Basic Exam (Fall 2004)

1) Consider the following two statements:
(A) The sequence (a_n) converges.
(B) The sequence $((a_1 + a_2 + ... + a_n)/n)$ converges.
Does (A) imply (B)? Does (B) imply (A)? Prove your answers.

2) State and prove Rolle's Theorem. (You can use without proof theorems about the maxima and minima of continuous or differentiable functions.)

3) Show that if $f_n \to f$ uniformly on the bounded closed interval $[a, b]$, then

$$\int_a^b f_n(x)dx \to \int_a^b f(x)dx.$$

4) Suppose that $(\mathcal{M}, \rho)$ is a metric space, $x, y \in \mathcal{M}$, and that $\{x_n\}$ is a sequence in this metric space such that $x_n \to x$. Prove that $\rho(x_n, y) \to \rho(x, y)$.

5) Prove that the space $C[0,1]$ of continuous functions from $[0,1]$ to $\mathbb{R}$ with the supremum norm, $||f||_\infty = \sup_{[0,1]} |f(x)|$, is complete. (You can use without proof the fact that a uniform limit of continuous functions is continuous.)

6) The Bolzano-Weirstrass Theorem in $\mathbb{R}^n$ states that if S is a bounded closed subset of $\mathbb{R}^n$ and (x_n) is a sequence which takes values in S, then (x_n) has a subsequence which converges to a point in S. Assume this statement known in case $n = 1$, and use it to prove the statement in case $n = 2$.

7) Observe that the point $P = (1, 1, 1)$ belongs to the set S of points in $\mathbf{R}^3$ satisfying the equations

$$x^4y^2 + x^2z + +yz^2 = 3.$$

Explain carefully how, in this case, the Implicit Function Theorem allows us to conclude that there exists a differentiable function $f(x, y)$ such that $(x, y, f(x, y))$ lie in S for all (x, y) in a small open set containing $(1, 1)$.

8) Let $A = (a_{ij})$ be a real, $n \times n$ symmetric matrix and let $Q(v) = v \cdot Av$ (ordinary dot product) be the associated quadratic form defined for $v = \langle v_1, \ldots, v_n \rangle \in \mathbf{R^n}$.

1. Show that $\nabla Q_v = 2Av$ where ∇Q_v is the gradient at v of the function Q.

2. Let M the minimum value of $Q(v)$ on on the unit sphere $S^n = \{v \in \mathbf{R^n} : ||\mathbf{v}|| = \mathbf{1}\}$ and let $u \in S^n$ be a vector such that $Q(u) = M$. Prove, using Lagrange multipliers, that u is an eigenvector of A with eigenvalue M.

9) Let $T : \mathbf{C^n} \to \mathbf{C^n}$ be a linear transformation and $P(X)$ a polynomial such that $P(T) = 0$. Prove that every eigenvalue of T is a root of $P(X)$.

10) Let $V = \mathbf{R^n}$ and let $T : V \to V$ be a linear transformation. For $\lambda \in \mathbf{C}$, the subspace

$$V(\lambda) = \{v \in V : (T - \lambda I)^N v \quad 0 \quad \text{for some } N \geq 1\}$$

is called a generalized eigenspace.

1. Prove that there exists a fixed number M such that $V(\lambda) \quad \ker((T \quad \lambda I)^M)$.

2. Prove that if $\lambda \neq \mu$, then $V(\lambda) \cap V(\mu) = \{0\}$. Hint: use the following equation by raising both sides to a high power.

$$\frac{T - \lambda I}{\mu - \lambda} + \frac{T - \mu I}{\lambda - \mu} \quad I$$