FEUP FACULDADE DE ENGENHARIA
UNIVERSIDADE DO PORTO

Rotating Cylinders

Complements of Machine Elements

Carlos Fernandes

2023

Mestrado em Engenharia Mecânica

It must be confessed that the inventors of the mechanical arts have been much more useful to men than the inventors of syllogisms.

Voltaire

Recommended bibliography

- Vullo, Vivio, “Rotors: stress analysis and design”, Springer, 2013.
- Timoshenko, “Strength of Materials. Part II: Advanced Theory and Problems”, Third Edition, CBS, 1986.
- Genta, “Kinetic energy storage”, Butterworths, 1985.

Hyperlink

Lecture 1

Lecture 2

Problems

References

Lecture 1

Summary

Introduction

For cylinders subjected to important angular speed, both radial and tangential stresses exist as in the theory for thick-walled cylinders. However the cause of the stresses is the inertial forces acting on all the particles of the cylinder [1].

Several rotating machine elements, such as flywheels and blowers can be simplified as a rotating cylinder.

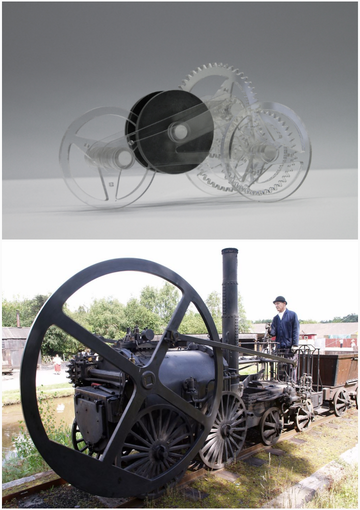


Figure 1: Rotating cylinders.

Introduction

The tangential and radial stresses found with the following equations are subjected to the following restrictions [2, 1]:

- the outside radius is large compared with thickness ($b > 10t$);
- the thickness is constant;
- the stresses are constant over the thickness.

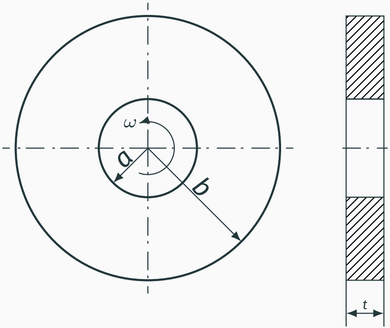


Figure 2: Rotating cylinder.

Stresses in a rotating cylinder

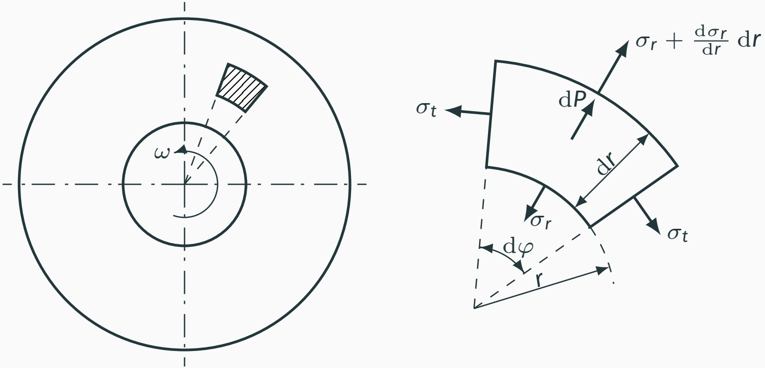


Figure 3: Stresses in a rotating cylinder [3, 4, 5].

Presentation of the topic taken from Timoshenko (pages 214–223) [4] and Féodossiev [3], similar to the Course Notes by Paulo M.S.T. Castro [6] .

Hoop and radial stress

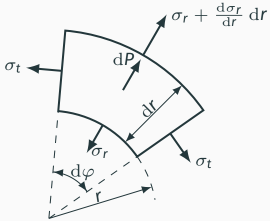


Figure 4: Infinitesimal element of the rotating thick wall cylinder [3, 4, 5, 6].

$$\sigma_t t\,\mathrm{d}r\,\mathrm{d}\varphi + \sigma_r t r\,\mathrm{d}\varphi - \left(\sigma_r + \frac{\mathrm{d}\sigma_r}{\mathrm{d}r}\,\mathrm{d}r\right)(r + \mathrm{d}r)\,t\,\mathrm{d}\varphi - \mathrm{d}P = 0$$

The inertia forces are the product of mass $\frac{\gamma}{g} t r\,\mathrm{d}\varphi\,\mathrm{d}r$ by the centrifugal acceleration $\omega^2 r$:

$$\mathrm{d}P = \frac{\gamma}{g}\omega^2 r^2 t\,\mathrm{d}\varphi\,\mathrm{d}r = \rho\omega^2 r^2 t\,\mathrm{d}\varphi\,\mathrm{d}r$$

Hoop and radial stress

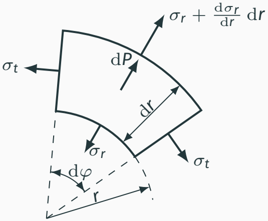


Figure 4: Infinitesimal element of the rotating thick wall cylinder [3, 4, 5, 6].

$$\sigma_t t\,\mathrm{d}r\,\mathrm{d}\varphi + \sigma_r r t\,\mathrm{d}\varphi - \left(\sigma_r + \frac{\mathrm{d}\sigma_r}{\mathrm{d}r}\,\mathrm{d}r\right)(r + \mathrm{d}r)\,t\,\mathrm{d}\varphi - \underbrace{\rho\omega^2 r^2 t\,\mathrm{d}\varphi\,\mathrm{d}r}_{\mathrm{d}P} = 0$$

Simplify the common terms t and $\mathrm{d}\varphi$:

$$\sigma_t\,\mathrm{d}r + \sigma_r r - \sigma_r r - \mathrm{d}\sigma_r\,r - \sigma_r\,\mathrm{d}r - \mathrm{d}\sigma_r\,\mathrm{d}r - \rho\omega^2 r^2\,\mathrm{d}r = 0$$

Hoop and radial stress

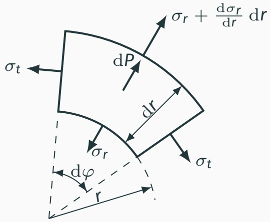


Figure 4: Infinitesimal element of the rotating thick wall cylinder [3, 4, 5, 6].

$$\sigma_t\,dr + \sigma_r r - \sigma_r r - d\sigma_r\,r - \sigma_r\,dr - d\sigma_r\,dr - \rho\omega^2 r^2\,dr = 0$$

Disregarding higher order terms:

$$\sigma_t - \sigma_r - r\frac{d\sigma_r}{dr} - \rho\omega^2 r^2 = 0$$

Hoop and radial stress

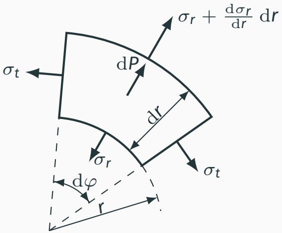


Figure 4: Infinitesimal element of the rotating thick wall cylinder [3, 4, 5, 6].

$$\frac{d}{dr}(\sigma_r r) - \sigma_t = -\rho\omega^2 r^2$$

Using the Hooke's law:

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \nu\left(\sigma_t + \sigma_l\right)\right]$$

$$\varepsilon_t = \frac{1}{E}\left[\sigma_t - \nu\left(\sigma_r + \sigma_l\right)\right]$$

Considering $\sigma_l = 0$:

$$\sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_t\right)$$

$$\sigma_t = \frac{E}{1-\nu^2}\left(\varepsilon_t + \nu\varepsilon_r\right)$$

Hoop and radial stress

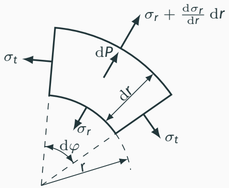


Figure 4: Infinitesimal element of the rotating thick wall cylinder [3, 4, 5, 6].

The circumferential strain is:

$$\varepsilon_t = \frac{2\pi\left(r+u\right) - 2\pi r}{2\pi r} = \frac{u}{r}$$

The radial strain is:

$$\varepsilon_r = \frac{\mathrm{d}u}{\mathrm{d}r} = r\frac{\mathrm{d}\varepsilon_t}{\mathrm{d}r} + \varepsilon_t$$

And substitute into radial and tangential stress equations:

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{\mathrm{d}u}{\mathrm{d}r} + \nu\frac{u}{r}\right)$$

$$\sigma_t = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{\mathrm{d}u}{\mathrm{d}r}\right)$$

Hoop and radial stress

$$\boxed{\sigma_t - \sigma_r - r\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} - \rho\omega^2 r^2 = 0}$$

Substitute each stress component as function of the radial displacement u:

$$\frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{\mathrm{d}u}{\mathrm{d}r}\right) - \frac{E}{1-\nu^2}\left(\frac{\mathrm{d}u}{\mathrm{d}r} + \nu\frac{u}{r}\right) - \frac{E}{1-\nu^2}\left(r\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \nu\frac{r}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \nu r\frac{u}{r^2}\right) - \rho\omega^2 r^2 = 0$$

$$\left(\frac{u}{r} + \nu\frac{\mathrm{d}u}{\mathrm{d}r}\right) - \left(\frac{\mathrm{d}u}{\mathrm{d}r} + \nu\frac{u}{r}\right) - \left(r\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \nu\frac{\mathrm{d}u}{\mathrm{d}r} - \nu\frac{u}{r}\right) - \frac{1-\nu^2}{E}\rho\omega^2 r^2 = 0$$

$$-\frac{u}{r^2} - \nu\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} + \nu\frac{u}{r^2} + \frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \nu\frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \nu\frac{u}{r^2} + \frac{1-\nu^2}{E}\rho\omega^2 r = 0$$

Finally, we get a second order non-homogeneous ordinary differential equation (ODE):

$$\boxed{\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r^2} + \frac{1-\nu^2}{E}\rho\omega^2 r = 0}$$

Solution for the non-homogeneous ODE

$$\boxed{\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r^2} + \frac{1-\nu^2}{E}\rho\omega^2 r = 0}$$

Making $\boxed{N = \frac{1-\nu^2}{E}\rho\omega^2}$:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r^2} + Nr = 0$$

$$u''(r) + \frac{u'(r)}{r} - \frac{u(r)}{r^2} + Nr = 0$$

Multiplying by r^2 both sides, we obtain an Euler-Cauchy equation:

$$r^2 u''(r) + r u'(r) - u(r) = -Nr^3$$

Solution for the non-homogeneous ODE

The solution for $r^2u'' + ru' - u = -Nr^3$ is:

$$u = u_h + u_p$$

u_h is the general solution for the homogeneous ODE $r^2u'' + ru' - u = 0$

u_p is the particular solution that satisfies the non-homogeneous ODE.

For $r^2u'' + ru' - u = 0$ and assuming a trial solution $u = r^m$:

$$r^2 \left(m\,(m-1)\,r^{m-2}\right) + r\left(mr^{m-1}\right) - r^m = 0$$

$$m\,(m-1)\,r^m + mr^m - r^m = \left(m^2 - m + m - 1\right) r^m = 0$$

Solution for the non-homogeneous ODE

$$\boxed{\left(m^2 - m + m - 1\right) r^m = 0}$$

For this equation $r^m = 0$ or $m = 1$ or $m = -1$ and the general solution is:

$$u_h = C_1 r^1 + C2 r^{-1} = C_1 r + \frac{C_2}{r}$$

Now we recall the non-homogeneous ODE:

$$r^2 u''(r) + r u'(r) - u(r) = -Nr^3$$

Recalling the original form of the non-homogeneous ODE:

$$\boxed{u''(r) + \frac{1}{r} u'(r) - \frac{1}{r^2} u(r) = -Nr}$$

Solution for the non-homogeneous ODE

$$u''(r) + \frac{1}{r}u'(r) - \frac{1}{r^2}u(r) = -Nr \quad \text{and} \quad u_h = C_1 r + \frac{C_2}{r}$$

Doing the Wronskians:

$$u_p = a_1 r + a_2 \frac{1}{r}$$

$$W = \begin{vmatrix} r & \frac{1}{r} \\ 1 & -\frac{1}{r^2} \end{vmatrix} = -\frac{1}{r} - \frac{1}{r} = -\frac{2}{r}$$

$$a_1' = \frac{W_1}{W} = -\frac{Nr}{2} \longleftrightarrow a_1 = -\frac{Nr^2}{4}$$

$$W_1 = \begin{vmatrix} 0 & \frac{1}{r} \\ -Nr & -\frac{1}{r^2} \end{vmatrix} = 0 + N = N$$

$$a_2' = \frac{W_2}{W} = \frac{Nr^3}{2} \longleftrightarrow a_2 = \frac{Nr^4}{8}$$

The particular solution is:

$$W_2 = \begin{vmatrix} r & 0 \\ 1 & -Nr \end{vmatrix} = -Nr^2 - 0 = -Nr^2$$

$$\boxed{u_p = -N\frac{r^2}{4}r + N\frac{r^4}{8}\frac{1}{r} = -N\frac{r^3}{8}}$$

Solution for the non-homogeneous ODE

$$\boxed{u = u_h + u_p = C_1 r + \frac{C_2}{r} - N\frac{r^3}{8}}$$

Recall again the radial stress equation as function of the radial displacement:

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{\mathrm{d}u}{\mathrm{d}r} + \nu\frac{u}{r}\right)$$

$$\sigma_r = \frac{E}{1-\nu^2}\left[C_1 - \frac{C_2}{r^2} - 3N\frac{r^2}{8} + \nu C_1 + \nu\frac{C_2}{r^2} - \nu N\frac{r^2}{8}\right]$$

$$\boxed{\sigma_r = \frac{E}{1-\nu^2}\left[C_1\left(1+\nu\right) - \left(1-\nu\right)\frac{C_2}{r^2} - \frac{3+\nu}{8}Nr^2\right]}$$

Radial and tangential stresses

$$\boxed{\sigma_r = \frac{E}{1-\nu^2}\left[C_1\left(1+\nu\right)-\left(1-\nu\right)\frac{C_2}{r^2}-\frac{3+\nu}{8}Nr^2\right]}$$

The boundary conditions for a disc with a hole at center:

$\sigma_r = 0 \quad \text{for} \quad r = a$

$$0 = \frac{E}{1-\nu^2}\left[C_1\left(1+\nu\right)-\left(1-\nu\right)\frac{C_2}{a^2}-\frac{3+\nu}{8}Na^2\right]$$

$\sigma_r = 0 \quad \text{for} \quad r = b$

$$0 = \frac{E}{1-\nu^2}\left[C_1\left(1+\nu\right)-\left(1-\nu\right)\frac{C_2}{b^2}-\frac{3+\nu}{8}Nb^2\right]$$

Radial and tangential stresses

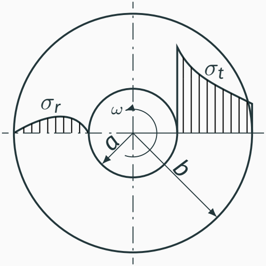


Figure 5: Hoop and radial stresses for a rotating thick wall cylinder.

The value of constants C_1 and C_2:

$$\begin{cases} C_1 = \frac{3+\nu}{8(1+\nu)}\left(a^2+b^2\right)N \\ \\ C_2 = \frac{3+\nu}{8(1+\nu)}\left(a^2b^2\right)N \end{cases}$$

With $\boxed{N = \frac{1-\nu^2}{E}\rho\omega^2}$

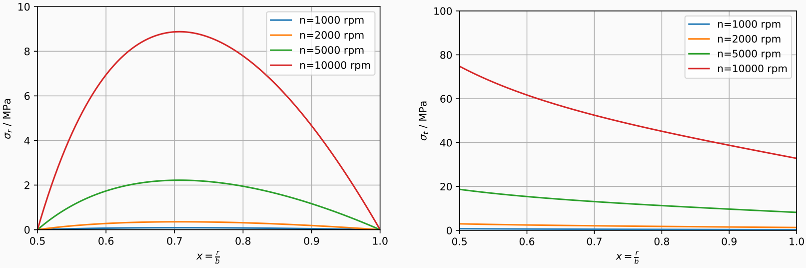
Finally the radial and hoop stresses equations are:

$$\begin{cases} \sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(a^2+b^2-\frac{a^2b^2}{r^2}-r^2\right) \\ \\ \sigma_t = \frac{3+\nu}{8}\rho\omega^2\left(a^2+b^2+\frac{a^2b^2}{r^2}-\frac{1+3\nu}{3+\nu}r^2\right) \end{cases}$$

Influence of angular speed

As expected, as the angular speed increases both the radial and tangential stress increase.

For a steel disc with $b = 100\,\mathrm{mm}$.



(a) Radial stress **(b)** Hoop stress

Figure 6: Stresses for a rotating disc made of steel with $K = \frac{b}{a} = 2$.

Location of maximum stresses

The radial stress equation presents a maximum for $r = \sqrt{ab}$:

$$\boxed{(\sigma_r)_{max} = \frac{3+\nu}{8}\rho\omega^2\,(b-a)^2}$$

The tangential stress is maximum at the inner edge of the disc $r = a$:

$$\boxed{(\sigma_t)_{max} = \frac{3+\nu}{4}\rho\omega^2\left(b^2 + \frac{1-\nu}{3+\nu}a^2\right)}$$

It is easy now to conclude that $(\sigma_t)_{max}$ is always larger than $(\sigma_r)_{max}$.

Thin wall cylinder

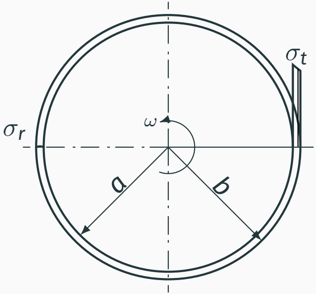


Figure 7: Hoop and radial stresses for a rotating thin wall cylinder.

According to the equations derived previously for thick wall cylinders:

$$\begin{cases} \sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(a^2 + b^2 - \frac{a^2b^2}{r^2} - r^2\right) \\ \sigma_t = \frac{3+\nu}{8}\rho\omega^2\left(a^2 + b^2 + \frac{a^2b^2}{r^2} - \frac{1+3\nu}{3+\nu}r^2\right) \end{cases}$$

Considering a wall thickness very small in comparison with the radius, and making $a \longrightarrow b$:

$$\boxed{\sigma_t = \rho\omega^2 b^2 = \rho\omega^2 r^2}$$

Thin wall cylinder

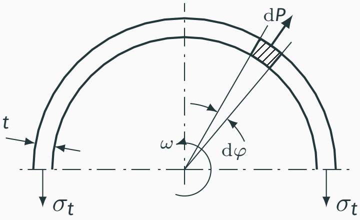


Figure 8: Hoop and radial stresses for a rotating thin wall cylinder.

$$\mathrm{d}P = r\,\mathrm{d}\varphi\, t\rho\omega^2 r$$

Using the thin wall theory, i.e. assuming constant hoop stress through the cylinder wall:

$$2\sigma_t t = 2\int_0^{\frac{\pi}{2}} r^2 t\rho\omega^2 \sin\varphi\,\mathrm{d}\varphi$$

$$2\sigma_t t = 2r^2 t\rho\omega^2\left[-\cos\varphi\right]_0^{\frac{\pi}{2}}$$

Using the thin wall theory, we find the same result as with the thick wall theory equations:

$$\boxed{\sigma_t = \rho\omega^2 r^2}$$

Solid disc

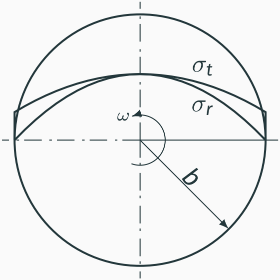


Figure 9: Hoop and radial stresses for a solid disc.

For $a = 0$, the equation becomes:

$$\begin{cases} \sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(b^2 - r^2\right) \\ \sigma_t = \frac{3+\nu}{8}\rho\omega^2\left(b^2 - \frac{1+3\nu}{3+\nu}r^2\right) \end{cases}$$

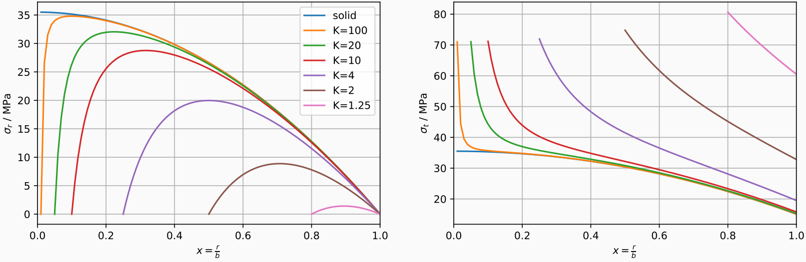
$$\boxed{(\sigma_r)_{max} = (\sigma_t)_{max} = \frac{3+\nu}{8}\rho\omega^2 b^2}$$

$$\boxed{(\sigma_t)_{min} = \frac{1-\nu}{4}\rho\omega^2 b^2}$$

Influence of hole size

As discussed for “Thick Cylinders”, the ratio K is $\frac{b}{a}$.

For a steel disc rotating at $n = 10\,000\,\mathrm{rpm}$ and $b = 100\,\mathrm{mm}$.



(a) Radial stress

(b) Hoop stress

Figure 10: Influence of cylinder wall thickness (or hole dimension) [4].

Finite Element Method solution

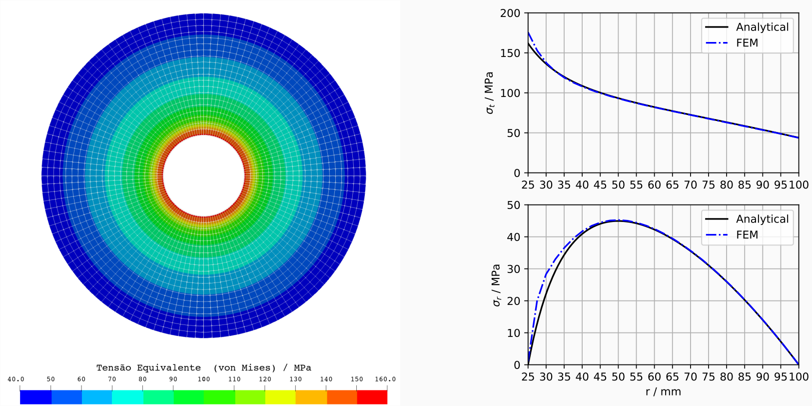


Figure 11: FEM 3D calculation with CalculiX [7] for a steel rotating cylinder $a = 25\,\mathrm{mm}$ and $b = 100\,\mathrm{mm}$ and $n = 15\,000\,\mathrm{rpm}$.

Rotating press fit

Two concentric AISI 1040 steel cylinders are press fit together with a radial interference of $0.025\,\mathrm{mm}$. The nominal sizes of the tubes are $c = 50\,\mathrm{mm}$ and $b = 40\,\mathrm{mm}$ for the outer tube and $b = 40\,\mathrm{mm}$ and $a = 30\,\mathrm{mm}$ for the inner tube [8].

How fast should the compound cylinder rotate in order to decrease the press-fit pressure to zero?

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(a^2 + c^2 - \frac{a^2c^2}{b^2} - b^2\right) = \frac{3+0.3}{8}7.85\times10^{-9}\omega^2\left(30^2 + 50^2 - \frac{30^2\times50^2}{40^2} - 40^2\right)$$

$$p_c = \frac{E\delta}{b}\left[\frac{\left(c^2-b^2\right)\left(b^2-a^2\right)}{2b^2\left(c^2-a^2\right)}\right] = \frac{206\times10^3\times0.025}{40}\left[\frac{\left(50^2-40^2\right)\left(40^2-30^2\right)}{2\times40^2\left(50^2-30^2\right)}\right] = 15.8\,\mathrm{MPa}$$

$$\sigma_r = p_c \Leftrightarrow \omega = 3525\,\mathrm{rad/s} = 33\,660\,\mathrm{rpm}$$

Lecture 2

Summary

Flywheels

A flywheel is an energy storage device. It stores energy by increasing its angular velocity and delivers energy by decreasing its angular velocity.

A flywheel is used to smooth the flow of energy between a power source and its load.

Typical applications are: internal combustion engines, reciprocating compressors and pumps, automobiles, punch presses.

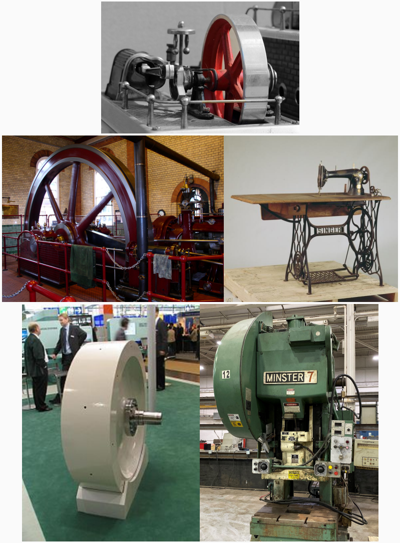


Figure 12: Flywheel applications.

Equation of motion of a system with flywheel

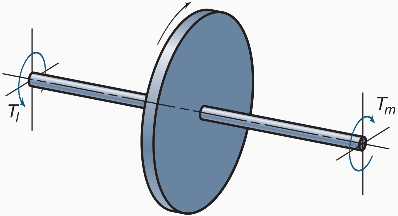


Figure 13: Flywheel with driving (mean) torque T_m and load torque T_l [8, 10].

Considering a rigid body analysis, the Newton's 2nd law is:

$$\sum M = I \cdot \frac{\mathrm{d}\omega}{\mathrm{d}t}$$

The equation of motion of the flywheel is:

$$I \cdot \frac{\mathrm{d}\omega}{\mathrm{d}t} = T_m - T_l$$

Recall that:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}\omega}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\omega}{\mathrm{d}\theta} = \omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta}$$

The design motor/engine torque should be equivalent to the average torque: $T_m = T_{avg}$

So, the equation of motion can be written as:

$$(T_{avg} - T_l) \cdot \mathrm{d}\theta = I \cdot \omega \cdot \mathrm{d}\omega$$

Turning moment diagram

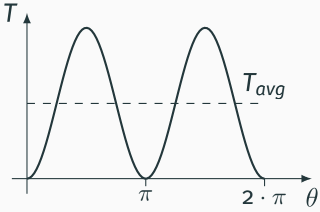


Figure 14: Example of a turning moment diagram of an engine

Typically, the **driven machine** exerts a fluctuating torque on the shaft.

The Figure shows the torque T for different values of the crank position θ, that represents a possible **driving machine**.

The area of the diagram T vs. θ is equal to the work done by the engine per cycle.

The useful work divided by the angle of the cycle ($2 \cdot \pi$) gives the average torque T_{avg}:

$$T_{avg} = \frac{\int T \, \mathrm{d}\theta}{\theta_{cycle}}$$

Influence of flywheel inertia

To show the influence of the flywheel inertia on the system angular acceleration, let's assume a constant resistant torque T_l and a very simple power source:

$$T_m\left(\theta\right) = T_l\left(1 + \sin\theta\right)$$

Recalling the equation of motion: $I \cdot \frac{\mathrm{d}\omega}{\mathrm{d}t} = T_m - T_l$

$$\boxed{\frac{\mathrm{d}\omega}{\mathrm{d}t} = \ddot{\theta} = \frac{T_l \sin\theta}{I}}$$

It becomes evident the influence of the mass moment of inertia of the flywheel

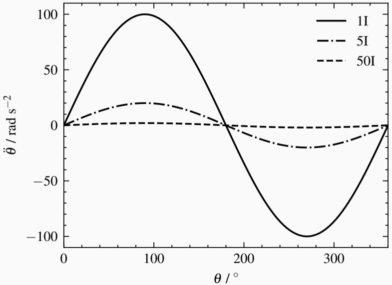
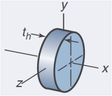
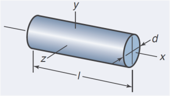
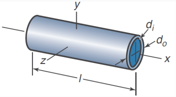


Figure 15: Influence of the flywheel inertia on the system acceleration.

Mass moment of inertia

$$\boxed{I = \int_m r^2 \, \mathrm{d}m}$$

Shape		Mass	Mass moment of inertia
	Disc	$m = \frac{\pi d^2 t_h \rho}{4}$	$I_x = \frac{md^2}{8}$
	Cylinder	$m = \frac{\pi d^2 l \rho}{4}$	$I_x = \frac{md^2}{8}$
	Hollow cylinder	$m = \frac{\pi \left(d_o^2 - d_i^2\right) l \rho}{4}$	$I_x = \frac{m\left(d_o^2 + d_i^2\right)}{8}$

Energy stored in a flywheel

Recalling again the equation of motion: $(T_{avg} - T_l) \cdot \mathrm{d}\theta = I \cdot \omega \cdot \mathrm{d}\omega$

The equation can be written in terms of a definite integral as:

$$\int_{\theta_{\omega_{min}}}^{\theta_{\omega_{max}}} (T_{avg} - T_l)\,\mathrm{d}\theta = \int_{\omega_{min}}^{\omega_{max}} I \cdot \omega\,\mathrm{d}\omega$$

The left side represents the change in kinetic energy between the maximum and minimum shaft speeds, so:

$$E_{max} - E_{min} = \frac{1}{2} \cdot I \cdot \left(\omega_{max}^2 - \omega_{min}^2\right)$$

This equation can be written in terms of the average angular speed $\omega_{avg} = \frac{\omega_{min} + \omega_{max}}{2}$ and the coefficient of speed fluctuation C_f:

$$\boxed{E_{max} - E_{min} = I \cdot C_f \cdot \omega_{avg}^2}$$

Coefficient of speed fluctuation

The coefficient of speed fluctuation is defined as:

$$C_f = \frac{\omega_{max} - \omega_{min}}{\omega_{avg}} = 2\frac{\omega_{max} - \omega_{min}}{\omega_{min} + \omega_{max}}$$

The reciprocal of the coefficient of fluctuation is known as *coefficient of steadiness* and is denoted by m:

$$m = \frac{1}{C_f}$$

Typical values [11, 6]:

Equipment	C_f
Crushing machinery	0.2
Electrical machinery	0.003
Electrical machinery, direct-driven	0.002
Engines with belt transmission	0.03
Flour-milling machinery	0.02
Gear transmission	0.02
Hammering machinery	0.2
Machine tools	0.03
Paper-making machinery	0.025
Pumping machinery	0.03-0.05
Shearing machinery	0.03-0.05
Spinning machinery	0.01-0.02
Textile machinery	0.025

Select the mass moment of inertia [1]

Consider torque displacement plot for one cycle in Figure 16. The average speed is to be $\omega_{avg} = 250\,\mathrm{rad/s}$.

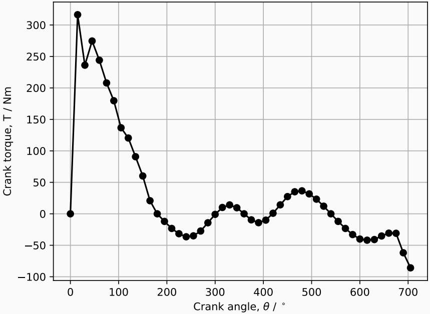


Figure 16: Torque displacement function for one cycle of a single-cylinder four-stroke engine (48 measurements with equal spacing) [10].

[1]See the example in Course Contents from "Design of Machine Elements" [11]

Select the mass moment of inertia

1. Integrate the torque-displacement function for one cycle, and find the energy that can be delivered to the load during the cycle;

   Using the trapezium rule and integrating the torque displacement function:

   $$E = 391.22\,\mathrm{N\,m}$$

2. Determine the average torque T_{avg};

   Considering the total displacement $\theta = 4\pi$, the average torque is:

   $$T_{avg} = \frac{E}{\theta} = \frac{391.22}{4\pi} = 31.79\,\mathrm{N\,m}$$

Select the mass moment of inertia

3. Find a suitable value for the flywheel mass moment of inertia considering a coefficient of speed fluctuation $C_f = 0.1$. Find ω_{max} and ω_{min}.

The maximum positive loop occurs between $\theta = 0$ and $\theta = 180$. The change in kinetic energy is given by:

$$E_{max} - E_{min} = \int_0^{\pi} T(\theta) - T_{avg} \, \mathrm{d}\theta = 399.84 \, \mathrm{N\,m}$$

$$E_{max} - E_{min} = C_f \cdot I \cdot \omega_{avg}^2 \Leftrightarrow I = \frac{E_{max} - E_{min}}{C_f \cdot \omega_{avg}^2} = 0.064 \, \mathrm{kg\,m^2}$$

Finally we solve simultaneously for ω_{max} and ω_{min}:

$$\begin{cases} \omega_{max} = \frac{1}{2}(2 + C_f) \cdot \omega_{avg} = 262.5 \, \mathrm{rad/s} \\ \omega_{min} = 2 \cdot \omega_{avg} - \omega_{max} = 237.5 \, \mathrm{rad/s} \end{cases}$$

Flywheel with rim and spokes

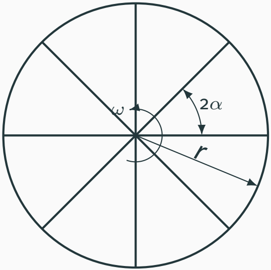


Figure 17: Flywheel with rim and spokes [12].

Nomenclature

r	radius of the center-line of the rim
A	cross-sectional area of the rim
A_1	cross-sectional area of a spoke
I	moment of inertia of the cross section of the rim
n	number of spokes
2α	angle between two consecutive spokes
q	weight of the rim per unit length of the center-line
q_1	weight of a spoke per unit length
ω	angular velocity of the flywheel

Due to the effect of the spokes, the rim of a rotating flywheel undergoes not only extension but also bending [2].

[2] Presentation of the topic taken from Timoshenko (pages 398-401) [12]. A more detailed discussion on the topic is “Remarks on rim and spokes flywheels” by Paulo M.S.T de Castro, available on the Course Contents.

Flywheel with rim and spokes

Considering as free body diagram a portion of the rim between two cross sections which bisect the angles between the spokes.

From the condition of symmetry, there can be no shearing stresses over the cross sections *A* and *B*. So, we just have longitudinal force N_o and the bending moment M_o.

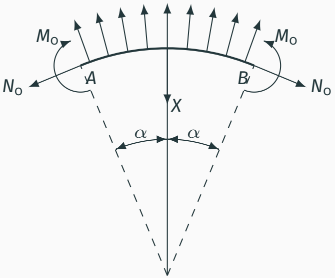


Figure 18: Free Body Diagram of the flywheel with rim and spokes [12].

Flywheel with rim and spokes

X denotes the force exerted by the spoke on the rim and the equation of equilibrium of the portion AB is:

$$2N_0 \sin\alpha + X - 2r^2 \frac{q}{g}\omega^2 \sin\alpha = 0$$

$$\boxed{N_0 = \frac{q}{g}\omega^2 r^2 - \frac{X}{2\sin\alpha}}$$

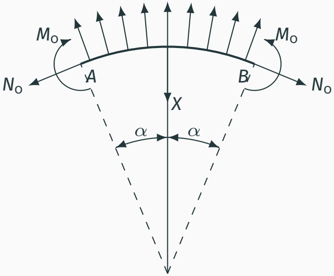


Figure 18: Free Body Diagram of the flywheel with rim and spokes [12].

Flywheel with rim and spokes

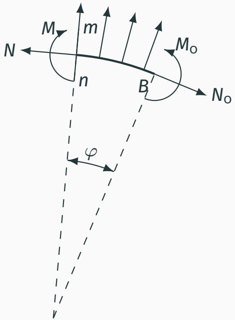


Figure 19: Free Body Diagram of the flywheel rim [12].

The longitudinal force N at any cross section mn is:

$$N = N_0 \cos\varphi + 2\frac{q}{g}\omega^2 r^2 \sin^2\frac{\varphi}{2} = 0$$

With:

$$N_0 = \frac{q}{g}\omega^2 r^2 - \frac{X}{2\sin\alpha}$$

Finally:

$$\boxed{N = \frac{q}{g}\omega^2 r^2 - \frac{X\cos\varphi}{2\sin\alpha}}$$

Flywheel with rim and spokes

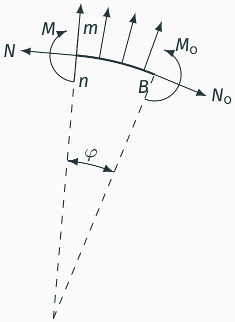


Figure 19: Free Body Diagram of the flywheel rim [12].

The bending moment M at any cross section mn is:

$$M = M_0 - N_0 r\,(1-\cos\varphi) + 2\frac{q}{g}\omega^2 r^3 \sin^2\frac{\varphi}{2}$$

$$M = M_0 + \frac{Xr}{\sin\alpha}\sin^2\frac{\varphi}{2}$$

Force X and the moment M_0 cannot be determined using the static equilibrium equations.

Flywheel with rim and spokes

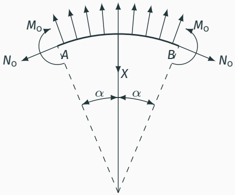


Figure 18: Free Body Diagram of the flywheel rim [12].

The strain energy of the portion AB of the rim is[3]:

$$U = 2\int_0^{\alpha} \frac{M^2}{2EI} r \,\mathrm{d}\varphi + 2\int_0^{\alpha} \frac{N^2}{2EA} r \,\mathrm{d}\varphi$$

The strain energy of the spoke is:

$$U_1 = \int_0^{r} \frac{N_1^2}{2EA_1} \,\mathrm{d}\rho$$

The tensile force N_1 at any cross section of the spoke at a distance ρ from the center of the flywheel is:

$$N_1 = X + \frac{q_1}{2g}\omega^2 \left(r^2 - \rho^2\right)$$

[3] cross-sectional area is small in comparison with r: just bending and tension is considered

Flywheel with rim and spokes

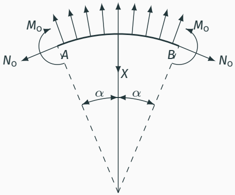


Figure 18: Free Body Diagram of the flywheel rim [12].

Applying the Castigliano's theorem, the equations for M_0 and X are:

$$\frac{\partial}{\partial M_0}(U + U_1) = 0$$

$$\frac{\partial}{\partial X}(U + U_1) = 0$$

After solving the equations, we get:

$$M_0 = -\frac{Xr}{2}\left(\frac{1}{\sin\alpha} - \frac{1}{\alpha}\right)$$

$$X = \frac{2}{3}\frac{q}{g}\omega^2 r^2 \frac{1}{\frac{Ar^2}{I}f_2(\alpha) + f_1(\alpha) + \frac{A}{A_1}}$$

with:

$f_1(\alpha) = \frac{1}{2\sin^2\alpha}\left(\frac{\sin 2\alpha}{4} + \frac{\alpha}{2}\right)$ and
$f_2(\alpha) = \frac{1}{2\sin^2\alpha}\left(\frac{\sin 2\alpha}{4} + \frac{\alpha}{2}\right) - \frac{1}{2\alpha}$

Flywheel with rim and spokes

$$f_1(\alpha) = \frac{1}{2\sin^2\alpha}\left(\frac{\sin 2\alpha}{4} + \frac{\alpha}{2}\right)$$

$$f_2(\alpha) = \frac{1}{2\sin^2\alpha}\left(\frac{\sin 2\alpha}{4} + \frac{\alpha}{2}\right) - \frac{1}{2\alpha}$$

The value of each function for different number of spokes *n* is:

n	4	6	8
$f_1(\alpha)$	0.643	0.957	1.274
$f_2(\alpha)$	0.00608	0.00169	0.00076

Problems

Summary

Assignments

Assignment A3

Available on the Course Contents:

“Assignments” proposed for Complements of Machine Elements

Flywheel with rim and 6 spokes

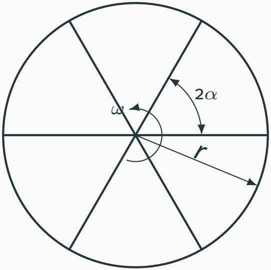


Figure 19: Flywheel with rim and 6 spokes [12].

Consider a 6 spoke steel flywheel rotating at 600 rpm, with radius $r = 1.5\,\mathrm{m}$, cross section of the rim a square $0.3 \times 0.3\,\mathrm{m}^2$ and the cross-sectional area of a spoke $A_1 = 0.015\,\mathrm{m}^2$. Calculate the maximum stress on the rim.

Flywheel stresses

A flywheel made of low-carbon steel has an outside radius of 150 mm and an inside radius of 25 mm. The flywheel is to be assembled (press fit) onto a shaft. The radial interference between the flywheel and shaft is 50 $\mu\mathrm{m}$, and the shaft will operate at a speed of 5000 rpm [8]. Calculate:

1. The circumferential and radial stresses on the flywheel inner radius.
2. The speed at which the flywheel will break loose from the shaft.

Flywheel design

The output, or load torque, of a flywheel used in a punch press for each revolution of the shaft is 12 $\mathrm{N\,m}$ from zero to π and from 3$\pi/$2 to 2π and 144 $\mathrm{N\,m}$ from π to 3$\pi/$2. The coefficient of fluctuation is $C_f = 0.05$ about an average speed of 600 rpm. Assume that the flywheel's solid disk is made of low-carbon steel of constant 25 mm thickness [8]. Determine the following:

1. The average load or output torque
2. The locations $\theta_{\omega_{min}}$ and $\theta_{\omega_{max}}$
3. The energy fluctuation required
4. The outside diameter of the flywheel

References

References i

[1] Budynas, Richard G.: *Shigley's mechanical engineering design*. **McGraw-Hill, 2014, ISBN 9789339221638.**

[2] Vullo, V. and F. Vivio: *Rotors: Stress Analysis and Design*. **Mechanical Engineering Series. Springer Milan, 2013, ISBN 9788847025622.**

[3] Féodossiev, V.: *Résistance des Matériaux*. **Moscou. Éditions MIR, 1971.**

[4] Timoshenko, Stephen: *Strength of Materials. Part 2: Advanced Theory and Problems*. **CBS, third edition, 1986, ISBN 81-239-1077-0.**

[5] Castro, Paulo M S Tavares de: *Tensões em peças cilíndricas*. **Órgãos de Máquinas, DEMec, Faculdade de Engenharia da Universidade do Porto, 2019.**

References ii

[6] Castro, Paulo M S Tavares de: *Tensões em discos giratórios. o caso dos volantes de inércia.*
Órgãos de Máquinas, DEMec, Faculdade de Engenharia da Universidade do Porto, 2020.

[7] Dhondt, Guido: *The Finite Element Method for Three-dimensional Thermomechanical Applications.*
Wiley, 2004, ISBN 0470857528.

[8] Schmid, Steven R., Bernard J. Hamrock, and Bo O. Jacobson: *Fundamentals of Machine Elements.*
CRC Press, 2014, ISBN 9781482247503.

[9] Gupta, R.S. and J.K. Gupta: *Theory of Machines.*
Eurasia Publishing House, 2008, ISBN 9788121925242.

[10] John J. Uicker, Jr., Gordon R. Pennock, and Joseph E. Shigley: *Theory of Machines and Mechanisms.* **Oxford University Press, fifth edition, 2017, ISBN 9780190264482.**

[11] Spotts, M.F.: *Design of Machine Elements.* **Prentice-Hall, 1953.**

[12] Timoshenko, Stephen: *Strength of Materials. Part 1: Elementary Theory and Problems.* **CBS, third edition, 1986, ISBN 81-239-1030-4.**