U.PORTO
FEUP FACULDADE DE ENGENHARIA
UNIVERSIDADE DO PORTO

1. Stresses and Strains

Complements of Machine Elements

Carlos M.C.G. Fernandes

2024

Mestrado em Engenharia Mecânica

Recomended bibliograhy

- Timoshenko, “Strength of Materials. Part I: Elementary Theory and Problems”, Third Edition, CBS, vol.1, 1986.
- Timoshenko, “Resistência de materiais”, Ao Livro Técnico, vol.1, 1969.
- Schmid, Hamrock, Jacobson; “Fundamentals of Machine Elements”, Third Edition, CRC, 2014.
- Robert C. Juvinall, Kurt M. Marshek; “Fundamentals of machine component design”, Wiley, 2017.

Contents

1.1. Curved Beams

1.2. Thick Cylinders

1.3. Rotating Cylinders

References

1.1. Curved Beams

Ein Mann, der konstruieren will, Der schau erst mal und denke!

A man who wants to design, He first looks and thinks!

From Gustav Niemann, Hans Winter, Bernd-Robert Höhn, “Maschinenelemente: Band I Konstruktion und Berechnung von Verbindungen”

Summary

1. Introduction 5
2. Pure bending loading - straight beams 6
3. Pure bending loading - curved beams 9
4. Particular cases of curved beams 18
5. Formulas for several cross sections of curved beams 23
6. Stress concentration factors for curved beams 25
7. Bending of curved beams due to forces 28
8. Deflection of thin curved beams 32
9. Proposed Assignment: Crane Hook 42
10. Proposed Exercises 45

Introduction

Machine frames, springs, clips and fasteners frequently are curved shapes [1].

A lifting crane hook as presented in Figure 1 is a typical curved beam.

Several lifting hook geometries are described in:

- DIN 15401-1, Lifting hooks for lifting appliances; Single hooks; Unmachined parts, November 1982.

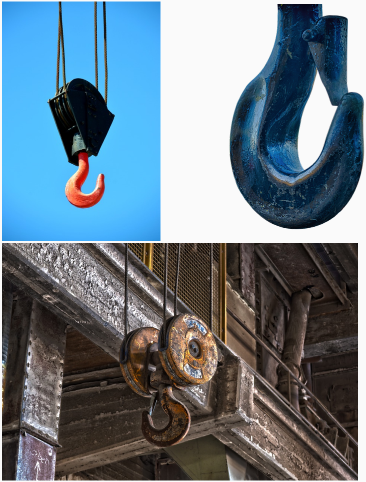


Figure 1: Lifting crane hook - typically treated as a curved beam.

Pure bending loading - straight beams

Figure 2 shows a straight beam loaded *only* in bending; hence the term, "pure bending" [2].

Figure 2 presents typical sections with two axes of symmetry.

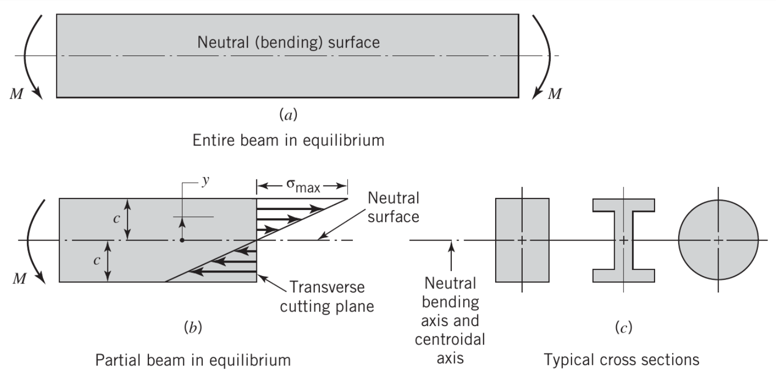


Figure 2: Pure bending of sections with two axes of symmetry [2].

Pure bending loading - straight beams

Figure 3 presents typical sections with one axis of symmetry.

Tensile stresses exist above the neutral axis and compressive stresses below.

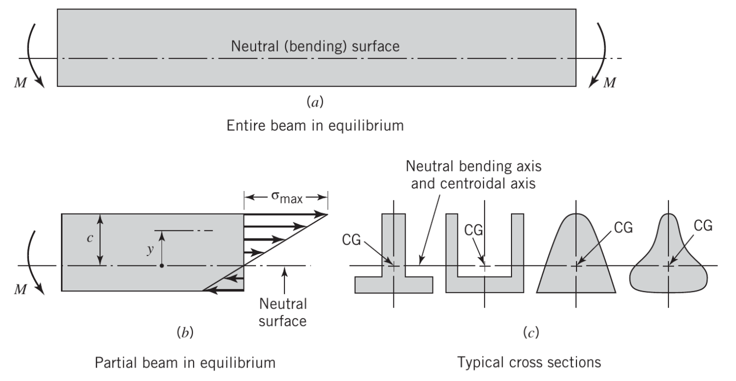


Figure 3: Pure bending of sections with one axis of symmetry [2].

Pure bending loading - straight beams

The bending stresses or normal stresses are given by:

$$\sigma = \frac{My}{I}$$

I – second moment of area of the cross section;
y – distance from the neutral axis.

Equation applies to any cross section with the following limitations [2]:

1. The bar must be initially straight and loaded in a plane of symmetry;
2. The material must be homogeneous, and all stresses must be within the elastic range;
3. The section for which stresses are calculated must not be too close to significant stress raisers or to regions where external loads are applied (Saint-Venant's Principle).

Pure bending loading - curved beams

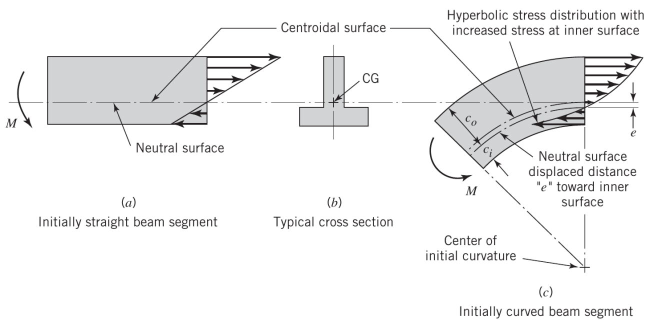


Figure 4: Effect of initial curvature, pure bending of sections with one axis of symmetry [2].

In a curved beam under pure bending in the plane of curvature, the bending stresses approximately follow $\sigma = \frac{M \cdot y}{I}$, but stress and strain are not linearly related. The centroidal and neutral surfaces are not coincident, with the distance between them being the eccentricity *e*.

Pure bending loading - curved beams

Hypothesis[1] [3]:

- *center line* of the beam is a plane curve;
- cross sections have an axis of symmetry in this plane;
- beam is submitted to the action of forces acting in the plane of symmetry;
- bending takes place in the plane of symmetry;
- cross sections originally plane and normal to the center line remain so after bending.

[1] Timoshenko, S., Strength of Materials. Part I: Elementary Theory and Problems, Third Edition, CBS, page 362.

Pure bending loading - curved beams

The strain on the fiber shown at distance y from the centroidal axis is[2]:

$$\epsilon = \frac{\delta}{l} = \frac{(y-e)\,\Delta\,\mathrm{d}\varphi}{(r-y)\,\mathrm{d}\varphi}$$

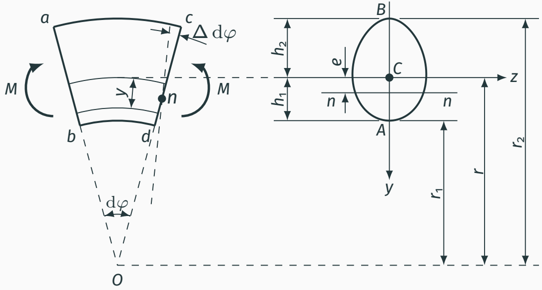


Figure 4: Curved beam under "pure bending" [3].

[2]The angle $\Delta\,\mathrm{d}\varphi$ and the bending moment M are taken positive is the initial curvature of the beam is dimished during bending.

Pure bending loading - curved beams

Applying Hooke's law[3]:

$$\sigma = E\frac{(y-e)\,\Delta\,\mathrm{d}\varphi}{(r-y)\,\mathrm{d}\varphi}$$

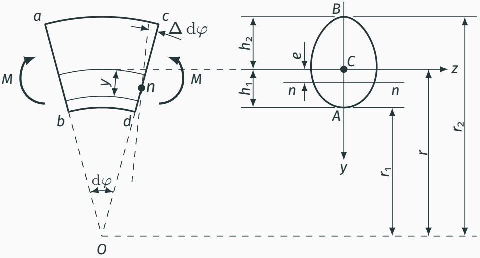


Figure 4: Curved beam under "pure bending" [3].

[3]Assuming no radial pressure between the longitudinal fibers.

Pure bending loading - curved beams

From statics, the sum of forces should be zero. Taking note that Young modulus $E \neq 0$:

$$\sum F = 0 : \int \sigma \,\mathrm{d}A = \underbrace{\frac{E\Delta \,\mathrm{d}\varphi}{\mathrm{d}\varphi}}_{\neq 0} \underbrace{\int \frac{y-e}{r-y} \,\mathrm{d}A}_{=0} = 0$$

$$\int \frac{y-e}{r-y} \,\mathrm{d}A = \underbrace{\int \frac{y}{r-y} \,\mathrm{d}A}_{\text{dimensions of area}} - e \int \frac{1}{r-y} \,\mathrm{d}A = 0$$

Expressing the first integral as a *modified area* of the cross section **mA**, where **m** is a number to be determined for each cross section:

$$\int \frac{y}{r-y} \,\mathrm{d}A = mA$$

Pure bending loading - curved beams

$$\int \frac{y-e}{r-y}\,\mathrm{d}A = \underbrace{\int \frac{y}{r-y}\,\mathrm{d}A}_{mA} - e\underbrace{\int \frac{1}{r-y}\,\mathrm{d}A}_{(m+1)\frac{A}{r}} = 0$$

The second integral may be transformed in the following way:

$$\int \frac{1}{r-y}\,\mathrm{d}A = \int \frac{y+r-y}{r\,(r-y)}\,\mathrm{d}A = \underbrace{\int \frac{y}{r\,(r-y)}\,\mathrm{d}A}_{\frac{mA}{r}} + \underbrace{\int \frac{1}{r}\,\mathrm{d}A}_{\frac{A}{r}} = (m+1)\,\frac{A}{r}$$

Finally, we can write $\boxed{mA - e\,(m+1)\,\frac{A}{r} = 0}$, so:

$$e = r\frac{m}{m+1} \quad \texttt{or} \quad m = \frac{e}{r-e}$$

Pure bending loading - curved beams

The sum of moments should also be zero:

$$\sum M_C = 0 : \int \sigma y \, \mathrm{d}A = \frac{E\Delta \, \mathrm{d}\varphi}{\mathrm{d}\varphi} \int \frac{(y-e)\, y}{r-y} \, \mathrm{d}A - M = 0$$

$$M = \frac{E\Delta \, \mathrm{d}\varphi}{\mathrm{d}\varphi} \left(\int \underbrace{\frac{y^2}{r-y}}_{-y+r\frac{y}{(r-y)}} \, \mathrm{d}A - \underbrace{e \int \frac{y}{r-y} \, \mathrm{d}A}_{meA} \right) = \frac{E\Delta \, \mathrm{d}\varphi}{\mathrm{d}\varphi} (mrA - meA)$$

$$\int \frac{y^2}{r-y} \, \mathrm{d}A = \underbrace{- \int y \, \mathrm{d}A}_{=0^4} + r \int \frac{y}{r-y} \, \mathrm{d}A = 0 + rmA$$

[4] The first moment of area about the centroidal axis is zero

Pure bending loading - curved beams

$$M = \frac{E\Delta \,\mathrm{d}\varphi}{\mathrm{d}\varphi}\left(mrA - meA\right) \Leftrightarrow \frac{\Delta \,\mathrm{d}\varphi}{\mathrm{d}\varphi} = \frac{M}{E\left(mrA - meA\right)}$$

Recall that:

$$\sigma = E\frac{\left(y - e\right)\Delta \,\mathrm{d}\varphi}{\left(r - y\right)\mathrm{d}\varphi} = E\frac{\left(y - e\right)}{\left(r - y\right)}\frac{M}{E\left(mrA - meA\right)} \quad \text{and} \quad m = \frac{e}{r - e}$$

$$\sigma = \frac{\left(y - e\right)}{\left(r - y\right)}\frac{M}{\frac{e}{r-e}rA - \frac{e}{r-e}eA} = \frac{M\left(y - e\right)\left(r - e\right)}{Ae\left(r - y\right)\left(r - e\right)}$$

$$\boxed{\sigma = \frac{M\left(y - e\right)}{Ae\left(r - y\right)}}$$

Pure bending loading - curved beams

$$\sigma_A = \frac{M\left(h_1 - e\right)}{eAr_1} \qquad \sigma_B = \frac{-M\left(h_2 + e\right)}{eAr_2}$$

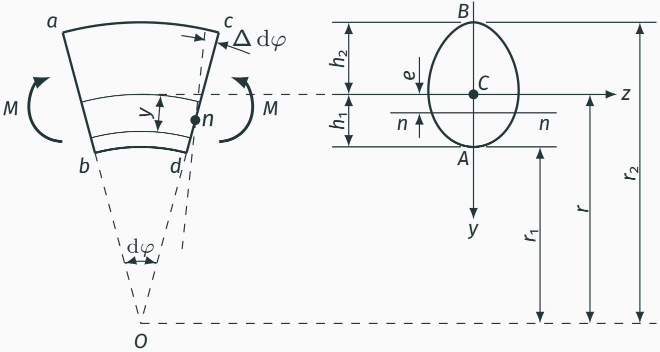


Figure 4: Curved beam under “pure bending” [3].

Particular cases of curved beams

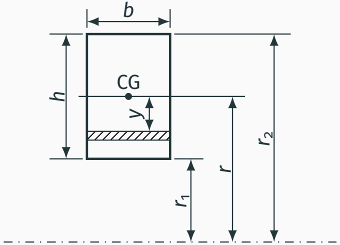


Figure 5: Curved beam with rectangular cross section [3].

$$mA = \int \frac{y}{r-y}\,\mathrm{d}A = b\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{y-r+r}{r-y}\,\mathrm{d}y = br\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{r-y}\,\mathrm{d}y - bh$$

$$mA = br\ln\frac{r_2}{r_1} - bh \Leftrightarrow m = \frac{r}{h}\ln\frac{r_2}{r_1} - 1$$

Particular cases of curved beams

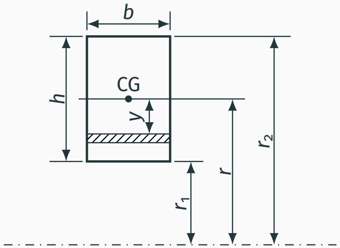


Figure 5: Curved beam with rectangular cross section [3].

$$m = \frac{r}{h} \ln \frac{r_2}{r_1} - 1$$

$$e = r \frac{m}{m+1} = r \frac{\frac{r}{h} \ln \frac{r_2}{r_1} - 1}{\frac{r}{h} \ln \frac{r_2}{r_1}} = r - \frac{h}{\ln \frac{r_2}{r_1}}$$

Particular cases of curved beams

For small values of $\frac{h}{r}$ the distance **e** is small in comparison with h. To have sufficient accuracy for e it is necessary to take $\ln \frac{r_2}{r_1}$ with a high degree of accuracy.

$$\ln \frac{r_2}{r_1} = \ln \frac{r + \frac{h}{2}}{r - \frac{h}{2}} = \ln \frac{\frac{2r+h}{2}}{\frac{2r-h}{2}} = \ln \frac{2r + h}{2r - h} = \ln \frac{1 + \frac{h}{2r}}{1 - \frac{h}{2r}}$$

A Taylor series expansion around zero (Maclaurin) can be employed:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(iv)}(0) + \ldots$$

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + O(x^6)$$

Particular cases of curved beams

Performing the same for $\ln(1-x)$:

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + O(x^6)$$

Recalling that $x = \frac{h}{2r}$:

$$\ln\left(1+\frac{h}{2r}\right) = \frac{h}{2r} - \frac{1}{2}\left(\frac{h}{2r}\right)^2 + \frac{1}{3}\left(\frac{h}{2r}\right)^3 - \frac{1}{4}\left(\frac{h}{2r}\right)^4 + \frac{1}{5}\left(\frac{h}{2r}\right)^5 + \ldots$$

$$\ln\left(1-\frac{h}{2r}\right) = -\frac{h}{2r} - \frac{1}{2}\left(\frac{h}{2r}\right)^2 - \frac{1}{3}\left(\frac{h}{2r}\right)^3 - \frac{1}{4}\left(\frac{h}{2r}\right)^4 - \frac{1}{5}\left(\frac{h}{2r}\right)^5 + \ldots$$

Particular cases of curved beams

Now remember that $\ln \frac{r_2}{r_1} = \ln r_2 - \ln r_1$:

$$\ln \frac{1+\frac{h}{2r}}{1-\frac{h}{2r}} = \frac{h}{2r} - \cancel{\frac{1}{2}\left(\frac{h}{2r}\right)^2} + \frac{1}{3}\left(\frac{h}{2r}\right)^3 - \cancel{\frac{1}{4}\left(\frac{h}{2r}\right)^4} + \frac{1}{5}\left(\frac{h}{2r}\right)^5 + \frac{h}{2r} + \cancel{\frac{1}{2}\left(\frac{h}{2r}\right)^2} + \frac{1}{3}\left(\frac{h}{2r}\right)^3 + \cancel{\frac{1}{4}\left(\frac{h}{2r}\right)^4} + \frac{1}{5}\left(\frac{h}{2r}\right)^5 + \dots$$

$$\ln \frac{1+\frac{h}{2r}}{1-\frac{h}{2r}} = \frac{h}{r}\left[1 + \frac{1}{3}\left(\frac{h}{2r}\right)^2 + \frac{1}{5}\left(\frac{h}{2r}\right)^4 + \dots\right]$$

And substituting in m:

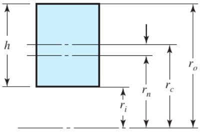
$$m = \frac{r}{h}\ln\frac{r_2}{r_1} - 1 = \frac{1}{3}\left(\frac{h}{2r}\right)^2 + \frac{1}{5}\left(\frac{h}{2r}\right)^4 + \dots$$

Taking only the first term:

$$m \approx \frac{h^2}{12r^2} \quad \text{and} \quad e \approx \frac{h^2}{12r}$$

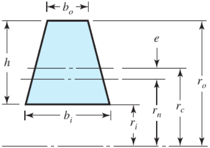
Formulas for several cross sections of curved beams

Formulas taken from [1]. See also [4, 5]



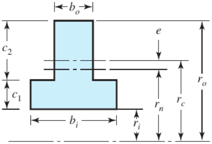
$$r_c = r_i + \frac{h}{2}$$

$$r_n = \frac{h}{\ln(r_o/r_i)}$$



$$r_c = r_i + \frac{h}{3}\,\frac{b_i + 2b_o}{b_i + b_o}$$

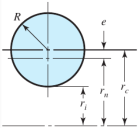
$$r_n = \frac{A}{b_o - b_i + [(b_i r_o - b_o r_i)/h]\ln(r_o/r_i)}$$



$$r_c = r_i + \frac{b_i c_1^2 + 2b_o c_1 c_2 + b_o c_2^2}{2(b_o c_2 + b_i c_1)}$$

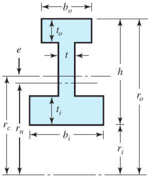
$$r_n = \frac{b_i c_1 + b_o c_2}{b_i \ln[(r_i + c_1)/r_i] + b_o \ln[r_o/(r_i + c_1)]}$$

Formulas for sections of Curved Beams



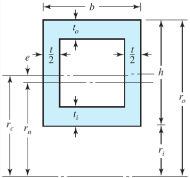
$$r_c = r_i + R$$

$$r_n = \frac{R^2}{2\left(r_c - \sqrt{r_c^2 - R^2}\right)}$$



$$r_c = r_i + \frac{\frac{1}{2}h^2 t + \frac{1}{2}t_i^2(b_i - t) + t_o(b_o - t)(h - t_o/2)}{t_i(b_i - t) + t_o(b_o - t) + ht}$$

$$r_n = \frac{t_i(b_i - t) + t_o(b_o - t) + ht_o}{b_i \ln\frac{r_i + t}{r_i} + t \ln\frac{r_o - t_o}{r_i + t_i} + b_o \ln\frac{r_o}{r_o - t_o}}$$



$$r_c = r_i + \frac{\frac{1}{2}h^2 t + \frac{1}{2}t_i^2(b - t) + t_o(b - t)(h - t_o/2)}{ht + (b - t)(t_i + t_o)}$$

$$r_n = \frac{(b - t)(t_i + t_o) + ht}{b\left(\ln\frac{r_i + t_i}{r_i} + \ln\frac{r_o}{r_o - t_o}\right) + t \ln\frac{r_o - t_o}{r_i + t_i}}$$

Stress concentration factors for curved beams

The bending stress can be determined based on stress concentration factors and the equation for straight beams:

$$\sigma = K_t \frac{Mc}{I}$$

As stated by [5], the concept was proposed in "A simple method of determining stress in curved flexural member" [6].

$$K_t = 1 + 0.5 \frac{I}{bc^2} \left[\frac{1}{r - c} + \frac{1}{r} \right]$$

For circular and elliptical sections use 1.05 instead of 0.5 in the formula.

I is the second moment of area as used in straight-beam formula
b is the maximum breadth of the cross section
$c = r - r_1$ is the distance from centroidal axis to inside fiber
r is the radius of curvature of centroidal axis of the beam

Graphical method:

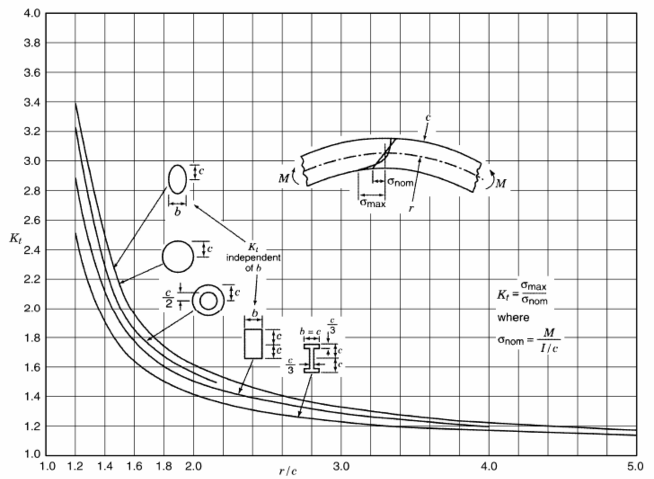


Figure 6: Stress concentration factors K_t for a curved beam in bending [7].

Stress concentration factors for curved beams

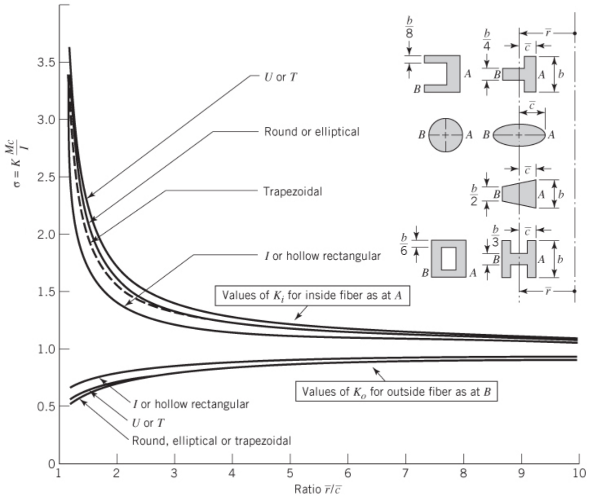


Figure 7: Effect of curvature on bending stresses: stress concentration factor for *inner* and *outer* fibers [2].

Bending of curved beams due to forces

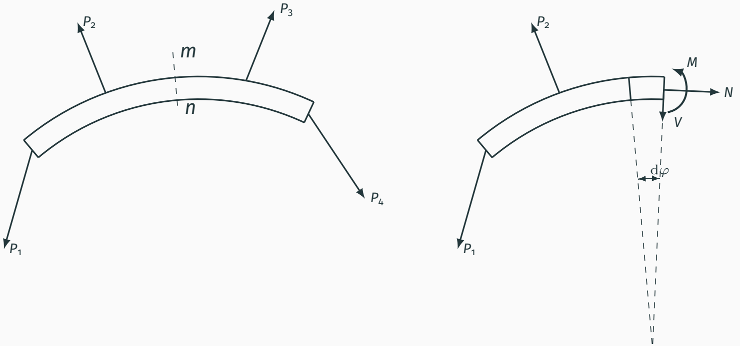


Figure 8: Bending due to forces acting in the plane of symmetry [3].

$P1, P_2, P_3, P_4$ act in the plane of symmetry of the beam. Removing the portion of the beam to the right of *mn*, its action on the left portion of the beam is replaced by *shearing force* (V) and the *longitudinal force* (N) and the bending moment (M).

Bending of curved beams due to forces

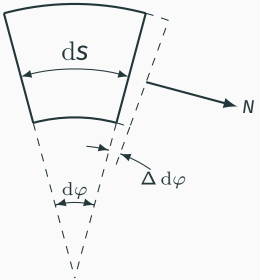


Figure 9: Bending due to forces [3].

The stresses and deformations caused by bending were already discussed considering pure bending conditions.

The initial angle $\mathrm{d}\varphi$ will increase by the amount:

$$\Delta\,\mathrm{d}\varphi = \frac{N}{AE}\,\mathrm{d}\varphi = \frac{N\,\mathrm{d}s}{AEr}$$

The center line length $\mathrm{d}s$ will increase by:

$$\Delta\,\mathrm{d}s = \frac{N\,\mathrm{d}s}{AE}$$

Bending of curved beams due to forces

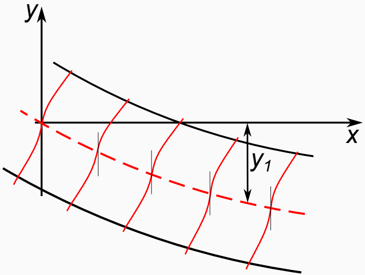


Figure 10: Effect of shear (only) [3].

The shear force V produces shear stresses and some warping of the cross section.

It is assumed that the distribution of shear stresses along the cross section is the same as for straight beams.

The relative radial displacement of two adjacent cross sections is:

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = \frac{(\tau_{xy})_{y=0}}{G} = \frac{\alpha V}{AG}$$

V/A is the average shearing stress τ_{xy}, G is the elastic modulus in shear, and α a coefficient to be determined depending on the cross section.

$\alpha = 3/2$ for rectangular cross section and $\alpha = 4/3$ for circular cross sections.

Bending of curved beams due to forces

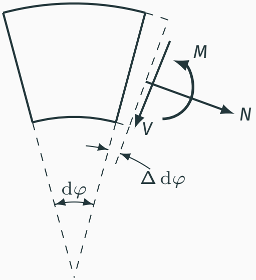


Figure 11: Bending due to forces [3].

The normal stresses are given by:

$$\sigma = \frac{N}{A}$$

Recalling the stresses due to pure bending:

$$\sigma = \frac{M\,(y-e)}{Ae\,(r-y)}$$

Combining bending stresses and normal load stresses:

$$\sigma = \frac{M\,(y-e)}{Ae\,(r-y)} + \frac{N}{A}$$

For the change $\Delta\,\mathrm{d}\varphi$ of the angle between two adjacent cross sections:

$$\Delta\,\mathrm{d}\varphi = \frac{M\,\mathrm{d}s}{AerE} - \frac{N\,\mathrm{d}s}{AEr}$$

Deflection of thin curved beams

The deflection of curved beams can be calculated with Castigliano's theorem.

Considering the simplest case in which the cross-sectional dimensions of the beam are small in comparison with the radius of its center line.

The strain energy of bending is given by the equation:

$$U = \int_0^s \frac{M^2}{2EI}\,\mathrm{d}s$$

The deflection of the point of application of any load P in the direction of the load is:

$$\delta = \frac{\partial U}{\partial P}$$

Example 1a: vertical deflection of thin curved beam

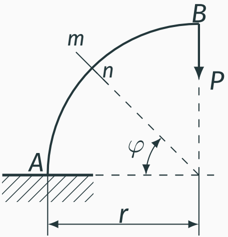


Figure 12: Curved beam example [3].

The bending moment at any cross section *mn* is:

$$M = -Pr\cos\varphi$$

$$U = \int_0^s \frac{M^2}{2EI}\,\mathrm{d}s = \int_0^{\frac{\pi}{2}} \frac{M^2}{2EI} \underbrace{r\,\mathrm{d}\varphi}_{\mathrm{d}s} = \int_0^{\frac{\pi}{2}} \frac{(-Pr\cos\varphi)^2}{2EI} r\,\mathrm{d}\varphi$$

Example 1a: vertical deflection of thin curved beam

$$U = \int_0^{\frac{\pi}{2}} \frac{(-Pr\cos\varphi)^2}{2EI} r\,\mathrm{d}\varphi$$

$$\delta = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\int_0^{\frac{\pi}{2}} \frac{(-Pr\cos\varphi)^2}{2EI} r\,\mathrm{d}\varphi$$

$$\delta = \frac{1}{EI}\int_0^{\frac{\pi}{2}} Pr^3\cos^2\varphi\,\mathrm{d}\varphi = \frac{Pr^3}{EI}\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\varphi}{2}\,\mathrm{d}\varphi$$

The vertical displacement of the end B is:

$$\delta = \frac{\pi}{4}\frac{Pr^3}{EI}$$

Example 1b: horizontal deflection of thin curved beam

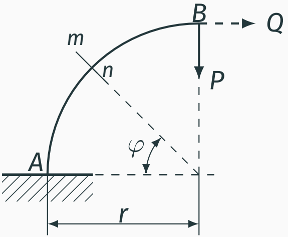


Figure 13: Curved beam example – fictitious horizontal load [3].

If required, the horizontal displacement of the end *B* can be determined using a fictitious load *Q*.

$$M = -\left[Pr\cos\varphi + Qr\left(1 - \sin\varphi\right)\right]$$

Example 1b: horizontal deflection of thin curved beam

$$U = \int_0^{\frac{\pi}{2}} \frac{M^2 r}{2EI} \, \mathrm{d}\varphi$$

$$\delta = \frac{\partial U}{\partial Q} = \frac{\partial}{\partial Q} \int_0^{\frac{\pi}{2}} \frac{M^2}{2EI} r \, \mathrm{d}\varphi = \frac{1}{EI} \int_0^{\frac{\pi}{2}} M \frac{\partial M}{\partial Q} r \, \mathrm{d}\varphi$$

Taking $Q = 0$ in M:

$$\delta = \frac{1}{EI} \int_0^{\frac{\pi}{2}} Pr^3 \cos\varphi \, (1 - \sin\varphi) \, \mathrm{d}\varphi$$

The horizontal displacement of end B is:

$$\delta = \frac{Pr^3}{2EI}$$

Example 2a: Thin ring

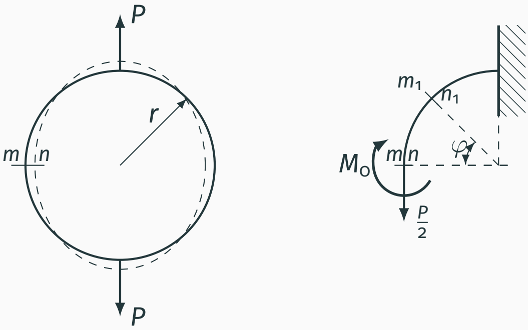


Figure 14: Thin ring [3].

Due to symmetry only one quadrant of the ring need be considered. There are no shear stresses over the cross section *mn* and the tensile force on this cross section is $\frac{P}{2}$.

Example 2a: thin ring

The magnitude of the bending moment acting on this cross section is statically indeterminate and may be found by the Castigliano theorem. From the condition of symmetry it is known that section *mn* does not rotate during the bending of the ring.

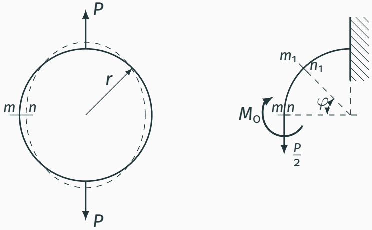


Figure 14: Thin ring [3].

$$\frac{\partial U}{\partial M_0} = 0$$

$$M = M_0 - \frac{P}{2} r\left(1 - \cos\varphi\right)$$

$$\frac{\mathrm{d}M}{\mathrm{d}M_0} = 1$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}M_0}\int_0^{\frac{\pi}{2}} \frac{M^2 r}{2EI}\,\mathrm{d}\varphi = \frac{1}{EI}\int_0^{\frac{\pi}{2}} M\frac{\mathrm{d}M}{\mathrm{d}M_0} r\,\mathrm{d}\varphi$$

Example 2a: Thin ring

$$0 = \frac{1}{EI}\int_0^{\frac{\pi}{2}} \left[M_0 - \frac{P}{2}r\left(1-\cos\varphi\right)\right] r\,\mathrm{d}\varphi \Leftrightarrow M_0 = \frac{Pr}{2}\left(1-\frac{2}{\pi}\right)$$

The equation for the bending moment for any cross section is:

$$M = M_0 - \frac{P}{2}r\left(1-\cos\varphi\right) \quad \text{with} \quad M_0 = \frac{Pr}{2}\left(1-\frac{2}{\pi}\right)$$

$$M = \frac{P}{2}r\left(\cos\varphi - \frac{2}{\pi}\right)$$

For $\frac{\pi}{2}$, point of application of force P:

$$M = \frac{Pr}{\pi} = -0.318Pr$$

Example 2b: Thin ring – change in vertical diameter

The minus sign indicate that the bending moments at the points of application of the forces P tend to increase the curvature. The total strain energy stored in the ring:

$$U = 4 \int_0^{\frac{\pi}{2}} \frac{M^2 r}{2EI} \, \mathrm{d}\varphi$$

The increase in the vertical diameter is:

$$\delta = \frac{\partial U}{\partial P} = \frac{4}{EI} \int_0^{\frac{\pi}{2}} M \frac{\mathrm{d}M}{\mathrm{d}P} r \, \mathrm{d}\varphi = \frac{Pr^3}{EI} \int_0^{\frac{\pi}{2}} \left(\cos\varphi - \frac{2}{\pi} \right)^2 \mathrm{d}\varphi$$

The increase in vertical diameter:

$$\delta = \left(\frac{\pi}{4} - \frac{2}{\pi} \right) \frac{Pr^3}{EI} = 0.149 \frac{Pr^3}{EI}$$

Example 2c: Thin ring – change in horizontal diameter

Applying two oppositely fictitious forces Q are applied at the ends of the horizontal diameter:

$$M = \frac{P}{2} r \left(\cos\varphi - \frac{2}{\pi} \right) - \frac{Q}{2} r \sin\varphi \quad \text{and} \quad \frac{\partial M}{\partial Q} = -\frac{r}{2} \sin\varphi$$

Taking $Q = 0$ in M:

$$\delta = \frac{4}{EI} \int_0^{\frac{\pi}{2}} M \frac{\partial M}{\partial Q} r \, \mathrm{d}\varphi = \frac{Pr^3}{EI} \int_0^{\frac{\pi}{2}} \left(-\cos\varphi \sin\varphi + \frac{2 \sin\varphi}{\pi} \right) \mathrm{d}\varphi$$

The decrease in horizontal diameter:

$$\delta = \left(\frac{2}{\pi} - \frac{1}{2} \right) \frac{Pr^3}{EI} = 0.137 \frac{Pr^3}{EI}$$

Proposed Assignment: Crane Hook

The theory of curved beams described above is applied in designing *crane hooks*.

$$\sigma = \frac{M\,(y-e)}{Ae\,(r-y)} + \frac{F}{A}$$

For circular cross sections:

$$\sigma_{max} = \frac{F}{A}\frac{h}{2mr_1}$$

$$\sigma_{min} = -\frac{F}{A}\frac{h}{2mr_2}$$

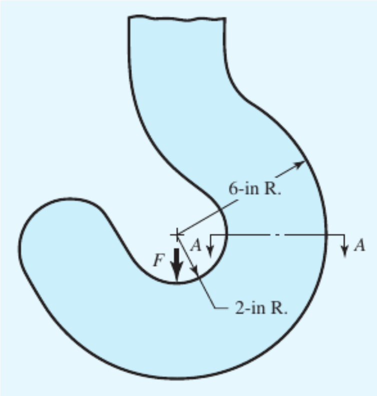
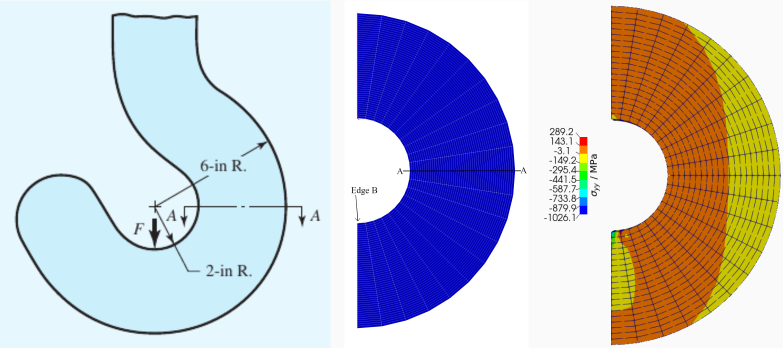


Figure 15: A crane hook [1].

Finite Element Method solution of a Crane Hook



(a) Crane hook geometry [1]. **(b)** FEM mesh. **(c)** Normal stress allong *yy* axis

Figure 16: FEM 3D calculation with CalculiX [8] for a crane hook (typically treated as a curved beam) - distributed load over the edge *B*.

Finite Element Method solutionof a Crane Hook

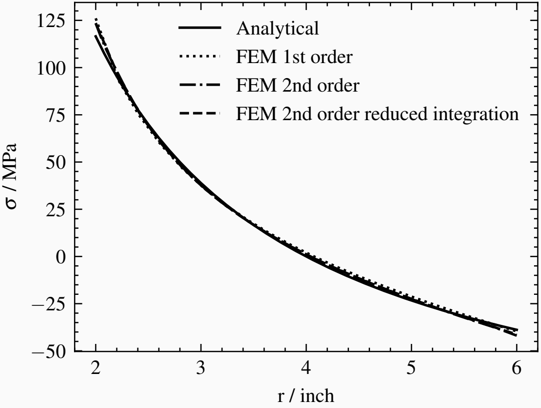


Figure 17: Normal stress along section $A - A$: comparison between analytical equation and FEM results.

Exercise 1: Thin curved beam

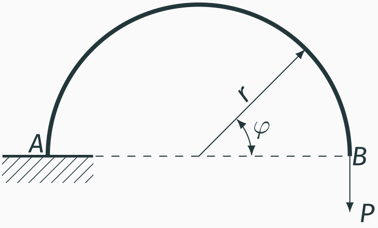


Figure 18: Thin curved beam [3].

Determine the vertical deflection of the end B of the thin curved beam of uniform cross section and semicircular center line.

Exercise 2: Increase in distance

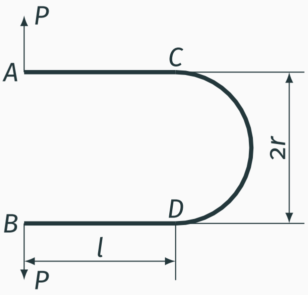


Figure 19: Increase in distance [3].

Determine the increase in distance between the ends A and B of a thin beam of uniform cross section consisting of a semicircular portion *CD* and two straight portions *AC* and *BD*.

Exercise 3: Piston ring

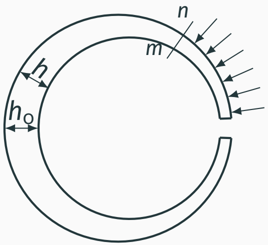


Figure 20: Piston ring [3].

A piston ring of a circular outer boundary has a rectangular cross section of constant width b and of a variable depth h. Determine the law of variation of h in order to obtain a ring which, when assembled with the piston in the cylinder, produces a uniformly distributed pressure on the cylinder wall.

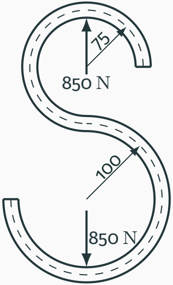


Figure 21: Open S link [4].

An open S link is made from a 25 mm diameter rod. Determine the maximum tensile stress and maximum shear stress.

Exercise 5: Offset bar

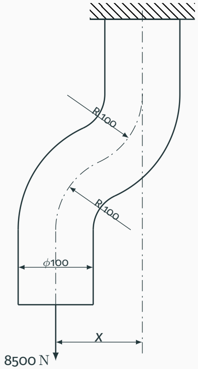


Figure 22: Curved beam geometry [4].

Consider the offset bar with circular cross section. The weight of the bar can be neglected. What is the maximum offset *x* if the allowable stress in tension is limited to 70 MPa.

End

Figure 23: Captain James B. Hook, author: woohnayoung, https://www.deviantart.com/woohnayoung/art/Captain-Hook-853386903

1.2. Thick Cylinders

In all things, success depends on previous preparation. And without such preparation there is sure to be failure.

Confucius, Analects - taken from Fundamentals of Machine Elements [9]

Summary

1. Introduction 53

2. Thin wall cylinders 54

3. Thick wall cylinders 60

4. Yielding criteria 69

5. Finite Element Method solution 76

6. Compound cylinders 78

7. Cylindrical interference fits 86

8. Proposed Exercises 91

Introduction

The circular cylinders are divided into two families: thin wall and thick wall cylinders [10]. The cylinder can have open or closed ends.

The main practical applications are: “pressure vessels such as gas storage cylinders, food and beverage cans and bottles, fuel tanks, hydraulic actuators and gun barrels; press and shrink fits; hydraulic and pneumatic tubing used for delivery of pressurized fluid” [9].

Figure 24: Typical applications of circular cylinders.

Thin wall cylinders

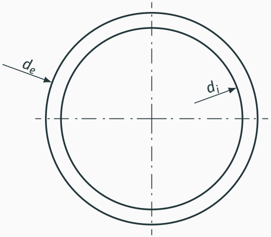


Figure 25: Thin wall cylinder: hoop stress.

$$t = \frac{d_e - d_i}{2}$$

If the wall thickness is small in comparison with the inner diameter, it is assumed that the stresses are uniformly distributed through it.

A thin wall cylinder should follow the following relation:

$$\frac{t}{d_i} \leq 0.051$$

$$K = \frac{d_e}{d_i} \leq 1.101$$

The reason behind such limits is presented ahead. Other sources state a value of $\frac{t}{d_i} < 0.1$ [11], $\frac{t}{d_i} < 0.083$ [10], $\frac{t}{d_i} < 0.05$ [1] and $\frac{t}{d_i} < 0.025$ [9].

Thin wall cylinders – hoop stress

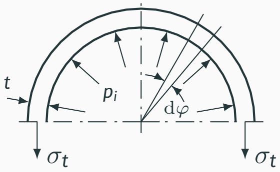


Figure 26: Thin wall cylinder with internal pressure[5] [11].

For internal pressure $p_i \neq 0$ and external pressure $p_o = 0$:

$$2\sigma_t t = 2\int_0^{\frac{\pi}{2}} p_i \sin\varphi r \,\mathrm{d}\varphi$$

$$2\sigma_t t = 2p_i r \left[-\cos\varphi\right]_0^{\frac{\pi}{2}} = 2p_i r$$

$$\sigma_t = \frac{p_i r}{t}$$

For internal pressure $p_i = 0$ and external pressure $p_o \neq 0$:

$$\sigma_t = -\frac{p_o r}{t}$$

These are the Mariotte's formulas for boilers.

Hoop stress is, in reality a function of the radius, but the differences between the values at the inner and outer radii are negligible.

[5]The inner radius is represented by r

Thin wall cylinders – axial stress

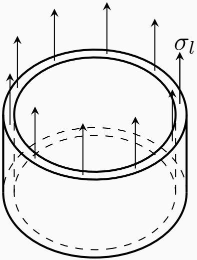


Figure 27: Thin wall cylinder: axial stress.

If the free ends are closed by two heads, the axial stress is:

For internal pressure $p_i \neq 0$ and external pressure $p_o = 0$:

$$2\pi r t \sigma_l = p_i \pi r^2$$

$$\sigma_l = \frac{p_i \pi r^2}{2\pi r t} = \frac{p_i r}{2t}$$

For internal pressure $p_i = 0$ and external pressure $p_o \neq 0$:

$$\sigma_t = -\frac{p_o r}{2t}$$

We can conclude that the hoop stress σ_t is approximately twice the axial stress σ_l.

Thin wall cylinders – radial stress

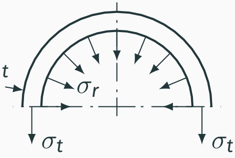


Figure 28: Thin wall cylinder: radial stress.

For internal pressure p_i:

The radial stress in the inner wall:

$$\sigma_r = -p_i$$

The radial stress in the outer wall:

$$\sigma_r = 0$$

For external pressure p_o:

The radial stress in the inner wall:

$$\sigma_r = 0$$

The radial stress in the outer wall:

$$\sigma_r = -p_o$$

Thin wall cylinders – principal stresses and strains

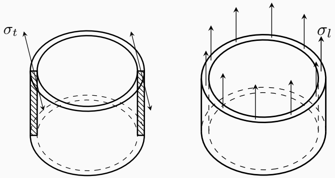


Figure 29: Stresses in a thin wall cylinder.

The axisymmetry and the uniform distribution of stress through the wall thickness means that the radial, tangential and axial are the principal directions of both stresses and strains [10].

$$\sigma_1 = \sigma_t$$

$$\sigma_2 = \sigma_l$$

$$\sigma_3 = \sigma_r$$

For cylinders with open ends: $\sigma_2 = \sigma_l = 0$

Thin wall cylinders – design

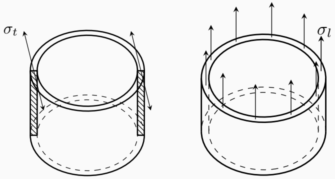


Figure 29: Stresses in a thin wall cylinder.

According to the SNCTTI code [11, 12], the radial stress is assumed to have the following value:

$$\sigma_r = \frac{-p_i + 0}{2} = -\frac{p_i}{2}$$

Using a failure criteria (Tresca):

$$\sigma_e = \sigma_1 - \sigma_3 = \sigma_y$$

With $\sigma_1 = \sigma_t = \frac{p_i r}{t}$ and $\sigma_3 = -\frac{p_i}{2}$:

$$\frac{p_i r}{t} + \frac{p_i}{2} = \frac{\sigma_y}{S_F}$$

S_F is a safety factor. The cylinder thickness t is given by:

$$t = \frac{p_i r}{\frac{\sigma_y}{S_F} - 0.5 p_i}$$

ASME Boiler & Pressure Vessel Code [13] suggests the following:

$$t = \frac{p_i r}{\frac{\sigma_y}{S_F} - 0.6 p_i}$$

Thick wall cylinders

Thick-walled circular cylinders, by convention, satisfy any one of the following equivalent conditions (Bickell and Ruiz 1967, Iurzolla 1981, Burr 1982, Ventsel and Krauthammer 2001, Ugural and Fenster 2003):

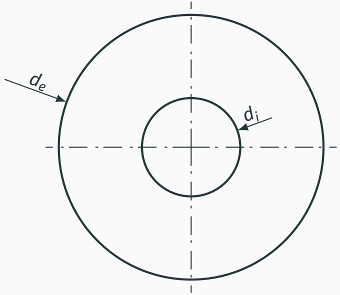


Figure 30: Thick wall cylinder.

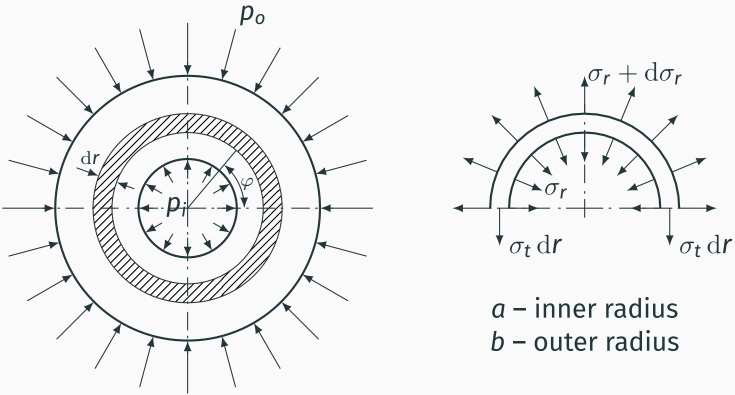
$$t = \frac{d_e - d_i}{2}$$

$$\frac{t}{d_i} > 0.051$$

$$K = \frac{d_e}{d_i} > 1.101$$

Other sources state a value of $\frac{t}{d_i} > 0.1$ [11], $\frac{t}{d_i} > 0.083$ [10], $\frac{t}{d_i} > 0.05$ [1] and $\frac{t}{d_i} > 0.025$ [9].

Thick wall cylinders



a – inner radius
b – outer radius

Figure 31: Thick wall cylinder with internal and external pressure [11].

Presentation of the topic taken from the Course Notes by Paulo M.S.T. Castro [11]. An equivalent treatment can be found in Féodossiev (pages 284–298) [14] or Timoshenko (pages 205–214) [15].

Thick wall cylinders – hoop and radial stress

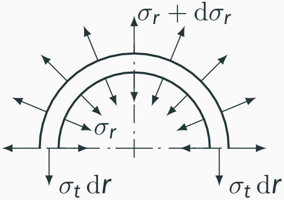


Figure 32: Infinitesimal element of the thick wall cylinder [11].

Static equilibrium written along the tangential direction:

$$2\sigma_t\,\mathrm{d}r + 2r\sigma_r - 2\left(\sigma_r + \mathrm{d}\sigma_r\right)\left(r + \mathrm{d}r\right) = 0$$

Disregarding higher order terms:

$$\sigma_t - \sigma_r - r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} = 0$$

Thick wall cylinders – hoop and radial stress

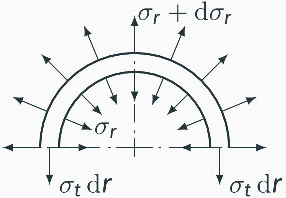


Figure 32: Infinitesimal element of the thick wall cylinder [11].

The equilibrium equation is:

$$\sigma_t - \sigma_r - r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} = 0$$

Using Hooke's law and the fact that the axial strain is constant:

$$\varepsilon_l = \frac{1}{E}\left[\sigma_l - \nu\left(\sigma_t + \sigma_r\right)\right]$$

In previous equation, if the axial stress $\sigma_l = 0$:

$$\sigma_t + \sigma_r = 2C_1 \Leftrightarrow \sigma_t = 2C_1 - \sigma_r$$

Replacing into equilibrium equation:

$$2C_1 - \sigma_r - \sigma_r - r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} = 0$$

$$2\sigma_r + r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} = 2C_1$$

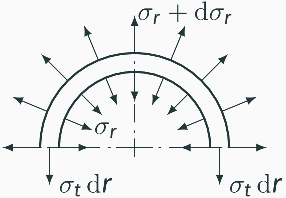


Figure 32: Infinitesimal element of the thick wall cylinder [11].

$$2\sigma_r + r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} = 2C_1$$

$$\underbrace{2r\sigma_r + r^2\frac{\mathrm{d}\sigma_r}{\mathrm{d}r}}_{\frac{\mathrm{d}}{\mathrm{d}r}(r^2\sigma_r)} = 2rC_1$$

$$2rC_1 = \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\sigma_r\right)$$

Integrating the equation:

$$r^2C_1 + C_2 = r^2\sigma_r$$

The radial and the hoop stress:

$$\begin{cases} \sigma_r = C_1 + \frac{C_2}{r^2} \\ \\ \sigma_t = C_1 - \frac{C_2}{r^2} \end{cases}$$

Thick wall cylinders – hoop and radial stress

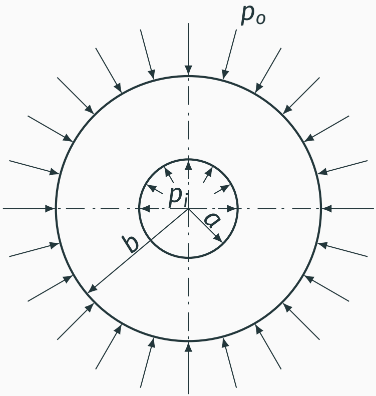


Figure 33: Thick wall cylinder with internal and external pressure.

$$\begin{cases} \sigma_r = C_1 + \frac{C_2}{r^2} \\ \sigma_t = C_1 - \frac{C_2}{r^2} \end{cases}$$

The pressure boundary conditions are then taken into account to determine the integration constants C_1 and C_2:

$$\begin{cases} \sigma_r = -p_i & r = a \\ \sigma_r = -p_o & r = b \end{cases}$$

$$\begin{cases} -p_i = C_1 + \frac{C_2}{a^2} \\ -p_o = C_1 + \frac{C_2}{b^2} \end{cases}$$

Thick wall cylinders – Lamé equations

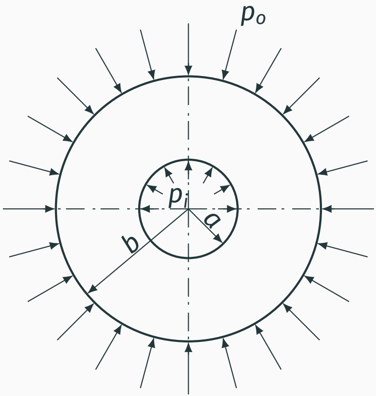


Figure 34: Thick wall cylinder with internal and external pressure.

The following solution was first presented by Lamé and Clapeyron[6]:

$$\begin{cases} \sigma_r = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} - \frac{(p_i - p_o) a^2 b^2}{r^2 (b^2 - a^2)} \\ \sigma_t = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} + \frac{(p_i - p_o) a^2 b^2}{r^2 (b^2 - a^2)} \end{cases}$$

[6] Lamé and Clapeyron, "Mémoire sur l'equilibre intérieur des corpes solides homogènes", Mém. divers savans, 1833.

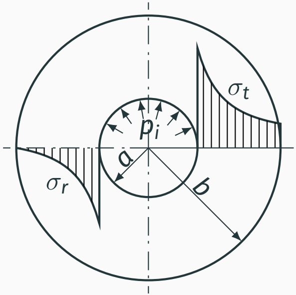


Figure 35: Hoop and radial stresses for a thick wall cylinder with internal pressure [11].

For internal pressure $p_i \neq 0$ and $p_o = 0$:

$$\begin{cases} \sigma_t = p_i \frac{a^2}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \\ \\ \sigma_r = p_i \frac{a^2}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \end{cases}$$

The radial stress at the inner radius is $-p_i$ and at the outer radius is $p_o = 0$.

The hoop stress at inner and outer radius:

$$\begin{cases} (\sigma_t)_{max} = p_i \frac{b^2 + a^2}{b^2 - a^2} & \text{inner radius} \\ (\sigma_t)_{min} = p_i \frac{2a^2}{b^2 - a^2} & \text{outer radius} \end{cases}$$

Thick wall cylinders – $p_i = 0$

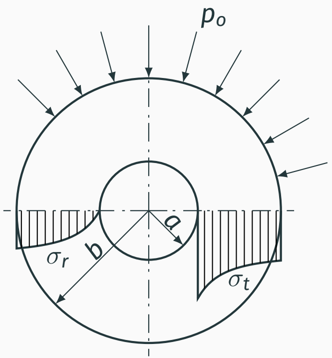


Figure 36: Hoop and radial stresses for a thick wall cylinder with external pressure [11].

For external pressure $p_o \neq 0$ and $p_i = 0$:

$$\begin{cases} \sigma_t = -p_o \frac{b^2}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right) \\ \\ \sigma_r = -p_o \frac{b^2}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right) \end{cases}$$

The radial stress at inner radius is $p_i = 0$ and at outer radius is $-p_o$.

The hoop stress at inner and outer radius:

$$\begin{cases} (\sigma_t)_{max} = -p_o \frac{2b^2}{b^2 - a^2} & \text{inner radius} \\ (\sigma_t)_{min} = -p_o \frac{b^2 + a^2}{b^2 - a^2} & \text{outer radius} \end{cases}$$

Yield criteria

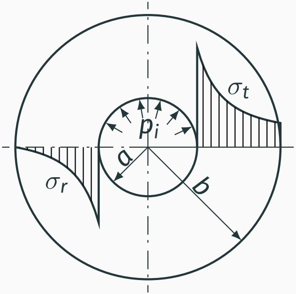


Figure 35: Hoop and radial stresses for a thick wall cylinder with internal pressure [11].

For a thick wall cylinder with internal pressure, the Yield starts at the inner radius.

Using a failure criteria (Tresca):

$$\sigma_e = \sigma_1 - \sigma_3 = \sigma_y$$

With $\sigma_1 = \sigma_{t_a} = p_i \frac{b^2+a^2}{b^2-a^2}$ and $\sigma_3 = -p_i$:

$$p_i \frac{b^2 + a^2}{b^2 - a^2} + p_i = \sigma_y$$

Using $K = \frac{b}{a}$:

$$\frac{p_i}{\sigma_y} = \frac{b^2 - a^2}{2b^2} = \frac{K^2 - 1}{2K^2}$$

Yield criteria

Other criteria for yielding of the internal wall due to internal pressure [10]:

Thick wall

Maximum normal stress

$\sigma_y = \sigma_{max} = \sigma_t = p_i \frac{b^2+a^2}{b^2-a^2}$ $\qquad \frac{p_i}{\sigma_y} = \frac{K^2-1}{K^2+1}$

Maximum principal strain:

$\sigma_y = E\varepsilon_{max} = \sigma_t - \nu\left(\sigma_r + \sigma_l\right)$ $\qquad \frac{p_i}{\sigma_y} = \frac{K^2-1}{(1-2\nu)+K^2(1+\nu)}$

Maximum distortion energy (von Mises):

$2\sigma_y^2 = \left(\sigma_t - \sigma_l\right)^2 + \left(\sigma_l - \sigma_r\right)^2 + \left(\sigma_t - \sigma_r\right)^2$ $\qquad \frac{p_i}{\sigma_y} = \frac{K^2-1}{\sqrt{3}K^2}$

Thin wall

Maximum normal stress (Hamburg formula):

$\sigma_y = \sigma_{max} = \sigma_t = \frac{p_i r}{t}$ $\qquad \frac{p_i}{\sigma_y} = K - 1$

Thin wall vs. thick wall

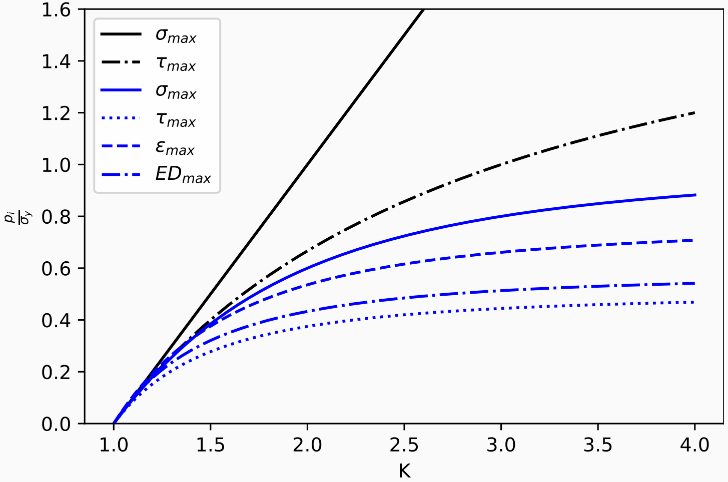


Figure 37: Several failure criteria for thin-wall (black) and thick-wall (blue) cylinders to define the beginning of yield due to internal pressure.

Thin wall vs. thick wall

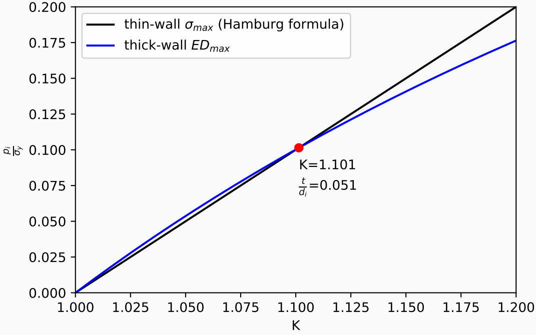


Figure 38: Maximum energy distortion for thick-wall vs. maximum normal stress (Hamburg formula) for thin-wall cylinder.

For $K \leq 1.101$ the maximum distortion energy criteria results in smaller wall thickness. This is the justification behind the international convention that sets the thickness limit for a thin-walled circular cylinder [10].

Thin wall vs. thick wall

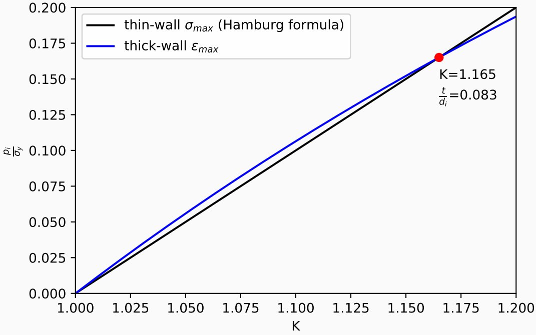


Figure 39: Maximum principal strain for thick-wall vs. maximum normal stress (Hamburg formula) for thin-wall cylinder.

Another thickness limit for a thin-walled circular cylinder can be set using the maximum strain theory: $K \leq 1.165$.

Autofrettage

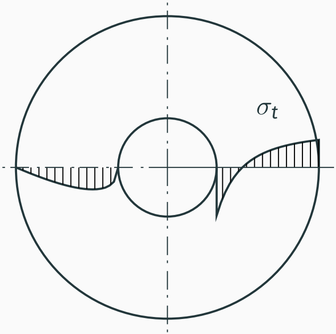


Figure 40: Permanent stress distribution after *autofrettage* process of a thick wall cylinder [14, 11].

In high pressure applications, apart from using compound cylinders, *autofrettage* is used, consisting of preloading the cylinder with an internal pressure greater than the yield, so as to obtain plastic deformations in the internal layers of the cylinder. When the pressure is gone, elastic extension stresses are retained in the outer layers and compressive stresses appear in the inner layers [14].

Fully plastic cylinder

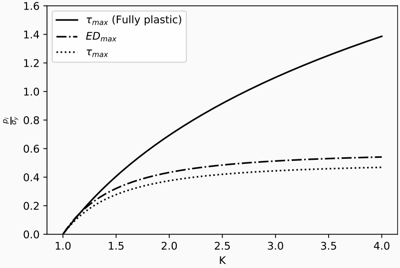


Figure 41: Internal pressure needed to reach a fully plastic cylinder using maximum shear stress theory [11].

Tresca criteria:

$$\sigma_y = \sigma_1 - \sigma_3 = \sigma_t - \sigma_r$$

Equilibrium equation:

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{\sigma_r - \sigma_t}{r} = 0$$

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} - \frac{\sigma_y}{r} = 0$$

After integration:

$$\sigma_r = \sigma_y \ln r + C_1$$

Boundary conditions: $\sigma_r = -p_i$ for $r = a$; $\sigma_r = 0$ for $r = b$.

$$p_i = \sigma_y \ln \frac{b}{a} \Leftrightarrow \frac{p_i}{\sigma_y} = \ln K$$

Finite Element Method solution

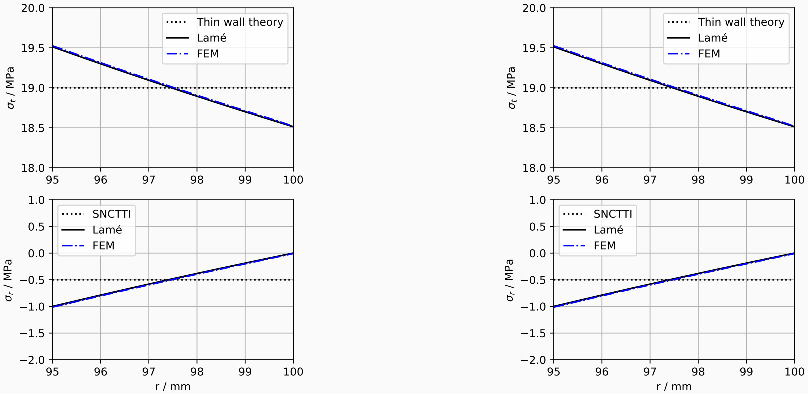


Figure 42: FEM 3D calculation with CalculiX [8] for a thin wall cylinder with internal pressure $p_i = 1\,\mathrm{MPa}$ (FEM calculation done with C3D20 elements for representation purposes)

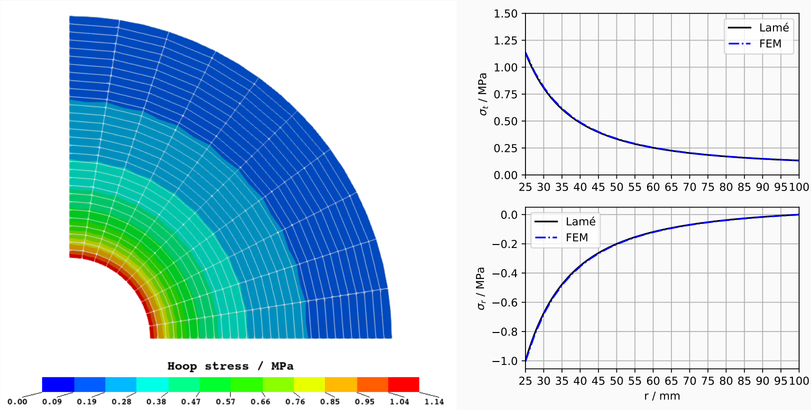


Figure 43: FEM 3D calculation with CalculiX [8] for a thick wall cylinder with internal pressure $p_i = 1\,\mathrm{MPa}$ (FEM calculation done with C3D20 elements for representation purposes).

Compound cylinders

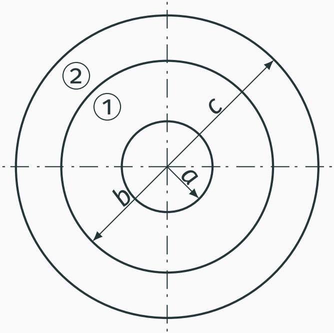


Figure 44: Compound cylinder.

The compound cylinders are structures consisting of concentric thick-walled cylinders assembled with a radial interference fit.

The cylinders can be:

- *press fit* assembled forcing them together with a press;
- *shrink fit*:
  - the outer cylinder **2** is preheated and expands to fit over the inner cylinder
  - the inner cylinder **1** is precooled and contracts.

Compound cylinders

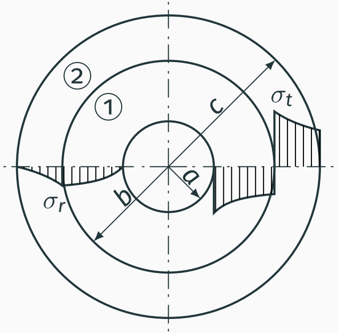


Figure 45: Compound cylinder.

Tangential stress:

- cylinder 1 outer radius:

$$\sigma_{t1} = -p_c \frac{b^2 + a^2}{b^2 - a^2}$$

- cylinder 2 inner radius:

$$\sigma_{t2} = p_c \frac{c^2 + b^2}{c^2 - b^2}$$

How to determine the contact pressure p_c?

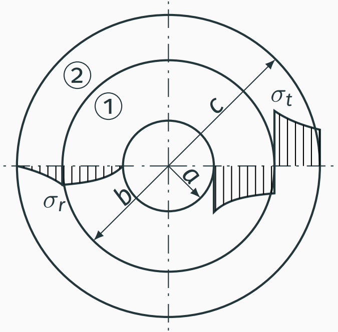


Figure 45: Compound cylinder [11].

The strain of the cylinder 2 inner radius is:

$$\varepsilon_{t2} = \frac{2\pi\left(b+\delta_2\right) - 2\pi b}{2\pi b} = \frac{\delta_2}{b}$$

The Hooke's law:

$$\varepsilon_{t2} = \frac{1}{E_2}\left(\sigma_{t2} - \nu_2\sigma_{r2}\right)$$

$$\delta_2 = \frac{bp_c}{E_2}\left(\frac{c^2+b^2}{c^2-b^2} + \nu_2\right)$$

$$\delta_1 = -\frac{bp_c}{E_1}\left(\frac{b^2+a^2}{b^2-a^2} - \nu_1\right)$$

Compound cylinder

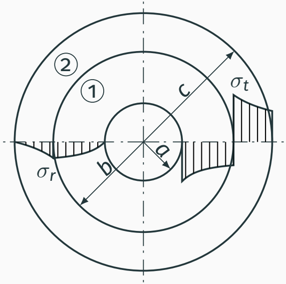


Figure 45: Compound cylinder [11, 14].

$$\delta_2 = \frac{bp_c}{E_2}\left(\frac{c^2+b^2}{c^2-b^2}+\nu_2\right)$$

$$\delta_1 = -\frac{bp_c}{E_1}\left(\frac{b^2+a^2}{b^2-a^2}-\nu_1\right)$$

Since $\delta = |\delta_2| + |\delta_1| = \delta_2 - \delta_1$ and for equal material $E_1 = E_2 = E$:

$$\boxed{p_c = \frac{E\delta}{b}\left[\frac{\left(c^2-b^2\right)\left(b^2-a^2\right)}{2b^2\left(c^2-a^2\right)}\right]}$$

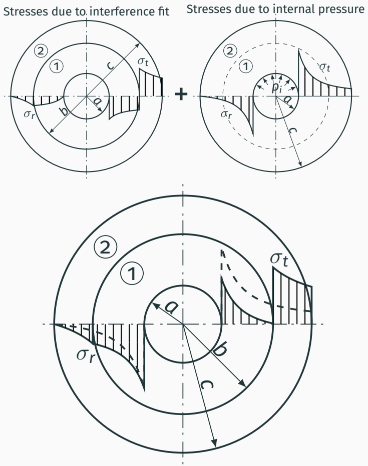


Figure 46: Compound cylinder with internal pressure [11].

The hoop stresses on the inner cylinder are now lower. So, the compound cylinder must withstand higher internal pressure than a single cylinder with the same dimensions.

The design is now based on the inner radius of the outer cylinder **2** where the stress state is more demanding, but still better than in the single cylinder with the same dimensions.

Optimizing two circular cylinders with interference fit

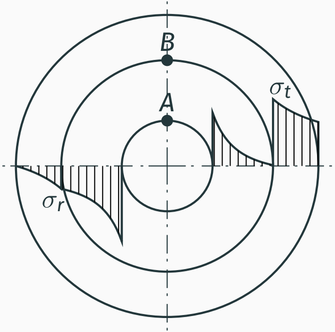


Figure 47: Compound cylinder with internal pressure [11, 14].

The stress state is now worse in the outer cylinder. Assuming the same material and the same yield strength for both cylinders.

Using Tresca criteria:

$$\sigma_e = \sigma_1 - \sigma_3 = \sigma_t - \sigma_r$$

In point A:

$$\sigma_{e_A} = \underbrace{p_i \frac{c^2 + a^2}{c^2 - a^2} - p_c \frac{2b^2}{b^2 - a^2}}_{\sigma_1} - \underbrace{(-p_i)}_{\sigma_3}$$

$$\sigma_{e_A} = p_i \frac{2c^2}{c^2 - a^2} - p_c \frac{2b^2}{b^2 - a^2}$$

Optimizing two circular cylinders with interference fit

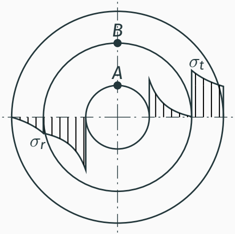


Figure 47: Compound cylinder with internal pressure [11, 14].

Using Tresca criteria in point B:

$$\sigma_{e_B} = \underbrace{p_c \frac{c^2 + b^2}{c^2 - b^2} + p_i \frac{a^2}{c^2 - a^2}\left(1 + \frac{c^2}{b^2}\right) - p_i \frac{a^2}{c^2 - a^2}\left(1 - \frac{c^2}{b^2}\right)}_{\sigma_1} - \underbrace{(-p_c)}_{\sigma_3}$$

$$\sigma_{e_B} = \frac{2c^2}{b^2}\left(p_i \frac{a^2}{c^2 - a^2} + p_c \frac{b^2}{c^2 - b^2}\right)$$

Optimizing two circular cylinders with interference fit

In order to have the same equivalent stress in both cylinders, we should do:

$$\sigma_{e_A} = \sigma_{e_B}$$

Replacing p_c into the equality $\sigma_{e_A} = \sigma_{e_B}$ we get the interference value:

$$\delta = \frac{2p_i}{E} \frac{bc^2\left(b^2 - a^2\right)}{b^2\left(c^2 - b^2\right) + c^2\left(b^2 - a^2\right)}$$

Recall that the contact pressure is:

$$p_c = \frac{E\delta}{b}\left[\frac{\left(c^2 - b^2\right)\left(b^2 - a^2\right)}{2b^2\left(c^2 - a^2\right)}\right]$$

Replacing p_c into σ_{e_A}:

$$\sigma_e = p_i \frac{2c^2}{c^2 - a^2}\left(1 - \frac{1}{\frac{b^2}{b^2 - a^2} + \frac{c^2}{c^2 - b^2}}\right)$$

The equation has a minimum for $b = \sqrt{ac}$ known as *Gadoline conditions*[7] [14, 11, 16]:

$$\sigma_e^{min} = p_i \frac{c}{c - a}$$

[7] Féodossiev, V., Résistance des Matériaux, 2 édition, Éditions MIR, Moscou, 1971, chpater IX, page 296.

Cylindrical interference fits

The maximum torque M_t that can be transmitted using a cylindrical interference fit is [11]:

$$M_t = F_c \cdot b = \overbrace{\underbrace{2 \cdot \pi \cdot b \cdot L}_{A} \cdot \underbrace{p_c \cdot \mu}_{\tau_c}}^{F_c} \cdot b$$

The axial force F_a needed to disassemble the parts is:

$$F_a \geq \underbrace{2 \cdot \pi \cdot b \cdot L}_{A} \cdot \underbrace{p_c \cdot \mu}_{\tau_a}$$

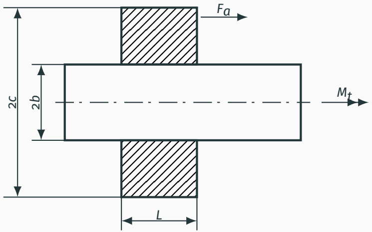


Figure 48: Cylindrical interference fit.

p_c is the contact pressure of the interference fit
A is the contact area of the interference fit
F_c is the circumferential force
b is the radius of the interference fit
c is the radius of the outer cylinder
L is the length of the shortest part on the interference fit
μ is the coefficient of adhesion/friction

Cylindrical interference fits

The maximum shear stress of the interference fit is [9]:

$$\tau_{max} = p_c \cdot \mu$$

The circumferential force causes the following shear stress:

$$\tau_c = \frac{F_c}{A} = \frac{F_c}{2 \cdot \pi \cdot a \cdot L}$$

The axial force causes the following shear stress:

$$\tau_a = \frac{F_a}{A} = \frac{F_a}{2 \cdot \pi \cdot a \cdot L}$$

The axial and circumferential shear stresses are related to the maximum:

$$\tau_{max} = \sqrt{\tau_a^2 + \tau_c^2}$$

Coefficient of adhesion according to DIN 7190

The coefficient of adhesion for longitudinal fits are presented for loosening μ_{ll} or μ_{rl} slipping conditions [17]:

Material		Coefficient of adhesion			
Used	New	Dry		Lubricated	
		μ_{ll}	μ_{rl}	μ_{ll}	μ_{rl}
St 60-2	E 335	0.11	0.08	0.08	0.07
GS-60	GE 300	0.11	0.08	0.08	0.07
RSt37-2	S 235JRG2	0.10	0.09	0.07	0.06
GG-25	EN-GJL-250	0.12	0.11	0.06	0.05
GGG-60	EN-GJS-600-3	0.10	0.09	0.06	0.05
G-AlSi12(Cu)	EN AB-44000	0.07	0.06	0.05	0.04
G-CuPb10Sn	CB495K	0.07	0.06	-	-
TiAl6V4	TiAl6V4	-	-	0.05	-

Coefficient of adhesion according to DIN 7190

The coefficient of adhesion for radial interference fits in the longitudinal (μ_{rl}) and tangential (μ_{ru}) directions subjected to slipping [17]:

Material pairing, lubrication and joining method	
Steel-steel pairing	
Oil pressure connection, normally joined with mineral oil	0.12
Oil pressure connection, degreased contact surfaces, joined with glycerin	0.18
Shrink fit, normally heating up to 300 $^\circ\mathrm{C}$ the external component in electrical oven	0.14
Shrink fit, degreased contact surfaces, heating up to 300 $^\circ\mathrm{C}$ in electrical oven	0.20
Steel-cast iron pairing	
Oil pressure connection, normally joined with mineral oil	0.10
Oil pressure connection, with degreased contact surfaces	0.16
Steel-MgAl pairing, dry	0.10-0.15
Steel-CuZn pairing, dry	0.17-0.25

Typical cylindrical interference fits

Just a brief review of ISO 286. Typical interference fits: H7/p6, H7/r6, H7/s6

Basic hole	Tolerance classes for shafts																	
	Clearance fits							Transition fits				Interference fits						
H 6						g5	h5	js5	k5	m5		n5	p5					
H 7					f6	**g6**	**h6**	**js6**	**k6**	m6	**n6**		**p6**	**r6**	**s6**	t6	u6	x6
H 8				e7	**f7**		**h7**	js7	k7	m7					s7		u7	
			d8	**e8**	f8		h8											
H 9			d8	**e8**	f8		h8											
H10	b9	c9	d9	e9			h9											
H 11	**b11**	**c11**	d10				h10											

(a) Basic hole (H)

Basic shaft	Tolerance classes for holes																	
	Clearance fits							Transition fits				Interference fits						
h 5						G6	H6	JS6	K6	M6		N6	P6					
h 6					F7	**G7**	**H7**	**JS7**	**K7**	M7	**N7**		**P7**	**R7**	**S7**	T7	U7	X7
h 7				E8	**F8**		**H8**											
h 8			D9	E9	F9		H9											
h 9				E8	**F8**		**H8**											
			D9	**E9**	F9		**H9**											
	B11	C10	**D10**				H10											

(b) Basic shaft (h)

Figure 49: Preferable fits according to ISO 286 [18].

Solved Exercise

Consider a cylindrical interference fit between a hub and a hollow shaft with a nominal diameter of 60 mm and H7/u6 interference fit according to ISO 286 standard. The shaft and the hub are made of C45 steel. The shaft inner diameter is 10 mm, the hub outer diameter is 90 mm.

Determine the minimum and maximum interference pressure p_c that can occur. What is the maximum, minimum and average transmittable torque if $\mu = 0.15$ and $L = 50\,\mathrm{mm}$?

$$\delta_{max} = D_{f_{shaft,max}} - D_{f_{hub,min}}$$

$$\delta_{min} = D_{f_{shaft,min}} - D_{f_{hub,max}}$$

$$\delta_{max} = e_s - E_i = 0.106\,\mathrm{mm}$$

$$\delta_{min} = e_i - E_s = 0.057\,\mathrm{mm}$$

$$p_{c,max} = 99.5\,\mathrm{MPa}$$

$$p_{c,min} = 53.2\,\mathrm{MPa}$$

Exercise 1: Thin wall cylinder

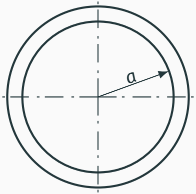


Figure 50: Thin wall cylinder [3].

Consider the thin wall cylinder in Figure 50. The inner radius is $a = 50\,\mathrm{mm}$. The yield strength of the material is $\sigma_y = 400\,\mathrm{MPa}$ and the internal pressure is $p_i = 2\,\mathrm{MPa}$. Use a safety factor of 3, consider that thin wall analysis is adequate and determine the wall thickness using the maximum shear stress theory. Repeat the calculation using the thick wall theory and quantify the difference on the wall thickness.

Exercise 2: Hoop stresses in a thick wall cylinder

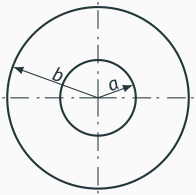


Figure 51: Thick wall cylinder [15].

Determine the tangential stresses at the inner and outer surfaces and at the middle thickness of the wall of a cylinder with inner radius $a = 100\,\mathrm{mm}$ and outer radius $b = 200\,\mathrm{mm}$, subjected to an internal pressure of $p_i = 200\,\mathrm{MPa}$. Repeat the calculation assuming $p_i = 0\,\mathrm{MPa}$ and $p_o = 200\,\mathrm{MPa}$ [15].

Exercise 3: Compound cylinder

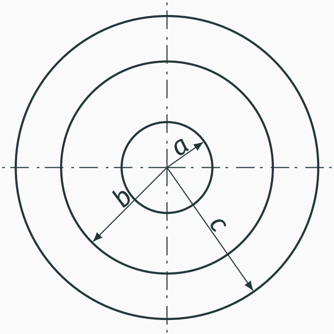


Figure 52: Compound cylinder [15].

Determine the stresses in the compound cylinder of Figure 52 subjected to an internal pressure $p_i = 200\,\mathrm{MPa}$ if $a = 100\,\mathrm{mm}$, $b = 150\,\mathrm{mm}$, $c = 200\,\mathrm{mm}$ and the interference is $\delta = 0.125\,\mathrm{mm}$. Consider that both materials are steel.

Exercise 4: Hub and shaft

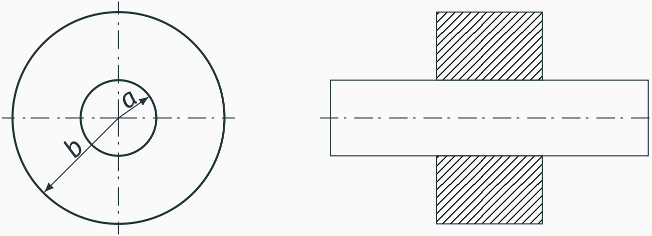


Figure 53: Hub and shaft [4].

For the hub and shaft in Figure 53, both in steel with $E = 206\,\mathrm{GPa}$, find the uniform pressure p_c if the radius of the shaft is $150\,\mathrm{mm}$ and the outer radius of the hub is $300\,\mathrm{mm}$. The initial difference in diameters between hub and shaft is $\delta = 0.3\,\mathrm{mm}$.

Exercise 5: Two stiff shafts

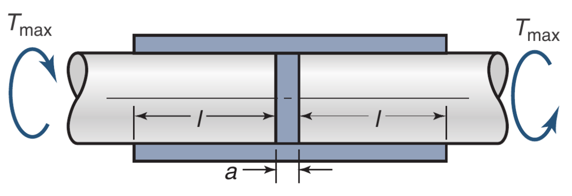


Figure 54: Two stiff shafts connected by a thin-wall tube [9].

Two stiff shafts are connected by a thin-walled elastic tube to a press-fit connection as shown in Figure 54. The contact pressure between the shafts and the tube is p_c.

The coefficient of friction is μ. Calculate the maximum torque T_{max} that can be transmitted through the press fit. Describe and calculate what happens if the torque decreases from T_{max} to θT_{max} where $0 < \theta < 1$.

1.3. Rotating Cylinders

It must be confessed that the inventors of the mechanical arts have been much more useful to men than the inventors of syllogisms.

Voltaire

Summary

1. Introduction 99

2. Stresses in a rotating cylinder 101

3. Solution for the non-homogeneous ODE 108

4. Tangential and radial stresses 113

5. Rotating press fit 122

Introduction

For cylinders subjected to important angular speed, both radial and tangential stresses exist as in the theory for thick-walled cylinders. However the cause of the stresses is the inertial forces acting on all the particles of the cylinder [1].

Several rotating machine elements, such as flywheels and blowers can be simplified as a rotating cylinder.

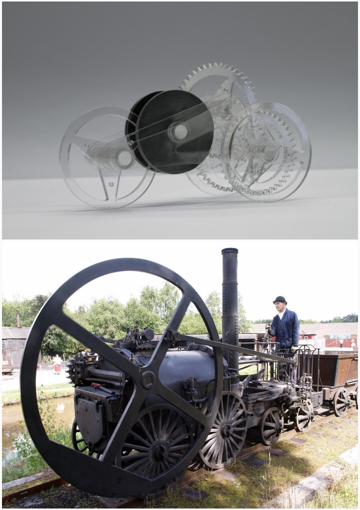


Figure 55: Rotating cylinders.

The tangential and radial stresses found with the following equations are subjected to the following restrictions [19, 1]:

- the outside radius is large compared with thickness ($b > 10t$);
- the thickness is constant;
- the stresses are constant over the thickness.

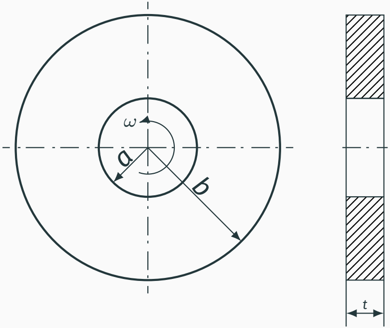


Figure 56: Rotating cylinder.

Stresses in a rotating cylinder

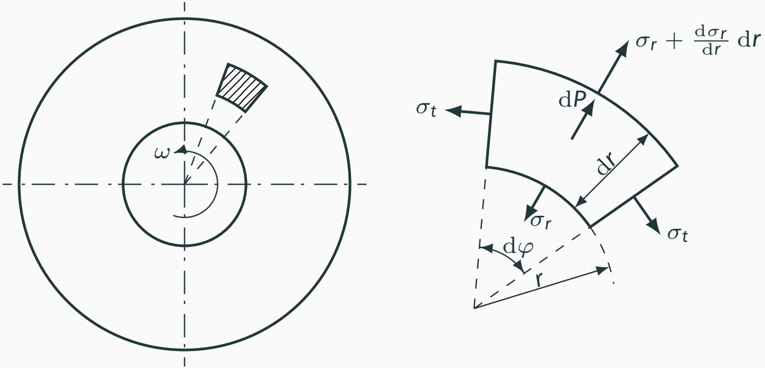


Figure 57: Stresses in a rotating cylinder [14, 15, 11].

Presentation of the topic taken from Timoshenko (pages 214–223) [15] and Féodossiev [14], similar to the Course Notes by Paulo M.S.T. Castro [20] .

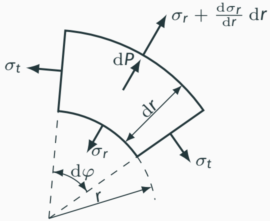


Figure 58: Infinitesimal element of the rotating thick wall cylinder [14, 15, 11, 20].

$$\sigma_t t\,\mathrm{d}r\,\mathrm{d}\varphi + \sigma_r tr\,\mathrm{d}\varphi - \left(\sigma_r + \frac{\mathrm{d}\sigma_r}{\mathrm{d}r}\,\mathrm{d}r\right)(r + \mathrm{d}r)\,t\,\mathrm{d}\varphi - \mathrm{d}P = 0$$

The inertia forces are the product of mass $\frac{\gamma}{g} tr\,\mathrm{d}\varphi\,\mathrm{d}r$ by the centrifugal acceleration $\omega^2 r$:

$$\mathrm{d}P = \frac{\gamma}{g}\omega^2 r^2 t\,\mathrm{d}\varphi\,\mathrm{d}r = \rho\omega^2 r^2 t\,\mathrm{d}\varphi\,\mathrm{d}r$$

Hoop and radial stress

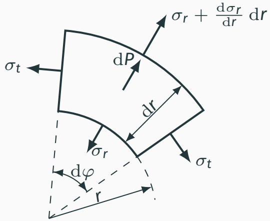


Figure 58: Infinitesimal element of the rotating thick wall cylinder [14, 15, 11, 20].

$$\sigma_t t\,\mathrm{d}r\,\mathrm{d}\varphi + \sigma_r r t\,\mathrm{d}\varphi - \left(\sigma_r + \frac{\mathrm{d}\sigma_r}{\mathrm{d}r}\,\mathrm{d}r\right)(r + \mathrm{d}r)\,t\,\mathrm{d}\varphi - \underbrace{\rho\omega^2 r^2 t\,\mathrm{d}\varphi\,\mathrm{d}r}_{\mathrm{d}P} = 0$$

Simplify the common terms t and $\mathrm{d}\varphi$:

$$\sigma_t\,\mathrm{d}r + \sigma_r r - \sigma_r r - \mathrm{d}\sigma_r\, r - \sigma_r\,\mathrm{d}r - \mathrm{d}\sigma_r\,\mathrm{d}r - \rho\omega^2 r^2\,\mathrm{d}r = 0$$

Hoop and radial stress

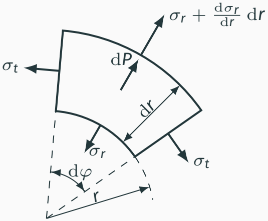


Figure 58: Infinitesimal element of the rotating thick wall cylinder [14, 15, 11, 20].

$$\sigma_t \, dr + \sigma_r r - \sigma_r r - d\sigma_r \, r - \sigma_r \, dr - d\sigma_r \, dr - \rho\omega^2 r^2 \, dr = 0$$

Disregarding higher order terms:

$$\sigma_t - \sigma_r - r\frac{d\sigma_r}{dr} - \rho\omega^2 r^2 = 0$$

Hoop and radial stress

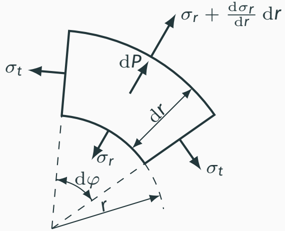


Figure 58: Infinitesimal element of the rotating thick wall cylinder [14, 15, 11, 20].

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\sigma_r r\right) - \sigma_t = -\rho\omega^2 r^2$$

Using the Hooke's law:

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \nu\left(\sigma_t + \sigma_l\right)\right]$$

$$\varepsilon_t = \frac{1}{E}\left[\sigma_t - \nu\left(\sigma_r + \sigma_l\right)\right]$$

Considering $\sigma_l = 0$:

$$\sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_t\right)$$

$$\sigma_t = \frac{E}{1-\nu^2}\left(\varepsilon_t + \nu\varepsilon_r\right)$$

Hoop and radial stress

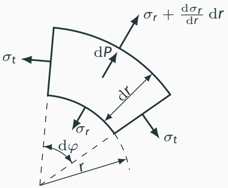


Figure 58: Infinitesimal element of the rotating thick wall cylinder [14, 15, 11, 20].

The circumferential strain is:

$$\varepsilon_t = \frac{2\pi\left(r + u\right) - 2\pi r}{2\pi r} = \frac{u}{r}$$

The radial strain is:

$$\varepsilon_r = \frac{\mathrm{d}u}{\mathrm{d}r} = r\frac{\mathrm{d}\varepsilon_t}{\mathrm{d}r} + \varepsilon_t$$

And substitute into radial and tangential stress equations:

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{\mathrm{d}u}{\mathrm{d}r} + \nu\frac{u}{r}\right)$$

$$\sigma_t = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{\mathrm{d}u}{\mathrm{d}r}\right)$$

Hoop and radial stress

$$\boxed{\sigma_t - \sigma_r - r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} - \rho\omega^2 r^2 = 0}$$

Substitute each stress component as function of the radial displacement u:

$$\frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{\mathrm{d}u}{\mathrm{d}r}\right) - \frac{E}{1-\nu^2}\left(\frac{\mathrm{d}u}{\mathrm{d}r} + \nu\frac{u}{r}\right) - \frac{E}{1-\nu^2}\left(r\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \nu\frac{r}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \nu r\frac{u}{r^2}\right) - \rho\omega^2 r^2 = 0$$

$$\left(\frac{u}{r} + \nu\frac{\mathrm{d}u}{\mathrm{d}r}\right) - \left(\frac{\mathrm{d}u}{\mathrm{d}r} + \nu\frac{u}{r}\right) - \left(r\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \nu\frac{\mathrm{d}u}{\mathrm{d}r} - \nu\frac{u}{r}\right) - \frac{1-\nu^2}{E}\rho\omega^2 r^2 = 0$$

$$-\frac{u}{r^2} - \nu\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} + \nu\frac{u}{r^2} + \frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \nu\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \nu\frac{u}{r^2} + \frac{1-\nu^2}{E}\rho\omega^2 r = 0$$

Finally, we get a second order non-homogeneous ordinary differential equation (ODE):

$$\boxed{\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r^2} + \frac{1-\nu^2}{E}\rho\omega^2 r = 0}$$

Solution for the non-homogeneous ODE

$$\boxed{\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r^2} + \frac{1-\nu^2}{E}\rho\omega^2 r = 0}$$

Making $\boxed{N = \frac{1-\nu^2}{E}\rho\omega^2}$:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r^2} + Nr = 0$$

$$u''(r) + \frac{u'(r)}{r} - \frac{u(r)}{r^2} + Nr = 0$$

Multiplying by r^2 both sides, we obtain an Euler-Cauchy equation:

$$r^2 u''(r) + r u'(r) - u(r) = -Nr^3$$

Solution for the non-homogeneous ODE

The solution for $r^2u'' + ru' - u = -Nr^3$ is:

$$u = u_h + u_p$$

u_h is the general solution for the homogeneous ODE
$r^2u'' + ru' - u = 0$

u_p is the particular solution that satisfies the non-homogeneous ODE.

For $r^2u'' + ru' - u = 0$ and assuming a trial solution $u = r^m$:

$$r^2\left(m\left(m-1\right)r^{m-2}\right) + r\left(mr^{m-1}\right) - r^m = 0$$

$$m\left(m-1\right)r^m + mr^m - r^m = \left(m^2 - m + m - 1\right)r^m = 0$$

Solution for the non-homogeneous ODE

$$\boxed{\left(m^2 - m + m - 1\right) r^m = 0}$$

For this equation $r^m = 0$ or $m = 1$ or $m = -1$ and the general solution is:

$$u_h = C_1 r^1 + C2r^{-1} = C_1 r + \frac{C_2}{r}$$

Now we recall the non-homogeneous ODE:

$$r^2 u''(r) + ru'(r) - u(r) = -Nr^3$$

Recalling the original form of the non-homogeneous ODE:

$$\boxed{u''(r) + \frac{1}{r}u'(r) - \frac{1}{r^2}u(r) = -Nr}$$

Solution for the non-homogeneous ODE

$$u''(r) + \frac{1}{r}u'(r) - \frac{1}{r^2}u(r) = -Nr \quad \text{and} \quad u_h = C_1 r + \frac{C_2}{r}$$

Doing the Wronskians:

$$u_p = a_1 r + a_2 \frac{1}{r}$$

$$W = \begin{vmatrix} r & \frac{1}{r} \\ 1 & -\frac{1}{r^2} \end{vmatrix} = -\frac{1}{r} - \frac{1}{r} = -\frac{2}{r}$$

$$a_1' = \frac{W_1}{W} = -\frac{Nr}{2} \Leftrightarrow a_1 = -\frac{Nr^2}{4}$$

$$W_1 = \begin{vmatrix} 0 & \frac{1}{r} \\ -Nr & -\frac{1}{r^2} \end{vmatrix} = 0 + N = N$$

$$a_2' = \frac{W_2}{W} = \frac{Nr^3}{2} \Leftrightarrow a_2 = \frac{Nr^4}{8}$$

The particular solution is:

$$W_2 = \begin{vmatrix} r & 0 \\ 1 & -Nr \end{vmatrix} = -Nr^2 - 0 = -Nr^2$$

$$\boxed{u_p = -N\frac{r^2}{4}r + N\frac{r^4}{8}\frac{1}{r} = -N\frac{r^3}{8}}$$

Solution for the non-homogeneous ODE

$$\boxed{u = u_h + u_p = C_1 r + \frac{C_2}{r} - N\frac{r^3}{8}}$$

Recall again the radial stress equation as function of the radial displacement:

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{\mathrm{d}u}{\mathrm{d}r} + \nu\frac{u}{r}\right)$$

$$\sigma_r = \frac{E}{1-\nu^2}\left[C_1 - \frac{C_2}{r^2} - 3N\frac{r^2}{8} + \nu C_1 + \nu\frac{C_2}{r^2} - \nu N\frac{r^2}{8}\right]$$

$$\boxed{\sigma_r = \frac{E}{1-\nu^2}\left[C_1\left(1+\nu\right) - \left(1-\nu\right)\frac{C_2}{r^2} - \frac{3+\nu}{8}Nr^2\right]}$$

Radial and tangential stresses

$$\boxed{\sigma_r = \frac{E}{1-\nu^2}\left[C_1\left(1+\nu\right) - \left(1-\nu\right)\frac{C_2}{r^2} - \frac{3+\nu}{8}Nr^2\right]}$$

The boundary conditions for a disc with a hole at center:

$\sigma_r = 0 \quad \text{for} \quad r = a$

$$0 = \frac{E}{1-\nu^2}\left[C_1\left(1+\nu\right) - \left(1-\nu\right)\frac{C_2}{a^2} - \frac{3+\nu}{8}Na^2\right]$$

$\sigma_r = 0 \quad \text{for} \quad r = b$

$$0 = \frac{E}{1-\nu^2}\left[C_1\left(1+\nu\right) - \left(1-\nu\right)\frac{C_2}{b^2} - \frac{3+\nu}{8}Nb^2\right]$$

Radial and tangential stresses

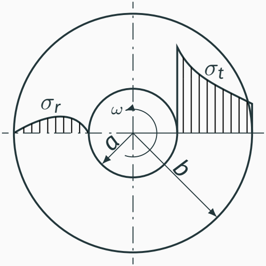


Figure 59: Hoop and radial stresses for a rotating thick wall cylinder.

The value of constants C_1 and C_2:

$$\begin{cases} C_1 = \frac{3+\nu}{8(1+\nu)} \left(a^2 + b^2\right) N \\ C_2 = \frac{3+\nu}{8(1+\nu)} \left(a^2 b^2\right) N \end{cases}$$

With $\boxed{N = \dfrac{1-\nu^2}{E}\rho\omega^2}$

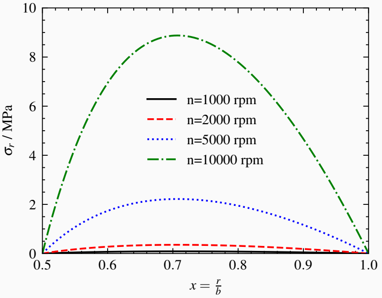
Finally the radial and hoop stresses equations are:

$$\begin{cases} \sigma_r = \frac{3+\nu}{8}\rho\omega^2 \left(a^2 + b^2 - \frac{a^2 b^2}{r^2} - r^2\right) \\ \sigma_t = \frac{3+\nu}{8}\rho\omega^2 \left(a^2 + b^2 + \frac{a^2 b^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2\right) \end{cases}$$

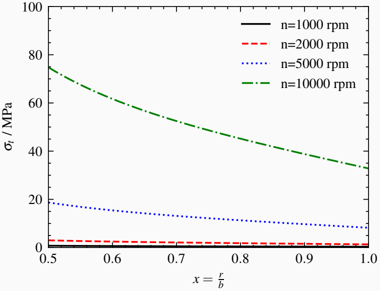
Influence of angular speed

As expected, as the angular speed increases both the radial and tangential stress increase.

For a steel disc with $b = 100\,\mathrm{mm}$.



(a) Radial stress



(b) Hoop stress

Figure 60: Stresses for a rotating disc made of steel with $K = \frac{b}{a} = 2$.

Location of maximum stresses

The radial stress equation presents a maximum for $r = \sqrt{ab}$:

$$\boxed{(\sigma_r)_{max} = \frac{3+\nu}{8}\rho\omega^2\left(b-a\right)^2}$$

The tangential stress is maximum at the inner edge of the disc $r = a$:

$$\boxed{(\sigma_t)_{max} = \frac{3+\nu}{4}\rho\omega^2\left(b^2 + \frac{1-\nu}{3+\nu}a^2\right)}$$

It is easy now to conclude that $(\sigma_t)_{max}$ is always larger than $(\sigma_r)_{max}$.

Thin wall cylinder

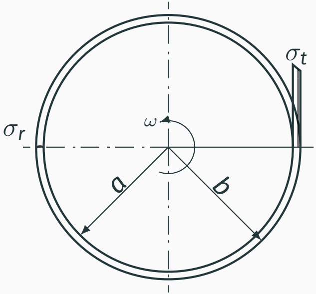


Figure 61: Hoop and radial stresses for a rotating thin wall cylinder.

According to the equations derived previously for thick wall cylinders:

$$\begin{cases} \sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(a^2 + b^2 - \frac{a^2b^2}{r^2} - r^2\right) \\ \sigma_t = \frac{3+\nu}{8}\rho\omega^2\left(a^2 + b^2 + \frac{a^2b^2}{r^2} - \frac{1+3\nu}{3+\nu}r^2\right) \end{cases}$$

Considering a wall thickness very small in comparison with the radius, and making $a \longrightarrow b$:

$$\boxed{\sigma_t = \rho\omega^2 b^2 = \rho\omega^2 r^2}$$

Thin wall cylinder

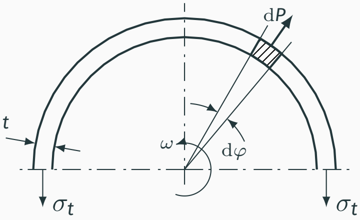


Figure 62: Hoop and radial stresses for a rotating thin wall cylinder.

$$\mathrm{d}P = r\,\mathrm{d}\varphi\, t\rho\omega^2 r$$

Using the thin wall theory, i.e. assuming constant hoop stress through the cylinder wall:

$$2\sigma_t t = 2\int_0^{\frac{\pi}{2}} r^2 t\rho\omega^2 \sin\varphi\,\mathrm{d}\varphi$$

$$2\sigma_t t = 2r^2 t\rho\omega^2 \left[-\cos\varphi\right]_0^{\frac{\pi}{2}}$$

Using the thin wall theory, we find the same result as with the thick wall theory equations:

$$\boxed{\sigma_t = \rho\omega^2 r^2}$$

Solid disc

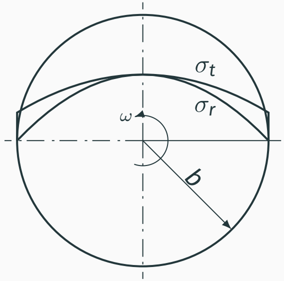


Figure 63: Hoop and radial stresses for a solid disc.

For $a = 0$, the equation becomes:

$$\begin{cases} \sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(b^2 - r^2\right) \\ \\ \sigma_t = \frac{3+\nu}{8}\rho\omega^2\left(b^2 - \frac{1+3\nu}{3+\nu}r^2\right) \end{cases}$$

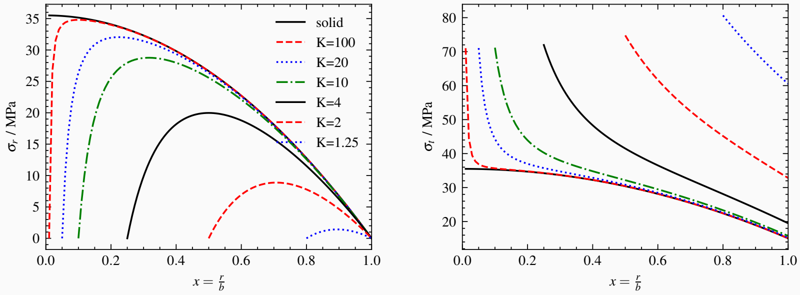
$$\boxed{(\sigma_r)_{max} = (\sigma_t)_{max} = \frac{3+\nu}{8}\rho\omega^2 b^2}$$

$$\boxed{(\sigma_t)_{min} = \frac{1-\nu}{4}\rho\omega^2 b^2}$$

Influence of hole size

As discussed for "Thick Cylinders", the ratio K is $\frac{b}{a}$.

For a steel disc rotating at $n = 10\,000\,\mathrm{rpm}$ and $b = 100\,\mathrm{mm}$.



(a) Radial stress

(b) Hoop stress

Figure 64: Influence of cylinder wall thickness (or hole dimension) [15].

Finite Element Method solution

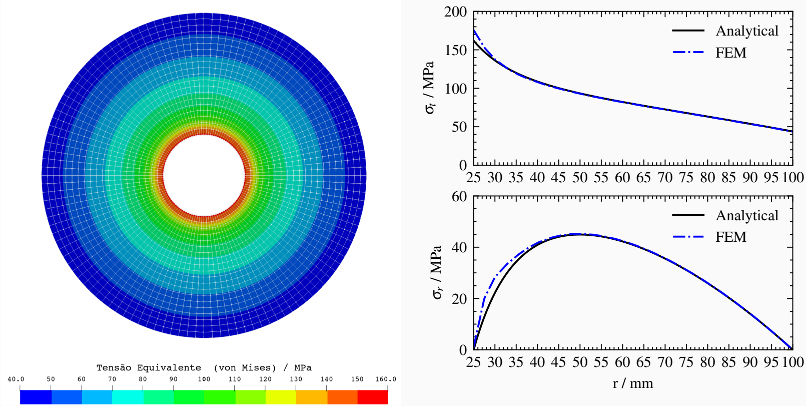


Figure 65: FEM 3D calculation with CalculiX [8] for a steel rotating cylinder $a = 25\,\mathrm{mm}$ and $b = 100\,\mathrm{mm}$ and $n = 15\,000\,\mathrm{rpm}$.

Rotating press fit

Two concentric AISI 1040 steel cylinders are press fit together with a radial interference of $0.025\,\mathrm{mm}$. The nominal sizes of the tubes are $c = 50\,\mathrm{mm}$ and $b = 40\,\mathrm{mm}$ for the outer tube and $b = 40\,\mathrm{mm}$ and $a = 30\,\mathrm{mm}$ for the inner tube [9].

How fast should the compound cylinder rotate in order to decrease the press-fit pressure to zero?

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(a^2 + c^2 - \frac{a^2c^2}{b^2} - b^2\right) = \frac{3+0.3}{8}7.85\times10^{-9}\omega^2\left(30^2 + 50^2 - \frac{30^2\times50^2}{40^2} - 40^2\right)$$

$$p_c = \frac{E\delta}{b}\left[\frac{\left(c^2-b^2\right)\left(b^2-a^2\right)}{2b^2\left(c^2-a^2\right)}\right] = \frac{206\times10^3\times0.025}{40}\left[\frac{\left(50^2-40^2\right)\left(40^2-30^2\right)}{2\times40^2\left(50^2-30^2\right)}\right] = 15.8\,\mathrm{MPa}$$

$$\sigma_r = p_c \Leftrightarrow \omega = 3525\,\mathrm{rad/s} = 33\,660\,\mathrm{rpm}$$

References

[1] Budynas, Richard G.: *Shigley's mechanical engineering design*.
McGraw-Hill, 2014, ISBN 9789339221638.

[2] Juvinall, Robert C. and Kurt M. Marshek: *FUNDAMENTALS OF MACHINE COMPONENT DESIGN*.
Wiley, 2017, ISBN 978-1-119-32153-8.

[3] Timoshenko, Stephen: *Strength of Materials. Part 1: Elementary Theory and Problems*.
CBS, third edition, 1986, ISBN 81-239-1030-4.

[4] Hall, Allen S., Alfred R. Holowenko, and Herman G. Laughlin: *Schaum's Outline of Theory and Problems of MACHINE DESIGN*.
McGraw-Hill, 1961.

References ii

[5] Castro, Paulo M S Tavares De: *Peças Curvas*.
Órgãos de Máquinas, DEMec, Faculdade de Engenharia da Universidade do Porto, 2016.

[6] Wilson, Benjamin James and John F. Quereau: *A simple method of determining stress in curved flexural member*.
Engineering Experiment Station, University of Illinois at Urbana-Champaign, 1928.

[7] Pilkey, Walter D., Deborah F. Pilkey, and Zhuming Bi: *Peterson's Stress Concentration Factors*.
Wiley, 2020, ISBN 9781119532514.

[8] Dhondt, Guido: *The Finite Element Method for Three-dimensional Thermomechanical Applications*.
Wiley, 2004, ISBN 0470857528.

[9] Schmid, Steven R., Bernard J. Hamrock, and Bo O. Jacobson: *Fundamentals of Machine Elements*. **CRC Press, 2014, ISBN 9781482247503.**

[10] Vullo, Vincenzo: *Circular Cylinders and Pressure Vessels*, volume 3 of *Springer Series in Solid and Structural Mechanics*. **Springer International Publishing, 2014, ISBN 978-3-319-00689-5.**

[11] Castro, Paulo M S Tavares de: *Tensões em peças cilíndricas*. **Órgãos de Máquinas, DEMec, Faculdade de Engenharia da Universidade do Porto, 2019.**

[12] *Code de Construction des Appareils à Pression: Régles de Calcul SNCTTI*. **Technical report, Paris, France, 1969.**

[13] ASME Boiler & Pressure Vessel Code, Section VIII, Division 1: *Rules for construction of pressure vessels.*
ASME, 2010.

[14] Féodossiev, V.: *Résistance des Matériaux.*
Moscou. Éditions MIR, 1971.

[15] Timoshenko, Stephen: *Strength of Materials. Part 2: Advanced Theory and Problems.*
CBS, third edition, 1986, ISBN 81-239-1077-0.

[16] Pilkey, Walter D.: *Formulas for stress, strain and structural matrices.*
John Wiley and Sons, second edition, 2005, ISBN 0471032212.

[17] *DIN 7190-1 Interference fits – Part 1: Calculation and design rules.*
Standard, Deutches Institut fur Normung, Germany, 2013.

[18] *ISO 286-1:2010 Geometrical product specifications (GPS) - ISO code system for tolerances on linear sizes - Part 1: Basis of tolerances, deviations and fits.*
Standard, International Organization for Standardization, Geneva, Switzerland, 2007.

[19] Vullo, V. and F. Vivio: *Rotors: Stress Analysis and Design.*
Mechanical Engineering Series. Springer Milan, 2013, ISBN 9788847025622.

References vi

[20] Castro, Paulo M S Tavares de: *Tensões em discos giratórios. o caso dos volantes de inércia.*
Órgãos de Máquinas, DEMec, Faculdade de Engenharia da Universidade do Porto, 2020.