Qualifying Exam, Fall 2001

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

ALL PROBLEMS HAVE EQUAL VALUE. There are 7 problems.

MA: Do any 5 problems.

Ph.D.: Do 5 problems and only 3 of them from 1, 2, 3, and 4.

[**1**] Consider the equation

$$3x + g(x) = 0 \tag{1}$$

with $g : R \to R$ and $g'(x^*) = 0$, where x^* is the unique solution of (1). Assume the following iteration is used to obtain the solution x^*

$$x^{n+1} = -\frac{g(x^n)}{3}$$

(a) Derive a recurrence relation for the error at each step of the iteration.

(b) What is the rate of convergence of this iteration?

(c) State and prove a local convergence theorem for this iteration.

[**2**] Consider the problem of fitting a function of the form

$$f(x) = a\cos(x) + b\sin(x) + c\cos(2x)$$

to the data

x	0	$\pi/2$	π	$3\pi/2$
y	1	1	1	0

(a) Give the set of linear equations whose solution determines the coefficients a, b, c so that $\sum_{i=1}^{4} |f(x_i) - y_i|^2$ is a minimum.

(b) Solve the linear system and give the coefficients a, b, c you obtain.

[3] Consider the following iterative method

$$\vec{x}^{k+1} = \mathrm{B}\vec{x}^k + \vec{c}$$

where B is the matrix

$$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

(a) Does this iteration converge for arbitrary initial vectors, $\vec{x}^0$? Justify your answer.

(b) Can α be chosen so that the following iteration converges for arbitrary initial vectors $\vec{x}^0$?

$$\vec{z}^0 = \vec{x}^0$$

$$\vec{x}^{k+1} = \mathrm{B}\vec{z}^k + \vec{c} \qquad\qquad \vec{z}^{k+1} = \alpha\,\vec{z}^k + (1-\alpha)\vec{x}^k$$

Justify your answer

[4] Consider the following Runge-Kutta method for solving the differential equation $y' = f(y)$

$$\begin{aligned} y^* &= y^n + \frac{\Delta t}{2}\, f(y^n) \\ y^{n+1} &= y^n + \Delta t\,\beta_1 f(y^n) + \Delta t\,\beta_2\, f(y^*) \end{aligned} \tag{2}$$

(a) How should β_1 and β_2 be chosen to obtain a first order method?

(b) How should β_1 and β_2 be chosen to obtain a second order method?

(c) The interval of absolute stability for all second order methods of the form (2) is (-2,0). Give an example of a first order method of the form (2) that has a larger interval of absolute stability. Justify your answer.

Qualifying Exam, Fall 2001

NUMERICAL ANALYSIS

[5] Consider

$$u_t = u_x + u_y$$

in $\Omega : 0 \le x, y \le 1$ with periodic boundary conditions for $t > 0$, with $u(x, y, 0)$ given to be smooth and periodic with period 1 in x and y.

(a) Construct a second order accurate convergent method that involves only approximations to the two one space dimensional problems $u_t = u_x$ and $u_t = u_y$.Justify your answer.

(b) What, if any, is the advantage of using this method over standard 2 space dimensional finite difference approximations? Explain your answer.

[6](a) For which values of the constants, a, b, c can you obtain stable and convergent approximations to:

$$u_t = au_{xx} + 2bu_{xy} + cu_{yy},$$

to be solved for $t > 0$, on $\Omega = 0 \le x, y \le 1$, $u(x, y, 0) = u_0(x, y)$, smooth and periodic with period 1 in x, y?

(b) Give such a scheme. Justify your answers.

[7](a) Show formally that u is the solution of the variational problem

$$\min_{u \,\in\, H_0^1(I)} \left[\frac{1}{2} \int_I k(x) (\frac{dv}{dx})^2 dx - \int_I v \right]$$

where $I = (0, 1)$ and

$$k(x) = \begin{matrix} 1 & \text{if } x \in I_1 = (0, \frac{1}{2}) \\ \frac{1}{2} & \text{if } x \in I_2 = (\frac{1}{2}, 1) \end{matrix}$$

if and only if u satisfies

(i) $-k(x)\dfrac{d^2u}{dx^2} = 1$ in I_1 and I_2

(ii) $u_1 = u_2$ and $\dfrac{2du_1}{dx} = \dfrac{du_2}{dx}$ at $x = \frac{1}{2}$. (Here $u_i = u|_{I_i}, i = 1, 2$)

(b) Formulate a finite element method for the problem (i) and (ii) using piecewise linear functions.

(c) Determine the corresponding linear system in the case of a uniform partition and interpret this system as a difference approximation to (i) and (ii), again using piecewise linear functions.