2. Shafting and Associated Parts

Complements of Machine Elements

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2024

Mestrado em Engenharia Mecânica

Recommended bibliography

- Vullo, Vivio, “Rotors: stress analysis and design”, Springer, 2013.
- Timoshenko, “Strength of Materials. Part II: Advanced Theory and Problems”, Third Edition, CBS, 1986.
- Genta, “Kinetic energy storage”, Butterworths, 1985.
- Haberhauer, Horst: Maschinenelemente. Springer Berlin Heidelberg, Berlin, Heidelberg, 2018, ISBN 978-3-662-53047-4

Contents

2.1. Flywheels

It must be confessed that the inventors of the mechanical arts have been much more useful to men than the inventors of syllogisms.

Voltaire

Summary

Flywheels

A flywheel is an energy storage device. It stores energy by increasing its angular velocity and delivers energy by decreasing its angular velocity.

A flywheel is used to smooth the flow of energy between a power source and its load.

Typical applications are: internal combustion engines, reciprocating compressors and pumps, automobiles, punch presses.

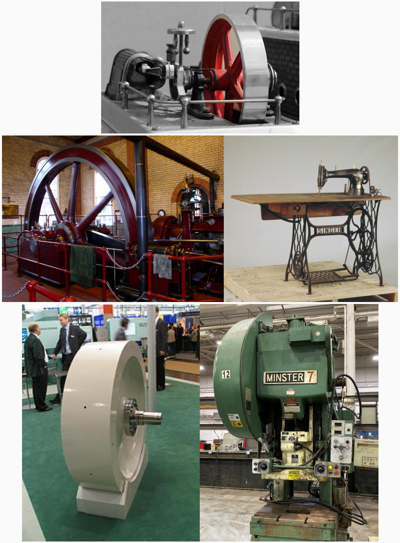


Figure 1: Flywheel applications.

Equation of motion of a system with flywheel

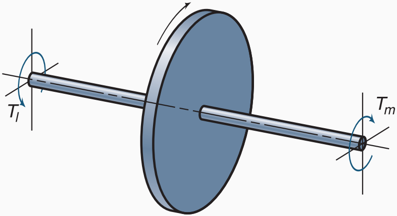


Figure 2: Flywheel with driving (mean) torque T_m and load torque T_l [1, 2].

Considering a rigid body analysis, the Newton's 2^{nd} law is:

$$\sum M = I \cdot \frac{\mathrm{d}\omega}{\mathrm{d}t}$$

The equation of motion of the flywheel is:

$$I \cdot \frac{\mathrm{d}\omega}{\mathrm{d}t} = T_m - T_l$$

Recall that:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}\omega}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\omega}{\mathrm{d}\theta} = \omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta}$$

The design motor/engine torque should be equivalent to the average torque: $T_m = T_{avg}$

So, the equation of motion can be written as:

$$(T_{avg} - T_l) \cdot \mathrm{d}\theta = I \cdot \omega \cdot \mathrm{d}\omega$$

Turning moment diagram

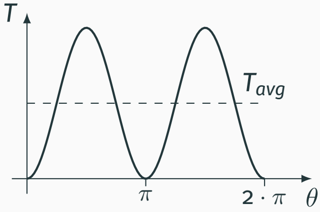


Figure 3: Example of a turning moment diagram of an engine

The Figure shows the torque T for different values of the crank position θ, that represents a possible **driving machine**.

The area of the diagram T vs. θ is equal to the work done by the engine per cycle.

The useful work divided by the angle of the cycle ($2 \cdot \pi$) gives the average torque T_{avg}:

$$T_{avg} = \frac{\int T \, \mathrm{d}\theta}{\theta_{cycle}}$$

Typically, the **driven machine** exerts a fluctuating torque on the shaft.

Influence of flywheel inertia

To show the influence of the flywheel inertia on the system angular acceleration, let's assume a constant resistant torque T_l and a very simple power source:

$$T_m(\theta) = T_l(1 + \sin\theta)$$

Recalling the equation of motion: $I \cdot \frac{\mathrm{d}\omega}{\mathrm{d}t} = T_m - T_l$

$$\boxed{\frac{\mathrm{d}\omega}{\mathrm{d}t} = \ddot{\theta} = \frac{T_l \sin\theta}{I}}$$

It becomes evident the influence of the mass moment of inertia of the flywheel:

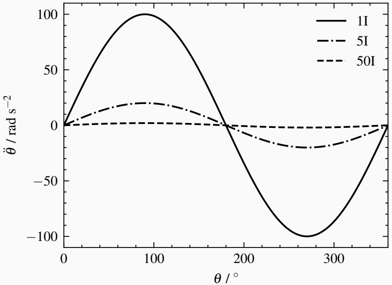
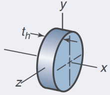
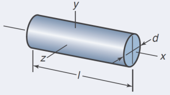
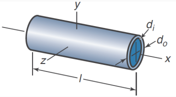


Figure 4: Influence of the flywheel inertia on the system acceleration.

Mass moment of inertia

$$\boxed{I = \int_m r^2 \, \mathrm{d}m}$$

Shape		Mass	Mass moment of inertia
	Disc	$m = \frac{\pi d^2 t_h \rho}{4}$	$I_x = \frac{md^2}{8}$
	Cylinder	$m = \frac{\pi d^2 l \rho}{4}$	$I_x = \frac{md^2}{8}$
	Hollow cylinder	$m = \frac{\pi \left(d_o^2 - d_i^2\right) l \rho}{4}$	$I_x = \frac{m\left(d_o^2 + d_i^2\right)}{8}$

Energy stored in a flywheel

Recalling again the equation of motion: $(T_{avg} - T_l) \cdot \mathrm{d}\theta = I \cdot \omega \cdot \mathrm{d}\omega$

The equation can be written in terms of a definite integral as:

$$\int_{\theta_{\omega_{min}}}^{\theta_{\omega_{max}}} (T_{avg} - T_l)\,\mathrm{d}\theta = \int_{\omega_{min}}^{\omega_{max}} I \cdot \omega\,\mathrm{d}\omega$$

The left side represents the change in kinetic energy between the maximum and minimum shaft speeds, so:

$$E_{max} - E_{min} = \frac{1}{2} \cdot I \cdot \left(\omega_{max}^2 - \omega_{min}^2\right)$$

This equation can be written in terms of the average angular speed $\omega_{avg} = \frac{\omega_{min} + \omega_{max}}{2}$ and the coefficient of speed fluctuation C_f:

$$\boxed{E_{max} - E_{min} = I \cdot C_f \cdot \omega_{avg}^2}$$

Coefficient of speed fluctuation

The coefficient of speed fluctuation is defined as:

$$C_f = \frac{\omega_{max} - \omega_{min}}{\omega_{avg}} = 2\frac{\omega_{max} - \omega_{min}}{\omega_{min} + \omega_{max}}$$

The reciprocal of the coefficient of fluctuation is known as *coefficient of steadiness* and is denoted by m:

$$m = \frac{1}{C_f}$$

Typical values [3, 4]:

Equipment	C_f
Crushing machinery	0.2
Electrical machinery	0.003
Electrical machinery, direct-driven	0.002
Engines with belt transmission	0.03
Flour-milling machinery	0.02
Gear transmission	0.02
Hammering machinery	0.2
Machine tools	0.03
Paper-making machinery	0.025
Pumping machinery	0.03-0.05
Shearing machinery	0.03-0.05
Spinning machinery	0.01-0.02
Textile machinery	0.025

Example 1: select the mass moment of inertia [2]

Consider torque displacement plot for one cycle in Figure 5. The average speed is to be $\omega_{avg} = 250\,\mathrm{rad/s}$.

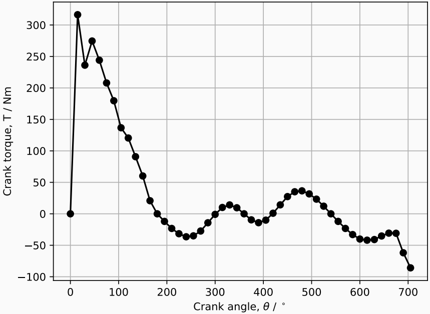


Figure 5: Torque displacement function for one cycle of a single-cylinder four-stroke engine (48 measurements with equal spacing) [1] [2].

[1]Python code available in Course Contents
[2]See the example in Course Contents from "Design of Machine Elements" [3]

Example 1: select the mass moment of inertia

1. Integrate the torque-displacement function for one cycle, and find the energy that can be delivered to the load during the cycle;

Using the trapezium rule and integrating the torque displacement function:

$$E = \int T \, \mathrm{d}\theta = 391.22 \, \mathrm{N\,m}$$

2. Determine the average torque T_{avg};

Considering the total displacement $\theta = 4\pi$, the average torque is:

$$T_{avg} = \frac{E}{\theta} = \frac{391.22}{4\pi} = 31.79 \, \mathrm{N\,m}$$

Select the mass moment of inertia

3. Find a suitable value for the flywheel mass moment of inertia considering a coefficient of speed fluctuation $C_f = 0.1$. Find ω_{max} and ω_{min}.

The maximum positive loop occurs between $\theta = 0$ and $\theta = 180$. The change in kinetic energy is given by:

$$E_{max} - E_{min} = \int_0^{\pi} T(\theta) - T_{avg}\, \mathrm{d}\theta = 399.84\,\mathrm{N\,m}$$

$$E_{max} - E_{min} = C_f \cdot I \cdot \omega_{avg}^2 \Leftrightarrow I = \frac{E_{max} - E_{min}}{C_f \cdot \omega_{avg}^2} = 0.064\,\mathrm{kg\,m^2}$$

Finally we solve simultaneously for ω_{max} and ω_{min}:

$$\begin{cases} \omega_{max} = \frac{1}{2}\left(2 + C_f\right) \cdot \omega_{avg} = 262.5\,\mathrm{rad/s} \\ \omega_{min} = 2 \cdot \omega_{avg} - \omega_{max} = 237.5\,\mathrm{rad/s} \end{cases}$$

Example 2: flywheel with rim and spokes

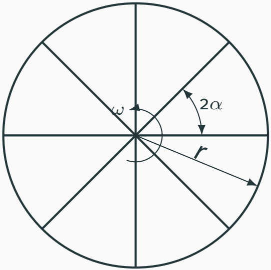


Figure 6: Flywheel with rim and spokes [5].

Nomenclature

r	radius of the center-line of the rim
A	cross-sectional area of the rim
A_1	cross-sectional area of a spoke
I	moment of inertia of the cross section of the rim
n	number of spokes
2α	angle between two consecutive spokes
q	weight of the rim per unit length of the center-line
q_1	weight of a spoke per unit length
ω	angular velocity of the flywheel

Due to the effect of the spokes, the rim of a rotating flywheel undergoes not only extension but also bending [3].

---

[3] Presentation of the topic taken from Timoshenko (pages 398-401) [5]. A more detailed discussion on the topic is “Remarks on rim and spokes flywheels” by Paulo M.S.T de Castro, available on the Course Contents.

Example 2: flywheel with rim and spokes

Considering as free body diagram a portion of the rim between two cross sections which bisect the angles between the spokes.

From the condition of symmetry, there can be no shearing stresses over the cross sections A and B. So, we just have longitudinal force N_0 and the bending moment M_0.

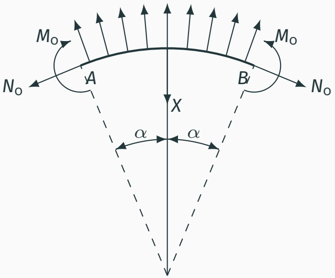


Figure 7: Free Body Diagram of the flywheel with rim and spokes [5].

Example 2: flywheel with rim and spokes

X denotes the force exerted by the spoke on the rim and the equation of equilibrium of the portion AB is:

$$2N_0 \sin\alpha + X - 2r^2 \frac{q}{g}\omega^2 \sin\alpha = 0$$

$$\boxed{N_0 = \frac{q}{g}\omega^2 r^2 - \frac{X}{2\sin\alpha}}$$

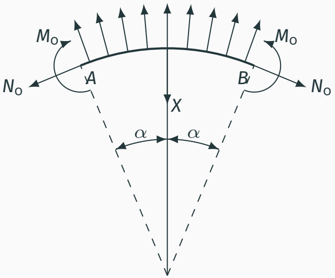


Figure 7: Free Body Diagram of the flywheel with rim and spokes [5].

Example 2: flywheel with rim and spokes

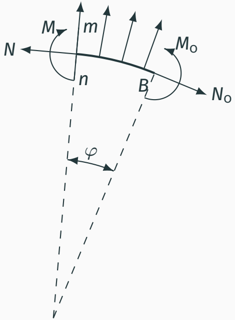


Figure 8: Free Body Diagram of the flywheel rim [5].

The longitudinal force N at any cross section mn is:

$$N = N_0 \cos\varphi + 2\frac{q}{g}\omega^2 r^2 \sin^2\frac{\varphi}{2} = 0$$

With:

$$N_0 = \frac{q}{g}\omega^2 r^2 - \frac{X}{2\sin\alpha}$$

Finally:

$$\boxed{N = \frac{q}{g}\omega^2 r^2 - \frac{X\cos\varphi}{2\sin\alpha}}$$

Example 2: flywheel with rim and spokes

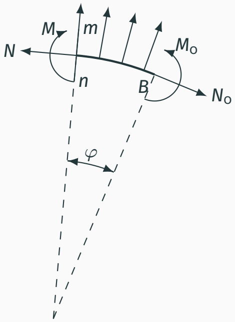


Figure 8: Free Body Diagram of the flywheel rim [5].

The bending moment M at any cross section mn is:

$$M = M_0 - N_0 r\,(1 - \cos\varphi) + 2\frac{q}{g}\omega^2 r^3 \sin^2\frac{\varphi}{2}$$

$$M = M_0 + \frac{Xr}{\sin\alpha}\sin^2\frac{\varphi}{2}$$

Force X and the moment M_0 cannot be determined using the static equilibrium equations.

Example 2: flywheel with rim and spokes

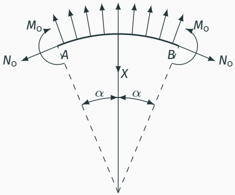


Figure 7: Free Body Diagram of the flywheel rim [5].

The strain energy of the portion AB of the rim is[4]:

$$U = 2\int_0^{\alpha} \frac{M^2}{2EI} r \, \mathrm{d}\varphi + 2\int_0^{\alpha} \frac{N^2}{2EA} r \, \mathrm{d}\varphi$$

The strain energy of the spoke is:

$$U_1 = \int_0^{r} \frac{N_1^2}{2EA_1} \mathrm{d}\rho$$

The tensile force N_1 at any cross section of the spoke at a distance ρ from the center of the flywheel is:

$$N_1 = X + \frac{q_1}{2g}\omega^2 \left(r^2 - \rho^2\right)$$

[4] cross-sectional area is small in comparison with r: just bending and tension is considered

Example 2: flywheel with rim and spokes

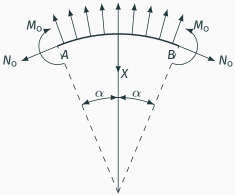


Figure 7: Free Body Diagram of the flywheel rim [5].

Applying the Castigliano's theorem, the equations for M_0 and X are:

$$\frac{\partial}{\partial M_0}\left(U + U_1\right) = 0$$

$$\frac{\partial}{\partial X}\left(U + U_1\right) = 0$$

After solving the equations, we get:

$$M_0 = -\frac{Xr}{2}\left(\frac{1}{\sin\alpha} - \frac{1}{\alpha}\right)$$

$$X = \frac{2}{3}\frac{q}{g}\omega^2 r^2 \frac{1}{\frac{Ar^2}{I} f_2(\alpha) + f_1(\alpha) + \frac{A}{A_1}}$$

with:

$$f_1(\alpha) = \frac{1}{2\sin^2\alpha}\left(\frac{\sin 2\alpha}{4} + \frac{\alpha}{2}\right) \text{ and}$$
$$f_2(\alpha) = \frac{1}{2\sin^2\alpha}\left(\frac{\sin 2\alpha}{4} + \frac{\alpha}{2}\right) - \frac{1}{2\alpha}$$

Example 2: flywheel with rim and spokes

$$f_1(\alpha) = \frac{1}{2\sin^2\alpha}\left(\frac{\sin 2\alpha}{4} + \frac{\alpha}{2}\right)$$

$$f_2(\alpha) = \frac{1}{2\sin^2\alpha}\left(\frac{\sin 2\alpha}{4} + \frac{\alpha}{2}\right) - \frac{1}{2\alpha}$$

The value of each function for different number of spokes *n* is:

n	4	6	8
$f_1(\alpha)$	0.643	0.957	1.274
$f_2(\alpha)$	0.00608	0.00169	0.00076

Exercise 1: Flywheel with rim and 6 spokes

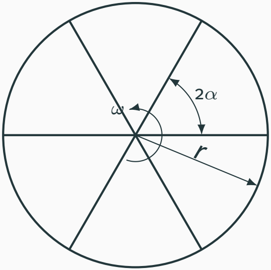


Figure 8: Flywheel with rim and 6 spokes [5].

Consider a 6 spoke steel flywheel rotating at 600 rpm, with radius $r = 1.5\,\mathrm{m}$, cross section of the rim a square $0.3 \times 0.3\ \mathrm{m}^2$ and the cross-sectional area of a spoke $A_1 = 0.015\,\mathrm{m}^2$. Calculate the maximum stress on the rim.

Exercise 2: Flywheel stresses

A flywheel made of low-carbon steel has an outside radius of $150\,\mathrm{mm}$ and an inside radius of $25\,\mathrm{mm}$. The flywheel is to be assembled (press fit) onto a shaft. The radial interference between the flywheel and shaft is $50\,\mathrm{\mu m}$, and the shaft will operate at a speed of $5000\,\mathrm{rpm}$ [1]. Calculate:

1. The circumferential and radial stresses on the flywheel inner radius.
2. The speed at which the flywheel will break loose from the shaft.

Exercise 3: Flywheel design

The output, or load torque, of a flywheel used in a punch press for each revolution of the shaft is $12\,\mathrm{N\,m}$ from zero to π and from $3\pi/2$ to 2π and $144\,\mathrm{N\,m}$ from π to $3\pi/2$. The coefficient of fluctuation is $C_f = 0.05$ about an average speed of 600 rpm. Assume that the flywheel's solid disk is made of low-carbon steel of constant $25\,\mathrm{mm}$ thickness [1]. Determine the following:

1. The average load or output torque
2. The locations $\theta_{\omega_{min}}$ and $\theta_{\omega_{max}}$
3. The energy fluctuation required
4. The outside diameter of the flywheel

2.2. Fatigue design according do DIN 743

All machine and structural designs are problems in fatigue because the forces of Nature are always at work and each object must respond in some fashion.

Carl Osgood

Summary

Introduction

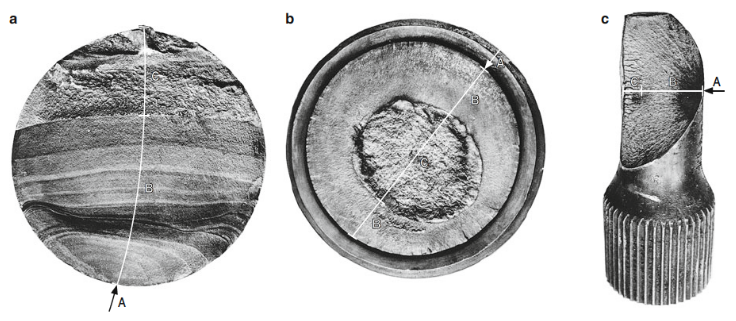


Figure 9: Typical fatigue fractures. **a.** bending fatigue fracture on the eccentric shaft of a crusher ($d = 230\,\mathrm{mm}$) at the transition from the shaft to the eccentric, **b.** Circular bending fatigue fracture on the driving axis of a tilting plow. Cause: too small fillet radius, **c.** torsional fatigue failure on a torsion bar spring [6].

A: First crack at the flaw or notch on the surface; B: Zone of progressive fatigue fracture; C: Final fracture (forced fracture)

Soderberg criterium: uniaxial stres

Figure 10: Soderberg criterium with a safety factor.

The equation should be used after:

- correct the fatigue limit strength $\sigma_{f0}^c = C_1 \cdot C_2 \cdot C_3 \cdot C_4 \cdot \sigma_{f0}$;
- add an adequate stress concentration factor for fatigue;
- consider a safety factor.

For uniaxial stress:

$$\boxed{\frac{K_f \cdot \sigma_a}{\sigma_{f0}^c} + \frac{\sigma_m}{\sigma_y} = \frac{1}{S}}$$

Soderberg criterium: biaxial stress

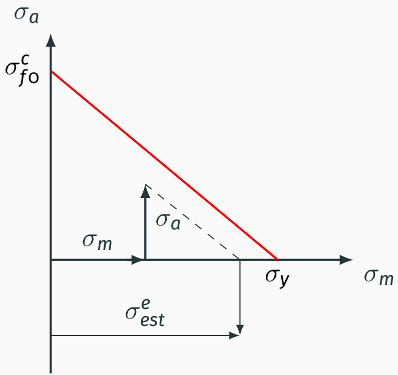


Figure 11: Determination of an equivalent static stress.

$$\frac{\sigma_a}{\sigma_{fo}^c} + \frac{\sigma_m}{\sigma_y} = 1$$

The Soderberg line slope is $\frac{\sigma_{fo}^c}{\sigma_y}$

For biaxial stress, an equivalent static stress may be computed each stress component:

$$\sigma_{est}^{eq} = \sigma_m + \frac{\sigma_y}{\sigma_{fo}^c} \cdot \sigma_a$$

And for practical applications:

$$\boxed{\sigma_{est}^{eq} = \sigma_m + \frac{\sigma_y}{\sigma_{fo}^c} \cdot K_f \cdot \sigma_a}$$

Fatigue design criteria based on Soderberg

For a solid shaft with circular cross section:

$$\sigma_a = K_f \frac{32 M_{f_a}}{\pi d^3}$$

$$\sigma_m = \frac{32 M_{f_m}}{\pi d^3}$$

$$\tau_a = K_f \frac{16 M_{t_a}}{\pi d^3}$$

$$\tau_m = \frac{16 M_{t_m}}{\pi d^3}$$

An equivalent static stress is calculated using the Soderberg criterium for normal stresses:

$$\sigma_{est}^{eq} = \sigma_m + K_f \frac{\sigma_y}{\sigma_{fo}^c} \sigma_a$$

And also for shear stresses:

$$\tau_{est}^{eq} = \tau_m + K_f \frac{\sigma_y}{\sigma_{fo}^c} \tau_a$$

ASME-elliptic criterium

Using Soderberg and maximum shear stress theory for a solid circular section:

$$\frac{\sigma_y}{S} = \sqrt{\left(\sigma_m + K_f \frac{\sigma_y}{\sigma^c_{fo}} \sigma_a\right)^2 + 4\left(\tau_m + K_f \frac{\sigma_y}{\sigma^c_{fo}} \tau_a\right)^2}$$

The section diameter can be estimated as:

$$d^3 = \frac{32S}{\pi} \sqrt{\left(\frac{M_{f_m}}{\sigma_y} + K_f \frac{M_{f_a}}{\sigma^c_{fo}}\right)^2 + \left(\frac{M_{t_m}}{\sigma_y} + K_f \frac{M_{t_a}}{\sigma^c_{fo}}\right)^2}$$

Also known as ASME-elliptic criterium [7, 8, 9]. S is the safety factor.

ASME criterium

Using Soderberg and maximum distortion energy theory for a solid circular section:

$$\frac{\sigma_y}{S} = \sqrt{\left(\sigma_m + K_f \frac{\sigma_y}{\sigma_{fo}^c} \sigma_a\right)^2 + 3 \left(\tau_m + K_f \frac{\sigma_y}{\sigma_{fo}^c} \tau_a\right)^2}$$

The section diameter can be estimated as:

$$d^3 = \frac{32S}{\pi} \sqrt{\left(\frac{M_{f_m}}{\sigma_y} + K_f \frac{M_{f_a}}{\sigma_{fo}^c}\right)^2 + \frac{3}{4} \left(\frac{M_{t_m}}{\sigma_y} + K_f \frac{M_{t_a}}{\sigma_{fo}^c}\right)^2}$$

Also known as ASME criterium [7, 8, 9].

Review on yield criteria: Tresca[5]

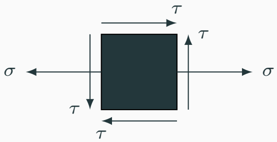


Figure 12: Element of a shaft under bending (σ) and torsion (τ).

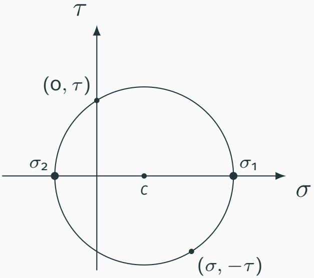


Figure 13: Mohr's Circle or circle of stress for combined stresses [5].

Principal stresses:

$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

$$\sigma_2 = \frac{\sigma}{2} - \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

Maximum shear stress theory:

$$\sigma_e = \sigma_1 - \sigma_2 = \sqrt{\sigma^2 + 4\tau^2}$$

[5]Topic mostly transcribed from Paulo M.S.T. Castro [10] based on a *must read* review of the german literature

Review on yield criteria: von-Mises

$$\sigma_e = \sqrt{{\sigma_1}^2 + {\sigma_2}^2 - \sigma_1\sigma_2}$$

$$\sigma_e = \sqrt{\sigma^2 + 3\tau^2}$$

The relation between the shear strength and the tensile strength is predicted by the maximum distortion energy theory as:

$$\tau_{adm} = \frac{\sigma_{adm}}{\sqrt{3}}$$

$$\frac{\sigma_{adm}}{\sqrt{3}\tau_{adm}} = 1$$

Which can be included into the maximum distortion energy theory equation:

$$\sigma_e = \sqrt{\sigma^2 + 3\left(\frac{\sigma_{adm}}{\sqrt{3}\tau_{adm}}\tau\right)^2}$$

According to literature [11, 6], the ratio $\frac{\sigma_{adm}}{\sqrt{3}\tau_{adm}} = 1$ is sometimes not verified experimentally.

Effort ratio

So, it is current to use a modification factor α_0, called *effort ratio*[6,7]. For the maximum distortion energy theory:

$$\boxed{\sigma_e = \sqrt{\sigma^2 + 3\left(\alpha_0 \tau\right)^2}}$$

The effort ratio is [11] [8]:

$$\boxed{\alpha_0 = \frac{\sigma_{Grenz}}{\varphi \tau_{Grenz}} = \frac{\sigma_{adm}}{\varphi \tau_{adm}}}$$

For the the maximum distortion energy theory, $\varphi = \sqrt{3} = 1.73$

[6] From german: *Anstrengungsverhältnis*
[7] In Niemman [6] it is described as *Schubfestigkeitsfaktor* which is "shear stress factor"
[8] *Grenz* is translated as limit, permissible stress

Equivalent stress

Criteria	Equivalent stress	Effort ratio
1. Maximum normal stress[9]	$\sigma_e = \frac{1}{2}\left[\sigma + \sqrt{\sigma^2 + 4\left(\alpha_0\tau\right)^2}\right]$	$\alpha_0 = \frac{\sigma_{adm}}{\tau_{adm}}$
2. Maximum shear stress[10]	$\sigma_e = \sqrt{\sigma^2 + 4\left(\alpha_0\tau\right)^2}$	$\alpha_0 = \frac{\sigma_{adm}}{2\tau_{adm}}$
3. Maximum distortion energy[11]	$\sigma_e = \sqrt{\sigma^2 + 3\left(\alpha_0\tau\right)^2}$	$\alpha_0 = \frac{\sigma_{adm}}{1.73\tau_{adm}}$
4. Maximum strain (Bach)	$\sigma_e = 0.35\sigma + 0.65\sqrt{\sigma^2 + 4\left(\alpha_0\tau\right)^2}$	$\alpha_0 = \frac{\sigma_{adm}}{(1+\nu)\tau_{adm}}$

1. To be used mainly for brittle materials;

2. For ductile materials in the event of failure (static) due to sliding fracture (e.g. with copper and copper alloys);

3. For ductile materials.

[9] NH *Normalspannungs-Hypothese*
[10] SH *Schubspannungs-Hypothese*
[11] GEH *Gestaltänderungsenergie-Hypothese*

Load cases

According to Bach [11, 6] there are essentially three basic load cases:

Load case I (*ruhende*): static load

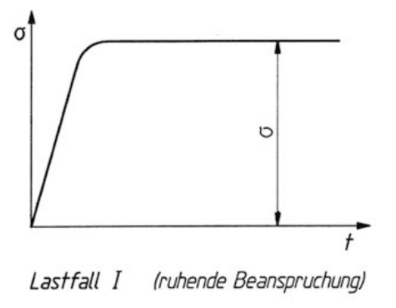


Figure 14: Lastfall I (*ruhende Beanspruchung* [11]

Load case II (*rein schwellende*): pulsating stress R=0

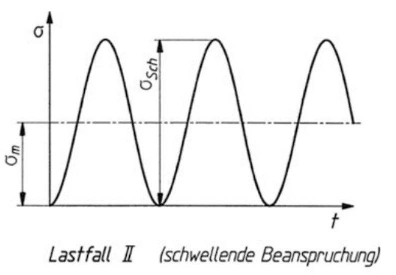


Figure 15: Lastfall II (*schwellende Beanspruchung* [11]

Load case III (*rein wechselnde*): fully reversed stress R=-1

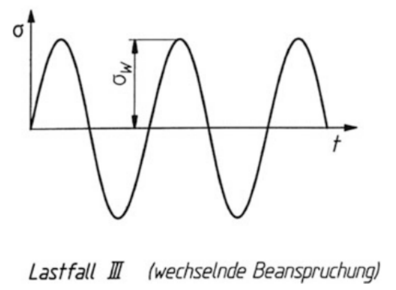


Figure 16: Lastfall III (*wechselnde Beanspruchung*) [11]

General load case (*allgemeiner*): oscilating stress – load case I plus load case III.

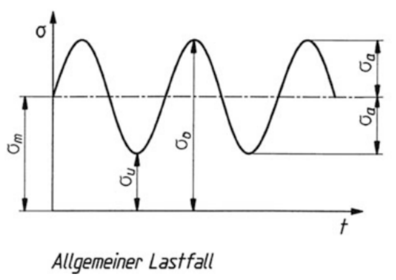


Figure 17: Allgemeiner Lastfall [11]

Effort ratio

Values for the *effort ratio* for different loading combinations:

Load case		[11]	[12]	[13]	[14]
σ_{III}	τ_{I}	0.7	0.7	0.5	0.7
σ_{II}	τ_{I}			0.7	
σ	τ	1	1	1	1
σ_{I}	τ_{III}	1.5	1.5	2	1.5
σ_{I}	τ_{II}			1.5	
σ_{III}	τ_{II}	0.7	0.7	0.75	
σ_{II}	τ_{III}		1.5	1.35	

Safety factor

$$\boxed{\sigma_e = \sqrt{\sigma^2 + 3\left(\alpha_0 \tau\right)^2}}$$

For the the maximum distortion energy theory, $\alpha_0 = \frac{1}{\sqrt{3}} \frac{\sigma_{adm}}{\tau_{adm}}$

$$\sigma_e^2 = \sigma^2 + 3\left(\frac{1}{\sqrt{3}} \frac{\sigma_{adm}}{\tau_{adm}} \tau\right)^2 = \sigma^2 + \left(\frac{\sigma_{adm}}{\tau_{adm}}\right)^2 \tau^2$$

Dividing by σ_{adm}^2:

$$\frac{\sigma_e^2}{\sigma_{adm}^2} = \frac{\sigma^2}{\sigma_{adm}^2} + \frac{\tau^2}{\tau_{adm}^2}$$

$$\frac{\sigma_e}{\sigma_{adm}} = \sqrt{\frac{\sigma^2}{\sigma_{adm}^2} + \frac{\tau^2}{\tau_{adm}^2}}$$

With $\sigma_e S = \sigma_{adm}$:

$$\frac{1}{S} = \sqrt{\frac{\sigma^2}{\sigma_{adm}^2} + \frac{\tau^2}{\tau_{adm}^2}}$$

$$\boxed{S = \frac{1}{\sqrt{\frac{\sigma^2}{\sigma_{adm}^2} + \frac{\tau^2}{\tau_{adm}^2}}}}$$

Smith diagram

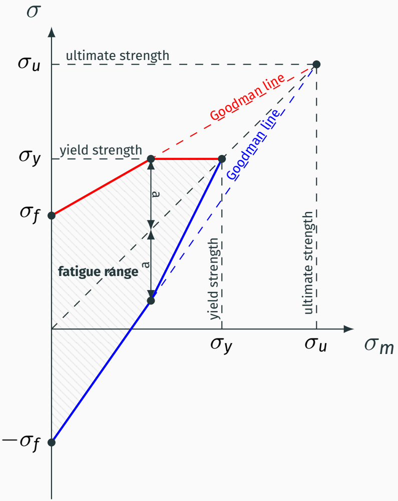


Figure 18: Smith diagram.

Smith Diagram for Traction

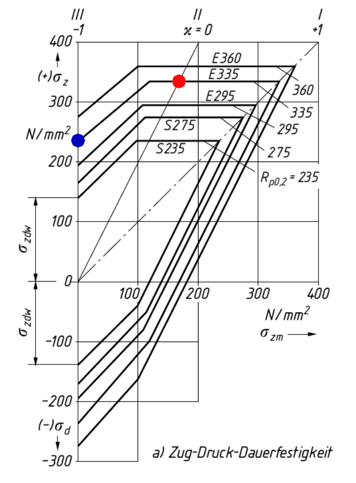


Figure 19: Smith diagram for traction for structural steels according to DIN EN 10025 [12]

- Fully reversed R=-1: $\sigma_{zdW} = 235\,\mathrm{MPa}$
- Pulsating R=0: $\sigma_{zdSch} = 335\,\mathrm{MPa}$

Yield strength: $\sigma_{zdF} = 335\,\mathrm{MPa}$

Smith Diagram for Bending

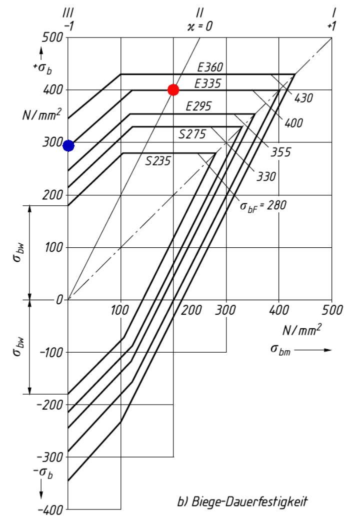


Figure 20: Smith diagram for bending for structural steels according to DIN EN 10025 [12]

- Fully reversed R=-1: $\sigma_{bW} = 290\,\mathrm{MPa}$
- Pulsating R=0: $\sigma_{bSch} = 400\,\mathrm{MPa}$

Yield strength: $\sigma_{bF} = 400\,\mathrm{MPa}$

Smith Diagram for Torsion

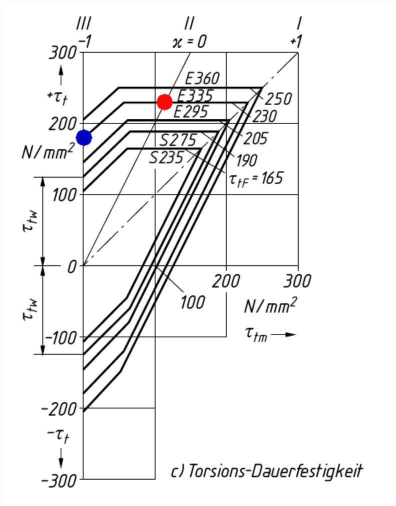


Figure 21: Smith diagram for torsion for structural steels according to DIN EN 10025 [12]

- Fully reversed R=-1: $\tau_{tW} = 180\,\mathrm{MPa}$
- Pulsating R=0: $\tau_{tSch} = 230\,\mathrm{MPa}$

Yield strength: $\tau_{tF} = 230\,\mathrm{MPa}$

Determine effort ratio from Smith diagram

Consider a shaft made of steel E335 according to DIN 10025 EN under bending and torsion[12]. Determine the *effort ratio* for:

- σ_{III} reversed bending ($\sigma_{bw} = 290\,\mathrm{MPa}$) and τ_{I} constant torsion ($\tau_{tF} = 230\,\mathrm{MPa}$):

$$\alpha_0 = \frac{\sigma_{bw}}{\sqrt{3}\tau_{tF}} = \frac{290}{230\sqrt{3}} = 0.728$$

- σ_{III} reversed bending ($\sigma_{bw} = 290\,\mathrm{MPa}$) and τ_{III} reversed torsion ($\tau_{tw} = 180\,\mathrm{MPa}$):

$$\alpha_0 = \frac{\sigma_{bw}}{\sqrt{3}\tau_{tW}} = \frac{290}{180\sqrt{3}} = 0.930$$

- σ_{I} constant bending ($\sigma_{bF} = 400\,\mathrm{MPa}$) and τ_{III} reversed torsion ($\tau_{tw} = 180\,\mathrm{MPa}$):

$$\alpha_0 = \frac{\sigma_{bF}}{\sqrt{3}\tau_{tW}} = \frac{400}{180\sqrt{3}} = 1.283$$

[12] Example transcribed from [8], *effort ratio* to be used with the maximum distortion energy (*Gestaltänderungsenergiehypothese – GEH*) theory with $\varphi = \sqrt{3}$

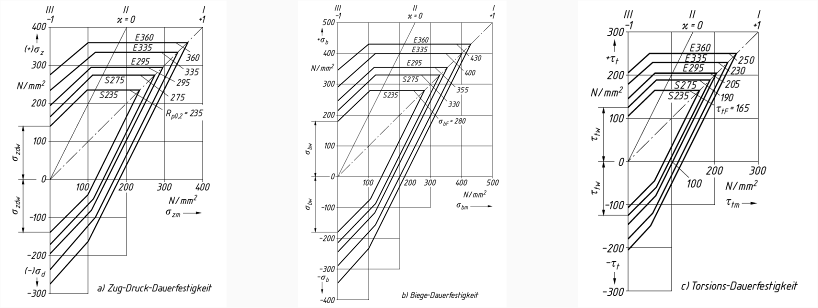
Comparison of *effort ratio* values

The following Table presents a comparison between the *effort ratio* calculated with the strength properties taken from Smith diagrams and values taken from literature.

The values taken from the literature are generic values, so, it is expected to find slight variations not only due to the loading conditions but also due to material specificity (ductility, anisotropy, manufacturing) [6].

Load case	Decker [13]	Roloff/Mattek [12]	Smith diagrams
σ_{III}, τ_{I}	0.5	0.7	**0.728**
σ_{III}, τ_{III}	1	1	**0.930**
σ_{I}, τ_{III}	2	1.5	**1.283**

Smith diagram



(a) Traction **(b)** Bending **(c)** Torsion

Figure 22: Smith diagrams for structural steels according to DIN EN 10025 [12]

Comparing the smith diagram for bending and the smith diagram for traction, it is possible to state that the yield strength $\sigma_{bF} > \sigma_{zdF}$ is bigger for bending.

Smith diagram

$$\boxed{S = \frac{1}{\sqrt{\frac{\sigma^2}{\sigma_{adm}^2} + \frac{\tau^2}{\tau_{adm}^2}}}}$$

From the Smith diagram for each loading type, it is evident that bending stresses and normal stresses should be treated separately [10, 12, 11, 6]:

$$\boxed{S = \frac{1}{\sqrt{\left(\frac{\sigma_{zd}}{\sigma_{adm,zd}} + \frac{\sigma_{bw}}{\sigma_{adm,bw}}\right)^2 + \frac{\tau_t^2}{\tau_{adm,t}^2}}}}$$

The shear stress due to shear forces is usually disregarded on these equations [6].

Equivalent moment

The equivalent stress can be transformed into an equivalent moment:

$$\sigma_e = \sqrt{\sigma^2 + 3\left(\alpha_0 \tau\right)^2}$$

Recall that for a round solid section:

$$\sigma = \frac{32M_f}{\pi d^3} \quad \text{and} \quad \tau = \frac{16M_t}{\pi d^3}$$

$$\sigma_e = \sqrt{\left(\frac{32M_f}{\pi d^3}\right)^2 + 3\left(\alpha_0 \frac{2 \times 16M_t}{2\pi d^3}\right)^2}$$

$$\sigma_e = \frac{32}{\pi d^3}\sqrt{{M_f}^2 + \frac{3}{4}\left(\alpha_0 M_t\right)^2}$$

$$\sigma_e \frac{\pi d^3}{32} = M_e = \sqrt{{M_f}^2 + \frac{3}{4}\left(\alpha_0 M_t\right)^2}$$

DIN 743

The DIN 743:2012 standard "Calculation of load capacity of shafts and axles" [13]:

- Part 1: General
- Part 2: Theoretical stress concentration factors and fatigue notch factors
- Part 3: Strength of materials
- Part 4: Fatigue limit, endurance limit - Equivalently damaging continuous stress
- Supplement 1: "Examples to part 1 to 3"
- Supplement 2: "Examples to part 4"

[13] In german: *Tragfähigkeitsberechnung von Wellen und Achsen*

DIN 743 - fatigue verification[14]

$$\boxed{S = \frac{1}{\sqrt{\left(\frac{\sigma_{zda}}{\sigma_{zdADK}} + \frac{\sigma_{ba}}{\sigma_{bADK}}\right)^2 + \left(\frac{\tau_{ta}}{\tau_{tADK}}\right)^2}}}$$

The safety factor should be:

$$S \geq S_{min} \geq 1.2$$

If the load quantification is uncertain, higher safety factors should be used.

The standard calculates the safety factor against fatigue failure for infinite life.

[14]A comprehensive step by step explanation about the calculation guideline of DIN 743 is given in Course Notes [10]

DIN 743 - plastic deformation verification

Verification for plastic deformation:

$$\boxed{S = \frac{1}{\sqrt{\left(\frac{\sigma_{zdmax}}{\sigma_{zdFK}} + \frac{\sigma_{bmax}}{\sigma_{bFK}}\right)^2 + \left(\frac{\tau_{tmax}}{\tau_{tFK}}\right)^2}}}$$

For the plastic deformation verification, the stress concentration factors is not need (already discussed in “Machine Elements”).

Strength of the part is calculated for any diameter d from the values for the specimen with d_B [15].

[15] To be discussed during KISSSoft class both for fatigue (two cases) and static verification

Comparison of different fatigue design criteria

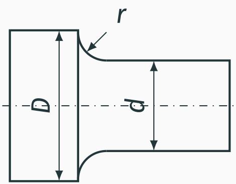


Figure 23: Shaft under alternated bending and alternating torsion.

$D = 50\,\mathrm{mm}$, $d = 42\,\mathrm{mm}$, $r = 5\,\mathrm{mm}$, Surface roughness $R_z = 4\,\mu\mathrm{m}$ [16]

Load at cross section with diameter d:

$\sigma_b = 500\,\mathrm{MPa} \pm 50\,\mathrm{MPa}$ and $\tau_t = 100\,\mathrm{MPa} \pm 30\,\mathrm{MPa}$

Material: 34CrMo4 (strength values according to DIN 743-3, $d_B \leq 16\,\mathrm{mm}$):

$\sigma_B = \sigma_R = 1000\,\mathrm{MPa}, \sigma_S = \sigma_y = 800\,\mathrm{MPa}, \sigma_{zdW} = 400\,\mathrm{MPa}, \sigma_{bW} = 500\,\mathrm{MPa}, \tau_{tW} = 300\,\mathrm{MPa}$

[16] Example and result taken from Supplement 1 of DIN 743

Comparison of different fatigue design criteria

The ASME and ASME elliptic follow the same procedure already discussed during “Machine Elements”:

- stress concentration factor K_t from Peterson [8, 15] assuming the notch sensitivity factor as $q = 1$:

$$K_f = 1 + q \cdot (K_t - 1)$$

- *Marine* factors (C_1, C_2 and C_3) as described on [8] or [7];
- Fatigue strength calculated as $\sigma_{fo}^{c} = 0.5 \cdot \sigma_R \cdot C_1 \cdot C_2 \cdot C_3$

Criteria	Safety factor
ASME	1.11
ASME Elliptic	1.08
DIN 743	2.61

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