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Basic Examination, F04

1. Consider the following two statements:

   (a) The sequence $\{a_n\}$ converges.

   (b) The sequence $\{(a_1 + a_2 + \cdots + a_n)/n\}$ converges.

   Does (a) imply (b)? Does (b) imply (a)? Prove your answers.

   Solution

   Let $a_n = (-1)^n$. Then $(a_1 + \cdots + a_n)/n \to 0$ but $\{a_n\}$ certainly doesn't converge. Therefore (b) does not imply (a).

   Suppose $a_n \to a$. Then given $\epsilon > 0$, there exists an N such that $|a_n - a| < \epsilon$ for $n \geq N$. Then for $m > N$,

   $$\frac{\sum_{i=1}^m a_i}{m} = \frac{1}{m}\sum_{i=1}^{N-1} a_i + \frac{1}{m}\sum_{i=N}^{m} a_i = c_m + \frac{1}{m}\sum_{i=N}^{m} a_i$$

   where $c_m \to 0$ as $m \to \infty$. Thus

   $$\left|\frac{\sum_{i=1}^m a_i}{m} - \frac{m-N+1}{m}a\right| \leq |c_m| + \frac{1}{m}\sum_{i=N}^{m}|a_i - a| < |c_m| + \frac{m-N+1}{m}\epsilon < |c_m| + \epsilon.$$

   Letting $m \to \infty$, and noting that ϵ was arbitrary, allows us to conclude that

   $$\lim_{m\to\infty}\frac{\sum_{i=1}^m a_i}{m} = a.$$

   Therefore (a) implies (b).

2. State and prove Rolle's Theorem. (You can use without proof theorems about the maxima and minima of continuous or differentiable functions.)

   Solution

   If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then $f'(x) = 0$ for some $x \in (a, b)$.

   If f is constant on $[a, b]$, then $f' \equiv 0$ on (a, b). Otherwise, suppose $f(t) > f(a)$ for some $t \in (a, b)$. Since f is continuous on $[a, b]$, f achieves its minimum and maximum value. Let $x \in (a, b)$ be a point at which f achieves its maximum value (must be on the interior of $[a, b]$ since $f(t) > f(a) = f(b)$). Then $f'(x) = 0$. Similarly, if $f(t) < f(a)$ for some $t \in (a, b)$, there again exists an $x \in (a, b)$ such that $f'(x) = 0$.

3. Show that if $f_n \to f$ uniformly on the bounded closed interval $[a, b]$, then

   $$\int_a^b f_n(x)dx \to \int_a^b f(x)dx.$$

   Solution

   (S04.3)

4. Suppose that $(\mathcal{M}, \rho)$ is a metric space, $x, y \in \mathcal{M}$, and that $\{x_n\}$ is a sequence in this metric space such that $x_n \to x$. Prove that $\rho(x_n, y) \to \rho(x, y)$.

   Solution

Given $\epsilon > 0$, let N be such that $\rho(x_n, x) < \epsilon$ for $n > N$. Then

$$|\rho(x_n, y) - \rho(x, y)| \leq \rho(x_n, x) < \epsilon$$

for $n > N$, hence $\rho(x_n, y) \to \rho(x, y)$.

5. Prove that the space $C[0,1]$ of continuous functions from $[0,1]$ to $\mathbb{R}$ with the supremum norm, $\|f\|_\infty = \sup_{[0,1]} |f(x)|$, is complete. (You can use without proof the fact that a uniform limit of continuous functions is continuous.)

   Solution

   (F03.7)

6. The Bolzano-Weirstrass Theorem in $\mathbb{R}^n$ states that if S is a bounded closed subset of $\mathbb{R}^n$ and $\{x_n\}$ is a sequence which takes values in S, then $\{x_n\}$ has a subsequence which converges to a point in S. Assume this statement known in case $n = 1$, and use it to prove the statement in case $n = 2$.

   Solution

   Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be a sequence in S, a bounded closed subset of $\mathbb{R}^2$. Let $T = \{x \mid \exists y \in \mathbb{R}, (x, y) \in S\}$ be the project of S onto the first coordinate. Then $\{x_n\}_{n=1}^{\infty}$ is a sequence in T, a bounded closed subset of $\mathbb{R}$, hence there exists some subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ which converges to $x^* \in T$. Now consider the sequence $\{(x_{n_i}, y_{n_i})\}_{i=1}^{\infty}$ in S. Similar as before, $\{y_{n_i}\}_{i=1}^{\infty}$ is a sequence in a bounded closed subset of $\mathbb{R}$, hence there exists a subsequence $\{y_{n_i'}\}_{i=1}^{\infty}$ which converges to y^*. It follows that $(x_{n_i'}, y_{n_i'}) \to (x^*, y^*)$. Further, $(x^*, y^*) \in S$ by the closedness of S, which proves the theorem in the case $n = 2$.

7. Observe that the point $P = (1, 1, 1)$ belongs to the set S of points in $\mathbb{R}^3$ satisfying the equation

   $$x^4y^2 + x^2z + yz^2 = 3.$$

   Explain carefully how, in this case, the Implicit Function Theorem allows us to conclude that there exists a differentiable function $f(x, y)$ such that $(x, y, f(x, y))$ lies in S for all (x, y) in a small open set containing $(1, 1)$.

   Solution

   Let $G : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ defined by

   $$G(x, y, z) = x^4y^2 + x^2z + yz^2 - 3.$$

   Then $G(1, 1, 1) = 0$ and $D_zG = x^2 + 2yz = 3 \neq 0$ at $P = (1, 1, 1)$. By the Implicit Function Theorem, there exist open sets U and V, $(1, 1) \in U \subset \mathbb{R}^2$, $1 \in V \subset \mathbb{R}$, and a differentiable function f such that $G(x, y, f(x, y)) = 0$ for each $(x, y) \in U$.

8. Let $A = (a_{ij})$ be a real, $n \times n$ symmetric matrix and let $Q(v) = v \cdot Av$ (ordinary dot product) be the associated quadratic form defined for $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$.

   (a) Show that $\nabla Q_v = 2Av$ where ∇Q_v is the gradient at v of the function Q.

   (b) Let M be the minimum value of $Q(v)$ on the unit sphere $S^n = \{v \in \mathbb{R}^n : \|v\| = 1\}$ and let $u \in S^n$ be a vector such that $Q(u) = M$. Prove, using Lagrange multipliers, that u is an eigenvector of A with eigenvalue M.

   Solution

   (a) We have that

   $$Q(v) = v \cdot Av = \sum_i \sum_j a_{ij} v_i v_j,$$

so

$$D_k Q(v) = \sum_i a_{ik} v_i + \sum_j a_{kj} v_j = 2 \sum_j a_{kj} v_i,$$

therefore

$$\nabla Q_v = (D_1 Q(v) \ \cdots \ D_n Q(v)) = 2Av.$$

(b) If we set

$$g(v) = \|v\|^2 - 1,$$

then Q attains its minimum and maximum values at points $v \in \mathbb{R}^n$ satisfying

$$\nabla Q_v = \lambda \nabla g_v,$$
$$g(v) = 0.$$

The first equality gives

$$2Av = \lambda(2v),$$

so λ is an eigenvalue of A, and v is a corresponding eigenvector. If M is the minimum value on S^n, and $Q(u) = M$ for $u \in S^n$, then by the previous statement, u is an eigenvector for A. Furthermore,

$$M = Q(u) = u \cdot Au = u \cdot (\lambda u) = \lambda \|u\|^2 = \lambda.$$

9. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation and P a polynomial such that $P(T) = 0$. Prove that every eigenvalue of T is a root of P.

Solution

Let $\lambda \in \mathbb{C}$, $x \in \mathbb{C}^n$ be an eigenvalue-eigenvector pair for T. Then, since

$$T^k x = \lambda^k x,$$

we have that

$$0 = P(T)x = P(\lambda)x.$$

Since $x \neq 0$, it must be that $P(\lambda) = 0$, i.e., λ is a root of P.

10. Let $V = \mathbb{R}^n$ and let $T : V \to V$ be a linear transformation. For $\lambda \in \mathbb{C}$, the subspace

$$V(\lambda) = \{v \in V : (T - \lambda I)^N v = 0 \text{ for some } N \geq 1\}$$

is called a generalized eigenspace.

(a) Prove that there exists a fixed number M such that $V(\lambda) = \ker((T - \lambda I)^M)$.

(b) Prove that if $\lambda \neq \mu$, then $V(\lambda) \cap V(\mu) = \{0\}$. Hint: use the following equation by raising both sides to a high power.

$$\frac{T - \lambda I}{\mu - \lambda} + \frac{T - \mu I}{\lambda - \mu} = I$$

Solution

(a) Without loss of generality, let $\lambda = 0$. Denote

$$K_n = \ker(T^n).$$

Clearly, $K_n \subset K_{n+1}$. We show that T maps K_{n+2}/K_{n+1} injectively into K_{n+1}/K_n. Indeed, for $v, w \in K_{n+2}/K_{n+1}$, if $Tv - Tw = T(v - w) \in K_n$, then $v - w \in K_{n+1}$ and, in fact, are equal in K_{n+2}/K_{n+1}. Thus the dimension of the quotient spaces are monotonically decreasing. Eventually, the quotient spaces must become trivial, for otherwise we could construct an infinite set of linearly independent vectors in V by selecting a nonzero vector from each of the quotient spaces, contradicting the fact that V is finite dimesional. Thus, there exists some N such that $K_n = K_N$ for all $n > N$, and $V(\lambda) = K_N$.

(b) Suppose $v \in V(\lambda) \cap V(\mu)$. Let N and M be such that $V(\lambda) = \ker((T - \lambda I)^N)$ and $V(\mu) = \ker((T - \mu I)^M)$. Set $R = N + M - 1$. Then

$$\begin{aligned} I &= I^R \\ &= \left(\tfrac{T-\lambda I}{\mu-\lambda} + \tfrac{T-\mu I}{\lambda-\mu}\right)^R \\ &= \textstyle\sum_{n=0}^{R} (-1)^n \frac{(T-\lambda I)^n (T-\mu I)^{R-n}}{(\lambda-\mu)^R} \end{aligned}.$$

Now $T - \lambda I$ and $T - \mu I$ commute; further, either $n \geq N$ or $R - n \geq M$, so $(T-\lambda I)^n (T-\mu I)^{R-n} v = 0$ for all $n = 0, \ldots, R$. Thus, applying the last summation to v gives 0, but applying I to v yields v, hence $v = 0$, proving the claim.