Complements of Machine Elements
Stresses and Strains

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1 Thin curved beam

Determine the vertical deflection of the end B of the thin curved beam of uniform cross section and semicircular center line.

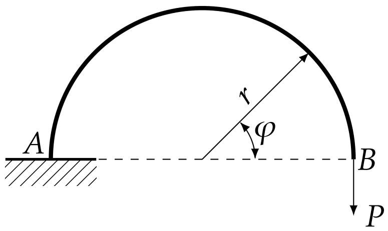

Figure 1: Thin curved beam [1].

1.1 Static equilibrium

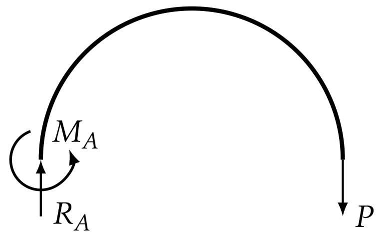

Figure 2: Free body diagram.

$$\begin{cases} \sum F_y = 0 \Leftrightarrow R_A - P = 0 \Leftrightarrow R_A = P \\ \sum M_A = 0 \Leftrightarrow M_A - P \cdot 2 \cdot r = 0 \Leftrightarrow M_A = P \cdot 2 \cdot r \end{cases} \tag{1}$$

1.2 Bending moment along the curved beam

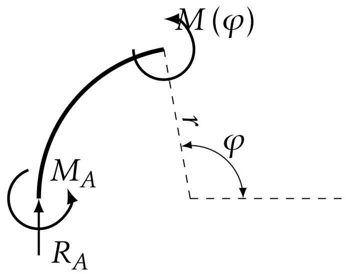

Figure 3: Thin curved beam [1].

$$\sum M = 0 \Leftrightarrow M(\varphi) + M_A - R_A \cdot r \cdot (1 + \cos(\varphi)) = 0 \tag{2}$$

$$M(\varphi) = P \cdot r \cdot (\cos(\varphi) - 1) \tag{3}$$

1.3 Strain energy due to bending

$$U = \int_0^s \frac{M^2}{2 \cdot E \cdot I} \mathrm{d}s = \int_0^\pi \frac{M(\varphi)^2}{2 \cdot E \cdot I} \mathrm{d}s = \int_0^\pi \frac{(P \cdot r \cdot (\cos(\varphi) - 1))^2}{2 \cdot E \cdot I} r \, \mathrm{d}\varphi \tag{4}$$

1.4 Castigliano theorem

$$\delta = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P} \int_0^\pi \frac{(P \cdot r \cdot (\cos(\varphi) - 1))^2}{2 \cdot E \cdot I} r \, \mathrm{d}\varphi \tag{5}$$

$$\delta = \frac{\partial}{\partial P} \int_0^\pi \frac{P^2 \cdot r^3 \cdot \left(\cos^2(\varphi) - 2\cos(\varphi) + 1\right)}{2 \cdot E \cdot I} \mathrm{d}\varphi \tag{6}$$

$$\delta = \frac{2 \cdot P \cdot r^3}{2 \cdot E \cdot I} \int_0^\pi \left(\cos^2(\varphi) - 2\cos(\varphi) + 1\right) \mathrm{d}\varphi \tag{7}$$

$$\delta = \frac{P \cdot r^3}{E \cdot I} \cdot \left[\frac{\varphi}{2} + \frac{1}{4}\sin(2\varphi) - 2\sin(\varphi) + \varphi\right]_0^\pi \tag{8}$$

$$\delta = \frac{P \cdot r^3}{E \cdot I} \cdot \left(\frac{\pi}{2} + \pi\right) = \frac{3 \cdot \pi}{2} \frac{P \cdot r^3}{E \cdot I} \tag{9}$$

2 Increase in distance

Determine the increase in distance between the ends A and B of a thin beam of uniform cross section consisting of a semicircular portion CD and two straight portions AC and BD.

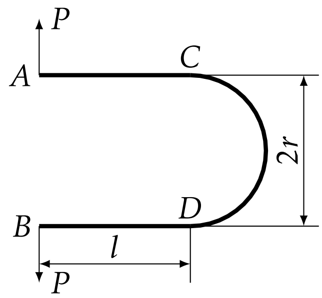

Figure 4: Increase in distance [1].

2.1 Bending moment along the curved beam

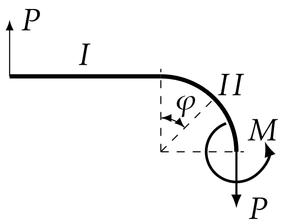

Figure 5: Free body diagram.

For the straight beam, the bending moment is:

$$M(x) - P \cdot x = 0 \Leftrightarrow M(x) = P \cdot x \tag{10}$$

For the curved beam:

$$M(\varphi) - P \cdot l - P \cdot r \cdot \sin(\varphi) = 0 \Leftrightarrow M(\varphi) = P \cdot (l + r \cdot \sin(\varphi)) \tag{11}$$

2.2 Strain energy due to bending

The free body diagram was done only for half of the structure. However, the total strain energy should include all the structure. By symmetry the strain energy of half of the structure should be multiplied by two.

$$U = 2\int_0^l \frac{M(x)^2}{2\cdot E\cdot I}\,\mathrm{d}x + 2\int_0^{\frac{\pi}{2}} \frac{M(\varphi)^2}{2\cdot E\cdot I} r\,\mathrm{d}\varphi \tag{12}$$

$$U = \int_0^l \frac{P^2\cdot x^2}{E\cdot I}\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \frac{[P\cdot(l + r\cdot\sin(\varphi))]^2}{E\cdot I} r\,\mathrm{d}\varphi \tag{13}$$

2.3 Castigliano theorem

Applying the Castigliano theorem:

$$\begin{aligned}\delta = \frac{\partial U}{\partial P} = &\frac{\partial}{\partial P}\int_0^l \frac{P^2\cdot x^2}{E\cdot I}\,\mathrm{d}x + \\ &\frac{\partial}{\partial P}\int_0^{\frac{\pi}{2}} \frac{P^2\cdot\left(l^2 + 2\cdot r\cdot l\cdot\sin(\varphi) + r^2\cdot\sin^2(\varphi)\right)}{E\cdot I} r\,\mathrm{d}\varphi\end{aligned} \tag{14}$$

$$\delta = \frac{2\cdot P}{E\cdot I}\cdot\left[\frac{x^3}{3}\right]_0^l + \frac{2\cdot P}{E\cdot I}\cdot\left[r\cdot l^2\cdot\varphi - 2\cdot r^2\cdot l\cdot\cos(\varphi) + r^3\left(\frac{\varphi}{2} - \frac{1}{4}\sin(2\varphi)\right)\right]_0^{\frac{\pi}{2}} \tag{15}$$

$$\delta = \frac{2\cdot P}{E\cdot I}\cdot\left(\frac{l^3}{3} + \frac{\pi\cdot r\cdot l^2}{2} + 2\cdot r^2\cdot l + \frac{\pi\cdot r^3}{4}\right) \tag{16}$$

The deflection in the vertical direction is:

$$\delta = \frac{2\cdot P}{E\cdot I}\cdot\left[\frac{l^3}{3} + r\cdot\left(\frac{\pi\cdot l^2}{2} + 2\cdot r\cdot l + \frac{\pi\cdot r^2}{4}\right)\right] \tag{17}$$

3 Flywheel stresses

A flywheel made of low-carbon steel has an outside radius of 150 mm and an inside radius of 25 mm. The flywheel is to be assembled (press fit) onto a shaft. The radial interference between the flywheel and shaft is 50 µm, and the shaft will operate at a speed of 5000 rpm [2]. Calculate:

3.1 The circumferential and radial stresses on the flywheel inner radius

The diameters of the composite cylinder are $a = 0\,\mathrm{mm}$, $b = 25\,\mathrm{mm}$ and $c = 150\,\mathrm{mm}$. For low-carbon steel, $E = 207\,\mathrm{GPa}$, $\nu = 0.3$, and $\rho = 7860\,\mathrm{kg\,m^{-3}}$. The interference pressure yields:

$$p_c = \frac{E \cdot \delta}{b} \cdot \left[\frac{\left(c^2 - b^2\right)\left(b^2 - a^2\right)}{2b^2\left(c^2 - a^2\right)} \right] = \frac{E \cdot \delta}{2b} \cdot \left(\frac{c^2 - b^2}{c^2} \right) \tag{18}$$

$$p_c = \frac{207 \times 10^3 \times 0.050}{25} \times \left(\frac{150^2 - 25^2}{150^2} \right) = 201.25\,\mathrm{MPa} \tag{19}$$

The angular speed of the system is:

$$\omega = \frac{2 \cdot \pi \cdot n}{60} = \frac{2 \times \pi \times 5000}{60} = 523.6\,\mathrm{rad\,s^{-1}} \tag{20}$$

For such angular speed ω and considering that the flywheel and shaft are both the same material, the hoop stress can be obtained by:

$$\sigma_t = \frac{3 + \nu}{8} \cdot \rho \cdot \omega^2 \cdot \left(a^2 + c^2 + \frac{a^2 \cdot c^2}{r^2} - \frac{1 + 3 \cdot \nu}{3 + \nu} \cdot r^2 \right) + p_c \cdot \frac{c^2 + b^2}{c^2 - b^2} \tag{21}$$

$\sigma_t = \frac{3+0.3}{8} \times 7.86 \times 10^{-9} \times 523.6^2 \times \left(150^2 - \frac{1+3\times0.3}{3+0.3} \times 25^2\right) + 201.25 \times \frac{150^2+25^2}{150^2-25^2} = 232.43\,\mathrm{MPa}$

$$\sigma_r = \frac{3 + \nu}{8} \cdot \rho \cdot \omega^2 \cdot \left(a^2 + c^2 - \frac{a^2 \cdot c^2}{r^2} - r^2 \right) - p_c \tag{22}$$

$$\sigma_r = \frac{3 + 0.3}{8} \times 7.86 \times 10^{-9} \times 523.6^2 \times \left(150^2 - 25^2\right) - 201.25 = -180.81\,\mathrm{MPa} \tag{23}$$

3.2 The speed at which the flywheel will break loose from the shaft

The flywheel breaks free when the radial stress at the inner radius b is reduced to zero at a certain angular speed ω:

$$\sigma_r = 0 = \frac{3 + 0.3}{8} \times 7.86 \times 10^{-9} \times \omega^2 \times \left(150^2 - 25^2\right) - 201.25 \tag{24}$$

After solving the equation, the angular speed $\omega = 1684\,\mathrm{rad\,s^{-1}}$, corresponding to 16 080 rpm.

References

[1] Timoshenko, Stephen: *Strength of Materials. Part 1: Elementary Theory and Problems*. CBS, third edition, 1986, ISBN 81-239-1030-4.

[2] Schmid, Steven R., Bernard J. Hamrock, and Bo O. Jacobson: *Fundamentals of Machine Elements*. CRC Press, 2014, ISBN 9781482247503.