

Math 269B, 2012 Winter, Homework 2 - Solutions

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1 Theory

1. (Strikwerda 2.1.9.) Finite Fourier Transforms. For a function v_m defined on the integers, $m = 0, 1, \dots, M-1$, we can define the Fourier transform as

$$\hat{v}_\ell = \sum_{m=0}^{M-1} e^{-2i\pi\ell m/M} v_m \quad \text{for } \ell = 0, \dots, M-1.$$

For this transform prove the Fourier inversion formula

$$v_m = \frac{1}{M} \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \hat{v}_\ell,$$

and the Parseval's relation

$$\sum_{m=0}^{M-1} |v_m|^2 = \frac{1}{M} \sum_{\ell=0}^{M-1} |\hat{v}_\ell|^2.$$

Note that v_m and \hat{v}_ℓ can be defined for all integers by making them periodic with period M .

Solution

Substituting in for \hat{v}_ℓ and swapping the order of summation yields

$$\begin{aligned} & \frac{1}{M} \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \hat{v}_\ell \\ &= \frac{1}{M} \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \sum_{m'=0}^{M-1} e^{-2i\pi\ell m'/M} v_{m'} \\ &= \frac{1}{M} \sum_{m'=0}^{M-1} v_{m'} \sum_{\ell=0}^{M-1} e^{2i\pi\ell(m-m')/M}. \end{aligned}$$

Now, if $m \neq m'$, then

$$\sum_{\ell=0}^{M-1} e^{2i\pi\ell(m-m')/M} = \frac{1 - e^{2i\pi M(m-m')/M}}{1 - e^{2i\pi(m-m')/M}} = 0;$$

on the other hand, if $m = m'$, then the summand is identically 1 and hence the sum is M . The result follows immediately:

$$\frac{1}{M} \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \hat{v}_\ell = v_m.$$

Regarding the Parseval's relation,

$$\begin{aligned}
\sum_{m=0}^{M-1} |v_m|^2 &= \frac{1}{M^2} \sum_{m=0}^{M-1} \left| \sum_{\ell=0}^{M-1} e^{2\pi i \ell m / M} \hat{v}_\ell \right|^2 \\
&= \frac{1}{M^2} \sum_{m=0}^{M-1} \left(\sum_{\ell=0}^{M-1} e^{2\pi i \ell m / M} \hat{v}_\ell \right) \overline{\left(\sum_{\ell=0}^{M-1} e^{2\pi i \ell m / M} \hat{v}_\ell \right)} \\
&= \frac{1}{M^2} \sum_{\ell=0}^{M-1} \sum_{\ell'=0}^{M-1} \sum_{m=0}^{M-1} e^{2\pi i (\ell - \ell') m / M} \hat{v}_\ell \overline{\hat{v}_{\ell'}} \\
&= \frac{1}{M} \sum_{\ell=0}^{M-1} |\hat{v}_\ell|^2,
\end{aligned}$$

as desired.

2. Prove convergence for the Beam-Warming scheme

$$u_m^{n+1} = u_m^n - \frac{ak}{2h} (3u_m^n - 4u_{m-1}^n + u_{m-2}^n) + \frac{a^2 k^2}{2h^2} (u_m^n - 2u_{m-1}^n + u_{m-2}^n)$$

used to approximate solutions to $u_t + au_x = 0$ for $a > 0$.

Solution

We rewrite the difference operator as

$$P_{k,h} u_m^n = \frac{1}{k} (u_m^{n+1} - u_m^n) + \frac{1}{2h} (3u_m^n - 4u_{m-1}^n + u_{m-2}^n) - \frac{a^2 k}{2h^2} (u_m^n - 2u_{m-1}^n + u_{m-2}^n).$$

Its symbol is thus

$$\begin{aligned}
p_{k,h}(s, \xi) &= P_{k,h}(e^{skn+imh\xi}) / e^{skn+imh\xi} \\
&= \frac{1}{k} (e^{sk} - 1) + \frac{a}{2h} (3 - 4e^{-ih\xi} + e^{-2ih\xi}) - \frac{a^2 k}{2h^2} (1 - 2e^{-ih\xi} + e^{-2ih\xi}) \\
&= \left(1 + \frac{k}{2} (s - ia\xi) \right) (s + ia\xi) + O(k^2 + kh + h^2);
\end{aligned}$$

since the symbol of the differential operator $P = \partial_t + a\partial_x$ is $p = s + ia\xi$, this demonstrates that $P_{k,h}$ is consistent with P (to second order if $k/h = O(1)$).

Regarding stability, we substitute $g = e^{sk}$ and solve $p_{k,h} = 0$ for g , giving

$$\begin{aligned}
g &= 1 + \frac{1}{2} a\lambda (3 - 4e^{-i\theta} + e^{-2i\theta}) - \frac{1}{2} a^2 \lambda^2 (1 - 2e^{-i\theta} + e^{-2i\theta}) \\
&= \dots
\end{aligned}$$

3. (Strikwerda 2.2.4.) Show that the box scheme

$$\frac{1}{2k} ((v_m^{n+1} + v_{m+1}^{n+1}) - (v_m^n + v_{m+1}^n)) + \frac{a}{2h} ((v_{m+1}^{n+1} - v_m^{n+1}) + (v_{m+1}^n - v_m^n)) = f_m^n$$

is consistent with the one-way wave equation $u_t + au_x = f$ and is stable for all values of λ .

Solution

The symbols corresponding to the difference operators $P_{k,h}$ and $R_{k,h}$ are

$$\begin{aligned}
p_{k,h}(s, \xi) &= P_{k,h}(e^{skn+imh\xi}) / e^{skn+imh\xi} \\
&= \frac{1}{2k} (e^{sk} - 1) (e^{ih\xi} + 1) + \frac{a}{2h} (e^{sk} + 1) (e^{ih\xi} - 1) \\
&= \frac{1}{2k} \left(sk + \frac{1}{2} s^2 k^2 + O(k^3) \right) (2 + ih\xi + O(h^2)) \\
&\quad + \frac{a}{2h} (2 + sk + O(k^2)) \left(ih\xi - \frac{1}{2} h^2 \xi^2 + O(h^3) \right) \\
&= \left(1 + \frac{1}{2} (sk + ih\xi) \right) (s + ia\xi) + O(h^2 + hk + k^2); \\
r_{k,h}(s, \xi) &\equiv 1.
\end{aligned}$$

4. (Strikwerda 2.2.6.) Determine the stability of the following scheme, sometimes called the Euler backward scheme, for $u_t + au_x = f$:

$$\begin{aligned}
v_m^{n+1/2} &= v_m^n - \frac{a\lambda}{2} (v_{m+1}^n - v_{m-1}^n) + k f_m^n, \\
v_m^{n+1} &= v_m^n - \frac{a\lambda}{2} (v_{m+1}^{n+1/2} - v_{m-1}^{n+1/2}) + k f_m^{n+1}.
\end{aligned}$$

The variable $v^{n+1/2}$ is a temporary variable, as is \tilde{v} in Example 2.2.5.

2 Programming

1. (Strikwerda 2.3.3.) Solve the initial value problem for equation

$$u_t + \left(1 + \frac{1}{4} (3 - x) (1 + x) \right) u_x = 0$$

on the interval $[-1, 3]$ with the Lax-Friedrichs scheme (2.3.1) with λ equal to 0.8. Demonstrate that the instability phenomena occur where $|a(t, x)\lambda|$ is greater than 1 and where there are discontinuities in the solution. Use the same initial data as in Exercise 2.3.1. Specify the solution to be 0 at both boundaries. Compute up to the time of 0.2 and use successively smaller values of h to show the location of the instability.

2. Investigate (via numerical evidence) the convergence (or lack thereof) of the forward-time central-space scheme

$$\frac{1}{k} (u_m^{n+1} - u_m^n) + \frac{a}{2h} (u_{m+1}^n - u_{m-1}^n) = 0$$

in the L^∞ -norm. Use the same scenarios from Homework 1, e.g., compare your results using smooth, continuous-but-non-smooth, and discontinuous initial conditions. Be sure to restrict the relation between k and h appropriately. Compare convergence in the L^∞ -norm with convergence in the L^2 -norm.