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Applied Differential Equations Qualifying Exam, Fall 2003

1. For the ODE

$$\begin{aligned} u_t &= v - u^2 \\ v_t &= u - v \end{aligned}$$

   (a) Find stationary points and their type.
   (b) Draw the plase plane and find all connections between the stationary points.

   Solution

   (a)
   (b)

2. (a) Let Ω_1 and Ω_2 be two smooth sets in $\mathbb{R}^2$ with Ω_1 a (strict) subset of Ω_2. Let $-\lambda_1$ and $-\lambda_2$ be the smallest (i.e., least negative) eigenvalues for the Dirichlet problem on Ω_1 and Ω_2, with eigenfunctions ϕ_1 and ϕ_2, respectively. That is,

$$\begin{aligned} \Delta\phi_1 &= -\lambda_1\phi_1 \text{ in } \Omega_1; \\ \Delta\phi_2 &= -\lambda_2\phi_2 \text{ in } \Omega_2; \\ \phi_1 &= 0 \text{ on } \partial\Omega_1; \\ \phi_2 &= 0 \text{ on } \partial\Omega_2. \end{aligned}$$

   Show that $\lambda_1 > \lambda_2 > 0$. Hint: Use the variational characterization of the smallest eigenvalue λ for a set Ω that $\lambda = \min_u \int_\Omega |\nabla u|^2 dx / \int_\Omega u^2 dx$.

   (b) Suppose Ω is a smooth set in $\mathbb{R}^2$ with mirror symmetry about the y axis, i.e., if $(x, y) \in \Omega$ then $(-x, y) \in \Omega$. Let ϕ be the eigenfunction for the Dirichlet problem on Ω with the smallest eigenvalue. Use the result in (a) to show that $\phi(x, y) = \phi(-x, y)$.

   Solution

   (a) Given $u_1 \in C^1(\Omega_1)$, let $u_2 \in C(\Omega_2)$ extend u_1 on $\Omega_2 \backslash \Omega_1$ by defining $u_2 = 0$ there. Then u_2 is weakly differentiable, and

$$\int_{\Omega_1} |\nabla u_1|^2 dx = \int_{\Omega_2} |\nabla u_2|^2 dx, \quad \int_{\Omega_1} u_1^2 dx = \int_{\Omega_2} u_2^2 dx,$$

   hence the Rayleigh Quotients are identical. Since

$$\lambda_j = -\max_{u=0 \text{ on } \partial\Omega_j} \frac{(\Delta u, u)}{(u, u)} = \min_{u=0 \text{ on } \partial\Omega_j} \frac{\int_{\Omega_j} |\nabla u|^2 dx}{\int_{\Omega_j} u^2 dx},$$

   we conclude that $0 < \lambda_2 < \lambda_1$, since the set of admissible functions with $u = 0$ on Ω_2 (strictly) contains those with $u = 0$ on Ω_1.

   (b)

3. The function

$$h(X, T) = (4\pi T)^{-1/2} \exp\left(-X^2/4T\right)$$

   satisfies (you do not need to show this)

$$h_T = h_{XX}.$$

Using this result, verify that for any smooth function U

$$u(x,t) = \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} U(\xi) h(x - t^2 - \xi, t) d\xi$$

satisfies

$$u_t + xu = u_{xx}.$$

Given that $U(x)$ is bounded and continuous everywhere on $-\infty \leq x \leq \infty$, establish that

$$\lim_{t \to 0} \int_{-\infty}^{\infty} U(\xi) h(x - \xi, t) d\xi = U(x)$$

and show that $u(x,t) \to U(x)$ as $t \to 0$. (You may use the fact that $\int_0^\infty e^{-\xi^2} d\xi = \sqrt{\pi}/2$.)

Solution

We compute

$$\begin{aligned} u_t(x,t) &= (t^2 - x) \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} U(\xi) h(x - t^2 - \xi, t) d\xi \\ &\quad + \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} U(\xi) \left(-2t h_X(x - t^2 - \xi, t) + h_T(x - t^2 - \xi, t)\right) d\xi; \end{aligned}$$

$$\begin{aligned} u_{xx}(x,t) &= t^2 \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} U(\xi) h(x - t^2 - \xi, t) d\xi \\ &\quad -2t \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} U(\xi) h_X(x - t^2 - \xi, t) d\xi \\ &\quad + \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} U(\xi) h_{XX}(x - t^2 - \xi, t) d\xi; \end{aligned}$$

so upon summing,

$$(u_t + xu_x - u_{xx})(x,t) = \exp\left(\frac{1}{3}t^3 - xt\right) \int_{-\infty}^{\infty} U(\xi) \left(h_T(x - t^2 - \xi, t) - h_{XX}(x - t^2 - \xi, t)\right) d\xi = 0$$

since h satisfies $h_T - h_{XX} = 0$.

We compute

$$\begin{aligned} \lim_{t \searrow 0} \int_{-\infty}^{\infty} U(\xi) h(x - \xi, t) d\xi &= \lim_{t \searrow 0} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4t} U(\xi) d\xi \\ &= \lim_{t \searrow 0} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} U\left(x + \eta\sqrt{4t}\right) d\eta \qquad \left[\xi = x + \eta\sqrt{4t}\right] \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} \left(\lim_{t \searrow 0} U\left(x + \eta\sqrt{4t}\right)\right) d\eta, \end{aligned}$$

by the Dominated Convergence Theorem, and

$$\lim_{t \searrow 0} U\left(x + \eta\sqrt{4t}\right) = U(x),$$

hence

$$\lim_{t \searrow 0} \int_{-\infty}^{\infty} U(\xi) h(x - \xi, t) d\xi = U(x) \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2} d\eta = U(x).$$

It follows that $u(x,t) \to U(x)$ as $t \to 0$.

4. Find the characteristics of the partial differential equation

$$xu_{xx} + (x-y)u_{xy} - yu_{yy} = 0,\ x > 0,\ y > 0,$$

and then show that it can be transformed into the canonical form

$$(\xi^2 + 4\eta)u_{\xi\eta} + \xi u_\eta = 0$$

whence ξ and η are suitably chosen canonical coordinates. Use this to obtain the general solution in the form

$$u(\xi,\eta) = f(\xi) + \int^{\eta} \frac{g(\eta')}{(\xi^2 + 4\eta')^{1/2}} d\eta'$$

where f and g are arbitrary functions of ξ and η.

Solution

Let $\gamma(s) = (f(s), g(s))$ be a curve in $\mathbb{R}^2$, and suppose we specify

$$u|_\gamma = h,\ u_x|_\gamma = \phi,\ u_y|_\gamma = \psi.$$

Then

$$\phi' = u_{xx}f' + u_{xy}g',\ \psi' = u_{xy}f' + u_{yy}g',$$

and, together with the fact that $au_{xx} + bu_{xy} + cu_{yy} = d$, we obtain

$$\begin{pmatrix} a & b & c \\ f' & g' & 0 \\ 0 & f' & g' \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix} = \begin{pmatrix} d \\ \phi' \\ \psi' \end{pmatrix}.$$

γ is characteristic if the above system is singular, i.e., if

$$0 = \begin{vmatrix} a & b & c \\ f' & g' & 0 \\ 0 & f' & g' \end{vmatrix} = a(g')^2 - bf'g' + c(f')^2.$$

Solving for $dy/dx = f'/g'$, and identifying $a = x$, $b = x - y$, $c = -y$, yields

$$\frac{dy}{dx} = \frac{f'}{g'} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a} = 1,\ -\frac{y}{x}.$$

Each of these solve to give $x - y = \text{const}$ and $xy = \text{const}$. We set $\xi = x - y$ and $\eta = xy$, and compute

$$\begin{aligned} u_x &= u_\xi + yu_\eta; \\ u_y &= -u_\xi + xu_\eta; \\ u_{xx} &= u_{\xi\xi} + 2yu_{\xi\eta} + y^2u_{\eta\eta}; \\ u_{xy} &= -u_{\xi\xi} + (x-y)u_{\xi\eta} + xyu_{\eta\eta} + u_\eta; \\ u_{yy} &= u_{\xi\xi} - 2xu_{\xi\eta} + x^2u_{\eta\eta}. \end{aligned}$$

Adding and cancelling terms then gives

$$0 = xu_{xx} + (x-y)u_{xy} - yu_{yy} = (\xi^2 + 4\eta)u_{\xi\eta} + \xi u_\eta.$$

This is a separable ODE in u_η, whose solution is

$$u_\eta = (\xi^2 + 4\eta)^{-1/2}g(\eta),$$

hence

$$u(\xi,\eta) = f(\xi) + \int^{\eta} (\xi^2 + 4\eta')^{-1/2}g(\eta')d\eta'.$$

5. State Parseval's relation for Fourier transforms. Find the Fourier transform $\widehat{f}(\xi)$ of

$$f(x) = \begin{cases} e^{i\alpha x}/2\sqrt{\pi y}, & |x| \le y \\ 0, & |x| > y \end{cases}$$

in which y and α are constants. Use this in Parseval's relation to show that

$$\int_{-\infty}^{\infty} \frac{\sin^2(\alpha - \xi)y}{(a-\xi)^2} d\xi = \pi y.$$

What does the transform $\widehat{f}(\xi)$ become in the limit $y \to \infty$?
Use Parseval's relation to show that

$$\frac{\sin(\alpha-\beta)y}{\alpha-\beta} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\alpha-\xi)y}{\alpha-\xi}\frac{\sin(\beta-\xi)y}{\beta-\xi} d\xi.$$

Solution

Recall that the Fourier transform is given by (formally at least)

$$\widehat{f}(\xi) = \mathcal{F}_x(f(x))(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx.$$

Parseval's relation (also known as Plancherel's Theorem) states that

$$\left\|\widehat{f}\right\|_{L^2}^2 = 2\pi \|f\|_{L^2}^2.$$

We compute

$$\begin{aligned} \widehat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx \\ &= \frac{1}{2\sqrt{\pi y}} \int_{|x|\le y} e^{-ix\xi} e^{i\alpha x} dx \\ &= \frac{1}{2\sqrt{\pi y}} \frac{e^{ix(\alpha-\xi)}}{i(\alpha-\xi)}\bigg|_{-y}^{y} \\ &= \frac{1}{\sqrt{\pi y}} \frac{\sin(y(\alpha-\xi))}{\alpha-\xi}, \end{aligned}$$

and so, by Parseval's relation,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\sin^2(y(\alpha-\xi))}{(\alpha-\xi)^2} d\xi &= \pi y \left\|\widehat{f}\right\|_{L^2}^2 \\ &= 2\pi^2 y \|f\|_{L^2}^2 \\ &= 2\pi^2 y \int_{-\infty}^{\infty} |f(x)|^2 dx \\ &= 2\pi^2 y \int_{-y}^{y} \frac{1}{4\pi y} dx \\ &= \pi y. \end{aligned}$$

In the limit as $y \to \infty$, $\widehat{f}(\xi) \to \delta(\alpha - \xi)$.

Let

$$g(x) = \begin{cases} e^{i\beta y}/2\sqrt{\pi y}, & |x| \le y \\ 0, & |x| > y \end{cases}.$$

Then by Parseval's relation,

$$\begin{aligned}
\left(\widehat{f}, \overline{\widehat{g}}\right)_{L^2} &= \frac{1}{2}\left(\left\|\widehat{f}+\widehat{g}\right\|_{L^2}^2 - \left\|\widehat{f}\right\|_{L^2}^2 - \|\widehat{g}\|_{L^2}^2\right) \\
&= \pi\left(\|f+g\|_{L^2}^2 - \|f\|_{L^2}^2 - \|g\|_{L^2}^2\right) \\
&= 2\pi\left(f, \overline{g}\right)_{L^2},
\end{aligned}$$

hence

$$\begin{aligned}
\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin(y(\alpha-\xi))}{\alpha-\xi}\frac{\sin(y(\beta-\xi))}{\beta-\xi}d\xi &= y\left(\widehat{f}, \overline{\widehat{g}}\right) \\
&= 2\pi y\left(f, \overline{g}\right)_{L^2} \\
&= 2\pi y\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx \\
&= 2\pi y\int_{-y}^{y} \frac{1}{4\pi y}e^{i(\alpha-\beta)x}dx \\
&= \frac{\sin(y(\alpha-\beta))}{\alpha-\beta}.
\end{aligned}$$

6. (a) For the cubic equation

$$\epsilon^3 x^3 - 2\epsilon x^2 + 2x - 6 = 0,$$

write the solution x in the asymptotic expansion $x = x_0 + \epsilon x_1 + O(\epsilon^2)$ as $\epsilon \to 0$. Find the first two terms x_0 and x_1 for all solutions x.

(b) For the ODE

$$\begin{aligned}
u_t &= u - \epsilon u^3, \\
u(0) &= 1,
\end{aligned}$$

write $u = u_0(t) + \epsilon u_1(t) + \epsilon^2 u_2(t) + O(\epsilon^3)$ as $\epsilon \to 0$. Find the first three terms u_0, u_1, and u_2.

Solution

(a) Clearly, to leading order, $x_0 = 3$. To determine x_1, let $x = x_0 + \epsilon x_1 + O(\epsilon^2)$ and substitute into the cubic equation:

$$\begin{aligned}
0 &= \epsilon^3 x^3 - 2\epsilon x^2 + 2x - 6 \\
&= \epsilon^3\left(x_0 + \epsilon x_1 + O(\epsilon^2)\right)^3 - 2\epsilon\left(x_0 + \epsilon x_1 + O(\epsilon^2)\right) + 2\left(x_0 + \epsilon x_1 + O(\epsilon^2)\right) - 6 \\
&= 2x_0 - 6 + \left(-2x_0 + 2x_1\right)\epsilon + O(\epsilon^2),
\end{aligned}$$

and so $x_1 = x_0 = 3$.

(b) We substitute into the differential equation:

$$\begin{aligned}
0 &= u_t - u + \epsilon u^3 \\
&= \left(u_0 + \epsilon u_1 + \epsilon^2 u_2 + O(\epsilon^3)\right)_t - \left(u_0 + \epsilon u_1 + \epsilon^2 u_2 + O(\epsilon^3)\right) + \epsilon\left(u_0 + \epsilon u_1 + \epsilon^2 u_2 + O(\epsilon^3)\right)^3 \\
&= (u_0)_t - u_0 + \left((u_1)_t - u_1 + u_0^3\right)\epsilon + \left((u_2)_t - u_2 + 3u_0^2 u_1\right)\epsilon^2 + O(\epsilon^3).
\end{aligned}$$

Thus,

$$
\begin{aligned}
0 = (u_0)_t - u_0,\ u_0(0) = 1 \quad &\Rightarrow \quad u_0(t) = e^t; \\
0 = (u_1)_t - u_1 + u_0^3 = (u_1)_t - u_1 + e^{3t},\ u_1(0) = 0 \quad &\Rightarrow \quad u_1(t) = \frac{1}{2}e^t - \frac{1}{2}e^{3t}; \\
0 = (u_2)_t - u_2 + 3u_0^2 u_1 = (u_2)_t - u_2 + \frac{3}{2}e^{3t} - \frac{3}{2}e^{5t},\ u_2(0) = 0 \quad &\Rightarrow \quad u_2(t) = \frac{3}{8}e^t - \frac{3}{4}e^{3t} + \frac{3}{8}e^{5t}.
\end{aligned}
$$