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Analysis Bootcamp, Homework 5

1. *Rudin, page 139. Problems 10 - 12, 15, 17, 19.*

10. *Let p and q be positive real numbers such that*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(a) *If $u \geq 0$ and $v \geq 0$, then*

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

(b) *If $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and*

$$\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha,$$

then

$$\int_a^b fg d\alpha \leq 1.$$

(c) *If f and g are complex functions in $\mathcal{R}(\alpha)$, then*

$$\left| \int_a^b fg d\alpha \right| \leq \left(\int_a^b |f|^p d\alpha \right)^{1/p} \left(\int_a^b |g|^q d\alpha \right)^{1/q}.$$

This is Hölder's inequality*. When $p = q = 2$ it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)*

(d) *Show that Hölder's inequality is also true for the "improper" integrals described in Exercises 7 and 8.*

Solution

(a) The inequality is trivial when either $u = 0$ or $v = 0$, so assume both are positive. $z \to e^z$ is a concave up, hence for any $x, y \in \mathbb{R}$ and $t \in [0, 1]$,

$$(1-t)e^x + te^y \geq e^{(1-t)x+ty} = (e^x)^{1-t} (e^y)^t.$$

Setting $x = \ln u^p$, $y = \ln v^q$, and $t = \frac{1}{q}$ establishes the inequality. Note that equality is achieved when either $t \in \{0, 1\}$ (which, for positive p, q, cannot happen) or when $x = y$, which happens if and only if $u^p = v^q$.

(b) Using the above inequality, for all $x \in [a, b]$,

$$\frac{f(x)^p}{p} + \frac{g(x)^q}{q} \geq f(x)g(x),$$

hence, given $\int f^p d\alpha = \int g^q d\alpha = 1$,

$$1 = \int_a^b \left(\frac{f^p}{p} + \frac{g^q}{q} \right) d\alpha \geq \int_a^b fg d\alpha.$$

(c) Let

$$c = \left(\int_a^b |f|^p d\alpha \right)^{1/p},$$

$$d = \left(\int_a^b |g|^q d\alpha \right)^{1/q}.$$

Then $\int \left|\frac{f}{c}\right|^p d\alpha = \int \left|\frac{g}{d}\right|^q d\alpha = 1$, hence the preceding inequality gives

$$\int_a^b \left|\frac{f}{c}\right| \left|\frac{g}{d}\right| d\alpha \le 1$$

from which it follows that

$$\int_a^b |f||g| d\alpha \le cd = \left(\int_a^b |f|^p d\alpha \right)^{1/p} \left(\int_a^b |g|^q d\alpha \right)^{1/q}.$$

(d)

11. *Let α be a fixed increasing function on $[a,b]$. For $u \in \mathcal{R}(\alpha)$, define*

$$\|u\|_2 = \left(\int_a^b |u|^2 d\alpha \right)^{1/2}.$$

Suppose $f, g, h \in \mathcal{R}(\alpha)$, and prove that triangle inequality

$$\|f-h\|_2 \le \|f-g\|_2 + \|g-h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Solution

$$\begin{aligned}
\|f-h\|_2^2 = \int |f-h|^2 d\alpha = \int \left(|f|^2 + |h|^2 - f\overline{h} - \overline{f}h\right) d\alpha \\
= \int \left(|f|^2 + |g|^2 - f\overline{g} - \overline{f}g\right) d\alpha + \int \left(|g|^2 + |h|^2 - g\overline{h} - \overline{g}h\right) d\alpha \\
+ \int \left(f\overline{g} + g\overline{h} - f\overline{h} - |g|^2\right) d\alpha + \int \left(\overline{f}g + \overline{g}h - \overline{f}h - |g|^2\right) d\alpha \\
= \int |f-g|^2 d\alpha + \int |g-h|^2 d\alpha + \int (f-g)\overline{(g-h)} d\alpha + \int \overline{(f-g)}(g-h) d\alpha \\
\le \|f-g\|_2^2 + \|g-h\|_2^2 + 2\left(\int |f-g|^2 d\alpha\right)^{1/2} \left(\int |g-h|^2 d\alpha\right)^{1/2} \\
= \|f-g\|_2^2 + \|g-h\|_2^2 + 2\|f-g\|_2 \|g-h\|_2 \\
= \left(\|f-g\|_2 + \|g-h\|_2\right)^2.
\end{aligned}$$

12. *With the notations of Exercise 11, suppose $f \in \mathcal{R}(\alpha)$ and $\epsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $\|f - g\|_2 < \epsilon$.*
Hint: *Let $P = \{x_0, \ldots, x_n\}$ be a suitable partition of $[a, b]$, define*

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if $x_{i-1} \le t \le x_i$.

Solution

Since $f \in \mathcal{R}(\alpha)$, f is bounded, say $|f| \le M$. Given $\epsilon > 0$, choose $P = \{x_0, \ldots, x_n\}$ a partition of $[a, b]$ according to Theorem 6.6 for $\epsilon/2M$:

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon/2M.$$

Set g as in the hint, i.e.,

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

for $t \in [x_{i-1}, x_i]$. Then g is continuous (since $g(x_i) = f(x_i)$ whether we consider $t = x_i$ to be in $[x_{i-1}, x_i]$ or $[x_i, x_{i+1}]$), and for $t \in [x_{i-1}, x_i]$, $g(t)$ is between $f(x_{i-1})$ and $f(x_i)$, hence $m_i \le g(t) \le M_i$. It follows that

$$m_i \Delta\alpha_i \le g(t)\Delta\alpha_i \le M_i \Delta\alpha_i, \ t \in [x_{i-1}, x_i],$$

hence

$$\int_a^b |f(t) - g(t)| d\alpha \le \sum_i (M_i - m_i)\Delta\alpha_i < \epsilon/2M,$$

and since $|f(t) - g(t)| \le 2M$ ($g(t)$ is bounded in absolute value by M as well since it is bounded by images of f over each interval in P), we have that

$$\|f - g\|_2^2 = \int_a^b |f(t) - g(t)|^2 d\alpha < \epsilon.$$

15. *Suppose f is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and*

$$\int_a^b f^2(x)dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$$

and that

$$\int_a^b (f'(x))^2 \, dx \cdot \int_a^b x^2 f^2(x) dx > \frac{1}{4}.$$

Solution

By Theorem 6.22 (Integration by Parts) for $F(x) = xf(x)$ and $g(x) = f'(x)$,

$$\int_a^b x f(x) f'(x) dx = x f(x) f(x)|_a^b - \int_a^b (f(x) + x f'(x)) f(x) dx = -1 - \int_a^b x f(x) f'(x),$$

hence

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}.$$

By Cauchy-Schwarz,

$$\int_a^b (f'(x))^2\, dx \cdot \int_a^b (xf(x))^2\, dx \geq \left(\int_a^b f'(x)xf(x)dx\right)^2 = \frac{1}{4}$$

with equality if and only if

$$f'(x) = cxf(x)$$

for some $c \in \mathbb{R}$. This is a separable differential equation with solution

$$f(x) = Ce^{cx^2/2}.$$

Now since $x \mapsto e^{cx^2/2}$ is never zero, the condition that $f(a) = f(b) = 0$ forces $C = 0$, hence $f \equiv 0$ and $\int f^2 dx = 0$ for any $[a, b]$, violating the givens. We conclude that equality is impossible, and the inequality is strict.

17. *Suppose α increases monotonically on $[a, b]$, g is continuous, and $g(x) = G'(x)$ for $a \leq x \leq b$. Prove that*

$$\int_a^b \alpha(x)g(x)dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G d\alpha.$$

Hint: *Take g real, without loss of generality. Given $P = \{x_0, x_1, \ldots, x_n\}$, choose $t_i \in (x_{i-1}, x_i)$ so that $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$. Show that*

$$\sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i.$$

Solution

Let $\epsilon > 0$ be given. Let $M = \max\{|\alpha(a)|, |\alpha(b)|\}$ (thus $\alpha \leq M$ on $[a, b]$). Since g is continuous on $[a, b]$, which is compact, g is uniformly continuous on $[a, b]$. Similarly, G is differentiable on $[a, b]$, hence continuous, hence uniformly continuous. Hence there exists a $\delta > 0$ such that $|g(x) - g(y)| < \epsilon/M(b - a)$ and $|G(x) - G(y)| < \epsilon/(\alpha(b) - \alpha(a))$ whenever $|x - y| < \delta$.

Let $P = \{x_0, \ldots, x_n\}$ be a partition as in Theorem 6.6, such that

$$U(P, \alpha g) - L(P, \alpha g) < \epsilon,$$

$$U(P, G, \alpha) - L(P, G, \alpha) < \epsilon,$$

and also such that $\Delta x_i < \delta$ for each i. As per the hint, we can choose $t_i \in [x_{i-1}, x_i]$ such that $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$. Then

$$\begin{aligned}
\sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i &= \sum_{i=1}^n \alpha(x_i)\left(G(x_i) - G(x_{i-1})\right) \\
&= \sum_{i=1}^n G(x_i)\alpha(x_i) - \sum_{i=1}^n G(x_{i-1})\alpha(x_i) \\
&= \sum_{i=2}^{n+1} G(x_{i-1})\alpha(x_{i-1}) - \sum_{i=1}^n G(x_{i-1})\alpha(x_i) \\
&= G(x_n)\alpha(x_n) - \sum_{i=1}^n G(x_{i-1})\left(\alpha(x_i) - \alpha(x_{i-1})\right) - G(x_0)\alpha(x_0)
\end{aligned}$$

$$= G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^{n} G(x_{i-1})\Delta\alpha_i.$$

Further,

$$\left|\sum_i \alpha(x_i)g(t_i)\Delta x_i - \sum_i \alpha(x_i)g(x_i)\Delta x_i\right| \leq \sum_i |\alpha(x_i)|\,|g(t_i) - g(x_i)|\,\Delta x_i$$

$$< \frac{\epsilon}{M(b-a)} \sum_i |\alpha(x_i)|\,\Delta x_i \leq \frac{\epsilon}{b-a}\sum_i \Delta x_i = \epsilon,$$

while

$$\left|\sum_i G(x_{i-1})\Delta\alpha_i - \sum_i G(x_i)\Delta\alpha_i\right| \leq \sum_i |G(x_{i-1}) - G(x_i)|\,\Delta\alpha_i < \frac{\epsilon}{\alpha(b)-\alpha(a)}\sum_i \Delta\alpha_i = \epsilon,$$

Hence we find that

$$\left|\int_a^b \alpha(x)g(x)dx - G(b)\alpha(b) + G(a)\alpha(a) + \int_a^b G d\alpha\right| < 4\epsilon,$$

and since ϵ was arbitrary, the equality is proved.

19. *Let γ_1 be a curve in $\mathbb{R}^k$, defined on $[a, b]$; let ϕ be a continuous 1-1 mapping of $[c, d]$ onto $[a, b]$, such that $\phi(c) = a$; and define $\gamma_2(s) = \gamma_1(\phi(s))$. Prove that γ_2 is an arc, a closed curve, or a rectifiable curve if and only if the same is true of γ_1. Prove that γ_2 and γ_1 have the same length.*

Solution

γ_1 is an arc if and only if γ_1 is one-to-one if and only if $\gamma_2 = \gamma_1 \circ \phi$ is one-to-one if and only if γ_2 is an arc.

γ_1 is a closed curve if and only if $\gamma_1(a) = \gamma_1(b)$ if and only if $\gamma_2(c) = \gamma_1(a) = \gamma_1(b) = \gamma_2(d)$ if and only if γ_2 is a closed curve.

Given $Q = \{c = x_0, x_1, \ldots, x_{n-1}, x_n = d\}$ a partition of $[c, d]$, there exists a corresponding partition $P = \phi(Q)$ of $[a, b]$. Further, if we let $y_i = \phi(x_i)$,

$$\Lambda(P, \gamma_1) = \sum_{i=1}^{n} |\gamma_1(y_i) - \gamma_1(y_{i-1})| = \sum_{i=1}^{n} |\gamma_2(x_i) - \gamma_2(x_{i-1})| = \Lambda(Q, \gamma_2).$$

Conversely, given a partition P of $[a, b]$, there exists a partition $Q = \phi^{-1}(P)$ of $[c, d]$ such that $\Lambda(Q, \gamma_2) = \Lambda(P, \gamma_1)$ (ϕ^{-1} exists since ϕ is one-to-one and onto). It follows that

$$\Lambda(\gamma_1) = \sup_P \Lambda(P, \gamma_1) = \sup_Q \Lambda(Q, \gamma_1) = \Lambda(\gamma_2).$$