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Numerical Analysis Qualifying Exam, Winter 2004

1. (5 Pts.) Let $g(x)$ be a continuously differentiable function and consider the fixed point problem

$$x = g(x).$$

(a) What conditions on $g(x)$ and α, $0 < \alpha \leq 1$, will guarantee convergence of the iteration

$$\begin{aligned} x^* &= g(x_n) \\ x_{n+1} &= \alpha x^* + (1-\alpha)x_n \end{aligned}$$

to the solution $\overline{x}$ of the fixed point problem?

(b) Prove that under the conditions that you derived in (a) the solution $\overline{x}$ of the fixed point problem is unique.

Solution

(a) By Taylor's Theorem,

$$g(x_n) = g(\overline{x}) + g'(y_n)(x_n - \overline{x}) = \overline{x} + g'(y_n)(x_n - \overline{x})$$

for some y_n between x_n and $\overline{x}$. Thus, the error in the $(n+1)^{st}$ iteration is related to the error in the n^{th} iteration by

$$\begin{aligned} e_{n+1} &= x_{n+1} - \overline{x} \\ &= \alpha g(x_n) + (1-\alpha)x_n - \overline{x} \\ &= \alpha(g(x_n) - \overline{x}) + (1-\alpha)(x_n - \overline{x}) \\ &= \alpha g'(y_n)(x_n - \overline{x}) + (1-\alpha)(x_n - \overline{x}) \\ &= (1 - \alpha(1 - g'(y_n)))\, e_n. \end{aligned}$$

Convergence will be guaranteed if

$$|1 - \alpha(1 - g'(y_n))| < 1$$

for all n, i.e.,

$$1 - \frac{2}{\alpha} < g'(y_n) < 1.$$

This is certainly satisfied if

$$1 - \frac{2}{\alpha} < \inf g', \ \sup g' < 1.$$

(b) Suppose some $\overline{x}' \neq \overline{x}$ with $g(\overline{x}') = \overline{x}'$. Then by the Mean Value Theorem,

$$g(\overline{x}) - g(\overline{x}') = g'(y)(\overline{x} - \overline{x}')$$

for some y between $\overline{x}$ and $\overline{x}'$. But this implies $g'(y) = 1$, contradictory to the conditions established in (a). It follows that the solution $\overline{x}$ is unique.

2. (5 Pts.) For a given value of $h > 0$ consider the two approximations to $f'(x)$:

$$D_h f = \frac{f(x+h) - f(x)}{h}; \ D_{2h} f = \frac{f(x+2h) - f(x)}{2h}.$$

Derive the coefficients β_1 and β_2 so that the combination of approximations $\beta_1 D_h f + \beta_2 D_{2h} f$ is a second-order approximation to $f'(x)$.

Solution

By Taylor's Theorem,

$$\begin{aligned} D_h f &= f'(x) + \frac{1}{2} f''(x)h + O(h^2); \\ D_{2h} f &= f'(x) + f''(x)h + O(h^2). \end{aligned}$$

Therefore,

$$2D_h f - D_{2h} f = f'(x) + O(h^2).$$

3. (5 Pts.) Assume the points $\{x_i\}$, for $i = 1, \ldots, n+1$, are distinct. Prove that the polynomial of degree less than or equal to n that interpolates the data $\{(x_i, f(x_i))\}$ is unique.

   Solution

   (W06.3)

4. (10 Pts.) Consider the following two-step numerical method for solving $\frac{dy}{dt} = f(t, y(t))$:

$$y_{i+2} = y_{i+1} + dt\left(\frac{3}{2} f(t_{i+1}, y_{i+1}) - \frac{1}{2} f(t_i, y_i)\right).$$

   (a) Is this method consistent? Explain.

   (b) What is the order of this method? Show your work.

   (c) Does this method converge? Explain.

   (d) Find a necessary and sufficient condition for the linear stability of the method (show your analysis, but without solving explicitly the obtained set of inequalities in the complex domain).

   Solution

   (a) Yes, as shown by the derivation of the order below.

   (b) Assume that $y_i = y(t_i)$ and $y_{i+1} = y(t_{i+1})$. Then by Taylor's Theorem,

$$\begin{aligned} y_{i+1} - y_i &= y'(t_{i+1})dt + O(dt^2) \\ &= f(t_{i+1}, y_{i+1})dt + O(dt^2), \end{aligned}$$

   so

$$\begin{aligned} f(t_i, y_i) &= f(t_{i+1}, y_{i+1}) - f_t(t_{i+1}, y_{i+1})dt - f_y(t_{i+1}, y_{i+1})(y_{i+1} - y_i) + O(dt^2) \\ &= f(t_{i+1}, y_{i+1}) - f_t(t_{i+1}, y_{i+1})dt - f_y(t_{i+1}, y_{i+1})f(t_{i+1}, y_{i+1})dt + O(dt^2), \end{aligned}$$

   hence

$$\begin{aligned} y_{i+2} &= y_{i+1} + dt\left(\frac{3}{2} f(t_{i+1}, y_{i+1}) - \frac{1}{2} f(t_i, y_i)\right) \\ &= y_{i+1} + f(t_{i+1}, y_{i+1})dt + \frac{1}{2}\left(f_t(t_{i+1}, y_{i+1}) + f_y(t_{i+1}, y_{i+1})f(t_{i+1}, y_{i+1})\right)dt^2 + O(dt^3) \\ &= y(t_{i+2}) + O(dt^3), \end{aligned}$$

   showing the method to be second-order.

   (c) The method will converge if f is Lipschitz.

(d) We apply the method to the model problem $y'(t) = \lambda y(t)$:

$$\begin{aligned} y_{i+2} &= y_{i+1} + dt\left(\frac{3}{2}\lambda y_{i+1} - \frac{1}{2}\lambda y_i\right) \\ \Rightarrow\quad & y_{i+2} - \left(1 + \frac{3}{2}\lambda dt\right) y_{i+1} + \frac{1}{2}\lambda dt y_i = 0, \end{aligned}$$

giving the characteristic polynomial

$$\rho(\theta) = \theta^2 - \left(1 + \frac{3}{2}\lambda dt\right)\theta + \frac{1}{2}\lambda dt$$

with roots

$$\zeta_{\pm} = 1 + \frac{3}{2}\lambda dt \pm \sqrt{\left(1 + \frac{3}{2}\lambda dt\right)^2 - 4\left(\frac{1}{2}\lambda dt\right)}.$$

The stability region are those complex λdt such that $|\zeta_{\pm}| \leq 1$.

5. (10 Pts.) Consider the hyperbolic equation

$$u_t + u_x + 2u_y = 0$$

for $t > 0$, (x, y) in the square $[-1, 1] \times [-1, 1]$, and initial data

$$u(x, y, 0) = \phi(x, y).$$

(a) Boundary conditions on u are imposed to be zero on which sides of the square? Why?

(b) Set up a finite difference approximation which converges to the correct solution. Justify your answer.

Solution

(a) We should have $u(-1, y) = u(x, -1) = 0$, since the waves will travel in the positive x and positive y direction.

(b) We consider using Crank-Nicolson:

$$\begin{aligned} P_{k,h_x,h_y} u^n_{\ell,m} &= D_{t+} u^n_{\ell,m} + \frac{1}{2}\left(D_{x0} u^{n+1}_{\ell,m} + D_{x0} u^n_{\ell,m}\right) + 2\frac{1}{2}\left(D_{y0} u^{n+1}_{\ell,m} + D_{y0} u^n_{\ell,m}\right) \\ &= \frac{u^{n+1}_{\ell,m} - u^n_{\ell,m}}{k} + \frac{u^{n+1}_{\ell+1,m} - u^{n+1}_{\ell-1,m} + u^n_{\ell+1,m} - u^n_{\ell-1,m}}{4h_x} \\ &+ \frac{u^{n+1}_{\ell,m+1} - u^{n+1}_{\ell,m-1} + u^n_{\ell,m+1} - u^n_{\ell,m-1}}{2h_x}; \\ R_{k,h_x,h_y} f^n_{\ell,m} &= \frac{f^{n+1}_{\ell,m} + f^n_{\ell,m}}{2}. \end{aligned}$$

The symbols $p_{k,h_x,h_y}(s,\xi,\eta)$ and $r_{k,h_x,h_y}(s,\xi,\eta)$ for these difference operators are

$$\begin{aligned}
p_{k,h_x,h_y}(s,\xi,\eta) &= P\left(e^{skn}e^{i(\xi h_x \ell+\eta h_y m)}\right) \Big/ e^{i(\xi h_x \ell+\eta h_y m)} \\
&= \frac{1}{k}\left(e^{sk}-1\right)+\frac{1}{4h_x}\left(e^{sk}+1\right)\left(e^{i\xi h_x}-e^{-i\xi h_x}\right) \\
&+ \frac{1}{2h_y}\left(e^{sk}+1\right)\left(e^{i\eta h_y}-e^{-i\eta h_y}\right) \\
&= \frac{1}{k}\left(e^{sk}-1\right)+i\left(e^{sk}+1\right)\left(\frac{\sin\xi h_x}{2h_x}+\frac{\sin\eta h_y}{h_y}\right); \\
r_{k,h_x,h_y}(s,\xi,\eta) &= R\left(e^{skn}e^{i(\xi h_x \ell+\eta h_y m)}\right) \Big/ e^{i(\xi h_x \ell+\eta h_y m)} \\
&= \frac{1}{2}\left(e^{sk}+1\right).
\end{aligned}$$

By Taylor's Theorem, these reduce to

$$\begin{aligned}
p_{k,h_x,h_y}(s,\xi,\eta) &= \left(1+\frac{1}{2}sk\right)s+i\left(1+\frac{1}{2}sk\right)(\xi+2\eta)+O(k^2)+O(h_x^2)+O(h_y^2) \\
r_{k,h_x,h_y}(s,\xi,\eta) &= 1+\frac{1}{2}sk+O(k^2).
\end{aligned}$$

We now note that the symbol of the differential operator $P=\partial_t+\partial_x+2\partial_y$ is

$$\begin{aligned}
p(s,\xi,\eta) &= P\left(e^{st}e^{i(\xi x+\eta y)}\right) \Big/ e^{st}e^{i(\xi x+\eta y)} \\
&= s+i\xi+2i\eta,
\end{aligned}$$

and so $p_{k,h_x,h_y}(s,\xi,\eta)-r_{k,h_x,h_y}(s,\xi,\eta)p(s,\xi,\eta)\in O(k^2)+O(h_x^2)+O(h_y^2)$, i.e., second-order accuracy.

We analyze stability by replacing $g=e^{sk}$ in $p_{k,h_x,h_y}(s,\xi,\eta)=0$ and solve for g to determine the root of the amplification polynomial:

$$\begin{aligned}
&\frac{1}{k}(g-1)+i(g+1)\left(\frac{\sin\xi h_x}{2h_x}+\frac{\sin\eta h_y}{h_y}\right)=0 \\
&\Rightarrow\quad g-1+i(g+1)\left(\frac{1}{2}\lambda_x\sin\theta+\lambda_y\sin\phi\right)=0 \\
&\Rightarrow\quad g=\frac{1-i\left(\frac{1}{2}\lambda_x\sin\theta+\lambda_y\sin\phi\right)}{1+i\left(\frac{1}{2}\lambda_x\sin\theta+\lambda_y\sin\phi\right)},
\end{aligned}$$

and we see that $|g|=1$ for all combinations of $\lambda_x,\lambda_y,\theta,\phi$, and hence the scheme is unconditionally stable. The Lax-Richtmyer Equivalence Theorem then implies that the scheme is convergent.

6. (10 Pts.) Consider the equation

$$u_t=u_{xx}$$

to be solved for $t>0$, $x\in[-1,1]$; with periodic initial data

$$u(x,0)\equiv\phi(x),\ \ \phi(x+2)\equiv\phi(x);$$

and $u(x,t)$ periodic in x for $t>0$. Give a fourth or higher order accurate convergent finite difference scheme. Justify your answer.

Solution

We consider using fourth-order Crank-Nicolson:

$$\begin{aligned}
P_{k,h}u_m^n &= D_{t+}u_m^n - \frac{1}{2}\left(D_x^{*2}u_m^{n+1} + D_x^{*2}u_m^n\right) \\
&= \frac{u_m^{n+1}-u_m^n}{k} - \frac{-u_{m+2}^{n+1}+16u_{m+1}^{n+1}-30u_m^{n+1}+16u_{m-1}^{n+1}-u_{m-2}^{n+1}}{24h^2} \\
&- \frac{-u_{m+2}^{n}+16u_{m+1}^{n}-30u_m^{n}+16u_{m-1}^{n}-u_{m-2}^{n}}{24h^2}; \\
R_{k,h}f_m^n &= \frac{f_m^{n+1}+f_m^n}{2}.
\end{aligned}$$

The symbols $p_{k,h}(s,\xi)$ and $r_{k,h}(s,\xi)$ for these difference operators are

$$\begin{aligned}
p_{k,h}(s,\xi) &= P\left(e^{skn}e^{i\xi mh}\right)/\,e^{skn}e^{i\xi mh} \\
&= \frac{1}{k}\left(e^{sk}-1\right) - \frac{1}{24h^2}\left(e^{sk}+1\right)\left(-e^{2i\xi h}+16e^{i\xi h}-30+16e^{-i\xi h}-e^{-2i\xi h}\right) \\
&= \frac{1}{k}\left(e^{sk}-1\right) + \frac{1}{24h^2}\left(e^{sk}+1\right)(30-32\cos\xi h+2\cos 2\xi h) \\
&= \frac{1}{k}\left(e^{sk}-1\right) + \frac{1}{12h^2}\left(e^{sk}+1\right)\left(15-16\cos\xi h+2\cos^2\xi h-1\right) \\
&= \frac{1}{k}\left(e^{sk}-1\right) + \frac{1}{6h^2}\left(e^{sk}+1\right)(\cos\xi h-7)(\cos\xi h-1); \\
r_{k,h}(s,\xi) &= R\left(e^{skn}e^{i\xi mh}\right)/\,e^{skn}e^{i\xi mh} \\
&= \frac{1}{2}\left(e^{sk}+1\right).
\end{aligned}$$

Using Taylor's Theorem to expand each symbol yields

$$\begin{aligned}
p_{k,h}(s,\xi) &= \frac{1}{k}\left(e^{sk}-1\right) + \frac{1}{6h^2}\left(e^{sk}+1\right)(\cos\xi h-7)(\cos\xi h-1) \\
&= \left(1+\frac{1}{2}sk\right)s + \left(1+\frac{1}{2}sk\right)\frac{1}{3h^2}\left(-6-\frac{1}{2}\xi^2h^2+\frac{1}{24}\xi^4h^4+O(h^6)\right) \\
&\quad \left(-\frac{1}{2}\xi^2h^2+\frac{1}{24}\xi^4h^4+O(h^6)\right)+O(k^2) \\
&= \left(1+\frac{1}{2}sk\right)\left(s+\xi^2\right)+O(k^2)+O(h^4); \\
r_{k,h}(s,\xi) &= 1+\frac{1}{2}sk+O(k^2).
\end{aligned}$$

Now noting that the symbol of the differential operator $P = \partial_t - \partial_x^2$ is $p(s,\xi) = s+\xi^2$, we see immediately that $p_{k,h}(s,\xi) - r_{k,h}(s,\xi)p(s,\xi) \in O(k^2)+O(h^4)$, verifying (2,4) order accuracy.

For the stability analysis, we replace $g = e^{sk}$ in $p_{k,h}(s,\xi) = 0$ and solve for g to determine the roots of the amplification polynomial:

$$\begin{aligned}
&\frac{1}{k}(g-1)+\frac{1}{6h^2}(g+1)(\cos\xi h-7)(\cos\xi h-1)=0 \\
&\Rightarrow\quad g-1+\frac{1}{6}\mu(g+1)(\cos\theta-7)(\cos\theta-1)=0 \\
&\Rightarrow\quad g=\frac{1-\frac{1}{6}\mu(\cos\theta-7)(\cos\theta-1)}{1+\frac{1}{6}\mu(\cos\theta-7)(\cos\theta-1)},
\end{aligned}$$

from which we see that $|g| \leq 1$ for all μ,θ, and hence the scheme is unconditionally stable. It follows by the Lax-Richtmyer Equivalence Theorem that the scheme is convergent.

7. (10 Pts.)

   (a)
   (b)
   (c)
   (d)

   Solution

   (a)
   (b)
   (c)
   (d)