Computer Methods in Engineering
Exercise 6

In this exercise we will work on using the conjugate gradient method to solve an elliptic equation. More specifically, we will solve the Laplace equation we encountered in Exercise 3 . Recall that Exercise 3 was considering flow in porous media as governed by the Darcy equation:

$$\vec{q} = -\frac{k}{\mu}\nabla p \quad ,$$

were q is the volumetric flow rate, k is the permeability (a measure for how well the porous medium allows for transport of fluids), μ is the viscosity of the fluid, and p is the fluid pressure. At steady state this gives the Laplace equation $\nabla^2 p = 0$. We considered a two-dimensional model, thus

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

Assume a sand-body connecting two fluid reservoirs at different pressure. The left reservoir has a pressure $p_l = 1 \times 10^5$ Pa, while the right reservoir has a pressure $p_r = 2 \times 10^5$ Pa. The sand-body has a shape between the two reservoirs as outlined in Fig. 1, where the grid cell size is $100\,\mathrm{m} \times 100\,\mathrm{m}$. Further, assume a viscosity of 1×10^{-3} Pa s, a permeability of $1 \times 10^{-10}\,\mathrm{m}^2$, and assume a sand body thickness of 10 m.

We saw in Exercise 3 that this gave the matrix representation for the pressure field as $A\vec{P} = \vec{b}$, where the A matrix was given as

$$A = \left[\begin{array}{ccccccccccccccccccccccccc}
-3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3
\end{array}\right]$$

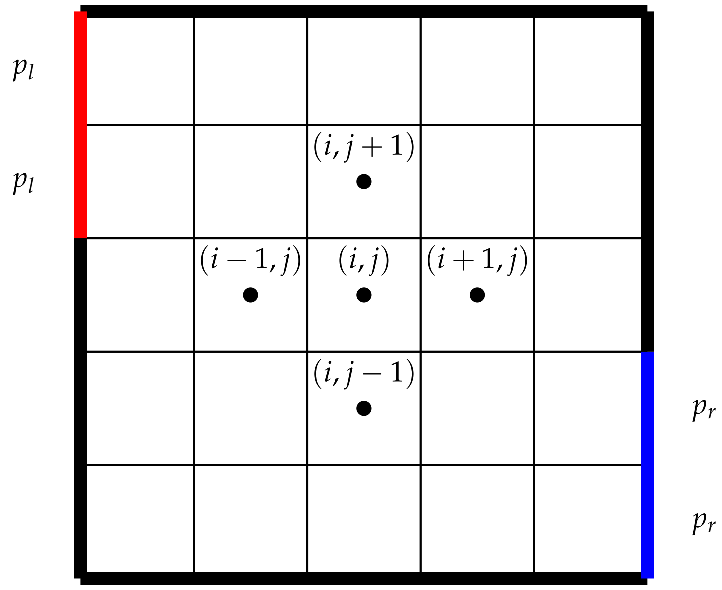

Figure 1: Grid representation of the sand-body connecting the two reservoirs. Thick black lines indicate no-flow boundaries, while red lines indicate boundaries connected to the left reservoir, and blue lines indicate boundaries connected to the right reservoir.

and the vector $\vec{b}$ as

$$\vec{b} = \begin{bmatrix} -100000 \\ 0 \\ 0 \\ 0 \\ 0 \\ -100000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -200000 \\ 0 \\ 0 \\ 0 \\ 0 \\ -200000 \end{bmatrix}$$

Problem 1

Write a Python code to solve for the pressure $\vec{P}$ using the steepest decent method. If you want a residual smaller than 10^{-8}, how many iterations do you need.

Solution:

```
import numpy as np
import matplotlib.pyplot as plt
from math import *
import time

def steepdesc(A,b,x,maxiter,eps) :
  success = False
```

```
  #Iterate over estimates of x
  for i in range(0,maxiter):
    #Find search direction:
    r = b-np.dot(A,x)
    res = sqrt(np.dot(r,r))
    fval = 0.5*np.dot(x,np.dot(A,x))-np.dot(b,x)
    #print("Iteration: ",i,"f value: ",fval,"Error: ",res,"Position: ...
        ",x,"Direction: ",r)
    if(res < eps):
      success = True
      break

    #Compute alfa
    alpha = np.dot(r,r)/np.dot(r,np.dot(A,r))

    #update solution
    x = x+alpha*r
  print('Stepest decent number of iterations:',i)
  return [x,success]


def cg(A,b,x,maxiter,eps) :
  success = False
  r = b-np.dot(A,x)
  d = r
  #Iterate over estimates of x
  for i in range(0,maxiter):
    #Evaluate error:
    rr = b-np.dot(A,x)
    res = sqrt(np.dot(rr,rr))
    #print("Iteration: ",i,"Error: ",res,"Position: ",x,"Direction: ",d)
    if(res < eps):
      success = True
      break
    #Compute alfa
    tmpr=np.dot(r,r) #store temporarily inner-product of r at previous step
    alpha = tmpr/np.dot(r,np.dot(A,d))
    #Update position
    x = x+alpha*d
    #Update r:
    r = r-alpha*np.dot(A,d)
    #Compute beta
    beta = np.dot(r,r)/tmpr
    d = r + beta*d

  return [x,success]

#Grid size
innSide=5
#innCells=(innSide,innSide)

#Boundary conditions
pl=1E5
pr=2E5

A=np.zeros((innSide**2,innSide**2))
b=np.zeros(innSide**2)

#Set up matrix A for all internal cells
```

```
for jj in range(0,innSide):
  for ii in range(0,innSide):
    cellNum=ii+innSide*jj
    if ii>0:
      A[cellNum,cellNum]-=1
      A[cellNum,cellNum-1]+=1
    if ii<innSide-1:
      A[cellNum,cellNum]-=1
      A[cellNum,cellNum+1]+=1
    if jj>0:
      A[cellNum,cellNum]-=1
      A[cellNum,ii+innSide*(jj-1)]+=1
    if jj<innSide-1:
      A[cellNum,cellNum]-=1
      A[cellNum,ii+innSide*(jj+1)]+=1

#Add boundaries to matrix A
#and to vector b
A[0,0]-=1
b[0]=-pl
A[5,5]-=1
b[5]=-pl
A[19,19]-=1
b[19]=-pr
A[24,24]-=1
b[24]=-pr

eps=1.0e-08
maxiter = 10000
x0=np.zeros(25)

x,success = steepdesc(A,b,x0,maxiter,eps)
print(x,success)

x0=np.zeros(25)
tic=time.time()
x,success = cg(A,b,x0,maxiter,eps)
toc=time.time()
print('Time of CG: ',toc-tic)
print(x,success)

tic=time.time()
x=np.dot(np.linalg.inv(A),b)
toc=time.time()
print('Time of numpy: ',toc-tic)

print(x)
```

Need 1191 iterations.

Problem 2

Write a Python code to solve for the pressure $\vec{P}$ using the conjugate gradient method. How does your solution compare with the solution you obtained in Exercise 3 . Compare the time used by your own python code for the conjugate gradient method to the numpy library `linalg.inv`.

Solution:

The solutions are exactly the same.

On my computer the CG used 4.8E-4s, while numpy used 3.0E-4s. This is quite comparable.