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Analysis Bootcamp, Homework 3

1. *Gamelin and Greene, page 45. Problems 1, 4, 5.*

   1. *Show that if Φ is a function from a nonempty compact metric space X to itself such that*

   $$d(\Phi(x), \Phi(y)) < d(x, y), \ \ x \neq y,$$

   then Φ has a unique fixed point.

   Solution

   Suppose that $x, y \in X$ are such that $\Phi(x) = x$ and $\Phi(y) = y$. Then, if $x \neq y$,

   $$d(x, y) = d(\Phi(x), \Phi(y)) < d(x, y),$$

   a contradiction, hence $x = y$. This establishes uniqueness.

   Note that Φ is continuous, hence $\Psi : X \to \mathbb{R}$ defined by

   $$\Psi(x) = d(x, \Phi(x))$$

   is a continuous function on a compact metric space, hence, by Exercise 6.8, assumes its minimum value, say, m, at $x = x^*$. Now if $m > 0$, we have that

   $$\Psi(\Phi(x^*)) < \Psi(x^*) = m,$$

   contradicting the fact that m is the minimum of Ψ. Hence $m = 0$ and $\Phi(x^*) = x^*$, establishing existence.

   4. *Let $\mathfrak{X}$ be a Banach space, let $m \geq 1$, and let T be a continuous linear operator on $\mathfrak{X}$ such that $\|T^m\| < 1$. Fix $u \in \mathfrak{X}$ and define*

   $$\Phi(v) = u + T(v), \ \ v \in \mathfrak{X}.$$

   (a) Show that Φ^m is a contraction.

   (b) Show that the equation

   $$v = u + T(v)$$

   has a unique solution $v \in \mathfrak{X}$.

   Solution

   (a) Let $v, w \in \mathfrak{X}$. Then

   $$d(\Phi(v), \Phi(w)) = \|T(v - w)\| \leq \|T\|\|v - w\| = \|T\|d(v, w),$$

   hence

   $$d(\Phi^m(v), \Phi^m(w)) \leq \|T\|d(\Phi^{m-1}(v), \Phi^{m-1}(w)) \leq \ldots \leq \|T\|^m d(v, w) < d(v, w),$$

   therefore Φ^m is a contraction.

   (b) Suppose v_1 and v_2 both satisfy $v = u + T(v)$. Then both v_1 and v_2 are fixed points of Φ, hence are fixed points of Φ^m. By Theorem 8.1 (Contraction Mapping Principle), since Φ^m is a contraction, $v_1 = v_2$. This establishes uniqueness.

   Let v be a fixed point of Φ^m. Then

   $$\Phi^m(\Phi(v)) = \Phi(\Phi^m(v)) = \Phi(v),$$

   hence $\Phi(v)$ is a fixed point of Φ^m as well, and by uniqueness, $v = \Phi(v)$, hence v is a fixed point of Φ as well. This establishes existence.

5. *Let $\mathfrak{X}$ be a Banach space and let T be a continuous linear operator on $\mathfrak{X}$. Let $u \in \mathfrak{X}$, let $\lambda \in \mathbb{C}$, and define $\Phi(v) = (u/\lambda) + T(v/\lambda)$, $v \in \mathfrak{X}$.*

   (a) By applying the Contraction Mapping Theorem to Φ, show that the equation

$$\lambda v = u + T(v)$$

   has a unique solution $v \in \mathfrak{X}$, provided $|\lambda| > \|T\|$.

   (b) Let $v_0 = 0$ and $v_m = \Phi(v_{m-1})$, $m \geq 1$. Show that

$$v_m = \sum_{k=0}^{m+1} \frac{T^k(u)}{\lambda^{k+1}}.$$

   (c) Show that if $|\lambda| > \|T\|$, then the series

$$\sum_{k=0}^{\infty} T^k/\lambda^{k+1}$$

   converges in norm to $(\lambda I - T)^{-1}$. How can this be used to solve (8.14)?

   (d) Show that if

$$|\lambda| > \limsup_{n\to\infty} \|T^n\|^{1/n},$$

   then the series (8.15) converges in norm to $(\lambda I - T)^{-1}$.

Solution

(a) Let $v, w \in \mathfrak{X}$. Then

$$d(\Phi(v), \Phi(w)) = \|T((v-w)/\lambda)\| \leq \|T\|\|v-w\|/|\lambda| \leq (\|T\|/|\lambda|)d(v,w) < d(v,w),$$

hence Φ is a contraction, so, by Theorem 8.1 (Contraction Mapping Principle), Φ has a unique fixed point v^*:

$$v^* = \Phi(v^*) = u/\lambda + T(v^*/\lambda) \;\Rightarrow\; \lambda v^* = u + T(v^*).$$

(b) We have that

$$v_1 = \Phi(v_0) = \Phi(0) = u/\lambda = \sum_{k=0}^{0} \frac{T^k(u)}{\lambda^{k+1}},$$

and given that

$$v_j = \sum_{k=0}^{j-1} \frac{T^k(u)}{\lambda^{k+1}},$$

for $j \geq 1$, we see that

$$v_{j+1} = \Phi(v_j) = \frac{u}{\lambda} + \frac{1}{\lambda}T(v_j) = \frac{u}{\lambda} + \frac{1}{\lambda}T\left(\sum_{k=0}^{j-1} \frac{T^k(u)}{\lambda^{k+1}}\right) = \frac{u}{\lambda} + \sum_{k=0}^{j-1} \frac{T^{k+1}(u)}{\lambda^{k+2}} = \sum_{k=0}^{j} \frac{T^k(u)}{\lambda^{k+1}},$$

which shows the claim by induction on m.

(c) Let

$$S_m = \sum_{k=0}^{m} \frac{T^k}{\lambda^{k+1}}.$$

We compute

$$(\lambda I - T)S_m = \sum_{k=0}^{m} \frac{T^k}{\lambda^k} - \sum_{k=0}^{m} \frac{T^{k+1}}{\lambda^{k+1}} = I - \frac{T^{m+1}}{\lambda^{m+1}},$$

hence

$$\|(\lambda I - T)S_m - I\| = \left\| \frac{T^{m+1}}{\lambda^{m+1}} \right\| \leq \left(\frac{\|T\|}{|\lambda|} \right)^{m+1} \to 0$$

as $m \to \infty$, so $S_m \to (\lambda I - T)^{-1}$ in norm as $m \to \infty$. Thus

$$(\lambda I - T)^{-1} = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}.$$

Going back to (8.14),

$$\lambda v = u + T(v),$$

we see that we wish to solve v such that

$$(\lambda I - T)(v) = u,$$

hence in this case,

$$v = (\lambda I - T)^{-1}(u).$$

(d) Given

$$|\lambda| > \limsup_{n\to\infty} \|T^n\|^{1/n},$$

then

$$|\lambda| > \|T^n\|^{1/n}$$

for all n sufficiently large. Thus

$$\frac{\|T^n\|^{1/n}}{|\lambda|} < 1$$

for all n sufficiently large, and

$$\frac{\|T^n\|}{|\lambda|^n} = \left(\frac{\|T^n\|^{1/n}}{|\lambda|} \right)^n \to 0$$

as $n \to \infty$. It follows that, as before, $S_m \to (\lambda I - T)^{-1}$ as $m \to \infty$.

2. *Gamelin and Greene, page 55. Problems 1, 2, 3, 8, 9.*

1. *Define* $G : \mathbb{R}^2 \to \mathbb{R}^1$ *by* $G(0,0) = 0$ *and* $G(x,y) = xy/(x^2+y^2)^{1/2}$ *if* $(x,y) \neq (0,0)$. *Show that* G *is continuous and the partial derivatives of* G *with respect to* x *and* y *exist everywhere, but* G *is not Frechet differentiable at* $(0,0)$.

Solution

G is certainly continuous off $(0,0)$. Around $(0,0)$, change to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. Then

$$G(x,y) = \frac{r^2 \cos\theta \sin\theta}{r} \to 0$$

as $r \to 0$, so since $G(0,0) = 0$, G is continuous everywhere.

One can compute that, off $(0,0)$,

$$G_1(x,y) = \frac{y^3}{(x^2+y^2)^{3/2}},$$

$$G_2(x,y) = \frac{x^3}{(x^2+y^2)^{3/2}},$$

both of which certainly exist. At $(0,0)$,

$$G_1(0,0) = \lim_{\delta \to 0} \frac{G(\delta,0) - G(0,0)}{\delta} = \lim_{\delta \to 0} \frac{0-0}{\delta} = 0,$$

and similarly $G_2(0,0) = 0$ as well. Therefore, the partial derivatives of G exist everywhere. Each $T \in \mathcal{B}(\mathbb{R}^2, \mathbb{R}^1)$ can be represented by

$$T(x,y) = T_1 x + T_2 y$$

for $T_1, T_2 \in \mathbb{R}$. Hence suppose $T = G'(0,0)$ existed. Then T satisfies

$$\lim_{(x,y)\to(0,0)} \frac{G(x,y) - G(0,0) - T(x,y)}{\|(x,y)\|} = 0.$$

The quotient simplifies to

$$\frac{xy - (T_1 x + T_2 y)\sqrt{x^2+y^2}}{x^2+y^2},$$

and changing to polar coordinates again, this simplifies further to

$$\cos\theta \sin\theta - T_1 \cos\theta - T_2 \sin\theta = \frac{1}{2}\sin(2\theta) - T_1 \cos\theta - T_2 \sin\theta,$$

which is never identically zero no matter the choice of T_1, T_2. Hence the limit as $r \to 0$ of the above expression, if it even exists, is, at the least, nonzero, meaning no such T can exist, as originally supposed. Therefore, G is not Frechet differentiable at $(0,0)$.

2. *Prove that if $G = (G_1, \ldots, G_m)$ maps $\mathbb{R}^n$ to $\mathbb{R}^m$ and if the partial derivatives $\partial G_j/\partial x_k$, $1 \le j \le m$, $1 \le k \le n$, exist everywhere and are continuous, then G is continuously differentiable in the sense of Frechet.*

Solution

Each $T \in \mathcal{B}(\mathbb{R}^n, \mathbb{R}^m)$ can be represented by a matrix (T_{jk}), $1 \le j \le m$, $1 \le k \le n$. Let $x_0 \in \mathbb{R}^n$ be given. Now if there exists a T such that

$$\lim_{x \to x_0} \frac{G(x) - G(x_0) - T(x - x_0)}{\|x - x_0\|} = 0,$$

then the corresponding quotients for each of the components of G,

$$\frac{G_j(x) - G_j(x_0) - T_{j\cdot}(x - x_0)}{\|x - x_0\|},$$

also tend to 0 as $x \to x_0$, and, conversely, if the component quotients tend to 0, the original quotient does as well. Further, $T(x_0)$ is continuous with respect to x_0 if and only if each component $T_{j\cdot}(x_0)$ is continuous with respect to x_0. Thus, without loss of generality, we may assume $m = 1$, i.e., $G = G_j$ and $T = T_{j\cdot}$ for some fixed j.

Without loss of generality, we may further assume $x_0 = 0$ and $G(0) = 0$. Let $x = \sum_{i=1}^n x_i e_i$, and set $y_0 = x_0 = 0$, $y_k = y_{k-1} + e_k x_k = \sum_{i=1}^k e_i x_i$, $1 \le k \le n$. Treating G as a mapping from $\mathbb{R} \to \mathbb{R}$ by varying only the k^{th} component, we obtain, from (9.2),

$$G(y_k) = G(y_{k-1}) + G_k(y_{k-1}) x_k + R_k(x_k),$$

where R_k is such that

$$\lim_{x_k \to 0} \frac{R_k(x_k)}{|x_k|} = 0.$$

We can rewrite this as

$$G(y_k) = G(y_{k-1}) + (G_k(0) - G_k(0) + G_k(y_{k-1}))x_k + R_k(x_k)$$
$$= G(y_{k-1}) + G_k(0)x_k + (G_k(y_{k-1}) - G_k(y_0))x_k + R_k(x_k)$$
$$= G(y_{k-1}) + G_k(0)x_k + R'_k(x),$$

where

$$R'_k(x) = (G_k(y_{k-1}) - G_k(0))x_k + R_k(x_k).$$

Now since $y_{k-1} \to 0$ as $x \to 0$, by the continuity of G_k, $G_k(y_{k-1}) \to G_k(0)$ as $x \to 0$, hence

$$\lim_{x\to 0} \left| \frac{R'_k(x)}{\|x\|} \right| \leq \lim_{x\to 0} \left| \frac{(G_k(y_{k-1}) - G_k(0))x_k + R_k(x_k)}{|x_k|} \right| = 0.$$

Using the recurrence relation of the $G(y_k)$'s, we can then write

$$G(x) = G(y_n) = \sum_{k=1}^{n} G_k(0)x_k + \sum_{k=1}^{n} R'_k(x),$$

where

$$\lim_{x\to 0} \frac{\sum_k R'_k(x)}{\|x\|} = 0,$$

from which it follows that

$$G'(0) = \left(G_1(0) \ \cdots \ G_n(0)\right),$$

and $G'(0)$ exists. Indeed, there was nothing special concerning 0, hence $G'(x)$ exists everywhere, and

$$G'(x) = \left(G_1(x) \ \cdots \ G_n(x)\right).$$

Since each $G_k(x)$ depends continuously on x, $G'(x)$ also depends continuously on x.

3. *Prove that if $F : \mathfrak{X} \to \mathfrak{Y}$ is differentiable at $x_0 \in \mathfrak{X}$ and if $G : \mathfrak{Y} \to \mathfrak{Z}$ is differentiable at $y_0 = F(x_0) \in \mathfrak{Y}$, then $G \circ F$ is differentiable at x_0 and*

$$(G \circ F)'(x_0) = G'(F(x_0))F'(x_0).$$

Solution

Choose $x \in X$ and set $y = F(x) \in Y$. (9.2) allows us to write

$$G(y) = G(y_0) + G'(y_0)(y - y_0) + R_Y(y),$$

where

$$\lim_{y\to y_0} \frac{R_Y(y)}{\|y - y_0\|} = 0.$$

We can evaluate $y - y_0$, again using (9.2), as

$$y - y_0 = F(x) - F(x_0) = F'(x_0)(x - x_0) + R_X(x),$$

where

$$\lim_{x\to x_0} \frac{R_X(x)}{\|x - x_0\|} = 0.$$

Substituting and expressing everything in x yields

$$G(F(x)) = G(F(x_0)) + G'(F(x_0))F'(x_0)(x - x_0) + R_Z(x)$$

where

$$R_Z(x) = G'(y_0)R_X(x) + R_Y(F(x)).$$

Now $y \to y_0$ whenever $x \to x_0$, by the continuity of F (implied by its differentiability, by Theorem 9.1). Further, since

$$y - y_0 = F'(x_0)(x - x_0) + R_X(x),$$

we have that

$$\|y - y_0\| \leq \|F'(x_0)\|\|x - x_0\| + \epsilon\|x - x_0\|$$

for some $\epsilon > 0$ and x sufficiently close to x_0. Hence

$$\lim_{x \to x_0} \left| \frac{R_Y(y)}{\|x - x_0\|} \right| \leq \lim_{x \to x_0} \left| \frac{R_Y(y)}{\|y - y_0\|} \right| (\|F'(x_0)\| + \epsilon) = 0,$$

and we see immediately that

$$\lim_{x \to x_0} \frac{R_Z(x)}{\|x - x_0\|} = 0.$$

Therefore,

$$(G \circ F)'(x_0) = G'(F(x_0))F'(x_0).$$

8. *Let U be an open subset of a Banach space $\mathfrak{X}$ and let F be a continuously differentiable function from U to a Banach space $\mathfrak{Y}$ such that $F'(x)$ is invertible for all $x \in U$. Show that F is an open map, that is, that the image under F of any open subset of U is an open subset of $\mathfrak{Y}$.*

   Solution

   Let $V \subset U$ be open, and consider $y \in F(V) \subset \mathfrak{Y}$. Let $x \in V$ such that $F(x) = y$. F' is invertible throughout $\mathfrak{Y}$, hence the hypotheses of Theorem 9.9 (Inverse Function Theorem) are met, and there exists an open neighborhood $W \subset V$ of x such that $F(W)$ is open in $\mathfrak{Y}$. But $y \in F(W) \subset F(V)$, hence y is an interior point of $F(V)$. Since y was arbitrary, $F(V)$ is open, hence F is an open map.

9. *A function F from $\mathbb{R}^l$ to $\mathbb{R}^m$ is a* C^k-function *if each of the component functions of F has continuous partial derivatives of all orders $\leq k$. Suppose that F is a C^k-function ($k \geq 1$) from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^m$ such that $F(0,0) = 0$ and the Jacobian matrix*

$$\left(\frac{\partial F_i}{\partial y_j}(0,0) \right)_{i,j=1}^{m}$$

   is invertible. Prove that there is a C^k-function f, defined in a neighborhood U of 0 in $\mathbb{R}^n$, such that $F(x, f(x)) = 0$, $x \in U$.

   Solution

   We use Theorem 9.8 (Implicit Function Theorem), with $\mathfrak{X} = \mathbb{R}^n$, $\mathfrak{Y} = \mathbb{R}^m$, $\mathfrak{Z} = \mathbb{R}^m$, and $(x_0, y_0) = (0, 0)$. This allows us to obtain a continuously differentiable function $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ for open U containing 0, with derivative at $0 \in \mathbb{R}^n$ of

$$f'(0) = -F_2(0,0)^{-1}F_1(0,0).$$

   Evidently, we can apply the Implicit Function Theorem at any point $(x, f(x))$ in a neighborhood of $(0,0)$ where the Jacobian is invertible, obtaining the derivative of f to be

$$f'(x) = -F_2(x, f(x))^{-1}F_1(x, f(x)).$$

   Since F_2 and hence F_2^{-1} are C^{k-1}, as is F_1, we see that in fact f is C^k in a sufficiently small neighborhood of 0.