Large Deviations

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1 Basic Definitions

Setting

- $(\Omega, \mathcal{F}, P)$: probabilty space.
- $\mathcal{X}$: a Polish space, i.e. complete separable metric space with metric d.
- $\{X_n\}_n$: sequence of random variables defined on $(\Omega, \mathcal{F}, P)$ and taking values on $\mathcal{X}$.

Definition 1.1 (Rate Function)

- The mapping $I : \mathcal{X} \to [0, \infty]$ is a **rate function** if it is lower semi-continuous.
- I is a **good rate function** if in addition, the level sets $\Psi_I(\alpha)$ are compact for all α.

Remark 1.2

- Lower semi-continuous means: $f(x) \leq \lim_{y \to x} f(y)$ for all x.
- Lower semi-continuous maps has closed level sets.
- The rate function being good implies its infimum is achieved on closed set.

Definition 1.3 (Large Deviation Principle)
We say $\{X_n\}_n$ satisfies a **large deviation principle** (LDP) with a (good) rate function $I : S \to [0, \infty]$ if the following hold:

- (LD lower bound) For each open set $A \in \mathcal{X}$
$$-\inf_{x \in A} I(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log P(X_n \in A)$$
- (LD upper bound) For each closed set $B \in \mathcal{X}$
$$\limsup_{n \to \infty} \frac{1}{n} \log P(X_n \in B) \leq -\inf_{x \in B} I(x)$$

2 LDP in Finite Dimensions

2.1 Chernoff

2.2 Sanov

2.3 Cramér

Theorem 2.1 (Cramér's Theorem)
The process $n^{-1} S_n$ obeys an LDP with rate function Λ^*.

2.4 Ellis-Gartner

Extend Cramér's Theorem to non-IID cases.

3 General Principles

4 Large Deviation for Processes

Definition 4.1 (Modulus of continuity in $\mathcal{C}$)
The **modulus of continuity** for $x \in \mathcal{C}[0,1]$ is defined as

$$w_x(\delta) = w(x,\delta) = \sup_{|s-t|<\delta} |x(s) - x(t)|, \quad \delta \in [0,1]$$

Definition 4.2 (Modulus of continuity in $\mathcal{D}$)
The **modulus of continuity** for $x \in \mathcal{D}[0,1]$ is defined as

$$w'_x(\delta) = w'(x,\delta) = \inf_{t_i} \max_{1 \le i \le v} w_x[t_{i-1}, t_i),$$

where infimum is taken over δ-spase sets $\{t_i\}$, i.e. $\min_{1\le i\le v}(t_i - t_{i-1}) > \delta$.

Theorem 4.3 (Exponential tightness in $\mathcal{C}$)
The family $\{X_n\}_n \subset \mathcal{C}$ is exponentially tight iff

1.
$$\lim_{A\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log P(|X_n(0))| \ge A) = -\infty$$
2. *for any $\eta > 0$ and any $T > 0$,*
$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P\left(w_T(X_n, \delta) \ge \eta\right) = -\infty.$$

Theorem 4.4 (Exponential tightness in $\mathcal{D}$)
The family $\{X_n\}_n \subset \mathcal{D}$ is exponentially tight iff

1. *For any $T > 0$,*
$$\lim_{A\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log P(\|X_n\|_T^* \ge A) = -\infty$$
2. *for any $\eta > 0$ and any $T > 0$,*
$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P\left(w'_T(X_n, \delta) \ge \eta\right) = -\infty.$$

5 Formulation via Laplace Principle

Theorem 5.1 (Varadhan)
Assume $\{X^n\}$ satisfies LDP on $\mathcal{X}$ with rate function I. Then $\forall h \in C_b(\mathcal{X})$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} \exp\left(-nh(X^n)\right) = -\inf_{x\in\mathcal{X}} \left[h(x) + I(x)\right]. \tag{5.1}$$

Remark 5.2

1. (DZ 4.3.1) The condition $h \in C_b(\mathcal{X})$ can be weakened to either
   - the tail condition
$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[e^{nh(X^n)} 1_{\{h(X^n)\ge M\}}\right]$$

- or the moment condition for some $\gamma > 1$

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[e^{nh(X^n)}\right] < \infty$$

2. More precisely, LD upper bound implies

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E} \exp\left(-nh(X^n)\right) \leq -\inf_{x\in\mathcal{X}} \left[h(x) + I(x)\right].$$

   and LD lower bound implies

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E} \exp\left(-nh(X^n)\right) \geq -\inf_{x\in\mathcal{X}} \left[h(x) + I(x)\right].$$