Math 269B, 2012 Winter, Homework 3 (Solutions)

Professor Joseph Teran   Jeffrey Lee Hellrung, Jr.

February 17, 2012

1 Theory

1. (Strikwerda 5.1.2.) Show that the modified leapfrog scheme (5.1.6) is stable for ϵ satisfying

$$0 < \epsilon \leq 1 \quad \text{if} \quad 0 < a^2\lambda^2 \leq \frac{1}{2}$$

and

$$0 < \epsilon \leq 4a^2\lambda^2 \left(1 - a^2\lambda^2\right) \quad \text{if} \quad \frac{1}{2} \leq a^2\lambda^2 < 1.$$

Note that these limits are not sharp. It is possible to choose ϵ larger than these limits and still have the scheme be stable.

Solution

Continuing from the text, we find the amplification factors to be

$$g_\pm(\theta) = -ia\lambda\sin\theta \pm \sqrt{1 - a^2\lambda^2\sin^2\theta - \epsilon\sin^4\frac{1}{2}\theta}.$$

If the expression under the $\sqrt{\cdot}$ is nonnegative, then

$$|g_\pm(\theta)|^2 = 1 - \epsilon\sin^4\frac{1}{2}\theta \leq 1,$$

hence the scheme is stable. We thus wish to satisfy

$$0 \leq 1 - a^2\lambda^2\sin^2\theta - \epsilon\sin^4\frac{1}{2}\theta =: \alpha(\theta).$$

We compute that

$$\alpha'(\theta) = -\frac{1}{2}\sin\theta\left(\left(4a^2\lambda^2 - \epsilon\right)\cos\theta + \epsilon\right),$$

and hence the extrema of α occur when $\sin\theta = 0$ or $\cos\theta = \epsilon/\left(\epsilon - 4a^2\lambda^2\right)$. Values of θ satisfying $\sin\theta = 0$ give $\alpha = 1$ or $\alpha = 1 - \epsilon$, requiring that $\epsilon \leq 1$. Values of θ satisfying $\cos\theta = \epsilon/\left(\epsilon - 4a^2\lambda^2\right)$ exist if and only if $\left|\epsilon/\left(\epsilon - 4a^2\lambda^2\right)\right| \leq 1$, which is equivalent to $\epsilon \leq 2a^2\lambda^2$. For such θ, we get $\alpha = 1 - 4a^4\lambda^4/\left(4a^2\lambda^2 - \epsilon\right)$, and for this to be nonnegative, we must have $\epsilon \leq 4a^2\lambda^2\left(1 - a^2\lambda^2\right)$. In particular, we must have $|a\lambda| < 1$.

So far, we have deduced that, at a minimum, $0 < \epsilon \leq 1$. Furthermore, if $\epsilon \leq 2a^2\lambda^2$, then we must additionally satisfy $\epsilon \leq 4a^2\lambda^2\left(1 - a^2\lambda^2\right)$. Now, in the instance that $2a^2\lambda^2 \leq 4a^2\lambda^2\left(1 - a^2\lambda^2\right)$, we would automatically satisfy the second condition, and this latter inequality is equivalent to $a^2\lambda^2 \leq \frac{1}{2}$. It follows that

- If $0 < a^2\lambda^2 \leq \frac{1}{2}$, it is sufficient to take $0 < \epsilon \leq 1$.

– If $\frac{1}{2} \le a^2\lambda^2 < 1$, it is sufficient to take $0 < \epsilon \le 4a^2\lambda^2\left(1 - a^2\lambda^2\right)$.

2. Derive the stability condition for the backward-time forward-space scheme

$$\frac{1}{k}\left(v_m^{n+1} - v_m^n\right) + \frac{a}{h}\left(v_{m+1}^{n+1} - v_m^{n+1}\right) = 0$$

used to approximate solutions to $u_t + au_x = 0$ with, say, $x \in [0,1]$ and periodic boundary conditions. Give an example of an initial condition v_m^0 and an explicit expression for v_m^n that demonstrate unstable behavior for a particular λ (your choice) which fails to satisfy the stability condition. Does the growth in your example agree with your theoretical amplification factor?

Solution

3. Prove that numerical solutions to the Lax-Friedrichs scheme

$$\frac{1}{k}\left(v_m^{n+1} - \frac{1}{2}\left(v_{m+1}^n + v_{m-1}^n\right)\right) + \frac{a}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) = 0$$

converge to solutions to the corresponding modified equation

$$u_t + au_x = \frac{h^2}{2k}\left(1 - \left(\frac{ak}{h}\right)^2\right)u_{xx}$$

to second order accuracy in L^∞. I.e., show that $\left|v_m^n - u_{k,h}\left(t_n, x_m\right)\right| \to 0$ as $h, k \to 0$ (according to the stability criterion), where the subscripts on $u_{k,h}$ only indicate that the solution to the modified equation is parameterized by k, h.

Solution

4. (Strikwerda 4.1.2.) Show that the $(2,2)$ leapfrog scheme for $u_t + au_{xxx} = f$ (see (2.2.15)) given by

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\delta^2\delta_0 v_m^n = f_m^n,$$

with $\nu = k/h^3$ constant, is stable if and only if

$$|a\nu| < \frac{2}{3^{3/2}}.$$

Solution

5. (Strikwerda 3.2.1.) Show that the (forward-backward) MacCormack scheme

$$\begin{aligned}\tilde{v}_m^{n+1} &= v_m^n - a\lambda\left(v_{m+1}^n - v_m^n\right) + kf_m^n,\\ v_m^{n+1} &= \frac{1}{2}\left(v_m^n + \tilde{v}_m^{n+1} - a\lambda\left(\tilde{v}_m^{n+1} - \tilde{v}_{m-1}^{n+1}\right) + kf_m^{n+1}\right)\end{aligned}$$

is a second-order accurate scheme for the one-way wave equation (1.1.1). Show that for $f = 0$ it is identical to the Lax-Wendroff scheme (3.1.1).

Solution

2 Programming

1. For the one-way wave equation $u_t + au_x = 0$, investigate how close the numerical solution to a finite difference scheme is to the solution to the corresponding modified equation. To be concrete, suppose a convenient initial condition for which you can solve the modified equation explicitly with periodic boundary conditions. Take $a = 1$, $k/h = 0.5$, and final time $T = 0.5$. Compare the following finite difference schemes: upwinding, Lax-Friedrichs, and Lax-Wendroff. Also, include a derivation of the respective corresponding modified equations.