Poisson Random Measure

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1 Random Measure

Let

- $(\Omega, \mathcal{H}, \mathbb{P})$ be probability space
- $(E, \mathcal{E})$ be a measurable space.

Definition 1.1 (Random Measure)
A **random measure** M is a transition kernel from $(\Omega, \mathcal{H}) \to (E, \mathcal{E})$.

That is, $M : \Omega \times \mathcal{E} \to \mathbb{R}_+$ satisfies

- $\forall A \in \mathcal{E}$, $M(\cdot, A) \equiv M(A)$ is a random variable.
- $\forall \omega \in \Omega$, $M(\omega, \cdot) \equiv M_\omega$ is a measure on $(E, \mathcal{E})$.

Example 1.2 (Poisson Process)
Let $E = \mathbb{R}_+$ be time, then $[0, t] \in \mathcal{E}$ is time interval. Consider a poisson process N with rate λ to model arrivals. One can define the so called Poisson Random Measure

- $N(t) \equiv N(\cdot, [0, t])$ is a poisson random variable with rate λt.
- If we fixed a path for the poisson process, $N(\omega, I)$ is a counting measure, counting the number of arrival over the time interval I.

Let

- M be a random measure on $(E, \mathcal{E})$.
- $\mathcal{E}_+$ be the set of positive $\mathcal{E}$-measurable functions. (i.e. $f^{-1}(B) \in \mathcal{E}$ for all Borel sets B)
  This will serve as the space of test functions.

Definition 1.3 (Mean Measure and Integral)
For all $A \in \mathcal{E}$, the **mean measure** $\mu(A)$ is a measure on $(E, \mathcal{E})$ defined by

$$\mu(A) \equiv \mathbb{E}[M(A)] = \int_\Omega M(\omega, A)\mathbb{P}(\mathrm{d}\omega)$$

For each $f \in \mathcal{E}_+$, we can define a random variable Mf by

$$Mf(\omega) = \int_E f(x) M(\omega, dx)$$

Denote

$$\mu f \equiv \int_E f(x) d\mu(x)$$

Proposition 1.4 (Relationship between mean measure and Integral)

$$\mathbb{E}[Mf] = \mu f, \quad \forall f \in \mathcal{E}_+.$$