Math 269B, 2012 Winter, Homework 1 (Solutions)

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January 30, 2012

1 Theory

1. (Strikwerda 1.1.3.) Solve the initial value problem for

$$u_t + \frac{1}{1 + \frac{1}{2}\cos x} u_x = 0$$

Show that the solution is given by $u(t, x) = u_0(\xi)$, where ξ is the unique solution of

$$\xi + \frac{1}{2}\sin\xi = x + \frac{1}{2}\sin x - t.$$

Solution

We use the method of characteristics, in which we use the change of variables $\tau = \tau(t, x)$, $\xi = \xi(t, x)$, and $\tilde{u}(\tau, \xi) = u\left(t(\tau, \xi), x(\tau, \xi)\right)$. Then

$$\tilde{u}_\tau = u_t t_\tau + u_x x_\tau = u_t + a(t, x) u_x \equiv 0$$

if

$$t_\tau \equiv 1, \quad x_\tau = a(t, x),$$

where

$$a(t, x) := \frac{1}{1 + \frac{1}{2}\cos x}.$$

We may solve the equation $t_\tau \equiv 1$ as $t = \tau$, while the equation for x_τ is an ordinary differential equation in $x = x(\tau)$:

$$\frac{dx}{d\tau} = \frac{1}{1 + \frac{1}{2}\cos x}, \quad x(\tau = 0) = \xi.$$

Via separation of variables, we obtain an implicit relation among x, $\tau = t$, and ξ:

$$x + \frac{1}{2}\sin x - \xi - \frac{1}{2}\sin\xi = \tau = t.$$

For each fixed ξ, we have $\tilde{u}_\tau = 0$, hence $\tilde{u}(\tau, \xi) = \tilde{u}(0, \xi) = u(0, \xi) = u_0(\xi)$. It follows that $u(t, x) = u_0(\xi)$, where $\xi = \xi(t, x)$ satisfies

$$\xi + \frac{1}{2}\sin\xi = x + \frac{1}{2}\sin x - t.$$

Note that the mapping $\xi \mapsto \xi + \frac{1}{2}\sin\xi$ is an automorphism on $\mathbb{R}$ (its derivative is uniformly bounded away from 0), hence such a ξ always exists and is unique.

2. Solve the initial value problem

$$u_t + (\sin t)\, u_x = \frac{1}{1+t^2}, \quad u(0,x) = u_0(x), \quad x \in \mathbb{R}, \quad t > 0.$$

We use the method of characteristics, in which we use the change of variables $\tau = \tau(t,x)$, $\xi = \xi(t,x)$, and $\tilde{u}(\tau,\xi) = u\left(t(\tau,\xi), x(\tau,\xi)\right)$. Then

$$\tilde{u}_\tau = u_t t_\tau + u_x x_\tau = u_t + a(t,x) u_x = f(t,x)$$

if

$$t_\tau \equiv 1, \quad x_\tau = a(t,x),$$

where

$$a(t,x) := \sin t, \quad f(t,x) := \frac{1}{1+t^2}.$$

We may solve the equation $t_\tau \equiv 1$ as $t = \tau$, while the equation for x_τ is an ordinary differential equation in $x = x(\tau)$:

$$\frac{dx}{d\tau} = \sin t = \sin \tau, \quad x(\tau = 0) = \xi.$$

This easily solves to

$$x(\tau,\xi) = \xi + \int_0^\tau \sin \tau' d\tau' = 1 - \cos\tau + \xi.$$

For each fixed ξ, we have

$$\tilde{u}_\tau = \frac{1}{1+t^2} = \frac{1}{1+\tau^2}, \quad \tilde{u}(0,\xi) = u(0,\xi) = u_0(\xi),$$

which easily solves to

$$\tilde{u}(\tau,\xi) = \tilde{u}(0,\xi) + \int_0^\tau \frac{1}{1+(\tau')^2} d\tau' = \arctan\tau + u_0(\xi).$$

It follows that

$$u(t,x) = \arctan t + u_0\left(\cos t + x + 1\right).$$

3. Consider the first order system of PDEs of the form

$$\vec{u}_t + A\vec{u}_x = 0, \quad \vec{u}(0,x) = \vec{u}_0(x), \quad x \in [0,1], \quad t > 0.$$

(a) Give the solution to the initial value problem when

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(b) Describe appropriate boundary conditions at $x = 0$ and/or $x = 1$, if possible, which make the initial boundary value problem in (a) well-posed. Try to be as general as possible. How should such boundary conditions be presented to put the solution in a simple form?

(c) Give the solution to the initial value problem when

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

(d) Describe appropriate boundary conditions at $x = 0$ and/or $x = 1$, if possible, which make the initial boundary value problem in (c) well-posed. Try to be as general as possible. How should such boundary conditions be presented to put the solution in a simple form?

Solution

(a) The eigenvalues and corresponding eigenvectors of A are

$$\lambda_\pm = 2 \pm 1, \quad \vec{u}_\pm = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix},$$

leading to the general solution

$$\vec{u}(t,x) = \begin{pmatrix} \vec{u}_+ & \vec{u}_- \end{pmatrix} \begin{pmatrix} c_+ (x - \lambda_+ t) \\ c_- (x - \lambda_- t) \end{pmatrix},$$

where the scalar functions $c_\pm$ are determined by the initial conditions and boundary conditions.

(b) Since both $\lambda_\pm > 0$, boundary conditions must only be specified at the $x = 0$ boundary. A simple way to present these boundary conditions is to simply specify $c_\pm$ when $x = 0$, i.e., $c_\pm(-\lambda_\pm t) = g_\pm^0(t)$ for $t > 0$.

(c) A here has the same eigenvectors as in (a) but now with eigenvalues $\lambda_\pm = 2 \pm 3$. Otherwise, the solution takes the same general form.

(d) This time, $\lambda_+ > 0$ but $\lambda_- < 0$, so boundary conditions must be specified at both the $x = 0$ boundary and the $x = 1$ boundary. A simple way to present these boundary conditions is to specify c_+ when $x = 0$ in terms of t and c_-, and vice versa, to specify c_- when $x = 1$ in terms of t and c_+:

$$\begin{aligned} c_+(-\lambda_+ t) &= g_+^0 (t, c_-(-\lambda_- t)), \\ c_-(1 - \lambda_- t) &= g_-^1 (t, c_+(1 - \lambda_+ t)), \end{aligned}$$

for $t > 0$.

4. Derive the leading term of the local truncation error for the following finite difference schemes used to approximate solutions to the equation $u_t + a u_x = 0$.

(a)

$$\frac{1}{k}\left(v_m^{n+1} - v_m^n\right) + a\frac{1}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) = 0.$$

(b)

$$\frac{1}{k}\left(v_m^{n+1} - \frac{1}{2}\left(v_{m+1}^n + v_{m-1}^n\right)\right) + a\frac{1}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) = 0.$$

Solution

Unless otherwise noted, all functions are evaluated at a common point (t, x).

(a) We use the following Taylor expansions about (t, x):

$$\begin{aligned} \phi(t+k, x) &= \phi + k\phi_t + \frac{1}{2}k^2\phi_{tt} + O\left(k^3\right); \\ \phi(t, x \pm h) &= \phi \pm h\phi_x + \frac{1}{2}h^2\phi_{xx} \pm \frac{1}{6}h^3\phi_{xxx} + \frac{1}{24}h^4\phi_{xxxx} + O\left(h^5\right). \end{aligned}$$

Thus,

$$\begin{aligned} P_{k,h}\phi &= \frac{1}{k}\left(\phi(t+k,x) - \phi\right) + a\frac{1}{2h}\left(\phi(t,x+h) - \phi(t,x-h)\right) \\ &= \phi_t + a\phi_x + \frac{1}{2}k\phi_{tt} + \frac{1}{6}h^2\phi_{xxx} + O\left(k^2 + h^4\right), \end{aligned}$$

which agrees with $P\phi = (\partial_t + a\partial_x)\phi$ up to a truncation error of

$$P_{k,h}\phi - P\phi = \frac{1}{2}k\phi_{tt} + \frac{1}{6}h^2\phi_{xxx} + O\left(k^2 + h^4\right).$$

(b) Using the same Taylor expansions as above,

$$\begin{aligned} P_{k,h}\phi &= \frac{1}{k}\left(\phi(t+k,x) - \frac{1}{2}\left(\phi(t,x+h)+\phi(t,x-h)\right)\right) + a\frac{1}{2h}\left(\phi(t,x+h)-\phi(t,x-h)\right) \\ &= \phi_t + a\phi_x + \frac{1}{2}k\phi_{tt} - \frac{1}{2}\frac{h^2}{k}\phi_{xx} + O\left(k^2+h^2\right), \end{aligned}$$

which agrees with $P\phi$ up to a truncation error of

$$P_{k,h}\phi - P\phi = \frac{1}{2}k\phi_{tt} - \frac{1}{2}\frac{h^2}{k}\phi_{xx} + O\left(k^2+h^2\right).$$

5. Determine the stability region Λ for each of the finite difference schemes in Problem 4.

Solution

(a) The amplification factor g satisfies the equation

$$\frac{1}{k}(g-1) + a\frac{1}{2h}\left(e^{i\theta} - e^{-i\theta}\right) = 0,$$

so

$$g = 1 - i\frac{ak}{h}\sin\theta,$$

so $|g|^2 = 1 + \frac{a^2k^2}{h^2}\sin^2\theta \le 1 + O(k)$ if and only if $k = O\left(h^2\right)$.

(b) The amplification factor g satisfies the equation

$$\frac{1}{k}\left(g - \frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)\right) + a\frac{1}{2h}\left(e^{i\theta}-e^{-i\theta}\right) = 0,$$

so

$$g = \cos\theta - i\frac{ak}{h}\sin\theta,$$

so $|g| \le 1$ if and only if $|ak/h| \le 1$.

2 Programming

1. Implement the finite difference schemes in Problem 4. in the Theory section for $x \in [0,1]$, $t \in [0,T]$ for some final time T, $u(x,0) = u_0(x)$, and *periodic* boundary conditions.

2. Investigate the convergence of each scheme for $a = 1$ and $T = 1$. Set $k/h =: \lambda$ to be constant, and demonstrate which values of λ cause the scheme to converge and which to diverge. If no such λ gives convergence, find an alternate relation between k and h which does ensure convergence (if possible). Try using both a smooth initial condition (e.g., $u_0(x) = \sin(2\pi x)$); a non-smooth initial condition (e.g., $u_0(x) = 1 - 2|x - 1/2|$); and a discontinuous initial condition (e.g., $u_0(x) = 0$ if $|x - 1/2| > 1/4$ and $u_0(x) = 1$ if $|x - 1/2| < 1/4$). Use the discrete L^2 norm to measure the error between your numerical solution and the true solution:

$$\|w\|_h = \left(h\sum_m |w_m|^2\right)^{1/2}.$$

[Note: Due to periodicity, be sure to avoid double-counting the contributions at $x = 0$ and $x = 1$!] Plot the numerical solutions from each scheme at $t = T$ when $h = 1/100$. Summarize your results. Which scheme do you think is better and why?

Solution

Using the included code, the results of the following statements investigate the convergence of the Lax-Friedrichs scheme with the various initial conditions and $\lambda = 0.99$.

```
test_convergence( ...
    1, 1, @(x) sin(2*pi*x), ...
    @lax_friedrichs, "Lax-Friedrichs", ...
    2.^(-(9:0.5:12)), @(h) 0.99*h);
test_convergence( ...
    1, 1, @(x) 1 - 2*abs(x - 1/2), ...
    @lax_friedrichs, "Lax-Friedrichs", ...
    2.^(-(9:0.5:12)), @(h) 0.99*h);
test_convergence( ...
    1, 1, @(x) (1 + sign(1/4 - abs(x - 1/2)))/2, ...
    @lax_friedrichs, "Lax-Friedrichs", ...
    2.^(-(9:0.5:12)), @(h) 0.99*h);
```

The numerical convergence rates, based on fitting a linear regression through a log-log plot of h vs. L^2-error, are found to be, respectively, 1.06, 0.80, and 0.19. For the smooth initial condition $u_0(x) = \sin(2\pi x)$, the numerical convergence rate, 1.06, agrees well with the theoretical convergence rate of 1. The fact that the numerical convergence rates for the non-smooth initial conditions are significantly less than 1 is not surprising given that our theoretical convergence analysis depended on the existence of some number of derivatives.

Additionally, as predicted by theory, numerical evidence indicates that the Lax-Friedrichs scheme diverges if $\lambda > 1$ (and, still, $a = 1$), while the forward-time central-space scheme diverges for any λ.

We do get the forward-time central-space scheme to converge if, e.g., we let $k = h^2$, as demonstrated by the results of the following statements.

```
test_convergence( ...
    1, 1, @(x) sin(2*pi*x), ...
    @ftcs, "forward-time central-space", ...
    2.^(-(5:0.5:8)), @(h) h^2);
test_convergence( ...
    1, 1, @(x) 1 - 2*abs(x - 1/2), ...
    @ftcs, "forward-time central-space", ...
    2.^(-(5:0.5:8)), @(h) h^2);
test_convergence( ...
    1, 1, @(x) (1 + sign(1/4 - abs(x - 1/2)))/2, ...
    @ftcs, "forward-time central-space", ...
    2.^(-(5:0.5:8)), @(h) h^2);
```

The numerical convergence rates are found to be 2.01, 1.01, and 0.23, respectively, the former of which agrees with the theoretical convergence rate of 2. (Note that the last convergence test with discontinuous u_0 does not give a strong linear relationship in the log-log plot of h vs. L^2-error.)