Basic Exam Spring 2006

PROBLEM 1

(A) Define precisely the notion of Riemann integrability for a function $f(x)$ on $[0,1]$.

(B) Suppose that $f_n(x)$ is a sequence of Riemann integrable functions on $[0,1]$ such that $\{f_n(x)\}$ converges uniformly to $f(x)$. Prove that $f(x)$ is Riemann integrable.

PROBLEM 2

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with $a_n \in \mathbf{R}$. Show that there exists a unique number $\rho \geq 0$ such that $F(x)$ converges if $|x| < \rho$ and $F(x)$ diverges if $|x| > \rho$.

PROBLEM 3

Prove that the series

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^{5/2}}$$

converges for all $x \in \mathbf{R}$ and that $f(x)$ is a continuous function on $\mathbf{R}$ with a continuous derivative. State clearly any facts you assume.

PROBLEM 4

The point $P = (1, 1, 1)$ lies on the surface S in $\mathbf{R}^3$ defined by

$$x^2y^3 + x^3z + 2yz^4 = 4$$

Prove that there exists a differentiable function $f(x, y)$ defined in an open neighborhood $\mathcal{N}$ of $(1, 1)$ in $\mathbf{R}^2$ such that $f(1, 1) = 1$ and $(x, y, f(x, y))$ lies in S for all $(x, y) \in \mathcal{N}$.

PROBLEM 5

(A) Define uniform continuity for a function f defined on a metric space X with distance function $\rho(x, y)$.

(B) Prove that if $0 < \alpha < 1$, then $F(x) = x^\alpha$ is uniformly continuous on $[0, \infty)$.

PROBLEM 6

Let W be the subset of the space $C[0, 1]$ of real-valued, continuous functions on $[0, 1]$ satisfying the conditions:

$$|f(x) - f(y)| < |x - y| \qquad \int_0^1 f(x)^2\,dx = 1$$

(A) Prove that W is uniformly bounded, i.e., there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [0, 1]$.
Hint: Show first that $|f(0)| \leq 2$ for all $f \in W$.

(B) Prove that W is a compact subset of $C[0, 1]$ under the sup norm $||f||_\infty = \sup_{x \in [0,1]} |f(x)|$.

PROBLEM 7

A matrix T (with entries, say, in the field $\mathbf{C}$ of complex numbers) is diagonalizable if there exists a non-singular matrix S such that STS^{-1} is diagonal. Prove that if $a, \lambda \in \mathbf{C}$ with $a \neq 0$, then the following matrix is not diagonalizable:

$$T = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & \lambda \end{pmatrix}$$

PROBLEM 8

A linear transformation T is called *orthogonal* if it is non-singular and ${}^tT = T^{-1}$. Prove that if $T : \mathbf{R}^{2n+1} \to \mathbf{R}^{2n+1}$ is orthogonal, then there exists a vector $v \in \mathbf{R}^{2n+1}$ such that $Tv = \pm v$.

PROBLEM 9

Let S be a real, $n \times n$-symmetric matrix S, i.e., ${}^tS = S$.

(A) Prove that the eigenvalues of S are real.

(B) State and prove the Spectral Theorem for S.

PROBLEM 10

Let Y is an arbitrary set of commuting matrices in $M_n(\mathbf{C})$ (i.e., $AB = BA$ for all $A, B \in Y$). Prove that there exists a non-zero vector $v \in \mathbf{C}^n$ which is a common eigenvector of all elements of Y.