Qualifying Examination on Applied Differential Equations

Tuesday, September 21 2004, 9.00 a.m.–1.00 p.m.

Solve all of the following 8 problems. In doing so, provide clear and concise arguments. Draw a figure when necessary.

Problem 1. Solve the following initial-boundary value problem for the wave equation with a potential term,

$$
\begin{aligned}
&(\partial_t^2 - \partial_x^2)\, u + u = 0, \quad 0 < x < \pi, \quad t > 0, \\
&u(0,t) = u(\pi, t) = 0, \quad t > 0, \\
&u(x,0) = f(x), \quad \partial_t u(x,0) = 0, \quad 0 < x < \pi,
\end{aligned}
$$

where

$$
f(x) = \begin{cases} x & \text{if } x \in (0, \pi/2) \\ \pi - x & \text{if } x \in (\pi/2, \pi). \end{cases}
$$

The answer should be given in terms of an infinite series of explicitly given functions.

Problem 2. Let $u(x,t)$ be a bounded solution to the Cauchy problem for the heat equation

$$
\begin{cases} \partial_t u = a^2 \partial_x^2 u, \quad t > 0, \quad x \in \mathbf{R}, \ a > 0, \\ u(x,0) = \varphi(x). \end{cases}
$$

Here $\varphi(x) \in C(\mathbf{R})$ satisfies

$$
\lim_{x \to +\infty} \varphi(x) = b, \quad \lim_{x \to -\infty} \varphi(x) = c.
$$

Compute the limit of $u(x,t)$ as $t \to +\infty$, $x \in \mathbf{R}$. Justify your argument carefully.

Problem 3. Let us consider a damped wave equation,

$$
\begin{cases} (\partial_t^2 - \Delta + a(x)\partial_t)\, u = 0, \quad (x,t) \in \mathbf{R}^3 \times \mathbf{R}, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1 \end{cases}
$$

Here the damping coefficient $a \in C_0^\infty(\mathbf{R}^3)$ is a non-negative function and u_0, $u_1 \in C_0^\infty(\mathbf{R}^3)$. Show that the energy of the solution $u(x,t)$ at time t,

$$
E(t) = \frac{1}{2} \int_{\mathbf{R}^3} \left(|\nabla_x u|^2 + |\partial_t u|^2 \right) dx
$$

is a decreasing function of $t \geq 0$.

Problem 4. Prove that each solution (except $x_1 = x_2 = 0$) of the autonomous system

$$\begin{cases} x_1' = x_2 + x_1(x_1^2 + x_2^2) \\ x_2' = -x_1 + x_2(x_1^2 + x_2^2) \end{cases}$$

blows up in finite time. What is the blow-up time for the solution which starts at the point $(1,0)$ when $t = 0$?

Problem 5. Let us consider a generalized Volterra-Lotka system in the plane, given by

$$x'(t) = f(x(t)), \quad x(t) \in \mathbf{R}^2, \tag{1}$$

where $f(x) = (f_1(x), f_2(x)) = (ax_1 - bx_1x_2 - ex_1^2, -cx_2 + dx_1x_2 - fx_2^2)$, and a, b, c, d, e, f are positive constants. Show that

$$\operatorname{div}(\varphi f) \neq 0 \quad x_1 > 0,\ x_2 > 0,$$

where $\varphi(x_1, x_2) = 1/(x_1x_2)$. Using this observation, prove that the autonomous system (1) has no closed orbits in the first quadrant.

Problem 6. Let $q \in C_0^1(\mathbf{R}^3)$. Prove that the vector field

$$u(x) = \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{q(y)(x-y)}{|x-y|^3}\,dy$$

enjoys the following properties:

1. $u(x)$ is conservative
2. $\operatorname{div} u(x) = q(x) \quad$ for all $x \in \mathbf{R}^3$
3. $|u(x)| = \mathcal{O}\left(|x|^{-2}\right) \quad$ for large x.

Furthermore, prove that the properties (1), (2), and (3) above determine the vector field $u(x)$ uniquely.

Problem 7. Consider the partial differential equation

$$uu_z + u_t + u = 0, \quad (z,t) \in \mathbf{R}^2.$$

- Find the particular solution that satisfies the condition $u(0,t) = e^{-2t}$.
- Show that at the point $(z,t) = (1/9, \ln 2)$, $u = 1/3$.

Problem 8. The function $y(x,t)$ satisfies the partial differential equation

$$x\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x \partial t} + 2y = 0,$$

and the boundary conditions

$$y(x,0) = 1, \quad y(0,t) = e^{-at},$$

where $a \geq 0$. Find the Laplace transform, $\overline{y}(x,s)$, of the solution, and hence derive an expression for $y(x,t)$ in the domain $x \geq 0$, $t \geq 0$.