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Basic Examination, S02

1. Prove that the closed interval $[0,1]$ is connected.

   Solution

   Suppose $[0,1]$ is not connected. Then $[0,1] = A \cup B$ for two nonempty sets A, B such that $A \cap \overline{B} = \overline{A} \cap B = \emptyset$. Suppose, without loss of generality, that $0 \in A$ and $1 \in B$. Let

   $$x = \sup A.$$

   Since $A \subset [0,1]$ is contained in a closed set, $x \in [0,1]$. Now certainly $x \in \overline{A}$, for otherwise x could not be a least upper bound for A. Hence $x \notin B$, since $\overline{A} \cap B = \emptyset$, so $x \in A$, since $x \in A \cup B$, thus $x \notin \overline{B}$, since $A \cap \overline{B} = \emptyset$. Hence there exists a neighborhood $U \subset [0,1]$ of x disjoint from B. Thus $U \subset A$. Since $1 \in B$ and $x \notin B$, $x < 1$. But then any neighborhood within $[0,1]$ of x contains points greater than x, hence some $y \in U \subset A$ is such that $y > x$. This contradicts the construction of x as the least upper bound for A, and we conclude that $[0,1]$ is connected.

2. Show that the set $\mathbb{Q}$ of rational numbers in $\mathbb{R}$ is not expressible as the intersection of a countable collection of open subset of $\mathbb{R}$.

   Solution

   Suppose $\bigcap_n E_n = \mathbb{Q}$ for some countable sequence $\{E_n\}$ of open subsets of $\mathbb{R}$. Each E_n must be dense in $\mathbb{R}$, since their intersection $\mathbb{Q} \subset E_n$ is dense in $\mathbb{R}$. Enumerate the elements of $\mathbb{Q}$ by x_n, and set $F_n = \mathbb{R} \backslash \{x_n\}$. Then each F_n is also open and dense in $\mathbb{R}$, hence by the Baire Category Theorem,

   $$\left(\bigcap_n E_n \right) \cap \left(\bigcap_n F_n \right)$$

   must be dense in $\mathbb{R}$, as it is a countable intersection of dense open sets in a complete metric space. Yet the intersection above is empty, hence certainly not dense in $\mathbb{R}$, a contradiction. It follows that $\mathbb{Q}$ cannot be expressed as the intersection of a countable collection of open subsets of $\mathbb{R}$.

3. Suppose that X is a compact metric space (in the covering sense of the word compact). Prove that every sequence $\{x_n : x_n \in X, n = 1, 2, 3, \ldots\}$ has a convergent subsequence. (Prove this directly. Do not just quote a theorem.)

   Solution

   Suppose that for each $y \in X$, there exists an $\epsilon = \epsilon_y > 0$ such that $B(y; \epsilon_y)$ contains only finitely many points of $\{x_n\}$. The family $\{B(y; \epsilon_y)\}_{y \in X}$ is an open cover of X, hence contains some finite subcover $\{B(y_i; \epsilon_{y_i})\}_{i=1}^m$ by compactness of X. Every x_n must lie within some $B(y_i; \epsilon_{y_i})$, thus, as there are only finitely many of the $B(y_i; \epsilon_{y_i})$'s, and each contains finitely many of the x_n's, there must be only finitely many of the x_n's, a contradiction. It follows that there exists some $y^* \in X$ such that $B(y^*; \epsilon)$ contains infinitely points of $\{x_n\}$ for all $\epsilon > 0$.

   We construct a convergent subsequence of $\{x_n\}$ converging to y^* as follows. Select $x_{n_j} \in B(y^*; 1/j)$ from $\{x_n\} \cap B(y^*; 1/j)$ and such that $n_j < n_{j+1}$, which is always possible due to the infinitude of the intersection for all $j = 1, 2, \ldots$. Then $d(y^*, x_{n_j}) < 1/j \to 0$ as $j \to \infty$, hence $x_{n_j} \to y^*$ and $\{x_{n_j}\}_{j=1}^\infty$ is a convergence subsequence of $\{x_n\}$.

4. (a) Define *uniform continuity* of a function $F : X \to \mathbb{R}$, X a metric space.

   (b) Prove that a function $f : (0,1) \to \mathbb{R}$ is the restriction to $(0,1)$ of a continuous function $F : [0,1] \to \mathbb{R}$ if and only if f is uniformly continuous on $(0,1)$.

   Solution

(a) F is uniformly continuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|F(x) - F(y)| < \epsilon$ whenever $x, y \in X$ with $d(x, y) < \delta$.

(b) Suppose $f : (0, 1) \to \mathbb{R}$ is the restriction to $(0, 1)$ of a continuous function $F : [0, 1] \to \mathbb{R}$. Then F is uniformly continuous (since $[0, 1]$ is compact), from which the uniform continuity of f follows immediately.

Now suppose that f is uniformly continuous on $(0, 1)$. Let δ_n be such that $|f(x) - f(y)| < 1/n$ whenever $x, y \in (0, 1)$ with $|x - y| < \delta_n$, and further ensure that $\delta_n \leq \delta_{n+1}$. Set $x_n = \delta_n/2$. Then for $n, m > N$, $x_n, x_m \in (0, \delta_N)$, hence $|f(x_n) - f(x_m)| < 1/N$, which shows that $\{f(x_n)\}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, the limit exists and we can set $a = \lim_{n\to\infty} f(x_n)$.

Now given an $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in (0, 1)$ with $|x - y| < \delta$. Now if $x \in (0, \delta)$, there exists an $x_n \in (0, \delta)$ such that $|a - f(x_n)| < \epsilon$, hence $|a - f(x)| < |a - f(x_n)| + |f(x_n) - f(x)| < 2\epsilon$, showing that, indeed $a = \lim_{x\to 0} f(x)$. Similarly, we can set $b = \lim_{x\to 1} f(x)$, and define $F : [0, 1] \to \mathbb{R}$ by

$$F(x) = \begin{cases} a, & x = 0 \\ f(x), & 0 < x < 1 \\ b, & x = 1 \end{cases}.$$

Evidently, F is continuous on $[0, 1]$, by construction, and f is the restriction of F to $(0, 1)$.

5. State some reasonable conditions under which a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

everywhere on $\mathbb{R}^2$ and prove this equality under the conditions you give.

Solution

(F01.5)

6. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a continuously differentiable function with $\operatorname{grad} f \neq \vec{0}$ at $\vec{0}$ ($\vec{0} = (0, 0, 0)$ in $\mathbb{R}^3$). Show that there are two continuously differentiable functions $g : \mathbb{R}^3 \to \mathbb{R}$, $h : \mathbb{R}^3 \to \mathbb{R}$ such that the function

$$(x, y, z) \mapsto (f(x, y, z), g(x, y, z), h(x, y, z))$$

from $\mathbb{R}^3$ to $\mathbb{R}^3$ is one-to-one on some neighborhood of $\vec{0}$.

Solution

Without loss of generality, suppose $\frac{\partial f}{\partial x} \neq 0$ at 0. Set $h(x, y, z) = y$ and $g(x, y, z) = z$. Set

$$F(x, y, z) = (f(x, y, z), g(x, y, z), h(x, y, z));$$

then

$$F'(x, y, z) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is nonsingular at 0, hence the Inverse Function Theorem guarantees open sets U and V of $\mathbb{R}^3$ with $0 \in U$, $F(0) \in V$, F is one-to-one on U, and $F(U) = V$.

7. Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable and that the Jacobian matrix of F is everywhere nonsingular. Suppose also that $F(\vec{0}) = \vec{0}$ and that $\|F((x, y))\| \geq 1$ for all (x, y) with $\|(x, y)\| = 1$.

Prove that $F(\{(x, y) : \|(x, y)\| < 1\}) \supset \{(x, y) : \|(x, y)\| < 1\}$.

(Hint: Show, with $U = \{(x, y) : \|(x, y)\| < 1\}$, that $F(U) \cap U$ is both open and closed in U.)

Solution

(W02.7)

8. Let V be a finite dimensional real vector space. Let $W \subset V$ be a subspace and $W^\circ = \{f : V \to \mathbb{F} \text{ linear} \mid f = 0 \text{ on } W\}$. Prove that

$$\dim(V) = \dim(W) + \dim(W^\circ).$$

Solution

We show an isomorphism between W° and $(V/W)^*$. Given $f \in W^\circ$, define $L \in (V/W)^*$ by

$$L\{x\} = f(x),$$

where $\{x\}$ is the equivalence class in V/W of x. It follows from the fact that $f \in W^\circ$ that L is well-defined, hence this defines a homomorphism from W° to $(V/W)^*$. Conversely, given $L \in (V/W)^*$, define $f \in V*$ by

$$f(x) = L\{x\}$$

for $x \in V$. Since $L\{x\} = 0$ for any $x \in W$, it follows that, in fact, $f \in W^\circ$, hence this defines a homomorphism from $(V/W)^*$ to W°. Thus $W^\circ \cong (V/W)^*$, and isomorphic vector spaces have equal dimension. Therefore,

$$\dim(V) = \dim(W) + \dim(V/W) = \dim(W) + \dim((V/W)^*) = \dim(W) + \dim(W^\circ).$$

9. Find the matrix representation in the standard basis for either rotation by an angle θ in the plane perpendicular to the subspace spanned by the vectors $(1, 1, 1, 1)$ and $(1, 1, 1, 0)$ in $\mathbb{R}^4$.

(You do not have to multiply the matrices out but must compute any inverses.)

Solution

Let

$$B_T = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & 0 & -\frac{2}{\sqrt{6}} \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then regarding the columns of B_T as an orthonormal basis, the matrix representation of T in this basis is

$$[T]_{B_T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix},$$

so the matrix representation of T in the standard basis is

$$T = B_T[T]_{B_T}B_T^{-1} = B_T[T]_{B_T}B_T^t.$$

10. Let V be a complex inner product space and W a finite dimensional subspace. Let $v \in V$. Prove that there exists a unique vector $v_W \in W$ such that

$$\|v - v_W\| \leq \|v - w\|$$

for all $w \in W$. Deduce that equality holds if and only if $w = v_W$.

Solution

Let $\{e_1, \ldots, e_k\}$ be an orthonormal basis of W with respect to the inner product of V, and set

$$v_W = \sum_{i=1}^{k} (v, e_i)e_i.$$

Then for any $w = \sum_i w_i e_i \in W$,

$$\begin{array}{rcl}\|v-w\|^2 & = & (v-w, v-w) \\ & = & (v,v) + (w,w) - (v,w) - (w,v) \\ & = & \|v\|^2 + \sum_i |w_i|^2 - \sum_i \overline{w_i}(v, e_i) - \sum_i w_i(e_i, v) \\ & = & \|v\|^2 + \sum_i \left(|w_i|^2 - \overline{w_i}(v,e_i) - w_i\overline{(v,e_i)}\right)\end{array} .$$

In particular,

$$\|v - v_W\|^2 = \|v\|^2 - \sum_i |(v, e_i)|^2,$$

hence

$$\begin{array}{rcl}\|v-w\|^2 - \|v-v_W\|^2 & = & \sum_i \left(|w_i|^2 + |(v,e_i)|^2 - \overline{w_i}(v,e_i) - w_i\overline{(v,e_i)}\right) \\ & = & \sum_i (w_i - (v,e_i))\left(\overline{w_i} - \overline{(v,e_i)}\right) \\ & = & \sum_i |w_i - (v,e_i)|^2 \\ & \geq & 0\end{array}$$

for all $w \in W$, with equality if and only if $w_i = (v, e_i)$ for each $i = 1, \ldots, k$.

11. Let V be a finite dimensional real inner product space and $T, S : V \to V$ two commuting hermitian linear operators. Show that there exists an orthonormal basis for V consisting of vectors that are simultaneously eigenvectors of T and S.

Solution

Let $\{\lambda_i\}_{i=1}^k$ be the eigenvectors of S, and consider the eigenspaces $E_i = \ker(S - \lambda_i I)$, $i = 1, \ldots, k$. Note that each pair of eigenspaces are orthogonal, since S is self-adjoint; indeed, the Spectral Theorem for self-adjoint matrices allows us to decompose V as

$$V = \bigoplus_i E_i.$$

Now for $x \in E_i$, $Sx = \lambda_i x$, so $S(Tx) = T(Sx) = T(\lambda_i x) = \lambda_i(Tx)$, hence $Tx \in E_i$ as well. Thus T is invariant on each of the subspaces E_i, so the restriction of T to E_i is a linear, self-adjoint operator. The Spectral Theorem for self-adjoint matrices then allows us to choose an orthonormal basis for E_i of eigenvectors of T, which also happen to be eigenvectors of S since they belong to E_i. Applying the Spectral Theorem to each restriction to E_i of T thus allows us to find an orthonormal basis for all of V of eigenvectors of both S and T.