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Applied Differential Equations Qualifying Exam, Fall 2006

1. Consider the second-order ODE

$$x''(t) + x^3(t) - 4x(t) = 0. \qquad (1)$$

- Find the conserved quantity for (1).
- Rewrite (1) as a first-order system.
- Find and classify the equilibrium points.
- Sketch the phase portrait of the systems.

Solution

- Multiplying by x' and integrating gives

$$C = (x')^2 + \frac{1}{4}x^4 - 2x^2.$$

- $$(x, x')' = (x', 4x - x^3) = F(x, x').$$

- Equilibrium points $(x, x')^*$ satisfy

$$F((x, x')^*) = 0 \;\Rightarrow\; (x, x')^* \in \{(0,0), (\pm 2, 0)\}.$$

  To classify the equilibrium points, we compute

$$DF(x, x') = \begin{pmatrix} 0 & 1 \\ 4 - 3x^2 & 0 \end{pmatrix}.$$

  - $(x, x')^* = (0, 0)$:

$$DF(0,0) = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$$

    has eigenvalues $\lambda_\pm = \pm 2$ and corresponding eigenvectors

$$v_\pm = \begin{pmatrix} \pm 1 \\ 2 \end{pmatrix}.$$

    This equilibrium point is a saddle.
  - $(x, x')^* = (2, 0)$:

$$DF(2,0) = \begin{pmatrix} 0 & 1 \\ -8 & 0 \end{pmatrix}$$

    has eigenvalues $\lambda_\pm = \pm 2\sqrt{2}i$. This equilibrium point is a center.
  - $(x, x')^* = (-2, 0)$: [Same as previous case.]

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2. Consider the equation

$$u_{tt} = c^2 u_{xx} \qquad (2)$$

for $-at < x < at$ and $0 \le t$, in which a and c are positive constants. For which boundary conditions on $x = \pm at$ is there existence and uniqueness for this problem? Hint: The answer depends on a.

Solution

3. Consider the PDE

$$\begin{aligned} u_t &= \Delta u; \quad (3) \\ u(x,y,t=0) &= u_0(x,y); \quad (4) \end{aligned}$$

in a half-plane $-\infty < x < \infty$ and $0 \le y < \infty$, with $u_0(x,y) \ge 0$. Compare the following two boundary conditions:

$$u(x,0,t) = 0 \quad (5)$$

and

$$u_y(x,0,t) = 0. \quad (6)$$

Denote the solution of (3),(4), and (5) as u^D; and the solution of (3), (4), and (6) as u^N. Show that $u^D \le u^N$ for all x, y and $t > 0$.

Solution

To obtain u^D, we can extend $u_0(x,y)$ for $y<0$ "oddly" by setting $u_0(x,y) = -u_0(x,-y)$, yielding

$$\begin{aligned} u^D(x,y) &= \frac{1}{4\pi t}\iint_{\mathbb{R}^2} e^{-\left((x-\xi)^2+(y-\eta)^2\right)/4t} u_0(\xi,\eta)d\xi d\eta \\ &= \frac{1}{4\pi t}\iint_{\eta\ge 0} \left(e^{-\left((x-\xi)^2+(y-\eta)^2\right)/4t} u_0(\xi,\eta) - e^{-\left((x-\xi)^2+(y+\eta)^2\right)/4t} u_0(\xi,\eta)\right) d\xi d\eta \\ &= \frac{1}{4\pi t}\iint_{\eta\ge 0} e^{-\left((x-\xi)^2+(y-\eta)^2\right)/4t}\left(1 - e^{-y\eta/t}\right) u_0(\xi,\eta)d\xi d\eta. \end{aligned}$$

Similarly, to obtain u^N, we can extend $u_0(x,y)$ for $y<0$ "evenly" by setting $u_0(x,y) = u_0(x,-y)$, yielding

$$\begin{aligned} u^N(x,y) &= \frac{1}{4\pi t}\iint_{\mathbb{R}^2} e^{-\left((x-\xi)^2+(y-\eta)^2\right)/4t} u_0(\xi,\eta)d\xi d\eta \\ &= \frac{1}{4\pi t}\iint_{\eta\ge 0} e^{-\left((x-\xi)^2+(y-\eta)^2\right)/4t}\left(1 + e^{-y\eta/t}\right) u_0(\xi,\eta)d\xi d\eta. \end{aligned}$$

Since $u_0 \ge 0$, it is easy to see that $u^N \ge u^D$.

4. Consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \;\; y > 0, \;\; x \in \mathbb{R}, \quad (7)$$

together with the boundary condition

$$\frac{\partial u}{\partial y}(x,0) - u(x,0) = f(x), \quad (8)$$

where $f(x) \in C_0^\infty(\mathbb{R})$ (i.e., f is smooth with compact support). Find a representation for a bounded solution $u(x,y)$ of (7), (8); and show that $u(x,y) \to 0$ as $y \to \infty$ uniformly in $x \in \mathbb{R}$.

Solution

We apply a Fourier transform in x. Recall that the Fourier transform, at least formally, is given by

$$\mathcal{F}_x(f(x))(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx,$$

and it is easy to verify that

$$\mathcal{F}_x(f'(x))(\xi) = i\xi\mathcal{F}_x(f(x))(\xi).$$

Denoting by $\widehat{u}(\xi, y) = \mathcal{F}_x(u(x,y))(\xi)$, we see that $\widehat{u}$ satisfies

$$-\xi^2 \widehat{u} + \widehat{u}_{yy} = 0$$

subject to the boundary condition

$$\widehat{u}_y(\xi, 0) - \widehat{u}(\xi, 0) = \mathcal{F}_x(f(x))(\xi) = \widehat{f}(\xi).$$

We can find the general solution for $\widehat{u}$:

$$\widehat{u}(\xi, y) = C_1(\xi) e^{-|\xi| y} + C_2(\xi) e^{|\xi| y}.$$

Boundedness requires $C_2 = 0$, while the boundary conditions require

$$C_1(\xi) = -\frac{\widehat{f}(\xi)}{1 + |\xi|},$$

and therefore

$$\widehat{u}(\xi, y) = -\frac{\widehat{f}(\xi)}{1 + |\xi|} e^{-|\xi| y}.$$

Since $f \in C_0^\infty \subset \mathcal{S}$, $\widehat{f} \in \mathcal{S}$, from which it follows easily that $\xi \mapsto \widehat{u}(\xi, y) \in \mathcal{S}$, so $x \mapsto u(x, y) \in \mathcal{S}$. The exponential decrease in y also implies that $y \mapsto u(x, y) \in \mathcal{S}$ as well, hence u is bounded. Indeed,

$$\begin{aligned} |u(x,y)| &= \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} \widehat{u}(\xi, y) d\xi \right| \\ &= \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} e^{ix\xi} \frac{\widehat{f}(\xi)}{1 + |\xi|} e^{-|\xi| y} d\xi \right| \\ &\leq \frac{1}{2\pi} \left\| \widehat{f} \right\|_{L^2} \left\| \frac{e^{-|\xi| y}}{1 + |\xi|} \right\|_{L^2_\xi} \end{aligned}$$

by the Cauchy-Schwarz Inequality. But

$$\left\| \widehat{f} \right\|_{L^2} = \frac{1}{2\pi} \|f\|_{L^2} < \infty,$$

by Plancherel's Theorem, while

$$\begin{aligned} \left\| \frac{e^{-|\xi| y}}{1 + |\xi|} \right\|_{L^2_\xi}^2 &= 2 \int_0^\infty \left(\frac{e^{-\xi y}}{1 + \xi} \right)^2 d\xi \\ &\leq 2 \int_0^\infty e^{-2\xi y} d\xi \\ &= \frac{1}{y}, \end{aligned}$$

from which it follows that

$$|u(x, y)| \leq \frac{1}{4\pi^2} \|f\|_{L^2} y^{-1} \to 0$$

as $y \to \infty$, uniformly in x.

5. Let $a \in \mathbb{R}$ be a positive constant and $f(t)$ a non-negative continuous function. Assume that $y(t)$ is a continuous function such that

$$0 \leq y(t) \leq a + \int_0^t f(s) y(s)^2 ds \quad \text{for } t \geq 0.$$

Show that

$$y(t) \leq \frac{a}{1 - a\int_0^t f(s)ds} \qquad (10)$$

for all $t \geq 0$ for which the denominator in the right hand side of (10) is positive.

Solution

Denote by

$$z(t) = a + \int_0^t f(s)y(s)^2 ds,$$

and note that

$$z'(t) = f(t)y(t)^2 \leq f(t)z(t)^2.$$

It follows that

$$\int_0^t \frac{z'(s)}{z(s)^2} ds \leq \int_0^t f(s)ds \;\Rightarrow\; \frac{1}{z(0)} - \frac{1}{z(t)} \leq \int_0^t f(s)ds \;\Rightarrow\; z(t) \leq \frac{a}{1 - a\int_0^t f(s)ds},$$

from which the claim follows.

6. Let $\phi \in C^1(\mathbb{C})$ be a function with compact support. When $\zeta \in \mathbb{C}$, let us write $\zeta = \xi + i\eta$, with $\xi, \eta \in \mathbb{R}$, and introduce the Cauchy-Riemann operator,

$$\frac{\partial}{\partial \bar{\zeta}} = \frac{1}{2}\left(\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial \eta}\right).$$

Let $z \in \mathbb{C}$. Show that

$$\phi(z) = -\frac{1}{\pi}\iint \frac{\partial \phi}{\partial \bar{\zeta}}(\zeta)(\zeta - z)^{-1} d\xi d\eta.$$

Solution

Notice that

$$\Delta = 4\frac{\partial}{\partial \zeta}\frac{\partial}{\partial \bar{\zeta}},$$

where

$$\frac{\partial}{\partial \zeta} = \frac{1}{2}\left(\frac{\partial}{\partial \xi} - i\frac{\partial}{\partial \eta}\right).$$

Further, if $K(\xi, \eta) = \frac{1}{4\pi}\log(\xi^2 + \eta^2)$ is the fundamental solution to the Laplacian, that is,

$$\Delta K = \delta,$$

then we can easily compute that

$$\frac{\partial}{\partial \zeta} K(\zeta) = \frac{1}{4\pi}\zeta^{-1},$$

so that, using integration by parts,

$$\begin{aligned}
\phi(z) &= \iint \phi(\zeta)\Delta K(z - \zeta) d\xi d\eta \\
&= -4\iint \frac{\partial \phi}{\partial \bar{\zeta}}(\zeta)\frac{\partial}{\partial \zeta}K(z - \zeta) d\xi d\eta \\
&= -\frac{1}{\pi}\iint \frac{\partial \phi}{\partial \bar{\zeta}}(\zeta)(\zeta - z)^{-1} d\xi d\eta.
\end{aligned}$$

7. Let u solve the heat equation in a two-dimensional channel, i.e.,

$$\begin{aligned} u_t &= \Delta u; \\ u(x,y,t=0) &= u_0(x,y); \\ u_y(x,0,t) = u_y(x,\pi,t) &= 0; \end{aligned}$$

for $-\infty < x < \infty$ and $0 \leq y \leq \pi$. The initial data u_0 is assumed to be smooth and vanish for $|x|$ large.

(a) Show that $u(x,y,t)$ can be expanded in a cosine series in y, i.e.,

$$u(x,y,t) = \sum_{k\geq 0} \widehat{u}(x,k,t)\cos(ky)$$

and find an equation for the k^{th} coefficient $\widehat{u}(x,k,t)$.

(b) Find the limit of $t^{1/2}u(x,y,t)$ as $t \to \infty$.

Solution

(a) We separate variables by assuming $u(x,y,t) = Y(y)Z(x,t)$ and substituting into the PDE, finding that

$$0 = u_t - \Delta u = YZ_t - YZ_{xx} - Y''Z \;\Rightarrow\; \frac{Z_t - Z_{xx}}{Z} = \frac{Y''}{Y} = \lambda$$

for some constant λ. Solving for Y yields the general solution $Y = C_1 e^{\sqrt{\lambda}y} + C_2 e^{\sqrt{\lambda}y}$. The boundary conditions $u_y(x,0,t) = u_y(x,\pi,t) = 0$ imply that $Y'(0) = Y'(\pi) = 0$, hence we conclude that $Y = \cos\left(\sqrt{-\lambda}y\right)$, with $\lambda \leq 0$ and $\sqrt{-\lambda} \in \mathbb{Z}$. Letting $\lambda = -k^2$ for $k \in \mathbb{Z}$, this gives simply $Y_k = \cos(ky)$. By linearity, then, u must be of the form

$$u(x,t) = \sum_{k\geq 0} Z_k(x,t)\cos(ky),$$

where Z_k satisfies

$$(Z_k)_t - (Z_k)_{xx} = -k^2 Z_k, \;\; x \in \mathbb{R}, \;\; t \geq 0.$$

This has the solution

$$Z_k(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-k^2 t} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4t} Z_k(\xi,0)d\xi.$$

We notice that $Z_k(x,0)$ is just the coefficient on $\cos(ky)$ when expanding $u_0(x,y)$ for fixed x:

$$Z_k(x,0) = \frac{2}{\pi}\int_0^{\pi} u_0(x,y)\cos(ky)dy.$$

(b) In the limit as $t \to \infty$, Z_0 dominates Z_k for $k > 0$, due to the presence of the exponentially decaying factor $e^{-k^2 t}$. Thus,

$$\lim_{t\to\infty} \sqrt{t}u(x,y,t) = \lim_{t\to\infty} \sqrt{t}Z_0(x,t).$$

Noting that

$$Z_0(x,0) = \frac{2}{\pi}\int_0^{\pi} u_0(x,y)dy,$$

we thus obtain

$$\begin{aligned} \sqrt{t}Z_0(x,t) &= \frac{1}{\pi^{3/2}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4t}\left(\int_0^{\pi} u_0(\xi,y)dy\right)d\xi \\ &\to \frac{1}{\pi^{3/2}}\int_{-\infty}^{\infty}\int_0^{\pi} u_0(\xi,y)dyd\xi \end{aligned}$$

as $t \to \infty$.

8. Suppose that u is a smooth solution of the initial boundary value problem

$$\begin{aligned} u_t &= u_{xx} + cu^2; \\ u(x,t=0) &= u_0(x); \\ u(0,t) &= u(1,t) = 0; \qquad (18) \end{aligned}$$

for $0 < x < 1$, in which c is a positive constant.

(a) Show that

$$\frac{d}{dt}\int_0^1 |u(x,t)|^2 dx \le -\left(\int_0^1 |u_x(x,t)|^2 dx\right)\left(1 - c\left(\int_0^1 |u(x,t)|^2 dx\right)^{1/2}\right).$$

Hint: First show that

$$\sup_x |u(x,t)|^2 \le \int_0^1 |u_x(x,t)|^2 dx.$$

(b) If the initial data u_0 satisfies

$$\int_0^1 |u_0(x)|^2 dx < \frac{1}{c^2},$$

show that u satisfies

$$\int_0^1 |u(x,t)|^2 dx < \frac{1}{c^2}$$

for all time. Hint: Show that

$$\frac{d}{dt}\int_0^1 |u(x,t)|^2 dx \le 0.$$

(c) If the boundary condition (18) is changed to $\partial_x u_0 = 0$ at $x = 0$ and $x = 1$, find a counterexample, i.e., find initial data u_0 for which the solution blows up in finite time.

Solution

(a) We first show the suggested hint:

$$\begin{aligned} u(x,t) &\le |u(x,t)| \\ &= \left|\int_0^x u_x(y,t)dy\right| \\ &\le \int_0^x |u_x(y,t)|dy \\ &\le \int_0^1 |u_x(x,t)|dx \\ &\le \left(\int_0^1 |u_x(x,t)|^2 dx\right)^{1/2}, \end{aligned}$$

where the last inequality is from an application of the Cauchy-Schwarz Inequality. It follows that

$$\begin{aligned}
\frac{d}{dt}\int_0^1 |u(x,t)|^2 dx &= \int_0^1 \frac{d}{dt}|u(x,t)|^2 dx \\
&= \int_0^1 2\Re\left(\overline{u(x,t)}u_t(x,t)\right) dx \\
&= 2\Re\left(\int_0^1 \overline{u(x,t)}u_t(x,t)dx\right) \\
&= 2\Re\left(\int_0^1 \overline{u(x,t)}\left(u_{xx}(x,t)+cu(x,t)^2\right) dx\right) \\
&= 2\Re\left(\int_0^1 \overline{u(x,t)}u_{xx}(x,t)dx + c\int_0^1 |u(x,t)|^2 u(x,t)dx\right) \\
&= 2\Re\left(-\int_0^1 |u_x(x,t)|^2 dx + c\int_0^1 |u(x,t)|^2 u(x,t)dx\right) \\
&\le 2c\int_0^1 |u(x,t)|^3 dx - 2\int_0^1 |u_x(x,t)|^2 dx \\
&\le 2\left(\int_0^1 |u_x(x,t)|^2 dx\right)\left(c\int_0^1 |u(x,t)|dx - 1\right) \\
&\le 2\left(\int_0^1 |u_x(x,t)|^2 dx\right)\left(c\left(\int_0^1 |u(x,t)|^2\right)^{1/2} - 1\right).
\end{aligned}$$

(b) Let

$$T = \inf\left\{t \ge 0 \;\middle|\; \int_0^1 |u(x,t)|^2 dx \ge \frac{1}{c^2}\right\}.$$

By continuity of u, $T > 0$, while for $t \le T$, the inequality from (a) gives that

$$\frac{d}{dt}\int_0^1 |u(x,t)|^2 dx \le 0.$$

This implies that

$$\int_0^1 |u(x,t)|^2 dx \le \int_0^1 |u(x,0)|^2 dx < \frac{1}{c^2}$$

for all $t \le T$, particularly for $t = T$ if $T < \infty$. It follows that we must have $T = \infty$, hence

$$\int_0^1 |u(x,t)|^2 dx < \frac{1}{c^2}$$

for all $t \ge 0$.

(c) If we take $u_0(x) = \alpha$, then it is easy to verify that the solution to the new boundary value problem is

$$u(x,t) = \frac{\alpha}{1 - c\alpha t},$$

which blows up at $t = (c\alpha)^{-1}$.