Math 269B, 2012 Winter, Homework 3 (Solutions)

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1 Theory

1. (Strikwerda 5.1.2.) Show that the modified leapfrog scheme (5.1.6) is stable for ϵ satisfying

$$0 < \epsilon \le 1 \quad \text{if} \quad 0 < a^2\lambda^2 \le \frac{1}{2}$$

and

$$0 < \epsilon \le 4a^2\lambda^2\left(1 - a^2\lambda^2\right) \quad \text{if} \quad \frac{1}{2} \le a^2\lambda^2 < 1.$$

Note that these limits are not sharp. It is possible to choose ϵ larger than these limits and still have the scheme be stable.

Solution

Continuing from the text, we find the amplification factors to be

$$g_{\pm}(\theta) = -ia\lambda\sin\theta \pm \sqrt{1 - a^2\lambda^2\sin^2\theta - \epsilon\sin^4\frac{1}{2}\theta}.$$

If the expression under the $\sqrt{\cdot}$ is nonnegative, then

$$|g_{\pm}(\theta)|^2 = 1 - \epsilon\sin^4\frac{1}{2}\theta \le 1,$$

hence the scheme is stable. We thus wish to satisfy

$$0 \le 1 - a^2\lambda^2\sin^2\theta - \epsilon\sin^4\frac{1}{2}\theta =: \alpha(\theta).$$

We compute that

$$\alpha'(\theta) = -\frac{1}{2}\sin\theta\left(\left(4a^2\lambda^2 - \epsilon\right)\cos\theta + \epsilon\right),$$

and hence the extrema of α occur when $\sin\theta = 0$ or $\cos\theta = \epsilon/\left(\epsilon - 4a^2\lambda^2\right)$. Values of θ satisfying $\sin\theta = 0$ give $\alpha = 1$ or $\alpha = 1 - \epsilon$, requiring that $\epsilon \le 1$. Values of θ satisfying $\cos\theta = \epsilon/\left(\epsilon - 4a^2\lambda^2\right)$ exist if and only if $\left|\epsilon/\left(\epsilon - 4a^2\lambda^2\right)\right| \le 1$, which is equivalent to $\epsilon \le 2a^2\lambda^2$. For such θ, we get $\alpha = 1 - 4a^4\lambda^4/\left(4a^2\lambda^2 - \epsilon\right)$, and for this to be nonnegative, we must have $\epsilon \le 4a^2\lambda^2\left(1 - a^2\lambda^2\right)$. In particular, we must have $|a\lambda| < 1$.

So far, we have deduced that, at a minimum, $0 < \epsilon \le 1$. Furthermore, if $\epsilon \le 2a^2\lambda^2$, then we must additionally satisfy $\epsilon \le 4a^2\lambda^2\left(1 - a^2\lambda^2\right)$. Now, in the instance that $2a^2\lambda^2 \le 4a^2\lambda^2\left(1 - a^2\lambda^2\right)$, we would automatically satisfy the second condition, and this latter inequality is equivalent to $a^2\lambda^2 \le \frac{1}{2}$. It follows that

- If $0 < a^2\lambda^2 \le \frac{1}{2}$, it is sufficient to take $0 < \epsilon \le 1$.

– If $\frac{1}{2} \leq a^2\lambda^2 < 1$, it is sufficient to take $0 < \epsilon \leq 4a^2\lambda^2 \left(1 - a^2\lambda^2\right)$.

2. Derive the stability condition for the backward-time forward-space scheme

$$\frac{1}{k}\left(v_m^{n+1} - v_m^n\right) + \frac{a}{h}\left(v_{m+1}^{n+1} - v_m^{n+1}\right) = 0$$

used to approximate solutions to $u_t + au_x = 0$ with, say, $x \in [0,1]$ and periodic boundary conditions. Give an example of an initial condition v_m^0 and an explicit expression for v_m^n that demonstrate unstable behavior for a particular λ (your choice) which fails to satisfy the stability condition. Does the growth in your example agree with your theoretical amplification factor?

Solution

Denoting the difference operator of the backward-time forward-space scheme by $P_{k,h}$, we find its corresponding symbol to be

$$\begin{aligned} p_{k,h}(s,\xi) &= P_{k,h}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi} \\ &= \frac{1}{k}\left(e^{sk} - 1\right) + \frac{a}{h}e^{sk}\left(e^{ih\xi} - 1\right). \end{aligned}$$

We determine stability by finding the roots of the symbol as a function of $g := e^{sk}$, yielding

$$g = \frac{1}{1 + a\lambda\left(e^{i\theta} - 1\right)}$$

where $\lambda := k/h$ and $\theta := h\xi$. We find that

$$|g|^{-2} = 1 + 2a\lambda\left(a\lambda - 1\right)\left(1 - \cos\theta\right),$$

hence the scheme is stable ($|g| \leq 1$) if and only if $a \leq 0$ or $a\lambda \geq 1$.

If, for example, $a\lambda = \frac{1}{4}$, then

$$|g|^{-2} = 1 - \frac{3}{8}\left(1 - \cos\theta\right) = \frac{5}{8} + \frac{3}{8}\cos\theta.$$

Choosing, for example, $\theta = \pi$ ought to give an amplication factor of exactly $g = 2$ of the pure mode $v_m = e^{i\theta m} = (-1)^m$. Indeed, one can quickly verify that $v_m^n = 2^n(-1)^m$ satisfies the difference equation:

$$\begin{aligned} kP_{k,h}v_m^n &= v_m^{n+1} - v_m^n + a\lambda\left(v_{m+1}^{n+1} - v_m^{n+1}\right) \\ &= 2^{n+1}(-1)^m - 2^n(-1)^m + \frac{1}{4}\left(2^{n+1}(-1)^{m+1} - 2^{n+1}(-1)^m\right) \\ &= 2^n(-1)^m\left(2 - 1 + \frac{1}{4}\left(-2 - 2\right)\right) \\ &= 0. \end{aligned}$$

One final remark: Notice that if $a\lambda = \frac{1}{2}$, $|g|$ is *unbounded* near $\theta = \pi$. This corresponds to a null space in the resulting system of equations for v^{n+1} induced by the difference operator, and this null space is spanned precisely by the mode corresponding to $\theta = \pi$, $v_m = (-1)^m$.

3. Prove that numerical solutions to the Lax-Friedrichs scheme

$$\frac{1}{k}\left(v_m^{n+1} - \frac{1}{2}\left(v_{m+1}^n + v_{m-1}^n\right)\right) + \frac{a}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) = 0$$

converge to solutions to the corresponding modified equation

$$u_t + au_x = \frac{h^2}{2k}\left(1 - \left(\frac{ak}{h}\right)^2\right) u_{xx}$$

to second order accuracy in L^∞. I.e., show that $\left|v^n - u_{k,h}\left(t_n, \cdot\right)\right|_\infty \to 0$ (with $t_n = T$) as $h, k \to 0$ (according to the stability criterion), where the subscripts on $u_{k,h}$ only indicate that the solution to the modified equation is parameterized by k, h.

Solution

Denoting the difference operator for the Lax-Friedrichs scheme by $P_{k,h}$, we find its corresponding symbol to be

$$\begin{aligned} p_{k,h}(s,\xi) &= P_{k,h}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi} \\ &= \frac{1}{k}\left(e^{sk} - \cos h\xi\right) + i\frac{a}{h}\sin h\xi \\ &= \frac{1}{k}\left(1 + sk + \frac{1}{2}s^2k^2 - 1 + \frac{1}{2}h^2\xi^2\right) + i\frac{a}{h}\left(h\xi\right) + O\left(k^2 + h^2 + h^4k^{-1}\right) \\ &= s + ia\xi + \frac{k}{2}s^2 + \frac{h^2}{2k}\xi^2 + O\left(k^2 + h^2 + h^4k^{-1}\right). \end{aligned}$$

4. (Strikwerda 4.1.2.) Show that the $(2,2)$ leapfrog scheme for $u_t + au_{xxx} = f$ (see (2.2.15)) given by

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\delta^2\delta_0 v_m^n = f_m^n,$$

with $\nu = k/h^3$ constant, is stable if and only if

$$|a\nu| < \frac{2}{3^{3/2}}.$$

Solution

Denoting the difference operator of the given leapfrog scheme by $P_{k,h}$, we find its corresponding symbol to be

$$\begin{aligned} p_{k,h}(s,\xi) &= P_{k,h}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi} \\ &= \frac{1}{2k}\left(e^{sk} - e^{-sk}\right) + i\frac{2a}{h^3}\left(\cos\theta - 1\right)\sin\theta \\ &= \frac{1}{2k}e^{-sk}\left(e^{2sk} - 4ia\nu\sin\theta\left(1 - \cos\theta\right)e^{sk} - 1\right), \end{aligned}$$

where $\nu := k/h^3$. We determine stability by finding the roots of the symbol as a function of $g := e^{sk}$, yielding

$$g_\pm = 2ia\nu\sin\theta\left(1 - \cos\theta\right) \pm \sqrt{1 - 4a^2\nu^2\sin^2\theta\left(1 - \cos\theta\right)^2}.$$

Clearly, if the expression under the $\sqrt{\cdot}$ is nonnegative, then $|g_\pm| \equiv 1$, so this would certainly be sufficient for stability. Furthermore, the expression under the $\sqrt{\cdot}$ should never be zero, as that would give $g_+(\theta) = g_-(\theta)$ (for some θ), leading to linear growth. Lastly, if the expression under the $\sqrt{\cdot}$ is ever *negative* (for some θ), then it is easy to see that at least one of $|g_+|$ or $|g_-|$ will be larger than 1, since, in that case, $|2a\nu\sin\theta\left(1 - \cos\theta\right)| > 1$. Stability is thus equivalent to

$$0 < 1 - 4a^2\nu^2\sin^2\theta\left(1 - \cos\theta\right)^2 =: \alpha(\theta)$$

for all θ. We compute that

$$\alpha'(\theta) = 2\sin\theta\,(1-\cos\theta)^2\,(1+2\cos\theta)\,,$$

and hence the extrema of α occur when $\sin\theta = 0$, $\cos\theta = 1$, or $\cos\theta = -\frac{1}{2}$. The former two cases give $\alpha = 1$, while the latter case gives $\alpha = 1 - \frac{27}{4}a^2\nu^2$. It follows that $\alpha > 0$ is equivalent to $|a\nu| < 2/3^{3/2}$, as desired.

5. (Strikwerda 3.2.1.) Show that the (forward-backward) MacCormack scheme

$$\begin{aligned}
\tilde{v}_m^{n+1} &= v_m^n - a\lambda\left(v_{m+1}^n - v_m^n\right) + kf_m^n,\\
v_m^{n+1} &= \frac{1}{2}\left(v_m^n + \tilde{v}_m^{n+1} - a\lambda\left(\tilde{v}_m^{n+1} - \tilde{v}_{m-1}^{n+1}\right) + kf_m^{n+1}\right)
\end{aligned}$$

is a second-order accurate scheme for the one-way wave equation (1.1.1). Show that for $f = 0$ it is identical to the Lax-Wendroff scheme (3.1.1).

Solution

Denoting the difference operators of the given MacCormack scheme by $P_{k,h}$ and $R_{k,h}$, we find the corresponding symbols to be

$$\begin{aligned}
p_{k,h}(s,\xi) &= P_{k,h}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi}\\
&= \frac{1}{k}\left(e^{sk} - \frac{1}{2}\left(1 + \tilde{v}^{n+1}\left(e^{imh\xi}, f^n \equiv 0\right)/e^{imh\xi}\right)\right)\\
&\quad + \frac{a}{2h}\left(1 - e^{-ih\xi}\right)\tilde{v}^{n+1}\left(e^{imh\xi}, f^n \equiv 0\right)/e^{imh\xi}\\
&= \frac{1}{k}\left(e^{sk} - \frac{1}{2}\left(2 - a\lambda\left(e^{ih\xi} - 1\right)\right)\right) + \frac{a}{2h}\left(1 - e^{-ih\xi}\right)\left(1 - a\lambda\left(e^{ih\xi} - 1\right)\right)\\
&= \frac{1}{k}\left(e^{sk} - 1\right) + i\frac{a}{h}\sin h\xi + \frac{a^2k}{h^2}\left(1 - \cos h\xi\right)\\
&= s + \frac{1}{2}ks^2 + ia\xi + \frac{1}{2}a^2k\xi^2 + O\left(k^2 + h^2\right)\\
&= \left(1 + \frac{1}{2}ks - \frac{1}{2}ika\xi\right)(s + ia\xi) + O\left(k^2 + h^2\right);\\
r_{k,h}(s,\xi) &= R_{k,h}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi}\\
&= \frac{1}{2}\left(\left(\frac{1}{k} - \frac{a}{h}\left(1 - e^{-ih\xi}\right)\right)\tilde{v}^{n+1}\left(v^n \equiv 0, e^{imh\xi}\right)/e^{imh\xi} + e^{sk}\right)\\
&= \frac{1}{2}\left(1 - a\lambda\left(1 - e^{-ih\xi}\right) + e^{sk}\right)\\
&= 1 + \frac{1}{2}ks - \frac{1}{2}ika\xi + O\left(k^2 + kh + h^2\right).
\end{aligned}$$

Given that the differential operator $P := \partial_t + a\partial_x$ for the one-way wave equation $Pu = f$ has symbol $p(s,\xi) = s + ia\xi$, we see that $p_{k,h} - r_{k,h}p = O\left(k^2 + kh + h^2\right)$, showing second order accuracy. Additionally, $P_{k,h}$ for this MacCormack scheme is identical to Lax-Wendroff, hence the stability condition is identical to that for Lax-Wendroff, $|a\lambda| \le 1$. By the Lax-Richtmyer Equivalence Theorem, for $|a\lambda| \le 1$, this MacCormack scheme is second order convergent.

2 Programming

1. For the one-way wave equation $u_t + au_x = 0$, investigate how close the numerical solution to a finite difference scheme is to the solution to the corresponding modified equation. To be concrete, suppose

a convenient initial condition for which you can solve the modified equation explicitly with periodic boundary conditions. Take $a = 1$, $k/h = 0.5$, and final time $T = 0.5$. Compare the following finite difference schemes: forward-time backward-space, Lax-Friedrichs, and Lax-Wendroff. Also, include a derivation of the respective corresponding modified equations.

Solution

– The forward-time backward-space scheme is given by

$$P_{k,h}v_m^n := \frac{1}{k}\left(v_m^{n+1} - v_m^n\right) + a\frac{1}{h}\left(v_m^n - v_{m-1}^n\right) = 0.$$

Its corresponding symbol is

$$\begin{aligned} p_{k,h}(s,\xi) &:= P_{k,h}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi} \\ &= \frac{1}{k}\left(e^{sk} - 1\right) + \frac{a}{h}\left(1 - e^{-ih\xi}\right) \\ &= s + ia\xi + \frac{1}{2}s^2k + \frac{1}{2}a\xi^2h + O\left(k^2 + h^2\right), \end{aligned}$$

suggesting a modified equation of

$$u_t + au_x + \frac{1}{2}ku_{tt} - \frac{1}{2}ahu_{xx} = 0.$$

Of course, if u satisfies the above, then

$$u_{tt} = a^2u_{xx} + O\left(k + h\right),$$

giving the modified equation

$$u_t + au_x - \frac{1}{2}a\left(h - ak\right)u_{xx} = 0.$$

To determine convenient initial conditions, let us suppose that u takes the form $u_K(t,x) = \alpha(t)e^{2\pi iKx}$ for some integer K (the $e^{2\pi iKx}$ term ensures we satisfy the periodic boundary conditions). Then α must satisfy the ordinary differential equation

$$\alpha' + \left(2\pi iKa + 2\pi^2K^2a\left(h - ak\right)\right)\alpha = 0,$$

giving

$$u_K(t,x) = e^{-2\pi^2K^2a(h-ak)t}e^{2\pi iK(x-at)}.$$

The included code tests the forward-time backward-space scheme against the following solution to the modified equation (where the parameters k, h have been suppressed for notational convenience):

$$u(t,x) = u_{K=1}(t,x) - 2u_{K=2}(t,x) + 3u_{K=3}(t,x).$$

The results of the following statement

```
test_convergence_ftbs(1, 0.5, 2.^(-(9:0.5:12)), 0.5);
```

give a numerical convergence rate of 2.97, which is an order of accuracy better than the theoretical convergence rate of 2. This is (most likely) due to a fortuitous choice of λ. For example, if $\lambda = 0.6$, the numerical convergence rate drops to 1.97, which is more in line with expectations.

– The Lax-Friedrichs scheme is given by

$$P_{k,h}v_m^n := \frac{1}{k}\left(v_m^{n+1} - \frac{1}{2}\left(v_{m+1}^n + v_{m-1}^n\right)\right) + a\frac{1}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) = 0.$$

Its corresponding symbol is

$$\begin{aligned}
p_{k,h}(s,\xi) &:= P_{k,h}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi} \\
&= \frac{1}{k}\left(e^{sk} - \frac{1}{2}\left(e^{ih\xi} + e^{-ih\xi}\right)\right) + \frac{a}{2h}\left(e^{ih\xi} - e^{-ih\xi}\right) \\
&= \frac{1}{k}\left(e^{sk} - \cos h\xi\right) + i\frac{a}{h}\sin h\xi \\
&= s + ia\xi + \frac{1}{2}s^2k + \frac{1}{2}\xi^2\frac{h^2}{k} + O\left(k^2 + h^2\right),
\end{aligned}$$

suggesting a modified equation of

$$u_t + au_x + \frac{1}{2}ku_{tt} - \frac{1}{2}\frac{h^2}{k}u_{xx} = 0.$$

Again, if u satisfies the above, then

$$u_{tt} = a^2u_{xx} + O\left(k + h\right),$$

giving the modified equation

$$u_t + au_x - \frac{1}{2}\left(\frac{h^2}{k} - a^2k\right)u_{xx} = 0.$$

To determine convenient initial conditions, we proceed as before and find that

$$u_K(t,x) = e^{-2\pi^2K^2\left(\frac{h^2}{k} - a^2k\right)t}e^{2\pi iK(x-at)}$$

satisfies the modified equation for each integer K.
The included code tests the Lax-Friedrichs scheme against the following solution to the modified equation (where the parameters k, h have been suppressed for notational convenience):

$$u(t,x) = u_{K=1}(t,x) - 2u_{K=2}(t,x) + 3u_{K=3}(t,x).$$

The results of the following statement

```
test_convergence_lax_friedrichs(1, 0.5, 2.^(-(9:0.5:12)), 0.5);
```

give a numerical convergence rate of 1.90, which is consistent with the theoretical convergence rate of 2.

– The Lax-Wendroff scheme is given by

$$P_{k,h}v_m^n := \frac{1}{k}\left(v_m^{n+1} - v_m^n\right) + a\frac{1}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) - \frac{a^2k}{2h^2}\left(v_{m+1}^n - 2v_m^n + v_{m-1}^n\right) = 0.$$

Its corresponding symbol is

$$\begin{aligned}
p_{k,h}(s,\xi) &:= P_{k,h}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi} \\
&= \frac{1}{k}\left(e^{sk} - 1\right) + \frac{a}{2h}\left(e^{ih\xi} - e^{-ih\xi}\right) - \frac{a^2k}{2h^2}\left(e^{ih\xi} - 2 + e^{-ih\xi}\right) \\
&= \frac{1}{k}\left(e^{sk} - 1\right) + i\frac{a}{h}\sin h\xi + \frac{a^2k}{h^2}\left(1 - \cos h\xi\right) \\
&= \left(1 + \frac{1}{2}k\left(s - ia\xi\right)\right)\left(s + ia\xi\right) + \frac{1}{6}s^3k^2 - i\frac{1}{6}a\xi^3h^2 + O\left(k^3 + h^3\right),
\end{aligned}$$

suggesting a modified equation of

$$u_t + au_x + \frac{1}{6}k^2 u_{ttt} + \frac{1}{6}ah^2 u_{xxx} = 0.$$

Similar to before, if u satisfies the above, then

$$u_{ttt} = -a^3 u_{xxx} + O\left(k + h\right),$$

giving the modified equation

$$u_t + au_x + \frac{1}{6}a\left(h^2 - a^2k^2\right)u_{xxx} = 0.$$

To determine convenient initial conditions, we proceed as before and find that

$$u_K(t,x) = e^{2\pi i K(x - a_K t)} \qquad \left[a_K := a\left(1 - \frac{2}{3}\pi^3 K^2 \left(h^2 - a^2k^2\right)\right)\right]$$

satisfies the modified equation for each integer K.
The included code tests the Lax-Wendroff scheme against the following solution to the modified equation (where the parameters k, h have been suppressed for notational convenience):

$$u(t,x) = u_{K=1}(t,x) - 2u_{K=2}(t,x) + 3u_{K=3}(t,x).$$

The results of the following statement

```
test_convergence_lax_wendroff(1, 0.5, 2.^(-(9:0.5:12)), 0.5);
```

give a numerical convergence rate of 3.00, which is identical (to the displayed precision) with the theoretical convergence rate of 3.