

Math 269B, 2012 Winter, Final

Professor Joseph Teran

Jeffrey Lee Hellrung, Jr.

March 14, 2012

1 Theory

1. Suppose $u: [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the inviscid Burger's equation,

$$0 = u_t + \frac{1}{2} (u^2)_x = u_t + uu_x, \quad u(0, x) = u_0(x). \quad (1)$$

Use the method of characteristics to show that u must then satisfy the implicit relation

$$u(t, x) = u_0(x - tu(t, x)). \quad (2)$$

[Hint: Begin by defining $\tilde{u}(t, X) := u(t, \varphi(t, X))$ for some to-be-determined change of variables $\varphi: [0, \infty) \times \{X\} \rightarrow \{x\}$, and choose φ such that $\tilde{u}_t \equiv 0$.]

2. Suppose u'_0 in (1) is bounded below, i.e., $u'_0 \geq c$ for some constant c . Determine the maximal T such that a solution to (2) is guaranteed to exist for $t \in [0, T)$ (possibly with $T = \infty$). [Hint: Determine when one can guarantee that the function $u \mapsto u - u_0(x - tu)$ has a root.]

3. Solve (1) for $u_0(x) = ax + b$, where a, b are constants. [Hint: Use (2).]

4. Denote the solution to (1) by $u = F[u_0]$. Express $F[x \mapsto au_0(x) + b]$ in terms of $F[u_0]$.

In other words, given u satisfying (1) for some u_0 , determine the solution v (in terms of the aforementioned u) to

$$v_t + vv_x = 0, \quad v(0, x) = v_0(x) := au_0(x) + b.$$

5. Suppose u_0 is given as

$$u_0(x) := \begin{cases} u_0^L(x) := a_L x + b_L, & x < 0 \\ u_0^R(x) := a_R x + b_R, & x > 0 \end{cases}.$$

Determine the path $t \mapsto (t, x_S(t))$ of the (physically correct) shock in the solution u to (1) emanating from $(t, x) = (0, 0)$. You may use the fact that

$$\frac{1}{2} \int \frac{b_L + b_R + (a_L b_R + a_R b_L)t}{((1 + a_L t)(1 + a_R t))^{3/2}} dt = \frac{(a_L b_R - a_R b_L)t + (b_R - b_L)}{(a_R - a_L) \sqrt{(1 + a_L t)(1 + a_R t)}} \quad [a_L \neq a_R].$$

[Hint: Recall that the shock speed $x'_S(t) = \frac{1}{2} (u^L + u^R)(t, x_S(t))$, thus allowing you to set up an ordinary differential equation for x_S .] Consider and explain the physical significance of the special cases $a_L = a_R$ and $b_L = b_R$.

6. Solve the weak form of (1) (i.e., give the entropy solution with rarefaction, and with any shocks propagating at the physically correct speed) on the *periodic* domain $[0, 4]$ with the “pulse” initial condition

$$u_0(x) := \begin{cases} 0, & 0 \leq x < 1 \\ 2, & 1 < x < 2 \\ 0, & 2 < x \leq 4 \end{cases}. \quad (3)$$

Identify key points in time t when the character of the solution changes. (It will be natural to express the solution $u(t, x)$ piecewise with respect to x and t .) Confirm that $\int u(t, x) dx$ is conserved (i.e., $\int u(t, x) dx = \text{constant}$ for all t), and determine $\lim_{t \rightarrow \infty} u(t, x)$.

7. (Strikwerda 6.3.9.) Consider a scheme for (6.1.1), $u_t = bu_{xx}$, of the form

$$v_m^{n+1} = (1 - 2\alpha_1 - 2\alpha_2) v_m^n + \alpha_1 (v_{m+1}^n + v_{m-1}^n) + \alpha_2 (v_{m+2}^n + v_{m-2}^n).$$

Show that when μ is constant, as k and h tend to zero, the scheme is inconsistent unless

$$\alpha_1 + 4\alpha_2 = b\mu.$$

Show that the scheme is fourth-order accurate in x if $\alpha_2 = -\alpha_1/16$.

2 Programming

1. Implement the following numerical schemes to solve (1) on the *periodic* domain $[0, 4]$:

- Godunov’s method. At time level n , solve the Riemann problem assuming a piecewise constant initial condition v^n , then resample to determine v^{n+1} .
- (Backward) Semi-Lagrangian. At time level $n + 1$ and grid vertex m , trace the characteristic $t \mapsto x_m + v_m^n (t - t_{n+1})$ *backward* to time level n and linearly interpolate v^n to determine v_m^{n+1} .
- (Forward) Semi-Lagrangian. Trace the characteristics $t \mapsto x_m + v_m^n (t - t_n)$ *forward* to time level $n + 1$ and linearly interpolate the nearest characteristics at a given grid vertex m to determine v_m^{n+1} .
- (Conservative) Lax-Friedrichs. Discretize the conservative form of (inviscid) Burger’s equation:

$$\frac{1}{k} \left(v_m^{n+1} - \frac{1}{2} (v_{m+1}^n - v_{m-1}^n) \right) + \frac{1}{2} \cdot \frac{1}{2h} \left((v_{m+1}^n)^2 - (v_{m-1}^n)^2 \right) = 0$$

- (Advective) Lax-Friedrichs. Discretize the advective form of (inviscid) Burger’s equation:

$$\frac{1}{k} \left(v_m^{n+1} - \frac{1}{2} (v_{m+1}^n - v_{m-1}^n) \right) + v_m^n \frac{1}{2h} (v_{m+1}^n - v_{m-1}^n) = 0$$

Use the initial condition (3). For those schemes that appear to converge to the exact solution (derived previously), compute a numerical convergence rate. For those schemes that don’t appear to converge to the exact solution, explain the discrepancy (e.g., incorrect rarefaction, non-physical shock speed, unstable). Which scheme do you think performs best for the given initial condition?

2. Use your implementation of the Thomas algorithm from Homework 4 to solve *periodic* tridiagonal systems:

$$a_i w_{i-1} + b_i w_i + c_i w_{i+1}, \quad i = 1, \dots, m,$$

with $w_0 = w_m$ and $w_{m+1} = w_1$. The following algorithm is described in Strikwerda. First, solve the following (non-periodic) tridiagonal systems:

$$\begin{aligned} a_i x_{i-1} + b_i x_i + c_i x_{i+1} &= d_i, & x_0 &= 0 \text{ and } x_{m+1} = 0; \\ a_i y_{i-1} + b_i y_i + c_i y_{i+1} &= 0, & y_0 &= 1 \text{ and } y_{m+1} = 0; \\ a_i z_{i-1} + b_i z_i + c_i z_{i+1} &= 0, & z_0 &= 0 \text{ and } z_{m+1} = 1; \end{aligned}$$

for $i = 1, \dots, m$. Then w_i is given by

$$w_i = x_i + r y_i + s z_i$$

where

$$\begin{aligned}r &:= \frac{1}{D} (x_m (1 - z_1) + x_1 z_m), \\s &:= \frac{1}{D} (x_m y_1 + x_1 (1 - y_m)), \\D &:= (1 - y_m) (1 - z_1) - y_1 z_m.\end{aligned}$$