1. *Consider the differential equation:*

$$u_{xx} + u_{yy} + \lambda u = 0$$

in the strip $\{(x,y) : 0 < y < \pi,\ -\infty < x < +\infty\}$ *with boundary conditions*

$$u(x,0) = 0, \quad u(x,\pi) = 0.$$

Find all bounded solutions of the boundary value problem when (a) $\lambda = 0$*, (b)* $\lambda > 0$*, and (c)* $\lambda < 0$*.*

By separation of variables, the solution has the form

$$u(x,y) = \sum_{n=1}^{\infty} X_n(x) \sin(ny).$$

For all n,

$$\begin{aligned} 0 &= X_n''(x)\sin(ny) - n^2 X_n(x)\sin(ny) + \lambda X_n(x)\sin(ny) \\ 0 &= X_n''(x) + (\lambda - n^2) X_n(x) \\ X_n(x) &= a_n \mathrm{e}^{\sqrt{n^2-\lambda}\,x} + b_n \mathrm{e}^{-\sqrt{n^2-\lambda}\,x}. \end{aligned}$$

(a) If $\lambda = 0$, then

$$X_n(x) = a_n \mathrm{e}^{nx} + b_n \mathrm{e}^{-nx}.$$

The only bounded solution is $a_n = b_n = 0$, so $u \equiv 0$.

(b) If $\lambda > 0$, then for $n^2 > \lambda$,

$$\begin{aligned} X_n(x) &= a_n \mathrm{e}^{i\sqrt{\lambda-n^2}\,x} + b_n \mathrm{e}^{-i\sqrt{\lambda-n^2}\,x} \\ &= a_n' \sin\sqrt{\lambda - n^2}\,x + b_n' \cos\sqrt{\lambda - n^2}\,x. \end{aligned}$$

Thus bounded solutions have the form

$$u(x,y) = \sum_{n^2<\lambda} \left(a_n' \sin\sqrt{\lambda - n^2}\,x + b_n' \cos\sqrt{\lambda - n^2}\,x \right) \sin ny.$$

(c) If $\lambda < 0$, then similar to part (a),

$$X_n(x) = a_n \mathrm{e}^{\sqrt{n^2-\lambda}\,x} + b_n \mathrm{e}^{-\sqrt{n^2-\lambda}\,x}.$$

Thus the only bounded solution is $a_n = b_n = 0$, $u \equiv 0$.

2. *Let $C^2(\overline{\Omega})$ be the space of all twice continuously differentiable functions in the bounded smooth closed domain $\overline{\Omega} \subset \mathbb{R}^2$. Let $u_0(x, y)$ be the function that minimizes the functional*

$$D(u) = \int_\Omega (u_x^2 + u_y^2 + fu) \; + \; \int_{\partial\Omega} au^2$$

where f and a are given continuous functions. Find the differential equation and boundary condition that u_0 satisfies.

In other words, we must find the Euler-Lagrange equations associated with the minimization problem. Let $u = u_0$, $v \in C^2(\overline{\Omega})$, and define $g(\epsilon) = D(u + \epsilon v)$. Then

$$\begin{aligned}
0 \; = \; g'(0) \quad &= \quad \int_\Omega (2u_x v_x + 2u_y v_y + fv) \; + \; \int_{\partial\Omega} a2u\,v \\
&= \quad 2\int_\Omega \nabla u \cdot \nabla v + \int_\Omega fv + 2\int_{\partial\Omega} a\,u\,v \\
&= \quad 2\left[-\int_\Omega \Delta u\,v + \int_{\partial\Omega} (n \cdot \nabla u)v\right] + \int_\Omega fv + 2\int_{\partial\Omega} a\,u\,v \\
&= \quad \int_\Omega (-2\Delta u + f)v + \int_{\partial\Omega} 2(n \cdot \nabla u + au)v.
\end{aligned}$$

Since this holds for any $v \in C^2(\overline{\Omega})$, it implies

$$\begin{cases} \Delta u = \frac{1}{2} f & \text{in } \Omega \\ n \cdot \nabla u + au = 0 & \text{on } \partial\Omega \end{cases}$$

3. *Let $f(x_1, x_2)$ be a continuous function with compact support. Define*

$$u(x_1, x_2) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} \frac{f(y_1, y_2)}{z - w} \, \mathrm{d}y_1 \, \mathrm{d}y_2$$

where $z = x_1 + ix_2$ and $w = y_1 + iy_2$. Prove that

$$\frac{\partial u}{\partial x_1} + i\frac{\partial u}{\partial x_2} = f(x_1, x_2) \qquad \textit{in } \mathbb{R}^2.$$

Consider $v_{x_1} + iv_{x_2} = f$. Then

$$\begin{aligned} f_{x_1} &= v_{x_1 x_1} + iv_{x_1 x_2} \\ f_{x_2} &= v_{x_1 x_2} + iv_{x_2 x_2}. \end{aligned}$$

Therefore, $\Delta v = f_{x_1} - if_{x_2}$. Let $g = \frac{1}{2\pi} \log \sqrt{x_1^2 + x_2^2}$, then since g is a fundamental solution of $\Delta v = f$,

$$\begin{aligned} v &= f_{x_1} * g - if_{x_2} * g \\ &= f * (g_{x_1} - ig_{x_2}) \\ &= \frac{1}{2\pi} f * \left(\frac{x_1}{x_1^2 + x_2^2} - \frac{ix_2}{x_1^2 + x_2^2} \right) \\ &= \frac{1}{2\pi} f * \frac{1}{x_1 + ix_2} \\ v(x_1, x_2) &= \frac{1}{2\pi} \iint_{\mathbb{R}^2} \frac{f(y_1, y_2)}{z - w} \, \mathrm{d}y_1 \, \mathrm{d}y_2. \end{aligned}$$

So $u = v$ and hence $u_{x_1} + iu_{x_2} = f$.

4. *Consider the boundary value problem on* $[0, \pi]$*:*

$$\begin{cases} y''(x) + p(x)y(x) = f(x), & 0 < x < \pi \\ y(0) = 0, \quad y'(\pi) = 0. \end{cases}$$

Find the smallest λ_0 *such that the boundary value problem has a unique solution whenever* $p(x) > \lambda_0$ *for all* x*. Justify your answer.*

Let y_1 and y_2 be two solutions and let $w = y_1 - y_2$, then

$$\begin{cases} w''(x) + p(x)w(x) = 0, & 0 < x < \pi \\ w(0) = 0, \quad w'(\pi) = 0. \end{cases}$$

So

$$0 \;=\; \int_0^\pi (-w'' + pw)w \;=\; -w'w\Big|_0^\pi + \int_0^\pi (w')^2 + pw^2 \;=\; \int_0^\pi (w')^2 + pw^2.$$

Therefore,

$$\int_0^\pi (w')^2 = -\int_0^\pi pw^2.$$

The left side is nonnegative. If $p > 0$, then the right side is nonpositive and $w \equiv 0$. Therefore, $\lambda_0 = 0$.

5. *Consider the Laplace equation*

$$u_{xx} + u_{yy} = 0, \qquad y > 0, \quad -\infty < x < +\infty$$

with the boundary condition

$$u_y(x,0) - u(x,0) = f(x),$$

where $f(x) \in C_0^\infty(\mathbb{R})$. *Find a bounded solution* $u(x,y)$ *and show that* $u(x,y) \to 0$ *when* $|x| + y \to \infty$.

Apply the Fourier transform in the x variable to obtain

$$\begin{cases} -\xi^2 \hat{u}(\xi,y) - \hat{u}(\xi,y) = 0 \\ \hat{u}_y(\xi,0) - \hat{u}(\xi,0) = \hat{f}(\xi) \end{cases}$$

where $\hat{(\cdot)}$ denotes the Fourier transform. For each fixed frequency, $-\xi^2\hat{u} + \hat{u}'' = 0$ is an ODE in y with solutions $\mathrm{e}^{\xi y}$ and $\mathrm{e}^{-\xi y}$. For bounded $\hat{u}$, we keep only the solution such that the exponent is negative,

$$\hat{u}(\xi,y) = c(\xi)\mathrm{e}^{-|\xi|y}.$$

The initial conditions determine $c(\xi)$:

$$\begin{aligned} \hat{u}_y(\xi,0) - \hat{u}(\xi,0) &= \hat{f}(\xi) \\ -|\xi|c(\xi) - c(\xi) &= \hat{f}(\xi) \\ c(\xi) &= -\frac{\hat{f}(\xi)}{1+|\xi|}, \end{aligned}$$

and we have

$$\hat{u}(\xi,y) = \underbrace{\frac{-\mathrm{e}^{-|\xi|y}}{1+|\xi|}}_{\hat{H}(\xi,y)} \hat{f}(\xi).$$

Let $\mathcal{S}$ denote the Schwartz space in one dimension. Since the Fourier transform maps $\mathcal{S}$ into itself, $f \in C_0^\infty(\mathbb{R}) \subset \mathcal{S}$ implies $\hat{f} \in \mathcal{S}$. Considering y as a fixed parameter, $\hat{H}$ is in $\mathcal{S}$ and hence $\hat{u}$ and u are in $\mathcal{S}$. Therefore, u is bounded and decays as $|x| + y \to \infty$.

6. *Consider the first order system $u_t - u_x = v_t + v_x = 0$ in the diamond shaped region $-1 < x + t < 1$, $-1 < x - t < 1$. For each of the following boundary value problems state whether this problem is well-posed. If it is well-posed, find the solution.*

(a) $$\begin{cases} u(x+t) &= u_0(x+t) \quad \text{on } x - t = -1 \\ v(x-t) &= v_0(x+t) \quad \text{on } x + t = -1 \end{cases}$$

(b) $$\begin{cases} v(x+t) &= v_0(x+t) \quad \text{on } x - t = -1 \\ u(x-t) &= u_0(x+t) \quad \text{on } x + t = -1 \end{cases}$$

By method of characteristics,

$$u : \begin{cases} t' &= 1 \\ x' &= -1 \\ u' &= 0 \end{cases} \quad \text{characteristics } x + t = c \qquad v : \begin{cases} t' &= 1 \\ x' &= 1 \\ v' &= 0 \end{cases} \quad \text{characteristics } x - t = c$$

(a) The boundary conditions are along noncharacteristic curves, $x - t = -1$ for u and $x + t = -1$ for v, so the problem is well-posed. The solution is

$$\begin{cases} u(x,t) &= u_0(x+t) \\ v(x,t) &= v_0(x-t) \end{cases} \quad \text{in } D.$$

(b) In this case, the boundary conditions are along characteristic curves, so it is not well-posed.

7. *For the two-point boundary value problem $Lf = f_{xx} - f$ on $-\infty < x < \infty$ with $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$, the Green's function $G(x, x')$ solves $LG = \delta(x - x')$ in which L acts on the variable x.*

(a) Show that $G(x, x') = G(x - x')$.

(b) For each x', show that

$$G(x, x') = \begin{cases} a_- \mathrm{e}^{x} & \text{for } x < x', \\ a_+ \mathrm{e}^{-x} & \text{for } x' < x, \end{cases}$$

in which $a_\pm$ are functions that depend only on x'.

(c) Using (a), find the x' dependence of $a_\pm$.

(d) Finish finding $G(x, x')$ by using the jump conditions to find the remaining unknowns in $a_\pm$.

(a) Define $\mathscr{G} f(x) = \int_{\mathbb{R}} G(x, x') f(x') \,\mathrm{d}x'$, then

$$\begin{aligned} f(x) &= \int_{\mathbb{R}} LG(x, x') f(x') \,\mathrm{d}x' \;=\; L\mathscr{G} f(x) \\ &= \int_{\mathbb{R}} G_{xx}(x, x') f(x') - G(x, x') f(x') \,\mathrm{d}x' \;=\; \int_{\mathbb{R}} G(x, x') f''(x') - G(x, x') f(x') \,\mathrm{d}x' \;=\; \mathscr{G} L f(x). \end{aligned}$$

Suppose f_0, f_τ are such that $f_\tau(x) = f_0(x - \tau)$ and let $g_0 = Lf_0$, $g_\tau = Lf_\tau$, then $g_\tau(x) = g_0(x - \tau)$ and $\mathscr{G} g_\tau(x) = f_\tau(x) = f_0(x - \tau) = \mathscr{G} g_0(x - \tau)$. So $\mathscr{G}$ is shift-invariant, which implies $G(x, x') = G(x - x')$.

(b) For $x < x'$, $LG(x, x') = G_{xx}(x, x') - G(x, x') = 0$, so

$$G(x, x') = a_-(x') \mathrm{e}^{x} + b_-(x') \mathrm{e}^{-x}.$$

The boundary condition $\lim_{x\to-\infty} G(x, x') = 0$ requires that $b_-(x') = 0$. Similarly for $x > x'$,

$$G(x, x') = b_+(x') \mathrm{e}^{x} + a_+(x') \mathrm{e}^{-x},$$

where $\lim_{x\to\infty} G(x, x') = 0$ implies $b_+(x') = 0$.

(c) If $x < x'$, then the shift-invariance of G implies $G(x + t, x' + t) = G(x, x')$ and

$$\begin{aligned} a_-(x' + t) \mathrm{e}^{x+t} &= a_-(x') \mathrm{e}^{x} \\ a_-(x' + t) &= a_-(x') \mathrm{e}^{-t} \\ a_-(t) &= c_- \mathrm{e}^{-t}, \quad c_- = a_-(0). \end{aligned}$$

Similarly, $a_+(t) = c_+ \mathrm{e}^{t}$.

(d) Notice that

$$f(x) = \mathscr{G} L f(x) = c_+ \int_{-\infty}^{x} \mathrm{e}^{x'-x} f''(x') \,\mathrm{d}x' + c_- \int_{x}^{\infty} \mathrm{e}^{x-x'} f''(x') \,\mathrm{d}x' - \mathscr{G} f(x).$$

Using integration by parts,

$$\begin{aligned} \int_{-\infty}^{x} \mathrm{e}^{x'-x} f''(x') \,\mathrm{d}x' &= \Big[\mathrm{e}^{x'-x} f'(x')\Big]_{-\infty}^{x} - \Big[\mathrm{e}^{x'-x} f(x')\Big]_{-\infty}^{x} + \int_{-\infty}^{x} \mathrm{e}^{x'-x} f(x') \,\mathrm{d}x' \\ \int_{x}^{\infty} \mathrm{e}^{x-x'} f''(x') \,\mathrm{d}x' &= \Big[\mathrm{e}^{x-x'} f'(x')\Big]_{x}^{\infty} - \Big[-\mathrm{e}^{x-x'} f(x')\Big]_{x}^{\infty} + \int_{x}^{\infty} \mathrm{e}^{x-x'} f(x') \,\mathrm{d}x' \end{aligned}$$

So the jump condition is $f(x) = (c_- - c_+) f'(x) + (-c_- - c_+) f(x)$, which yields $c_- = c_+ = -\frac{1}{2}$. Therefore, the Green's function is $G(x, x') = -\frac{1}{2} \mathrm{e}^{-|x-x'|}$.

8. *For the ODE*

$$\begin{cases} u_t &= u - v^2 \\ v_t &= v - u^2 \end{cases}$$

do all of the following:

(a) *Find all stationary points.*

(b) *Analyze their type.*

(c) *Show that $u = v$ is an invariant set for this ODE; i.e., if $u(0) = v(0)$, then $u(t) = v(t)$ for all t.*

(d) *Draw the phase plane for this system.*

(a) The stationary points satisfy $u = v^2$ and $v = u^2$, they are $(0, 0)$ and $(1, 1)$.

(b) Linearize the system about a stationary point:

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 1 & -2v \\ -2u & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

At $(0, 0)$, $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with eigenvalues $1, 1$. So $(0, 0)$ is an unstable proper node.

At $(1, 1)$, $J = \left(\begin{smallmatrix} 1 & -2 \\ -2 & 1 \end{smallmatrix}\right)$ with eigenvalues $3, -1$ and eigenvectors $\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$. So $(1, 1)$ is a saddle node.

(c) Notice that $|u_t - v_t| = |u - v^2 - v + u^2| = |u - v||1 + u + v|$. If $u(0) = v(0)$, we have by Gronwall's inequality

$$|u(t) - v(t)| \;=\; \int_0^t |u(s) - v(s)||1 + u(s) + v(s)|\, \mathrm{d}s \;\leq\; 0, \qquad t \geq 0.$$

(d)