1. *(a) Find a radially symmetric solution u to the equation in $\mathbb{R}^2$*

$$\Delta u = \frac{1}{2\pi}\log|x|,$$

and show that u is a fundamental solution for Δ^2, that is, show

$$\phi(0) = \int_{\mathbb{R}^2} u\Delta^2\phi \,\mathrm{d}x$$

for any smooth ϕ which vanishes for $|x|$ large.

(b) Explain how to construct the Green's function for the following boundary value in a bounded domain $D \subset \mathbb{R}^2$ with smooth boundary ∂D

$$w = 0 \text{ and } \frac{\partial w}{\partial n} = 0 \text{ on } \partial D, \quad \Delta^2 w = f \text{ on } D,$$

where $\partial/\partial n$ denotes the normal derivative.

(a) Since u is radially symmetric, the equation can be written in polar form without the $\frac{\partial^2}{\partial\theta^2}$ term.

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) &= \frac{1}{2\pi}\log r \\
r\frac{\partial u}{\partial r} &= \int \frac{r}{2\pi}\log r\,\mathrm{d}r + C_1 \\
&= \frac{1}{2\pi}\left(\int \frac{-r^2}{2}\frac{1}{r}\,\mathrm{d}r + \frac{r^2}{2}\log r\right) + C_1 && \text{integration by parts} \\
&= \frac{1}{2\pi}\left(\frac{-r^2}{4} + \frac{r^2}{2}\log r\right) + C_1 && \text{choose } C_1 = 0 \\
u &= \frac{1}{2\pi}\int\left(\frac{-r}{4} + \frac{r}{2}\log r\right)\mathrm{d}r + C_2 && \text{choose } C_2 = 0 \\
&= \frac{1}{2\pi}\left[\frac{-r^2}{8} + \tfrac{1}{2}\left(\frac{-r^2}{4} + \frac{r^2}{2}\log r\right)\right] \\
&= \frac{1}{2\pi}\left(\frac{-r^2}{4} + \frac{r^2}{4}\right) = \frac{r^2}{8\pi}(\log r - 1).
\end{aligned}$$

We already know that $\frac{1}{2\pi}\log|x|$ is a fundamental solution of Laplace's equation in 2D. And since $\Delta u = \frac{1}{2\pi}\log|x|$, we can show that this is a fundamental solution with

$$\begin{aligned}
\int_{\mathbb{R}^2} u\Delta^2\phi &= \lim_{R\to\infty}\int_{B_R(0)} u\Delta^2\phi \overset{\text{Green's}}{=} \lim_{R\to\infty}\int_{B_R(0)} \Delta u\Delta\phi + \underbrace{\int_{\partial B_R(0)}\left(u\tfrac{\partial\Delta\phi}{\partial n} - \Delta\phi\tfrac{\partial u}{\partial n}\right)}_{\to 0} \\
&= \int_{\mathbb{R}^2}\Delta u\Delta\phi = \phi(0).
\end{aligned}$$

(b) Since D is bounded, there exist orthonormal eigenfunctions (ϕ_n) and eigenvalues (λ_n) such that

$$\begin{cases} \Delta^2\phi_n = -\lambda_n\phi & \text{on } D \\ \phi_n = \frac{\partial\phi_n}{\partial n} = 0 & \text{on } \partial D \end{cases}$$

The Green's function is $\sum_n \langle u, \phi_n\rangle\phi_n$.

2. *(a) Given a continuous function f on $\mathbb{R}$ which vanishes for $|x| > R$, solve the initial value problem $u_{tt} - u_{xx} = f(x)\cos t$, $u(x,0) = u_t(x,0) = 0$, $x \in \mathbb{R}$, $0 \le t < \infty$ by first finding a particular solution by separation of variables and then adding the appropriate solution of the homogeneous PDE.*

(b) Since the particular solution is not unique, it will not be obvious that the solution to the initial value problem that you have found in part (a) is unique. Prove that it is unique.

By Duhamel's principle, the solution is $u(x,t) = u_h(x,t) + \int_0^t U(x, t-s, s)\,\mathrm{d}s$, where U is the solution to

$$\begin{cases} U_{tt} - U_{xx} = 0 \\ U(x,0,s) = 0, \quad U_t(x,0,s) = f(x)\cos s \end{cases}$$

and where u_h is the solution to the homogeneous PDE: $u_{tt} - u_{xx} = 0$, $u(x,0) = u_t(x,0) = 0$. By d'Alembert's solution, $u_h \equiv 0$.

Separation of variables with $U(x,t) = X(x)T(t)$ yields solutions $X(x) = a(\lambda)\mathrm{e}^{i\lambda x} + b(\lambda)\mathrm{e}^{-i\lambda x}$ and $T(t) = c(\lambda)\mathrm{e}^{i\lambda t} + d(\lambda)\mathrm{e}^{-i\lambda t}$ for any $\lambda \ge 0$. The boundary condition $U(x,0,s) = 0$ implies $T(t) = \sin \lambda t$. Therefore,

$$U(x,t,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\lambda, s)\sin\lambda t\,\mathrm{e}^{i\lambda x}\,\mathrm{d}\lambda.$$

The other boundary condition gives $U_t(x,0,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lambda C(\lambda,s)\,\mathrm{e}^{i\lambda x}\,\mathrm{d}\lambda = f(x)\cos s$, so $\lambda C(\lambda, s)$ is the Fourier transform of $f(x)\cos s$. From here we reproduce d'Alembert's solution:

$$\begin{aligned} U(x,t,s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} C(\lambda,s)\sin\lambda t\,\mathrm{e}^{i\lambda x}\,\mathrm{d}\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\lambda}\int_{-\infty}^{\infty} f(x')\cos s\,\mathrm{e}^{-i\lambda x'}\,\mathrm{d}x'\ \sin\lambda t\,\mathrm{e}^{i\lambda x}\,\mathrm{d}\lambda \\ &= \frac{\cos s}{2\cdot 2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{i\lambda t} - \mathrm{e}^{-i\lambda t}}{i\lambda}\int_{-\infty}^{\infty} f(x')\,\mathrm{e}^{-i\lambda x'}\,\mathrm{d}x'\,\mathrm{e}^{i\lambda x}\,\mathrm{d}\lambda \end{aligned}$$

now let $F(x)$ denote an antiderivative of $f(x)$, then

$$U(x,t,s) = \frac{\cos s}{2}\big(F(x+t) - F(x-t)\big) = \frac{1}{2}\int_{x-t}^{x+t} f(x')\cos s\,\mathrm{d}x'.$$

Since f is continuous with compact support, U is bounded. Therefore, u is well-defined.

(b) Let u_1 and u_2 be two solutions and let $w = u_1 - u_2$. Then $w_{tt} - w_{xx} = 0$ and $w(x,0) = w_t(x,0) = 0$. By d'Alembert's solution, $w \equiv 0$.

3. *Steady viscous flow in a cylindrical pipe is described by the equation*

$$(\vec{u}\cdot\nabla)\vec{u}+\frac{1}{\rho}\nabla p-\frac{\eta}{\rho}\Delta\vec{u}=0$$

on the domain $-\infty<x_1<\infty$, $x_2^2+x_3^2\le R^2$, *where* $\vec{u}=(u_1,u_2,u_3)=(U(x_2,x_3),0,0)$ *is the velocity vector,* $p(x_1,x_2,x_3)$ *is the pressure, and* η *and* ρ *are constants.*

(a) Show that $\frac{\partial p}{\partial x_1}$ *is a constant* c*, and that* $\Delta U=c/\eta$*.*

(b) Assuming further that U *is radially symmetric and* $U=0$ *on the surface of the pipe, determine the mass* Q *of fluid passing through a cross-section of pipe per unit time in terms of* c,ρ,η *and* R*. Note that*

$$Q=\rho\int_{\{x_2^2+x_3^2\le R^2\}}U\,\mathrm{d}x_2\,\mathrm{d}x_3.$$

(a) Since $u_2=u_3=0$, the equation reduces to

$$u_1\frac{\partial}{\partial x_1}U(x_2,x_3)+\frac{1}{\rho}\frac{\partial p}{\partial x_1}-\frac{\eta}{\rho}\Delta u_1=0,\qquad \frac{\partial p}{\partial x_2}=\frac{\partial p}{\partial x_3}=0.$$

So $\frac{\partial p}{\partial x_1}=\eta\Delta u_1=\eta\Delta U(x_2,x_3)=\eta\left(\frac{\partial^2}{\partial {x_2}^2}U+\frac{\partial^2}{\partial {x_3}^2}U\right)$, and

$$\begin{aligned}
\frac{\partial}{\partial x_1}\left(\frac{\partial p}{\partial x_1}\right) &= \eta\Big(\frac{\partial^2}{\partial {x_2}^2}\underbrace{\frac{\partial}{\partial x_1}U}_{=0}+\frac{\partial^2}{\partial {x_3}^2}\underbrace{\frac{\partial}{\partial x_1}U}_{=0}\Big) \;=\; 0\\
\frac{\partial}{\partial x_2}\left(\frac{\partial p}{\partial x_1}\right) &= \frac{\partial}{\partial x_1}\left(\frac{\partial p}{\partial x_2}\right)\;=\;0\\
\frac{\partial}{\partial x_3}\left(\frac{\partial p}{\partial x_1}\right) &= \frac{\partial}{\partial x_1}\left(\frac{\partial p}{\partial x_3}\right)\;=\;0.
\end{aligned}$$

Thus $\frac{\partial p}{\partial x_1}=c$, and since $\frac{\partial p}{\partial x_1}=\eta\Delta U(x_2,x_3)=\eta\left(\frac{\partial^2}{\partial {x_2}^2}U+\frac{\partial^2}{\partial {x_3}^2}U\right)$, we have $\Delta U=\frac{c}{\eta}$.

(b) The flow rate is

$$Q\;=\;\rho\int_{\{x_2^2+x_3^2\le R^2\}}U\,\mathrm{d}x_2\,\mathrm{d}x_3\;=\;\rho\int_0^{2\pi}\!\!\int_0^R Ur\,\mathrm{d}r\,\mathrm{d}\theta\;=\;2\pi\rho\int_0^R Ur\,\mathrm{d}r.$$

Since $\Delta U=\frac{c}{\eta}$,

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)=\frac{c}{\eta}\qquad\Longrightarrow\qquad r\frac{\partial U}{\partial r} &= \int\frac{c}{\eta}r\,\mathrm{d}r+d_0=\frac{c}{2\eta}r^2+d_0\\
U &= \int\frac{c}{2\eta}r+\frac{d_0}{r}\,\mathrm{d}r+d_1\\
&= \frac{c}{4\eta}r^2+\underbrace{d_0}_{=0}\log r+d_1.
\end{aligned}$$

and $U(R)=0$ implies $d_1=-\frac{c}{4\eta}R^2$. So

$$\begin{aligned}
Q &= 2\pi\rho\int_0^R\left(\frac{c}{4\eta}r^3-\frac{c}{4\eta}R^2r\right)\mathrm{d}r\\
&= \frac{\pi\rho c}{2\eta}\left[\frac{1}{4}r^4-\frac{R^2}{2}r^2\right]_0^2\;=\;\frac{\pi\rho c}{2\eta}\left(\frac{1}{4}R^4-\frac{1}{2}R^4\right)\;=\;\frac{-\pi\rho c}{8\eta}R^4.
\end{aligned}$$

4. *Use the Fourier transform on $L^2(\mathbb{R})$ to show that*

$$\frac{\mathrm{d}u}{\mathrm{d}x} + cu(x) + u(x-1) = f$$

has a unique solution $u \in L^2(\mathbb{R})$ for each $f \in L^2(\mathbb{R})$ when $|c| > 1$. You may assume that $c \in \mathbb{R}$.

In the Fourier domain,

$$\begin{aligned} i\xi\hat{u}(\xi) + c\hat{u}(\xi) + \mathrm{e}^{-i\xi}\hat{u}(\xi) &= \hat{f}(\xi) \\ \hat{u}(\xi) &= \frac{\hat{f}(\xi)}{i\xi + c + \mathrm{e}^{-i\xi}}. \end{aligned}$$

So $u(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\hat{f}(\xi)\mathrm{e}^{ix\xi}}{i\xi+c+\mathrm{e}^{-i\xi}}\,\mathrm{d}\xi$. Since $|i\xi + c + \mathrm{e}^{-i\xi}|^2 = (c+\cos\xi)^2 + (\xi - \sin\xi)^2 \geq (|c|-1)^2 > 0$, we have u exists and is unique by the Fourier inversion theorem.

5. *The following equation (Fisher's equation) arises in the study of population genetics:* $u_t = u(1-u) + u_{xx}$ *on* $-\infty < x < \infty$, $t > 0$. *The solutions of physical interest satisfy* $0 \le u \le 1$, *and*

$$\lim_{x\to-\infty} u(x,t) = 0, \quad \lim_{x\to\infty} u(x,t) = 1.$$

One class of solutions is the set of "wavefront" solutions. These have the form $u(x,t) = \phi(x+ct)$, $c \ge 0$.

Determine the ordinary differential equation and boundary conditions which ϕ *must satisfy to be of physical interest. Carry out a phase plane analysis of this equation, and show that physically interesting wavefront solutions are possible if* $c \ge 2$, *but not if* $0 \le c < 2$.

Let $\xi = x + ct$, then ϕ must satisfy

$$\begin{cases} c\phi' = \phi(1-\phi) + \phi'', \\ \lim\limits_{\xi\to-\infty} \phi(\xi) = 0, \quad \lim\limits_{\xi\to\infty} \phi(\xi) = 0. \end{cases}$$

Let $y_1 = \phi$ and $y_2 = \phi'$. Then we have the first-order autonomous system

$$\begin{cases} y_1' = y_2, \\ y_2' = cy_2 + y_1(y_1 - 1), \end{cases}$$

with equilibrium points where $y_2 = 0$ and $y_1(y_1 - 1) = 0$, that is, at $(0,0)$ and $(1,0)$.

Linearizing the system about $(a,0)$, $a = 0$ or 1,

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (2a-1) & c \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

At $(0,0)$, $\lambda = \frac{1}{2}(c \pm \sqrt{c^2-4})$.
If $c \ge 2$, $\lambda > 0$ $\Rightarrow$ unstable node
If $0 < c < 2$, $\lambda = \frac{1}{2}(c \pm i\sqrt{4-c^2})$ $\Rightarrow$ unstable spiral

At $(1,0)$, $\lambda = \frac{1}{2}(c \pm \sqrt{c^2+4})$, it is a saddle node for all values of c.

6. *Consider the equation* $u_x + u_x u_y = 1$ *in* $\mathbb{R}^2$ *with* u *prescribed on* $y = 0$*, that is,* $u(x,0) = f(x)$*. Assuming that* f *is differentiable, what conditions on* f *insure that the problem is noncharacteristic? If* f *satisfies those conditions, show that the solution is*

$$u(x,y) = f(r) - y + \frac{2y}{f'(r)}$$

where r *must satisfy* $y = (f'(r))^2(x-r)$*. Finally, show that one can solve the equation for* (x,y) *in a sufficiently small neighborhood of* $(x_0,0)$ *with* $r(x_0,0) = x_0$*.*

Let $p = u_x$ and $q = u_y$, then the PDE is $F = 0$ with $F(x,y,u,p,q) = p + pq - 1$. First, we find an initial parameterization $\Gamma = (s, 0, f(s), \phi(s), \psi(s))$, where ϕ and ψ are such that

$$\phi + \phi\psi - 1 = 0, \qquad f'(s) = \phi(s).$$

So $\phi(s) = f'(s)$ and $\psi(s) = \frac{1}{f'(s)} - 1$ (we must require that $f'(s) \neq 0$). Since the characteristic equations for p and q are $p' = 0$ and $q' = 0$, we have $p = f'(s)$ and $q = \frac{1}{f'(s)} - 1$. Solving the rest of the characteristic equations,

$$\begin{array}{rclcrclcrcl} x' &=& 1+q & & y' &=& p & & u' &=& 2pq+p \\ x(0) &=& s & & y(0) &=& 0 & & u(0) &=& f(s) \\ x &=& \frac{1}{f'(s)}t + s & & y &=& f'(s)t & & u &=& (2-f'(s))t + f(s) \end{array}$$

So $x = \frac{1}{f'(s)}\frac{y}{f'(s)} + s \;\Rightarrow\; y = f'(s)^2(x-s)$ and

$$\begin{aligned} u(x,y) \;=\; (2-f'(s))t + f(s) &= 2t - y + f(s) \\ &= f(s) - y - 2\frac{y}{f'(s)}. \end{aligned}$$

Let $G(x,y,s) = f'(s)^2(x-s) - y$, $G = 0$ on Γ. Since $G_s(x,y,s) = -f'(s)^2 + (x-s)[f'(s)^2]'$, we have $G_s(x_0,0,x_0) = -f'(x_0)^2 \neq 0$. Thus by the implicit function theorem, there is a neighborhood of $(x_0,0,x_0)$ such that $G(x,y,s) = 0$.

7. *Consider the system*

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix}.$$

Find an explicit solution for the following mixed problem for the system:

$$(u(x,0), v(x,0)) = (f(x), 0) \text{ for } x > 0, \quad u(t,0) = 0 \text{ for } t > 0.$$

You may assume that the function f is smooth and vanishes on a neighborhood of $x = 0$.

Let $A = \left(\begin{smallmatrix} -1 & 2 \\ 2 & 2 \end{smallmatrix}\right)$. Its eigenvalues are $(\lambda+1)(\lambda-2) - 4 = \lambda^2 - \lambda - 6 = 0 \Rightarrow \lambda = 3$ and -2. So this is a hyperbolic system. The eigenvectors are $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right)$.

Let $P = \left(\begin{smallmatrix} 1 & -2 \\ 2 & 1 \end{smallmatrix}\right)$, then $P^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} = P^{-1}AP$. Diagonalize the system with the change of variables $V = P^{-1}u$, then $V_t = DV_x$,

$$\begin{cases} v_{1t} &= 3v_{1x} \\ v_{2t} &= -2v_{2x} \end{cases}$$

which is the one-way wave equation in each coordinate. The initial conditions give $V_{|t=0} = P^{-1}u_{|t=0} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} u_1(x,0) \\ u_2(x,0) \end{pmatrix} = \frac{1}{5}\begin{pmatrix} f(x) \\ -2f(x) \end{pmatrix}$. Therefore, the solution is

$$v_1(x,t) = \tfrac{1}{5}f(x+3t), \qquad v_2(x,t) = -\tfrac{2}{5}f(x-2t)$$

and

$$u = PV = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} f(x+3t) \\ -2f(x-2t) \end{pmatrix} = \frac{1}{5}\begin{pmatrix} f(x+3t) + 4f(x-2t) \\ 2f(x+3t) - 2f(x-2t) \end{pmatrix}.$$

The boundary condition $u(t,0) = 0$ requires $f(3t) = -4f(-2t)$ and $f(3t) = 2f(-2t)$. Since f vanishes on a neighborhood of $x = 0$, this is already satisfied.

8. *(a) Assume that D is a bounded domain in $\mathbb{R}^n$ with smooth boundary ∂D and outer unit normal ν. Find a variational formula for the lowest eigenvalue of $-\Delta u$ in D with the boundary condition $\frac{\partial u}{\partial \nu} + au = 0$ on ∂D, and show that the lowest eigenvalue will be positive or negative depending on the sign of a.*

(b) For the values of a which make the lowest eigenvalue positive, derive the following estimate for the solution u of the boundary value problem $-\Delta u + k^2 u = 0$ in D, $\frac{\partial u}{\partial \nu} + au = g$ on ∂D:

$$\max_D |u| \leq C_a \max_{\partial D} |g|,$$

where C_a does not depend on k. Use maximum principle arguments.

(a) Consider $\langle -\Delta u, u \rangle = \lambda \langle u, u \rangle$ for $u \in H^2(D)$. By the divergence theorem,

$$\begin{aligned} 0 = \int_D \Delta u \cdot u + \lambda u^2 &= \int_D -\nabla u \cdot \nabla u + \int_{\partial D} \frac{\partial u}{\partial \nu} u + \int_D \lambda u^2 \\ &= -\int_D |\nabla u|^2 - \int_{\partial D} a u^2 + \lambda \int_D u^2 \\ \Rightarrow \quad \lambda &= \frac{\int_D |\nabla u|^2 + a \int_{\partial D} u^2}{\int_D u^2}. \end{aligned}$$

So the smallest eigenvalue is $\lambda_* = \min\limits_{u \in H^2(D)} \dfrac{\int_D |\nabla u|^2 + a \int_{\partial D} u^2}{\int_D u^2}$. If $a > 0$, then $\lambda_* > 0$. If $a < 0$, then λ_* minimizes the $\int_D |\nabla u|^2$ term and $\lambda_* < 0$.

(b) Assume $a > 0$. If u is not constant in D, and $u(x_0) = \max\limits_{\overline{D}} u$ for $x_0 \in \partial D$, then we have $\frac{\partial u}{\partial \nu}(x_0) > 0$ by the maximum principle. Therefore,

$$\begin{aligned} \frac{\partial u}{\partial \nu}(x_0) + a u(x_0) &= g(x_0) \\ a u(x_0) &< g(x_0) \\ u(x_0) &< \frac{g(x_0)}{a}. \end{aligned}$$

So $\max\limits_{\overline{D}} u(x) \leq \max\limits_{\partial D} \{\frac{g(x)}{a}, 0\}$. Similarly, $\max\limits_{\overline{D}} -u(x) \leq \max\limits_{\partial D} \{\frac{-g(x)}{a}, 0\}$. Therefore,

$$\max_{\overline{D}} |u| \leq C_a \max_{\partial D} |g|, \qquad \text{where } C_a = \tfrac{1}{a}.$$