Brilliant Question

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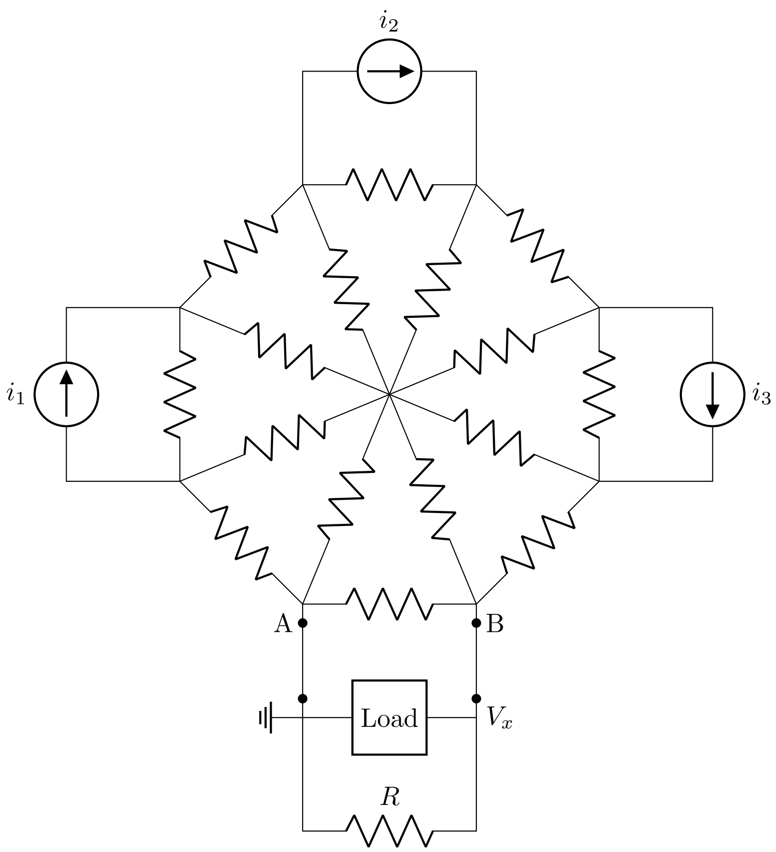
Consider the following circuit which consists of 3 current sources, 16 resistors arranged in an octagon, and a load in parallel with a resistor. Furthermore, consider three different version of this circuit which have three different loads.

1. A resistor of resistance $R_L = 1\text{k}\Omega$. Determine $V_1 = V_x$.
2. A capacitor of capacitance C. Determine $V_2 = V_x(\tau)$, the voltage $V_x(t)$ after a single time constant, if switches A and B are opened at $t = 0$.
3. An inductor of inductance L. If the switches are opened at $t = 0$, determine $V_3 = V_x(\tau)$.

Calculate $V_1 + V_2 + V_3$ to 1 decimal place.

Details and assumptions:

- All of the resistors are of identical resistance $R = 50\Omega$.
- $i_1 = 1\text{A}$, $i_2 = 2\text{A}$, $i_3 = 3\text{A}$.
- Assume that switches A and B have been closed for a very long time so the circuit is in steady state prior to the opening of the switches.



Solution:

The voltages across the loads are all *terminal characteristics*. Thus, it will be very useful to find a **Thevenin equivalent** circuit as seen from the terminals of the load. Let's find the Thevenin voltage V_{TH} first by disconnecting the load from the circuit. Notice that if we can find the current labelled i_x, the Thevenin voltage falls out trivially as $V_{TH} = Ri_x$.

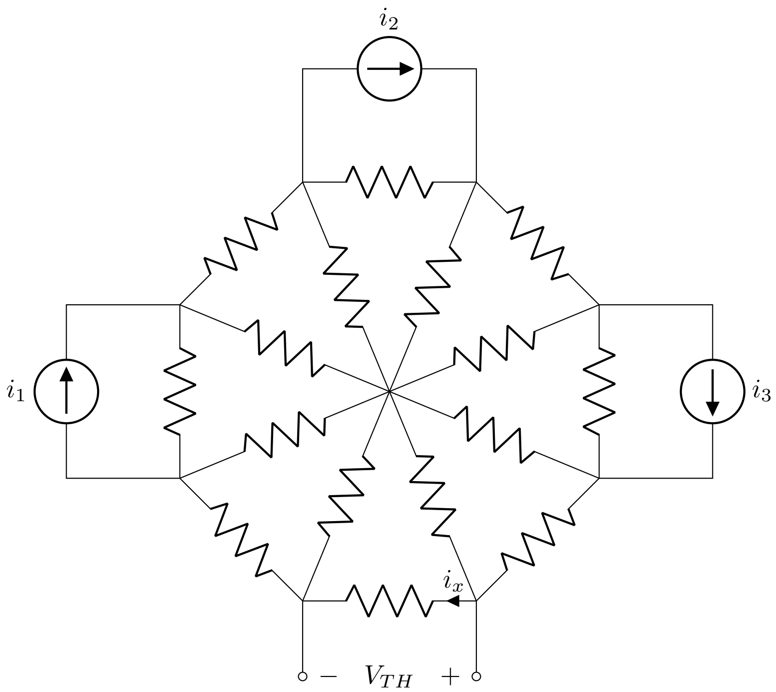


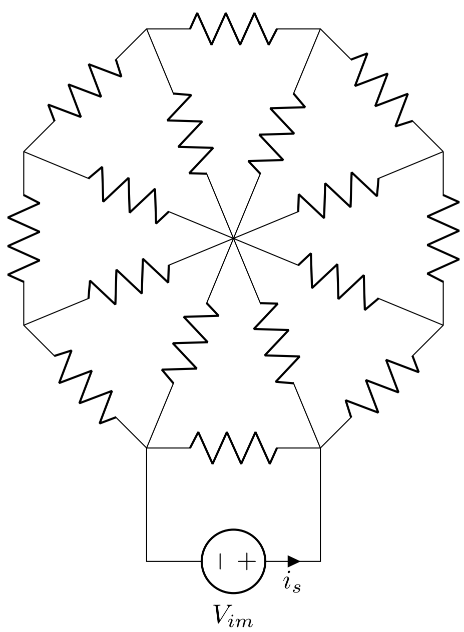
Figure 1: Circuit with load disconnected

Use superposition to calculate i_x.

$$i_x(\text{due to } i_1) = 66.7\text{mA}$$
$$i_x(\text{due to } i_2) = 38.1\text{mA}$$
$$i_x(\text{due to } i_3) = 200\text{mA}$$
$$i_x = 66.7\text{mA} + 38.1\text{mA} + 200\text{mA} = 304.8\text{mA}$$
$$V_{TH} = Ri_X = 50\Omega \times 304.8\text{mA} = 15.24\text{V}$$

Alternatively, you could bypass using superposition and just get the equations directly. It will probably be easier and faster.

To calculate the Thevenin resistance R_{TH} attach an imaginary voltage source to the terminals of the load, in addition to setting the current sources to zero. Use a similar method to the previous.



$$i_s = 0.036207V_{im} \longrightarrow R_{TH} = \frac{V_{im}}{i_s} = \frac{V_{im}}{0.036207V_{im}} = \frac{1}{0.036207} = 27.619\Omega$$

Now we can set up the Thevenin equivalent circuits for each load.

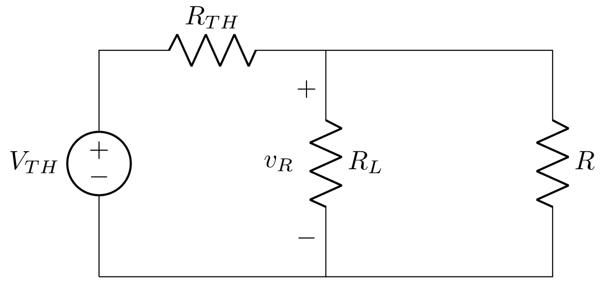


Figure 2: Thevenin equivalent circuit with resistive load

The equivalent parallel resistance of R_L and R is

$$R_L||R = \frac{R_L R}{R_L + R} = \frac{1000 \times 50}{1000 + 50} = 47.619\Omega$$

By the voltage divider principle:

$$v_R = V_{TH} \times \frac{R_L||R}{R_{TH} + R_L||R} = 15.24 \times \frac{47.619}{27.619 + 47.619} = \boxed{9.646\text{V}}$$

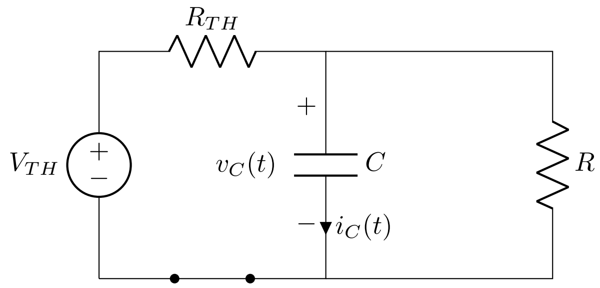


Figure 3: Thevenin equivalent circuit with capacitive load

At steady state, $\frac{dv_C}{dt} = 0$, so $i_C(t) = C\frac{dv_C}{dt} = 0$. Thus, at steady state, the capacitor voltage will be just

$$v_C(0^-) = v_C(0^+) = V_{TH} \times \frac{R}{R_{TH} + R} = 15.24 \times \frac{50}{27.619 + 50} = 9.817\text{V}$$

When the switch is opened at $t = 0$, the voltage source can no longer supply current to the circuit. All of the current through R came from the source at steady state. However, since the capacitor voltage cannot change instantaneously, the voltage across R must remain the same due to KVL. Thus, the current through it must remain the same as well.

$$i_C(0^+) = \frac{v_C(0^+)}{R} = \frac{9.817}{50} = 196.34\text{mA}$$

We can use KVL to formulate a differential equation to solve for $v_C(t)$.

$$v_C(t) + Ri_C(t) = 0 \longrightarrow RC\frac{dv_C}{dt} + v_C(t) = 0$$

$$RC\lambda + 1 = 0 \longrightarrow \lambda = \frac{-1}{RC}$$

$$v_C(t) = Ae^{-\frac{t}{RC}}$$

Using the initial condition $v_C(0^+) = 9.817$V:

$$9.817 = Ae^0 = A \longrightarrow v_C(t) = 9.817e^{-\frac{t}{RC}}$$

Notice that RC is the time constant $\tau = RC$. After 1 time constant,

$$v_C(3\tau) = 9.817e^{-\frac{RC}{RC}} = 9.817e^{-1} = \boxed{3.611\text{V}}$$

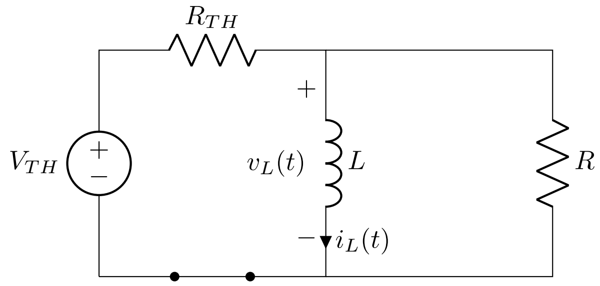


Figure 4: Thevenin equivalent circuit with inductive load

At steady state $\frac{di_L}{dt} = 0$, so $v_L(t) = L\frac{di_L}{dt} = 0$. Since the inductor is in parallel with R, R has zero voltage drop across it. Consequently, no current flows through R.

$$i_L(0^-) = i_L(0^+) = \frac{V_{TH}}{R_{TH}} = \frac{15.24}{27.619} = 551.8\text{mA}$$

Immediately after the switch is opened, the inductor current remains the same since it cannot change instantaneously. In order to satisfy KVL:

$$v_L(0^+) = Ri_L(0^+) = 50\Omega \times 551.8\text{mA} = 27.59\text{V}$$

As with the capacitor:

$$v_L(t) + Ri_L(t) = 0 \longrightarrow v_L(t) + R\int \frac{v_L(t)}{L}dt = 0$$

Differentiating this equation:

$$\frac{dv_L}{dt} + \frac{R}{L}v_L(t) = 0$$

$$\lambda + \frac{R}{L} = 0 \longrightarrow \lambda = -\underbrace{\frac{R}{L}}_{\tau}$$

$$v_L(t) = Ae^{-\frac{t}{\tau}}$$

Using the initial condition:

$$27.59 = Ae^0 = A \longrightarrow v_L(t) = 27.59e^{-\frac{t}{\tau}}$$

After 1 time constant:

$$v_L(\tau) = 27.59e^{-\frac{\tau}{\tau}} = 27.59e^{-1} = \boxed{10.15\text{V}}$$

Thus,

$$V_1 + V_2 + V_3 = 9.646 + 3.611 + 10.15 = \boxed{23.4\text{V}}$$