# Math 269B, 2012 Winter, Final

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## 1  Theory

1. Suppose $u\colon [0, \infty) \times \mathbb{R} \to \mathbb{R}$ satisfies the inviscid Burger's equation,

$$0 = u_t + \frac{1}{2}\left(u^2\right)_x = u_t + uu_x, \quad u(0, x) = u_0(x). \tag{1}$$

Use the method of characteristics to show that $u$ must then satisfy the implicit relation

$$u(t, x) = u_0\left(x - tu(t, x)\right). \tag{2}$$

[Hint: Begin by defining $\tilde{u}(t, X) := u\left(t, \varphi(t, X)\right)$ for some to-be-determined change of variables $\varphi\colon [0, \infty) \times \{X\} \to \{x\}$, and choose $\varphi$ such that $\tilde{u}_t \equiv 0$.]

2. Suppose $u_0'$ in (1) is bounded below, i.e., $u_0' \geq c$ for some constant $c$. Determine the maximal $T$ such that a solution to (2) is guaranteed to exist for $t \in [0, T)$ (possibly with $T = \infty$). [Hint: Determine when one can guarantee that the function $u \mapsto u - u_0(x - tu)$ has a unique root.)

3. Solve (1) for $u_0(x) = ax + b$, where $a, b$ are constants. [Hint: Use (2).]

4. Denote the solution to (1) by $u = F\left[u_0\right]$. Express $F\left[x \mapsto au_0(x) + b\right]$ in terms of $F\left[u_0\right]$.

   In other words, given $u$ satisfying (1) for some $u_0$, determine the solution $v$ (in terms of the aforementioned $u$) to

$$v_t + vv_x = 0, \quad v(0, x) = v_0(x) := au_0(x) + b.$$

5. Suppose $u_0$ is given as

$$u_0(x) := \begin{cases} u_0^L(x) := a_L x + b_L, & x < 0 \\ u_0^R(x) := a_R x + b_R, & x > 0 \end{cases}.$$

   Determine the path $t \mapsto (t, x_S(t))$ of the (physically correct) shock in the solution $u$ to (1) eminating from $(t, x) = (0, 0)$ (assume $b_L \geq b_R$). You may use the fact that

$$\frac{1}{2}\int \frac{b_L + b_R + (a_L b_R + a_R b_L)\,t}{((1 + a_L t)\,(1 + a_R t))^{3/2}}\,dt = \frac{(a_L b_R - a_R b_L)\,t + (b_R - b_L)}{(a_L - a_R)\,\sqrt{(1 + a_L t)\,(1 + a_R t)}} \quad [a_L \neq a_R].$$

   [Hint: Recall that the shock speed $x_S'(t) = \frac{1}{2}\left(u^L + u^R\right)(t, x_S(t))$, thus allowing you to set up an ordinary differential equation for $x_S$.] Consider and explain the physical significance of the special cases $a_L = a_R$ and $b_L = b_R$.

6. Solve the weak form of (1) (i.e., give the entropy solution with rarefaction, and with any shocks propagating at the physically correct speed) on the *periodic* domain $[0, 4]$ with the "pulse" initial condition

$$u_0(x) := \begin{cases} 0, & 0 \leq x < 1 \\ 2, & 1 < x < 2 \\ 0, & 2 < x \leq 4 \end{cases}. \tag{3}$$

Identify key points in time $t$ when the character of the solution changes. (It will be natural to express the solution $u(t, x)$ piecewise with respect to $x$ *and* $t$.) Confirm that $\int u(t, x)dx$ is conserved (i.e., $\int u(t, x)dx = $ constant for all $t$), and determine $\lim_{t\to\infty} u(t, x)$.

7. (Strikwerda 6.3.9.) Consider a scheme for (6.1.1), $u_t = bu_{xx}$, of the form

$$v_m^{n+1} = (1 - 2\alpha_1 - 2\alpha_2) v_m^n + \alpha_1 \left(v_{m+1}^n + v_{m-1}^n\right) + \alpha_2 \left(v_{m+2}^n + v_{m-2}^n\right).$$

Show that when $\mu$ is constant, as $k$ and $h$ tend to zero, the scheme is inconsistent unless

$$\alpha_1 + 4\alpha_2 = b\mu.$$

Show that the scheme is fourth-order accurate in $x$ if $\alpha_2 = -\alpha_1/16$.

# 2   Programming

1. Implement the following numerical schemes to solve (1) on the *periodic* domain $[0, 4]$:

   - Godunov's method. At time level $n$, solve the Riemann problem assuming a piecewise constant initial condition $v^n$, then resample to determine $v^{n+1}$.

   - (Backward) Semi-Lagrangian. At time level $n + 1$ and grid vertex $m$, trace the characteristic $t \mapsto x_m + v_m^n (t - t_{n+1})$ *backward* to time level $n$ and linearly interpolate $v^n$ to determine $v_m^{n+1}$.

   - (Forward) Semi-Lagrangian. Trace the characteristics $t \mapsto x_m + v_m^n (t - t_n)$ *forward* to time level $n + 1$ and linearly interpolate the nearest characteristics at a given grid vertex $m$ to determine $v_m^{n+1}$.

   - (Conservative) Lax-Friedrichs. Discretize the conservative form of (inviscid) Burger's equation:

   $$\frac{1}{k} \left(v_m^{n+1} - \frac{1}{2} \left(v_{m+1}^n - v_{m-1}^n\right)\right) + \frac{1}{2} \cdot \frac{1}{2h} \left(\left(v_{m+1}^n\right)^2 - \left(v_{m-1}^n\right)^2\right) = 0$$

   - (Advective) Lax-Friedrichs. Discretize the advective form of (inviscid) Burger's equation:

   $$\frac{1}{k} \left(v_m^{n+1} - \frac{1}{2} \left(v_{m+1}^n - v_{m-1}^n\right)\right) + v_m^n \frac{1}{2h} \left(v_{m+1}^n - v_{m-1}^n\right) = 0$$

   Use the initial condition (3). For those schemes that appear to converge to the exact solution (derived previously), compute a numerical convergence rate. For those schemes that don't appear to converge to the exact solution, explain the discrepancy (e.g., incorrect rarefaction, non-physical shock speed, unstable). Which scheme do you think performs best for the given initial condition?

2. Use your implementation of the Thomas algorithm from Homework 4 to solve *periodic* tridiagonal systems:

$$a_i w_{i-1} + b_i w_i + c_i w_{i+1}, \quad i = 1, \ldots, m,$$

with $w_0 = w_m$ and $w_{m+1} = w_1$. The following algorithm is described in Strikwerda. First, solve the following (non-periodic) tridiagonal systems:

$$\begin{aligned}
a_i x_{i-1} + b_i x_i + c_i x_{i+1} &= d_i, & x_0 = 0 \text{ and } x_{m+1} = 0; \\
a_i y_{i-1} + b_i y_i + c_i y_{i+1} &= 0, & y_0 = 1 \text{ and } y_{m+1} = 0; \\
a_i z_{i-1} + b_i z_i + c_i z_{i+1} &= 0, & z_0 = 0 \text{ and } z_{m+1} = 1;
\end{aligned}$$

for $i = 1, \ldots, m$. Then $w_i$ is given by

$$w_i = x_i + r y_i + s z_i$$

where

$$r := \frac{1}{D}\left(x_m\left(1 - z_1\right) + x_1 z_m\right),$$
$$s := \frac{1}{D}\left(x_m y_1 + x_1\left(1 - y_m\right)\right),$$
$$D := \left(1 - y_m\right)\left(1 - z_1\right) - y_1 z_m.$$