Title

1. Viscosity Solutions

1.1. Abstract Dynamic Programming Principle. Setup

- Σ be a closed subset of Banach space
- $\mathcal{C}$ be family of functions on Σ that are closed under addition.
  - When $\Sigma \subset \mathbb{R}^n$, require $\mathcal{C}$ contains $\mathcal{M}(\Sigma) \cap C_p(\Sigma)$.

Nonlinear operator semigroup

$$V(t,x) = \mathcal{T}_{t,r} V(r,\cdot)(x) \tag{1.1}$$

Assumption 1.1 (Semigroup)**.**

Assumption 1.2. There exist

- $\Sigma' \subset \Sigma$
- family of test functions $\mathcal{D} \subset C([t_0,t_1),\Sigma')$
- family of nonlinear operators $\{\mathcal{G}_t\}$ called generators such that
  1. $\frac{\partial}{\partial t} w(t,x)$ and $\mathcal{G}_t w(t,\cdot)(x)$ continuous on $(t,x) \in Q$, and $w(t,\cdot) \in \mathcal{C}$ for all $t \in [t_0,t_1]$.
  2. $w, \tilde{w} \in \mathcal{D}$, $\lambda \geq 0$ implies that $w + \tilde{w}, \lambda w \in \mathcal{D}$
  3. $\lim_{h\to 0} \frac{1}{h}[\mathcal{T}_{t,t+h} w(t+h,\cdot)(x) - w(t,x)] = \frac{\partial}{\partial t} w(t,x) - \mathcal{G}_t w(t,\cdot)(x)$.

The dynamic programming equation has the form:

$$-\frac{\partial}{\partial t} V(t,x) + \mathcal{G}_t V(t,\cdot)(x) = 0, \quad (t,x) \in Q. \tag{1.2}$$

Definition 1.3 (Classical Solutions)**.** $V \in \mathcal{D}$ is called classical solution to (1.2) if it satisfies (1.2) at all $(t,x) \in Q$.

1.2. On Definitions of Viscosity Solutions.

Definition 1.4 (Viscosity Solution for Abstract DPE)**.** Let $W \in C([t_0,t_1] \times \Sigma)$. Then

- W is viscosity subsolution of (1.2) in Q if $\forall w \in \mathcal{D}$, and $(t^*,x^*) \in Q$ that maximizes $W - w$ with $W(t^*,x^*) = w(t^*,x^*)$, we have
$$-\frac{\partial}{\partial t} w(t^*,x^*) + \mathcal{G}_{t^*} w(t^*,\cdot)(x^*) \leq 0.$$
- W is viscosity supersolution of (1.2) in Q if $\forall w \in \mathcal{D}$, and $(t^*,x^*) \in Q$ that minimizes $W - w$ with $W(t^*,x^*) = w(t^*,x^*)$, we have
$$-\frac{\partial}{\partial t} w(t^*,x^*) + \mathcal{G}_{t^*} w(t^*,\cdot)(x^*) \geq 0.$$
- W is viscosity solution if it is both viscosity sub and super solution.

Definition 1.5 (Maximum Principle For Operator)**.** A general operator $\mathcal{G}_t$ has the maximum principal if for all $t \in [t_0, t_1]$, ψ, ϕ in domain of $\mathcal{G}_t$, we have

$$\mathcal{G}_t\phi(\bar{x}) \geq \mathcal{G}_t\psi(\bar{x})$$

for all $\bar{x}$ that satisfies

$$\bar{x} \in \arg\max\{(\phi - \psi)(x) | x \in \Sigma\} \text{ and } \phi(\bar{x}) = \psi(\bar{x})$$

Remark 1.6. The maximum principle for operator $\mathcal{G}_t$ holds if

- $\mathcal{G}_t$ is infinitesimal generator of $\mathcal{T}_{t,r}$
- $\mathcal{G}_t\phi(x) = H(t, x, D\phi(x))$, where H is continuous function.
- (and only if) $\mathcal{G}_t\phi(x) = F(t, x, D\phi(x), D^2\phi(x), \phi(x))$, where F is continuous and elliptic (see definition below). *For a proof see [FS] II.4 p69.*

Let O open subset of $\mathbb{R}^n$, $Q = [t_0, t_1) \times O$, $W = C(\bar{Q})$, F continuous and elliptic. Consider the PDE

$$-\frac{\partial}{\partial t}W(t, x) + F(t, x, D_xW(t, x), D_x^2W(t, x), W(t, x)) = 0.$$

Definition 1.7 (Viscosity Solution for 2nd order nonlinear PDE's)**.**

1.3. DPP and Viscosity Solutions.

Theorem 1.8 (Continuous Values Functions are Viscosity Solutions)**.** *Let assumptions 1.1 and 1.2 hold. Consider the value function from the abstract DPP*

$$V(t, x) = \mathcal{T}_{t,t_1}\psi(x).$$

If $V \in C(\bar{Q})$, then V is viscosity solution to the DPE (1.2) in Q.

Proof. Sub and supersolution are proved the same way. Use monotonicity assumptions for the semigroup. See [FS Thm 5.1]. ☐

Lemma 1.9 (Test Functions as Solutions)**.** *Suppose $W \in \mathcal{D}$. Then W is viscosity of DPE (1.2) in $Q \iff$ it is classical solution of DPE.*

Proof. For necessity, take test function $w \equiv W$. For sufficiency, use the assumptions on the generator and monotonicity. See [FS Lemma 5.1]. ☐

1.4. Results for Partial Differential Operator.

Assumption 1.10 (On Space of Value Functions and Test Functions)**.** Let $\Sigma' = O$ be open subset of $\mathbb{R}^n$ and $\Sigma = \bar{O}$. Also assume

(1) $C_p(\bar{O}) \cap \mathcal{M}(\bar{O}) \subset \mathcal{C}$.
(2) $C_p^\infty(\bar{Q}) \cap \mathcal{M}(\bar{Q}) \subset \mathcal{D}$.

Consider the case when the generator is given by a partial differential operator

$$\mathcal{G}_t\phi(x) = F(t, x, D\phi(x), D^2\phi(x), \phi(x))$$

Definition 1.11 (Ellipticity)**.**

Lemma 1.12 (Consider only strict extremums)**.** *Let Assumption 1.10 hold, In the definitions of viscosity solutions, it suffice to only consider strict extrema of $W - w$.*

Proof. Consider test function $w^\varepsilon = w - \varepsilon\xi$, where

$$\xi(t,x) = e^{-|t-\bar{t}|^2+|x-\bar{x}|^2} - 1, \quad (t,x) \in \bar{Q}.$$

See [FS Lemma 6.1]. $\square$

Theorem 1.13. *Let Assumption 1.10 hold. Let* $W \in C_p(\bar{Q}) \cap \mathcal{M}(\bar{Q})$, $\mathcal{D} \subset C^{1,2}(Q)$, *and*

$$\mathcal{G}_t\phi(x) = F(t,x,D\phi(x),D^2\phi(x),\phi(x))$$

Then W *is a viscosity subsolution (or a supersolution) of*

$$-\frac{\partial}{\partial t}V(t,x) + (\mathcal{G}_t V(t,\cdot))(x) = 0, \quad (t,x) \in Q.$$

in the sense of Definition 1.4, $\iff$ W *is a viscosity solution (or a supersolution, respectively) in the sense of Definition 1.7.*

Proof. See [FS Theorem 6.1]. $\square$

Remark 1.14. The main takeaway from previous theorem and its proof is that

- When showing viscosity property, we have the freedom to choose the test function w from $\mathcal{D}$ or $C^\infty(Q)$ or any other dense subset of $C^{1,2}(Q)$ (for example $C^{1,2}(\bar{Q})$)
- The equivalence of definition also holds for first order partial differential operator

  $$\mathcal{G}_t\phi(x) = H(t,x,D\phi(x)), \quad (t,x) \in \bar{Q},\ \phi \in C^1(\bar{O}).$$

  Similarly we can choose test function w from $\mathcal{D}$ or $C^\infty(Q)$ or any other dense subset of $C^1(Q)$

2. Differential Games

2.1. Fleming & Soner.

Appendix A. Notations

Below are set of notations used in this note:

- Σ a Banach space
- $\mathcal{M}(\Sigma)$ = set of real-valued functions which are bounded below.
- $C_p(\Sigma)$ = set of all continuous, real-valued functions that are polynomially growing. re