Qualifying Examination on Applied Differential Equations

Saturday, April 1, 2006, 2:00 p.m.–6:00 p.m.

Solve all of the following 8 problems. In doing so, provide clear and concise arguments. Draw a figure when necessary.

Problem 1. Solve the following initial value problem and verify your solution:

$$u_x + u_y = u^2, \quad u(x,0) = h(x).$$

Problem 2. Consider an initial value problem for the *Korteweg–de Vries equation*

$$u_t + u_{xxx} + 6uu_x = 0, \quad x \in \mathbf{R}, \quad t > 0, \quad u(x,0) = \varphi(x). \tag{1}$$

Show that the following are conserved quantities for (1) (you may assume that the function $u(x,t)$ vanishes as $|x| \to \infty$, together with all of its derivatives):

- Mass:
$$\int_{-\infty}^{\infty} u(x,t)\,dx,$$
- Momentum:
$$\int_{-\infty}^{\infty} u^2(x,t)\,dx,$$
- Energy:
$$\int_{-\infty}^{\infty} \left(\frac{1}{2}u_x(x,t)^2 - u(x,t)^3\right)\,dx.$$

Problem 3. Let $0 < L < \infty$ and let $0 < p(x) \in C^\infty([0,L])$. Consider the following initial-boundary value problem on $(0,L) \times (0,\infty)$:

$$\begin{cases} \partial_t u = \partial_x(p(x)\partial_x u), & (x,t) \in (0,L) \times (0,\infty), \\ u(x,0) = \varphi(x), & \partial_x u(0,t) = \partial_x u(L,t) = 0. \end{cases} \tag{2}$$

Here $\varphi \in C^\infty([0,L])$. Compute the limit of $u(x,t)$ as $t \to \infty$.

Problem 4. Consider the initial value problem of the form

$$\frac{dy}{dt} = f(y), \quad y(0) = 0. \tag{3}$$

Show that there exists a continuous function $f : \mathbf{R} \to \mathbf{R}$ with $f(y) = 0$ precisely when $y = 0$ and such that f does not satisfy the Lipschitz condition in any neighborhood of 0, while the uniqueness for the initial value problem (3) holds.

Problem 5. Consider the second order ODE

$$x''(t) + x(t) + 2x^2(t) = 0. \tag{4}$$

- Find the conserved quantity for (4),
- Rewrite (4) as a 2×2 system of the first order,
- Find and classify the equilibrium points,
- Sketch the phase portrait of the equation.

Problem 6. Let $\Omega \subset \mathbf{R}^n$ be a bounded, open, and connected set. Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is a solution of

$$\Delta u + \sum_{k=1}^{n} a_k(x) \frac{\partial u}{\partial x_k} + c(x)u = 0 \quad \text{in} \quad \Omega$$

where $a_k(x)$, $1 \le k \le n$, and $c(x)$ are continuous in $\overline{\Omega}$, with $c(x) < 0$ in Ω. Show that $u = 0$ on $\partial\Omega$ implies that $u = 0$ in Ω.

Hint. Show that $\max u(x) \le 0$ and $\min u(x) \ge 0$.

Problem 7. Let $\Omega \subset \mathbf{R}^n$ be a smooth bounded domain and let $f \in C(\overline{\Omega})$. Find the minimum of the functional

$$E(u) = \int_\Omega \left(\frac{1}{2} \sum_{k=1}^{n} \left(\frac{\partial u}{\partial x_k} \right)^2 - f(x)u(x) \right) dx$$

on the space of smooth functions in $\overline{\Omega}$, subject to the constraints

$$u|_{\partial\Omega} = 0, \qquad \int_\Omega u(x)\, dx = A,$$

where A is a given constant. You may assume that a smooth solution of this problem exists. You may also regard the solution of

$$\Delta w = h \text{ in } \Omega, \quad w|_{\partial\Omega} = 0$$

as known, for any $h \in C(\overline{\Omega})$.

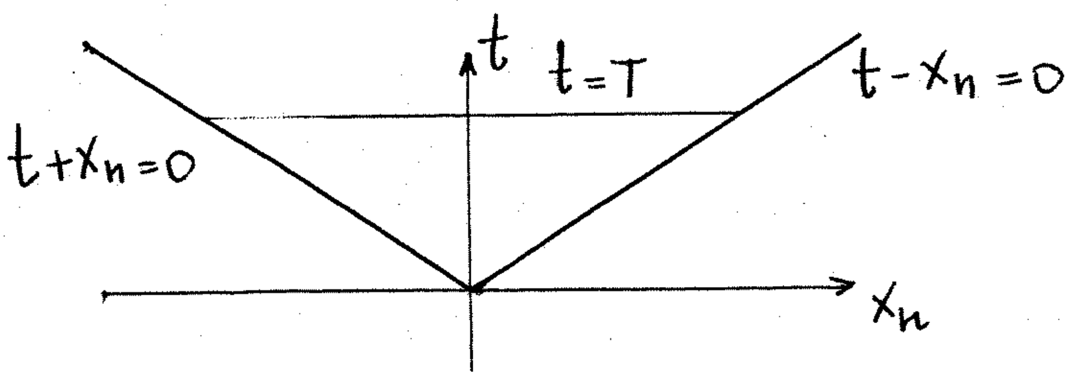
Hint. Use Lagrange multipliers.

Problem 8. Let $u(x,t) \in C^2(\mathbf{R}^n \times \mathbf{R})$ be a solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = 0$$

in the domain

$$\mathcal{D} = \{(x,t); x' = (x_1, \dots x_{n-1}) \in \mathbf{R}^{n-1},\ t \geq |x_n|\}.$$



In the picture, the variable $x' = (x_1, \dots x_{n-1})$ has been suppressed.

Assume for simplicity that $u = 0$ for $|x'| \geq R$ for some $R > 1$. Suppose that $u|_{\Gamma_1} = 0$ and $u|_{\Gamma_2} = 0$, where

$$\Gamma_1 = \{(x,t); x' \in \mathbf{R}^{n-1},\ t - x_n = 0,\ t > 0\},$$

and

$$\Gamma_2 = \{(x,t); x' \in \mathbf{R}^{n-1},\ t + x_n = 0,\ t > 0\}.$$

Prove that $u \equiv 0$.

Hint. Integrate by parts in

$$0 = \int \left(\frac{\partial^2 u}{\partial t^2} - \Delta u \right) \frac{\partial u}{\partial t}\, dx\, dt,$$

the integration being performed over the domain $\mathcal{D} \cap \{t \leq T\}$, where $T > 0$ is arbitrary. You may find it useful to make a change of variables $s = t - x_n$, $\tau = t + x_n$, $y' = x'$.