Basic Exam (S04)

In several problems you will need the usual "norm" terminology. If V is a real vector space, then a norm on V is a map $\| \, \| : V \to [0, \infty)$ such that $\|v + w\| \leq \|v\| + \|w\|$, $\|cv\| = |c|\|v\|$, and $\|v\| = 0$ if and only if $v = 0$. Each norm determines a metric d on V via the relation $d(v, w) = \|v - w\|$. The Euclidean norm (also called the "inner product" norm) on $\mathbb{R}^n$ is given by

$$\left\|\sum_{k=1}^{n} x_k e_k\right\|_2 = [\sum_{k=1}^{n} |x_k|^2]^{1/2}$$

where e_k is the usual vector basis. Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ we define

$$\|T\| = \sup\{\|T(x)\|_2 : \|x\|_2 \leq 1\}$$

For all x, $\|T(x)\| \leq \|T\|\|x\|$.

1. Let S denote the set of sequences $a = (a_1, a_2, \quad)$, with $a_k \quad 0$ or 1 Show that the mapping $\theta : S \to \mathbb{R}$ defined by

$$\theta((a_1, a_2, \quad)) = \frac{a_1}{10} + \frac{a_2}{10^2} +$$

is an injection. Include an explanation of why the infinite series converges. Hint: if $a \neq b$, you may assume that

$$\begin{aligned} a &= (a_1, \quad , a_{n-1}, 0, a_{n+1}, . \\ b &= (a_1, \quad , a_{n-1}, 1, b_{n+1}, . \end{aligned}$$

2. Is $f(x) = \sqrt{x}$ uniformly continuous on $[0, \infty)$? Prove your assertion.

3. a) Carefully define when a function f on $[0, 1]$ is Riemann integrable.
b) Show that if f_n are Riemann integrable functions on $[0, 1]$ and f_n converges to f uniformly, then f is Riemann integrable.

4. Are there infinite compact subsets of $\mathbb{Q}$? Prove your assertion.

5. Suppose that G is an open set in $\mathbb{R}^n$, $f : G \to \mathbb{R}^m$ is a function, and that $x_0 \in G$.
a) Carefully define what is meant by $f'(x_0) : \mathbb{R}^n \to \mathbb{R}^m$.
b) Suppose that I is a line segment in G such that $f'(x)$ is defined for all $x \in I$. Show that if f is differentiable at all the points of I, then for some point c in I

$$\|f(q) - f(p)\|_2 \leq \|f'(c)\| \, \|q - p\|_2$$

let w be a unit vector with $||f(q) \quad f(p)||_2 \qquad f(q) \quad f(p)) \cdot u$

6. Let $|| \quad |$ be any norm on $\mathbb{R}^n$.

a) Prove that there exists a constant d with $||x| \leq d\,||x||_2$ for all $x \in \mathbb{R}^n$, and use this to show that $N(x) = \ |x|$ is continuous in the usual topology on $\mathbb{R}^n$.

b) Prove that there exists a constant c with $||x|| \geq c\,||x||_2$ (Hint: use the fact that N is continuous on the sphere $\{x : ||x||_2 = 1\}$).

c) Show that if L is an n-dimensional subspace of an arbitrary normed vector space V, then L is closed.

7. Let V be a finite dimensional real vector space Let $W \quad W_2 \subset V$ be subspaces. Show both of the following:

a) $W_1^0 \cap W_2^0 = (W_1 + W_2)^0$

b) $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$

[Note: W_i^0 is the annihilator of W_i.]

8. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a rotation about the axis $(1, 0 - 1)$ by an angle of 30^o (you can use either orientation).

a) Find the matrix representation $A \in \mathbf{M}_3(\mathbf{R})$ of T in the standard basis. (You do not have to multiply out matrices but must evaluate inverses.)

b) Find all the eigenvalues of $A \in \mathbf{M}_3(\mathbf{R})$.

c) Find all the eigenvalues of $A \in \mathbf{M}_3(\mathbf{C})$.

9. Let V be a finite dimensional real inner product space under (, and $V \to V$ a linear operator. Show the following are equivalent:

a) $(Tx, Ty) = (x, y)$ for all $x, y \in V$.

b) $||T(x)|| = ||x||$ for all $x \in V$.

c) $T^*T = Id_V$, where T^* is the adjoint of T.

d) $TT^* = Id_V$.

10. Let T be a real symmetric matrix Show that T is similar to a diagonal matrix.

[You cannot use the Spectral Theorem.]