Math 269B, 2012 Winter, Homework 4

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1 Theory

1. Solve the heat equation $u_t = bu_{xx}$ on a interval $I \subset \mathbb{R}$ with *periodic* boundary conditions. How does $\int_I u(t,x)dx$ vary with time t?

2. (Strikwerda 6.1.4.) Use the representation (6.1.3) to verify the following estimates on the norms of $u(t,x)$:
$$\|u(t,\cdot)\|_1 \leq \|u_0\|_1 ,$$
$$\|u(t,\cdot)\|_\infty \leq \|u_0\|_\infty .$$
Show that if u_0 is nonnegative, then
$$\|u(t,\cdot)\|_1 = \|u_0\|_1 .$$

3. Determine the stability and accuracy of the following combination of the Lax-Wendroff and backward-time central-space schemes to solve $u_t + au_x = bu_{xx}$ (with $b > 0$):
$$\begin{aligned} 0 &= P_{k,h} v_m^n \\ &= \frac{1}{k}\left(v_m^{n+1} - v_m^n\right) + \frac{a}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) - \frac{a^2 k}{2h^2}\left(v_{m+1}^n - 2v_m^n + v_{m-1}^n\right) \\ &\quad - \frac{b}{h^2}\left(v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}\right) . \end{aligned}$$

2 Programming

1. Solve $u_t + au_x = 0$ numerically using the Lax-Friedrichs scheme. Take $a = 1$, $T = 1$, $x \in [0,1]$ with periodic boundary conditions, and $u_0(x) = \sin 2\pi x$. For each fixed λ within a decreasing sequence of λs (each satisfying the stability criterion), demonstrate convergence with $k/h =: \lambda$ by plotting the logarithm of the L^2-norm of the error (between the analytic solution and the numerical solution) versus the logarithm of h. Verify that the slope suggested by your plot agrees with theory, and estimate the error constant C_λ in the relation error $= C_\lambda h^p$. Use enough values of λ to estimate the relation between C_λ and λ. What appears to happen to C_λ as $\lambda \to 0+$, i.e., as you shrink k relative to h? What happens if, instead of taking $k = \lambda h$, you take $k = h^2$? Explain your numerical results in the context of the theoretical convergence analysis of the Lax-Friedrichs scheme.

2. Implement the scheme from problem 3 in the Theory section and confirm numerically the theoretical rate of convergence. Use convenient (but non-trivial) initial and boundary conditions such that the solution takes a simple form.

3. Write a function implementing the Thomas algorithm presented in Strikwerda 3.5. Specifically, we solve the system of equations
$$a_i w_{i-1} + b_i w_i + c_i w_{i+1} = d_i, \quad i = 1, \ldots, m-1,$$

with $w_0 = \beta_0$ and $w_m = \beta_m$. The solution is given by

$$w_i = p_{i+1} w_{i+1} + q_{i+1}$$

where p_{i+1} and q_{i+1} are defined recursively by

$$\begin{aligned} p_{i+1} &= -\left(a_i p_i + b_i\right)^{-1} c_i, \\ q_{i+1} &= \left(a_i p_i + b_i\right)^{-1} \left(d_i - a_i q_i\right), \end{aligned}$$

and with p_1 and q_1 determined by the boundary conditions. For the next homework, be prepared to utilize your function implementing the Thomas algorithm to write a function which solves *periodic* tridiagonal systems.