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Basic Examination, F03

1. Prove that $\mathbb{R}$ is uncountable. If you like to use the Baire Category Theorem, you have to prove it.

   Solution

   It suffices to show that the interval $(0, 1]$ is uncountable. Consider the following isomorphism between $(0, 1]$ and a subset of $S_{0,1}$, the set of sequences composed only of 0's and 1's. Let $x \in (0, 1]$ and express

$$x = \sum_{n=1}^{\infty} \frac{d_n}{2^n},$$

   where each $d_n \in \{0, 1\}$ and $d_n = 1$ infinitely often. Note this is always possible, as it is an expression of the following process. Subdivide $(0, 1]$ into 2^m half-open equal-lengthed subintervals of length 2^{-m}; thus each subinterval has the form $(x_i, x_{i+1}]$ with each

$$x_i = \sum_{n=1}^{m} \frac{d_n^i}{2^n} = \frac{i}{2^m},$$

   $i = 0, \ldots, 2^m$. Now note the left endpoint of the subinterval within which x falls; this is the m^{th} partial sum of the above infinite sum, which is always bounded above by x and within 2^{-m} of x. Further, in choosing the left endpoint of the half-open interval, we guarantee that $d_n = 1$ infinitely often (if we could represent x such that $d_n = 0$ after some point, then x would be an endpoint of some fine subinterval and x could never be a partial sum of the above infinite sum, since we always choose the *left* endpoint, which is not included in the half-open interval).

   This thus provides an injective mapping ϕ from $(0, 1]$ into $S_{0,1}$ by mapping x to $(d_1, d_2, \ldots)$. A diagonalization argument shows that $S_{0,1}$ is itself uncountable, so we have only to show that $S_{0,1} \backslash \phi((0, 1])$ is countable to establish that $(0, 1]$ is uncountable. Note that $S_{0,1} \backslash \phi((0, 1])$ consists of those sequences of 0's and 1's which are constantly 0 after some point, for all other sequences may be mapped to an $x \in (0, 1]$ by the infinite summation. Therefore, let $T_n \subset S_{0,1}$ be the set of sequences which are constantly 0 from the n^{th} term on. Then each T_n is finite (indeed, it has cardinality 2^{n-1}). Thus

$$S_{0,1} \backslash \phi((0, 1]) = \bigcup_{n=1}^{\infty} T_n,$$

   which is certainly at most countable. This establishes the claim.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely often differentiable function. Assume that for each element $x \in [0, 1]$ there is a positive integer m such that the m^{th} derivative of f at x is not zero.

   Prove that there exists an integer M such that the following stronger statement holds:

   For each element $x \in [0, 1]$, there is a positive integer m with $m \leq M$ such that the m^{th} derivative of f at x is not zero.

   Solution

   As given, for each $x \in [0, 1]$, there exists an m_x such that the m_x^{th} derivative of f at x is nonzero; by the continuity of $f^{(m_x)}$, there exists a δ_x such that $f^{(m_x)}$ is nonzero in the neighborhood $B(x; \delta_x)$ of x. The family of open sets $\{B(x; \delta_x)\}_{x \in [0,1]}$ is an open cover of $[0, 1]$, which is compact, hence there exists a finite subcover $\{B(x_i; \delta_{x_i})\}_{i=1}^{n}$. Set $M = \max_i m_{x_i}$. Then each $x \in [0, 1]$ lies in some $B(x_i; \delta_{x_i})$, hence the $m_{x_i}^{th}$ derivative of f at x is nonzero, and $m_{x_i} \leq M$ by construction.

3. Prove that the sequence $a_1, a_2, \ldots$ with

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

converges as $n \to \infty$.

Solution

$$\begin{array}{rcl} a_n & = & \left(1+\frac{1}{n}\right)^n \\ & = & \sum_{i=0}^n \binom{n}{i} \frac{1}{n^i} \\ & = & \sum_{i=0}^n \frac{n!}{(n-i)!i!} \frac{1}{n^i} \\ & = & \sum_{i=0}^n \frac{n(n-1)(n-2)\cdots(n-i+1)}{(n)(n)(n)\cdots(n)} \frac{1}{i!} \\ & < & \sum_{i=0}^n \frac{1}{i!} \\ & < & \sum_{i=0}^\infty \frac{1}{i!} \\ & = & e \end{array},$$

hence the a_n's are a sequence of positive real numbers bounded above by e.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. State the definition of the Riemann integral

$$\int_0^1 f(x)dx$$

and prove that it exists.

Solution

Let $P = \{0 = x_0 \leq x_1 \leq \cdots \leq x_n = 1\}$ be a partition of $[0,1]$ and define $\Delta x_i = x_i - x_{i-1}$, $m_i = \inf_{[x_{i-1},x_i]} f$, and $M_i = \sup_{[x_{i-1},x_i]} f$, $i = 1, \ldots, n$. Then we denote

$$U(P,f) = \sum_{i=1}^n M_i \Delta x_i,$$

$$L(P,f) = \sum_{i=1}^n m_i \Delta x_i,$$

and set

$$\overline{\int_0^1} f dx = \inf_{P \subset [0,1]} U(P,f),$$

$$\underline{\int_0^1} f dx = \sup_{P \subset [0,1]} L(P,f).$$

If the upper Riemann integral and lower Riemann integral are equal, then f is Riemann integrable and

$$\int_0^1 f dx$$

is their common value.

Now if f is continuous on $[0,1]$, then f is uniformly continuous. Thus, given an $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in [0,1]$ with $|x - y| < \delta$. Let P be a partition of $[0,1]$ such that $\Delta x_i < \delta/2$. Then $M_i - m_i < \epsilon$ and

$$U(P,f) - L(P,f) = \sum_{i=1}^n (M_i - m_i)\Delta x_i < \epsilon \sum_{i=1}^n \Delta x_i = \epsilon,$$

and since ϵ was aribtrary, we conclude that f is Riemann integrable on $[0,1]$.

5. Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that all partial derivatives of order 3 exist and are continuous. Write down (explicitly in terms of partial derivatives of f) a quadratic polynomial $P(x, y)$ in x and y such that

$$|f(x, y) - P(x, y)| \leq C(x^2 + y^2)^{3/2}$$

for all (x, y) in some small neighborhood of $(0, 0)$, where C is a number that may depend on f but not on x and y. Then prove the above estimate.

Solution

Let U be some open ball of $(0, 0)$, and let M be a bound for each of the order 3 partial derivatives of f on U (which exists by the continuity of these partials). Let

$$P(x, y) = f(0, 0) + (D_1 f)x + (D_2 f)y + (D_{11} f)\frac{x^2}{2} + (D_{12} f)xy + (D_{22} f)\frac{y^2}{2},$$

where each partial derivative is evaluated at $(0, 0)$. Then

$$g(x, y) = f(x, y) - P(x, y)$$

has vanishing partial derivatives of order up to 2 at $(0, 0)$. Fix $(x, y) \in U$ and define $h : [0, 1] \to \mathbb{R}$ by

$$h(t) = g(tx, ty).$$

Then

$$h(0) = h'(0) = h''(0) = 0$$

and

$$h^{(3)}(t) = (D_{111} g)x^3 + 3(D_{112} g)x^2 y + 3(D_{122} g)xy^2 + (D_{222} g)y^3,$$

where each partial derivative is evaluated at $(tx, ty) \in U$. Thus

$$\left|h^{(3)}(t)\right| \leq M \left(\max\{x, y\}\right)^3 \leq M \left(x^2 + y^2\right)^{3/2}.$$

By Taylor's Theorem,

$$h(1) = h(0) + h'(0) + h''(0)/2 + h^{(3)}(t)/6$$

for some $0 \leq t \leq 1$, hence

$$|f(x, y) - P(x, y)| = |g(x, y)| = |h(1)| \leq M/6 \left(x^2 + y^2\right)^{3/2}$$

for all $(x, y) \in U$.

6. Let $U = \{(x, y) : x^2 + y^2 < 1\}$ be the standard unit ball in $\mathbb{R}^2$ and let ∂U denotes its boundary.

Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable and that the Jacobian determinant of F is everywhere non-zero. Suppose also that $F(x, y) \in U$ for some $(x, y) \in U$ and $F(x, y) \notin U \cup \partial U$ for all $(x, y) \in \partial U$. Prove that $U \subset F(U)$.

Solution

(W02.7)

7. Prove that the space of continuous functions on the closed interval $[0, 1]$ with the metric

$$\operatorname{dist}(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)| = \|f - g\|_\infty$$

is complete. You do not need to show that this is a metric space.

Solution

Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence of continuous functions on $[0,1]$ with the metric dist. For each $x \in [0,1]$, it follows that $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, hence define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

We show now that f is uniformly continuous on $[0,1]$, hence continuous. To this end, let $\epsilon > 0$ be given; then there exists an N such that $\|f_n - f_m\|_\infty < \epsilon$ for all $n, m \geq N$, that is, $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in [0,1]$ and $n, m \geq N$. Letting $m \to \infty$, we see that $|f_n(x) - f(x)| \leq \epsilon$ for all $x \in [0,1]$ and $n \geq N$. f_N is continuous on $[0,1]$, hence uniformly continuous, so there exists a δ such that $|f_N(x) - f_N(y)| < \epsilon$ whenever $|x - y| < \delta$. It follows that

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < 3\epsilon,$$

proving that f is uniformly continuous.

8. Prove the following three statements. You certainly may choose an order of these statements and then use the earlier statements to prove the later statements.

   (a) If $T : V \to W$ is a linear transformation between two finite dimensional real vector spaces V, W, then
   $$\dim(\operatorname{im}(T)) = \dim(V) - \dim(\ker(T)).$$

   (b) If $T : V \to V$ is a linear transformation on a finite dimensional real inner product space and T^* denotes its adjoint, then $\operatorname{im}(T^*)$ is the orthogonal complement of $\ker(T)$ in V.

   (c) Let A be an n by n real matrix; then the maximal number of linearly independent rows (row rank) in the matrix equals the maximal number of linearly independent columns (column rank).

   Solution

   (a) Let U be any subspace of V, and let $\{u_1, \ldots, u_k\}$ be a basis for U. Extend this to a basis of V with $\{u_{k+1}, \ldots, u_n\}$. Then $\{\{u_{k+1}\}, \ldots, \{u_n\}\}$ is evidently a basis for V/U, hence
   $$\dim(U) + \dim(V/U) = k + (n-k) = n = \dim(V).$$
   Let $U = \ker(T)$, and define T acting on V/U by setting
   $$T\{x\} = Tx.$$
   Then T provides an isomorphism between V/U and $\operatorname{im}(T)$, hence
   $$\dim(V) = \dim(U) + \dim(V/U) = \dim(\ker(T)) + \dim(\operatorname{im}(T)).$$

   (b) We show first that $\operatorname{im}(T^*)^\perp = \ker(T)$:
   $$\begin{array}{rcll} v \in \operatorname{im}(T^*)^\perp & \Leftrightarrow & (v,x) = 0 \ \forall x \in \operatorname{im}(T^*) & \\ & \Leftrightarrow & (v, T^*y) = 0 \ \forall y \in V^* & \\ & \Leftrightarrow & (Tv, y) = 0 \ \forall y \in V^* & \\ & \Leftrightarrow & Tv = 0 & \Leftrightarrow \ v \in \ker(T) \end{array}.$$
   We next show that, for a subspace W of $V*$, $W^{\perp\perp} = W$. Indeed,
   $$x \in W \ \Leftrightarrow \ (v,x) = 0 \ \forall v \in W^\perp \ \Leftrightarrow \ x \in W^{\perp\perp},$$
   hence
   $$\operatorname{im}(T^*) = \operatorname{im}(T^*)^{\perp\perp} = \ker(T)^\perp.$$

(c) We have that

$$n = \dim(\mathrm{im}(A^t)) + \dim(\ker(A^t)) = \mathrm{rowrank}(A^t) + \mathrm{nullity}(A^t).$$

But

$$\mathrm{rowrank}(A^t) = \mathrm{colrank}(A)$$

and

$$\mathrm{nullity}(A^t) = n - \dim(\ker(A^t)^\perp) = n - \dim(\mathrm{im}(A)) = n - \mathrm{rowrank}(A),$$

from which it follows that $\mathrm{colrank}(A) = \mathrm{rowrank}(A)$.

9. Consider a 3 by 3 real symmetric matrix with determinant 6. Assume $(1,2,3)$ and $(0,3,-2)$ are eigenvectors with eigenvalues 1 and 2. Give answers to (a) and (b) below and justify the answers.

   (a) Give an eigenvector of the form $(1, x, y)$ for some real x, y which is linearly independent of the two vectors above.

   (b) What is the eigenvalue of this eigenvector.

   Solution

   Since the product of all (real and complex) eigenvalues of a matrix A is equal to the determinant A, it follows that if $A \in M_{3\times 3}(\mathbb{R})$ with eigenvalues 1 and 2 and $\det(A) = 6$, then the third eigenvalue is $\lambda = 3$. If, additionally, A is symmetric, then the eigenspace associated with the eigenvalue 3 is orthogonal to the other 2 eigenspaces; it follows that $(1, x, y)$ is an eigenvector with eigenvalue 3 if and only if

   $$0 = (1, x, y) \cdot (1, 2, 3) = 1 + 2x + 3y,$$
   $$0 = (1, x, y) \cdot (0, 3, -2) = 3x - 2y,$$

   from which we solve $x = -2/13$, $y = -3/13$.

10. (a) Let $t \in \mathbb{R}$ such that t is not an integer multiple of π. For the matrix

   $$A = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

   prove that there does not exist a real valued matrix B such that BAB^{-1} is a diagonal matrix.

   (b) Do the same for the matrix

   $$A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

   where $\lambda \in \mathbb{R}\backslash\{0\}$.

   Solution

   (a) Suppose $BAB^{-1} = D$ is diagonal for some real-valued matrix B, and let its diagonal entries be λ_1 and λ_2. Then each λ_i is an eigenvalue of D, hence an eigenvalue of A since

   $$\begin{aligned} 0 &= \det(\lambda_i I - D) \\ &= \det(\lambda_i BB^{-1} - BAB^{-1}) \\ &= \det(B(\lambda_i I - A)B^{-1}) \\ &= \det(\lambda_i I - A) \end{aligned}.$$

   But any λ satisfying

   $$\det(\lambda I - A) = 0$$

is not real, since

$$0 = \det(\lambda I - A) = (\lambda - \cos(t))^2 + \sin(t)^2$$

implies that

$$\lambda = \cos(t) \pm i \sin(t),$$

and $\sin(t) \neq 0$ as t is not a multiple of π. But $D = BAB^{-1}$ must be real-valued since it is a product of real-valued matrices, a contradiction. This proves there can exist no such B.

(b) Suppose $BAB^{-1} = D$ is diagonal for some real-valued matrix B. Again, the diagonal entries of D correspond to eigenvalues of A, so we compute

$$0 = \det(\mu I - A) = (\mu - 1)^2,$$

thus $\mu = 1$ is the only eigenvalue of A, so we conclude that $D = I$, and $A = B^{-1}DB = B^{-1}B = I$, which is contradiction to the given A for $\lambda \neq 0$. This proves there can exist no such B.