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Basic Examination, W02

1. (a) State some reasonably general conditions under which this "differentiation under the integral sign" formula is valid:

$$\frac{d}{dx}\int_c^d f(x,y)dy = \int_c^d \frac{\partial f}{\partial x}(x,y)dy.$$

(b) Prove that the formula is valid under the conditions you gave in part (a).

Solution

(a) i. f is defined on $[a,b]\times[c,d]$.

ii. $f(x,\cdot) = f^x \in \mathcal{R}$ for all $x \in [a,b]$.

iii. $D_1 f$ is continuous on $[a,b]\times[c,d]$.

(b) Since $D_1 f$ is continuous on $[a,b]\times[c,d]$, $D_1 f$ is uniformly continuous, hence given an $\epsilon > 0$, there exists a $\delta > 0$ such that, in particular,

$$|D_1 f(x,y) - D_1 f(t,y)| < \epsilon$$

whenever $y \in [c,d]$ and $x, t \in (a,b)$ with $|x-t| < \delta$. Fix $x \in (a,b)$ and let

$$g_x(t,y) = \frac{f(x,y) - f(t,y)}{x-t}$$

for $t \in (a,b)$, $y \in [c,d]$. Now to each such t, the Mean Value Theorem provides a $u = u_y(t)$ between x and t such that

$$g_x(t,y) = D_1 f(u,y).$$

If we restrict $|x-t| < \delta$, then the uniform continuity of $D_1 f$ implies that

$$|g_x(t,y) - D_1 f(x,y)| < \epsilon,$$

for all $y \in [c,d]$. Thus $g_x(t,\cdot) \to D_1 f(x,\cdot)$ uniformly on $[c,d]$ as $t \to x$. Now if we set

$$I(s) = \int_c^d f(s,y)dy,$$

then

$$\frac{I(x)-I(t)}{x-t} = \int_c^d g_x(t,y)dy$$

hence

$$\frac{d}{dx}\int_c^d f(x,y)dy = \lim_{t\to x}\frac{I(x)-I(t)}{x-t} = \lim_{t\to x}\int_c^d g_x(t,y)dy = \int_c^d \lim_{t\to x} g_x(t,y)dy = \int_c^d D_1 f(x,y)dy,$$

as the limit and the integral may be interchanged since $g_x(t,\cdot) \to D_1 f(x,\cdot)$ uniformly.

We prove, for completeness, that, given $f_n \to f$ uniformly on $[a,b]$, with each $f_n \in \mathcal{R}$, then

$$\int_a^b f(x)dx = \lim_{n\to\infty}\int_a^b f_n(x)dx.$$

To this end, let

$$\epsilon_n = \sup_{[a,b]} |f_n - f|.$$

Then

$$f_n - \epsilon_n \le f \le f_n + \epsilon_n,$$

hence

$$\int_a^b (f_n - \epsilon_n) dx \le \underline{\int_a^b} f dx \le \overline{\int_a^b} f dx \le \int_a^b (f_n + \epsilon_n) dx,$$

so

$$0 \le \overline{\int_a^b} f dx - \underline{\int_a^b} f dx \le \int_a^b 2\epsilon_n dx = 2\epsilon_n(b-a).$$

As $n \to \infty$, $\epsilon_n \to 0$, and the upper and lower integrals of f converge. Further,

$$\left| \int_a^b f dx - \int_a^b f_n dx \right| \le \epsilon_n(b-a),$$

from which the claim follows.

2. Prove that the unit interval $[0, 1]$ is sequentially compact, i.e., that every infinite sequence has a convergent subsequence.

   (Prove this directly. Do not just quote general theorems like Heine-Borel.)

   Solution

   Given a sequence $\{x_n\}_{n=1}^{\infty}$, define $x_n^0 = x_n$ for $n \ge 1$. We begin with the closed unit interval $I_0 = [a, b] = [0, 1]$. Consider the subintervals $L = [a, m]$ and $R = [m, b]$ for $m = (a + b)/2$. Since $L \cup R = I_0$, infinitely many of the points in $\{x_n^0\}_{n=1}^{\infty}$ must lie in either L or R (or both). If L contains infinitely many such points, let $\{x_n^1\}$ be the subsequence of $\{x_n^0\}$ of those points lying in L, and set $I_1 = L$; otherwise, let $\{x_n^1\}$ be the subsequence of $\{x_n^0\}$ of those points lying in R, and set $I_1 = R$. Repeat this process to obtain the subsequence $\{x_n^2\}$ and I_2, and so on. The i^{th} step will generate a subsequence $\{x_n^i\}$ contained within some closed interval I_i, which will have length 2^{-i}.

   As the I_i's are a nested sequence of closed intervals, their intersection must be nonempty. Indeed, the left endpoints of the I_i's form a bounded, monotonically increasing sequence in $[0, 1] \subset \mathbb{R}$, hence have some limit $\ell \in [0, 1]$ (since $[0, 1]$ is closed), and since the left endpoint of any I_i is less than or equal to ℓ, $\ell \in I_i$ for each i (note that, by the nesting property of the I_i's, ℓ must be strictly less than the right endpoint of any I_i), hence $\ell \in \bigcap_i I_i$. Similarly, the limit $r \in [0, 1]$ of the right endpoints is in the intersection, hence we conclude that $[l, r] = \bigcap_i I_i$. As the length of the I_i's tend to zero, we conclude that $\ell = r$ and the intersection consists of precisely one point, $x^* = \ell = r$.

   We now construct a subsequence of $\{x_n\}$ converging to x^* as follows. Select x_{n_i} from $\{x_n^i\}$ and such that $n_i < n_{i+1}$. Since both $x^* \in I_i$ and $x_{n_i} \in I_i$, $|x^* - x_{n_i}| \le 2^{-i} \to 0$ as $i \to \infty$, showing that $x_{n_i} \to x^*$.

3. Prove that open unit ball in $\mathbb{R}^2$

   $$\left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\right\}$$

   is connected.

   (You may assume that intervals in $\mathbb{R}$ are connected. You should not just quote other general results, but give a direct proof.)

   Solution

   Set

   $$U = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\right\}.$$

   We show first that U is path-connected. Indeed, taking two points $p, q \in U$, let

   $$\gamma(t) = p + t(q - p) = (1 - t)p + tq, \ t \in [0, 1].$$

Then

$$\|\gamma(t)\|_2 \leq (1-t)\|p\|_2 + t\|q\|_2 \leq \max\{\|p\|_2, \|q\|_2\} < 1$$

for all $t \in [0,1]$. Hence γ is a path from p to q within U.

Now suppose that U is not connected. Then there exists nonempty subsets A and B such that $U = A \cup B$ but $A \cap \overline{B} = \overline{A} \cap B = \emptyset$. Let $p \in A$ and $q \in B$ and let $\gamma : [0,1] \to U$ be a path from p to q. Set $A' = A \cap \gamma([0,1])$ and $B' = B \cap \gamma([0,1])$. It follows that $\gamma([0,1])$ is not connected, as it can be partitioned into A' and B' such that $A' \cap \overline{B'} = \overline{A'} \cap B = \emptyset$ (in the subspace topology of $\gamma([0,1])$). But $\gamma([0,1])$ is homeomorphic to the unit interval $[0,1]$, which is connected, and homeomorphisms of connected sets are connected. This is a contradiction, hence U must be connected.

4. Prove that the set of irrational numbers in $\mathbb{R}$ is not a countable union of closed sets.

Solution

Suppose $\bigcup_n E_n = \mathbb{R}\backslash\mathbb{Q}$ for some countable sequence $\{E_n\}$ of closed subsets of $\mathbb{R}$. Each E_n must have empty interior (else would contain some of $\mathbb{Q}$, since $\mathbb{Q}$ is dense in $\mathbb{R}$), hence each E_n is nowhere dense. Enumerate $\mathbb{Q}$ by x_n, so that $\bigcup_n \{x_n\} = \mathbb{Q}$. Each $\{x_n\}$ is also nowhere dense, hence by a corollary to the Baire Category Theorem,

$$\left(\bigcup_n E_n\right) \cup \left(\bigcup_n \{x_n\}\right)$$

must have empty interior, as it is a countable union of nowhere dense sets of a complete metric space. Yet the union above is equal to all of $\mathbb{R}$, whose interior is certainly all of $\mathbb{R}$, a contradiction. It follows that $\mathbb{R}\backslash\mathbb{Q}$ cannot be expressed as a countable union of closed sets.

5. (a) Let $f : U \to \mathbb{R}^k$ be a function on an open set U in $\mathbb{R}^n$. Define what it means for f to be differentiable at a point $x \in U$.
   (b) State carefully the Chain Rule for the composition of differentiable functions of several variables.
   (c) Prove the Chain Rule you stated in part (b).

Solution

(a) f is differentiable at $x \in U$ if there exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^k$ such that

$$\lim_{t\to x} \frac{f(t) - f(x) - T(t-x)}{\|t-x\|} = 0,$$

or, equivalently, the "remainder function" R defined by

$$f(t) - f(x) - T(t-x) = R(t)$$

is such that

$$\lim_{t\to x} \frac{R(t)}{\|t-x\|} = 0.$$

We denote T by $f'(x)$.

(b) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$, f differentiable at $x \in \mathbb{R}^n$, and g differentiable at $f(x) \in \mathbb{R}^m$. Then $g \circ f : \mathbb{R}^n \to \mathbb{R}^k$ is differentiable at x, with derivative

$$(g \circ f)'(x) = g'(f(x)) \circ f'(x).$$

(c) Set $y = f(x)$. Then the remainder function $R(t)$ defined by

$$f(t) - f(x) - f'(x)(t-x) = R(t)$$

is such that $R(t)/\|t-x\| \to 0$ as $t \to x$. Also, the remainder function $S(s)$ defined by

$$g(s) - g(y) - g'(y)(s-y) = S(s)$$

is such that $S(s)/\|s-y\| \to 0$ as $s \to y$. If we let $s = f(t)$, then

$$\begin{array}{rcl} (g\circ f)(t) - (g\circ f)(x) & = & g(s) - g(y) \\ & = & g'(y)(s-y) + S(s) \\ & = & g'(f(x))(f(t) - f(x)) + S(f(t)) \\ & = & g'(f(x))(f'(x)(t-x) + R(t)) + S(f(t)) \\ & = & g'(f(x))(f'(x)(t-x)) + g'(f(x))(R(t)) + S(f(t)) \end{array}.$$

Let

$$P(t) = g'(f(x))(R(t)) + S(f(t)).$$

Then

$$\|P(t)\| \le \|g'(f(x))\|\|R(t)\| + \|S(f(t))\|$$

so

$$\lim_{t\to x} \frac{\|P(t)\|}{\|t-x\|} \le \lim_{t\to x} \frac{\|S(f(t))\|}{\|t-x\|}.$$

Now

$$s - y = f'(x)(t-x) + R(t),$$

hence

$$\frac{1}{\|t-x\|} \le \frac{\|f'(x)\| + \|R(t)\|/\|t-x\|}{\|s-y\|},$$

and since $s \to y$ as $t \to x$,

$$\lim_{t\to x} \frac{\|S(f(t))\|}{\|t-x\|} = \lim_{t\to x} \frac{\|S(s)\|}{\|s-y\|}\left(\|f'(x)\| + \frac{\|R(t)\|}{\|t-x\|}\right) = 0,$$

which proves that

$$(g\circ f)'(x) = g'(f(x)) \circ f'(x).$$

6. (a) State some reasonably general conditions on a function $f : \mathbb{R}^2 \to \mathbb{R}$ under which

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

(b) Prove the formula under the conditions you stated.

Solution

(a) i. f is continuous.
ii. The partial derivatives $D_1 f$ and $D_2 f$ exist and are continuous.
iii. The mixed partials $D_{21} f$ and $D_{12} f$ exist and are continuous.

(b) (F01.5)

7. Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is everywhere differentiable and that its first derivative (Jacobian) matrix

$$\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}$$

is continuous everywhere and nonsingular everywhere.

(Here we use the notation $F(x,y) = (F_1(x,y), F_2(x,y)) \in \mathbb{R}^2$.)

Suppose also that

$$\|F((x,y))\| \geq 1 \ \text{ if } \ \|(x,y)\| = 1 \ \text{ and that } \ F((0,0)) = (0,0).$$

Prove that

$$F\left(\left\{(x,y) : x^2 + y^2 < 1\right\}\right) \supset \left\{(x,y) : x^2 + y^2 < 1\right\}.$$

(Hint: With $U = \left\{(x,y) : x^2 + y^2 < 1\right\}$, prove that $F(U) \cap U$ is open and is closed in U.)

Solution

Let $y \in F(U) \cap U$. Then there exists some $x \in U$ such that $F(x) = y$, and $F'(x)$ is nonsingular. The Inverse Function Theorem then gives us open sets $V \subset U$ and $W \subset \mathbb{R}^2$, $x \in V$, $y \in W$, F is one-to-one on V, and $F(V) = W$. Thus $W \cap U \subset F(U) \cap U$ is an open (with respect to U) neighborhood of y, hence $F(U) \cap U$ is open in U.

Now suppose $y \in U$ is a limit point of $F(U)$. Then there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset F(U)$ such that $y_n \to y$. For each n, let $x_n \in U$ such that $F(x_n) = y_n$. Then $\{x_n\}_{n=1}^{\infty}$ is a sequence within $\overline{U}$, a compact set, hence there exists a convergent subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ with $x_{n_i} \to x^* \in \overline{U}$. By the continuity of F, then, $F(x^*) = y$. Now since $F(\partial U) \cap U = \emptyset$, $x^* \notin \partial U$, as $F(x^*) = y \in U$. Thus $x^* \in U$ and $y = F(x^*) \in F(U)$, showing that $F(U) \cap U$ is closed in U.

Finally, $F(U) \cap U \neq \emptyset$, since $F((0,0)) = (0,0)$. Thus, as $F(U) \cap U$ is both open and closed in U, $F(U) \cap U = U$ and $F(U) \supset U$.

8. Let $T : V \to W$ and $S : W \to X$ be linear transformations of finite dimensional real vector spaces. Prove that

$$\operatorname{rank}(T) + \operatorname{rank}(S) - \dim(W) \leq \operatorname{rank}(S \circ T) \leq \min\left\{\operatorname{rank}(T), \operatorname{rank}(S)\right\}.$$

(The rank of a linear transformation is the dimension of its image.)

Solution

The rank of the restriction of S to any subspace of W can be no larger than $\operatorname{rank}(S)$, hence

$$\operatorname{rank}(S) \geq \operatorname{rank}\left(S|_{\operatorname{im}(T)}\right) = \operatorname{rank}(S \circ T).$$

Further, the dimension of the image of a subspace under a linear transformation is no larger than the dimension of the subspace, hence

$$\operatorname{rank}(T) = \dim(\operatorname{im}(T)) \geq \dim(S(\operatorname{im}(T))) = \dim(\operatorname{im}(S \circ T)) = \operatorname{rank}(S \circ T).$$

Thus the right inequality is established.

We have that

$$\dim(V) = \operatorname{rank}(T) + \dim(\ker(T)),$$
$$\dim(W) = \operatorname{rank}(S) + \dim(\ker(S)),$$
$$\dim(V) = \operatorname{rank}(S \circ T) + \dim(\ker(S \circ T)),$$

hence

$$\operatorname{rank}(T) + \operatorname{rank}(S) - \dim(W) = \operatorname{rank}(S \circ T) + \dim(\ker(S \circ T)) - \dim(\ker(T)) - \dim(\ker(S)).$$

Now $T(\ker(S \circ T)) \subset \ker(S)$ (since $v \in \ker(S \circ T)$ implies $S(Tv) = 0$, hence $Tv \in \ker S$), hence

$$\dim\left(\operatorname{im}\left(T|_{\ker(S \circ T)}\right)\right) \leq \dim(\ker(S)).$$

Further, $\ker\left(T|_{\ker(S\circ T)}\right) = \ker(T)$ ($v \in \ker(T)$ implies $v \in \ker(S \circ T)$, and the opposite inclusion is obvious), hence

$$\dim(\ker(S\circ T)) = \dim\left(\operatorname{im}\left(T|_{\ker(S\circ T)}\right)\right) + \dim\left(\ker\left(T|_{\ker(S\circ T)}\right)\right) \leq \dim(\ker(S)) + \dim(\ker(T))$$

and it follows that

$$\operatorname{rank}(T) + \operatorname{rank}(S) - \dim(W) \leq \operatorname{rank}(S \circ T),$$

which establishes the left inequality.

9. Let V be a real vector space and $T : V \to V$ be a linear transformation. Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of T. Let $0 \neq v_i$ be an eigenvector of T with eigenvalue λ_i for $1 \leq i \leq m$. Show that $\{v_1, \ldots, v_m\}$ is linearly independent.

   Solution

   (F01.10)

10. Let V be a finite dimensional complex inner product space and $f : V \to \mathbb{C}$ a linear functional. Show that there exists a vector $w \in V$ such that $f(v) = \langle v, w\rangle$ for all $v \in V$.

    Solution

    Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V with respect to its inner product, let

    $$w_i = \overline{f(e_i)}$$

    for $i = 1, \ldots, n$, and set $w = \sum_{i=1}^n w_i e_i$. Then for $v = \sum_{i=1}^n v_i e_i$,

    $$(v, w) = \left(\sum_i v_i e_i, \sum_j w_j e_j\right) = \sum_i \sum_j v_i \overline{w_j}(e_i, e_j) = \sum_i v_i \overline{w_i} = \sum_i v_i f(e_i) = f\left(\sum_i v_i e_i\right) = f(v).$$

11. Let V be a finite dimensional complex inner product space and $T : V \to V$ a linear transformation. Prove that there exists an orthonormal ordered basis for V such that the matrix representation A in this basis is upper triangular, i.e., $A_{ij} = 0$ if $i > j$.

    (Hint: First show if $S : V \to V$ is a linear transformation and W is a subspace then W is S-invariant if and only if $W^\perp$ is S^*-invariant, where S^* is the adjoint of S.)

    Solution

    Let $\lambda \in \mathbb{C}$ be an eigenvector of T^* (whose existence is guaranteed by the presence of roots in the characteristic polynomial), and x a corresponding eigenvector; that is, $T^*x = \lambda x$. Let $y \in x^\perp$. Then

    $$(Ty, x) = (y, T^*x) = (y, \lambda x) = \overline{\lambda}(y, x) = 0,$$

    hence $Ty \in x^\perp$. It follows that T is invariant on $x^\perp$, i.e., the restriction of T to $x^\perp$ is a linear transformation. By induction, there exists an orthonormal basis $\{e_1, \ldots, e_{n-1}\}$ such that the matrix representation of $T|_{x^\perp}$ is upper triangular. If we set $e_n = x/\|x\|$, then e_n is a unit vector orthogonal to $\{e_1, \ldots, e_{n-1}\} \subset x^\perp$, hence $\{e_1, \ldots, e_n\}$ is an orthonormal basis, and the matrix representation of T in this basis is upper triangular (since, by repeated application of the inductive step, $T(\operatorname{span}\{e_1, \ldots, e_k\}) \subset \operatorname{span}\{e_1, \ldots, e_k\}$).