Single Server Queue

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1 Basic Quantities

Stochastic primitives: $\{u_i\}_{i\in\mathbb{N}}, \{v_i\}_{i\in\mathbb{N}}$ for interarrival and service times.

2 1D Reflection

2.1 Reflected Brownian Motion

$$Z = \mathsf{RBM}_x(\theta, \sigma^2)$$

2.2 Stationary Distribution of RBM

Definition 2.1
We say π is a stationary distribution if for all BDD, CTS function f, we have

$$\mathbb{E}_\pi\left[f(Z(t))\right] = \int_0^\infty \mathbb{E}_z\left[f(Z(t))\right] \mathrm{d}z = \mathbb{E}_\pi\left[f(Z(0))\right].$$

Theorem 2.2
Z has SD iff $\theta < 0$, in which case

$$Z(\infty) \sim \mathsf{Exp}(\eta), \quad \eta = -\frac{2\theta}{\sigma^2}.$$

The Basic Adjoint Relationship (BAR) Approach

BAR is used to characterize the stationary distribution of Markov processes.

See Chen, Hong. "Basic adjoint relation for transient and stationary analysis of some Markov processes", 1999

Basic Procedure:

1. Obtain Ito Formula in integral form.
2. Take expectation with respect to the stationary distribution.
3. Take advantage of martingales and derive an expression for the stationary distribution.

Proof for SD of RBM when $\theta < 0$. SDE for the $Z = \mathsf{RBM}_x(\theta, \sigma^2)$:

$$dZ(t) = \theta dt + \sigma dW(t) + dY(t)$$

Ito's Formula:

$$f(Z_t) - f(Z_0) = \int_0^t \left[\frac{1}{2}\sigma^2 f''(Z_s) + \theta f'(Z_s)\right] \mathrm{d}s + \sigma \int_0^t f'(Z_s)\mathrm{d}W_s + \int_0^t f'(Z_s)\mathrm{d}Y_s$$

Take expectation wrt SD π on both sides, we obtain the BAR:

$$\begin{aligned} 0 = \mathbb{E}_\pi[f(Z_t) - f(Z_0)] &= \mathbb{E}_\pi \int_0^t \left[\frac{1}{2}\sigma^2 f''(Z_s) + \theta f'(Z_s)\right] \mathrm{d}s + f'(0)\mathbb{E}_\pi\left[Y_t\right] \\ &= t \cdot \int_0^\infty \left[\frac{1}{2}\sigma^2 f''(z) + \theta f'(z)\right] \mathrm{d}\pi(z) + f'(0)\mathbb{E}_\pi\left[Y_t\right] \end{aligned}$$

It is worth recalling the generator for 1D BM: $\frac{1}{2}\sigma^2 f'' + \theta f'$. To solve the BAR, we need to take clever choices of f:

1. Take $f(z) = z$, obtain $\mathbb{E}_\pi Y_t = -\theta t$. Then BAR becomes

$$0 = \int_0^\infty \left[\frac{1}{2}\sigma^2 f''(z) + \theta f'(z)\right] \mathrm{d}\pi(z) - \theta f'(0)$$

2. Take $f(z) = e^{-\alpha z}$, obtain the Laplace transform of SD π:

$$\mathbb{E}\left[e^{-\alpha Z_\infty}\right] = \frac{-\theta}{\frac{1}{2}\alpha\sigma^2 - \theta} = \frac{\eta}{\alpha + \eta}$$

This matches the Laplace transform of the exponential distribution with parameter η. $\square$