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Analysis Bootcamp, Homework 2

1. *Let (X, d) and (Y, ρ) be metric spaces and let $\{f_n\}$ be a sequence of continuous functions $f_n : X \to Y$. Assume $\{f_n\}$ converges uniformly on X (defined on page 10 of Gamelin-Greene) to $f : X \to Y$. Prove that f is continuous.*

   Solution

   Let $x \in X$ and $\epsilon > 0$ be given. Since $\{f_n\}$ converges uniformly on X, there exists an N such that $\rho(f_n(y), f(y)) < \epsilon$ for all $n \geq N$. Further, since each f_n is continuous, there exists a $\delta > 0$ such that $\rho(f_N(y), f_N(x)) < \epsilon$ for $d(y, x) < \delta$. Hence, for $d(y, x) < \delta$,

   $$\rho(f(y), f(x)) \leq \rho(f(y), f_N(y)) + \rho(f_N(y), f_N(x)) + \rho(f_N(x), f(x)) < 3\epsilon,$$

   which proves that f is continuous.

2. *Gamelin and Greene, page 25. 4, 5, 8.*

   4. *Let $\{U_\alpha\}_{\alpha \in A}$ be a finite open cover of a compact metric space X.*

      (a) Show that there exists $\epsilon > 0$ such that for each $x \in X$, the open ball $B(x; \epsilon)$ is contained in one of the U_α's.
      Remark: *Such an ϵ is called a* Lebesgue number *of the cover.*

      (b) Show that if at least one of the U_α's is a proper subset of X, then there is a largest Lebesgue number for the cover.

      Solution

      (a) Suppose no such ϵ exists. Then, for each integral $n \geq 1$, there exists some x_n such that $B\left(x_n; \frac{1}{n}\right)$ is contained in no U_α. Thus $\{x_n\}_{n=1}^{\infty}$ is a sequence in the compact space X, hence by Theorem 5.1, there exists a convergent subsequence, say, $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \to x \in X$ as $k \to \infty$. Now $x \in U_\alpha$ for some $\alpha \in A$, and since U_α is open, there exists some $r > 0$ such that $B(x; r) \subset U_\alpha$. Let K be large enough such that $1/n_k < r/2$ and $d(x_{n_k}, x) < r/2$ for $k > K$. Then $B\left(x_{n_k}; \frac{1}{n_k}\right) \subset B(x; r) \subset U_\alpha$ for $k > K$, contradicting the construction of $\{x_n\}$. This establishes the existence of such an ϵ.

      (b) Let E be the set of Lebesgue numbers for the cover $\{U_\alpha\}_{\alpha \in A}$. Since X is compact, by Theorem 5.1, X is totally bounded, hence X is bounded by Lemma 5.2 and there exists an R such that $X = B(x; r)$ for all $x \in X$ and $r \geq R$. It follows that if some U_α is such that $U_\alpha = X$, $U_\alpha = B(x; r)$ for all $x \in X$ and $r \geq R$, hence E is unbounded. On the other hand, if all the U_α's are a proper subset of X, $B(x; R) \not\subset U_\alpha$ for any α and for any x, and so $R \notin E$. By the definition of a Lebesgue number, $r \in E$ implies $(r, 0) \subset E$, hence R must be an upper bound for E. Let ϵ be the least upper bound for $E \subset \mathbb{R}$. Then, evidently, $[\epsilon, 0) \supset E \supset (\epsilon, 0)$. We claim that $\epsilon \in E$, i.e., the second inclusion is strict. Indeed, given some $x \in X$, we know that $B(x; r)$ is contained in some $U_{\alpha(r)}$ for each positive $r < \epsilon$. Note that $\alpha(r)$ may be dependent on r (as the notation suggests), but we show that need not to be the case. For take $\alpha_n = \alpha\left(\epsilon - \frac{1}{n}\right)$. Then $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence of elements of A, a finite set, hence there exists some constant subsequence $\{\alpha_{n_k}\}_{k=1}^{\infty} = \{\alpha, \alpha, \ldots\}$. It follows that U_α contains $B(x; r)$ for each positive $r < \epsilon$. Now consider some $y \in B(x; \epsilon)$. Since $d(x, y) < \epsilon$, there exists some $r < \epsilon$ such that $d(x, y) < r$ as well, hence $y \in B(x; r) \subset U_\alpha$. Since y was arbitrary, we conclude that, in fact, $B(x; \epsilon) \subset U_\alpha$, and since x was arbitrary, $\epsilon \in E$, as was to be shown. ϵ then corresponds to the largest Lebesgue number for the cover.

   5. *Prove that the product of a finite number of compact metric spaces is compact.*

      Solution

Let $(X_1, d_1), \ldots, (X_n, d_n)$ be compact metric spaces, set $X = X_1 \times \cdots \times X_n$, and let $d = \max\{d_1, \ldots, d_n\}$ be the product metric. Let $\epsilon > 0$ be given, and let $\left\{B(x_j^i; \epsilon)\right\}_{i=1}^{m_j}$ be a finite covering of X_j of open balls of radius ϵ (possible since each X_j is compact, hence by Theorem 5.1, is totally bounded). Then the set of open sets

$$\left\{B\left(x_1^{i_1}; \epsilon\right) \times \cdots \times B\left(x_n^{i_n}; \epsilon\right)\right\},$$

where the indices range independently as

$$\begin{gathered} 1 \leq i_1 \leq m_1, \\ \vdots \\ 1 \leq i_n \leq m_n, \end{gathered}$$

is a finite covering (there are $\prod_{j=1}^n m_j < \infty$ open sets) of X. Further, in the product metric d above, each of these sets is equal to the open ball $B\left(\left(x_1^{i_1}, \ldots, x_n^{i_n}\right); \epsilon\right)$, hence X can be covered by a finite number of open balls of radius ϵ, and since ϵ was arbitrary, this means that X is totally bounded. By Theorem 5.1, X is compact.

8. *Let (X, d) be a bounded metric space and let $\mathcal{E}$ be the family of nonempty closed subsets of X. Show that*

$$\rho(E, F) = \max\left(\sup_{x \in E} d(x, F), \sup_{y \in F} d(y, E)\right)$$

defines a metric on $\mathcal{E}$. Show that $\mathcal{E}$ is compact whenever X is compact.

Solution

Clearly, $\rho(E, F) \geq 0$.

$d(x, E) = 0$ for any $x \in E \in \mathcal{E}$, hence $\rho(E, E) = 0$ for any $E \in \mathcal{E}$. Conversely, suppose $\rho(E, F) = 0$ for $E, F \in \mathcal{E}$. Then $\sup_{x \in E} d(x, F) = 0$, i.e., $d(x, F) = 0$ for all $x \in E$, implying that every point of E is adherent to F, hence $E \subset \overline{F} = F$. A similar argument shows that $F \subset E$ as well, hence $E = F$.

The definition is symmetrical with respect to E and F, hence $\rho(E, F) = \rho(F, E)$ for any $E, F \in \mathcal{E}$.

Let $E, F, G \in \mathcal{E}$ and $\epsilon > 0$. Fix $x \in E$, and choose $y \in F$ such that $d(x, y) \leq d(x, F) + \epsilon$. Then $d(x, y) \leq d(x, F) \leq \rho(E, F)$. Choose $z \in G$ such that $d(y, z) \leq d(y, G) + \epsilon$. Then $d(y, z) \leq d(y, G) \leq \rho(F, G)$. Hence

$$d(x, G) \leq d(x, z) \leq d(x, y) + d(y, z) \leq \rho(E, F) + \rho(F, G) + 2\epsilon.$$

Letting $\epsilon \to 0$ and taking the supremum over x yields

$$\sup_{x \in E} d(x, G) \leq \rho(E, F) + \rho(F, G).$$

A similar argument establishes the same bound for $\sup_{y \in F} d(y, G)$, which proves the triangle inequality.

3. *Gamelin and Greene, page 27. Problems 2, 8, 9, 11.*

2. *Prove that two metrics d and ρ for X are equivalent if and only if the identity map $(X, d) \to (X, \rho)$ is bicontinuous (that is, it and its inverse are continuous).*

Solution

According to Exercise 1.12, d and ρ are equivalent if and only if they have the same convergent sequences. Let $f : (X, d) \to (X, \rho)$ be the identity mapping; then d and ρ are equivalent if and only if $\{x_n\}$ converges whenever $\{f(x_n)\}$ converges. But this is precisely the condition of continuity for f. Reversing d and ρ establishes the continuity of the inverse mapping as well.

8. *Prove that a continuous real-valued function on a compact metric space assumes its maximum value and its minimum value.*

   Solution

   Let $f : X \to \mathbb{R}$ for X a compact metric space. We show first that $f(X)$ is a compact subspace of $\mathbb{R}$. Indeed, let $\{U_\alpha\}$ be an open cover of $f(X)$. Then by Theorem 6.2, $\{f^{-1}(U_\alpha)\}$ is an open cover of X, hence the compactness of X implies there exists some finite subcover $\left\{f^{-1}\left(U_{\alpha_n}\right)\right\}$. It follows that $\{U_{\alpha_n}\}$ is a finite subcover of $\{U_\alpha\}$, which establishes that $f(X)$ is compact. By Theorem 5.5, $f(X)$ is closed and bounded, hence has a maximum and minimum value.

9. *Prove that a metric space X is compact if and only if every continuous real-valued function on X is bounded.*

   Solution

   That X being compact implies that every continuous real-valued function on X is bounded is established in Exercise 6.8.

   Suppose every continuous real-valued function on X is bounded. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in X. Now if X is not complete, then the function $f : X \to \mathbb{R}$ defined by $f(x) = \lim_{n\to\infty} 1/d(x, x_n)$ is defined for all $x \in X$, and is continuous. However, given any $M > 0$, there exists an N such that $d(x_n, x_m) < \frac{1}{M}$ for $n, m \geq N$, hence, in particular, $d(x_N, x_n) < \frac{1}{M}$ for $n > N$, and it follows that $f(x_N) > M$. Thus f is unbounded, contradicting the hypothesis for X. This establishes that X is complete.

   Suppose that X is not totally bounded. Then there exists an $\epsilon > 0$ and a sequence $\{x_n\}_{n=1}^{\infty}$ such that $d(x_n, x_m) > \epsilon$ for all $n \neq m$. Define $f : X \to \mathbb{R}$ by $f(x) = n(1 - 2d(x, x_n)/\epsilon)$ if $d(x, x_n) < \epsilon/2$ (at most one such x_n will be within $\epsilon/2$ of x), and $f(x) = 0$ otherwise. f is continuous and unbounded, again contradicting the hypothesis for X, establishing that X is totally bounded. By Theorem 5.1, therefore, X is compact.

11. *Show that there exists a continuous real-valued function h on $[0, 1]$ such that*

$$\limsup_{t\to 0+} \left| \frac{h(x+t) - h(x)}{t} \right| = \infty$$

   whenever $0 \leq x < 1$.

   Hint: *Consider the space $C([0, 2])$ of continuous real-valued functions on the interval $[0, 2]$, with the metric of uniform convergence. Let E_m be the set of $f \in C([0, 2])$ for which there exists $x \in [0, 1]$ satisfying $|f(x + t) - f(x)|/|t| \leq m$ for $t > 0$, $x + t \leq 2$. Show that E_m is a closed nowhere-dense subset of $C([0, 2])$.*

   Solution

4. *Gamelin and Greene, page 35. Problems 10, 12.*

10. *Show that $\|f\|_1 = \int_0^1 |f(s)|ds$ defines a norm on $f \in C[0, 1]$.*

    Solution

    Let $f, g \in C[0, 1]$, $c \in \mathbb{R}$.

    Clearly $\|f\|_1 \geq 0$ as $|f| \geq 0$. Equally clear is that $\|0\|_1 = 0$. Further, if $\|f\|_1 = 0$, the continuity of f allows us to conclude that $f = 0$.

    $\|cf\|_1 = \int_0^1 |(cf)(s)|dx = |c| \int_0^1 |f(s)|ds = |c|\|f\|_1$.

    $|f + g| \leq |f| + |g|$, hence $\|f + g\|_1 = \int_0^1 |(f + g)(s)|ds \leq \int_0^1 (|f(s)| + |g(s)|)\, ds = \int_0^1 |f(s)|ds + \int_0^1 |g(s)|ds = \|f\|_1 + \|g\|_1$.

12. *(a) Prove that if f and g are continuous real-valued functions on $[0, 1]$, then*

$$\int_0^1 f(s)g(s)ds \leq \left(\int_0^1 f(s)^2 ds\right)^{1/2} \left(\int_0^1 g(s)^2 ds\right)^{1/2}.$$

Remark: *This is an integral version of the Cauchy-Schwarz inequality. For the proof, refer back to Exercise 1.3 and consider the integral*

$$\int_0^1 (f(s) - \lambda g(s))^2 \, ds,$$

which is nonnegative for all values of the real parameter λ.

(b) *Using (a), show that the formula*

$$\|f\|_2 = \left(\int_0^1 f(s)^2 ds \right)^{1/2}$$

defines a norm on the space of continuous real-valued functions on $[0,1]$.

(c) *Prove that if* f *and* g *are continuous complex-valued functions on* $[0,1]$, *then*

$$\int f(s)\overline{g(s)}ds \leq \left(\int |f(s)|^2 ds \right)^{1/2} \left(\int |g(s)|^2 ds \right)^{1/2}.$$

Remark: *This is an integral form of the complex version of the Cauchy-Schwarz inequality.*

(d) *Show that the formula*

$$\|f\|_2 = \left(\int_0^1 |f(s)|^2 ds \right)^{1/2}$$

defines a norm on the space of continuous complex-valued functions on $[0,1]$.

Solution

(a) If $\int g(s)^2 ds = 0$, then the continuity of g implies that $g \equiv 0$, and the conclusion follows. So suppose $\int g(s)^2 ds > 0$. Expand the polynomial in λ

$$\int (f(s) - \lambda g(s))^2 \, ds \geq 0$$

and substitute $\lambda = \int f(s)g(s)ds / \int g(s)^2 ds$. Rearranging quickly yields

$$\int f(s)^2 ds \int g(s)^2 ds \geq \left(\int f(s)g(s)ds \right)^2.$$

(b) The positivity and scalar factoring of $\|\cdot\|_2$ are immediate from the definition. Let $f, g \in C([0,1])$. Then

$$\|f+g\|_2^2 = \int (f+g)(s)^2 ds = \int f(s)^2 ds + \int g(s)^2 ds + 2 \int f(s)g(s)ds$$
$$\leq \|f\|_2^2 + \|g\|_2^2 + 2\|f\|_2\|g\|_2 = (\|f\|_2 + \|g\|_2)^2.$$

(c) The procedure in (a) is still valid, except we start with

$$\int |f(s) - \lambda g(s)|^2 ds \geq 0$$

and substitute

$$\lambda = \frac{\int f(s)\overline{g(s)}ds}{\int |g(s)|^2 ds}$$

to obtain

$$\int |f(s)|^2 ds \int |g(s)|^2 ds \geq \left| \int f(s)\overline{g(s)}ds \right|^2.$$

(d) The positivity and scalar factoring of $\|\cdot\|_2$ are immediate from the definition. Let $f, g \in C([0,1] \to \mathbb{C})$. Then

$$\|f+g\|_2^2 = \int (f+g)(s)\overline{(f+g)}(s)ds = \int |f(s)|^2 + \int |g(s)|^2 + 2\Re \int f(s)\overline{g(s)}ds$$

$$\leq \|f\|_2^2 + \|g\|_2^2 + 2\left|\int f(s)\overline{g(s)}ds\right| \leq \|f\|_2^2 + \|g\|_2^2 + 2\|f\|_2\|g\|_2 = (\|f\|_2 + \|g\|_2)^2.$$

5. *Let (X, d_X) be a compact metric space and let $C(X)$ be the set of continuous functions from X to the real numbers $\mathbb{R}$. Give $C(X)$ the metric*

$$d(f,g) = \sup_X |f(x) - g(x)|$$

of uniform convergence. Then by Exercise 1, $C(X)$ is complete.

Now let $A \subset C(X)$. Prove that A is compact (relative to the metric of $C(X)$) if and only if (a), (b), and (c) hold:

(a) A is closed.

(b) A is uniformly bounded: There is an $M < \infty$ such that $d(f,0) \leq M$ for all $f \in A$ (i.e., $|f(x)| \leq M$ for all $f \in A$ and all $x \in X$).

(c) A is equicontinuous*: For every $\epsilon > 0$ there is a $\delta > 0$ (not depending on $f \in A$) such that*

$$d_X(x,y) < \delta \;\Rightarrow\; \sup_{f \in A} |f(x) - f(y)| < \epsilon.$$

This is the Arzela-Ascoli theorem. Hint: Use Theorem 5.1 and figure out what it means for A to be totally bounded (With respect to the metric of $C(X)$).

Solution

We show first that *(a)* – *(c)* imply the compactness of A. First, A is closed by *(a)*, $C(X)$ is complete, and a closed subspace of a complete metric space is complete, by Theorem 2.3. Thus A is complete.

We next show that A is totally bounded. Let $2\epsilon > 0$ be given. Let $\delta > 0$ be as in *(c)* concerning the equicontinuity of A, i.e., $|f(x) - f(y)| < \epsilon$ whenever $f \in A$ and $d_X(x,y) < \delta$. By Theorem 5.1, there exists a finite covering $\{B_X(x_i,\delta)\}_{i=1}^n$ of X of δ-balls. Let M be as in *(b)* concerning the uniform boundedness of A, i.e., $|f(x)| < M$ whenever $f \in A$ and $x \in X$. Set $y_j = j\epsilon$, $j \in \{-\lceil \frac{M}{\epsilon} \rceil, \ldots, 0, \ldots, \lceil \frac{M}{\epsilon} \rceil\} = J$.

Now consider the family of functions

$$\Phi = \left\{ x \mapsto \frac{\sum_{x_i \in B_X(x,\delta)} y_{j_i}(\delta - d_X(x,x_i))}{\sum_{x_i \in B_X(x,\delta)} \delta - d_X(x,x_i)} \right\},$$

where each $j_i \in J$ for $i = 1, \ldots, n$. We first note that Φ is finite, for it has at most $|J|^n = \left(2\lceil \frac{M}{\epsilon} \rceil + 1\right)^n$ elements. Secondly, any $\phi \in \Phi$ is continuous (which is left as an exercise to the reader!). We claim that any $f \in A$ is at most 2ϵ from some $\phi \in \Phi$, thus showing that A can be covered by a finite number of 2ϵ-balls in $C(X)$, from which it follows that A can be covered by a finite number of 2ϵ-balls in A, i.e., A is totally bounded.

Let $f \in A$ and choose $\phi \in \Phi$ by specifying $|y_{j_i} - f(x_i)| < \epsilon$ for $i = 1, \ldots, n$ (possible due to the uniform boundedness of A and the construction of the y_j's). Additionally, fix $x \in X$. By the equicontinuity of f, $|f(x_i) - f(x)| < \epsilon$ for any $x_i \in B_X(x,\delta)$, hence $|y_{j_i} - f(x)| < 2\epsilon$ for any $x_i \in B_X(x,\delta)$. It follows that

$$|f(x) - \phi(x)| = \frac{\sum_{x_i \in B_X(x,\delta)} |f(x) - y_{i_j}|(\delta - d_X(x,x_i))}{\sum_{x_i \in B_X(x,\delta)} \delta - d_X(x,x_i)} < \frac{\sum \epsilon(\delta - d_X(x,x_i))}{\sum \delta - d_X(x,x_i)} = 2\epsilon.$$

Since x was arbitrary, we see that $d(f, \phi) < 2\epsilon$, as desired. Therefore, since A is both complete and totally bounded, by Theorem 5.1, A is compact.

Now we show that if A is compact, then *(a)* – *(c)* hold. First, a compact subspace of a compact metric space is closed, by Thereom 2.4, establishing *(a)*. Secondly, compact spaces are totally bounded, by Theorem 5.1, hence bounded, by Lemma 5.2, establishing *(b)*. Lastly, let $3\epsilon > 0$ be given. By Theorem 5.1, there exists a finite covering $\{B(f_i, \epsilon)\}$ of A of ϵ-balls. By Theorem 6.3, each f_i, being a function from a compact metric space (X), is uniformly continuous. Thus, for each f_i, there exists a δ_i such that $|f_i(x) - f_i(y)| < \epsilon$ whenever $d_X(x, y) < \delta_i$. Set $\delta = \min_i \delta_i$ (which exists and is positive, since there are finitely many δ_i's). Then given any $f \in A$ and $x, y \in X$ such that $d_X(x, y) < \delta$, there exists an f_i such that $d(f, f_i) < \epsilon$, hence

$$|f(x) - f(y)| < |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < 3\epsilon,$$

establishing *(c)*.

(Note: The first part of the solution is made easier if, instead of constructing each $\phi \in \Phi$, one simply picks $f \in A$ that is "close" to a given ϕ, but the idea is the same.)