Math 269B, 2012 Winter, Homework 3

Professor Joseph Teran   Jeffrey Lee Hellrung, Jr.

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1 Theory

1. (Strikwerda 5.1.2.) Show that the modified leapfrog scheme (5.1.6) is stable for ϵ satisfying
$$0 < \epsilon \leq 1 \quad \text{if} \quad 0 < a^2\lambda^2 \leq \frac{1}{2}$$
and
$$0 < \epsilon \leq 4a^2\lambda^2\left(1 - a^2\lambda^2\right) \quad \text{if} \quad \frac{1}{2} \leq a^2\lambda^2 < 1.$$
Note that these limits are not sharp. It is possible to choose ϵ larger than these limits and still have the scheme be stable.

2. Derive the stability condition for the backward-time forward-space scheme
$$\frac{1}{k}\left(v_m^{n+1} - v_m^n\right) + \frac{a}{h}\left(v_{m+1}^{n+1} - v_m^{n+1}\right) = 0$$
used to approximate solutions to $u_t + au_x = 0$ with, say, $x \in [0,1]$ and periodic boundary conditions. Give an example of an initial condition v_m^0 and an explicit expression for v_m^n that demonstrate unstable behavior for a particular λ (your choice) which fails to satisfy the stability condition. Does the growth in your example agree with your theoretical amplification factor?

3. Prove that numerical solutions to the Lax-Friedrichs scheme
$$\frac{1}{k}\left(v_m^{n+1} - \frac{1}{2}\left(v_{m+1}^n + v_{m-1}^n\right)\right) + \frac{a}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) = 0$$
converge to solutions to the corresponding modified equation
$$u_t + au_x = \frac{h^2}{2k}\left(1 - \left(\frac{ak}{h}\right)^2\right)u_{xx}$$
to second order accuracy in L^∞. I.e., show that $\left|v_m^n - u_{k,h}\left(t_n, x_m\right)\right| \to 0$ as $h, k \to 0$ (according to the stability criterion), where the subscripts on $u_{k,h}$ only indicate that the solution to the modified equation is parameterized by k, h.

4. (Strikwerda 4.1.2.) Show that the $(2,2)$ leapfrog scheme for $u_t + au_{xxx} = f$ (see (2.2.15)) given by
$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\delta^2\delta_0 v_m^n = f_m^n,$$
with $\nu = k/h^3$ constant, is stable if and only if
$$|a\nu| < \frac{2}{3^{3/2}}.$$

5. (Strikwerda 3.2.1.) Show that the (forward-backward) MacCormack scheme

$$\begin{aligned}
\tilde{v}_m^{n+1} &= v_m^n - a\lambda\left(v_{m+1}^n - v_m^n\right) + k f_m^n, \\
v_m^{n+1} &= \frac{1}{2}\left(v_m^n + \tilde{v}_m^{n+1} - a\lambda\left(\tilde{v}_m^{n+1} - \tilde{v}_{m-1}^{n+1}\right) + k f_m^{n+1}\right)
\end{aligned}$$

is a second-order accurate scheme for the one-way wave equation (1.1.1). Show that for $f = 0$ it is identical to the Lax-Wendroff scheme (3.1.1).

2 Programming

1. Solve $u_t + au_x = 0$ numerically using the Lax-Friedrichs scheme. Take $a = 1$, $T = 1$, $x \in [0,1]$ with periodic boundary conditions, and $u_0(x) = \sin 2\pi x$. Demonstrate convergence for various decreasing values of $\lambda := k/h$ (satisfying the stability criterion) by plotting the logarithm of the L^2-norm of the error (between the analytic solution and the numerical solution) versus the logarithm of h. Verify that the slope suggested by your plot agrees with theory, and estimate the error constant C_λ in the relation error $= C_\lambda h^p$. Use enough values of λ to estimate the relation between C_λ and λ. What appears to happen to C_λ as $\lambda \to 0+$, i.e., as you shrink k relative to h? What happens if, instead of taking $k = \lambda h$, you take $k = h^2$? Explain your numerical results in the context of the theoretical convergence analysis of the Lax-Friedrichs scheme.

2. For the one-way wave equation $u_t + au_x = 0$, investigate how close the numerical solution to a finite difference scheme is to the solution to the corresponding modified equation. To be concrete, suppose a pulse initial condition $u_0(x) = \frac{1}{2}\left(1 + |x|/x\right)$, $x \in [-1,1]$, and periodic boundary conditions. Take $a = 1$, $k/h = 0.5$, and final time $T = 0.5$. Compare the following finite difference schemes: upwinding, Lax-Friedrichs, and Lax-Wendroff. Also, include a derivation of the respective corresponding modified equations. You may find solutions to the modified equations using any appropriate method (i.e., analytically or to a sufficiently high accuracy numerically).