Operator Semigroups

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1 Operator Semigroups

- Let E be a locally compact Hausdorf space. (that is $\forall x \in E$, x has a compact neighborhood).
- Let $(\mathcal{X}, \|\cdot\|)$ be a Banach Space. And denote $\mathcal{L}(\mathcal{X})$ to be the space of bounded linear operators on $\mathcal{X}$.

Topologies on $\mathcal{L}(\mathcal{X})$

(1) Weak Operator Topology (WOT)
  - Equip $\mathcal{X}$ with the weak topolgy. (recall weak convergence: $f^* \in \mathcal{X}^*$, $\langle f^*, T_n f\rangle \to \langle f^*, Tf\rangle$)
  - The coarsest topology on $\mathcal{L}(\mathcal{X})$ such that all maps: $T \mapsto Tf$ are continuous, where $T \in \mathcal{L}(\mathcal{X})$ and $f \in \mathcal{X}$.

(2) Strong Operator Topology (SOT)
  - Equip $\mathcal{X}$ with the strong topolgy.
  - The coarsest topology on $\mathcal{L}(\mathcal{X})$ such that all maps: $T \mapsto Tf$ are continuous, where $T \in \mathcal{L}(\mathcal{X})$ and $f \in \mathcal{X}$.
  - In other words, this is the topology with pointwise convergence: $T_n \to T$ if $T_n f \to Tf$ in $\mathcal{X}$, for all $f \in \mathcal{X}$.

(3) Norm Topology (Uniform Topology)
  - Topology induced by the operator norm ($\|T\| = \sup\{\|Tf\| : \|f\| \leq 1\}$).
  - $T_n \to T$ if $\|T_n - T\| \to 0$.

Definition 1.1 (Monoid)**.**

- An algebraic semigroup is a pairing $(M, \circ)$, where M is an non-empty set and $\circ$ is an associative binary operation $M \times M \to M$.
- M is a monoid if it is an algebraic semigroup and have an unit element. i.e. $\exists e \in M$ s.t. $e \circ a = a \circ e = a$, $\forall a \in M$.
- A topological monoid is a monoid with a topology, in which $\circ$ is continuous.

Definition 1.2 (Algebraic Representation)**.** let M be a monoid. A map $T : M \to \mathcal{L}(\mathcal{X})$ is called an algebraic representation if

(1) $T(e) = Id$
(2) $T(a \circ b) = T(a)T(b)$, for all $a, b \in M$.

If in addition, M is a topological monoid and $a \mapsto T(a)$ is continuous when $\mathcal{L}(\mathcal{X})$ is given the strong operator topology, then we say T is strongly continuous represenatiton.