Qualifying Exam, Winter 2005

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NUMBER ON YOUR EXAM.

Do all 7 problems.

Problems 1-3 are worth 5 points; problems 4-7 are worth 10 points.

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[1] **(5 Pts.)** Let $f(x) = \cos(x) - x$.

(a) Prove that $f(x)$ has exactly one root in the interval $[0, \frac{\pi}{2}]$.

(b) Give a good estimate of the minimum number of bisection iterations required to obtain an approximation that is within $10^{-6}\left(\frac{\pi}{2}\right)$ of this root when the initial interval used is $[0, \frac{\pi}{2}]$.

[2] **(5 Pts.)** Let I_h be the composite trapezoidal rule approximation to the integral $\int_0^1 f(s)\,ds$ using N panels of size h (e.g. $h = \frac{1}{N}$).

(a) Give a derivation of the formula that combines I_h and $I_{\frac{h}{2}}$ to obtain an approximation to the integral that is fourth order accurate.

(b) When the trapezoidal method is applied to the function $f(x) = x^{\frac{3}{2}}$ the rate of convergence is ≈ 1.7. What is the expected rate of convergence when the formula you derived in **(a)** is applied to $f(x) = x^{\frac{3}{2}}$?

[3] **(5 Pts.)** Let A be an n $\times$ n non-singular matrix, and consider iterative methods of the form

$$M x^{n+1} = b + N x^n$$

where A = M - N

(a) Assuming M is non-singular, state a sufficient condition that insures convergence of the iterates to the solution of Ax=b for any starting vector x^0.

(b) Describe the matrices M and N for (i) Jacobi iteration and (ii) Gauss-Seidel iteration

(c) If A is strictly diagonally dominant, prove that Jacobi's method converges.

[4] **(10 Pts.)** Consider the following finite difference scheme for solving $y' = f(y)$

$$y_{n+1} = y_n + hf\left((1-\theta)y_n + \theta y_{n+1}\right),$$

where $\theta \in [0,1]$ is a parameter.

(a) Find the order of the scheme, for $\theta \in [0,1]$.

(b) Determine the region of linear stability.

(c) Determine all the values of $\theta \in [0,1]$ for which the method is A-stable.

[5] **(10 Pts.)** Consider the equation

$$u_{tt} = u_{xx} + u_x$$

to be solved for $t > 0,\ 0 \le x \le 1$.

(a) Give initial data and boundary data that make this a well posed problem. Do not assume periodicity in x.

(b) Give a stable and convergent finite difference approximation to this initial-boundary value problem. Justify your answers.

[6] **(10 Pts.)** Consider the equation

$$u_t = u_{xx} + u_{yy} + 2au_{xy},$$

where a is real number, to be solved for $0 \le x \le 1,\ 0 \le y \le 1,\ t \ge 0$, with initial data $u(x,y,0) = u_0(x,y)$ and periodicity in x and y: $u(x+1,y,t) \equiv u(x,t)$, $u(x,y+1,t) \equiv u(x,y,t)$.

(a) For which values of a would you expect good behavior of the solution?

(b) Write a stable and convergent finite difference approximation to this problem. Justify your answers.

[7] **(10 Pts.)** Consider the boundary value problem

$$\begin{aligned} \Delta u + u &\quad f(x,y), \quad (x,y) \in \Omega \quad [0,1] \times [0,1], \\ u &\quad 0 \text{ for } (x,y) \in \partial\Omega, \ x \quad 0, \\ u_y &\quad 0 \text{ for } (x,y) \in \partial\Omega, \ y = 0, \ 1. \end{aligned}$$

(a) Give a weak variational formulation of the problem.

(b) Analyze the existence and uniqueness of the solution to this problem. Justify your answers (assume $f \in L^2(\Omega)$).

(c) Formulate a finite element approximation of the elliptic problem using piecewise-linear elements. Discuss the form and properties of the stiffness matrix and the existence and uniqueness of the solution of the linear system thus obtained. Justify your answers.