Operator Semigroups

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1 Spaces and Topologies

- Let E be a locally compact Hausdorf space. (that is $\forall x \in E$, x has a compact neighborhood).
- $(\mathcal{X}, \|\cdot\|)$: Banach Space.
- $\mathcal{L}(\mathcal{X})$: space of bounded linear operators on $\mathcal{X}$.

Topologies on $\mathcal{L}(\mathcal{X})$

(1) Weak Operator Topology (WOT)
  - coarest topology s.t. $\forall f \in \mathcal{X}$, the map $\mathcal{L}(\mathcal{X}) \ni T \mapsto T(f) \in \mathcal{X}$ is continuous, when X is equipped with the weak topology.
  - In other words, $T_n \to T$ if $\langle f^*, T_n f\rangle \to \langle f^*, Tf\rangle$, $\forall f^* \in \mathcal{X}^*$

(2) Strong Operator Topology (SOT)
  - coarest topology s.t. $\forall f \in \mathcal{X}$, the map $\mathcal{L}(\mathcal{X}) \ni T \mapsto T(f) \in \mathcal{X}$ is continuous, when X is equipped with the strong topology.
  - In other words, $T_n \to T$ if $T_n f \to Tf$ in $\mathcal{X}$, for all $f \in \mathcal{X}$ (topology of pointwise convergence).

(3) Norm Topology (Uniform Topology)
  - Topology induced by the operator norm ($\|T\| = \sup\{\|Tf\| : \|f\| \leq 1\}$).
  - $T_n \to T$ if $\|T_n - T\| \to 0$.

It turns out for semigroups, weak and strong topology are equivalent in some sense. The norm topology is usually to strong. So we shall work with SOT.

Choices for $\mathcal{X}$:

Shall use the sup norm

- $C_b(E)$: cts bdd functions on E
- $C_0(E)$: cts bdd functions on E that vanish at ∞,
  i.e. $\{f \in C_b(E) : \lim_{\|x\| \to \infty} f(x) = 0\}$.

Often, $C_b(E)$ is not enough to get SCS, need $C_0(E)$ instead.

2 Definition of Semigroups

Definition 2.1 (Monoid)
- An **algebraic semigroup** is a pairing $(M, \circ)$, where

- M: nonempty set
- $\circ$ is associative binary operation $M \times M \to M$.

- M is a **monoid** if it is an algebraic semigroup and have an unit element.
  i.e. $\exists e \in M$ s.t. $e \circ a = a \circ e = a$, $\forall a \in M$.
- A **topological monoid** is a monoid with a topology, in which $\circ$ is continuous.

Definition 2.2 (Algebraic Representation)
let M be a monoid. A map $T : M \to \mathcal{L}(\mathcal{X})$ is called an **algebraic representation** if

(1) $T(e) = Id$
(2) $T(a \circ b) = T(a)T(b)$, for all $a, b \in M$.

If in addition, M is a topological monoid and $a \mapsto T(a)$ is continuous when $\mathcal{L}(\mathcal{X})$ is given the strong operator topology, then we say T is **strongly continuous representation**.

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Definition 2.3 (Operator Semigroups)

- A family $\{T_t, t \geq 0\}$ of bounded linear operators on $\mathcal{X}$ is a **semigroup** if
  1. $T_0 = Id$
  2. $T_{s+t} = T_s T_t$, $\forall s, t \geq 0$.
- $\{T_t\}$ is **strongly continuous semigroup (SCS)** if $\lim_{t \downarrow 0} T_t f = f$, $\forall f \in \mathcal{X}$.
  (notice this is simply right continuity at 0)
- $\{T_t\}$ is **contraction semigroup** if $\|T_t\| \leq 1$, $\forall t \geq 0$.

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A useful inequality for SCS:

Proposition 2.4
Let $\{T_t\}$ be SCS on $\mathcal{X}$, then $\exists M \geq 1, w \geq 0$ s.t.

$$\|T_t\| \leq M e^{wt}, \quad t \geq 0.$$

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In the definition of SCS, it should be $\lim_{s \to t} T_s f = T_t f$, $\forall f \in \mathcal{X}, t \geq 0$.
Why is right continuity at 0 sufficient for its definition?

Corollary 2.5 (EK1.2)
Let $\{T_t\}$ be SCS, then $\forall f \in \mathcal{X}$, $t \mapsto T_t f$ is continuous.

3 Examples

3.1 Translation Semigroups

Take $\mathcal{X} = C_b(\mathbb{R})$.

For fixed speed $c > 0$, define

$$T_t f(x) = f(x + ct)$$

$\{T_t\}$ is not SCS.

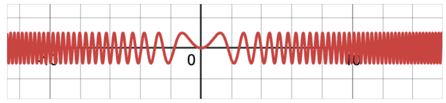

Figure 1: title

To make $\{T_t\}$ a SCS, need to take $\mathcal{X} = C_0(\mathbb{R})$.

Also note that $\{T_t\}$ is not uniformly continuous! ($T_s \not\to T_t$ in operator norm, which is too strong).

- for each $t > 0$, find $f \in C_0$ with $\|f\| \leq 1$ but $|f(0) - f(-ct)| \geq 1$. Then we see $\|T_t - T_0\| \geq 1$, $\forall t > 0$.

3.2 Flow Semigroups

Consider ODE

$$\begin{cases} X'(t) = F(X(t)), \\ X(0) = x \in \mathbb{R}^d, \end{cases}$$

Assume F is nice enough (for example Lipschitz continuous), such that the ODE has unique solution, denoted X^x.

By uniqueness, we have

$$X^{X^x(s)}(t) = X^x(s+t)$$

Let

$$T_t f(x) = f(X^x(t))$$

3.3 Heat Semigroup

3.4 Poisson Semigroup

Take $\mathcal{X} = l^\infty$, the space of bounded sequences $\{x_n\}_{n \in \mathbb{N}_0}$.
For $t \geq 0$, define $T_t : l^\infty \to l^\infty$ by

$$(T_t f)_n := \sum_{m \in \mathbb{N}_0} e^{\lambda t} \frac{(\lambda t)^m}{m!} f_{n+m} = \mathbb{E}\left[f_{n+Pois(\lambda t)}\right]$$

4 Infinitesimal Generators

Definition 4.1 (Operator on Banach Spaces)

- An **operator** on Banach space $\mathcal{X}$ is a pair $(A, \mathcal{D}(A))$ where
  - $A : \mathcal{D}(A) \to \mathcal{X}$ is a linear map (not necessarily cts)
  - $\mathcal{D}(A)$ is a subspace of $\mathcal{X}$.
- The **graph** of A is the linear subspace

$$\Gamma(A) = \{(f, g) \in \mathcal{X} \times \mathcal{X} : f \in \mathcal{D}(A), g = Af\}.$$

  And the **graph norm** is the map $\|f\|_{\mathcal{D}(A)} := \|f\| + \|Af\|$, $f \in \mathcal{D}(A)$.

- The operator $(A, \mathcal{D}(A))$ is
  - **closed** if its graph $\Gamma(A)$ is closed.
  - **closable** if the closure of $\Gamma(A)$ defines the graph of a operator $\bar{A} : \mathcal{D}(\bar{A}) \to \mathcal{X}$, (which is necessarily unique and closed).

Recall some basic functional analysis facts:

Proposition 4.2

1. *A is closable $\iff$ for all sequences $\{f_n\}_n \subset \mathcal{D}(A)$ s.t. $f_n \to 0$, the existence of limit $Af_n \to g$ implies $g = 0$.*
2. *If A is closed, then $(D(A), \|\cdot\|_{\mathcal{D}(A)})$ is a Banach space.*
3. *(Closed Graph Thm) If A is closed and $\mathcal{D}(A) = \mathcal{X}$, then A is bounded (i.e. cts).*

Example 4.3 (Dense but not bdd operators)

Definition 4.4 (Infinitesimal Generator)
The **infinitesimal generator** of SCS $(T_t)_{t\geq 0}$ is the operator $(A, \mathcal{D}(A))$ defined by

$$Af := \lim_{t\to 0} \frac{T_t f - f}{t}, \quad f \in \mathcal{D}(A),$$

where the domain $\mathcal{D}(A)$ is all $f \in \mathcal{X}$ for which the limit exists.