Qualifying Exam, Spring 2002

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

ALL PROBLEMS HAVE EQUAL VALUE. There are 7 problems.

MA: Do any 5 problems.

Ph.D.: Do 5 problems and only 3 of them from 1, 2, 3, and 4.

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[1] Consider the task of finding solutions to $f(x) = 0$ where $f \quad \Re \to \Re$.

(a) What properties must $f(x)$ possess in order that Newton's method converge *cubically* to a particular solution x^* of $f(x^*) = 0$?

(b) Give an example of a function, $f(x)$, and an associated solution value, x^*, for which Newton's method will converge cubically to x^* if an initial guess is chosen sufficiently close to x^*.

[2] Consider the problem of finding a quadratic polynomial $p(x)$ that satisfies the conditions

$$p(0) = p_0 \qquad p(1) = p_1 \qquad p'(s) = p_2$$

For which values of $s \in [0, 1]$ will there exist a unique quadratic polynomial that satisfies these conditions? Justify your answer. (Don't forget to demonstrate uniqueness.)

[3] Suppose a value L is computed with a numerical procedure $\phi(h)$ and that $\lim_{h \to 0} \phi(h) = L$. Assume that there exists an asymptotic error expansion for $\phi(h)$ of the form

$$L = \phi(h) + c_1 h + c_3 h^3 + c_5 h^5 +$$

(a) How should the values $\phi(h)$ and $\phi(\frac{h}{2})$ be combined to yield an approximation to L that is $O(h^3)$?

(b) How should the values $\phi(h)$, $\phi(\frac{h}{2})$ and $\phi(\frac{h}{4})$ be combined to yield an approximation to L that is $O(h^5)$?

[4] Consider a predictor-corrector scheme for ordinary differential equations based on the Euler and trapezoidal methods,

$$\begin{array}{ll} y^*_{n+1} & y_n + hf_n \\ y_{n+1} & y_n + \frac{h}{2}(f_n + f^*_n) \end{array}$$

(a) Determine the total order of accuracy.

(b) Determine the region of absolute stability for the combined method.

(c) Describe how an adaptive step size control can be derived from the predictor-corrector computations.

[5] Consider the following partial differential equations

$$\frac{\partial}{\partial x}\left(a(x,y)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(b(x,y)\frac{\partial u}{\partial y}\right) + c(x,y)u = f(x,y),\ (x,y) \in \Omega$$

$$u = 1, \quad (x,y) \in \partial\Omega_1$$

$$\frac{\partial u}{\partial y} = 0, \quad (x,y) \in \partial\Omega_2$$

$$\Omega = \{(x,y),\ |x| < 1,\ |y| < 1\}$$

$$\partial\Omega_1 = \{(x,y),\ |x| = 1,\ |y| \le 1\}$$

$$\partial\Omega_2 = \{(x,y),\ |y| = 1\ |x| < 1\}$$

(a) Set up a finite element method based on a weak form of the problem above.

(b) Give conditions on a, b and c such that the method will converge. Give the convergence estimate and motivate your answer.

(c) Give the finite element method for the corresponding eigenvalue problem ($c(x,y) \to \lambda,\ f(x,y) \to 0,\ u = 1 \to u = 0$).

[6] We approximate the scalar differential equation

$$u_t + f(u)_x = 0$$

to be solved for, $t > 0$, u periodic with period 2π in x, $u(x,0) = u_0(x)$, a given periodic smooth function, and f a given smooth function of u, by the two step finite difference scheme

$$\begin{array}{ll} \hat{u}^{n+1}_j & u^n_j - \lambda(f(u^n_{j+1}) - f(u^n_j)) \\ u^{n+1}_j & \frac{1}{2}(u^n_j + \hat{u}^{n+1}_j) - \frac{\lambda}{2}(f(\hat{u}^n_j) \quad f(\hat{u}^n_{j-1})) \end{array}$$

For what values of $\lambda = \frac{\Delta t}{\Delta x}$ does this converge as $\Delta t, \Delta x \to 0$? This convergence is generally valid only for a small interval $0 \le t \le T$, for some $T > 0$, why? What is the rate of convergence?

Justify your answers.

[7] Consider the equation

$$u_t = u_{xx} + u_{yy} + u_{zz}$$

to be solved for $t > 0$ on the cube $0 < x, y, z \le 1$ with $u = 0$ for the boundary of the cube and $u(x, y, t, 0) = u_0(x, y, z)$ given and smooth.

Devise an unconditionally stable scheme that involves only inverting $n \times n$ matrices if there are n grid points per direction in our discretization.

What is the highest order of accuracy you can get doing this?

Justify your answers.