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Numerical Analysis Qualifying Exam, Fall 2003

1. (5 Pts.) For a single panel, Simpson's rule

$$\int_a^{a+2h} f(x)dx \approx \frac{h}{3}\left(f(a) + 4f(a+h) + f(a+2h)\right)$$

is fifth-order accurate.

(a) What is the order of accuracy of the composite Simpson's rule?

(b) I have written a program that implements composite Simpson's rule for integrating functions over the interval $[0,1]$. In checking the program for correctness, I test the routine on the integral $\int_0^1 x^5 dx$ and I obtain the following results:

M	approximation	error
8	0.24169	0.04169
16	0.22083	0.02083
32	0.21041	0.01041
64	0.20520	0.00520

What is the factor by which the errors should decrease as the number of panels, M, is doubled?

(c) On the basis of the above computational results, can I conclude that my program is incorrect? Explain your answer.

Solution

(a) Composite Simpson's rule would be fourth-order accurate.

(b) The errors should decrease by a factor of $(1/2)^4 = 1/16$ for each doubling of M.

(c) Yes, the program is probably incorrect. The errors are only decreasing by a factor of $1/2$ for each doubling of M.

2. (5 Pts.) Consider the following iterative method:

$$Ax^{k+1} = Bx^k + c$$

where c is the vector $(1\ 1)^t$ and A and B are the matrices

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(a) Assume the iteration converges; to what vector x does the iteration converge?

(b) Does this iteration converge for arbitrary initial vectors x^0? Justify your answer.

Solution

(a) The iteration converging implies that

$$Ax = Bx + c \;\Rightarrow\; x = (A-B)^{-1}c = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

(b) Since

$$\begin{aligned} A(x^{k+1} - x) &= Bx^k + c - Ax \\ &= Bx^k + (A - B)x - Ax \\ &= B(x^k - x), \end{aligned}$$

the iteration converges for all initial iterates x^0 if and only if the eigenvalues of $A^{-1}B$ are less than 1 in magnitude. Indeed, we have

$$A^{-1}B = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

with characteristic equation

$$p(\lambda) = \begin{vmatrix} \lambda - 1 & -\frac{1}{2} \\ -\frac{1}{2} & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - \frac{1}{4}$$

whose roots are

$$\lambda_{\pm} = 1 \pm \frac{1}{2}.$$

In particular, then $|\lambda_+| \geq 1$, so the iteration will not converge if, e.g., x^0 is an eigenvector corresponding to the eigenvalue λ_+.

3. (5 Pts.)

(a) Give the cubic polynomial that interpolates the function $f(x) = 2^x$ at the points $x = 0$, $x = 1$, $x = 2$, and $x = 3$.

(b) Give the value of your interpolant at $x = \frac{1}{2}$, and hence derive an approximation to $\sqrt{2} = 2^{1/2}$.

Solution

(a) We use the Lagrange polynomial form:

$$\begin{aligned} p(x) &= \frac{(x-1)(x-2)(x-3)}{(-1)(-2)(-3)} f(0) + \frac{x(x-2)(x-3)}{(1)(-1)(-2)} f(1) \\ &+ \frac{x(x-1)(x-3)}{(2)(1)(-1)} f(2) + \frac{x(x-1)(x-2)}{(3)(2)(1)} f(3) \\ &= -\frac{1}{6}\left(x^3 - 6x^2 + 11x - 6\right)(1) + \frac{1}{2}\left(x^3 - 5x^2 + 6x\right)(2) \\ &- \frac{1}{2}\left(x^3 - 4x^2 + 3x\right)(4) + \frac{1}{6}\left(x^3 - 3x^2 + 2x\right)(8) \\ &= \frac{1}{6}x^3 + \frac{5}{6}x + 1 \end{aligned}$$

(b) We have

$$\begin{aligned} p\left(\frac{1}{2}\right) &= \frac{1}{6}\left(\frac{1}{2}\right)^3 + \frac{5}{6}\left(\frac{1}{2}\right) + 1 \\ &= \frac{1}{48} + \frac{5}{12} + 1 \\ &= \frac{69}{48}. \end{aligned}$$

4. (10 Pts.)

(a) Construct a two-stage second-order Runge-Kutta method for the ODE

$$y' = f(y), \ y(0) = y_0;$$

and find its region of absolute stability.

(b) Give an equivalent first-order system for the second-order differential equation

$$y'' - 21y' + 20y = 0.$$

(c) Give the stability time-step restriction if second-order Runge-Kutta is used to compute solutions to the first-order system.

Solution

(a) We use the method described by

$$y_{n+1} = y_n + hf\left(y_n + \frac{1}{2}hf(y_n)\right).$$

To show second-order accuracy, we use Taylor's Theorem:

$$f\left(y_n + \frac{1}{2}hf(y_n)\right) = f(y_n) + \frac{1}{2}f'(y_n)f(y_n)h + O(h^2)$$

so then, if $y(t_n) = y_n$,

$$\begin{aligned} y_{n+1} &= y_n + f(y_n)h + \frac{1}{2}f'(y_n)f(y_n)h^2 + O(h^3) \\ &= y(t_n + h) + O(h^3). \end{aligned}$$

We find the region of absolute stability by applying the method to the model problem $y' = f(y) = \lambda y$:

$$\begin{aligned} y_{n+1} &= y_n + h\lambda\left(y_n + \frac{1}{2}h\lambda y_n\right) \\ &= \left(1 + \lambda h + \frac{1}{2}(\lambda h)^2\right) y_n, \end{aligned}$$

giving the characteristic polynomial

$$\rho(\theta) = \theta - \left(1 + \lambda h + \frac{1}{2}(\lambda h)^2\right)$$

which has the single root

$$\zeta = 1 + \lambda h + \frac{1}{2}(\lambda h)^2.$$

The region of absolute stability is the set of complex λh such that

$$\left|1 + \lambda h + \frac{1}{2}(\lambda h)^2\right| < 1.$$

We note that this region contains the interval $\left(-\frac{1}{2}, 0\right)$.

(b) We have

$$Y' = \begin{pmatrix} 0 & 1 \\ -20 & 21 \end{pmatrix} Y = AY$$

where

$$Y(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}.$$

(c) The characteristic polynomial of A is

$$\begin{aligned} p_A(\lambda) &= \begin{vmatrix} \lambda & -1 \\ 20 & \lambda - 21 \end{vmatrix} \\ &= \lambda(\lambda - 21) + 20 \\ &= \lambda^2 - 21\lambda + 20 \end{aligned}$$

which has roots

$$\lambda_\pm = \frac{21}{2} \pm \frac{1}{2}\sqrt{361},$$

both of which are strictly positive. Thus there is no stability time-step restriction.

5. (10 Pts.) Consider the differential equation

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)^2 \phi - c^2 \frac{\partial^2 \phi}{\partial x^2} = 0$$

to be solved for $0 \le x \le 1$, $t > 0$; with periodic boundary conditions in x; and initial data

$$\begin{aligned} \phi(x, 0) &= \phi_0(x) \\ \phi_t(x, 0) &= \phi_1(x) \end{aligned}$$

Here $u, c > 0$ are constants. Give a convergent second-order accurate finite difference approximation to this equation. Be sure to justify that your approximation is second-order accurate and convergent.

Solution

We expand the differential equation, obtaining

$$P\phi = \phi_{tt} + 2u\phi_{tx} - (c^2 - u^2)\phi_{xx} = 0.$$

We now rewrite the equation as a system. Introduce

$$\Phi(t, x) = \begin{pmatrix} \phi(x, t) \\ \phi_t(x, t) \\ \phi_x(x, t) \end{pmatrix};$$

then we find that

$$\Phi_t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2u & c^2 - u^2 \\ 0 & 1 & 0 \end{pmatrix} \Phi_x + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Phi.$$

(F04.5)

6. (10 Pts.) Consider the one-dimensional diffusion equation

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2}, \ \alpha > 0$$

to be solved for $0 \le x \le 1$, $t > 0$; with periodic boundary conditions in x; and initial data

$$v(x,0) = v_0(x).$$

Assume one uses the Dufort Frankel method:

$$\frac{v_m^{n+1} - v_m^{n-1}}{2\Delta t} - \alpha \left(\frac{v_{m+1}^n - (v_m^{n+1} + v_m^{n-1}) + v_{m-1}^n}{\Delta x^2} \right)$$

as a means of computing approximate solutions to this equation.

(a) Determine the truncation error associated with this approximation. Under what conditions does the scheme provide a consistent approximation to the diffusion equation? Would the condition required for consistency be difficult to satisfy in a set of computational experiments where Δx is repeatedly halved?

(b) Surprisingly, this scheme is explicit and unconditionally stable. Show this, and explain why this does not violate the CFL condition.

Solution

(a) The symbol $p(s,\xi)$ of the differential operator $P = \partial_t - \alpha \partial_x^2$ is

$$\begin{aligned} p(s,\xi) &= P\left(e^{st} e^{i\xi x}\right) / e^{st} e^{i\xi x} \\ &= s + \alpha \xi^2, \end{aligned}$$

while the symbol $p_{\Delta t, \Delta x}(s,\xi)$ of the difference operator $P_{\Delta t,\Delta x}$ given by

$$P_{\Delta t,\Delta x} v_m^n = \frac{v_m^{n+1} - v_m^{n-1}}{2\Delta t} - \alpha \left(\frac{v_{m+1}^n - (v_m^{n+1} + v_m^{n-1}) + v_{m-1}^n}{\Delta x^2} \right)$$

is

$$\begin{aligned} p_{\Delta t,\Delta x}(s,\xi) &= P_{\Delta t,\Delta x}\left(e^{sn\Delta t} e^{i\xi m \Delta x}\right) / e^{sn\Delta t} e^{i\xi m\Delta x} \\ &= \frac{1}{2\Delta t}\left(e^{s\Delta t} - e^{-s\Delta t}\right) - \frac{\alpha}{\Delta x^2}\left(e^{i\xi\Delta x} - \left(e^{s\Delta t} + e^{-s\Delta t}\right) e^{-i\xi\Delta x}\right) \\ &= \frac{1}{\Delta t}\sinh s\Delta t - \frac{2\alpha}{\Delta x^2}\left(\cos \xi\Delta x - \cosh s\Delta t\right) \\ &= s + O(\Delta t^2) - \frac{2\alpha}{\Delta x^2}\left(1 - \frac{1}{2}\xi^2\Delta x^2 + O(\Delta x^4) - 1 + O(\Delta t^2)\right) \\ &= s + \alpha\xi^2 + O(\Delta x^2) + O\left(\left(\frac{\Delta t}{\Delta x}\right)^2\right) \\ &= p(s,\xi) + O(\Delta x^2) + O(\lambda^2), \end{aligned}$$

hence the scheme provides a consistent approximation to the differential equation if $\lambda = \frac{\Delta t}{\Delta x} \to 0$ as $\Delta t, \Delta x \to 0$. This isn't any more difficult to satisfy than if we used the simplest explicit scheme and wished to retain stability. For example, if we take $\Delta t = \Delta x^2$ to retain second-order spatial accuracy, then we'd need to decrease Δt by a factor of $\frac{1}{4}$ for each halving of Δx. However, there are certainly more efficient schemes.

(b) The scheme is explicit since it (ultimately) expresses v_m^{n+1} in terms of v_{m+1}^n, v_{m-1}^n, and v_m^{n-1}.

To analyze stability, we replace $g = e^{s\Delta t}$ in $p_{\Delta t,\Delta x}(s,\xi) = 0$ and solve for g to determine the roots of the amplification polynomial:

$$
\begin{aligned}
&\frac{1}{2\Delta t}\left(g - g^{-1}\right) - \frac{\alpha}{\Delta t^2}\left(2\cos\xi\Delta x - \left(g + g^{-1}\right)\right) = 0 \\
&\Rightarrow \quad \frac{1}{2}\left(g - g^{-1}\right) - \alpha\mu\left(2\cos\theta - \left(g + g^{-1}\right)\right) \\
&\Rightarrow \quad (2\alpha\mu + 1)g^2 - 4\alpha\mu\cos\theta g + (2\alpha\mu - 1) = 0 \\
&\Rightarrow \quad g_\pm = \frac{4\alpha\mu\cos\theta \pm \sqrt{(4\alpha\mu\cos\theta)^2 - 4(4\alpha^2\mu^2 - 1)}}{2(2\alpha\mu + 1)} \\
&\Rightarrow \quad g_\pm = \frac{2\alpha\mu\cos\theta \pm \sqrt{1 - 4\alpha^2\mu^2\sin^2\theta}}{2\alpha\mu + 1}.
\end{aligned}
$$

Now if $1 - 4\alpha^2\mu^2\sin^2\theta \geq 0$,

$$
|g_\pm| \leq \frac{2\alpha\mu|\cos\theta| + 1}{2\alpha\mu + 1} \leq 1,
$$

while if $1 - 4\alpha^2\mu^2\sin^2\theta < 0$,

$$
|g_\pm|^2 = \frac{4\alpha^2\mu^2\cos^2\theta + 4\alpha^2\mu^2\sin^2\theta - 1}{4\alpha^2\mu^2 + 4\alpha\mu + 1} = \frac{4\alpha^2\mu^2 - 1}{4\alpha^2\mu^2 + 4\alpha\mu + 1} < 1.
$$

It follows that the scheme is unconditionally stable. This does not violate the CFL condition, since consistency (rather than stability) requires that $\lambda \to 0$ as $\Delta t, \Delta x \to 0$

7. (10 Pts.)

Solution