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Basic Examination, S03

1. (a) Suppose $f : (0,1) \to \mathbb{R}$ is a continuous function. Define what it means for f to be *uniformly continuous*.

   (b) Show that if $f : (0,1) \to \mathbb{R}$ is uniformly continuous, then there is a continuous function $F : [0,1] \to \mathbb{R}$ with $F(x) = f(x)$ for all $x \in (0,1)$.

   Solution

   (S02.4)

2. Prove: If $a_1, a_2, a_3, \ldots$ is a sequence of real numbers with

$$\sum_{j=1}^{\infty} |a_j| < +\infty,$$

   then $\lim_{N\to+\infty} \sum_{j=1}^{N} a_j$ exists.

   Solution

   Let

$$S_n = \sum_{j=1}^{n} |a_j|,$$

$$T_n = \sum_{j=1}^{n} a_j.$$

   Given $\epsilon > 0$, let N be such that $|S_n - S_m| < \epsilon$ for all $n, m > N$. Then for $N < n < m$,

$$|T_n - T_m| = \left| \sum_{j=n+1}^{m} a_j \right| \le \sum_{j=n+1}^{m} |a_j| < \epsilon,$$

   hence $\{T_n\}_{n=1}^{\infty}$ is a Cauchy sequence, so has a limit since $\mathbb{R}$ is complete.

3. Find a subset S of the real numbers $\mathbb{R}$ such that both (a) and (b) hold for S:

   (a) S is not the countable union of closed sets.

   (b) S is not the countable intersection of open sets.

   Solution

   Let $S = (\mathbb{Q} \cap (0,\infty)) \cup ((-\infty,0)\backslash\mathbb{Q})$. Then $S^C = (\mathbb{Q} \cap (-\infty,0]) \cup ((0,\infty)\backslash\mathbb{Q})$. If S is a countable intersection of open sets, then $S \cap (0,\infty) = \mathbb{Q} \cap (0,\infty)$ would be a countable intersection of open sets, which is impossible, following from the Baire Category Theorem. Similarly, if S is the countable union of closed sets, S^C is the countable intersection of open sets, which again is impossible for the same reasons. Hence S satisfies the given conditions.

4. Consider the following equations for a function $F(x,y)$ on $\mathbb{R}^2$:

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2} \quad (*)$$

   (a) Show that if a function F has the form $F(x,y) = f(x+y) + g(x-y)$ where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are twice differentiable, then F satisfies the equation $(*)$.

(b) Show that if $F(x,y) = ax^2 + bxy + cy^2$, $a, b, c \in \mathbb{R}$, satisfies $(*)$, then $F(x,y) = f(x+y) + g(x-y)$ for some polynomials f and g in one variable.

Solution

(a) Given F as above, then

$$\frac{\partial^2 F}{\partial x^2} = f''(x+y) + g''(x-y),$$
$$\frac{\partial^2 F}{\partial y^2} = f''(x+y) + g''(x-y),$$

hence F satisfies the differential equation.

(b) Given F as above, then

$$\frac{\partial^2 F}{\partial x^2} = 2a,$$
$$\frac{\partial^2 F}{\partial y^2} = 2c,$$

hence $a = c$ and

$$F(x,y) = \frac{a}{2}\left((x+y)^2 + (x-y)^2\right) + \frac{b}{4}\left((x+y)^2 - (x-y)^2\right) = f(x+y) + g(x-y)$$

for

$$f(z) = \frac{a}{2}z^2 + \frac{b}{4}z^2,$$
$$g(z) = \frac{a}{2}z^2 - \frac{b}{4}z^2.$$

5. Consider the function $F(x,y) = ax^2 + 2bxy + cy^2$ on the set $A = \{(x,y) : x^2 + y^2 = 1\}$.

(a) Show that F has a maximum and minimum on A.

(b) Use Lagrange multipliers to show that if the maximum of F on A occurs at a point (x_0, y_0), then the vector (x_0, y_0) is an eigenvector of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

Solution

(a) F is continuous on A, which is compact, hence F achieves its maximum and minimum.

(b) Let

$$g(x,y) = x^2 + y^2 - 1.$$

Then F achieves its maximum and minimum values whenever $(x,y) \in \mathbb{R}^2$ simultaneously satisfy

$$\nabla F(x,y) = \lambda \nabla g(x,y),$$
$$g(x,y) = 0$$

for some $\lambda \in \mathbb{R}$. Thus we compute

$$\nabla F(x,y) = (2ax + 2by, 2bx + 2cy),$$
$$\nabla g(x,y) = (2x, 2y),$$

and substituting into the first condition gives the system

$$2ax + 2by = 2\lambda x,$$

$$2bx + 2cy = 2\lambda y,$$

which is equivalent to

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix},$$

hence $\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

6. Formulate some reasonably general conditions on a function $f : \mathbb{R}^2 \to \mathbb{R}$ which guarantee that

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

and prove that your conditions do in fact guarantee that this equality holds.

Solution

(F01.5)

7. Let V be a finite dimensional real vector space. If $W \subset V$ is a subspace, let $W^\circ = \{f : V \to \mathbb{F} \text{ linear} \mid f = 0 \text{ on } W\}$. Let $W_i \subset V$ be subspaces for $i = 1, 2$. Prove that

$$W_1^\circ \cap W_2^\circ = (W_1 + W_2)^\circ.$$

Solution

Suppose $f \in W_1^\circ \cap W_2^\circ$. Let $v \in W_1 + W_2$. Then $v = w_1 + w_2$ for some $w_i \in W_i$. Thus

$$f(v) = f(w_1 + w_2) = f(w_1) + f(w_2) = 0 + 0 = 0$$

since $f \in W_1^\circ$ as well as $f \in W_2^\circ$. It follows that $f \in (W_1 + W_2)^\circ$ and $W_1^\circ \cap W_2^\circ \subset (W_1 + W_2)^\circ$.

Now suppose $f \in (W_1 + W_2)^\circ$. Then any $w \in W_1$ can be expressed as $w_1 + 0 \in W_1 + W_2$ ($0 \in W_2$ since W_2 is a subspace), hence $f(w) = 0$ and $f \in W_1^\circ$. Similarly, $f \in W_2^\circ$ as well, so $f \in W_1^\circ \cap W_2^\circ$ and $(W_1 + W_2)^\circ \subset W_1^\circ \cap W_2^\circ$. This completes the proof of the claim.

8. Let V be an n-dimensional complex vector space and $T : V \to V$ a linear operator. Suppose that the characteristic polynomial of T has n distinct roots. Show that there is a basis B for V such that the matrix representation of T in the basis B is diagonal. (Make sure that you prove your choice of B is in fact a basis.)

Solution

Let the roots of the characteristic polynomial of T be $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $\lambda_i \neq \lambda_j$ for $i \neq j$. Let $x_1, \ldots, x_n \in \mathbb{R}^n$ be corresponding eigenvectors, respectively; i.e., $Tx_i = \lambda_i x_i$. We first show by induction that $\{x_1, \ldots, x_n\}$ is linearly independent. For suppose $\{x_1, \ldots, x_{n-1}\}$ is linearly independent, and suppose some trivial linear relation

$$\sum_{i=1}^{n} c_i x_i = 0,$$

$c_i \in \mathbb{R}$. Then

$$0 = T\sum_i c_i x_i = \sum_i c_i T x_i = \sum_i c_i \lambda_i x_i.$$

If we multiply the first equation by λ_n and subtract the second, we obtain

$$\sum_{i=1}^{n-1} c_i(\lambda_n - \lambda_i)x_i = 0.$$

Since $\lambda_n \neq \lambda_i$ for $i = 1, \ldots, n-1$ and $\{x_1, \ldots, x_{n-1}\}$ is linearly independent, it follows that $c_i = 0$ for each $i = 1, \ldots, n-1$, hence $c_n = 0$ as well. Thus $\{x_1, \ldots, x_n\}$ is linearly independent as well, as claimed.

Let $B = \{x_1, \ldots, x_n\}$. Then as B is a linearly independent set of vectors within an n-dimensional vector space, B must be a basis for that vector space. It is evident as well that the matrix representation of T in the basis B is diagonal, with diagonal terms $[T]_{ii} = \lambda_i$.

9. Let $A \in M_3(\mathbb{R})$ satisfy $\det(A) = 1$ and $A^t A = I = AA^t$ where I is the 3×3 identity matrix. Prove that the characteristic polynomial of A has 1 as a root.

   Solution

   Let λ_i, $i = 1, 2, 3$ be the roots of the characteristic polynomial. Then $\lambda_1 \lambda_2 \lambda_3 = \det A = 1$, hence at least one of the λ_i's is real. Without loss of generality, suppose $\lambda_1 \in \mathbb{R}$.

   Let x_i be an associated eigenvector for λ_i. Then

$$(x_i, x_i) = (A^t A x_i, x_i) = (Ax_i, Ax_i) = (\lambda_i x_i, \lambda_i x_i) = |\lambda_i|^2 (x_i, x_i),$$

   so $|\lambda_i|^2 = 1$, hence $|\lambda_i| = 1$.

   Now if $\lambda_2 \notin \mathbb{R}$, then $\lambda_3 = \overline{\lambda_2}$ and

$$1 = \lambda_1 \lambda_2 \lambda_3 = \lambda_1 \lambda_2 \overline{\lambda_2} = \lambda_1 |\lambda_2|^2 = \lambda_1,$$

   which proves the claim. On the other hand, if $\lambda_2 \in \mathbb{R}$, $\lambda_3 \in \mathbb{R}$ as well, hence $\lambda_i \in \{-1, 1\}$ and either 1 or 3 of the λ_i's will be equal to 1 (since their product is 1).

10. Let V be a finite dimensional real inner product space and $T : V \to V$ a hermitian linear operator. Suppose the matrix representation of T^2 in the standard basis has trace zero. Prove that T is the zero operator.

    Solution

    In the standard basis $\{e_1, \ldots, e_n\}$, we have that

$$0 = \operatorname{tr} T^2 = \sum_{i=1}^{n} (T^2 e_i, e_i) = \sum_{i=1}^{n} (Te_i, Te_i) = \sum_{i=1}^{n} \|Te_i\|^2,$$

    hence $Te_i = 0$ for $i = 1, \ldots, n$, from which it follows that $T = 0$.