

Math 269B, 2012 Winter, Homework 4 (Solutions)

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1 Theory

1. Solve the heat equation $u_t = bu_{xx}$ on a interval $I \subset \mathbb{R}$ with *periodic* boundary conditions. How does $\int_I u(t, x) dx$ vary with time t ?

Solution

Let us first suppose that $I = [-\pi, +\pi]$. We proceed as in the text for the heat equation over \mathbb{R} , but this time use the Fourier transform defined over periodic functions on $[-\pi, +\pi]$:

$$\hat{u}_m = (\mathcal{F}u)_m := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} e^{-imx} u(x) dx.$$

Applying this Fourier transform in space to $u_t = bu_{xx}$ yields $(\hat{u}_m)_t = -bm^2 \hat{u}_m$, an ordinary differential equation in $t \mapsto \hat{u}_m(t)$ which easily solves to $\hat{u}_m(t) = e^{-bm^2 t} \hat{u}_m(0)$. An inverse Fourier transform thus yields

$$\begin{aligned} u(t, x) &= (\mathcal{F}^{-1} \hat{u}(t))(x) \\ &:= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{+\infty} e^{imx} \hat{u}_m(t) \\ &= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{+\infty} e^{imx} e^{-bm^2 t} \hat{u}_m(0). \end{aligned}$$

A similar formula holds for general $I = [\alpha, \beta]$. First, let $\tilde{u} : [-\pi, \pi] \rightarrow \mathbb{R}$ be defined as

$$\tilde{u}(t, \xi) := u(t, \gamma(\xi)), \quad \gamma : [-\pi, \pi] \rightarrow I, \xi \mapsto \frac{\alpha(\pi - \xi) + \beta(\xi + \pi)}{2\pi}.$$

Then \tilde{u} and satisfies

$$b\tilde{u}_{\xi\xi} = b(\gamma')^2 u_{xx} = (\gamma')^2 u_t = (\gamma')^2 \tilde{u}_t \Rightarrow \tilde{u}_t = \tilde{b}\tilde{u}_{\xi\xi}, \quad \tilde{b} = (\gamma')^{-2} b.$$

(Note that $\gamma' \equiv (\beta - \alpha)/2\pi$.) Thus,

$$\tilde{u}(t, \xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{+\infty} e^{im\xi} e^{-\tilde{b}m^2 t} (\mathcal{F}\tilde{u}(t=0, \cdot))_m$$

where

$$\begin{aligned} (\mathcal{F}\tilde{u}(t=0, \cdot))_m &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} e^{-im\xi} \tilde{u}(0, \xi) d\xi \\ &= \frac{\sqrt{2\pi}}{\beta - \alpha} \int_I e^{-im\gamma^{-1}(x)} u_0(x) dx \\ &=: (\mathcal{F}u_0)_m. \end{aligned}$$

This finally gives us

$$u(t, x) = \tilde{u}(t, \gamma^{-1}(x))$$

with \tilde{u} as above.

Clearly, with the representation above, since $\int_{-\pi}^{+\pi} e^{im\xi} d\xi = 0$ except when $m = 0$,

$$\int_I u(t, x) dx = \frac{\beta - \alpha}{2\pi} \int_{-\pi}^{+\pi} \tilde{u}(t, \xi) d\xi = \frac{\beta - \alpha}{\sqrt{2\pi}} (\mathcal{F}u_0)_0 = \int_I u_0(x) dx.$$

2. (Strikwerda 6.1.4.) Use the representation (6.1.3) to verify the following estimates on the norms of $u(t, x)$:

$$\begin{aligned} \|u(t, \cdot)\|_1 &\leq \|u_0\|_1, \\ \|u(t, \cdot)\|_\infty &\leq \|u_0\|_\infty. \end{aligned}$$

Show that if u_0 is nonnegative, then

$$\|u(t, \cdot)\|_1 = \|u_0\|_1.$$

Solution

For the first inequality, we make use of exchanging the order of integration:

$$\begin{aligned} \|u(t, \cdot)\|_1 &= \int_{-\infty}^{+\infty} |u(t, x)| dx \\ &= \int_{-\infty}^{+\infty} \left| \frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4bt} u_0(y) dy \right| dx \\ &\leq \int_{-\infty}^{+\infty} |u_0(y)| \left(\frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4bt} dx \right) dy \\ &= \int_{-\infty}^{+\infty} |u_0(y)| dy \\ &= \|u_0\|_1. \end{aligned}$$

Note that if $u_0 \geq 0$, then the inequality above is actually an equality, we have that $\|u(t, \cdot)\|_1 = \|u_0\|_1$.

The second inequality is derived similarly:

$$\begin{aligned} \|u(t, \cdot)\|_\infty &= \sup_{x \in \mathbb{R}} |u(t, x)| \\ &= \sup_{x \in \mathbb{R}} \left| \frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4bt} u_0(y) dy \right| \\ &\leq \sup_{x \in \mathbb{R}} \frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4bt} \|u_0\|_\infty dy \\ &= \|u_0\|_\infty \sup_{x \in \mathbb{R}} \frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4bt} dy \\ &= \|u_0\|_\infty. \end{aligned}$$

3. Determine the stability and accuracy of the following combination of the Lax-Wendroff and backward-time central-space schemes to solve $u_t + au_x = bu_{xx}$ (with $b > 0$):

$$\begin{aligned} 0 &= P_{k,h} v_m^n \\ &= \frac{1}{k} (v_m^{n+1} - v_m^n) + \frac{a}{2h} (v_{m+1}^n - v_{m-1}^n) - \frac{a^2 k}{2h^2} (v_{m+1}^n - 2v_m^n + v_{m-1}^n) \\ &\quad - \frac{b}{h^2} (v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}). \end{aligned}$$

Solution

The symbol corresponding to the differential operator $P_{k,h}$ is

$$\begin{aligned} p_{k,h}(s, \xi) &:= P_{k,h}(e^{skn+imh\xi})/e^{skn+imh\xi} \\ &= \frac{1}{k}(e^{sk} - 1) + i\frac{a}{h}\sin h\xi + \frac{a^2k}{h^2}(1 - \cos h\xi) + \frac{2b}{h^2}e^{sk}(1 - \cos h\xi) \\ &= s + ia\xi + b\xi^2 + O(k + h^2) \\ &= p(s, \xi) + O(k + h^2), \end{aligned}$$

where p is the symbol of the differential operator $P := \partial_t + a\partial_x - b\partial_x^2$. Thus, this scheme is accurate to order $(1, 2)$.

Regarding stability, we need to find the roots of $p_{k,h}$ with respect to $g := e^{sk}$, giving

$$g = \frac{1 - ia\lambda \sin \theta - a^2\lambda^2(1 - \cos \theta)}{1 + 2b\mu(1 - \cos \theta)},$$

where $\lambda := k/h$ and $\mu := k/h^2$. Note that the numerator is precisely the amplification factor of the Lax-Wendroff scheme applied to $u_t + au_x = 0$, which we know is stable for $|a\lambda| \leq 1$, while the denominator is always at least 1. Hence, $|a\lambda| \leq 1$ is certainly sufficient for stability. However, notice that the numerator is $1 + O(\lambda + \lambda^2)$ while the denominator is $1 + O(\mu)$, and since $\lambda + \lambda^2 = (h + k)\mu$, we see that the denominator will bound the numerator (uniformly in θ) as $h, k \rightarrow 0$. Thus, this scheme is actually unconditionally stable.

2 Programming

1. Solve $u_t + au_x = 0$ numerically using the Lax-Friedrichs scheme. Take $a = 1$, $T = 1$, $x \in [0, 1]$ with periodic boundary conditions, and $u_0(x) = \sin 2\pi x$. For each fixed λ within a decreasing sequence of λ s (each satisfying the stability criterion), demonstrate convergence with $k/h =: \lambda$ by plotting the logarithm of the L^2 -norm of the error (between the analytic solution and the numerical solution) versus the logarithm of h . Verify that the slope suggested by your plot agrees with theory, and estimate the error constant C_λ in the relation $\text{error} = C_\lambda h^p$. Use enough values of λ to estimate the relation between C_λ and λ . What appears to happen to C_λ as $\lambda \rightarrow 0+$, i.e., as you shrink k relative to h ? What happens if, instead of taking $k = \lambda h$, you take $k = h^2$? Explain your numerical results in the context of the theoretical convergence analysis of the Lax-Friedrichs scheme.

Solution

The results of the following statements

```
lambdas = 2.^(-(3:0.5:7));
C = lax_friedrichs_convergence_constant(1, 1, @(x) sin(2*pi*x), lambdas);
r = corr(log(lambdas), log(C));
p = polyfit(log(lambdas), log(C), 1);
plot(log(lambdas), log(C), "o", log(lambdas), polyval(p, log(lambdas)));
```

yields a correlation coefficient of $r = -0.99999$ between $\log \lambda$ and $\log C_\lambda$, with the linear regression giving the relation $C_\lambda = 2.8266\lambda^{-0.61853}$. Clearly, $C_\lambda \rightarrow +\infty$ as $\lambda \rightarrow 0+$.

Using code from Homework 1 allows one to easily see that Lax-Friedrichs does not converge if $k = h^2$. This is consistent with the truncation error being $O(h^2/k)$.

2. Implement the scheme from problem 3 in the Theory section and confirm numerically the theoretical rate of convergence. Use convenient (but non-trivial) initial and boundary conditions such that the solution takes a simple form.

Solution

Using the included code, the results of the following statements

```
test_convergence_lax_wendroff_btcs(1, 1/128, 1, 2.^(-(10:0.5:13)), @(h) 0.5*h);
test_convergence_lax_wendroff_btcs(1, 1/128, 1, 2.^(-(10:0.5:13)), @(h) 2*h);
```

give numerical convergence rates of 1.06 and 1.05, respectively, consistent with the theoretical convergence rate of 1. Note that we set the diffusion coefficient b to be $1/128$ to avoid a nearly zero solution once $T = 1$, and that we get convergence even when $|a\lambda| > 1$.

3. Write a function implementing the Thomas algorithm presented in Strikwerda 3.5. Specifically, we solve the system of equations

$$a_i w_{i-1} + b_i w_i + c_i w_{i+1} = d_i, \quad i = 1, \dots, m-1,$$

with $w_0 = \beta_0$ and $w_m = \beta_m$. The solution is given by

$$w_i = p_{i+1} w_{i+1} + q_{i+1}$$

where p_{i+1} and q_{i+1} are defined recursively by

$$\begin{aligned} p_{i+1} &= -(a_i p_i + b_i)^{-1} c_i, \\ q_{i+1} &= (a_i p_i + b_i)^{-1} (d_i - a_i q_i), \end{aligned}$$

and with p_1 and q_1 determined by the boundary conditions. For the next homework, be prepared to utilize your function implementing the Thomas algorithm to write a function which solves *periodic* tridiagonal systems.

Solution

One can test the included code as follows.

```
N = 10;
a = rand([N 1]); b = rand([N 1]); c = rand([N 1]);
A = spdiags([a(2:end);0] b [0;c(1:end-1)]), [-1 0 +1], N, N);
x = rand([N 1]);
d = A*x;
y = solve_tridiag(a,b,c,d);
norm(x-y, "inf")
```