoir, and blue lines indicate boundaries connected to the right reservoir.

where we have used the index notation $P_{i,j} = P(i\Delta x, j\Delta y)$.

Assume a sand-body connecting two fluid reservoirs at different pressure. The left reservoir has a pressure $p_l = 1 \times 10^5$ Pa, while the right reservoir has a pressure $p_r = 2 \times 10^5$ Pa. The sand-body has a shape between the two reservoirs as outlined in Fig. 1, where the grid cell size is 100 m × 100 m. Further, assume a viscosity of 1×10^{-3} Pa s, a permeability of 1×10^{-10} m^2, and assume a sand body thickness of 10 m.

Problem 1

Assume the porous medium is filled with water, and fully connected to the left and right boundary at the indicated boundaries. To simplify, we can assume that the porous medium stretches a half grid block into the two reservoirs where the connections are indicated.

- **a)** Set up the linear equations describing this system.
- **b)** We want to solve the system of equations using matrix form, $\mathbf{A}\vec{p} = \vec{b}$. Set up the matrix $\mathbf{A}$ and vector $\vec{b}$ in Python.
- **c)** Solve for the pressure field $\vec{p}$, and plot the resulting pressure field.
- **d)** Solve for the pressure using the Gauss-Seidel relaxation method, with relaxation parameter $\alpha = 1$ and using 10 iterations. Plot the resulting pressure field.
- **e)** Compare the solution with the implicit method above. Assuming the implicit method gives the correct answer, what is the error when using 10 and 100 iterations.
- **f)** Use the Gauss-Seidel relaxation method with relaxation parameter $\alpha = 1.8$. How does the change in α value affect the error?
- **g)** Calculate the fluid flow from the right to the left reservoir.