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Basic Examination, S05

1. Given $n \geq 1$, let $\text{tr} : M_n(\mathbb{C}) \to \mathbb{C}$ denote the trace of a matrix:

$$\text{tr}(A) = \sum_{k=1}^{n} A_{k,k}.$$

   (a) Determine a basis for the kernel (or null-space) of tr.

   (b) For $X \in M_n(\mathbb{C})$, show that $\text{tr}(X) = 0$ if and only if there exists an integer m and matrices $A_1, \ldots, A_m, B_1, \ldots, B_m \in M_n(\mathbb{C})$ so that

$$X = \sum_{j=1}^{m} A_j B_j - B_j A_j.$$

Solution

   (a) Let E_{ij} be the matrix with ij entry equal to 1 and the rest 0. Then a basis for the kernel of tr includes E_{ij} for $i \neq j$ and $E_{11} - E_{ii}$ for $i > 1$. Note that this is a linearly independent set of matrices, and they number $(n^2 - n) + (n - 1) = n^2 - 1$, which is the dimension of ker(tr) (since $\dim(\text{im}(\text{tr})) = 1$ and $\dim(M_n(\mathbb{C})) = n^2$), so must span ker(tr).

   (b) Note that

$$E_{ij}E_{k\ell} - E_{k\ell}E_{ij} = E_{i\ell}\delta_{jk} - E_{kj}\delta_{\ell i}.$$

   Thus for $\ell \neq i$,

$$E_{ij}E_{j\ell} - E_{j\ell}E_{ij} = E_{i\ell},$$

   and for $i > 1$,

$$E_{i1}E_{1i} - E_{1i}E_{i1} = E_{ii} - E_{11}.$$

   It follows that the set of matrices $\{E_{ij}E_{k\ell} - E_{k\ell}E_{ij}\}$ spans ker(tr), from which we can deduce the given representation.

2. Let V be a finite-dimensional vector space, and let V^* denote the dual space; that is, the space of linear maps $\phi : V \to \mathbb{C}$. For a set $W \subset V$, let

$$W^\perp = \{\phi \in V^* : \phi(w) = 0 \ \forall w \in W\}.$$

For a subset $U \subset V^*$, let

$${}^\perp U = \{v \in V : \phi(v) = 0 \ \forall \phi \in U\}.$$

   (a) Show that for any subset $W \subset V$, ${}^\perp(W^\perp) = \text{span}(W)$.
   Recall that the span of a set of vectors is the smallest vector subspace that contains these vectors.

   (b) Let $W \subset V$ be a linear subspace. Give an explicit isomorphism between $(V/W)^*$ and $W^\perp$. Show that it is an isomorphism.

Solution

   (a) We first show that $W^\perp = \text{span}(W)^\perp$. Indeed, given any $\phi \in W^\perp$, any $w \in \text{span}(W)$ can be expressed as a linear combination of vectors in W, hence $\phi(w) = 0$ and $\phi \in \text{span}(W)^\perp$. Conversely, $W \subset \text{span}(W)$, so certainly any ϕ vanishing on $\text{span}(W)$ will likewise vanish on W. This shows the claim. We are thus reduced to the case of showing ${}^\perp(W^\perp) = W$ for a subspace W of V.

   Let $\{w_1, \ldots, w_k\}$ be a basis for W, and complete this with $\{v_{k+1}, \ldots, v_n\}$ to give a basis for V. Then $\{v_i^*\}_{i=k+1}^n$ is a basis for $W^\perp$. Certainly $w \in W$ if and only if $v_i^*(w) = 0$ for $i = k+1, \ldots, n$, hence ${}^\perp(W^\perp) = W$, as claimed.

(b) Let $\phi \in W^\perp$; then define $\Phi \in (V/W)^*$ by

$$\Phi\{v\} = \phi(v).$$

It follows from $\phi \in W^\perp$ that this is well-defined, hence provides an injection from $W^\perp$ to $(V/W)^*$. Conversely, given $\Phi \in (V/W)^*$, we can define $\phi \in V^*$ by

$$\phi(v) = \Phi\{v\},$$

and since $\Phi\{v\} = \Phi\{0\} = 0$ for any $v \in W$, we have that $\phi \in W^\perp$. This provides an injection in the other direction, hence the two spaces are isomorphic.

3. Let A be a Hermitian-symmetric $n \times n$ complex matrix. Show that if $(Av, v) \geq 0$ for all $v \in \mathbb{C}^n$, then there exists an $n \times n$ matrix T so that $A = T^*T$.

   Solution

4. Let $\mathcal{A} = M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices with complex entries.

   We say that $\mathcal{I} \subset \mathcal{A}$ is a *two-sided ideal* in $\mathcal{A}$ if

   (a) for all $A, B \in \mathcal{I}$, $A + B \in \mathcal{I}$

   (b) for all $A \in \mathcal{I}$ and $B \in \mathcal{A}$, AB and BA belong to $\mathcal{I}$

   Show that the only two-side ideals in $\mathcal{A}$ are $\{0\}$ and $\mathcal{A}$ itself.

   Solution

   Suppose $\mathcal{I} \neq \{0\}$. Then there exists some $A \in \mathcal{I}$ with $A \neq 0$. Let $\alpha = (A)_{rc} \neq 0$ be the rc^{th} entry of A.

   Let $E_{ij} \in \mathcal{A}$ have ij^{th} entry equal to 1 and the rest 0. Then by iterative applications of (b),

$$E_{ir} A E_{cj} \left(\frac{c}{\alpha} I \right) = c E_{ij} \in \mathcal{I}$$

   for any i, j and $c \in \mathbb{C}$. It follows from (a) and the fact that $\operatorname{span}\{E_{ij}\} = \mathcal{A}$ that $\mathcal{I} = \mathcal{A}$.

5. For a subset $X \subset \mathbb{R}$, we say that X is *algebraic*, if there exists a family $\mathcal{F}$ of polynomials with rational coefficients, so that $x \in X$ if and only if $p(x) = 0$ for some $p \in \mathcal{F}$.

   (a) Show that the set $\mathbb{Q}$ of rational numbers is algebraic.

   (b) Show that the set $\mathbb{R} \backslash \mathbb{Q}$ of irrational numbers is not algebraic.

   Solution

   (a) Let $\mathcal{F} = \{x \mapsto qx - p \mid q, p \in \mathbb{Z}\}$. Then for each $x = \frac{p}{q} \in \mathbb{Q}$, with $p, q \in \mathbb{Z}$, $q \neq 0$, x is a root of $qx - p$, so it follows that $\mathbb{Q}$ is algebraic.

   (b) The set of all polynomials P with rational coefficients is countable. Indeed, if

$$P_n = \left\{ x \mapsto \sum_{i=0}^{n} a_i x^i \;\middle|\; a_i \in \mathbb{Q} \right\}$$

   is the set of all polynomials of degree n or less, we see that each $p \in P_n$ can be associated with the n-tuple of its coefficients, hence $P_n \cong \mathbb{Q}^n$, which is countable.

   Now each $p \in P_n$ has at most n roots, hence the set of $x \in \mathbb{R}$ for which x is a root of some $p \in P_n$ is countable for each n, and it follows that the set of all algebraic numbers themselves, which is the union of all such x over n, is countable. Since $\mathbb{R} \backslash \mathbb{Q}$ is uncountable, it therefore can't be algebraic.

6. Let X be the set of all infinite sequences $\{\sigma_n\}_{n=1}^{\infty}$ of 1's and 0's endowed with the metric

$$\text{dist}\left(\{\sigma_n\}_{n=1}^{\infty}, \{\sigma'_n\}_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left|\sigma_n - \sigma'_n\right|.$$

Give a direct proof that every infinite subset of X has an accumulation point.

Solution

Let S be an infinite subset of X. Construct nested infinite subsets $S_0, S_1, S_2, \ldots$ of S as follows. Set $S_0 = S$, and given S_n infinite, select σ_{n+1} such that infinitely many of the sequences in S_n have $(n+1)^{th}$ term equal to σ_{n+1}. Then set S_{n+1} to be the set of sequences of S_n with $(n+1)^{th}$ term equal to σ_{n+1}. Thus S_{n+1} is also infinite, and the process can continue.

The claim is that the sequence $s = \{\sigma_i\}_{i=1}^{\infty}$ is an accumulation point of S. Indeed, any $s' = \{\sigma'_i\}_{i=1}^{\infty} \in S_n$ agrees with s in the first n terms, hence

$$\text{dist}(s, s') = \sum_{i=n+1}^{\infty} \frac{1}{2^i} \left|\sigma_i - \sigma'_i\right| \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n}$$

which tends to 0 as $n \to \infty$. Since each S_n has infinitely many sequences, it follows that s is an accumulation point of S.

7. Let X, Y be two topological spaces. We say that a continuous function $f : X \to Y$ is *proper* if $f^{-1}(K)$ is compact for any compact set $K \subset Y$.

   (a) Give an example of a function that is proper but not a homeomorphism.

   (b) Give an example of a function that is continuous but not proper.

   (c) Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1 (that is, has a continuous derivative) and

   $$|f'(x)| \geq 1 \ \text{ for all } x \in \mathbb{R}.$$

   Show that f is proper.

Solution

(a) Let $f : \mathbb{R} \to [0, \infty)$ be the function $x \mapsto |x|$. Then if $K \subset [0, \infty)$ is compact, $f^{-1}(K) = K \cup -K$ is certainly compact, where $-K = \{x \mid -x \in K\}$, hence f is proper, but f is not injective hence not a homeomorphism.

(b) Let $f : \mathbb{R} \to \{0\}$ be the function $x \mapsto 0$. Then $f^{-1}(\{0\}) = \mathbb{R}$, which is not compact, hence f is not proper.

(c) We first show that f is a homeomorphism, i.e., bijective. Let $x, y \in \mathbb{R}$. Then by the Mean Value Theorem,

$$|f(x) - f(y)| = |f'(c)||x - y| \geq |x - y|$$

for some c between x and y, which shows that f is injective. Now f' is continuous, hence either $f' \geq 1$ or $f' \leq -1$ on all of $\mathbb{R}$. If $f' \geq 1$, we have

$$f(y) - f(0) \geq y$$

for any $y \geq 0$, so, since f is continuous, by the Intermediate Value Theorem there exists an $x \in [0, y]$ such that $f(x) = f(0) + y$. A similar argument can be made for $y \leq 0$ and for when $f' \leq -1$, and we conclude that f is surjective, hence a homeomorphism.

Now let $K \subset \mathbb{R}$ be compact. Then K is closed, hence $f^{-1}(K)$ is closed by the continuity of f. Further, K is bounded, say, $|y| < M$ for all $y \in K$. Then $f^{-1}(-M)$ and $f^{-1}(M)$ bound $f^{-1}(K)$, for take any $x \in f^{-1}(K)$ and set $y = f(x)$. Then y is bounded between $-M$ and M, hence x is bounded between $f^{-1}(-M)$ and $f^{-1}(M)$ (either f or $-f$ preserves ordering). Thus $f^{-1}(K)$ is closed and bounded, hence compact, and it follows that f is proper.

8. Suppose $f : \mathbb{R} \to \mathbb{R}$ is C^1 (i.e., continuously differentiable). Show that

$$\lim_{n\to\infty} \sum_{j=1}^{n} \left| f\left(\frac{j-1}{n}\right) - f\left(\frac{j}{n}\right) \right|$$

is equal to

$$\int_0^1 |f'(t)|dt.$$

Solution

f' is continuous on $[0,1]$, a compact set, hence uniformly continuous. Let $\epsilon > 0$ be given. Then there exists an N such that, if $n > N$, $|f'(x)-f'(y)| < \epsilon$ whenever $|x-y| < \frac{1}{n}$. By the Mean Value Theorem,

$$f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) = f'(x_j)\frac{1}{n},$$

$x_j \in \left(\frac{j}{n}, \frac{j-1}{n}\right)$, for each $j = 1, \dots, n$. Let

$$P = \left\{\frac{j}{n}\right\}_{j=0}^{n}$$

be a partition of $[0,1]$, and set

$$M_j = \sup_{\left[\frac{j-1}{n},\frac{j}{n}\right]} |f'|,$$
$$m_j = \inf_{\left[\frac{j-1}{n},\frac{j}{n}\right]} |f'|$$

for $j = 1, \dots, n$. Then

$$M_j - |f'(x_j)| < \epsilon,$$
$$|f'(x_j)| - m_j < \epsilon,$$

thus

$$U(P,|f'|) - \sum_{j=1}^{n} \left| f\left(\frac{j-1}{n}\right) - f\left(\frac{j}{n}\right) \right| < \sum_{j=1}^{n} \epsilon\frac{1}{n} = \epsilon,$$

and similarly

$$\sum_{j=1}^{n} \left| f\left(\frac{j-1}{n}\right) - f\left(\frac{j}{n}\right) \right| - L(P,|f'|) < \epsilon,$$

hence

$$U(P,|f'|) - L(P,|f'|) < 2\epsilon,$$

so $|f'|$ is Riemann integrable on $[0,1]$. Further,

$$\left| \int_0^1 |f'(t)|dt - \sum_{j=1}^{n} \left| f\left(\frac{j-1}{n}\right) - f\left(\frac{j}{n}\right) \right| \right| < \epsilon$$

for all $n > N$, which proves the claim.

9. (a) Suppose

$$\lim_{n\to\infty} a_n = A.$$

Show that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n = A.$$

(b) Show by example that the converse is false.

Solution

(F04.1)

10. Consider the set of $f : [0,1] \to \mathbb{R}$ that obey

$$|f(x) - f(y)| \leq |x - y| \quad \text{and} \quad \int_0^1 f(x)dx = 1.$$

Show that this is a compact subset of $C([0,1])$.

Solution

By the Arzela-Ascoli Theorem, $A \subset C([0,1])$ is compact if and only if

(a) A is closed.

(b) A is uniformly bounded.

(c) A is equicontinuous.

Let A be the subset described above. To show (a), suppose $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence of functions in A. Then, for each $x \in [0,1]$, $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, hence converges. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = \lim_{n\to\infty} f_n(x)$. Then given $\epsilon > 0$, there exists an N such that $\sup_{[0,1]} |f_n - f_m| < \epsilon$ for $n, m > N$, hence, letting $m \to \infty$, we see that $\sup_{[0,1]} |f_n - f| \leq \epsilon$ for $n > N$. Thus

$$|f(x) - f(y)| \leq |f_n(x) - f_n(y)| + 2\epsilon \leq |x - y| + 2\epsilon$$

for any $x, y \in [0,1]$, and since ϵ was arbitrary, f satisfies the first condition of being in A. Note that $f_n \to f$ uniformly on $[0,1]$, hence

$$\int_0^1 f dx = \lim_{n\to\infty} \int_0^1 f_n dx = 1,$$

and f satisfies the second condition of being in A. Therefore, $f \in A$ and A is closed.

To show (b), let $f \in A$. Then the first condition implies that

$$|f(x) - f(0)| \leq x \leq 1$$

for all $x \in [0,1]$, hence

$$f(0) - 1 \leq f(x) \leq f(0) + 1.$$

Now using the second condition,

$$1 = \int_0^1 f(x)dx \geq f(0) - 1,$$

so $f(0) \leq 2$ and $f \leq 3$ on $[0,1]$. On the other hand,

$$1 = \int_0^1 f(x)dx \leq f(0) + 1,$$

so $f(0) \geq 0$ and $f \geq -1$ on $[0,1]$. Thus f is bounded in absolute value by 3, and since f was arbitrary, A is uniformly bounded.

(c) follows directly from the first condition, so we conclude that A is compact.

11. Let us make $M_n(\mathbb{C})$ into a metric space in the following fashion:

$$\operatorname{dist}(A, B) = \left(\sum_{i,j} |A_{ij} - B_{ij}|^2 \right)^{1/2}$$

(which is just the usual metric on $\mathbb{R}^{n^2}$).

(a) Suppose $F : \mathbb{R} \to M_n(\mathbb{C})$ is continuous. Show that the set

$$\{x \in \mathbb{R} : F(x) \text{ is invertible}\}$$

is open (in the usual topology on $\mathbb{R}$).

(b) Show that on the set given above, $x \mapsto F(x)^{-1}$ is continuous.

Solution

(a) Since $\det A$ is a continuous function of the entries of A (indeed, it is a polynomial in the entries of A), $x \mapsto \det F(x)$ is a continuous mapping, hence the inverse image of $\mathbb{R}\backslash\{0\}$ under this mapping is open, which is precisely those x such that $F(x)$ is invertible.

(b) Since the entries of A^{-1} are continuous functions of the entries of A (indeed, rational functions of the entries of A), it follows that $x \mapsto F(x)^{-1}$ is continuous.

12. Let (X, d) be a metric space. Prove that the following are equivalent:

(a) There is a countable dense set.

(b) There is a countable basis for the topology.

Recall that a collection of open sets $\mathcal{U}$ is called a basis if every open set can be written as a union of elements of $\mathcal{U}$.

Solution

Let Y be a countable dense set of X. Consider the family of open sets

$$\mathcal{U} = \{B_d(y; r) \mid y \in Y, r \in \mathbb{Q}\}.$$

Note that $\mathcal{U}$ is countable. Further, given some open set $U \subset X$, every $x \in U$ is covered by some $B_d(y_x; r_x) \subset U$ (since Y is dense and U is open), hence

$$U = \bigcup_{x \in U} B_d(y_x; r_x),$$

from which it follows that $\mathcal{U}$ is a countable basis for (X, d).

Conversely, let $\mathcal{U} = \{U_n\}_{n=1}^{\infty}$ be a countable basis for (X, d), and choose $x_n \in U_n$. Then since every $B(x; r)$ contains some U_n for every $x \in X$, $r \in \mathbb{R}$, $\{x_n\}_{n=1}^{\infty}$ is a countable dense subset of X.