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Numerical Analysis Qualifying Exam, Fall 2005

1. (5 Pts.) Let $\{x_n\}$ be a sequence such that $x_n \geq \overline{x}$ for all n and $\lim_{n\to\infty} x_n = \overline{x}$. Assume there exist constants α and $p > 0$ such that, for sufficiently large n,

$$x_{n+1} - \overline{x} \approx \alpha \left(x_n - \overline{x}\right)^p .$$

   (a) Assuming $\overline{x}$ is known, give a derivation of a formula that estimates p in terms of $\overline{x}$ and some number of consecutive iterates of the sequence $\{x_n\}$.

   (b) Assuming $\overline{x}$ is *unknown*, give a derivation of a formula that estimates p in terms of some number of consecutive iterates of the sequence $\{x_n\}$.

Solution

   (a) Denote $e_n = x_n - \overline{x}$. Then the givens stipulate that, for large enough n,

$$e_{n+1} \approx \alpha e_n^p,$$

   in which case we also have

$$e_{n+2} \approx \alpha e_{n+1}^p,$$

   so

$$\frac{e_{n+1}}{e_{n+2}} \approx \left(\frac{e_n}{e_{n+1}}\right)^p \Rightarrow p \approx \frac{\ln e_{n+1} - \ln e_{n+2}}{\ln e_n - \ln e_{n+1}}.$$

   (b) We suppose that $e_n = x_n - \overline{x} \approx x_n - x_{n+1}$, and similarly for e_{n+1} and e_{n+2}, obtaining

$$p \approx \frac{\ln(x_{n+1} - x_{n+2}) - \ln(x_{n+2} - x_{n+3})}{\ln(x_n - x_{n+1}) - \ln(x_{n+1} - x_{n+2})}.$$

2. (5 Pts.) Consider the forward and backward difference operators D^+ and D^- defined by

$$D^+ f(x) = \frac{f(x+h) - f(x)}{h} \quad D^- f(x) = \frac{f(x) - f(x-h)}{h}.$$

   (a) Assuming f is smooth, derive asymptotic error expansions for each of these operators.

   (b) What combination of $D^+ f(x)$ and $D^- f(x)$ gives a second order accurate approximation to the derivative $f'(x)$? Justify your answer.

Solution

   (a) By Taylor's Theorem,

$$\begin{aligned} f(x+h) &= f(x) + f'(x)h + \frac{1}{2} f''(\alpha_1)h^2, \\ f(x-h) &= f(x) - f'(x)h + \frac{1}{2} f''(\alpha_2)h^2, \end{aligned}$$

   where α_1 and α_2 are between $x, x+h$ and $x, x-h$, respectively. It follows that

$$\begin{aligned} D^+ f(x) &= f'(x) + \frac{1}{2} f''(\alpha_1)h \\ D^- f(x) &= f'(x) + \frac{1}{2} f''(\alpha_2)h. \end{aligned}$$

(b) Further using Taylor's Theorem,

$$\begin{aligned} f(x+h) &= f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{6}f^{(3)}(\alpha_3)h^3, \\ f(x-h) &= f(x) - f'(x)h + \frac{1}{2}f''(x)h^2 - \frac{1}{6}f^{(3)}(\alpha_4)h^3, \end{aligned}$$

so that

$$\frac{1}{2}\left(D^+ + D^-\right)f(x) = \frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{1}{12}\left(f^{(3)}(\alpha_3) + f^{(3)}(\alpha_4)\right)h^2.$$

3. (5 Pts.) Consider the following factorization of a tri-diagonal matrix A:

$$A = \begin{pmatrix} a_1 & c_1 & & & \\ b_2 & a_2 & c_2 & & \\ & * & * & * & \\ & & * & * & c_{n-1} \\ & & & b_n & a_n \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ d_2 & 1 & & & \\ & * & * & & \\ & & * & * & \\ & & & d_n & 1 \end{pmatrix} \begin{pmatrix} e_1 & c_1 & & & \\ & e_2 & c_2 & & \\ & & * & * & \\ & & & * & c_{n-1} \\ & & & & e_n \end{pmatrix}.$$

(a) Derive the recurrence relations that determine the values of the d_k's and e_k's in terms of the values of the a_k's, b_k's, and c_k's.

(b) Give a condition on the matrix A which ensures your recurrence relations won't break down.

Solution

(a) By simply multiplying out the matrices on the right,

$$\begin{aligned} e_1 &= a_1 \\ d_k &= b_k/e_{k-1}, \ 2 \le k \le n \\ e_k &= a_k - c_{k-1}d_k, \ 2 \le k \le n. \end{aligned}$$

(b) Errors using machine-precision approximations won't propagate if, e.g., A is diagonally dominant, i.e., $|a_i| \ge |b_i| + |c_i|$.

4. (10 Pts.)

(a) Find conditions on the coefficients a_1, a_2, p_1, p_2 so that the following Runge-Kutta method for $y'(t) = f(t, y(t))$ is of order $m \ge 2$:

$$y_{n+1} = y_n + h\left(a_1 f(t_n, y_n) + a_2 f(t_n + p_1 h, y_n + p_2 h f(t_n, y_n))\right).$$

(b) Show by an example that the order cannot exceed two.

(c) Analyze the linear stability of the scheme when $a_1 = 0$, $a_2 = 1$, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{2}$.

Solution

(a) Suppose $y(t_n) = y_n$. By Taylor's Theorem,

$$\begin{aligned} y(t_n + h) &= y_n + y'(t_n)h + \frac{1}{2}y''(t_n)h^2 + O(h^3) \\ &= y_n + f(t_n, y_n)h + \frac{1}{2}\left(f_t(t_n, y_n) + f_y(t_n, y_n)f(t_n, y_n)\right)h^2 + O(h^3). \end{aligned}$$

Also by Taylor's Theorem,

$$f(t_n + p_1 h, y_n + p_2 h f(t_n, y_n)) = f(t_n, y_n) + p_1 f_t(t_n, y_n)h + p_2 f_y(t_n, y_n) f(t_n, y_n) h + O(h^2),$$

hence

$$\begin{aligned} y_{n+1} &= y_n + h\left(a_1 f(t_n, y_n) + a_2 f(t_n + p_1 h, y_n + p_2 h f(t_n, y_n))\right) \\ &= y_n + h\left(a_1 f(t_n, y_n) + a_2\left(f(t_n, y_n) + p_1 f_t(t_n, y_n)h + p_2 f_y(t_n, y_n) f(t_n, y_n) h + O(h^2)\right)\right) \\ &= y_n + (a_1 + a_2) f(t_n, y_n) h + a_2\left(p_1 f_t(t_n, y_n) + p_2 f_y(t_n, y_n) f(t_n, y_n)\right) h^2 + O(h^3) \end{aligned}$$

which agrees with the expression for $y(t_n+h)$ to $O(h^3)$ if $a_1+a_2 = 1$, $p_1 = p_2$, and $a_2 p_1 = a_2 p_2 = \frac{1}{2}$.

(b) If f is simply a function of t, i.e., $f(t, y(t)) = f(t)$, then the scheme reduces to (again, applying Taylor's Theorem)

$$\begin{aligned} y_{n+1} &= y_n + h\left(a_1 f(t_n) + a_2 f(t_n + p_1 h)\right) \\ &= y_n + h\left(a_1 f(t_n) + a_2\left(f(t_n) + p_1 f'(t_n)h + \frac{1}{2}p_1^2 f''(t_n)h^2 + O(h^3)\right)\right) \\ &= y_n + (a_1 + a_2) f(t_n) h + a_2 p_1 f'(t_n) h^2 + \frac{1}{2} a_2 p_1^2 f''(t_n) h^3 + O(h^4). \end{aligned}$$

Matching coefficients on h up with the Taylor expansion of $y(t_n + h)$ about t_n up to h^3 yields

$$\begin{aligned} a_1 + a_2 &= 1; \\ a_2 p_1 &= \frac{1}{2}; \\ \frac{1}{2} a_2 p_1^2 &= \frac{1}{6}. \end{aligned}$$

The last two equalities give $p_1 = \frac{2}{3}$, from which we determine that $a_2 = \frac{3}{4}$ and $a_1 = \frac{1}{4}$. Thus, in this special case, the order can actually exceed two.

(c) We analyze stability by applying the method to the model problem $y'(t) = f(t, y(t)) = \lambda y(t)$:

$$\begin{aligned} y_{n+1} &= y_n + hf\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(t_n, y_n)\right) \\ &= y_n + h\lambda\left(y_n + \frac{1}{2}h\lambda y_n\right) \\ &= \left(1 + \lambda h + \frac{1}{2}(\lambda h)^2\right) y_n \end{aligned}$$

giving the root of the characteristic polynomial to be

$$\zeta = 1 + \lambda h + \frac{1}{2}(\lambda h)^2.$$

The stability region is the set of complex λh such that

$$\left|1 + \lambda h + \frac{1}{2}(\lambda h)^2\right| < 1.$$

In particular, this is satisfied by $\lambda h \in (-2, 0)$.

5. (10 Pts.) Let $a(x, y)$ and $b(x, y)$ be smooth positive functions. Consider the equation

$$u_t = \left(a(x, y) u_x\right)_x + \left(b(x, y) u_y\right)_y$$

be solved for $t > 0$, $(x, y) \in [0, 1] \times [0, 1]$, with smooth initial data $u(x, y, 0) = u_0(x, y)$ and periodic boundary conditions in x and y: $u(0, y, t) \equiv u(1, y, t)$, $u(x, 0, t) \equiv u(x, 1, t)$.

(a) Construct a second-order accurate unconditionally stable scheme for this equation. Justify the accuracy and stability properties of your scheme.

(b) Construct a second-order accurate unconditionally stable scheme for this equation that only requires the inversion of one dimensional operators. Justify the accuracy and stability properties of your scheme.

Solution

(a) We consider using Crank-Nicolson:

$$
\begin{aligned}
P_{k,h_x,h_y} u^n_{\ell,m} &= \frac{1}{k}\left(u^{n+1}_{\ell,m} - u^n_{\ell,m}\right) \\
&- \frac{1}{2h_x}\left(a\left(x+\frac{1}{2}h_x, y\right)\frac{u^{n+1}_{\ell+1,m} - u^{n+1}_{\ell,m}}{h_x} - a\left(x-\frac{1}{2}h_x, y\right)\frac{u^{n+1}_{\ell,m} - u^{n+1}_{\ell-1,m}}{h_x}\right) \\
&- \frac{1}{2h_x}\left(a\left(x+\frac{1}{2}h_x, y\right)\frac{u^{n}_{\ell+1,m} - u^{n}_{\ell,m}}{h_x} - a\left(x-\frac{1}{2}h_x, y\right)\frac{u^{n}_{\ell,m} - u^{n}_{\ell-1,m}}{h_x}\right) \\
&- \frac{1}{2h_y}\left(b\left(x, y+\frac{1}{2}h_y\right)\frac{u^{n+1}_{\ell,m+1} - u^{n+1}_{\ell,m}}{h_y} - b\left(x, y-\frac{1}{2}h_y\right)\frac{u^{n+1}_{\ell,m} - u^{n+1}_{\ell,m-1}}{h_y}\right) \\
&- \frac{1}{2h_y}\left(b\left(x, y+\frac{1}{2}h_y\right)\frac{u^{n}_{\ell,m+1} - u^{n}_{\ell,m}}{h_y} - b\left(x, y-\frac{1}{2}h_y\right)\frac{u^{n}_{\ell,m} - u^{n}_{\ell,m-1}}{h_y}\right) \\
&= \frac{1}{k}\left(u^{n+1}_{\ell,m} - u^n_{\ell,m}\right) \\
&- \frac{1}{2h_x^2}\left(a\left(x+\frac{1}{2}h_x, y\right)\left(u^{n+1}_{\ell+1,m} - u^{n+1}_{\ell,m} + u^{n}_{\ell+1,m} - u^{n}_{\ell,m}\right)\right. \\
&\qquad \left. -a\left(x-\frac{1}{2}h_x, y\right)\left(u^{n+1}_{\ell,m} - u^{n+1}_{\ell-1,m} + u^{n}_{\ell,m} - u^{n}_{\ell-1,m}\right)\right) \\
&- \frac{1}{2h_y^2}\left(b\left(x, y+\frac{1}{2}h_y\right)\left(u^{n+1}_{\ell,m+1} - u^{n+1}_{\ell,m} + u^{n}_{\ell,m+1} - u^{n}_{\ell,m}\right)\right. \\
&\qquad \left. -b\left(x, y-\frac{1}{2}h_y\right)\left(u^{n+1}_{\ell,m} - u^{n+1}_{\ell,m-1} + u^{n}_{\ell,m} - u^{n}_{\ell,m-1}\right)\right); \\
R_{k,h_x,h_y} f^n_{\ell,m} &= \frac{1}{2}\left(f^{n+1}_{\ell,m} + f^n_{\ell,m}\right).
\end{aligned}
$$

The symbols $p_{k,h_x,h_y}(s,\xi,\eta)$ and $r_{k,h_x,h_y}(s,\xi,\eta)$ for these difference operators are given by

$$
\begin{aligned}
p_{k,h_x,h_y}(s,\xi,\eta) &= P_{k,h_x,h_y}\left(e^{skn}e^{i(\xi\ell h_x + \eta m h_y)}\right)\Big/ e^{skn}e^{i(\xi\ell h_x + \eta m h_y)} \\
&= \frac{1}{k}\left(e^{sk} - 1\right) \\
&- \frac{1}{2h_x^2}\left(e^{sk}+1\right)\left(a\left(x+\frac{1}{2}h_x, y\right)\left(e^{i\xi h_x} - 1\right) - a\left(x-\frac{1}{2}h_x, y\right)\left(1 - e^{-i\xi h_x}\right)\right) \\
&- \frac{1}{2h_y^2}\left(e^{sk}+1\right)\left(b\left(x, y+\frac{1}{2}h_y\right)\left(e^{i\eta h_y} - 1\right) - b\left(x, y-\frac{1}{2}h_y\right)\left(1 - e^{-i\eta h_y}\right)\right); \\
r_{k,h_x,h_y}(s,\xi,\eta) &= R_{k,h_x,h_y}\left(e^{skn}e^{i(\xi\ell h_x + \eta m h_y)}\right)\Big/ e^{skn}e^{i(\xi\ell h_x + \eta m h_y)} \\
&= \frac{1}{2}\left(e^{sk}+1\right).
\end{aligned}
$$

We can simplify p_{k,h_x,h_y} to order $O(k^2) + O(h_x^2) + O(h_y^2)$ as follows. First, utilizing Taylor's Theorem,

$$\begin{aligned}
& a\left(x + \frac{1}{2}h_x, y\right)\left(e^{i\xi h_x} - 1\right) - a\left(x - \frac{1}{2}h_x, y\right)\left(1 - e^{-i\xi h_x}\right) \\
= \; & a\left(x + \frac{1}{2}h_x, y\right)\left(i\xi h_x - \frac{1}{2}\xi^2 h_x^2 - \frac{1}{6}i\xi^3 h_x^3 + O(h_x^4)\right) \\
- \; & a\left(x - \frac{1}{2}h_x, y\right)\left(i\xi h_x + \frac{1}{2}\xi^2 h_x^2 - \frac{1}{6}i\xi^3 h_x^3 + O(h_x^4)\right) \\
= \; & i\xi a_x(x,y)h_x^2 - \xi^2 a(x,y)h_x^2 + O(h_x^4),
\end{aligned}$$

and similarly

$$\begin{aligned}
& b\left(x, y + \frac{1}{2}h_y\right)\left(e^{i\eta h_y} - 1\right) - b\left(x, y - \frac{1}{2}h_y\right)\left(1 - e^{-i\eta h_y}\right) \\
= \; & i\eta b_y(x,y)h_y^2 - \eta^2 b(x,y)h_y^2 + O(h_y^4).
\end{aligned}$$

Also, we have that

$$\begin{aligned}
\frac{1}{k}\left(e^{sk} - 1\right) &= \frac{1}{k}\left(sk + \frac{1}{2}s^2k^2 + O(k^3)\right) \\
&= s + \frac{1}{2}s^2 k + O(k^2) \\
&= \frac{1}{2}\left(1 + 1 + sk + O(k^2)\right)s \\
&= \frac{1}{2}\left(e^{sk} + 1\right)s + O(k^2)
\end{aligned}$$

Thus,

$$\begin{aligned}
p_{k,h_x,h_y}(s,\xi,\eta) &= \frac{1}{k}\left(e^{sk} - 1\right) \\
&- \frac{1}{2h_x^2}\left(e^{sk} + 1\right)\left(i\xi a_x h_x^2 - \xi^2 a h_x^2 + O(h_x^4)\right) \\
&- \frac{1}{2h_y^2}\left(e^{sk} + 1\right)\left(i\eta b_y h_y^2 - \eta^2 b h_y^2 + O(h_y^4)\right) \\
&= \frac{1}{2}\left(e^{sk} + 1\right)\left(s - i\xi a_x + \xi^2 a - i\eta b_y + \eta^2 b\right) + O(k^2) + O(h_x^2) + O(h_y^2).
\end{aligned}$$

The symbol $p(s,\xi,\eta)$ of the differential operator $P = \partial_t - \partial_x(a\partial_x) - \partial_y(b\partial_y)$ is

$$\begin{aligned}
p(s,\xi,\eta) &= P\left(e^{sk}e^{i(\xi x + \eta y)}\right) \Big/ e^{sk}e^{i(\xi x + \eta y)} \\
&= s - i\xi a_x + \xi^2 a - i\eta b_x + \eta^2 b,
\end{aligned}$$

from which we see that $p_{k,h_x,h_y}(s,\xi,\eta)$ agrees with $r_{k,h_x,h_y}(s,\xi,\eta)p(s,\xi,\eta)$ to $O(k^2) + O(h_x^2) + O(h_y^2)$, as required for second-order accuracy.

Stability is analyzed by replacing $g = e^{sk}$ in $p_{k,h_x,h_y}(s,\xi,\eta) = 0$ and solving for g to determine the roots of the amplification polynomial:

$$
\begin{aligned}
&\frac{1}{k}(g-1)+\frac{1}{2}(g+1)\left(\frac{1}{h_x^2}\left(a\left(x+\frac{1}{2}h_x,y\right)\left(1-e^{i\xi h_x}\right)+a\left(x-\frac{1}{2}h_x,y\right)\left(1-e^{-i\xi h_x}\right)\right)\right.\\
&\quad+\quad\left.\frac{1}{h_y^2}\left(b\left(x,y+\frac{1}{2}h_y\right)\left(1-e^{i\eta h_y}\right)+b\left(x,y-\frac{1}{2}h_y\right)\left(1-e^{-i\eta h_y}\right)\right)\right)=0\\
&\quad\Rightarrow\quad g-1+\frac{1}{2}(g+1)c=0\\
&\quad\Rightarrow\quad g=\frac{2-c}{2+c},
\end{aligned}
$$

where we have let

$$
\begin{aligned}
c &= \mu_x\left(a\left(x+\frac{1}{2}h_x,y\right)\left(1-e^{i\xi h_x}\right)+a\left(x-\frac{1}{2}h_x,y\right)\left(1-e^{-i\xi h_x}\right)\right)\\
&+\ \mu_y\left(b\left(x,y+\frac{1}{2}h_y\right)\left(1-e^{i\eta h_y}\right)+b\left(x,y-\frac{1}{2}h_y\right)\left(1-e^{-i\eta h_y}\right)\right).
\end{aligned}
$$

Observing that $\Re(c)\geq 0$ for all ξ,η,x,y (here the nonnegativity of a and b is required), we get that $|g|\leq 1$, hence the scheme is unconditionally stable.

(b) Abbreviate the operators

$$\partial_x\left(a\partial_x\right)=A,\ \partial_y\left(b\partial_y\right)=B.$$

Then $P=\partial_t-A-B$. By Taylor's Theorem, if $u_t=Au+Bu$,

$$\frac{u^{n+1}-u^n}{k}=\frac{1}{2}\left(Au^{n+1}+Au^n\right)+\frac{1}{2}\left(Bu^{n+1}+Bu^n\right)+O(k^2).$$

Rearranging gives

$$\left(I-\frac{k}{2}A-\frac{k}{2}B\right)u^{n+1}=\left(I+\frac{k}{2}A+\frac{k}{2}B\right)u^n+O(k^3),$$

so

$$\left(I-\frac{k}{2}A\right)\left(I-\frac{k}{2}B\right)u^{n+1}=\left(I+\frac{k}{2}A\right)\left(I+\frac{k}{2}B\right)u^n+\frac{k^2}{4}AB(u^{n+1}-u^n)+O(k^3),$$

and since $u^{n+1}-u^n\in O(k)$, we obtain

$$\left(I-\frac{k}{2}A\right)\left(I-\frac{k}{2}B\right)u^{n+1}=\left(I+\frac{k}{2}A\right)\left(I+\frac{k}{2}B\right)u^n+O(k^3).$$

It follows that if A_{h_x} and B_{h_y} approximate A and B to $O(h_x^2)$ and $O(h_y^2)$, respectively, then

$$\left(I-\frac{k}{2}A_{h_x}\right)\left(I-\frac{k}{2}B_{h_y}\right)u^{n+1}=\left(I+\frac{k}{2}A_{h_x}\right)\left(I+\frac{k}{2}B_{h_y}\right)u^n+O(k^3)+O(kh_x^2)+O(kh_y^2).$$

This suggests the second-order ADI scheme

$$\left(I-\frac{k}{2}A_{h_x}\right)\left(I-\frac{k}{2}B_{h_y}\right)u^{n+1}=\left(I+\frac{k}{2}A_{h_x}\right)\left(I+\frac{k}{2}B_{h_y}\right)u^n.$$

We would use the A_{h_x} and B_{h_y} suggested by the scheme in (a), as we have already shown these to be second-order accurate. The operators may be split according to

$$
\begin{aligned}
\left(I-\frac{k}{2}A_{h_x}\right)\widetilde{u}^{n+1/2} &= \left(I+\frac{k}{2}B_{h_y}\right)u^n,\\
\left(I-\frac{k}{2}B_{h_y}\right)u^{n+1} &= \left(I+\frac{k}{2}A_{h_x}\right)\widetilde{u}^{n+1/2}.
\end{aligned}
$$

We show stability in a way similar as before. To simplify the notation, set

$$\begin{aligned} c_x &= \mu_x \left(a\left(x + \frac{1}{2}h_x, y\right)\left(1 - e^{i\xi h_x}\right) + a\left(x - \frac{1}{2}h_x, y\right)\left(1 - e^{-i\xi h_x}\right)\right); \\ c_y &= \mu_y \left(b\left(x, y + \frac{1}{2}h_y\right)\left(1 - e^{i\eta h_y}\right) + b\left(x, y - \frac{1}{2}h_y\right)\left(1 - e^{-i\eta h_y}\right)\right). \end{aligned}$$

As before, $\Re(d_x), \Re(d_y) \geq 0$. We can now quickly compute the amplification factor:

$$\begin{aligned} \left(1 + \frac{1}{2}c_x\right)\widetilde{g} &= 1 - \frac{1}{2}c_y, \\ \left(1 + \frac{1}{2}c_y\right) g &= \left(1 - \frac{1}{2}c_x\right)\widetilde{g}, \end{aligned}$$

so

$$g = \frac{\left(1 - \frac{1}{2}c_x\right)\left(1 - \frac{1}{2}c_y\right)}{\left(1 + \frac{1}{2}c_x\right)\left(1 + \frac{1}{2}c_y\right)},$$

from which it is evident that $|g| \leq 1$, hence the scheme is unconditionally stable.

6. (10 Pts.) Consider the initial boundary value problem

$$u_t + au_x = 0,$$

where a is a real number, to be solved for $x \geq 0$ and $t \geq 0$, with smooth initial data $u(x,0) = u_0(x)$.

(a) For a given value of the constant a, what boundary conditions, if any, are needed to solve this problem?

(b) Suppose the Lax-Wendroff scheme

$$u_j^{n+1} = u_j^n - \frac{a\lambda}{2}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{a^2\lambda^2}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right);$$

where $\lambda = \frac{\Delta t}{\Delta x}$, $j = 1, 2, \ldots$, and $n = 0, 1, 2, \ldots$; is used to approximate solutions to this equation. Give stable boundary conditions for u_0^n. Justify your statements.

Solution

(a) If $a < 0$, the solution to the boundary value problem is simply $u(x,t) = u_0(x - at)$. In this case, the characteristics travel to the left, and no boundary conditions are necessary. On the other hand, if $a > 0$, the characteristics travel to the right, and boundary conditions would be necessary to determine $u(x,t)$ for $t > x/a$.

(b)

7. The following elliptic problem is approximated by the finite element method:

$$\begin{aligned} -\nabla \cdot (a(x)\nabla u(x)) &= f(x), \; x \in \Omega \subset \mathbb{R}^2, \\ u(x) &= u_0(x), \; x \in \Gamma_1, \\ \frac{\partial u}{\partial x_1}(x) + u(x) &= 0, \; x \in \Gamma_2, \\ \frac{\partial u}{\partial x_2}(x) &= 0, \; x \in \Gamma_3; \end{aligned}$$

where

$$\begin{aligned}
\Omega &= \{(x_1,x_2) \mid 0 < x_1 < 1,\ 0 < x_2 < 1\}\,, \\
\Gamma_1 &= \{(x_1,x_2) \mid x_1 = 0,\ 0 \le x_2 \le 1\}\,, \\
\Gamma_2 &= \{(x_1,x_2) \mid x_1 = 1,\ 0 \le x_2 \le 1\}\,, \\
\Gamma_3 &= \{(x_1,x_2) \mid 0 < x_1 < 1,\ x_2 = 0,1\}\,;
\end{aligned}$$

$$0 < A \le a(x) \le B \text{ for a.e. } x \in \Omega,\ f \in L^2(\Omega);$$

and $u_0|_{\Gamma_1}$ is the trace of a function $u_0 \in H^1(\Omega)$.

(a) Determine an appropriate weak variational formulation of the problem.

(b) Prove conditions on the corresponding linear and bilinear forms which are needed for existence and uniqueness of the solution.

(c) Setup a finite element approximation using P_1 elements and a set of basis functions such that the associated linear system is sparse and of band structure. Discuss the linear system thus obtained, and give the rate of convergence.

Solution

(a) We set $w = u - u_0$ (so $u = w + u_0$) and reformulate the problem in terms of w to obtain homogeneous boundary conditions:

$$\begin{aligned}
-\nabla \cdot (a(x)\nabla w(x)) &= f(x) + \nabla \cdot (a(x)\nabla u_0(x)) = g(x),\ x \in \Omega, \\
w(x) &= 0,\ x \in \Gamma_1, \\
\frac{\partial w}{\partial x_1}(x) + w(x) &= -\frac{\partial u_0}{\partial x_1}(x) - u_0(x) = h(x),\ x \in \Gamma_2, \\
\frac{\partial w}{\partial x_2}(x) &= -\frac{\partial u_0}{\partial x_2}(x) = k(x),\ x \in \Gamma_3.
\end{aligned}$$

Let $V = \{v \in H^1(\Omega) \mid v|_{\Gamma_1} \equiv 0\}$ equipped with the norm $\|\cdot\|_{H^1(\Omega)}$. We determine a weak variational formulation by multiplying the differential equation by $v \in V$, applying integration by parts, and noting that $v|_{\Gamma_1} \equiv 0$:

$$\begin{aligned}
&-\nabla \cdot (a\nabla w)\, v = gv \\
\Rightarrow\ & \int_\Omega -\nabla \cdot (a\nabla w)\, v = \int_\Omega gv \\
\Rightarrow\ & -\int_{\partial\Omega} av\frac{\partial w}{\partial \nu} + \int_\Omega a\nabla w \cdot \nabla v = \int_\Omega gv \\
\Rightarrow\ & -\int_{\Gamma_2} av\frac{\partial w}{\partial x_1} + \int_{\Gamma_3, x_2=0} av\frac{\partial w}{\partial x_2} - \int_{\Gamma_3, x_2=1} av\frac{\partial w}{\partial x_2} + \int_\Omega a\nabla w \cdot \nabla v = \int_\Omega gv \\
\Rightarrow\ & -\int_{\Gamma_2} av\,(h - w) + \int_{\Gamma_3, x_2=0} akv - \int_{\Gamma_3, x_2=1} akv + \int_\Omega a\nabla w \cdot \nabla v = \int_\Omega gv \\
\Rightarrow\ & \int_\Omega a\nabla w \cdot \nabla v + \int_{\Gamma_2} awv = \int_\Omega gv + \int_{\Gamma_2} ahv - \int_{\Gamma_3, x_2=0} akv + \int_{\Gamma_3, x_2=1} akv.
\end{aligned}$$

Let

$$\begin{aligned}
a(w,v) &= \int_\Omega a\nabla w \cdot \nabla v + \int_{\Gamma_2} awv \\
Lv &= \int_\Omega gv + \int_{\Gamma_2} ahv - \int_{\Gamma_3, x_2=0} akv + \int_{\Gamma_3, x_2=1} akv
\end{aligned}$$

such that the weak variational formulation is to find $w \in V$ such that

$$a(w, v) = Lv \text{ for all } v \in V.$$

(b) The Lax-Milgram Lemma provides sufficient conditions the bilinear form a and the linear form L must satisfy for existence and uniqueness of w:

- *a is symmetric.* Clearly $a(v_1, v_2) = a(v_2, v_1)$ for all $v_1, v_2 \in V$.
- *a is continuous.* For $v_1, v_2 \in V$, by the Cauchy-Schwarz Inequality,

$$\begin{aligned}
|a(v_1, v_2)| &= \left| \int_\Omega a \nabla v_1 \cdot \nabla v_2 + \int_{\Gamma_2} a v_1 v_2 \right| \\
&\leq B \|\nabla v_1\|_{L^2(\Omega)} \|\nabla v_2\|_{L^2(\Omega)} + B \|v_1\|_{L^2(\Gamma_2)} \|v_2\|_{L^2(\Gamma_2)} \\
&\leq B \|v_1\|_{H^1(\Omega)} \|v_2\|_{H^2(\Omega)} + B \|v_1\|_{L^2(\Gamma_2)} \|v_2\|_{L^2(\Gamma_2)}.
\end{aligned}$$

  But

$$\|v_i\|_{L^2(\Gamma_2)} \leq C \|v_i\|_{H^1(\Omega)}$$

  for some $C > 0$, so, in fact,

$$|a(v_1, v_2)| \leq B(1 + C) \|v_1\|_{H^1(\Omega)} \|v_2\|_{H^2(\Omega)},$$

  and we conclude that a is continuous.
- *a is V-elliptic.* For $v \in V$,

$$\begin{aligned}
a(v, v) &= \int_\Omega a |\nabla v|^2 + \int_{\Gamma_2} a v^2 \\
&\geq A \int_\Omega |\nabla v|^2 + A \int_{\Gamma_2} v^2 \\
&= A \|\nabla v\|^2_{L^2(\Omega)} + A \|v\|^2_{L^2(\Gamma_2)} \\
&\geq A \|\nabla v\|^2_{L^2(\Omega)}.
\end{aligned}$$

  Now since $v|_{\Gamma_1} = 0$ and Γ_1 has positive length,

$$\|\nabla v\|_{L^2(\Omega)} \geq C' \|v\|_{H^1(\Omega)}$$

  for some $C' > 0$, so

$$a(v, v) \geq AC'^2 \|v\|^2_{H^1(\Omega)},$$

  and so a is V-elliptic.
- *L is continuous.* For $v \in V$, by the Cauchy-Schwarz Inequality,

$$\begin{aligned}
|Lv| &= \left| \int_\Omega g v + \int_{\Gamma_2} a h v - \int_{\Gamma_3, x_2 = 0} a k v + \int_{\Gamma_3, x_2 = 1} a k v \right| \\
&\leq \|g\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + B \|h\|_{L^2(\Gamma_2)} \|v\|_{L^2(\Gamma_2)} + B \|k\|_{L^2(\Gamma_3)} \|v\|_{L^2(\Gamma_3)}.
\end{aligned}$$

  As before,

$$\|v\|_{L^2(\Gamma_3)} \leq C'' \|v\|_{H^1(\Omega)}$$

  for some $C'' > 0$, so that

$$|Lv| \leq \left(\|g\|_{L^2(\Omega)} + BC \|h\|_{L^2(\Gamma_2)} + BC'' \|k\|_{L^2(\Gamma_3)} \right) \|v\|_{H^1(\Omega)},$$

  hence L is continuous.

(c) (W06.7(c))