Jeffrey Hellrung
Applied Differential Equations Qualifying Exam, Fall 2004

1. Solve the following initial-boundary value problem for the wave equation with a potential term,

$$\begin{aligned}&(\partial_t^2 - \partial_x^2)u + u = 0,\ 0 < x < \pi,\ t > 0,\\ &u(0,t) = u(\pi,t) = 0,\ t > 0,\\ &u(x,0) = f(x),\ \partial_t u(x,0) = 0,\ 0 < x < \pi,\end{aligned}$$

where

$$f(x) = \begin{cases} x, & \text{if } x \in (0, \pi/2) \\ \pi - x, & \text{if } x \in (\pi/2, \pi) \end{cases}.$$

The answer should be given in terms of an infinite series of explicitly given functions.

Solution

Supposing $u(x,t) = X(x)T(t)$, we separate variables:

$$XT'' - X''T + XT = 0 \ \Rightarrow\ \frac{X''}{X} = \frac{T'' + T}{T} = \lambda$$

for some constant λ. The boundary conditions on X are $X(0) = X(\pi) = 0$, hence we find that $X = \sin(kx)$ and $\lambda = \lambda_k = -k^2$ for $k \geq 1$ integral. Using the boundary condition $T'(0) = 0$, we then solve T to be $T = \cos\left(\sqrt{1+k^2}t\right)$, so, by linearity,

$$u(x,t) = \sum_{k\geq 1} c_k \cos\left(\sqrt{1+k^2}t\right)\sin(kx),$$

where, by orthogonality of the c_k's,

$$c_k = \frac{2}{\pi}\int_0^\pi u(x,0)\sin(kx)dx.$$

Since f is even about $\pi/2$, we find that $c_k = 0$ for even k since $\sin(kx)$ is odd about $\pi/2$. For odd k, $\sin(kx)$ is even about $\pi/2$, so

$$\begin{aligned} c_k &= \frac{2}{\pi}\int_0^\pi f(x)\sin(kx)dx \\ &= \frac{4}{\pi}\int_0^\pi x\sin(kx)dx \\ &= -\frac{4}{k\pi}x\cos(kx)\Big|_0^{\pi/2} + \frac{4}{k\pi}\int_0^{\pi/2}\cos(kx)dx \\ &= \frac{4}{k^2\pi}\sin(kx)\Big|_0^{\pi/2} \\ &= (-1)^{(k-1)/2}\frac{4}{\pi k^2}. \end{aligned}$$

Therefore,

$$u(x,t) = \sum_{k\geq 1,\ k \text{ odd}} (-1)^{(k-1)/2}\frac{4}{\pi k^2}\cos\left(\sqrt{1+k^2}t\right)\sin(kx).$$

2. Let $u(x,t)$ be a bounded solution to the Cauchy problem for the heat equation

$$\begin{cases} \partial_t u = a^2\partial_x^2 u,\ t > 0,\ x \in \mathbb{R},\ a > 0, \\ u(x,0) = \phi(x). \end{cases}$$

Here $\phi(x) \in C(\mathbb{R})$ satisfies

$$\lim_{x\to\infty} \phi(x) = b, \quad \lim_{x\to-\infty} \phi(x) = c.$$

Compute the limit of $u(x,t)$ as $t \to \infty$, $x \in \mathbb{R}$. Justify your argument carefully.

Solution

We have that u is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(ax-y)^2/4t}\phi(y)dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2}\phi(y)dz,$$

where $y = ax + \sqrt{4t}z$. Given $\epsilon > 0$, let $B > 0$, $C < 0$ be large enough (in absolute value) such that $|\phi(x) - b| < \epsilon$ for $x > B$ and $|\phi(x) - c| < \epsilon$ for $x < C$. Let

$$\beta = \frac{B - ax}{\sqrt{4t}}, \ \gamma = \frac{-C - ax}{\sqrt{4t}},$$

such that $z > \beta$ if and only if $y > B$, and $z < \gamma$ if and only if $y < C$. In preparing to make some estimations, we decompose the above integral as follows, and estimate each part separately:

$$\int_{-\infty}^{\infty} e^{-z^2}\phi(y)dz = \int_{-\infty}^{\gamma} e^{-z^2}\phi(y)dz + \int_{\gamma}^{\beta} e^{-z^2}\phi(y)dz + \int_{\beta}^{\infty} e^{-z^2}\phi(y)dz.$$

We estimate the first integral as

$$\int_{-\infty}^{\gamma} e^{-z^2}\phi(y)dz = \int_{-\infty}^{\gamma} ce^{-z^2}dz + \int_{-\infty}^{\gamma} e^{-z^2}(\phi(y) - c)dz.$$

Since $\gamma \to 0$ as $t \to \infty$,

$$\left|\int_{-\infty}^{\gamma} ce^{-z^2}dz - \frac{\sqrt{\pi}}{2}c\right| \le \epsilon$$

for large enough t, while, since $|\phi(y) - c| < \epsilon$ for $z < \gamma$,

$$\left|\int_{-\infty}^{\gamma} e^{-z^2}(\phi(y) - c)dz\right| \le \sqrt{\pi}\epsilon.$$

It follows that

$$\left|\int_{-\infty}^{\gamma} e^{-z^2}\phi(y)dz - \frac{\sqrt{\pi}}{2}c\right| \le \left(\sqrt{\pi} + 1\right)\epsilon.$$

Similarly, the third integral admits the estimate

$$\left|\int_{\beta}^{\infty} e^{-z^2}\phi(y)dz - \frac{\sqrt{\pi}}{2}b\right| \le \left(\sqrt{\pi} + 1\right)\epsilon$$

for large enough t. Finally, the middle integral vanishes as $t \to \infty$, since $\beta, \gamma \to 0$:

$$\left|\int_{\gamma}^{\beta} e^{-z^2}\phi(y)dy\right| < \epsilon$$

for large enough t. We thus obtain the estimate

$$\left|\int_{-\infty}^{\infty} e^{-z^2}\phi(y)dz - \frac{\sqrt{\pi}}{2}(b + c)\right| \le \left(2\sqrt{\pi} + 3\right)\epsilon$$

for large enough t. It follows that

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2}\phi(y)dz \to \frac{1}{2}(b + c)$$

as $t \to \infty$.

3. Let us consider a damped wave equation,

$$\begin{cases} (\partial_t^2 - \Delta + a(x)\partial_t)u = 0, \ (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ u|_{t=0} = u_0, \ \partial_t u|_{t=0} = u_1. \end{cases}$$

Here the damping coefficient $a \in C_0^\infty(\mathbb{R}^3)$ is a non-negative function with $u_0, u_1 \in C_0^\infty(\mathbb{R}^3)$. Show that the energy of the solution $u(x,t)$ at time t,

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla_x u|^2 + |\partial_t u|^2 \right) dx,$$

is a decreasing function of $t \geq 0$.

Solution

Since $u_0, u_1 \in C_0^\infty(\mathbb{R}^3)$, we also have $u \in C_0^\infty(\mathbb{R}^3)$ by finite propagation speed. Thus, when integrating by parts in what follows, boundary terms vanish. With this in mind, we simply compute

$$\begin{aligned} E'(t) &= \frac{d}{dt}\left(\frac{1}{2} \int_{\mathbb{R}^3} \left(|\nabla u|^2 + u_t^2 \right) dx \right) \\ &= \int_{\mathbb{R}^3} (\nabla u \cdot \nabla u_t + u_t u_{tt})\, dx \\ &= -\int_{\mathbb{R}^3} (\Delta u) u_t dx + \int_{\mathbb{R}^3} u_t\left(\Delta u - a u_t \right) dx \\ &= -\int_{\mathbb{R}^3} a(u_t)^2 dx \\ &\leq 0, \end{aligned}$$

hence E is a decreasing function of t.

4. Prove that each solution (except $x_1 = x_2 = 0$) of the autonomous system

$$\begin{cases} x_1' = x_2 + x_1 \left(x_1^2 + x_2^2 \right) \\ x_2' = -x_1 + x_2(x_1^2 + x_2^2) \end{cases}$$

blows up in finite time. What is the blow-up time for the solution which starts at the point $(1,0)$ when $t = 0$?

Solution

Let $r^2 = x_1^2 + x_2^2$. Then

$$rr' = x_1 x_1' + x_2 x_2' = r^4 \ \Rightarrow \ r' = r^3,$$

which solves to give

$$r(t) = \frac{r_0}{\sqrt{1 - 2r_0^2 t}},$$

where $r_0 = r(0)$. Thus, solutions will blow up at $t = 1/2r_0^2$. For the initial point $(1,0)$, $r_0 = 1$, so the blow-up time is $t = 1/2$.

5. Let us consider a generalized Volterra-Lotka system in the plane, given by

$$x'(t) = f(x(t)), \ x(t) \in \mathbb{R}^2, \qquad (1)$$

where $f(x) = (f_1(x), f_2(x)) = (ax_1 - bx_1x_2 - ex_1^2, -cx_2 + dx_1x_2 - fx_2^2)$, and a, b, c, d, e, f are positive constants. Show that

$$\operatorname{div}(\phi f) \neq 0, \ x_1 > 0, \ x_2 > 0,$$

where $\phi(x_1, x_2) = 1/(x_1 x_2)$. Using this observation, prove that the autonomous system (1) has no closed orbits in the first quadrant.

Solution

Since

$$(\phi f)(x_1, x_2) = \left(\frac{a}{x_2} - b - e\frac{x_1}{x_2}, -\frac{c}{x_1} + d - f\frac{x_2}{x_1} \right),$$

we have simply

$$\operatorname{div}(\phi f)(x_1, x_2) = -\frac{e}{x_2} - \frac{f}{x_2} < 0.$$

Now let $C \subset \mathbb{R}^2$ be a simple closed C^1-curve in the first quadrant enclosing a region Ω, such that $C = \partial\Omega$ as subsets of $\mathbb{R}^2$. From the above, we have that

$$0 > \int_\Omega \operatorname{div}(\phi f) dx = \int_C \phi f \cdot \nu ds.$$

Along trajectories of (1), $f \cdot \nu = 0$, so it follows that C cannot be a trajectory. But since C is arbitrary, we conclude that (1) has no closed orbits.

6. Let $q \in C_0^1(\mathbb{R}^3)$. Prove that the vector field

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{q(y)(x-y)}{|x-y|^3} dy$$

enjoys the following properties:

(a) $u(x)$ is conservative.
(b) $\operatorname{div} u(x) = q(x)$ for all $x \in \mathbb{R}^3$.
(c) $|u(x)| = \mathcal{O}(|x|^{-2})$ for large x.

Furthermore, prove that the properties (a), (b), and (c) above determine the vector field $u(x)$ uniquely.

Solution

Define

$$f(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{q(y)}{|x-y|} dy.$$

Then it is easy to see that $\nabla f = u$, hence u is conservative.

To compute the divergence, we first make a change of variables, expressing

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} q(x-z) \frac{z}{|z|^3} dz,$$

so that

$$\operatorname{div} u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \nabla_x q(x-z) \cdot \frac{z}{|z|^3} dz = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \nabla_z q(x-z) \cdot \frac{z}{|z|^3} dz.$$

Since $q \in C_0^1(\mathbb{R}^3)$ and $z/|z|^3$ is integrable near 0, we can write

$$\operatorname{div} u(x) = \lim_{\epsilon \searrow 0} -\frac{1}{4\pi} \int_{|z| \geq \epsilon} \nabla_z q(x-z) \cdot \frac{z}{|z|^3} dz.$$

Integrating by parts, and using the fact that q vanishes for large enough $|z|$, we get

$$\begin{aligned} & -\frac{1}{4\pi} \int_{|z| \geq \epsilon} \nabla_x q(x-z) \cdot \frac{z}{|z|^3} dz \\ = \quad & -\frac{1}{4\pi} \int_{|z|=\epsilon} q(x-z) \frac{z}{|z|^3} \cdot \nu dS(z) + \frac{1}{4\pi} \int_{|z| \geq \epsilon} q(x-z) \nabla_z \cdot \left(\frac{z}{|z|^3} \right) dz. \end{aligned}$$

It is not hard to show that $\nabla_z \cdot \left(z/|z|^3\right)$ vanishes (for z away from 0):

$$\begin{aligned}
\nabla_z \cdot \left(\frac{z}{|z|^3}\right) &= \frac{\nabla_z(z)}{|z|^3} + z \cdot \nabla_z \left(|z|^{-3}\right) \\
&= \frac{3}{|z|^3} + z \cdot \left(\frac{-3z}{|z|^5}\right) \\
&= 0.
\end{aligned}$$

This takes care of the second integral. We evaluate the first integral by noticing that $\nu = -z/|z|$ (the inward pointing normal) and $|z| = \epsilon$ on the domain of integration:

$$-\frac{1}{4\pi}\int_{|z|=\epsilon} q(x-z)\frac{z}{|z|^3} \cdot \nu dS(z) = \frac{1}{4\pi\epsilon^2}\int_{|z|=\epsilon} q(x-z)dS(x) \to q(x)$$

as $\epsilon \searrow 0$, by continuity of q. The claim then immediately follows:

$$\operatorname{div} u(x) = \lim_{\epsilon \searrow 0} -\frac{1}{4\pi}\int_{|z|\geq\epsilon} \nabla_z q(x-z) \cdot \frac{z}{|z|^3}dz = q(x).$$

For the decay claim, let $R > 0$ be large enough such that $B_R(0) \supset \operatorname{supp} q$, and let $M > 0$ be such that $q < M$. Then for large x, $|x-y| \geq |x| - R$ for $y \in \operatorname{supp} q$, so

$$|u(x)| = \left|\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{q(y)(x-y)}{|x-y|^3}dy\right| < \frac{M}{4\pi}\int_{\mathbb{R}^3} \frac{|x|+R}{(|x|-R)^3}dy = \mathcal{O}(|x|^{-2}).$$

We now address the uniqueness claim. First, u being conservative implies that $u = \nabla f$ for some f, and hence $q = \operatorname{div} u = \Delta f$, i.e., f satisfies a Poisson equation. The decay of u guarantees the solution f to be unique, and we know that f is given by convolution with the fundamental solution:

$$f(x) = (K * q)(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{q(y)}{|x-y|}dy.$$

It follows that u is as given.

7. Consider the partial differential equation

$$uu_x + u_t + u = 0, \ (z,t) \in \mathbb{R}^2.$$

- Find the particular solution that satisfies the condition $u(0,t) = e^{-2t}$.
- Show that at the point $(z,t) = (1/9, \log 2)$, $u = 1/3$.

Solution

- We let $x = z$ and $y = t$, for notational convenience in applying the method of characteristics, giving the PDE $uu_x + u_y + u = 0$. The initial condition curve may be parametrized by $s \mapsto (0, s, e^{-2s}) = (x_0, y_0, z_0)$. The method of characteristics yields the system of ODEs

$$\begin{aligned}
x' &= z; \\
y' &= 1; \\
z' &= -z.
\end{aligned}$$

  y and z may be solved immediately:

$$\begin{aligned}
y &= t + y_0 = t + s; \\
z &= z_0 e^{-t} = e^{-2s}e^{-t}.
\end{aligned}$$

x may now be found:

$$x = e^{-2s}\left(1 - e^{-t}\right) + x_0 = e^{-2s}\left(1 - e^{-t}\right).$$

We can solve for s, t in terms of x, y, giving the relations

$$t = y - s,\ e^{-s} = \frac{1}{2}e^{-y}\left(1 + \sqrt{1 + 4xe^{2y}}\right).$$

The solution is thus

$$u(x, y) = z = e^{-y}e^{-s} = \frac{1}{2}e^{-2y}\left(1 + \sqrt{1 + 4xe^{2y}}\right).$$

- We compute

$$u(1/9, \log 2) = \frac{1}{2}\left(\frac{1}{4}\right)\left(1 + \sqrt{1 + 4\left(\frac{1}{9}\right)(4)}\right) = \frac{1}{3}.$$

8. The function $y(x, t)$ satisfies the partial differential equation

$$x\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x \partial t} + 2y = 0,$$

and the boundary conditions

$$y(x, 0) = 1,\ y(0, t) = e^{-at},$$

where $a > 0$. Find the Laplace transform, $\overline{y}(x, s)$, of the solution, and hence derive an expression for $y(x, t)$ in the domain $x \geq 0$, $t \geq 0$.

Solution

Recall that the Laplace transform is given by

$$\mathcal{L}_t(f(t))(s) = \int_0^\infty e^{-st}f(t)dt.$$

It is easy to derive that

$$\mathcal{L}_t(f'(t))(s) = f(0+) + s\mathcal{L}_t(f(t))(s),$$

and so applying the Laplace transform to the PDE results in

$$x\overline{y}_x + s\overline{y}_x + 2\overline{y} = 0,$$

where $\overline{y} = \overline{y}(x, s) = \mathcal{L}_t(y(x, t))(s)$. The boundary condition transforms to

$$\overline{y}(0, s) = \frac{1}{s + a},$$

and so this solves easily to

$$\overline{y}(x, s) = \frac{s^2}{(x + s)^2(s + a)}.$$

We can recover y by the inverse Laplace transform via contour integration:

$$\begin{aligned}
y(x,t) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\overline{y}(x,s)e^{st}ds \\
&= \operatorname{Res}\left(\overline{y}(x,s)e^{st}; s=-a\right)+\operatorname{Res}\left(\overline{y}(x,s)e^{st}; s=-x\right) \\
&= \left(\lim_{s\to -a}(s+a)\overline{y}(x,s)e^{st}\right)+\left(\lim_{s\to -x}\frac{\partial}{\partial s}(s+x)^2\overline{y}(x,s)e^{st}\right) \\
&= \left(\lim_{s\to -a}\frac{s^2}{(x+s)^2}e^{st}\right)+\left(\lim_{s\to -x}\frac{\partial}{\partial s}\frac{s^2}{s+a}e^{st}\right) \\
&= \frac{a^2}{(x-a)^2}e^{-at}+\left(\lim_{s\to -x}\frac{s+2a+s^2t+ast}{(s+a)^2}se^{st}\right) \\
&= \frac{a^2}{(x-a)^2}e^{-at}+\frac{-x+2a+x^2t-axt}{(x-a)^2}(-x)e^{-xt} \\
&= \frac{1}{(x-a)^2}\left(a^2e^{-at}+\left(x-2a-x^2t+axt\right)xe^{-xt}\right).
\end{aligned}$$