Numerically Solving Burgers and KPZ Equations

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1 Introduction

The Burgers equation is a nonlinear PDE that arises in a variety of contexts (including as a one-dimensional equivalent of the incompressible Navier-Stokes equations). In this homework we solve the Burgers equation as well as the related Kardar-Parisi-Zhang equation (a stochastic PDE describing phenomena such as interface growth).

2 Problem

The Burgers equation is given by

$$\partial_t u + u\partial_x u = \nu \partial_{xx} u$$

and the KPZ equation is given by

$$\partial_t h = \frac{\lambda}{2} \left(\nabla h\right)^2 + \nu \nabla^2 h + r\xi$$

where ξ is unit-variance Gaussian white noise and r is a scaling factor. We saw in class that the two equations are equivalent (up to stochasticity) by the substitution $u = -\partial_x h$. The aim of this assignment is to solve these equations numerically.

3 Methods

We primarily use finite difference methods to solve the equations. That is, for the Burgers equation, we have

$$u_j^{n+1} = u_j^n - \frac{u_j^n \Delta t}{2\Delta x} \left(u_{j+1}^n - u_{j-1}^n\right) + \frac{\nu \Delta t}{\Delta x^2} \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

To check our solutions we perform standard convergence tests diffusive scaling and check the order of the rror. For the KPZ equation we have

$$h_j^{n+1} = h_j^n + \frac{\lambda \Delta t}{8\Delta x^2}\left(h_{j+1}^n - h_{j-1}^n\right)^2 + \frac{\nu \Delta t}{\Delta x^2}\left(h_{j+1}^n - 2h_j^n + h_{j-1}^n\right) + \xi_j^n$$

in one dimension, and

$$\begin{aligned} h_{j,k}^{n+1} = h_{j,k}^n &+ \frac{\lambda \Delta t}{8\Delta x^2}\left(h_{j+1,k}^n - h_{j-1,k}^n + h_{j,k+1}^n - h_{j,k-1}^n\right)^2 \\ &+ \frac{\nu \Delta t}{\Delta x^2}\left(h_{j+1,k}^n + h_{j-1,k}^n + h_{j,k+1}^n + h_{j,k-1}^n - 4h_{j,k}^n\right) + \xi_{j,k}^n \end{aligned}$$

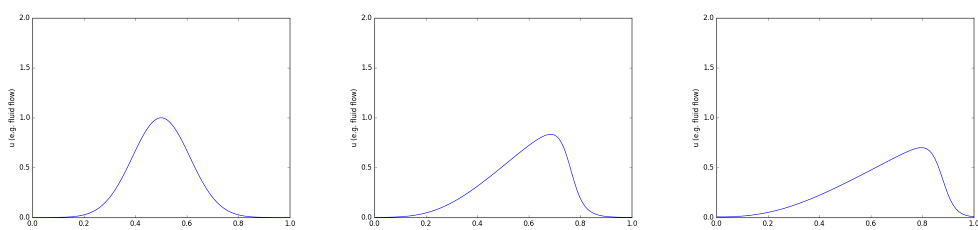
in two dimensions. To validate, we check scaling laws for the numerical solutions.

4 Experiments

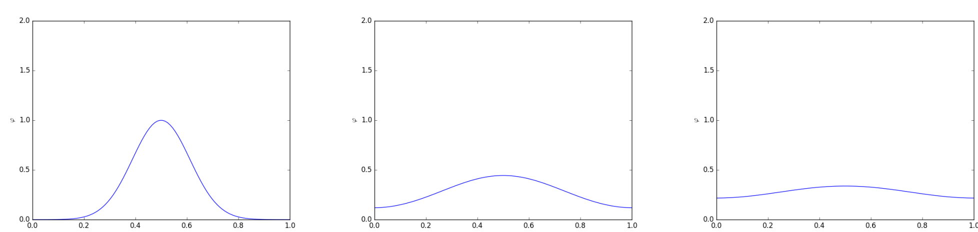
4.1 Burgers Equation

4.1.1 Simulation

The results of the simulation for $N = 100$ grid points, $\mathrm{Re} = \frac{|u_j^n| L}{\nu} = |u_j^n| \times 10^2$, with Gaussian initial conditions, look as follows (at $n\Delta t = 0, 25, 50$):



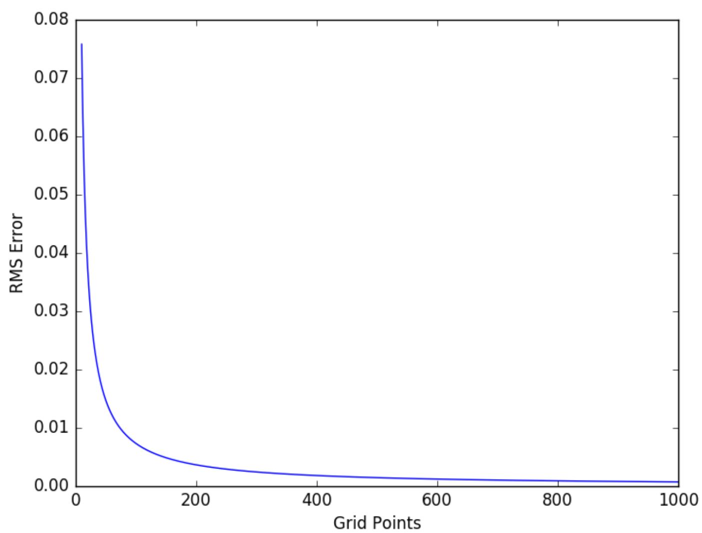
For the alternative choice of parameters such that $\mathrm{Re} = \frac{|u_j^n| L}{\nu} = |u_j^n| \times 10^{-2}$, we get the following (now with $n\Delta t = 0, .025, .05$):



Thus we get far more diffusive solutions at low Reynolds numbers, as we might expect.

4.1.2 Validation

We check the validity of the method by testing grid convergence with diffusive scaling as usual, with the following results:



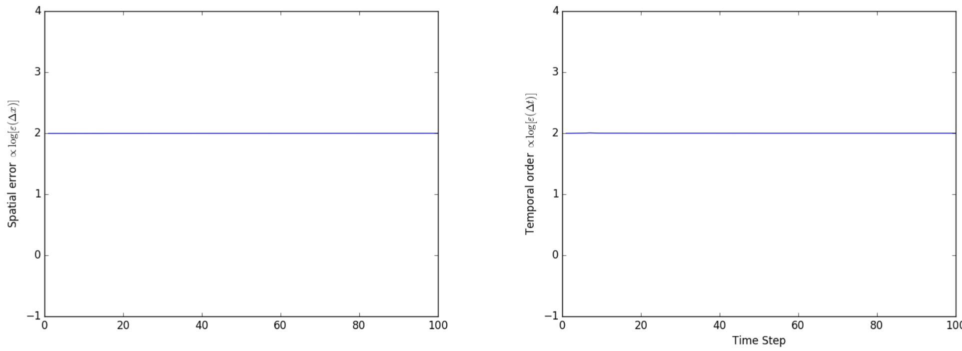
We also check the order of our method by computing

$$\frac{\varphi_{\Delta x} - \varphi_{\Delta x/2}}{\varphi_{\Delta x/2} - \varphi_{\Delta x/4}} = \frac{\varepsilon\left(\Delta x^n - \Delta x^n/2^n\right)}{\varepsilon\left(\Delta x^n/2^n - \Delta x^n/4^n\right)} = 2^n$$

where $\varphi_{\Delta x}$ is the numerical solution with spatial interval Δx, and we have assumed that

$$\varphi = \varphi_{\Delta x} + \varepsilon \Delta x^n$$

for exact solution φ. An identical computation can be used to find the order of the method in time. The results are as follows:

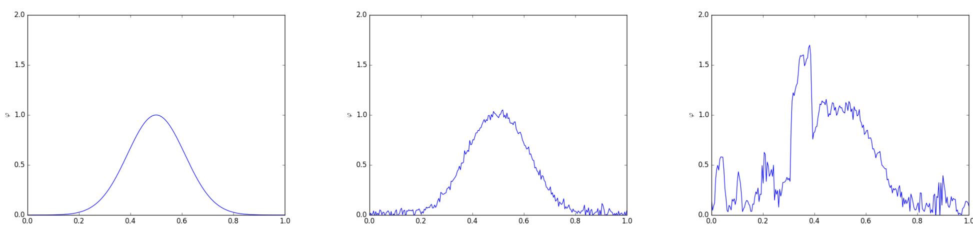


That is, the method seems to be first-order in both space and time (I'm not sure I have a good explanation for this since it seems as though the method ought to be second order in space).

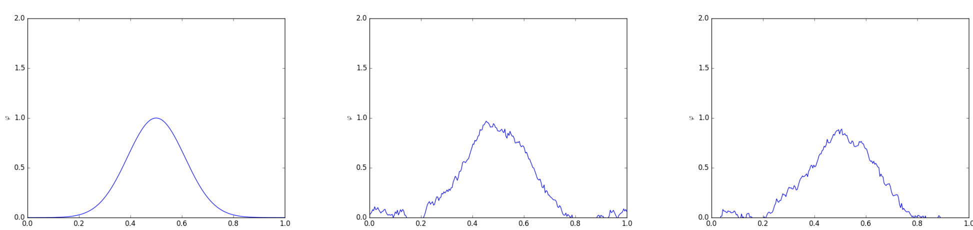
4.2 KPZ Equation

One Dimension:

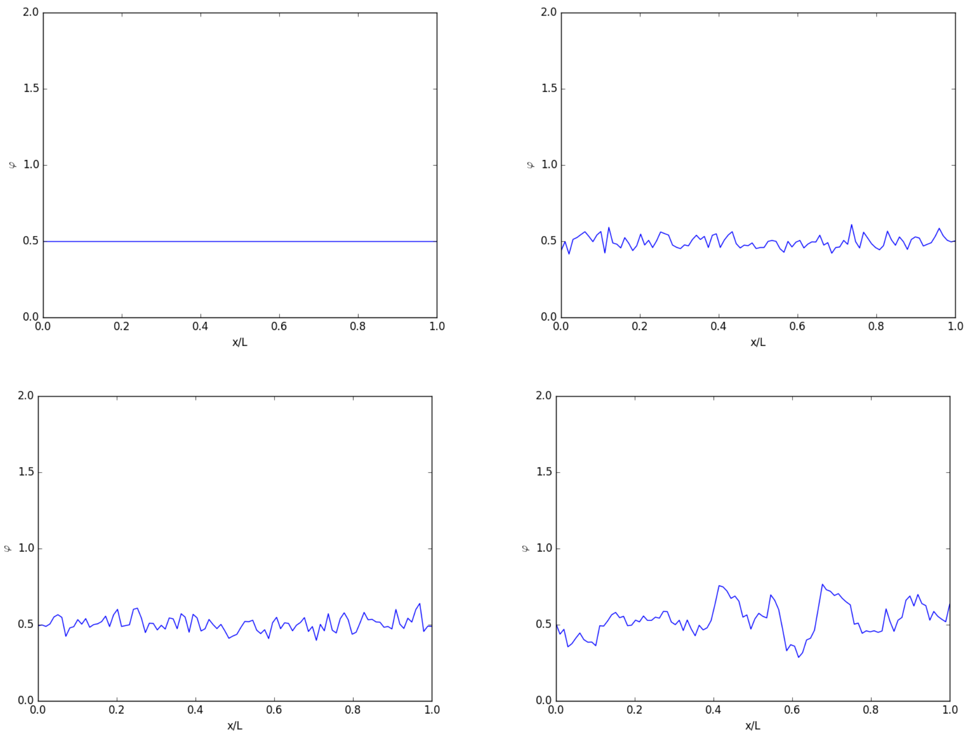
With $\lambda \Delta t/\nu = 1 \times 10^4$, the results of the 1-dimensional simulation are as follows (at time steps $t = 0, 25, 50$):



On the other hand, $\lambda \Delta t/\nu = 1 \times 10^3$ yields:

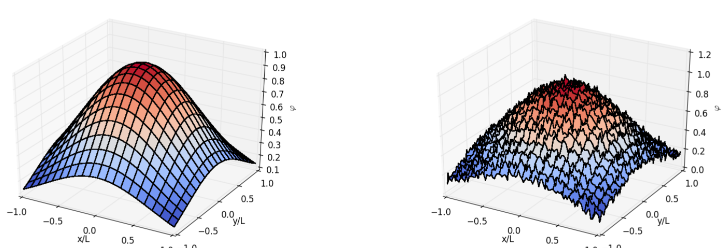


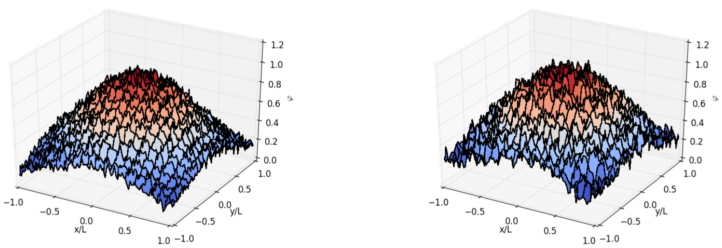
As expected, this solution is more diffusive. We also solve the equation for uniform initial conditions, which make the growth phenomena even more apparent ($\lambda \Delta t / \nu = 1$, $n \Delta t = 0, 2.5, 5.0, 50.0$):



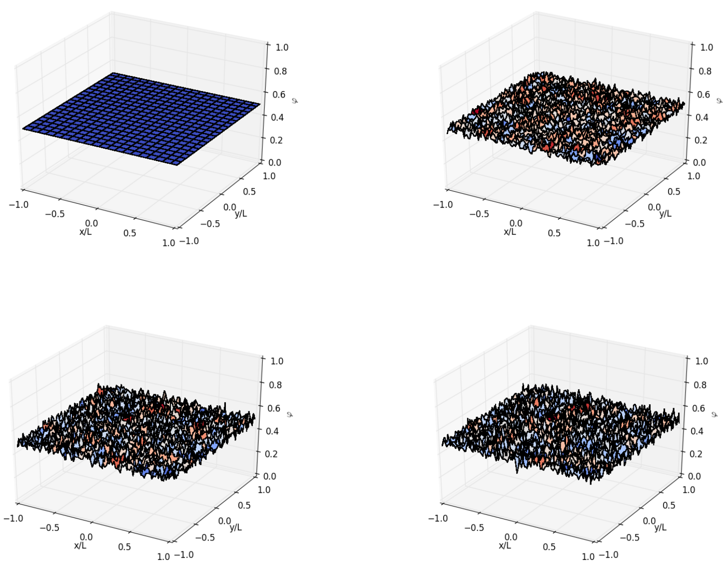
Two Dimensions:

We perform similar experiments in two dimensions. With $N = 100^2$ total grid points and $\lambda \Delta t / \nu = .1$, we get (at time $n \Delta t = 0, 1.0, 2.0, 50.0$).





We perform similar experiments in two dimensions. With $N = 100^2$ total grid points and $\lambda \Delta t / \nu = 1$, we get (at time $n\Delta t = 0, .5, 1.0, 1.5$).



4.2.1 Validation

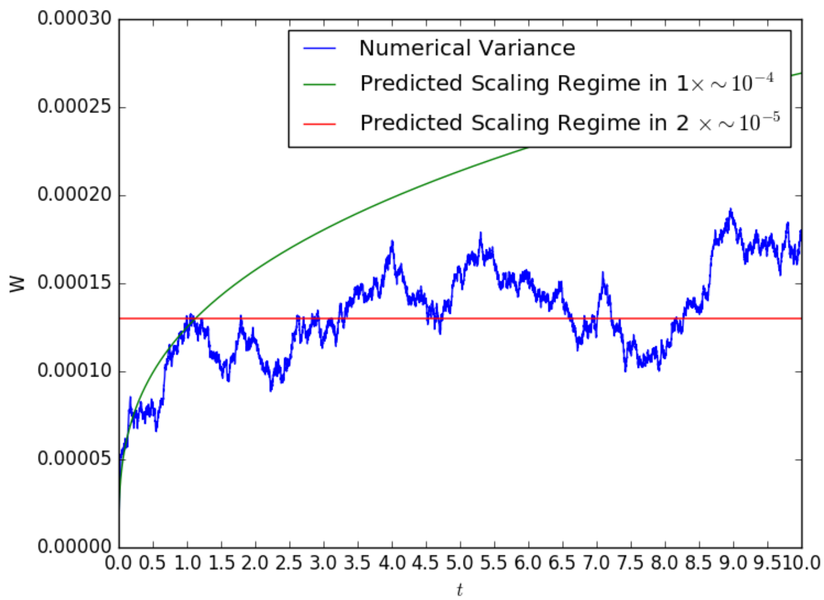
We validate our solutions by checking the scaling laws seen in lecture. That is, for a solution of the KPZ equation, we expect that

$$L^a f(t/L^z)$$

where

$$f(x) = \begin{cases} x^b & \text{if } x \ll 1 \\ 1 & \text{if } x \gg 1 \end{cases}$$

and $a = 1/2, b = 1/3, z = a/b$ in the one dimensional case. Plotting the variance against our prediction from the scaling laws, we get the following:



Note that we had to scale the expected scaling by small constant factors for the scales to match— this may be due to a difference in how noise was computed from the slides (we use arbitrarily scaled Gaussian noise). However, the essential features of the scaling can be seen clearly: the variance obeys a power law until t becomes significantly large, at which point it becomes approximately constant.