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Numerical Analysis Qualifying Exam, Winter 2006

1. (5 Pts.) Consider the fixed point problem $x = G(x)$ with a solution α. Assume that $G(x)$ is two times continuously differentiable and that $G'(\alpha) = 0$ but $G''(\alpha) = K \neq 0$.

   (a) Show that if the initial iterate x^0 is sufficiently close to α then the fixed point iteration $x^{n+1} = G(x^n)$ converges to α quadratically.

   (b) Give an estimate of the size of ϵ that ensures the iteration $x^{n+1} = G(x^n)$ converges to α if $x^0 \in [\alpha - \epsilon, \alpha + \epsilon]$.

   Solution

   (a) Denote the error of the n^{th} iterate by $e_n = |x^n - \alpha|$. By Taylor's Theorem,

   $$\begin{aligned} G(x) &= G(\alpha) + G'(\alpha)(x - \alpha) + G''(\beta)(x - \alpha)^2 \\ &= \alpha + G''(\beta)(x - \alpha)^2 \end{aligned}$$

   for some β between x and α. It follows that, for some β_n between x^n and α,

   $$\begin{aligned} e_{n+1} &= \left|x^{n+1} - \alpha\right| \\ &= |G(x^n) - \alpha| \\ &= \left|\left(\alpha + G''(\beta_n)(x^n - \alpha)^2\right) - \alpha\right| \\ &= |G''(\beta_n)|\, e_n^2, \end{aligned}$$

   establishing the desired quadratic convergence.

   (b) Choose $\epsilon > 0$ small enough such that
   - $|x - \alpha| < \epsilon$ implies $|G''(x) - K| < \frac{1}{2}|K|$ (possible by the continuity of G'');
   - $\epsilon < 1$; and
   - $\epsilon < \frac{1}{2|K|}$.

   It follows then that if $e_n < \epsilon$,

   $$\frac{e_{n+1}}{e_n} = |G''(\beta_n)|\, e_n < \left(\frac{3}{2}|K|\right)\left(\frac{1}{2|K|}\right) < \frac{3}{4},$$

   which also implies that $e_{n+1} < \epsilon$ as well. By induction, if $e_0 < \epsilon$, then $e_n < (3/4)^n \epsilon \to 0$, i.e., $x^n \to \alpha$.

2. (5 Pts.) Let A be a square non-singular matrix and x be the solution to $Ax = b$. Assume one has an approximate solution z with an associated residual $r = b - Az$. Give a derivation of the following relation between the norm of the error and the norm of the residual:

   $$\frac{\|x - z\|_2}{\|x\|_2} \leq \|A\|_2 \|A^{-1}\|_2 \frac{\|r\|_2}{\|b\|_2}.$$

   Solution

   We have immediately that

   $$b = Ax \;\Rightarrow\; \|b\|_2 \leq \|A\|_2 \|x\|_2,$$

   and

   $$x - z = A^{-1}b - A^{-1}(b - r) = A^{-1}r \;\Rightarrow\; \|x - z\|_2 \leq \|A^{-1}\|_2 \|r\|_2,$$

   so combining these two inequalities gives

   $$\|x - z\|_2 \|b\|_2 \leq \|A^{-1}\|_2 \|r\|_2 \|A\|_2 \|x\|_2,$$

   from which the claim follows (assuming $b \neq 0$).

3. (5 Pts.) Given data points (x_i, y_i) for $i = 1, \ldots, N+1$ with distinct ordinates, prove that the interpolating polynomial of degree at most N is unique.

Solution

Let p and q each be such interpolating polynomials, and consider $r = p - q$. Then r is a polynomial of degree at most N with $N+1$ distinct roots (each x_i is a root). This implies that the $N+1$ degree polynomial $\prod_{i=1}^{N+1}(x - x_i)$ divides into r, which is impossible if r has degree at most N, unless $r \equiv 0$. It follows that $p \equiv q$ and the interpolating polynomial is unique.

4. (10 Pts.) Consider the ordinary differential equation

$$y'(t) = f(t, y(t)), \ y(t_0) = y_0.$$

(a) Give a derivation of the trapezoidal method in a manner analogous to the derivation of general linear multistep methods.

(b) Find the leading term of the local truncation error of the trapezoidal method. What is the global error of the method?

(c) Analyze the linear stability for the trapezoidal method and show that the method is A-stable.

Solution

(a) By Taylor's Theorem,

$$\begin{aligned} y(t+h) &= y(t) + y'(t)h + \frac{1}{2}y''(t)h^2 + \frac{1}{6}y^{(3)}(\alpha_1)h^3 \\ y(t) &= y(t+h) - y'(t+h)h + \frac{1}{2}y''(t+h)h^2 - \frac{1}{6}y^{(3)}(\alpha_2)h^3 \end{aligned}$$

for some $\alpha_1, \alpha_2 \in [t, t+h]$. Combining the above two equalities gives

$$y(t+h) = y(t) + \frac{1}{2}(y'(t) + y'(t+h))h + \frac{1}{4}(y''(t) - y''(t+h))h^2 + \frac{1}{12}(y^{(3)}(\alpha_1) + y^{(3)}(\alpha_2))h^3.$$

This suggests the trapezoidal method

$$y_{k+1} = y_k + \frac{1}{2}(f(t, y_k) + f(t+h, y_{k+1}))h.$$

(b) To compute the local truncation error, we suppose $y_k = y(t)$. Then, noting that $y'(t) = f(t, y(t))$,

$$\begin{aligned} e &= y(t+h) - y_{k+1} \\ &= \frac{1}{2}(f(t+h, y(t+h)) - f(t+h, y_{k+1}))h \\ &+ \frac{1}{4}(y''(t) - y''(t+h))h^2 + \frac{1}{12}\left(y^{(3)}(\alpha_1) + y^{(3)}(\alpha_2)\right)h^3. \end{aligned}$$

Now by the Mean Value Theorem,

$$f(t+h, y(t+h)) - f(t+h, y_{k+1}) = f_y(t+h, \beta)(y(t+h) - y_{k+1}) = f_y(t+h, \beta)e$$

for some β between $y(t+h)$ and y_{k+1}. Also by the Mean Value Theorem,

$$y''(t) - y''(t+h) = -y^{(3)}(\alpha_3)h$$

for some $\alpha_3 \in [t, t+h]$. Thus,

$$e = \frac{1}{2}f_y(t+h, \beta)he + \frac{1}{12}\left(y^{(3)}(\alpha_1) + y^{(3)}(\alpha_2) - 3y^{(3)}(\alpha_3)\right)h^3$$

and solving for e yields

$$e = \frac{y^{(3)}(\alpha_1) + y^{(3)}(\alpha_2) - 3y^{(3)}(\alpha_3)}{6\,(2 - f_y(t+h,\beta)h)} h^3 \in O(h^3).$$

(c) The stability of the method is determined by analyzing the method's behavior when applied to the model problem $y'(t) = f(t, y(t)) = \lambda y(t)$. In this case,

$$y_{k+1} = y_k + \frac{1}{2}(f(t, y_k) + f(t+h, y_{k+1}))h = y_k + \frac{1}{2}\lambda(y_k + y_{k+1})h$$

$$\Rightarrow \frac{1}{2}\left((2 - \lambda h)y_{k+1} - (2 + \lambda h)y_k\right) = 0,$$

so the characteristic polynomial is given by

$$\rho(\theta) = \frac{1}{2}\left((2 - \lambda h)\theta - (2 + \lambda h)\right)$$

which has the single root

$$\zeta = \frac{2 + \lambda h}{2 - \lambda h}.$$

The stability region is the set of complex λh such that

$$\left|\frac{2 + \lambda h}{2 - \lambda h}\right| < 1.$$

Since this is the case for $\Re(\lambda h) < 0$, the method is A-stable.

5. (10 Pts.) Consider the second order partial differential equation

$$u_{tt} = u_{xx} + u_{yy} + 2bu_{xy}$$

to be solved for $0 \le x, y \le 1$, periodic boundary conditions, and smooth initial data

$$\begin{aligned} u(x, y, 0) &= u_0(x, y) \\ u_t(x, y, 0) &= u_1(x, y) \end{aligned}$$

(a) For which real values of b is this a well-posed problem? Why?

(b) Set up a second-order accurate convergent finite difference scheme. Justify your answer.

Solution

(a) We first compute the symbol $p(s, \xi, \eta)$ of the differential operator $P = \partial_t^2 - \partial_x^2 - \partial_y^2 - 2b\partial_{xy}$:

$$\begin{aligned} p(s, \xi, \eta) &= P\left(e^{st}e^{i(\xi x + \eta y)}\right) \Big/ e^{st}e^{i(\xi x + \eta y)} \\ &= s^2 + \xi^2 + \eta^2 + 2b\xi\eta. \end{aligned}$$

The roots of the symbol (as a function of s) are then

$$q_\pm(\xi, \eta) = \pm\sqrt{-\xi^2 - \eta^2 - 2b\xi\eta}.$$

Well-posedness requires that $\Re\left(q_\pm\right)$ be bounded above for all ξ, η, i.e., in this case, due to homogeneity of $q_\pm$ in ξ, η,

$$\xi^2 + \eta^2 + 2b\xi\eta \ge 0.$$

Considering $\xi = \eta$ gives the requirement $b \geq -1$; considering $\xi = -\eta$ gives the requirement $b \leq 1$. We now show that these bounds are not only necessary, but also sufficient. So suppose $-1 \leq b \leq 1$; then

$$\begin{aligned} \xi^2 + \eta^2 + 2b\xi\eta &\geq \xi^2 + \eta^2 - 2|\xi\eta| \\ &= (|\xi| - |\eta|)^2 \\ &\geq 0, \end{aligned}$$

as desired. Therefore, the problem is well-posed for $-1 \leq b \leq 1$.

(b) We consider using centered differences to approximate all derivatives:

$$\begin{aligned} P_{k,h_x,h_y} u^n_{\ell,m} &= D_t^2 u^n_{\ell,m} - D_x^2 u^n_{\ell,m} - D_y^2 u^n_{\ell,m} - 2b D_{x0} D_{y0} u^n_{\ell,m} \\ &= \frac{u^{n+1}_{\ell,m} - 2u^n_{\ell,m} + u^{n-1}_{\ell,m}}{k^2} - \frac{u^n_{\ell+1,m} - 2u^n_{\ell,m} + u^n_{\ell-1,m}}{h_x^2} \\ &- \frac{u^n_{\ell,m+1} - 2u^n_{\ell,m} + u^n_{\ell,m-1}}{h_y^2} - 2b\frac{u^n_{\ell+1,m+1} - u^n_{\ell+1,m-1} - u^n_{\ell-1,m+1} + u^n_{\ell-1,m-1}}{4h_x h_y}; \\ R_{k,h_x,h_y} f^n_{\ell,m} &= f^n_{\ell,m}. \end{aligned}$$

The symbols $p_{k,h_x,h_y}(s,\xi,\eta)$ and $r_{k,h_x,h_y}(s,\xi,\eta)$ for these difference operators are

$$\begin{aligned} p_{k,h_x,h_y}(s,\xi,\eta) &= P_{k,h_x,h_y}\left(e^{snk} e^{i(\xi\ell h_x + \eta m h_y)}\right) \Big/ e^{snk} e^{i(\xi\ell h_x + \eta m h_y)} \\ &= \frac{1}{k^2}\left(e^{sk} - 2 + e^{-sk}\right) \\ &- \frac{1}{h_x^2}\left(e^{i\xi h_x} - 2 + e^{-i\xi h_x}\right) \\ &- \frac{1}{h_y^2}\left(e^{i\eta h_y} - 2 + e^{-i\eta h_y}\right) \\ &- \frac{b}{2h_x h_y}\left(e^{i\xi h_x} - e^{-i\xi h_x}\right)\left(e^{i\eta h_y} - e^{-i\eta h_y}\right) \\ &= \frac{2}{k^2}(\cosh sk - 1) \\ &+ \frac{2}{h_x^2}(1 - \cos\xi h_x) \\ &+ \frac{2}{h_y^2}(1 - \cos\eta h_y) \\ &+ \frac{2b}{h_x h_y}\sin\xi h_x \sin\eta h_y; \\ r_{k,h_x,h_y}(s,\xi,\eta) &= R_{k,h_x,h_y}\left(e^{snk} e^{i(\xi m h_x + \eta\ell h_y)}\right) \Big/ e^{snk} e^{i(\xi m h_x + \eta\ell h_y)} \\ &= 1. \end{aligned}$$

Second-order accuracy is quickly verified by noting that $p_{k,h_x,h_y}(s,\xi,\eta) - r_{k,h}(s,\xi,\eta)p(s,\xi,\eta) \in O(k^2) + O(h_x^2) + O(h_y^2)$. This follows easily by applying Taylor's Theorem to each of the terms in

$p_{k,h_x,h_y}(s,\xi,\eta)$:

$$\begin{aligned}
\frac{2}{k^2}(\cosh sk - 1) &= s^2 + O(k^2) \\
\frac{2}{h_x^2}(1-\cos\xi h_x) &= \xi^2 + O(h_x^2) \\
\frac{2}{h_y^2}(1-\cos\eta h_y) &= \eta^2 + O(h_y^2) \\
\frac{2b}{h_x h_y}\sin\xi h_x \sin\eta h_y &= 2b\xi\eta + O(h_x^2) + O(h_y^2)
\end{aligned}$$

and noticing that each term matches with a term in $p(s,\xi,\eta)$.
Convergence will be implied by stability via the Lax-Richtmyer Equivalence Thereom. As such, we replace $g = e^{sk}$ in $p_{k,h_x,h_y}(s,\xi,\eta) = 0$ and solve for g to determine the roots of the amplification polynomial:

$$\begin{aligned}
&\frac{2}{k^2}\left(g - 2 + g^{-1}\right) + \frac{2}{h_x^2}(1-\cos\xi h_x) + \frac{2}{h_y^2}(1-\cos\eta h_y) + \frac{2b}{h_x h_y}\sin\xi h_x \sin\eta h_y = 0 \\
&\Rightarrow \quad g - 2 + g^{-1} + 2\lambda_x^2(1-\cos\theta) + 2\lambda_y^2(1-\cos\phi) + 2b\lambda_x\lambda_y \sin\theta\sin\phi = 0.
\end{aligned}$$

Let $4c = 2\lambda_x^2 \ldots \sin\theta\sin\phi$ to simplify the notation. Then

$$\begin{aligned}
&g - 2 + g^{-1} + 4c = 0 \\
&\quad\Rightarrow \quad \left(g^{1/2} - g^{-1/2}\right)^2 = -4c \\
&\quad\Rightarrow \quad g^{1/2} - g^{-1/2} = \pm\sqrt{-4c} \\
&\quad\Rightarrow \quad g \pm \left(\sqrt{-4c}\right) g^{1/2} - 1 = 0 \\
&\quad\Rightarrow \quad g_\pm^{1/2} = \pm\sqrt{-c} \pm \sqrt{1-c}.
\end{aligned}$$

Noting that $|g_\pm| \le 1$ if and only if $\left|g_\pm^{1/2}\right| \le 1$, we see that $|g_\pm| \le 1$ if and only if $0 \le c \le 1$ (indeed, in such case $|g_\pm| = 1$). The lower bound on c can be established as follows:

$$\begin{aligned}
4c &= 2\lambda_x^2(1-\cos\theta) + 2\lambda_y^2(1-\cos\phi) + 2b\lambda_x\lambda_y\sin\theta\sin\phi \\
&= 2(\lambda_x\sin\theta)^2\frac{1-\cos\theta}{\sin^2\theta} + 2(\lambda_y\sin\phi)^2\frac{1-\cos\phi}{\sin^2\phi} + 2b(\lambda_x\sin\theta)(\lambda_y\sin\phi) \\
&= (\lambda_x\sin\theta)^2\frac{2}{1+\cos\theta} + (\lambda_y\sin\phi)^2\frac{2}{1+\cos\phi} + 2b(\lambda_x\sin\theta)(\lambda_y\sin\phi) \\
&\ge (\lambda_x\sin\theta)^2 + (\lambda_y\sin\phi)^2 + 2b(\lambda_x\sin\theta)(\lambda_y\sin\phi) \\
&\ge 0
\end{aligned}$$

if $-1 \le b \le 1$ (same argument as in (a)). We can get the upper bound on c by restricting $\lambda_x + \lambda_y \le 1$, for in this case (remembering that $|b| \le 1$),

$$\begin{aligned}
c &= \frac{1}{2}\left(\lambda_x^2(1-\cos\theta) + \lambda_y^2(1-\cos\phi) + b\lambda_x\lambda_y\sin\theta\sin\phi\right) \\
&\le \lambda_x^2 + \lambda_y^2 + \frac{1}{2}\lambda_x\lambda_y \\
&\le \lambda_x^2 + \lambda_y^2 + 2\lambda_x\lambda_y \\
&= (\lambda_x + \lambda_y)^2 \\
&\le 1,
\end{aligned}$$

as desired. It follows that for $-1 \le b \le 1$ and $\lambda_x + \lambda_y \le 1$, the scheme is stable, hence convergent.

6. (10 Pts.) Consider the convection-diffusion equation

$$u_t + au_x = bu_{xx}, \ b > 0, \ a \neq 0 \text{ for } 0 \leq x \leq 1.$$

(a) Construct a second-order accurate unconditionally stable scheme.

(b) Do you think it is uniformly stable in the maximum norm as $b \downarrow 0$? Justify your answers.

Solution

(a) We consider using Crank-Nicolson to approximate u_x and u_{xx}:

$$\begin{aligned}
P_{k,h}u_m^n &= D_{t+}u_m^n + \frac{1}{2}a\left(D_{x0}u_m^{n+1} + D_{x0}u_m^n\right) - \frac{1}{2}b\left(D_x^2u_m^{n+1} + D_x^2u_m^n\right) \\
&= \frac{u_m^{n+1} - u_m^n}{k} + \frac{1}{2}a\left(\frac{u_{m+1}^{n+1} - u_{m-1}^{n+1}}{2h} + \frac{u_{m+1}^n - u_{m-1}^n}{2h}\right) \\
&- \frac{1}{2}b\left(\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2} + \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2}\right); \\
R_{k,h}f_m^n &= \frac{1}{2}\left(f_m^{n+1} + f_m^n\right).
\end{aligned}$$

The symbols $p_{k,h}(s,\xi)$ and $r_{k,h}(s,\xi)$ for these difference operators are

$$\begin{aligned}
p_{k,h}(s,\xi) &= P_{k,h}\left(e^{snk}e^{i\xi mh}\right)/\,e^{snk}e^{i\xi mh} \\
&= \frac{e^{sk} - 1}{k} + \frac{1}{2}a\left(e^{ks}\frac{e^{i\xi h} - e^{-i\xi h}}{2h} + \frac{e^{i\xi h} - e^{-i\xi h}}{2h}\right) \\
&- \frac{1}{2}b\left(e^{ks}\frac{e^{i\xi h} - 2 + e^{-i\xi h}}{h^2} + \frac{e^{i\xi h} - 2 + e^{-i\xi h}}{h^2}\right) \\
&= \frac{e^{sk} - 1}{k} + i\frac{a}{2h}\left(e^{sk} + 1\right)\sin\xi h + \frac{b}{h^2}\left(e^{sk} + 1\right)(1 - \cos\xi h); \\
r_{k,h}(s,\xi) &= R_{k,h}\left(e^{snk}e^{i\xi mh}\right)/\,e^{snk}e^{i\xi mh} \\
&= \frac{1}{2}\left(e^{sk} + 1\right).
\end{aligned}$$

The symbol $p(s,\xi)$ of the differential operator $P = \partial_t + a\partial_x - b\partial_x^2$ is

$$\begin{aligned}
p(s,\xi) &= P\left(e^{st}e^{i\xi x}\right)/\,e^{st}e^{i\xi x} \\
&= s + ia\xi + b\xi^2.
\end{aligned}$$

To show second-order accuracy, we verify that $p_{k,h}(s,\xi) - r_{k,h}(s,\xi)p(s,\xi) \in O(k^2) + O(h^2)$:

$$
\begin{aligned}
p_{k,h}(s,\xi) - r_{k,h}(s,\xi)p(s,\xi) &= \frac{e^{sk}-1}{k} + i\frac{a}{2h}\left(e^{sk}+1\right)\sin\xi h + \frac{b}{h^2}\left(e^{sk}+1\right)(1-\cos\xi h) \\
&- \frac{1}{2}\left(e^{sk}+1\right)\left(s + ia\xi + b\xi^2\right) \\
&= s + \frac{1}{2}s^2 k + O(k^2) \\
&+ i\frac{a}{2h}\left(2 + sk + O(k^2)\right)\left(\xi h + O(h^3)\right) \\
&+ \frac{b}{h^2}\left(2 + sk + O(k^2)\right)\left(\frac{1}{2}\xi^2 h^2 + O(h^4)\right) \\
&- \frac{1}{2}\left(2 + sk + O(k^2)\right)\left(s + ia\xi + b\xi^2\right) \\
&= s + \frac{1}{2}s^2 k + O(k^2) \\
&+ ia\xi + \frac{1}{2}ias\xi k + O(k^2) + O(h^2) \\
&+ b\xi^2 + \frac{1}{2}bs\xi^2 k + O(k^2) + O(h^2) \\
&- \left(1 + \frac{1}{2}sk\right)\left(s + ia\xi + b\xi^2\right) + O(k^2) + O(h^2) \\
&= O(k^2) + O(h^2),
\end{aligned}
$$

as desired.

To show stability, we replace $g = e^{sk}$ in $p_{k,h} = 0$ and solve for g to determine the root of the amplification polynomial:

$$
\begin{aligned}
&\frac{g-1}{k} + (g+1)\left(i\frac{a}{2h}\sin\xi h + \frac{b}{h^2}(1-\cos\xi h)\right) = 0 \\
\Rightarrow\quad & g - 1 + (g+1)\left(i\frac{1}{2}a\lambda\sin\theta + b\mu(1-\cos\theta)\right) = 0 \\
\Rightarrow\quad & g(\theta) = \frac{1 - i\frac{1}{2}a\lambda\sin\theta - b\mu(1-\cos\theta)}{1 + i\frac{1}{2}a\lambda\sin\theta + b\mu(1-\cos\theta)}
\end{aligned}
$$

Since $|1 + b\mu(1-\cos\theta)| \geq |1 - b\mu(1-\cos\theta)|$ for all θ (recall that $b > 0$), we see that $|g(\theta)| \leq 1$ for all θ, hence the scheme is unconditionally stable.

(b) Yes, since $|g| \uparrow 1$ as $b \downarrow 0$.

7. (10 Pts.) Consider the problem

$$
\begin{aligned}
-\Delta u + u = f(x,y) \qquad & (x,y) \in \Omega, \\
u = 1 \qquad & (x,y) \in \partial\Omega_1, \\
\frac{\partial u}{\partial \nu} + u = x \qquad & (x,y) \in \partial\Omega_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega &= \left\{(x,y) \mid x^2 + y^2 < 1\right\}, \\
\partial\Omega_1 &= \left\{(x,y)\ |x^2 + y^2 = 1,\ x \leq 0\right\}, \\
\partial\Omega_2 &= \left\{(x,y) \mid x^2 + y^2 = 1,\ x > 0\right\},
\end{aligned}
$$

and $f \in L^2(\Omega)$.

(a) Determine an appropriate weak variational formulation.

(b) Verify conditions on the corresponding linear and bilinear forms needed for existence and uniqueness of the solution.

(c) Assume that the boundary $\partial\Omega$ is approximated by a polygonal curve. Describe a finite element approximation using P_1 elements. Discuss the form and properties of the stiffness matrix and the existence and uniqueness of the solution of the linear system thus obtained. Give a rate of convergence.

Solution

(a) We set $w = u-1$ (so $u = w+1$) and reformulate the problem in terms of w to obtain homogeneous boundary conditions:

$$\begin{aligned}
-\Delta w + w = f(x,y) - 1 = g(x,y) &\qquad (x,y) \in \Omega, \\
w = 0 &\qquad (x,y) \in \partial\Omega_1, \\
\frac{\partial w}{\partial \nu} + w = x - 1 = h(x,y) &\qquad (x,y) \in \partial\Omega_2.
\end{aligned}$$

Let $V = \{v \in H^1(\Omega) \mid v|_{\partial\Omega_1} \equiv 0\}$ equipped with the norm $\|\cdot\|_{H^1(\Omega)}$. We determine a weak variational formulation by multiplying the differential equation by $v \in V$, applying integration by parts, and noting that $v|_{\partial\Omega_1} \equiv 0$:

$$\begin{aligned}
&(-\Delta w + w)v = fv \\
&\Rightarrow \int_\Omega (-\Delta w + w)v = \int_\Omega fv \\
&\Rightarrow -\int_{\partial\Omega} v\frac{\partial w}{\partial \nu} + \int_\Omega \nabla w \cdot \nabla v + \int_\Omega wv = \int_\Omega fv \\
&\Rightarrow -\int_{\partial\Omega_2} v(h-w) + \int_\Omega (\nabla w \cdot \nabla v + wv) = \int_\Omega fv \\
&\Rightarrow \int_\Omega (\nabla w \cdot \nabla v + wv) + \int_{\partial\Omega_2} wv = \int_\Omega fv + \int_{\partial\Omega_2} hv.
\end{aligned}$$

Let

$$\begin{aligned}
a(w,v) &= \int_\Omega (\nabla w \cdot \nabla v + wv) + \int_{\partial\Omega_2} wv \\
Lv &= \int_\Omega fv + \int_{\partial\Omega_2} hv
\end{aligned}$$

such that the weak variational formulation is to find $w \in V$ such that

$$a(w,v) = Lv \text{ for all } v \in V.$$

(b) The Lax-Milgram Lemma provides sufficient conditions the bilinear form a and the linear form L must satisfy for existence and uniqueness of w:

- *a is symmetric.* Clearly $a(v_1,v_2) = a(v_2,v_1)$ for $v_1, v_2 \in V$.
- *a is continuous.* For $v_1, v_2 \in V$, by the Cauchy-Schwarz Inequality,

$$\begin{aligned}
|a(v_1,v_2)| &= \left| \int_\Omega (\nabla v_1 \cdot \nabla v_2 + v_1 v_2) + \int_{\partial\Omega_2} v_1 v_2 \right| \\
&\le \|v_1\|_{H^1(\Omega)}\|v_2\|_{H^1(\Omega)} + \|v_1\|_{L^2(\partial\Omega_2)}\|v_2\|_{L^2(\partial\Omega_2)}.
\end{aligned}$$

But

$$\|v_i\|_{L^2(\partial\Omega_2)} \le C\|v_i\|_{H^1(\Omega)}$$

for some $C > 0$, so, in fact,

$$|a(v_1, v_2) \le (1 + C)\|v_1\|_{H^1(\Omega)}\|v_2\|_{H^1(\Omega)},$$

and we conclude that a is continuous.

- *a is V-elliptic.* For $v \in V$,

$$\begin{aligned} a(v,v) &= \int_\Omega \left(|\nabla v|^2 + v^2\right) + \int_{\partial\Omega_2} v^2 \\ &\ge \int_\Omega \left(|\nabla v|^2 + v^2\right) \\ &= \|v\|^2_{H^1(\Omega)}, \end{aligned}$$

and so a is indeed V-elliptic.

- *L is continuous.* For $v \in V$, by the Cauchy-Schwarz Inequality,

$$\begin{aligned} |Lv| &= \left|\int_\Omega fv + \int_{\partial\Omega_2} hv\right| \\ &\le \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|h\|_{L^2(\partial\Omega_2)}\|v\|_{L^2(\partial\Omega_2)} \\ &\le \left(\|f\|_{L^2(\Omega)} + C\|h\|_{L^2(\partial\Omega_2)}\right)\|v\|_{H^1(\Omega)}, \end{aligned}$$

hence L is continuous.

Therefore, we have existence and uniqueness of the solution w.

(c) For the finite element approximation, we suppose some triangulation $\{K\}_h$, where h denotes the fineness of the triangulation mesh, with nodes denoted by $\{N_j\}$. Let

$$V_h = \{v \in V \mid v|_K \in P_1(K) \text{ for each } K\}.$$

The approximate variational formulation then becomes to find $w_h \in V_h$ such that $a(w_h, v) = Lv$ for all $v \in V_h$. By linearity, if $\{\phi_i\}$ is a basis for V_h, this is equivalent to finding $w_h \in V_h$ such that $a(w_h, \phi_i) = L\phi_i$ for all ϕ_i. We take ϕ_i such that $\phi_i(N_j) = \delta_{ij}$. Now we can also express $w_h = \sum_j \xi_j\phi_j$, thus obtaining the linear system

$$\sum_j \xi_j a(\phi_j, \phi_i) = L\phi_i \;\Rightarrow\; A\xi = b,$$

where the entries of the stiffness matrix are $A_{ij} = a(\phi_j, \phi_i)$ and the entries of the load vector are $b_i = L\phi_i$. If the enumeration of the N_j's is done efficiently, A will be sparse (since $a(\phi_j, \phi_i) = 0$ if $|i - j|$ is too large) and banded, allowing for efficient solving of the system. Further, A is positive definite (since a is an inner product), hence is nonsingular, so the system has a unique solution.

If w is the solution to the weak variational formulation and w_h is the solution to the approximate variational formulation, then we have the bound $\|w - w_h\|_a \le \|w - v\|_a$ for any $v \in V_h$, where $\|\cdot\|_a$ is the norm induced by the inner product $a(\cdot,\cdot)$. In particular, we can take the linear interpolant $\pi_h w \in V_h$ of w, and we know that $\|w - \pi_h w\|_a \le C_w h^2$ for some constant C_w (dependent on w but independent of h), from which we obtain the convergence rate estimate $\|w - w_h\|_a \le C_w h^2$.