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Analysis Bootcamp, Homework 4

1. *Rudin, page 138. Problems 1 - 6, 8.*

   1. *Suppose α increases on $[a,b]$, $a \le x_0 \le b$, α continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \ne x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and that $\int f d\alpha = 0$.*

      Solution

      Clearly, if $a = b$, $\int f d\alpha = 0$, so suppose $a < b$. Further suppose that $x_0 < b$ (a similar argument works for $x_0 = b$).

      Given $\epsilon > 0$, let $\delta > 0$ be such that $|\alpha(x) - \alpha(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$ (possible since α is continuous at x_0) and such that $x_0 + \delta/2 \le b$. Let P be the partition $\{a, x_0, x_0 + \delta/2, b\}$. Then, evidently, $0 \le U(P, f, \alpha) = \alpha(x_0 + \delta/2) - \alpha(x_0) < \epsilon$, and since ϵ was arbitrary, we conclude that

      $$\overline{\int} f d\alpha = \inf U(P, f, \alpha) = 0.$$

      Further, by the nonnegativity of f,

      $$0 \le \underline{\int} f d\alpha \le \overline{\int} f d\alpha$$

      and we conclude the upper and lower integrals are equal to each other and to 0, hence $\int f d\alpha = 0$.

   2. *Suppose $f \ge 0$, f is continuous on $[a,b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$. (Compare this with Exercise 1.)*

      Solution

      Suppose $f(x_0) > 0$ for some $x_0 \in [a,b]$. Then given some $\epsilon > 0$, there exists some neighborhood $[c,d] \subset [a,b]$, $c < d$, containing x_0 and such that $f(x) > \epsilon$ for $x \in [c,d]$, by continuity of f. By the nonnegativity of f, then,

      $$\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx \ge \epsilon(d-c) > 0,$$

      a contradiction. It follows that $f \equiv 0$ on $[a,b]$.

   3. *Define three functions $\beta_1, \beta_2, \beta_3$ as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.*

      (a) Prove that $f \in \mathcal{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then

      $$\int f\beta_1 = f(0).$$

      (b) State and prove a similar result for β_2.

      (c) Prove that $f \in \mathcal{R}(\beta_3)$ if and only if f is continuous at 0.

      (d) If f is continuous at 0 prove that

      $$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0).$$

      Solution

(a) Suppose that $f \in \mathcal{R}(\beta_1)$, and let $\epsilon > 0$. Then, by Theorem 6.6, there exists a partition P such that

$$U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon.$$

Let $P^* = P \cup \{0\}$ and δ the minimum positive element of P. Then

$$U(P, f, \beta_1) = (\beta_1(\delta) - \beta_1(0)) \sup_{x \in [0,\delta]} f(x) = \sup_{x \in [0,\delta]} f(x),$$

$$L(P, f, \beta_1) = (\beta_1(\delta) - \beta_1(0)) \inf_{x \in [0,\delta]} f(x) = \inf_{x \in [0,\delta]} f(x).$$

In particular, then, for any $x \in [0, \delta]$,

$$|f(x) - f(0)| < \epsilon,$$

from which it follows that $f(0+) = f(0)$.

Now suppose that $f(0+) = f(0)$, and let $2\epsilon > 0$. Then there exists a $\delta > 0$ such that

$$|f(x) - f(0)| < \epsilon$$

whenever $x \in [0, \delta]$. We then take $P = \{a, 0, \delta, b\}$ ($a \leq 0$ and $b \geq \delta$ defining the relevant interval) and see that, as before,

$$U(P, f, \beta_1) = \sup_{x \in [0,\delta]} f(x),$$

$$L(P, f, \beta_1) = \inf_{x \in [0,\delta]} f(x),$$

and since $f(x)$ is within ϵ of $f(0)$ for all $x \in [0, \delta]$,

$$U(P, f, \beta_1) - L(P, f, \beta_1) < 2\epsilon,$$

implying that $f \in \mathcal{R}(\beta_1)$, by Theorem 6.6.

Further, $U(P, f, \beta_1)$ and $L(P, f, \beta_1)$ can be made within ϵ of $f(0)$, for ϵ arbitrary, hence $\int f \beta_1 = f(0)$.

(b) The proof of the following claim is almost precisely the same as above: $f \in \mathcal{R}(\beta_2)$ if and only if $f(0-) = f(0)$, and in that case $\int f d\beta_2 = f(0)$.

(c)

$$\int_0^b f d\beta_3 = \frac{1}{2} \int_0^b f d\beta_1$$

exists if and only if $f(0+) = f(0)$, and

$$\int_a^0 f d\beta_3 = \frac{1}{2} \int_a^0 f d\beta_2$$

exists if and only if $f(0-) = f(0)$, hence

$$\int_a^b f d\beta_3$$

exists if and only if f is continuous at 0.

(d) We know that

$$\int_a^0 f d\beta_1 = \int_0^b f d\beta_2 = 0,$$

hence, from the above equalities,

$$\int_a^b f d\beta_1 = \int_a^b f d\beta_2 = \int_a^b f d\beta_3 = f(0).$$

4. *If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.*

   Solution

   For any partition $P = \{x_i\}$,

   $$U(P, f) = \sum M_i \Delta x_i = \sum (b - a)\Delta x_i = b - a,$$

   while

   $$L(P, f) = \sum m_i \Delta x_i = \sum (0)\Delta x_i = 0,$$

   hence $f \notin \mathcal{R}$.

5. *Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume that $f^3 \in \mathcal{R}$?*

   Solution

   No; take f such that $f(x) = 1$ for x rational and $f(x) = -1$ for x irrational. $f^2 \equiv 1$, hence $f^2 \in \mathcal{R}$, but $f \notin \mathcal{R}$.

   Yes; apply Theorem 6.11 to $\phi(y) = y^{1/3}$, which is continuous everywhere.

6. *Let P be the Cantor set constructed in Sec. 2.44. Let f be a bounded real function on $[0, 1]$ which is continuous at every point outside P. Prove that $f \in \mathcal{R}$ on $[0, 1]$.* Hint: *P can be covered by finitely many segmenets whose total length can be made as small as desired. Proceed as in Theorem 6.10.*

   Solution

8. *Suppose $f \in \mathcal{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define*

   $$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

   if this limit exists (and is finite). In that case, we say that the integral on the left converges. *If it also converges after f has been replaced by $|f|$, it is said to converge* absolutely.

   Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that

   $$\int_1^\infty f(x)dx$$

   converges if and only if

   $$\sum_{n=1}^\infty f(n)$$

   converges. (This is the so-called "integral test" for convergence of series.)

   Solution

   The inequalities

   $$\int_1^N f(x)dx \leq \sum_{n=1}^{N-1} f(n)$$

   and

   $$\sum_{n=1}^N f(n) \leq f(1) + \int_1^N f(x)dx,$$

   for integral $N \geq 1$, establish the convergence result (these follow from the fact that f is monotonically decreasing).