Math 269B, 2012 Winter, Homework 1

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1 Theory

1. (Strikwerda 1.1.3) Solve the initial value problem for
$$u_t + \frac{1}{1 + \frac{1}{2}\cos x} u_x = 0$$
Show that the solution is given by $u(t, x) = u_0(\xi)$, where ξ is the unique solution of
$$\xi + \frac{1}{2}\sin\xi = x + \frac{1}{2}\sin x - t.$$

2. Solve the initial value problem
$$u_t + (\sin t)\, u_x = \frac{1}{1 + t^2}, \quad u(0, x) = u_0(x), \quad x \in \mathbb{R}, \quad t > 0.$$

3. Consider the first order system of PDEs of the form
$$\vec{u}_t + A\vec{u}_x = 0, \quad \vec{u}(0, x) = \vec{u}_0(x), \quad x \in [0, 1], \quad t > 0.$$
   (a) Give the solution to the initial value problem when
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$
   (b) Describe appropriate boundary conditions at $x = 0$ and/or $x = 1$, if possible, which make the initial boundary value problem in (a) well-posed. Try to be as general as possible. How should such boundary conditions be presented to put the solution in a simple form?
   (c) Give the solution to the initial value problem when
$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$
   (d) Describe appropriate boundary conditions at $x = 0$ and/or $x = 1$, if possible, which make the initial boundary value problem in (c) well-posed. Try to be as general as possible. How should such boundary conditions be presented to put the solution in a simple form?

4. Derive the leading term of the local truncation error for the following finite difference schemes used to approximate solutions to the equation $u_t + au_x = 0$.
   (a)
$$\frac{1}{k}\left(v_m^{n+1} - v_m^n\right) + a\frac{1}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) = 0.$$
   (b)
$$\frac{1}{k}\left(v_m^{n+1} - \frac{1}{2}\left(v_{m+1}^n + v_{m-1}^n\right)\right) + a\frac{1}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) = 0.$$

5. Determine the stability region Λ for each of the finite difference schemes in Problem 4.

2 Programming

1. Implement the finite difference schemes in Problem 4. in the Theory section for $x \in [0,1]$, $t \in [0,T]$ for some final time T, $u(x,0) = u_0(x)$, and *periodic* boundary conditions.

2. Investigate the convergence of each scheme for $a = 1$ and $T = 1$. Set $k/h =: \lambda$ to be constant, and demonstrate which values of λ cause the scheme to converge and which to diverge. If no such λ gives convergence, find an alternate relation between k and h which does ensure convergence (if possible). Try using both a smooth initial condition (e.g., $u_0(x) = \sin(2\pi x)$); a non-smooth initial condition (e.g., $u_0(x) = 1 - 2\,|x - 1/2|$); and a discontinuous initial condition (e.g., $u_0(x) = 0$ if $|x - 1/2| > 1/4$ and $u_0(x) = 1$ if $|x - 1/2| < 1/4$). Use the discrete L^2 norm to measure the error between your numerical solution and the true solution:
$$\|w\|_h = \left(h \sum_m |w_m|^2 \right)^{1/2}.$$
[Note: Due to periodicity, be sure to avoid double-counting the contributions at $x = 0$ and $x = 1$!] Plot the numerical solutions from each scheme at $t = T$ when $h = 1/100$. Summarize your results. Which scheme do you think is better and why?