Ph.D Qualifying Exam
APPLIED DIFFERENTIAL EQUATIONS
Fall 2003

Do all of the following 6 problems

1. For the ODE

$$\begin{aligned} u_t &= v - u^3 \\ v_t &= u - v \end{aligned}$$

(a) Find stationary points and their type.
(b) Draw the phase plane and find all connections between the stationary points.

2. (a) Let Ω_1 and Ω_2 be two smooth sets in R^2 with Ω_1 a (strict) subset of Ω_2. Let $-\lambda_1$ and $-\lambda_2$ be the smallest (i.e. least negative) eigenvalues for the Dirichlet problem on Ω_1 and Ω_2, with eigenfunction ϕ_1 and ϕ_2, respectively. That is

$$\begin{aligned} \Delta\phi_1 &= -\lambda_1\phi_1 \quad \text{in } \Omega_1 && (1) \\ \Delta\phi_2 &= -\lambda_2\phi_2 \quad \text{in } \Omega_2 && (2) \\ \phi_1 &= 0 \quad \text{on } \partial\Omega_1 && (3) \\ \phi_2 &= 0 \quad \text{on } \partial\Omega_2 && (4) \end{aligned}$$

Show that $\lambda_1 > \lambda_2 > 0$. Hint: Use the variational characterization of the smallest eigenvalue λ for a set Ω that $\lambda = \min_u \int_\Omega (\nabla u)^2 dxdy / \int_\Omega u^2 dxdy$

(b) Suppose Ω is a smooth set in R^2 with mirror symmetry about the y axis; i.e. if $(x, y) \in \Omega$ then $(-x, y) \in \Omega$. Let ϕ be the eigenfunction for the Dirichlet problem on Ω with the smallest eigenvalue. Use the result in (a) to show that $\phi(x) = \phi(-x)$.

3. The function

$$h(X, T) = (4\pi T)^{-\frac{1}{2}} \exp(-X^2/4T)$$

satisfies (you do not need to show this)

$$h_T = h_{XX}.$$

Using this result, verify that for any smooth function U

$$u(x,t) = \exp\left(\frac{1}{3}t^3 - xt\right)\int_{-\infty}^{\infty} U(\xi)h(x - t^2 - \xi, t)d\xi$$

satisfies

$$u_t + xu = u_{xx}. \tag{6}$$

Given that $U(x)$ is bounded and continuous everywhere on $-\infty \le x \le \infty$, establish that

$$\lim_{t\to 0}\int_{-\infty}^{\infty} U(\xi)h(x-\xi,t)d\xi = U(x) \tag{7}$$

and show that $u(x,t) \to U(x)$ as $t \to 0$. (You may use the fact that $\int_0^\infty e^{-\xi^2}d\xi = \frac{1}{2}\sqrt{\pi}$.)

4. Find the characteristics of the partial differential equation

$$xu_{xx} + (x-y)u_{xy} - yu_{yy} = 0, \quad x > 0,\ y > 0, \tag{8}$$

and then show that it can be transformed into the canonical form

$$(\xi^2 + 4\eta)u_{\xi\eta} + \xi u_\eta = 0 \tag{9}$$

whence ξ and η are suitably chosen canonical coordinates. Use this to obtain the general solution in the form

$$u(\xi,\eta) = f(\xi) + \int^{\eta} \frac{g(\eta')d\eta'}{(\xi^2 + 4\eta')^{\frac{1}{2}}}$$

where f and g are arbitrary functions of ξ and η.

5. State Parseval's relation for Fourier transforms. Find the Fourier transform $\hat{f}(\xi)$ of

$$f(x) = \begin{cases} e^{i\alpha x}/2\sqrt{\pi y}, & |x| \le y \\ 0, & |x| > y \end{cases} \tag{11}$$

in which y and α are constants. Use this in Parseval's relation to show that

$$\int_{-\infty}^{\infty} \frac{\sin^2(\alpha - \xi)y}{(\alpha-\xi)^2}d\xi = \pi y$$

What does the transform $\hat{f}(\xi)$ become in the limit $y \to \infty$?

Use Parseval's relation to show that

$$\frac{\sin(\alpha-\beta)y}{\alpha-\beta} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\alpha-\xi)y}{(\alpha-\xi)} \frac{\sin(\beta-\xi)y}{(\beta-\xi)} d\xi.$$

6.(a) For the cubic equation

$$\varepsilon^3 x^3 - 2\varepsilon x^2 + 2x - 6 = 0$$

write the solutions x in the asymptotic expansion $x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$ as $\varepsilon \to 0$. Find the first two terms x_0 and x_1 for all solutions x.

(b) For the ODE

$$\begin{aligned} u_t &= u - \varepsilon u^3 \qquad (15)\\ u(0) &= 1 \qquad (16) \end{aligned}$$

write $u = u_0(t) + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + O(\varepsilon^3)$ as $\varepsilon \to 0$. Find the first three terms u_0, u_1 and u_2.