Basic Examination September, 2005

Do all problems

1. A real number α is said to be *algebraic* if for some finite set of integers $a_0, \ldots, a_n$, not all 0,
$$a_0 + a_1\alpha + \cdots + a_n\alpha^n = 0\,.$$
Prove that the set of algebraic real numbers is countable.

2. State some reasonable conditions on a real-valued function $f(x, y)$ on $\mathbb{R}^2$ which guarantee that $\dfrac{\partial^2 f}{\partial x dy} = \dfrac{\partial^2 f}{\partial y \partial x}$ at every point of $\mathbb{R}^2$. Then prove that your conditions do in fact guarantee this equality.

3. (a) Prove that if $f_j : [0,1] \to \mathbb{R}$ is a sequence of continuous functions which converges uniformly on $[0,1]$ to a (necessarily continuous) function $F : [0,1] \to \mathbb{R}$ then
$$\int_0^1 F^2(x)dx = \lim \int_0^1 f_j^2(x)dx\,.$$
(b) Give an example of a sequence $f_j : [0,1] \to \mathbb{R}$ of continuous functions which converges to a continuous function $F : [0,1] \to \mathbb{R}$ pointwise and for which
$$\lim \int_0^1 f_j^2(x)dx \quad \text{exists but}$$
$$\lim \int_0^1 f_j^2(x)dx \neq \int_0^1 F^2(x)dx$$
(f_j converges to F "pointwise" means that for each $x \in [0,1]$, $F(x) = \lim f_j(x)$).

4. Suppose $F : [0,1] \to [0,1]$ is a C^2 function with $F(0) = 0$, $F(1) = 0$, and $F''(x) < 0$ for all $x \in [0,1]$. Prove that the arc length of the curve $\{(x, F(x)) : x \in [0,1]\}$ is less than 3. (Suggestion: Remember that $\sqrt{a^2 + b^2} < |a| + |b|$ when you are looking at the arc length formula - and at picture of what $\{(x, f(x)\}$ could look like.)

5. Prove carefully that $\mathbb{R}^2$ is not a (countable) union of sets $S_i, i = 1, 2 \ldots$ with each S_i being a subset of some straight line L_i in $\mathbb{R}^2$.

6. (a) Prove that if P is a real-coefficient polynomial and if A is a real symmetric matrix, then the eigenvalues of $P(A)$ are exactly the numbers $P(\lambda)$, where λ is an eigenvalue of A.
(b) Use part (a) to prove that if A is a real symmetric matrix, then A^2 is nonnegative definite.
(c) Check part (b) by verifying directly that det A^2 and trace(A^2) are nonnegative when A is real symmetric.

7. Let A be a real $n \times m$ matrix. Prove that the maximum number of linearly independent rows of A = the maximum number of linearly independent columns. ("Row rank = column rank").

8. For a real $n \times n$ matrix A, let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be the associated linear mapping. Set $\|A\| = \sup_{\vec{x} \in \mathbb{R}^n} \|, \|\vec{x}\| = 1$ (here $\|\vec{x}\|$ = usual euclidean norm, i.e.

$$\|(x_1, \ldots, x_n)\| = (x_1^2 + \cdots + x_n^2)^{1/2}).$$

   (a) Prove that $\|A + B\| \leq A\| + \|B\|$

   (b) Use part (a) to check that the set M of all $n \times n$ matrices is a metric space if the distance function d is defined by

$$d(A, B) = \|B - A\|.$$

   (c) Prove that M is a complete metric space with this "distance function". (Suggestion: The ijth element of $A = \langle T_A e_j, e_i \rangle$ where $e_i = (0, \ldots, 1 \ldots 0)$, 1 in ith position.)

9. Suppose V_1 and V_2 are subspaces of a finite-dimensional vector space V.

   (a) Show that

$$\dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2) - \dim(\operatorname{span}(V_1, V_2))$$

   where $\operatorname{span}(V_1, V_2)$ is by definition the smallest subspace that contains both V_1 and

   (b) Let $n = \dim V$. Use part (a) to show that, if $k < n$, then an intersection of k subspaces of dimension $n - 1$ always has dimension at least $n - k$. (Suggestion: Do induction on k

10. (a) For each $n = 2, 3, 4, \ldots$, is there an $n \times n$ matrix A with $A^{n-1} \neq 0$ but $A^n = 0$? (Give example or proof of nonexistence.)

    (b) Is there an $n \times n$ upper triangular matrix A with $A^n \neq 0$ but $A^{n+1} = 0$? (Give example or proof of nonexistence.)
    [Note: A square matrix is *upper triangular* if all the entries below the main diagonal are 0.]