Operator Semigroups

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1 Spaces and Topologies

- Let E be a locally compact Hausdorf space. (that is $\forall x \in E$, x has a compact neighborhood).
- $(\mathcal{X}, \|\cdot\|)$: Banach Space.
- $\mathcal{L}(\mathcal{X})$: space of bounded linear operators on $\mathcal{X}$.

Topologies on $\mathcal{L}(\mathcal{X})$

(1) Weak Operator Topology (WOT)
  - coarest topology s.t. $\forall f \in \mathcal{X}$, the map $\mathcal{L}(\mathcal{X}) \ni T \mapsto T(f) \in \mathcal{X}$ is continuous, when X is equipped with the weak topology.
  - In other words, $T_n \to T$ if $\langle f^*, T_n f\rangle \to \langle f^*, Tf\rangle$, $\forall f^* \in \mathcal{X}^*$

(2) Strong Operator Topology (SOT)
  - coarest topology s.t. $\forall f \in \mathcal{X}$, the map $\mathcal{L}(\mathcal{X}) \ni T \mapsto T(f) \in \mathcal{X}$ is continuous, when X is equipped with the strong topology.
  - In other words, $T_n \to T$ if $T_n f \to Tf$ in $\mathcal{X}$, for all $f \in \mathcal{X}$ (topology of pointwise convergence).

(3) Norm Topology (Uniform Topology)
  - Topology induced by the operator norm ($\|T\| = \sup\{\|Tf\| : \|f\| \leq 1\}$).
  - $T_n \to T$ if $\|T_n - T\| \to 0$.

It turns out for semigroups, weak and strong topology are equivalent in some sense. The norm topology is usually to strong. So we shall work with SOT.

Choices for $\mathcal{X}$:

Shall use the sup norm

- $C_b(E)$: cts bdd functions on E
- $C_0(E)$: cts bdd functions on E that vanish at ∞,
  i.e. $\{f \in C_b(E) : \lim_{\|x\| \to \infty} f(x) = 0\}$.

Often, $C_b(E)$ is not enough to get SCS, need $C_0(E)$ instead.

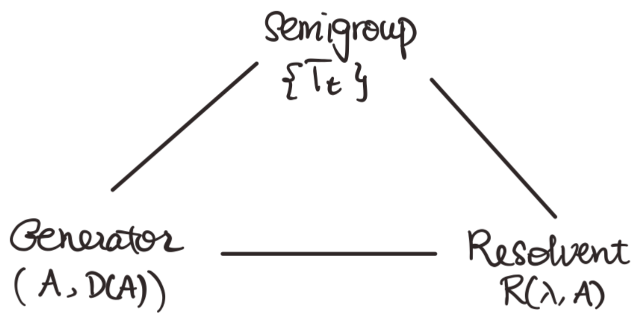

Figure 1: title

2 Definition of Semigroups

Definition 2.1 (Monoid)

- An **algebraic semigroup** is a pairing $(M, \circ)$, where
  - M: nonempty set
  - $\circ$ is associative binary operation $M \times M \to M$.
- M is a **monoid** if it is an algebraic semigroup and have an unit element.
  i.e. $\exists e \in M$ s.t. $e \circ a = a \circ e = a$, $\forall a \in M$.
- A **topological monoid** is a monoid with a topology, in which $\circ$ is continuous.

Definition 2.2 (Algebraic Representation)
let M be a monoid. A map $T : M \to \mathcal{L}(\mathcal{X})$ is called an **algebraic representation** if

(1) $T(e) = Id$
(2) $T(a \circ b) = T(a)T(b)$, for all $a, b \in M$.

If in addition, M is a topological monoid and $a \mapsto T(a)$ is continuous when $\mathcal{L}(\mathcal{X})$ is given the strong operator topology, then we say T is **strongly continuous representation**.

❖

The operator semigroup that we consider is actually an algebraic representation, with the monoid $M = [0, \infty)$ and addition as the binary operation.

Definition 2.3 (Operator Semigroups)

- A family $\{T_t, t \geq 0\}$ of bounded linear operators on $\mathcal{X}$ is a **semigroup** if
  1. $T_0 = Id$
  2. $T_{s+t} = T_s T_t$, $\forall s, t \geq 0$.
- $\{T_t\}$ is **strongly continuous semigroup (SCS)** if $\lim_{t \downarrow 0} T_t f = f$, $\forall f \in \mathcal{X}$.
  (notice this is simply right continuity at 0)
- $\{T_t\}$ is **contraction semigroup** if $\|T_t\| \leq 1$, $\forall t \geq 0$.

❖

A useful inequality for SCS:

Proposition 2.4 (Growth Bound)
Let $\{T_t\}$ be SCS on $\mathcal{X}$, then $\exists M \geq 1, w \geq 0$ s.t.

$$\|T_t\| \leq M e^{wt}, \quad t \geq 0.$$

❖

In the definition of SCS, it should be $\lim_{s \to t} T_s f = T_t f$, $\forall f \in \mathcal{X}, t \geq 0$.
Why is right continuity at 0 sufficient for its definition?

Corollary 2.5 (EK1.2)
Let $\{T_t\}$ be SCS, then $\forall f \in \mathcal{X}$, $t \mapsto T_t f$ is continuous.

3 Examples

3.1 Translation Semigroups

Take $\mathcal{X} = C_b(\mathbb{R})$.

For fixed speed $c > 0$, define

$$T_t f(x) = f(x + ct)$$

$\{T_t\}$ is not SCS.

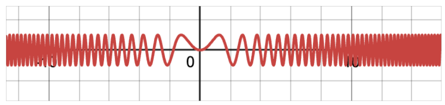

Figure 2: title

To make $\{T_t\}$ a SCS, need to take $\mathcal{X} = C_0(\mathbb{R})$.

Also note that $\{T_t\}$ is not uniformly continuous! ($T_s \not\to T_t$ in operator norm, which is too strong).

- for each $t > 0$, find $f \in C_0$ with $\|f\| \leq 1$ but $|f(0) - f(-ct)| \geq 1$. Then we see $\|T_t - T_0\| \geq 1$, $\forall t > 0$.

3.2 Flow Semigroups

Consider ODE

$$\begin{cases} X'(t) = F(X(t)), \\ X(0) = x \in \mathbb{R}^d, \end{cases}$$

Assume F is nice enough (for example Lipschitz continuous), such that the ODE has unique solution, denoted X^x.

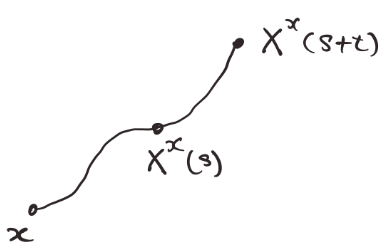

Figure 3: title

By uniqueness, we have

$$X^{X^x(s)}(t) = X^x(s + t)$$

Let

$$T_t f(x) = f(X^x(t))$$

3.3 Heat Semigroup

3.4 Poisson Semigroup

Take $\mathcal{X} = l^\infty$, the space of bounded sequences $\{x_n\}_{n\in\mathbb{N}_0}$.
For $t \geq 0$, define $T_t : l^\infty \to l^\infty$ by

$$(T_t f)_n := \sum_{m\in\mathbb{N}_0} e^{-\lambda t} \frac{(\lambda t)^m}{m!} f_{n+m} = \mathbb{E}\left[f_{n+Pois(\lambda t)}\right]$$

4 Infinitesimal Generators

Definition 4.1 (Operator on Banach Spaces)

- An **operator** on Banach space $\mathcal{X}$ is a pair $(A, \mathcal{D}(A))$ where
  - $A : \mathcal{D}(A) \to \mathcal{X}$ is a linear map (not necessarily cts)
  - $\mathcal{D}(A)$ is a subspace of $\mathcal{X}$.
- The **graph** of A is the linear subspace

$$\Gamma(A) = \{(f, g) \in \mathcal{X} \times \mathcal{X} : f \in \mathcal{D}(A), g = Af\}.$$

  And the **graph norm** is the map $\|f\|_{\mathcal{D}(A)} := \|f\| + \|Af\|$, $f \in \mathcal{D}(A)$.
- The operator $(A, \mathcal{D}(A))$ is
  - **closed** if its graph $\Gamma(A)$ is closed.
  - **closable** if the closure of $\Gamma(A)$ defines the graph of a operator $\bar{A} : \mathcal{D}(\bar{A}) \to \mathcal{X}$, (which is necessarily unique and closed).

Recall some basic functional analysis facts:

Proposition 4.2

1. *A is closable $\iff$ for all sequences $\{f_n\}_n \subset \mathcal{D}(A)$ s.t. $f_n \to 0$, the existence of limit $Af_n \to g$ implies $g = 0$.*
2. *If A is closed, then $(D(A), \|\cdot\|_{\mathcal{D}(A)})$ is a Banach space.*
3. *(Closed Graph Thm) If A is closed and $\mathcal{D}(A) = \mathcal{X}$, then A is bounded (i.e. cts).*

Example 4.3 (Dense but not bdd operators)

Definition 4.4 (Infinitesimal Generator)
The **infinitesimal generator** of SCS $(T_t)_{t \geq 0}$ is the operator $(A, \mathcal{D}(A))$ defined by

$$Af := \lim_{t \to 0} \frac{T_t f - f}{t}, \quad f \in \mathcal{D}(A),$$

where the domain $\mathcal{D}(A)$ is all $f \in \mathcal{X}$ for which the limit exists.

Example 4.5 (Generator for Poisson Semigroup)
Taking the derivative $\frac{d}{dt}(T_t A)_n$ formally yields

$$(Af)_n = \lambda(f_{n+1} - f_n), \quad f \in l^\infty.$$

- A is a scaled difference operator.
- $\mathcal{D}(A)$ is the entire space l^∞.
- Hence A is bounded and $T_t = e^{tA}$.

Example 4.6 (Generator for Flow Semigroup)

$$Af(x) = Df(x) \cdot F(x)$$

- A is a differential operator of first order with state dependent coefficient $F(x)$.
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Example 4.7 (Generator for Heat Semigroup)
Recall the representation $T_t f = \mathbb{E}[f(x + B_t)]$.

The generator of the heat semigroup is the Laplacian:

$$Af = \frac{1}{2} \nabla f, \quad f \in C_c^2(\mathbb{R}^d).$$

4.1 Integrating Banach-valued functions

specifically, want to integrate continuous maps $[a, b] \to \mathcal{X}$. Turns out, the theory closely parallels Riemann integrals.

Lemma 4.8
Let $f : [0, t] \to \mathcal{X}$ be continuous, T be BDD linear operator on $\mathcal{X}$. Then the following holds:

1. $T \int_0^t f(u)\mathrm{d}u = \int_0^t Tf(u)\mathrm{d}u$.
2. $\left\| \int_0^t f(u)\mathrm{d}u \right\| \leq \int_0^t \|f(u)\| \,\mathrm{d}u$.

4.2 Main Results on Generators of SCS

Proposition 4.9
Let $(A, \mathcal{D}(A))$ be generator of SCS $\{T_t\}_t$ on $\mathcal{L}(\mathcal{X})$. Then

1. *for $f \in \mathcal{X}$, $t > 0$,*
   (a) $\int_0^t T_u f \mathrm{d}u \in \mathcal{D}(A)$ *and*
   (b) $A \int_0^t T_u f \mathrm{d}u = T_t f - f$.
2. $\mathcal{D}(A)$ *is dense in* $\mathcal{X}$.
3. *For $t \geq 0$, $f \in \mathcal{D}(A)$*
   (a) $T_t f \in \mathcal{D}(A)$ *and*
   (b) $T_t A f = A T_t f$ *and*
   (c) $T_t f - f = \int_0^t A T_u f \mathrm{d}u$.
4. *A is closed.*

Corollary 4.10 (Generator has unique semigroup)
Let $\{T_t\}$ and $\{S_t\}$ be two semigroups on $\mathcal{L}(\mathcal{X})$ with the same generator $(A, \mathcal{D}(A))$. Then $T = S$.

4.3 Core of Generator

Often, the full domain $\mathcal{D}(A)$ is hard to describe, instead, work with the core of the generator.

Definition 4.11
We say $D \subset \mathcal{D}(A)$ is a **core** of generator A if D is dense in $\mathcal{D}(A)$ in the graph norm.

Proposition 4.12
The following are cores:

1. $\mathcal{D}(A^\infty) := \bigcap_n \mathcal{D}(A^n)$
2. *Any linear subspace D of $\mathcal{D}(A)$ such that*
   - *D is norm-dense in $\mathcal{X}$ and*
   - *$T_t D \subseteq D$, for all $t \geq 0$.*

5 Resolvent Operators

5.1 Basic Definition and Properties

Definition 5.1
Let $(A, \mathcal{D}(A))$ be an operator.

- $\lambda \in \mathbb{R}$ is in the **resolvent set** $\rho(A)$ if the operator $\lambda I - A : \mathcal{D}(A) \to \mathcal{X}$
  1. is bijective, and
  2. its inverse $(\lambda I - A)^{-1} : \mathcal{X} \to \mathcal{D}(A) \subseteq \mathcal{X}$ is continuous (bdd).
- Let $\lambda \in \rho(A)$, the operator $R(\lambda, A) = (\lambda I - A)^{-1}$ is the **resolvent** of A at λ.
- The spectrum of A is the set $\sigma(A) = \mathbb{R} \setminus \rho(A)$.

Remark 5.2

- When $\{T_t\}$ is SCS, recall that A is closed. Hence by the closed graph theorem, requirement 2 in the definition of resolvent set is unnecessary.
- Aware there are alternative definitions that use the density of the range of $\lambda - A$.

Proposition 5.3 (The Resolvent Equation)
Let A be an operator and $\lambda, \mu \in \rho(A)$. Then

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A).$$

In particular, $\{R(\lambda, A) : \lambda \in \rho(A)\}$ is a commutative family.

Proposition 5.4 (Analyticity of the Resolvent)
Let $\lambda_0 \in \rho(A)$, $\lambda \in \mathbb{R}$ s.t.

$$|\lambda - \lambda_0| < \|R(\lambda_0, A)\|^{-1},$$

then $\lambda \in \rho(A)$ and $\rho(A)$ is open.
Further, $\lambda \mapsto R(\lambda, A)$ is analytic on $\rho(A)$ and

$$R(\lambda, A) = \sum_{k=0}^{\infty} (-1)^k (\lambda - \lambda_0)^k R^{k+1}(\lambda_0, A).$$

5.2 Resolvent as Laplace Transform

Use the convenient heuristic $T_t = e^{tA}$,

$$\int_0^\infty e^{-\lambda t} T_t \mathrm{d}t = \int_0^\infty e^{-\lambda t + At} \mathrm{d}t = \frac{1}{\lambda - A} = R(\lambda, A).$$

Proposition 5.5
let $(A, \mathcal{D}(A))$ be generator of SCS $\{T_t\}_t$ s.t. $\|T_t\| \leq M e^{ct}$ for some $M > 0$ and $c \in \mathbb{R}$.
Then

- *$(c, \infty) \subseteq \rho(A)$ and*
- *$R(\lambda, A) f = \int_0^\infty e^{-\lambda u} T_u f \mathrm{d}u$ for $\lambda > 0$.*

And hence $\|(\lambda - c) R(\lambda, A)\| \leq M$, for all $\lambda > c$.

6 Hille-Yosida Theorem