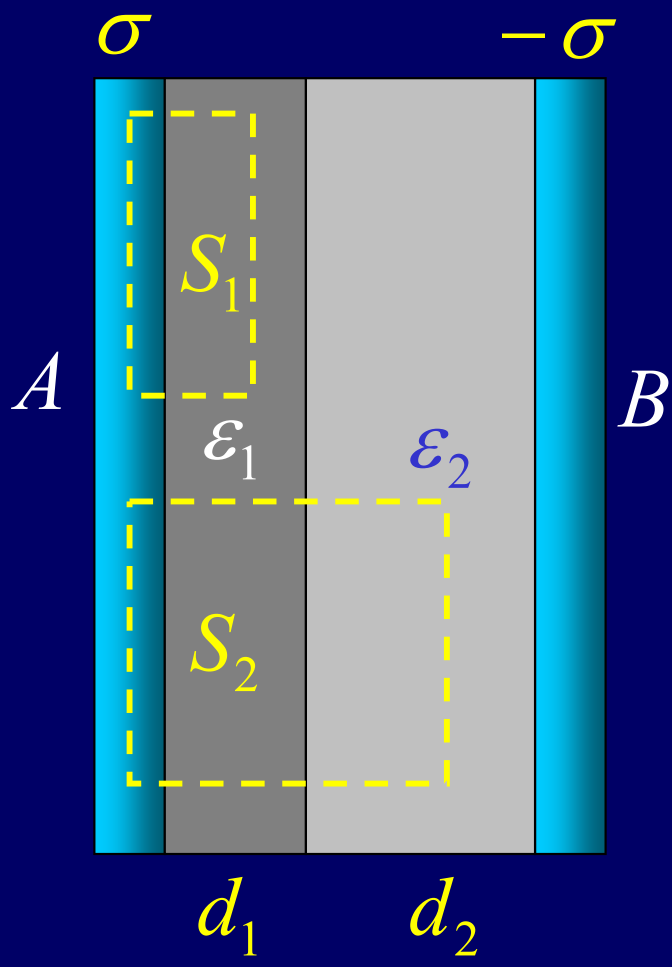
例　平行板电容器，其中充有两种均匀电介质。

求　(1) 各电介质层中的场强

　　(2) 极板间电势差



解　做一个圆柱形高斯面 S_1

$$\oint_{S_1} \vec{D}\cdot \mathrm{d}\vec{S} = \sum q_i (S_1\text{内})$$

$$D_1 \Delta S_1 = \sigma \Delta S_1 \qquad D_1 = \sigma$$

同理，做一个圆柱形高斯面 S_2

$$\int_{S_2} \vec{D}\cdot \mathrm{d}\vec{S} = \sum q_i (S_2\text{内}) \qquad D_2 = \sigma$$

$$D_1 = D_2 \qquad E_1 = \frac{\sigma}{\varepsilon_1} \neq E_2 = \frac{\sigma}{\varepsilon_2}$$

$$E_1=\frac{\sigma}{\varepsilon_1}\qquad E_2=\frac{\sigma}{\varepsilon_2}$$

$$\Delta u=\int_A^B\vec{E}\cdot\mathrm{d}\vec{r}=\int_0^{d_1}\vec{E}_1\cdot\mathrm{d}\vec{r}+\int_{d_1}^{d_1+d_2}\vec{E}_2\cdot\mathrm{d}\vec{r}$$

$$=\frac{\sigma}{\varepsilon_o\varepsilon_{r1}}d_1+\frac{\sigma}{\varepsilon_o\varepsilon_{r2}}d_2$$

$$C=q/\Delta u\quad=\left(\frac{d_1}{S\varepsilon_1}+\frac{d_2}{S\varepsilon_2}\right)^{-1}=\frac{\varepsilon_1\varepsilon_2 S}{\varepsilon_1 d_2+\varepsilon_2 d_1}$$

- 各电介质层中的场强不同
- 相当于电容器的串联

平板电容器中充介质的另一种情况

$$\Delta u_1 = \Delta u_2$$

$$E_1 = \frac{\Delta u_1}{d} = E_2 = \frac{\Delta u_2}{d}$$

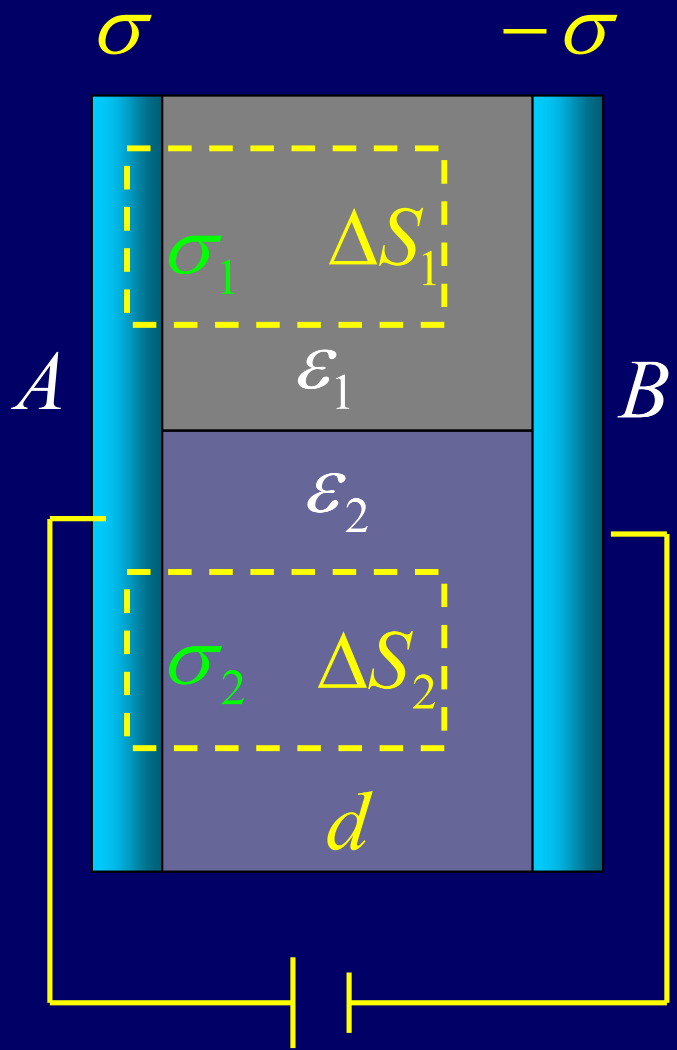
$$D_1 = \varepsilon_0 \varepsilon_{r1} E_1 \neq D_2 = \varepsilon_0 \varepsilon_{r2} E_2$$

$$D_1 = \sigma_1 \qquad D_2 = \sigma_2$$

考虑到 $q = \sigma_1 S_1 + \sigma_2 S_2$

$$\Delta u = \frac{\sigma_2}{\varepsilon_2} d = \frac{\sigma_1}{\varepsilon_1} d = \frac{qd}{\varepsilon_1 S_1 + \varepsilon_2 S_2}$$

$$C = \frac{q}{\Delta u} = \frac{\varepsilon_1 S_1 + \varepsilon_2 S_2}{d} = C_1 + C_2$$



- **各电介质层中的场强相同**
- **相当于电容器的并联**

静电场习题课

库仑定律 $\vec{F} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1 q_2}{r^2}\vec{r}^{\,0}$

场强叠加原理：$\vec{E} = \int \mathrm{d}\vec{E}$

$$\boxed{\Phi_e = \oint_S \vec{E}\cdot \mathrm{d}\vec{S} = \frac{1}{\varepsilon_0}\sum_i q_i(\text{内})}$$

$$\boxed{\Phi_D = \oint_S \vec{D}\cdot \mathrm{d}\vec{S} = \sum_i q_i(\text{内})}$$

有源场

环路定理 $\boxed{\oint_L \vec{E}\cdot \mathrm{d}\vec{l} = 0}$ 无旋场

电势 $\boxed{u_a = \dfrac{W_a}{q_0} = \int_a^{\text{"0"}} \vec{E}\cdot \mathrm{d}\vec{l}}$

$$\boxed{\vec{E} = -\left(\frac{\partial u}{\partial x}\vec{i} + \frac{\partial u}{\partial y}\vec{j} + \frac{\partial u}{\partial z}\vec{k}\right) = -\mathrm{grad}(u) = -\nabla u}$$

电介质的高斯定理 $$\oint_S \vec{D}\cdot \mathrm{d}\vec{S} = \sum_i q_{0i,\text{内}}$$

电位移矢量 $$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon\ \vec{E}$$

电场能量密度 $$w_e = \frac{1}{2}\varepsilon_0 \varepsilon_r E^2 = \frac{1}{2}DE$$

例 圆锥底面半径R，侧面均匀带电面密度σ。

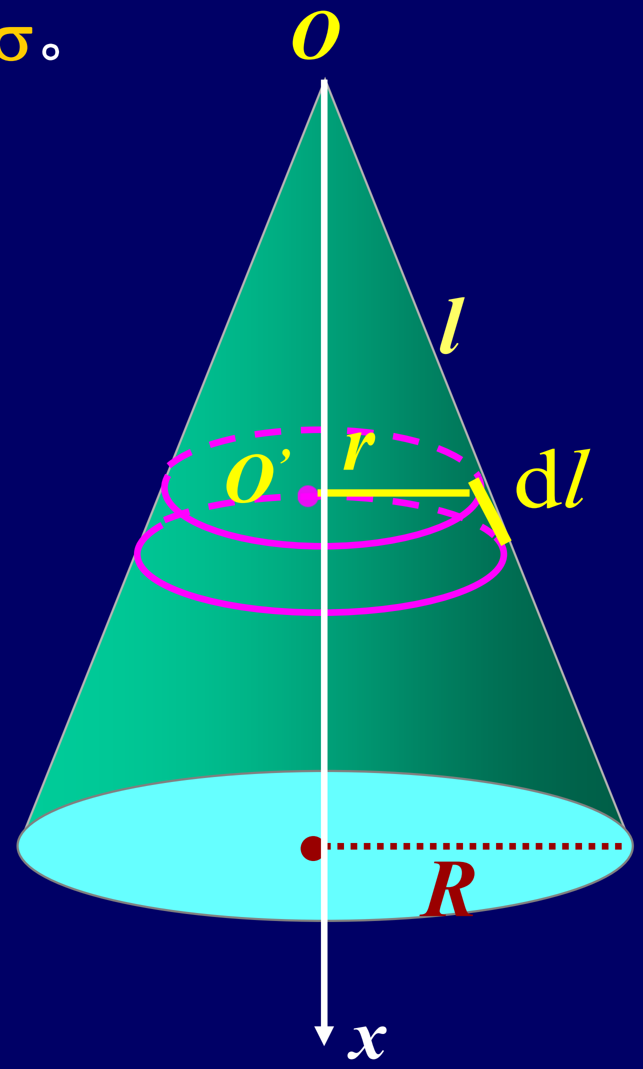
证明： 圆锥顶点O的电势与圆锥高度无关

$$\mathrm{d}q = 2\pi r \mathrm{d}l \cdot \sigma$$

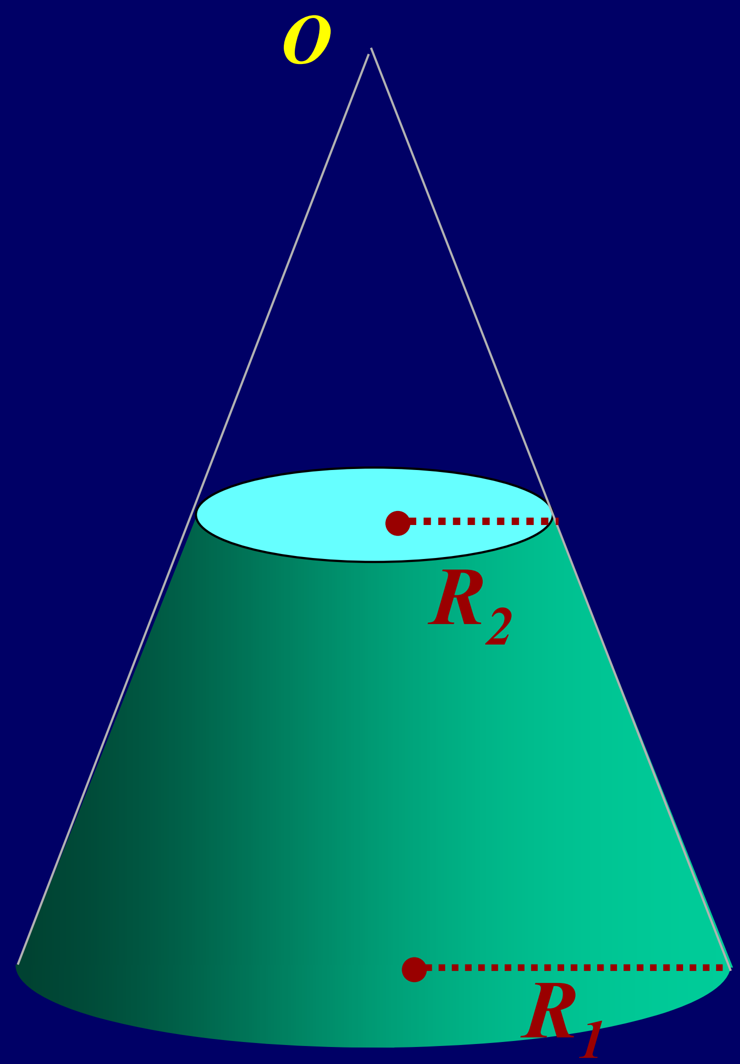
$$\mathrm{d}u = \frac{\mathrm{d}q}{4\pi\varepsilon_0 l} = \frac{2\pi r \sigma \mathrm{d}l}{4\pi\varepsilon_0 l} = \frac{r\sigma \mathrm{d}l}{2\varepsilon_0 l}$$

$$= \frac{R\sigma \mathrm{d}l}{2\varepsilon_0 L}$$

$$u_o = \int \mathrm{d}u = \int \frac{R\sigma \mathrm{d}l}{2\varepsilon_0 L} = \frac{R\sigma}{2\varepsilon_0}$$



圆台上底面半径R_2，下底面半径R_1，侧面均匀带电面密度σ。



顶点O的电势

$$U_O = \frac{\sigma R_1}{2\varepsilon_0} - \frac{\sigma R_2}{2\varepsilon_0}$$

例 均匀带电Q的球体，半径为R

求 球内任一点的电势

解

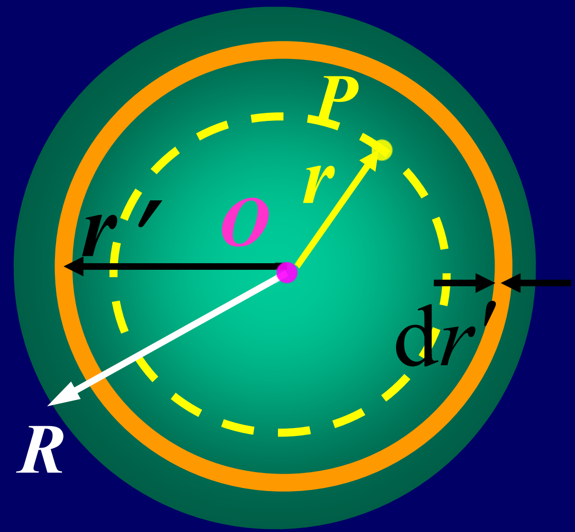
$$r \geq R \qquad E_2 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \frac{\rho}{3\varepsilon_0}\frac{R^3}{r^2}$$

$$r < R \qquad E_1 = \frac{\rho}{3\varepsilon_0} r$$

$$u_{内} = \int_r^R E_1 \mathrm{d}r + \int_R^\infty E_2 \mathrm{d}r$$

内部电荷在*p*点产生的电势

$$u_1 = \frac{\sum q(内)}{4\pi\varepsilon_0 r} = \frac{Q}{4\pi\varepsilon_0 R^3} r^3$$



$$\sum q(内) = \frac{Q}{R^3} r^3$$

外部电荷在*p*点产生的电势

$$\mathrm{d}q = \rho \mathrm{d}V = 4\pi r'^2 \mathrm{d}r' \frac{Q}{4\pi R^3/3} = \frac{3Qr'^2}{R^3}\mathrm{d}r'$$

$$\mathrm{d}u = \frac{\mathrm{d}q}{4\pi\varepsilon_0 r'} \qquad u_2 = \int_r^R \frac{\mathrm{d}q}{4\pi\varepsilon_0 r'} \qquad u = u_1 + u_2 =$$

导体静电平衡的条件 $\vec{E}_{内} = \vec{E}_0 + \vec{E}' = 0$

导体静电平衡时，导体是等势体，表面是等势面。

导体的内部处处不带电，净电荷只分布在导体表面。

$$\vec{E}_{表} = \frac{\sigma}{\varepsilon_0}\vec{n}$$

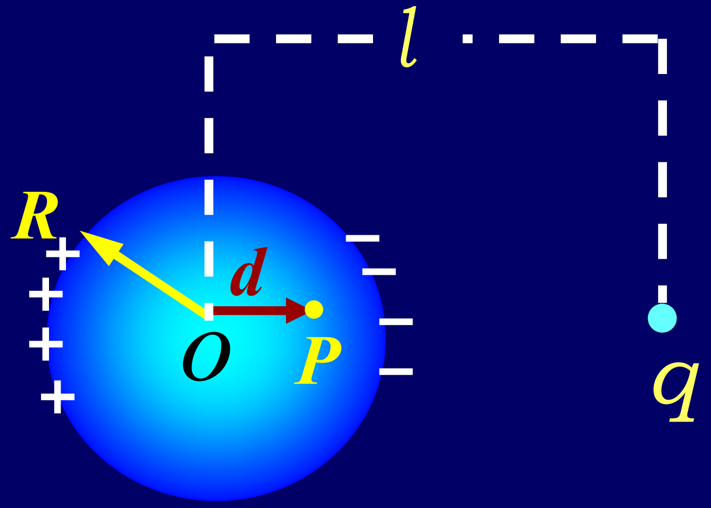
例　如图所示, 原先不带电的导体球附近有一点电荷q 。

求　P点处感应电荷产生的电场

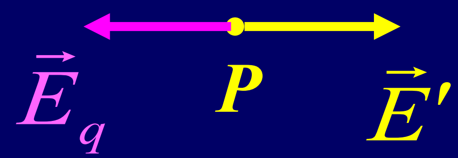
$$\vec{E}=\vec{E}_q+\vec{E}'=0$$

$$\vec{E}'=-\vec{E}_q$$

$$=-\frac{q}{4\pi\varepsilon_0(l-d)^2}\vec{r}^{\,0}$$



求　导体球的电势



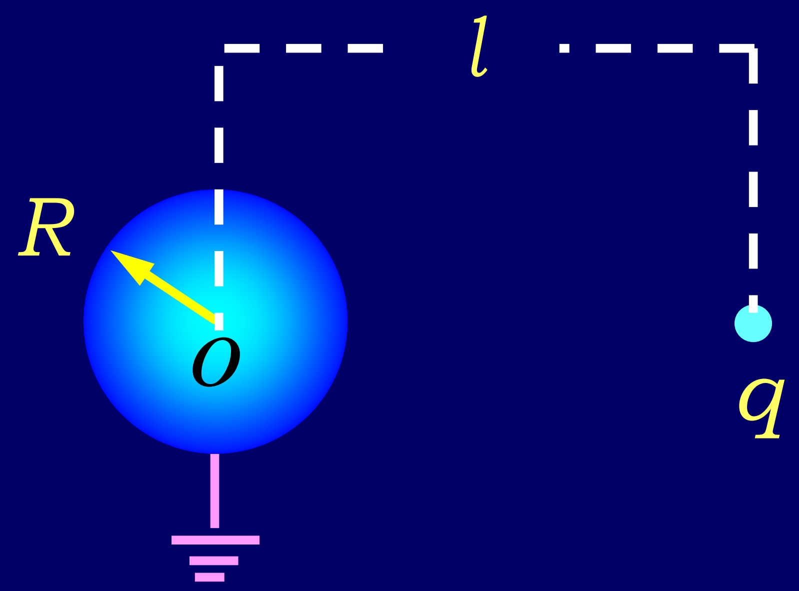
$$u_{球}=u_o=u_q+u_{q'}=\frac{q}{4\pi\varepsilon_0(l-d)}$$

$$u_{q'}=\frac{\sum \mathrm{d}q'}{4\pi\varepsilon_0 R}=0$$

求 接地后导体上感应电荷的电量

解 设感应电量为Q

$$Q=\begin{cases}-q\\ 0\end{cases}$$



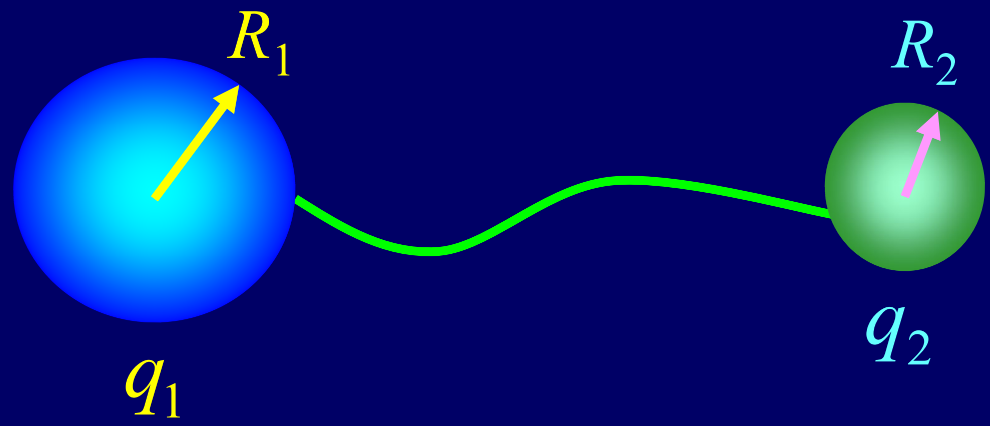
接地 即 $U=0$

由导体是个等势体

O点的电势为0 则

$$\frac{Q}{4\pi\varepsilon_0 R}+\frac{q}{4\pi\varepsilon_0 l}=0 \Rightarrow Q=-\frac{R}{l}q$$

例　两球半径分别为R_1、R_2，带电量q_1、q_2，设两球相距很远，当用导线将彼此连接时，电荷将如何 分布？



解　设用导线连接后，两球带电量为 q_1'　q_2'

$$q_1' + q_2' = q_1 + q_2$$

$$\left.\begin{aligned} u_1 &= \frac{q_1'}{4\pi\varepsilon_0 R_1} \\ u_2 &= \frac{q_2'}{4\pi\varepsilon_0 R_2} \end{aligned}\right\} \xrightarrow{u_1 = u_2} \frac{\sigma_1 \cdot 4\pi R_1^{\ 2}}{\sigma_2 \cdot 4\pi R_2^{\ 2}} = \frac{R_1}{R_2} \Rightarrow \frac{\sigma_1}{\sigma_2} = \frac{R_2}{R_1}$$

思考　**如果两球相距较近，结果怎样?**

例 已知导体球壳 A 带电量为 Q，导体球 B 带电量为 q

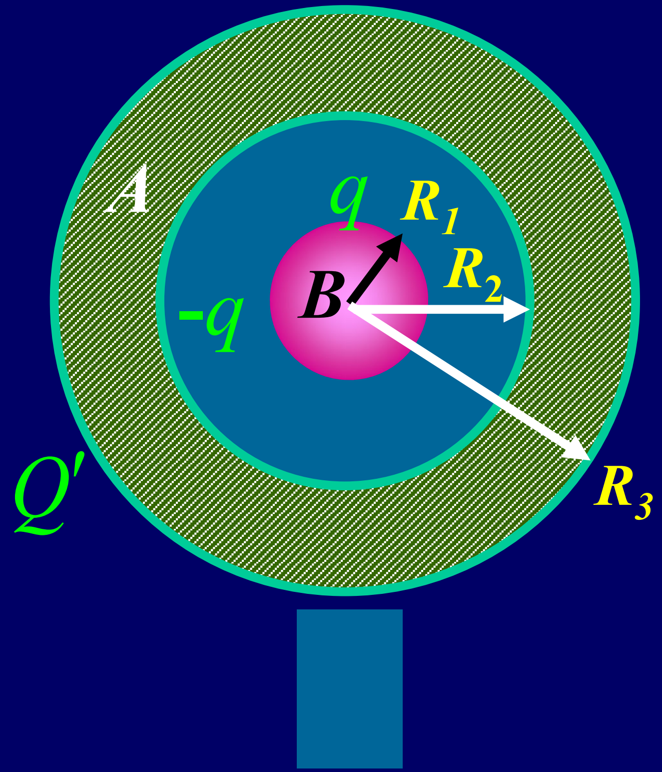
求 (1) 将 A 接地后再断开，电荷和电势的分布；

(2) 再将 B 接地，电荷和电势的分布。

解 (1) A 接地时,外表面电荷设为 Q'

$$U_A = \frac{Q'}{4\pi\varepsilon_0 R_3} = 0 \qquad Q' = 0$$

A 与地断开后， $Q_A = -q$



$R_1 < r < R_2$

$$u = \int_r^{R_2} \frac{q}{4\pi\varepsilon_0 r^2}\mathrm{d}r = \frac{q}{4\pi\varepsilon_0 r} + \frac{-q}{4\pi\varepsilon_0 R_2}$$

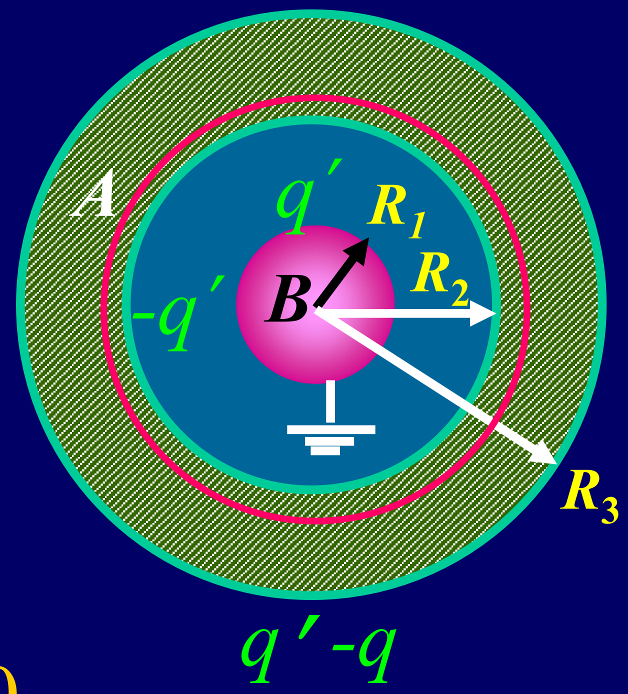
$$u_B = \frac{q}{4\pi\varepsilon_0 R_1} + \frac{-q}{4\pi\varepsilon_0 R_2}$$

(2) 设B上的电量为 q'

$$Q_{内}=-q'$$

根据孤立导体电荷守恒

$$Q_{内}+Q_{外}=-q \qquad Q_{外}=q'-q$$



B 球圆心处的电势

$$U_B=\frac{q'}{4\pi\varepsilon_0 R_1}+\frac{-q'}{4\pi\varepsilon_0 R_2}+\frac{q'-q}{4\pi\varepsilon_0 R_3}=0$$

$$q'=\frac{qrR_1}{R_1r-R_2r+R_1R_2}$$

$$U_A=\frac{q'-q}{4\pi\varepsilon_0 R_2}$$

总结（有导体存在时静电场的计算方法）

1. 静电平衡的条件和性质: $E_{内} = 0$ $U_{导体} = C$

2. 电荷守恒定律

3. 确定电荷分布,然后求解

非孤立导体接地后，电荷分布由场分布决定，与原先的带电量无关。

例　已知导体球壳 A 带电量为Q，导体球 B 带电量为q

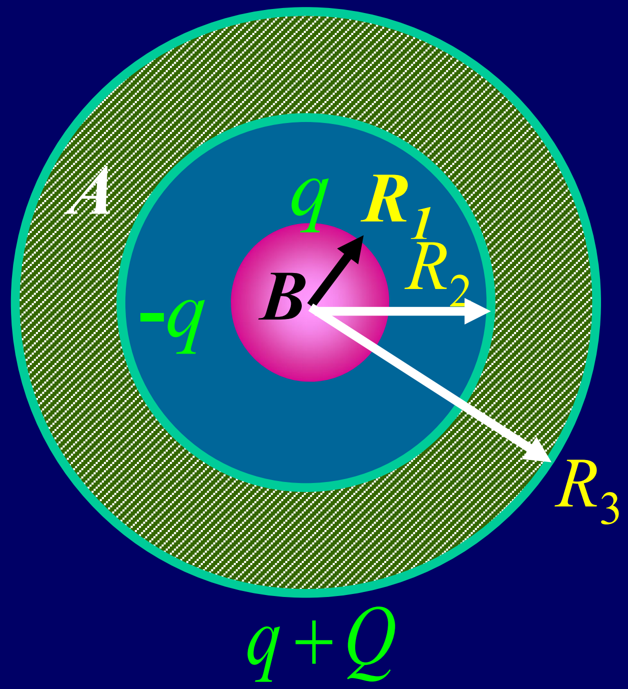
求　它所产生的电场中储藏的电场能量

$r < R_1$　　$E_1 = 0$

$R_1 < r < R_2$　　$E_2 = \dfrac{q}{4\pi\varepsilon_0 r^2}$

$R_2 < r < R_3$　　$E_3 = 0$

$r > R_3$　　$E_4 = \dfrac{Q+q}{4\pi\varepsilon_0 r^2}$



取体积元 $\mathrm{d}V = 4\pi r^2 \mathrm{d}r$

$$W = \int_0^\infty \frac{1}{2}\varepsilon_0 E^2 \mathrm{d}V = \int_{R_1}^{R_2} \frac{1}{2}\varepsilon_0 E_2{}^2 \mathrm{d}V + \int_{R_3}^{\infty} \frac{1}{2}\varepsilon_0 E_4{}^2 \mathrm{d}V$$

$$= \int_{R_1}^{R_2} \frac{1}{2}\varepsilon_0 \left(\frac{q}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 \mathrm{d}r + \int_{R_3}^{\infty} \frac{1}{2}\varepsilon_0 \left(\frac{Q+q}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2 \mathrm{d}r$$

$$= \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{(q+Q)^2}{8\pi\varepsilon_0 R_3}$$

$$W_e = \frac{q^2}{8\pi\varepsilon_0}\frac{R_2 - R_1}{R_1 R_2} + \frac{(q+Q)^2}{8\pi\varepsilon_0 R_3}$$

球形电容器 $$C_1 = \frac{4\pi\varepsilon_0 R_1 R_2}{R_1 - R_2}$$

孤立导体球 $$C_2 = 4\pi\varepsilon_0 R_3$$

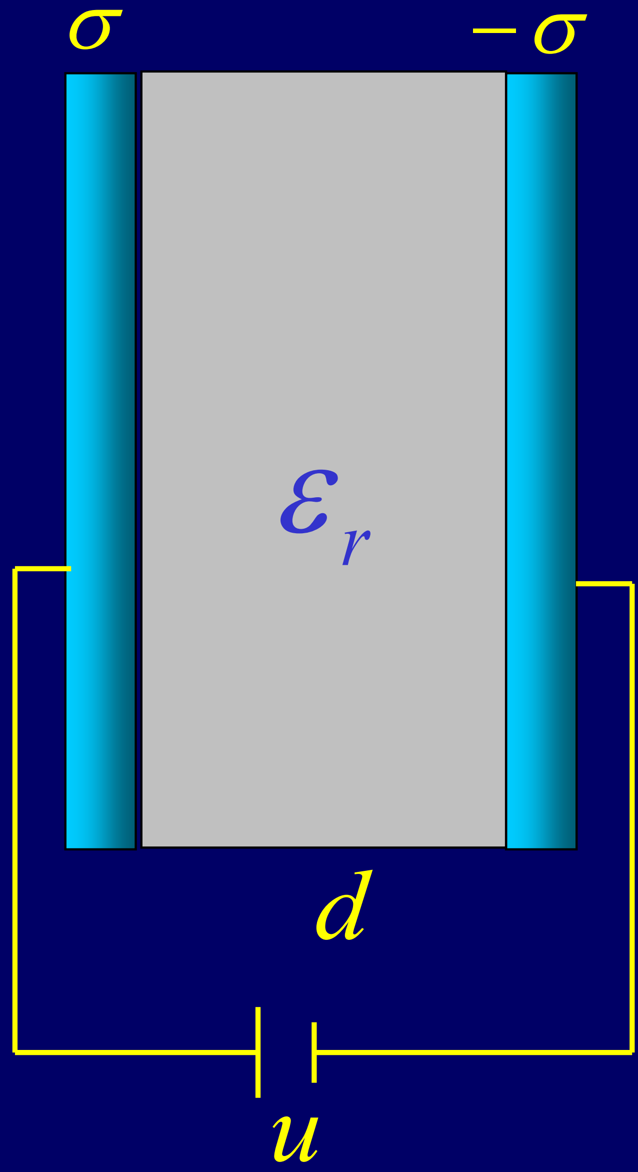
$$W_e = \frac{q^2}{2C_1} + \frac{(Q+q)^2}{2C_2}$$

例 平板电容器，u不变，将一厚d的介质板插入电容器

求 （1）电场能量变化

（2）电源的功

（3）电场对介质板作的功



解

$$W_0=\frac{1}{2}C_0u^2 \qquad W=\frac{1}{2}Cu^2$$

$$\Delta W=\frac{1}{2}u^2\left(C-C_0\right)=\frac{1}{2}u^2C_0(\varepsilon_r-1)$$

$$=\frac{u^2\varepsilon_0S}{2d}(\varepsilon_r-1)$$

$$\Delta W = \frac{u^2 \varepsilon_0 S}{2d}(\varepsilon_r - 1)$$

$$Q = Cu \quad > Q_0 = C_0 u$$

$$A_{电源} = u\Delta Q = u^2\left(C - C_0\right) = \frac{\varepsilon_0 S u^2}{d}(\varepsilon_r - 1)$$

$$A_{电源} - A_{电场} = \Delta W$$

$$A_{电场} = A_{电源} - \Delta W = \frac{1}{2}\frac{\varepsilon_0 S U^2}{d}(\varepsilon_r - 1) > 0$$

例 均匀带电球面（R、Q），均匀带电直线段（l、λ）沿径向放置

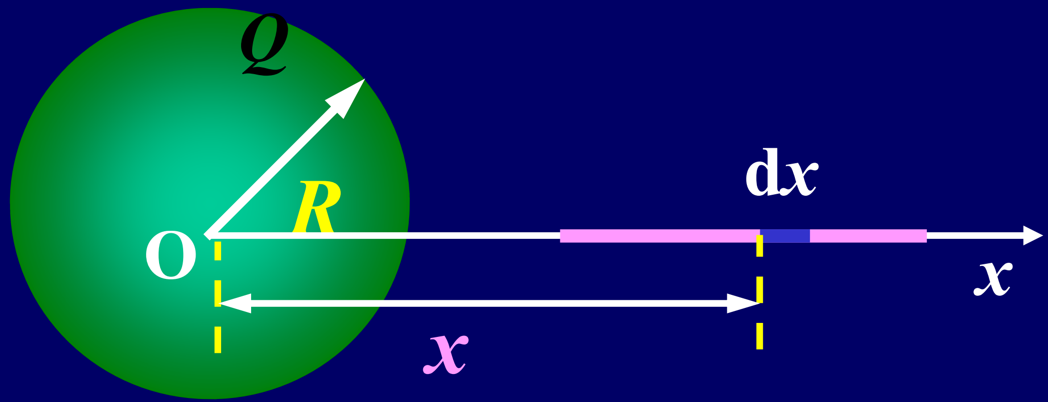
求 均匀带电直线段在均匀带电球面电场中的电势能

解

$$\mathrm{d}q = \lambda \mathrm{d}x$$

均匀带电球面在x处的电势

$$u = \frac{Q}{4\pi\varepsilon_0 x}$$



$$dW = u dq = \frac{Q}{4\pi\varepsilon_0 x}\lambda dx$$

$$W = \int \mathrm{d}W = \int_l^{2l} \frac{Q}{4\pi\varepsilon_0 x}\lambda dx = \frac{\lambda Q}{4\pi\varepsilon_0}\ln 2$$

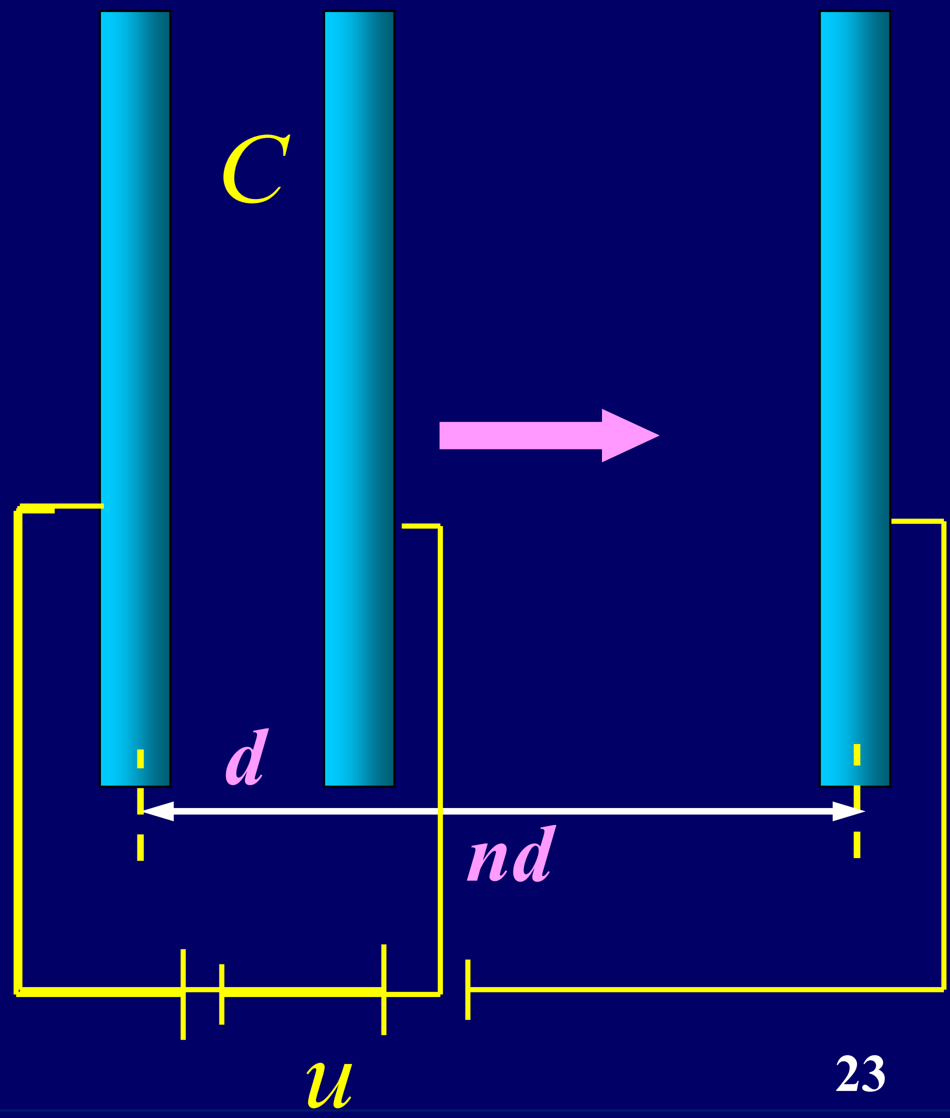
例　平板电容器，电容为C，与电压为u的电源相连

求　两板间距由 d 变为nd过程中，外力作的功

解

$$A_{外力}=-\frac{1}{2}U^2\Delta C$$

$$=\frac{1}{2}U^2C\left(1-\frac{1}{n}\right)$$



$$\sigma_1 = \sigma_4 = 0$$

$$\sigma_2 = -\sigma_3 = \frac{q_A}{S}$$