Math 269B, 2012 Winter, Homework 2

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1 Theory

1. (Strikwerda 2.1.9.) Finite Fourier Transforms. For a function v_m defined on the integers, $m = 0, 1, \ldots, M-1$, we can define the Fourier transform as

$$\hat{v}_\ell = \sum_{m=0}^{M-1} e^{-2i\pi\ell m/M} v_m \quad \text{for } \ell = 0, \ldots, M-1.$$

For this transform prove the Fourier inversion formula

$$v_m = \frac{1}{M} \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \hat{v}_\ell,$$

and the Pareseval's relation

$$\sum_{m=0}^{M-1} |v_m|^2 = \frac{1}{M} \sum_{\ell=0}^{M-1} |\hat{v}_\ell|^2 .$$

Note that v_m and $\hat{v}_\ell$ can be defined for all integers by making them periodic with period M.

2. Prove convergence for the Beam-Warming scheme

$$u_m^{n+1} = u_m^n - \frac{ak}{2h}\left(3u_m^n - 4u_{m-1}^n + u_{m-2}^n\right) + \frac{a^2k^2}{2h^2}\left(u_m^n - 2u_{m-1}^n + u_{m-2}^n\right)$$

used to approximate solutions to $u_t + au_x = 0$ for $a > 0$.

3. (Strikwerda 2.2.4.) Show that the box scheme

$$\frac{1}{2k}\left(\left(v_m^{n+1} + v_{m+1}^{n+1}\right) - \left(v_m^n + v_{m+1}^n\right)\right) + \frac{a}{2h}\left(\left(v_{m+1}^{n+1} - v_m^{n+1}\right) + \left(v_{m+1}^n - v_m^n\right)\right) = f_m^n$$

is consistent with the one-way wave equation $u_t + au_x = f$ and is stable for all values of λ.

4. (Strikwerda 2.2.6.) Determine the stability of the following scheme, sometimes called the Euler backward scheme, for $u_t + au_x = f$:

$$\begin{aligned} v_m^{n+1/2} &= v_m^n - \frac{a\lambda}{2}\left(v_{m+1}^n - v_{m-1}^n\right) + kf_m^n, \\ v_m^{n+1} &= v_m^n - \frac{a\lambda}{2}\left(v_{m+1}^{n+1/2} - v_{m-1}^{n+1/2}\right) + kf_m^{n+1}. \end{aligned}$$

The variable $v^{n+1/2}$ is a temporary variable, as is $\tilde{v}$ in Example 2.2.5.

2 Programming

1. (Strikwerda 2.3.3.) Solve the initial value problem for equation

$$u_t + \left(1 + \frac{1}{4}(3-x)(1+x)\right) u_x = 0$$

on the interval $[-1, 3]$ with the Lax-Friedrichs scheme (2.3.1) with λ equal to 0.8. Demonstrate that the instability phenomena occur where $|a(t,x)\lambda|$ is greater than 1 and where there are discontinuities in the solution. Use the same initial data as in Exercise 2.3.1. Specify the solution to be 0 at both boundaries. Compute up to the time of 0.2 and use successively smaller values of h to show the location of the instability.

2. Investigate (via numerical evidence) the convergence (or lack thereof) of the forward-time central-space scheme

$$\frac{1}{k}\left(u_m^{n+1} - u_m^n\right) + \frac{a}{2h}\left(u_{m+1}^n - u_{m-1}^n\right) = 0$$

in the L^∞-norm. Use the same scenarios from Homework 1, e.g., compare your results using smooth, continuous-but-non-smooth, and discontinuous initial conditions. Be sure to restrict the relation between k and h appropriately. Compare convergence in the L^∞-norm with convergence in the L^2-norm.