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Basic Examination, F02

1. Let K be a compact subset and F be a closed subset in the metric space X. Suppose $K \cap F = \emptyset$. Prove that
$$0 < \inf\{d(x, y) : x \in K, y \in F\}.$$

   Solution

   Let $f : K \to \mathbb{R}$ be defined by
$$f(x) = \inf_{y \in F} d(x, y).$$
   It is evident that f is continuous, hence must achieve its minimum m for some $x \in K$. Now if $m = 0$, this implies that $x \in \overline{F} = F$, hence $K \cap F \neq \emptyset$, which is a contradiction to the given. Thus
$$0 < m \leq \inf_{x \in K} f(x) = \inf_{x \in K, y \in F} d(x, y).$$

2. Show why the Least Upper Bound Property (every set bounded above has a least upper bound) implies the Cauchy Completeness Property (every Cauchy sequence has a limit) of the real numbers.

   Solution

   Let $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be a Cauchy sequence. Let N_i be large enough such that $|x_n - x_m| < 1/i$ for all $n, m \geq N_i$, and also such that $N_i < N_{i+1}$, $i \geq 1$. Let
$$E_i = \bigcup_{n=N_i}^{\infty} \{x_n\},$$
$$\alpha_i = \sup E_i,$$
   whose existence is guaranteed by the Least Upper Bound Property of $\mathbb{R}$, since E_i is bounded above by $x_{N_i} + 1/i$. Then $\{\alpha_i\}_{i=1}^{\infty}$ is a monotonically decreasing sequence bounded below by $x_{N_1} - 1$. Since the Least Upper Bound Property and the Greatest Lower Bound Property are equivalent for $\mathbb{R}$ (by the isomorphism $x \leftrightarrow -x$), it follows that $\bigcup_{i=1}^{\infty}\{\alpha_i\}$ has a greatest lower bound, say, $\alpha^* \in \mathbb{R}$.

   Now given $\epsilon > 0$, we can choose i such that $\alpha_i - \alpha^* < \epsilon$ and such that $1/i < \epsilon$. Then for $n \geq N_i$, $x_n \in E_i$, hence $\alpha_i - x_n \leq 1/i < \epsilon$, hence
$$|x_n - \alpha^*| \leq |x_n - \alpha_i| + |\alpha_i - \alpha^*| < 2\epsilon,$$
   showing that $x_n \to \alpha^*$.

3. Show that there is a subset of the real numbers which is not the countable intersection of open subsets.

   Solution

   $\mathbb{Q}$ cannot be expressed as a countable intersection of open sets.

   (S02.2)

4. By integrating the series
$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + x^8 \cdots$$
   prove that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} \cdots$. Justify carefully all the steps (especially taking the limit as $x \to 1$ from below).

   Solution

We first note that

$$\int_0^\alpha \frac{1}{1+x^2}dx = \arctan(\alpha) \to \frac{\pi}{4}$$

as $\alpha \uparrow 1$. Also, for all $\alpha \in [0,1)$,

$$\int_0^\alpha \frac{1}{1+x^2}dx = \int_0^\alpha \left(\sum_{i=0}^\infty (-1)^i x^{2i}\right) dx = \sum_{i=0}^\infty (-1)^i \int_0^\alpha x^{2i}dx = \sum_{i=0}^\infty (-1)^i \frac{\alpha^{2i+1}}{2i+1},$$

where the exchange of the integral and summation is justified by the uniform convergence of the series on $[0,\alpha]$. Set

$$S_n(\alpha) = \sum_{i=0}^n (-1)^i \frac{\alpha^{2i+1}}{2i+1},$$
$$S(\alpha) = \lim_{n\to\infty} S_n(\alpha)$$

for $\alpha \in [0,1]$. We know that $S(\alpha) = \arctan(\alpha)$ for $\alpha \in [0,1)$, hence we just need to show that $S(\alpha) \to S(1)$ as $\alpha \uparrow 1$. Indeed, for any fixed $n \geq 0$,

$$\begin{array}{rcl} |S(1) - S(\alpha)| & \leq & |S(1) - S_n(1)| + |S_n(1) - S_n(\alpha)| + |S_n(\alpha) - S(\alpha)| \\ & < & \frac{1}{2n+3} + |S_n(1) - S_n(\alpha)| + \frac{\alpha^{2n+3}}{2n+3} \\ & < & \frac{2}{2n+3} + |S_n(1) - S_n(\alpha)| \end{array}.$$

As $\alpha \uparrow 1$, $S_n(\alpha) \to S_n(1)$, hence

$$\lim_{\alpha\uparrow 1} |S(1) - S(\alpha)| \leq \frac{2}{2n+3},$$

and since n was arbitrary, we conclude that

$$\lim_{\alpha\uparrow 1} S(\alpha) = S(1),$$

hence

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = S(1) = \lim_{\alpha\uparrow 1} S(\alpha) = \lim_{\alpha\uparrow 1} \arctan(\alpha) = \frac{\pi}{4}.$$

5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has partial derivatives at every point bounded by $A > 0$.

   (a) Show that there is an $M > 0$ such that

   $$|f((x,y)) - f((x_1,y_1))| \leq M((x-x_1)^2 + (y-y_1)^2)^{1/2}.$$

   (b) What is the smallest value of M (in terms of A) for which this always works?

   (c) Give an example where that value of M makes the inequality an equality.

   Solution

   (a) Given $(x,y), (x_1,y_1) \in \mathbb{R}^2$, two applications of the Mean Value Theorem yields points c between x and x_1 and d between y and y_1 such that

   $$f(x_1,y) - f(x,y) = \frac{\partial f}{\partial x}(c,y)(x_1 - x),$$

   $$f(x_1,y_1) - f(x_1,y) = \frac{\partial f}{\partial y}(x_1,d)(y_1 - y),$$

   hence

   $$|f(x_1,y_1) - f(x,y)| \leq A|x_1 - x| + A|y_1 - y| \leq \sqrt{2}A\sqrt{(x_1-x)^2 + (y_1-y)^2}.$$

(b) The above inequality is tight for general f and $(x,y),(x_1,y_1)\in\mathbb{R}^2$, i.e., $M\geq A/\sqrt{2}$.

(c) Let

$$f(x,y)=A(x+y).$$

Then

$$|f(x+t,y+t)-f(x,y)|=2A|t|=\sqrt{2}A\sqrt{t^2+t^2}.$$

6. Suppose $F:\mathbb{R}^3\to\mathbb{R}^2$ is continuously differentiable. Suppose for some $v_0\in\mathbb{R}^3$ and $x_0\in\mathbb{R}^2$ that $F(v_0)=x_0$ and $F'(v_0):\mathbb{R}^3\to\mathbb{R}^2$ is onto. Show that there is a continuously differentiable function γ, $\gamma:(-\epsilon,\epsilon)\to\mathbb{R}^3$ for some $\epsilon>0$, such that

   (a) $\gamma'(0)\neq\vec{0}\in\mathbb{R}^3$, and

   (b) $F(\gamma(t))=x_0$ for all $t\in(-\epsilon,\epsilon)$.

   Solution

7. Let $T:V\to W$ be a linear transformation of finite dimensional real vector spaces. Define the transpose of T and then prove both of the following:

   (a) $\operatorname{im}(T)^\circ=\ker(T^t)$, where $\operatorname{im}(T)^\circ$ is the annihilator of $\operatorname{im}(T)$, the image (range of T, and $\ker(T^t)$ is the kernel (null space) of T^t.

   (b) $\operatorname{rank}(T)=\operatorname{rank}(T^t)$, where the rank of a linear transformation is the dimension of its image.

   Solution

   $T^t:W^*\to V^*$ is defined such that $T^t(g)=g\circ T\in V^*$ for $g\in W^*$.

   (F01.7)

8. Let T be the rotation of an angle $60°$ counterclockwise about the origin in the plane perpendicular to $(1,1,2)$ in $\mathbb{R}^3$.

   (a) Find the matrix representation of T in the standard basis. Find all eigenvalues and eigenspaces of T.

   (b) What are the eigenvalues and eigenspaces of T if $\mathbb{R}^3$ is replaced by $\mathbb{C}^3$.

   (You do not have to multiply any matrices out but must compute any inverses.)

   Solution

   (a) Let

$$B_T=\begin{pmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}\\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}}\end{pmatrix}.$$

   Then regarding the columns of B_T as an orthonormal basis, the matrix representation of T in this basis is

$$[T]_{B_T}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos 60° & \sin 60°\\ 0 & -\sin 60° & \cos 60°\end{pmatrix},$$

   so the matrix representation of T in the standard basis is

$$T=B_T[T]_{B_T}B_T^{-1}=B_T[T]_{B_T}B_T^t.$$

   The only eigenvalue of T is 1, with corresponding eigenspace $\operatorname{span}\{(1\ 1\ 2)\}$.

(b) If we consider complex eigenvalues and eigenspaces, then T additionally has eigenvalues e^{i60° and e^{-i60° with corresponding eigenspaces span$\{(1\ i)\}$ and span$\{(1\ \ -i)\}$, respectively.

9. Let V be a complex inner product space. State and prove the Cauchy-Schwarz inequality.

Solution

Given $u, v \in V$, the Cauchy-Schwarz inequality states that

$$|(u, v)| \leq \|u\|\|v\|,$$

with equality if and only if u and v are linearly dependent.

To prove the claim, consider

$$0 \geq (u - tv, u - tv) = \|u\|^2 + |t|^2\|v\|^2 - t\overline{(u, v)} - \bar{t}(u, v)$$

for any $t \in \mathbb{C}$. Now if $v = 0$, the claim is trivial, so suppose $v \neq 0$ and set $t = (u, v)/\|v\|^2$ to yield

$$0 \geq \|u\|^2 + \frac{|(u, v)|^2}{\|v\|^2} - \frac{|(u, v)|^2}{\|v\|^2} - \frac{|(u, v)|^2}{\|v\|^2} = \|u\|^2 - \frac{|(u, v)|^2}{\|v\|^2},$$

from which the first part of the claim follows immediately. Clearly, equality holds if and only if $u - tv = 0$ for some value of t.

10. Let A be an $n \times n$ complex matrix satisfying $A^*A = AA^*$, where A^* is the adjoint of A. Let $V = \mathbb{C}^{\{n \times 1\}}$, the $n \times 1$ complex column matrices, be an inner product space under the dot product. View $A : V \to V$ as a linear map. Prove that there exists an orthonormal basis of V consisting of eigenvectors of A, i.e., prove this form of the Spectral Theorem for normal operators.

Solution

Let λ be an eigenvalue of A (whose existence is given by the roots of the characteristic polynomial), and let $E_\lambda = \{x \in V \mid Ax = \lambda x\}$ be the corresponding eigenspace. Then given any $x \in E_\lambda$,

$$A(A^*x) = A^*(Ax) = A^*(\lambda x) = \lambda(A^*x),$$

thus $A^*x \in E_\lambda$ and A^* is invariant on E_λ. Now given any $y \in E_\lambda^\perp$, if $x \in E_\lambda$, then

$$(Ay, x) = (y, A^*x) = 0,$$

since $y \in E_\lambda^\perp$ and $A^*x \in E_\lambda$. Thus $Ay \in E_\lambda^\perp$ and A is invariant on $E_\lambda^\perp$. It follows that the restriction of A to $E_\lambda^\perp$ is a normal linear transformation, hence, by induction, has an orthonormal basis consisting of eigenvectors of A. Combine this orthonormal basis with an orthonormal basis for E_λ and we arrive at the desired conclusion.