# Complements of Machine Elements
# Curved Beams

Carlos M. C. G. Fernandes

## 1 Thin curved beam

Determine the vertical deflection of the end $B$ of the thin curved beam of uniform cross section and semicircular center line.

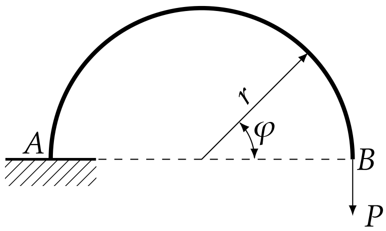


Figure 1: Thin curved beam [1].

### 1.1 Static equilibrium

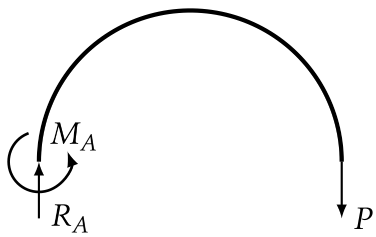


Figure 2: Free body diagram.

$$\begin{cases} \sum F_y = 0 \Leftrightarrow R_A - P = 0 \Leftrightarrow R_A = P \\ \sum M_A = 0 \Leftrightarrow M_A - P \cdot 2 \cdot r = 0 \Leftrightarrow M_A = P \cdot 2 \cdot r \end{cases} \tag{1}$$

## 1.2 Bending moment along the curved beam

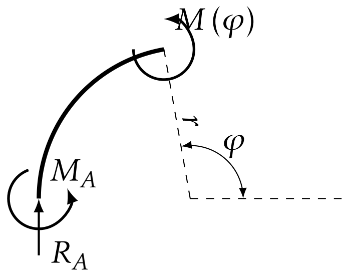


Figure 3: Thin curved beam [1].

$$\sum M = 0 \Leftrightarrow M(\varphi) + M_A - R_A \cdot r \cdot (1 + \cos(\varphi)) = 0 \tag{2}$$

$$M(\varphi) = P \cdot r \cdot (\cos(\varphi) - 1) \tag{3}$$

## 1.3 Strain energy due to bending

$$U = \int_0^s \frac{M^2}{2 \cdot E \cdot I} \mathrm{d}s = \int_0^\pi \frac{M(\varphi)^2}{2 \cdot E \cdot I} \mathrm{d}s = \int_0^\pi \frac{(P \cdot r \cdot (\cos(\varphi) - 1))^2}{2 \cdot E \cdot I} r \,\mathrm{d}\varphi \tag{4}$$

## 1.4 Castigliano theorem

$$\delta = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P} \int_0^\pi \frac{(P \cdot r \cdot (\cos(\varphi) - 1))^2}{2 \cdot E \cdot I} r \,\mathrm{d}\varphi \tag{5}$$

$$\delta = \frac{\partial}{\partial P} \int_0^\pi \frac{P^2 \cdot r^3 \cdot \left(\cos^2(\varphi) - 2\cos(\varphi) + 1\right)}{2 \cdot E \cdot I} \mathrm{d}\varphi \tag{6}$$

$$\delta = \frac{2 \cdot P \cdot r^3}{2 \cdot E \cdot I} \int_0^\pi \left(\cos^2(\varphi) - 2\cos(\varphi) + 1\right) \mathrm{d}\varphi \tag{7}$$

$$\delta = \frac{P \cdot r^3}{E \cdot I} \cdot \left[\frac{\varphi}{2} + \frac{1}{4}\sin(2\varphi) - 2\sin(\varphi) + \varphi\right]_0^\pi \tag{8}$$

$$\delta = \frac{P \cdot r^3}{E \cdot I} \cdot \left(\frac{\pi}{2} + \pi\right) = \frac{3 \cdot \pi}{2} \frac{P \cdot r^3}{E \cdot I} \tag{9}$$

# 2 Increase in distance

Determine the increase in distance between the ends A and B of a thin beam of uniform cross section consisting of a semicircular portion $CD$ and two straight portions $AC$ and $BD$.

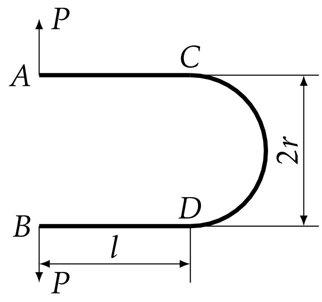


Figure 4: Increase in distance [1].

## 2.1 Bending moment along the curved beam

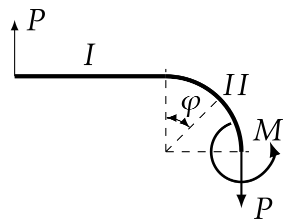


Figure 5: Free body diagram.

For the straight beam, the bending moment is:

$$M(x) - P \cdot x = 0 \Leftrightarrow M(x) = P \cdot x \tag{10}$$

For the curved beam:

$$M(\varphi) - P \cdot l - P \cdot r \cdot \sin(\varphi) = 0 \Leftrightarrow M(\varphi) = P \cdot (l + r \cdot \sin(\varphi)) \tag{11}$$

## 2.2 Strain energy due to bending

The free body diagram was done only for half of the structure. However, the total strain energy should include all the structure. By symmetry the strain energy of half of the structure should be multiplied by two.

$$U = 2\int_0^l \frac{M(x)^2}{2\cdot E\cdot I}\,\mathrm{d}x + 2\int_0^{\frac{\pi}{2}} \frac{M(\varphi)^2}{2\cdot E\cdot I}\,r\,\mathrm{d}\varphi \tag{12}$$

$$U = \int_0^l \frac{P^2\cdot x^2}{E\cdot I}\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \frac{[P\cdot(l + r\cdot\sin(\varphi))]^2}{E\cdot I}\,r\,\mathrm{d}\varphi \tag{13}$$

## 2.3 Castigliano theorem

Applying the Castigliano theorem:

$$\begin{aligned}\delta = \frac{\partial U}{\partial P} = &\frac{\partial}{\partial P}\int_0^l \frac{P^2\cdot x^2}{E\cdot I}\,\mathrm{d}x + \\ &\frac{\partial}{\partial P}\int_0^{\frac{\pi}{2}} \frac{P^2\cdot\left(l^2 + 2\cdot r\cdot l\cdot\sin(\varphi) + r^2\cdot\sin^2(\varphi)\right)}{E\cdot I}\,r\,\mathrm{d}\varphi\end{aligned} \tag{14}$$

$$\delta = \frac{2\cdot P}{E\cdot I}\cdot\left[\frac{x^3}{3}\right]_0^l + \frac{2\cdot P}{E\cdot I}\cdot\left[r\cdot l^2\cdot\varphi - 2\cdot r^2\cdot l\cdot\cos(\varphi) + r^3\left(\frac{\varphi}{2} - \frac{1}{4}\sin(2\varphi)\right)\right]_0^{\frac{\pi}{2}} \tag{15}$$

$$\delta = \frac{2\cdot P}{E\cdot I}\cdot\left(\frac{l^3}{3} + \frac{\pi\cdot r\cdot l^2}{2} + 2\cdot r^2\cdot l + \frac{\pi\cdot r^3}{4}\right) \tag{16}$$

The deflection in the vertical direction is:

$$\delta = \frac{2\cdot P}{E\cdot I}\cdot\left[\frac{l^3}{3} + r\cdot\left(\frac{\pi\cdot l^2}{2} + 2\cdot r\cdot l + \frac{\pi\cdot r^2}{4}\right)\right] \tag{17}$$

# 3 Piston ring

A piston ring of a circular outer boundary has a rectangular cross section of constant width *b* and of a variable depth *h*. Determine the law of variation of *h* in order to obtain a ring which, when assembled with the piston in the cylinder, produces a uniformly distributed pressure on the cylinder wall.

# 4 Open S link

An open S link is made from a 25 mm diameter rod. Determine the maximum tensile stress and maximum shear stress.

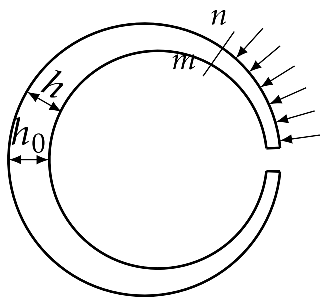


Figure 6: Piston ring [1].

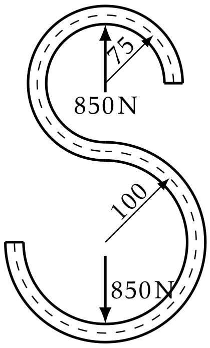


Figure 7: Open S link [2].

# References

[1] Timoshenko, Stephen: *Strength of Materials. Part 1: Elementary Theory and Problems*. CBS, third edition, 1986, ISBN 81-239-1030-4.

[2] Hall, Allen S., Alfred R. Holowenko, and Herman G. Laughlin: *Schaum's Outline of Theory and Problems of MACHINE DESIGN*. McGraw-Hill, 1961.