

Math 269B, 2012 Winter, Homework 2 (Solutions)

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1 Theory

1. (Strikwerda 2.1.9.) Finite Fourier Transforms. For a function v_m defined on the integers, $m = 0, 1, \dots, M-1$, we can define the Fourier transform as

$$\hat{v}_\ell = \sum_{m=0}^{M-1} e^{-2i\pi\ell m/M} v_m \quad \text{for } \ell = 0, \dots, M-1.$$

For this transform prove the Fourier inversion formula

$$v_m = \frac{1}{M} \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \hat{v}_\ell,$$

and the Parseval's relation

$$\sum_{m=0}^{M-1} |v_m|^2 = \frac{1}{M} \sum_{\ell=0}^{M-1} |\hat{v}_\ell|^2.$$

Note that v_m and \hat{v}_ℓ can be defined for all integers by making them periodic with period M .

Solution

Substituting in for \hat{v}_ℓ and swapping the order of summation yields

$$\begin{aligned} & \frac{1}{M} \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \hat{v}_\ell \\ &= \frac{1}{M} \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \sum_{m'=0}^{M-1} e^{-2i\pi\ell m'/M} v_{m'} \\ &= \frac{1}{M} \sum_{m'=0}^{M-1} v_{m'} \sum_{\ell=0}^{M-1} e^{2i\pi\ell(m-m')/M}. \end{aligned}$$

Now, if $m \neq m'$, then

$$\sum_{\ell=0}^{M-1} e^{2i\pi\ell(m-m')/M} = \frac{1 - e^{2i\pi M(m-m')/M}}{1 - e^{2i\pi(m-m')/M}} = 0;$$

on the other hand, if $m = m'$, then the summand is identically 1 and hence the sum is M . The result follows immediately:

$$\frac{1}{M} \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \hat{v}_\ell = v_m.$$

Regarding the Parseval's relation,

$$\begin{aligned}
\sum_{m=0}^{M-1} |v_m|^2 &= \frac{1}{M^2} \sum_{m=0}^{M-1} \left| \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \hat{v}_\ell \right|^2 \\
&= \frac{1}{M^2} \sum_{m=0}^{M-1} \left(\sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \hat{v}_\ell \right) \overline{\left(\sum_{\ell'=0}^{M-1} e^{2i\pi\ell' m/M} \hat{v}_{\ell'} \right)} \\
&= \frac{1}{M^2} \sum_{\ell=0}^{M-1} \sum_{\ell'=0}^{M-1} \sum_{m=0}^{M-1} e^{2i\pi(\ell-\ell')m/M} \hat{v}_\ell \overline{\hat{v}_{\ell'}} \\
&= \frac{1}{M} \sum_{\ell=0}^{M-1} |\hat{v}_\ell|^2,
\end{aligned}$$

as desired (since, when $\ell \neq \ell'$, the inner sum over m sums out to 0). Alternatively, we may derive Parseval's relation via the inversion formula:

$$\begin{aligned}
\sum_{m=0}^{M-1} |v_m|^2 &= \sum_{m=0}^{M-1} v_m \overline{v_m} \\
&= \frac{1}{M} \sum_{m=0}^{M-1} \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \hat{v}_\ell \overline{v_m} \\
&= \frac{1}{M} \sum_{\ell=0}^{M-1} \hat{v}_\ell \sum_{m=0}^{M-1} e^{-2i\pi\ell m/M} v_m \\
&= \frac{1}{M} \sum_{\ell=0}^{M-1} \hat{v}_\ell \overline{\hat{v}_\ell} \\
&= \frac{1}{M} \sum_{\ell=0}^{M-1} |\hat{v}_\ell|^2.
\end{aligned}$$

2. Prove convergence for the Beam-Warming scheme

$$u_m^{n+1} = u_m^n - \frac{ak}{2h} (3u_m^n - 4u_{m-1}^n + u_{m-2}^n) + \frac{a^2k^2}{2h^2} (u_m^n - 2u_{m-1}^n + u_{m-2}^n)$$

used to approximate solutions to $u_t + au_x = 0$ for $a > 0$.

Solution

We rewrite the difference operator as

$$P_{k,h} u_m^n = \frac{1}{k} (u_m^{n+1} - u_m^n) + \frac{a}{2h} (3u_m^n - 4u_{m-1}^n + u_{m-2}^n) - \frac{a^2k}{2h^2} (u_m^n - 2u_{m-1}^n + u_{m-2}^n).$$

Its symbol is thus

$$\begin{aligned}
p_{k,h}(s, \xi) &= P_{k,h} (e^{skn+imh\xi}) / e^{skn+imh\xi} \\
&= \frac{1}{k} (e^{sk} - 1) + \frac{a}{2h} (3 - 4e^{-ih\xi} + e^{-2ih\xi}) - \frac{a^2k}{2h^2} (1 - 2e^{-ih\xi} + e^{-2ih\xi}) \\
&= \frac{1}{k} \left(sk + \frac{1}{2} s^2 k^2 + O(k^3) \right) + \frac{a}{2h} (2ih\xi + O(h^3)) - \frac{a^2k}{2h^2} (-h^2\xi^2 + O(h^3)) \\
&= \left(1 + \frac{k}{2} (s - ia\xi) \right) (s + ia\xi) + O(k^2 + kh + h^2).
\end{aligned}$$

Since the symbol of the differential operator $P = \partial_t + a\partial_x$ is $p = s + ia\xi$, this demonstrates that $P_{k,h}$ is consistent with P to second order in k, h . Incidentally, it also suggests that the symbol $r_{k,h}$ of the difference operator $R_{k,h}$ should be (for example)

$$\begin{aligned} r_{k,h}(s, \xi) &= 1 + \frac{k}{2}(s - ia\xi) + O(k^2 + kh + h^2) \\ &= \frac{1}{2}(e^{sk} + 1) - \frac{ak}{2h}(1 - e^{-ih\xi}), \end{aligned}$$

giving

$$R_{k,h}f_m^n = \frac{1}{2}(f_m^{n+1} + f_m^n) - \frac{ak}{2h}(f_m^n - f_{m-1}^n).$$

Regarding stability, we substitute $g := e^{sk}$ and solve $p_{k,h} = 0$ for g , giving

$$\begin{aligned} g &= 1 - \frac{ak}{2h}(3 - 4e^{-i\theta} + e^{-2i\theta}) + \frac{a^2k^2}{2h^2}(1 - 2e^{-i\theta} + e^{-2i\theta}) \\ &= 1 - e^{-i\theta}\frac{ak}{2h}\left(\left(3 - \frac{ak}{h}\right)e^{i\theta} - 2\left(2 - \frac{ak}{h}\right) + \left(1 - \frac{ak}{h}\right)e^{-i\theta}\right) \\ &= 1 + e^{-i\theta}\frac{ak}{h}\left(\left(2 - \frac{ak}{h}\right)(1 - \cos\theta) - i\sin\theta\right) \\ &= e^{-i\theta}\left(\cos\theta + \frac{ak}{h}\left(2 - \frac{ak}{h}\right)(1 - \cos\theta) + i\left(1 - \frac{ak}{h}\right)\sin\theta\right) \\ &= e^{-i\theta}\left(1 - \left(1 - \frac{ak}{h}\right)^2(1 - \cos\theta) + i\left(1 - \frac{ak}{h}\right)\sin\theta\right). \end{aligned}$$

If we let $\alpha = 1 - ak/h$, then

$$\begin{aligned} |g|^2 &= |1 - \alpha^2(1 - \cos\theta) + i\alpha\sin\theta|^2 \\ &= 1 + \alpha^4(1 - \cos\theta)^2 + \alpha^2\sin^2\theta - 2\alpha^2(1 - \cos\theta) \\ &= 1 - \alpha^2(1 - \alpha^2)(1 - \cos\theta)^2, \end{aligned}$$

hence $|g| \leq 1$ if and only if $|\alpha| \leq 1$, which is equivalent to $0 \leq ak/h \leq 2$.

3. (Strikwerda 2.2.4.) Show that the box scheme

$$\frac{1}{2k}((v_m^{n+1} + v_{m+1}^{n+1}) - (v_m^n + v_{m+1}^n)) + \frac{a}{2h}((v_{m+1}^{n+1} - v_m^{n+1}) + (v_{m+1}^n - v_m^n)) = f_m^n$$

is consistent with the one-way wave equation $u_t + au_x = f$ and is stable for all values of λ .

Solution

The symbols corresponding to the difference operators $P_{k,h}$ and $R_{k,h}$ are

$$\begin{aligned} p_{k,h}(s, \xi) &= P_{k,h}(e^{skn+imh\xi})/e^{skn+imh\xi} \\ &= \frac{1}{2k}(e^{sk} - 1)(e^{ih\xi} + 1) + \frac{a}{2h}(e^{sk} + 1)(e^{ih\xi} - 1) \\ &= \frac{1}{2k}\left(sk + \frac{1}{2}s^2k^2 + O(k^3)\right)(2 + ih\xi + O(h^2)) \\ &\quad + \frac{a}{2h}(2 + sk + O(k^2))\left(ih\xi - \frac{1}{2}h^2\xi^2 + O(h^3)\right) \\ &= \left(1 + \frac{1}{2}(sk + ih\xi)\right)(s + ia\xi) + O(h^2 + hk + k^2); \\ r_{k,h}(s, \xi) &\equiv 1. \end{aligned}$$

Since the symbol of the differential operator $P = \partial_t + a\partial_x$ is $p = s + ia\xi$, this shows that the box scheme as given by the difference operators $P_{k,h}$ and $R_{k,h}$ is consistent with P to first order in k, h . It also shows that we can actually achieve second order accuracy by modifying $r_{k,h}$ to

$$\begin{aligned} r'_{k,h}(s, \xi) &= 1 + \frac{1}{2}(sk + ih\xi) + O(k^2 + kh + h^2) \\ &= \frac{1}{4}(1 + e^{sk})(1 + e^{ih\xi}), \end{aligned}$$

giving

$$R'_{k,h}f_m^n = \frac{1}{4}(f_m^n + f_m^{n+1} + f_{m+1}^n + f_{m+1}^{n+1}).$$

Regarding stability, we substitute $g := e^{sk}$ and solve $p_{k,h} = 0$ for g , giving

$$\begin{aligned} g &= \frac{e^{i\theta} + 1 - a\lambda(e^{i\theta} - 1)}{e^{i\theta} + 1 + a\lambda(e^{i\theta} - 1)} \\ &= \frac{e^{i\theta/2} + e^{-i\theta/2} - a\lambda(e^{i\theta/2} - e^{-i\theta/2})}{e^{i\theta/2} + e^{-i\theta/2} + a\lambda(e^{i\theta/2} - e^{-i\theta/2})}, \end{aligned}$$

which clearly shows that $|g| \equiv 1$ (the numerator and denominator are complex conjugates).

4. (Strikwerda 2.2.6.) Determine the stability of the following scheme, sometimes called the Euler backward scheme, for $u_t + au_x = f$:

$$\begin{aligned} v_m^{n+1/2} &= v_m^n - \frac{a\lambda}{2}(v_{m+1}^n - v_{m-1}^n) + kf_m^n, \\ v_m^{n+1} &= v_m^n - \frac{a\lambda}{2}(v_{m+1}^{n+1/2} - v_{m-1}^{n+1/2}) + kf_m^{n+1}. \end{aligned}$$

The variable $v^{n+1/2}$ is a temporary variable, as is \tilde{v} in Example 2.2.5.

Solution

The symbol $p_{k,h}^{1/2}$ of the difference operator $P_{k,h}^{1/2}$ of the first half-step is

$$\begin{aligned} p_{k,h}^{1/2}(s, \xi) &= P_{k,h}^{1/2}(e^{skn+imh\xi})/e^{skn+imh\xi} \\ &= \frac{1}{k}(e^{sk} - 1) + \frac{a}{2h}(e^{ih\xi} - e^{-ih\xi}); \end{aligned}$$

the symbol $p_{k,h}$ of the full difference operator $P_{k,h}$ is then

$$\begin{aligned} p_{k,h}(s, \xi) &= P_{k,h}(e^{skn+imh\xi})/e^{skn+imh\xi} \\ &= \frac{1}{k}(e^{sk} - 1) + \frac{a}{2h}(e^{ih\xi} - e^{-ih\xi})v^{n+1/2}(e^{imh\xi}) \\ &= \frac{1}{k}(e^{sk} - 1) + \frac{a}{2h}(e^{ih\xi} - e^{-ih\xi})\left(1 - \frac{a\lambda}{2}(e^{ih\xi} - e^{-ih\xi})\right). \end{aligned}$$

Substituting $g := e^{sk}$ and solving $p_{k,h} = 0$ for g yields

$$\begin{aligned} g &= 1 - \frac{a\lambda}{2}(e^{i\theta} - e^{-i\theta})\left(1 - \frac{a\lambda}{2}(e^{i\theta} - e^{-i\theta})\right) \\ &= 1 - a\lambda i \sin \theta - a^2\lambda^2 \sin^2 \theta, \end{aligned}$$

hence

$$\begin{aligned} |g|^2 &= (1 - a^2\lambda^2 \sin^2 \theta)^2 + a^2\lambda^2 \sin^2 \theta \\ &= 1 - a^2\lambda^2 \sin^2 \theta (1 - a^2\lambda^2 \sin^2 \theta) \end{aligned}$$

from which we conclude that $|g| \leq 1$ (independent of θ) if and only if $|a\lambda| \leq 1$.

2 Programming

1. (Strikwerda 2.3.3.) Solve the initial value problem for equation

$$u_t + \left(1 + \frac{1}{4}(3-x)(1+x)\right)u_x = 0$$

on the interval $[-1, 3]$ with the Lax-Friedrichs scheme (2.3.1) with λ equal to 0.8. Demonstrate that the instability phenomena occur where $|a(t, x)\lambda|$ is greater than 1 and where there are discontinuities in the solution. Use the same initial data as in Exercise 2.3.1. Specify the solution to be 0 at both boundaries. Compute up to the time of 0.2 and use successively smaller values of h to show the location of the instability.

Solution

2. Investigate (via numerical evidence) the convergence (or lack thereof) of the forward-time central-space scheme

$$\frac{1}{k}(u_m^{n+1} - u_m^n) + \frac{a}{2h}(u_{m+1}^n - u_{m-1}^n) = 0$$

in the L^∞ -norm. Use the same scenarios from Homework 1, e.g., compare your results using smooth, continuous-but-non-smooth, and discontinuous initial conditions. Be sure to restrict the relation between k and h appropriately. Compare convergence in the L^∞ -norm with convergence in the L^2 -norm.

Solution

We need only run the previous convergence tests for the forward-time central-space scheme with use of the L^∞ -norm rather than the L^2 -norm:

```
test_convergence( ...
    1, 1, @(x) sin(2*pi*x), ...
    @ftcs, "forward-time central-space", ...
    2.^(-(5:0.5:8)), @(h) h*h, "inf");
test_convergence( ...
    1, 1, @(x) 1 - 2*abs(x - 1/2), ...
    @ftcs, "forward-time central-space", ...
    2.^(-(5:0.5:8)), @(h) h*h, "inf");
test_convergence( ...
    1, 1, @(x) (1 + sign(1/4 - abs(x - 1/2)))/2, ...
    @ftcs, "forward-time central-space", ...
    2.^(-(5:0.5:8)), @(h) h*h, "inf");
```

The numerical convergence rates are found to be 2.51, 1.19, and 0.36, respectively. (Note that the last convergence test with discontinuous u_0 does not give a strong linear relationship in the log-log plot of h vs. L^2 -error.) Interestingly, these numerical convergence rates are all non-negligibly better than those for the L^2 -error.