Jeffrey Hellrung
Applied Differential Equations Qualifying Exam, Spring 2006

1. Solve the following initial value problem and verify your solution:

$$u_x + u_y = u^2, \; u(x,0) = h(x).$$

Solution

We use the method of characteristics, and as such parametrize the initial curve Γ by $s \mapsto (s, 0, h(s)) = (x_0, y_0, z_0)$, and solve the system

$$\begin{aligned} x'(t) &= 1; \\ y'(t) &= 1; \\ z'(t) &= z^2. \end{aligned}$$

We can solve all 3 equations immediately and independently:

$$\begin{aligned} x(t) &= t + x_0 = t + s; \\ y(t) &= t + y_0 = t; \\ z(t) &= \frac{z_0}{1 - tz_0} = \frac{h(s)}{1 - th(s)}. \end{aligned}$$

We solve for s, t in terms of x, y to obtain

$$\begin{aligned} s &= x - y; \\ t &= y. \end{aligned}$$

It follows that

$$u(x,y) = z = \frac{h(x-y)}{1 - yh(x-y)}.$$

To verify this solves the initial value problem, we note that $u(x,0) = h(x)$, while

$$\begin{aligned} u_x(x,y) &= \frac{h'(x-y)}{(1 - yh(x-y))^2}, \\ u_y(x,y) &= \frac{h(x-y)^2 - h'(x-y)}{(1 - yh(x-y))^2}, \end{aligned}$$

from which we easily verify that, indeed, $u_x + u_y = u^2$.

2. Consider an initial value problem for the *Kortewig-deVries equation*

$$u_t + u_{xxx} + 6uu_x = 0, \; x \in \mathbb{R}, \; t > 0, \; u(x,0) = \phi(x). \quad (1)$$

Show that the following are conserved quantities for (1) (you may assume that the function $u(x,t)$ vanishes as $|x| \to \infty$, together with all of its derivatives):

- Mass:

$$\int_{-\infty}^{\infty} u(x,t)dx,$$

- Momentum:

$$\int_{-\infty}^{\infty} u(x,t)^2 dx,$$

- Energy:
$$\int_{-\infty}^{\infty}\left(\frac{1}{2}u_x(x,t)^2-u(x,t)^2\right)dx.$$

Solution

All integrals below are over $\mathbb{R}$, and we keep in mind that u and all its x-derivatives vanish at $\pm\infty$.

- We compute
$$\begin{aligned}\frac{d}{dt}\int u dx &= \int u_t dx\\ &= \int -\left(u_{xxx}+6uu_x\right)dx\\ &= -u_{xx}-3u^2\big|_{-\infty}^{\infty}\\ &= 0.\end{aligned}$$
It follows that mass is conserved.
- We compute
$$\begin{aligned}\frac{d}{dt}\int u^2 dx &= \int 2uu_t dx\\ &= \int -2u\left(u_{xxx}+6uu_x\right)dx\\ &= -2\int uu_{xxx}dx-12\int u^2u_x dx\\ &= -2u_xu_{xx}\big|_{-\infty}^{\infty}+2\int u_xu_{xx}dx-12\int u^2u_xdx\\ &= u_x^2-4u^3\big|_{-\infty}^{\infty}\\ &= 0.\end{aligned}$$
It follows that momentum is conserved.
- We compute
$$\begin{aligned}\frac{d}{dt}\int\left(\frac{1}{2}u_x^2-u^3\right)dx &= \int\left(u_x(u_x)_t-3u^2u_t\right)dx\\ &= \int\left(u_x(u_t)_x-3u^2u_t\right)dx\\ &= \int\left(-u_x\left(u_{xxx}+6uu_x\right)_x+3u^2\left(u_{xxx}+6uu_x\right)\right)dx\\ &= \int\left(-u_xu_{xxxx}-6u_x^3-6uu_xu_{xx}+3u^2u_{xxx}+18u^3u_x\right)dx.\end{aligned}$$
Now,
$$\begin{aligned}\int u_xu_{xxxx}dx &= -\int u_{xx}u_{xxx}dx=-u_{xx}^2\big|_{-\infty}^{\infty}=0;\\ \int u_x^3dx &= \int u_x^2u_xdx=-\int 2uu_xu_{xx}dx;\\ \int u^2u_{xxx}dx &= -\int 2uu_xu_{xx}dx;\\ \int u^3u_xdx &= \frac{1}{4}u^4\bigg|_{-\infty}^{\infty}=0.\end{aligned}$$

Thus, the outer terms in the integrand vanish outright, while the inner 3 terms additively cancel, leaving 0. It follows that energy is conserved.

3. Let $0 < L < \infty$ and let $0 < p(x) \in C^\infty([0, L])$. Consider the following intial-boundary value problem on $(0, L) \times (0, \infty)$:

$$\begin{cases} \partial_t u = \partial_x \left(p(x)\partial_x u\right), \ (x,t) \in (0,L) \times (0,\infty); \\ u(x,0) = \phi(x), \ \partial_x u(0,t) = \partial_x u(L,t) = 0. \end{cases}$$

Here $\phi \in C^\infty([0, L])$. Compute the limit of $u(x, t)$ as $t \to \infty$.

Solution

We separate variables, assuming $u(x, t) = X(x)T(t)$, giving

$$XT' = (pX')'T \ \Rightarrow \ \frac{T'}{T} = \frac{(pX')'}{X} = \lambda$$

for some constant λ. T solves easily to $T(t) = e^{\lambda t}$. We are thus left to analyze

$$(pX')' = \lambda X$$

subject to the boundary conditions $X'(0) = X'(L) = 0$. Let M denote the linear differential operator (with boundary conditions) on the left-hand side. Then λ is an eigenvalue for M and X is an eigenfunction. We show that $\lambda \leq 0$:

$$\begin{aligned} \lambda(X,X) &= (\lambda X, X) \\ &= (MX, X) \\ &= \int_0^L (MX)X dx \\ &= \int_0^L (pX')'X dx \\ &= -\int_0^L p(X')^2 dx \\ &\leq 0, \end{aligned}$$

and hence $\lambda \leq 0$, as claimed. Further, note that M is self-adjoint in the usual L^2-inner product:

$$\begin{aligned} (Mu, v) &= \int_0^L (Mu)v dx \\ &= \int_0^L (pu')'v dx \\ &= -\int_0^L pu'v' dx \\ &= \int_0^L u(pv')dx \\ &= (u, Mv); \end{aligned}$$

it follows that the eigenfunctions of M form an orthogonal basis. Denoting the eigenvalues by λ_k and the corresponding (normalized) eigenfunctions by X_k, by linearity the solution to the PDE is

$$u(x,t) = \sum_k c_k e^{\lambda_k t} X_k(x),$$

where the c_k's are the Fourier coefficients of $u(x,0) = \phi(x)$:

$$c_k = \int_0^L \phi(x) X_k(x) dx.$$

Now in the limit as $t \to \infty$, due to the $e^{\lambda_k t}$'s, the term in the expression of u corresponding to the largest λ_k dominates the rest of the terms in the sum. We see that $\lambda = 0$ is an eigenvalue of M with eigenfunction $X \equiv 1$, and this must be the largest eigenvalue by the nonpositivity of all eigenvalues, hence, based on this discussion,

$$\lim_{t\to\infty} u(x,t) = \int_0^L \phi(x) dx.$$

4. Consider the initial value problem of the form

$$\frac{dy}{dt} = f(y), \ \ y(0) = 0. \qquad (3)$$

Show that there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f(y) = 0$ precisely when $y = 0$ and such that f does not satisfy the Lipschitz condition in any neighborhood of 0, while the uniqueness of the initial value problem (3) holds.

Solution

Let

$$f(y) = -y^{1/3}.$$

Notice that f is continuous; $f(y) = 0$ precisely when $y = 0$; and f' is unbounded in any neighborhood of 0, hence cannot be Lipschitz in any neighborhood of 0. Further, $y(t) \equiv 0$ is the unique solution to $y'(t) = f(y(t))$ with $y(0) = 0$, since 0 is a stable fixed point for f.

5. Consider the second-order ODE

$$x''(t) + x(t) + 2x(t)^2 = 0. \qquad (4)$$

- Find the conserved quantity for (4).
- Rewrite (4) as a 2×2 system of the first order.
- Find and classify the equilibrium points.
- Sketch the phase portrait of the equation.

Solution

- Multiplying the equation by x' and integrating gives

$$C = \frac{1}{2}\left(x'\right)^2 + \frac{1}{2}x^2 + \frac{2}{3}x^3.$$

- $$(x, x')' = (x', -x - 2x^2) = F(x, x').$$

- Equilibrium points $(x, x')^*$ satisfy

$$0 = F((x, x')*) \ \Rightarrow \ (x, x')^* \in \left\{(0,0), \left(-\frac{1}{2}, 0\right)\right\}.$$

  We also compute

$$DF(x, x') = \begin{pmatrix} 0 & 1 \\ -1 - 4x & 0 \end{pmatrix}.$$

- $(x, x')^* = (0, 0)$. We have

$$DF(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with eigenvalues $\lambda_\pm = \pm i$. It follows that $(0, 0)$ is a center.

- $(x, x')^* = (-1/2, 0)$. We have

$$DF\left(-\frac{1}{2}, 0\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with eigenvalues $\lambda_\pm = \pm 1$ and corresponding eigenvalues

$$v_\pm = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}.$$

It follows that $(-1/2, 0)$ is a saddle.

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6. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open, and connected set. Suppose that $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ is a solution of

$$\Delta u + \sum_{k=1}^{n} a_k(x) \frac{\partial u}{\partial x_k} + c(x) u = 0 \text{ in } \Omega,$$

where $a_k(x)$, $1 \leq k \leq n$, and $c(x)$ are continuous in $\overline{\Omega}$, with $c(x) < 0$ in Ω. Show that $u = 0$ on $\partial\Omega$ implies that $u = 0$ in Ω.

Hint: Show that $\max u(x) \leq 0$ and $\min u(x) \geq 0$.

Solution

Assume that $u = 0$ on $\partial\Omega$, and suppose u attains its maximum at $x^* \in \Omega$; immediately we have $u(x^*) \geq 0$. Then $\Delta u(x^*) \leq 0$ and $(\partial u / \partial x_k)(x^*) = 0$ for $1 \leq k \leq n$, thus it follows (from the PDE that u satisfies) that $c(x^*) u(x^*) \geq 0$. The fact that $c < 0$ implies then that $u(x^*) \leq 0$, so in fact we must have $\max_{\overline{\Omega}} u = u(x^*) = 0$. A completely analogous argument allows us to conclude that $\min_{\overline{\Omega}} u = 0$ as well, hence $u \equiv 0$.

7. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and let $f \in C\left(\overline{\Omega}\right)$. Find the minimum of the functional

$$E(u) = \int_\Omega \left(\frac{1}{2} \sum_{k=1}^{n} \left(\frac{\partial u}{\partial x_k} \right)^2 - f(x) u(x) \right) dx$$

on the space of smooth functions in $\overline{\Omega}$, subject to the constraints

$$u|_{\partial\Omega} = 0, \quad \int_\Omega = u(x) dx = A,$$

where A is a given constant. You may assume that a smooth solution of this problem exists. You may also regard the solution of

$$\Delta w = h \text{ in } \Omega, \; w|_{\partial\Omega} = 0$$

as known, for any $h \in C\left(\overline{\Omega}\right)$.

Hint. Use Lagrange multipliers.

Solution

Let u be the solution to $\Delta u = -f$ in Ω with $u = 0$ on $\partial\Omega$. We claim that $E(u)$ is the minimum of E. Indeed, for any $v \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ with $v = 0$ on $\partial\Omega$,

$$\begin{aligned}
E(u+v) &= \int_\Omega \left(\frac{1}{2}\|\nabla u + \nabla v\|^2 - f(u+v)\right) dx \\
&= E(u) + \frac{1}{2}\int_\Omega \|\nabla v\|^2 dx + \int_\Omega \nabla u \cdot \nabla v dx - \int_\Omega f v dx \\
&= E(u) + \frac{1}{2}\int_\Omega \|\nabla v\|^2 dx - \int_\Omega \Delta u v dx - \int_\Omega f v dx \\
&= E(u) + \frac{1}{2}\int_\Omega \|\nabla v\|^2 dx - \int_\Omega (\Delta u + f)\, v dx \\
&= E(u) + \frac{1}{2}\int_\Omega \|\nabla v\|^2 dx \\
&\geq E(u),
\end{aligned}$$

where we have used the fact that $v = 0$ on $\partial\Omega$ and that $\Delta u = -f$ in Ω. Since the set of such $u + v$ is exactly the space which is the domain of E, we conclude that u, indeed, minimizes E.

8. Let $u(x,t) \in C^2(\mathbb{R}^n \times \mathbb{R})$ be a solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = 0$$

in the domain

$$\mathcal{D} = \left\{(x,t) \;\middle|\; x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1},\ t \geq |x_n|\right\}.$$

In the picture, the variable $x' = (x_1, \ldots, x_{n-1})$ has been suppressed.

Assume for simplicity that $u = 0$ for $|x'| \geq R$ for some $R > 1$. Suppose that $u|_{\Gamma_1} = 0$ and $u|_{\Gamma_2} = 0$, where

$$\Gamma_1 = \left\{(x,t) \;\middle|\; x' \in \mathbb{R}^{n-1},\ t - x_n = 0,\ t > 0\right\}$$

and

$$\Gamma_2 = \left\{(x,t) \;\middle|\; x' \in \mathbb{R}^{n-1},\ t + x_n = 0,\ t > 0\right\}.$$

Prove that $u \equiv 0$.

Hint. Integrate by parts in

$$0 = \int \left(\frac{\partial^2 u}{\partial t^2} - \Delta u\right) \frac{\partial u}{\partial t} dx dt,$$

the integration being performed over the domain $\mathcal{D} \cap \{t \leq T\}$, where $T > 0$ is arbitrary. You may find it useful to make a change of variables $s = t - x_n$, $\tau = t + x_n$, $y' = x'$.

Solution

As suggested by the hint, we attempt to examine

$$0 = \int_{\mathcal{D}\cap\{t\leq T\}} (u_{tt} - \Delta u)\, u_t dx dt = \int_0^T \int_{\mathbb{R}^{n-1}} \int_{-t}^{t} (u_{tt}(x,t) - \Delta u(x,t))\, u_t(x,t) dx_n dx' dt = I.$$

We first split I into the $n+1$ pieces below:

$$\begin{aligned}
I_t &= \int_0^T \int_{\mathbb{R}^{n-1}} \int_{-t}^{t} u_{tt}(x,t) u_t(x,t) dx_n dx' dt; \\
I_j &= \int_0^T \int_{\mathbb{R}^{n-1}} \int_{-t}^{t} u_{x_k x_k}(x,t) u_t(x,t) dx_n dx' dt,\ j = 1, \ldots, n;
\end{aligned}$$

such that

$$0 = I = I_t - \sum_{j=1}^{n} I_j.$$

We will integrate by parts and permute orders of integration in what follows without much comment. We first examine I_t:

$$\begin{aligned}
I_t &= \int_0^T \int_{\mathbb{R}^{n-1}} \int_{-t}^{t} u_{tt}(x,t)u_t(x,t)dx_n dx' dt \\
&= \int_{\mathbb{R}^{n-1}} \int_{-T}^{T} \int_{|x_n|}^{T} u_{tt}(x,t)u_t(x,t)dt dx_n dx' \\
&= \int_{\mathbb{R}^{n-1}} \int_{-T}^{T} \int_{|x_n|}^{T} \frac{\partial}{\partial t}\left(\frac{1}{2}u_t(x,t)^2\right) dt dx_n dx' \\
&= \frac{1}{2}\int_{\mathbb{R}^{n-1}} \int_{-T}^{T} \left(u_t(x,T)^2 - u_t(x,|x_n|)^2\right) dx_n dx' \\
&= \frac{1}{2}\int_{\mathbb{R}^{n-1}} \left(\int_{-T}^{T} u_t(x',y,T)^2 dy - \int_0^T \left(u_t(x',y,y)^2 + u_t(x',-y,y)^2\right) dy\right) dx'.
\end{aligned}$$

Second, we examine I_n:

$$\begin{aligned}
I_n &= \int_0^T \int_{\mathbb{R}^{n-1}} \int_{-t}^{t} u_{x_n x_n}(x,t)u_t(x,t)dx_n dx' dt \\
&= \int_{\mathbb{R}^{n-1}} \int_0^T \left(u_{x_n}(x,t)u_t(x,t)|_{-t}^{t} - \int_{-t}^{t} u_{x_n}(x,t)u_{tx_n}(x,t)dx_n\right) dt dx' \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_0^T \left(u_{x_n}(x',t,t)u_t(x',t,t) - u_{x_n}(x',-t,t)u_t(x',-t,t)\right) dt \right. \\
&\quad \left. - \int_{-T}^{T} \int_{|x_n|}^{T} u_{x_n}(x,t)u_{x_n t}(x,t)dt dx_n\right) dx' \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_0^T \left(u_{x_n}(x',y,y)u_t(x',y,y) - u_{x_n}(x',-y,y)u_t(x',-y,y)\right) dy \right. \\
&\quad \left. - \int_{-T}^{T} \int_{|x_n|}^{T} \frac{\partial}{\partial t}\left(\frac{1}{2}u_{x_n}(x,t)^2\right) dt dx_n\right) dx' \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_0^T \left(u_{x_n}(x',y,y)u_t(x',y,y) - u_{x_n}(x',-y,y)u_t(x',-y,y)\right) dy \right. \\
&\quad \left. -\frac{1}{2}\int_{-T}^{T} \left(u_{x_n}(x,T)^2 - u_{x_n}(x,|x_n|)^2\right) dx_n\right) dx' \\
&= \int_{\mathbb{R}^{n-1}} \left(\int_0^T \left(u_{x_n}(x',y,y)u_t(x',y,y) - u_{x_n}(x',-y,y)u_t(x',-y,y)\right) dy \right. \\
&\quad \left. -\frac{1}{2}\int_{-T}^{T} u_{x_n}(x',y,T)^2 dy + \frac{1}{2}\int_0^T \left(u_{x_n}(x',y,y)^2 + u_{x_n}(x',-y,y)^2\right) dy\right) dx'.
\end{aligned}$$

The I_j's for $j = 1, \ldots, n-1$ are similar to I_n, with the exception that we don't pick any boundary terms when integrating by parts to move the $\partial/\partial x_j$:

$$I_j = -\frac{1}{2}\int_{\mathbb{R}^{n-1}}\left(\int_{-T}^{T} u_{x_j}(x',y,T)^2 dy - \int_0^T \left(u_{x_j}(x',y,y)^2 + u_{x_j}(x',-y,y)^2\right) dy\right) dx'.$$

We can simplify this further by noting that, since $u(x',y,y) = 0$ for all $x_j \in \mathbb{R}$ (and $y \in [0,T]$),

$$0 = \frac{\partial}{\partial x_j}\left(u(x',y,y)\right) = u_{x_j}(x',y,y),$$

and similarly for $u_{x_j}(x',-y,y)$. Thus,

$$I_j = -\frac{1}{2}\int_{\mathbb{R}^{n-1}}\int_{-T}^{T} u_{x_j}(x',y,T)^2 dydx'.$$

Next, we simplify the difference $I_t - I_n$:

$$\begin{aligned}
I_t - I_n &= \frac{1}{2}\int_{\mathbb{R}^{n-1}}\Bigg(\int_{-T}^{T}\left(u_t(x',y,T)^2 + u_{x_n}(x',y,T)^2\right) dy \\
&\quad - \int_0^T \big(u_t(x',y,y)^2 + u_{x_n}(x',y,y)^2 + 2u_t(x',y,y)u_{x_n}(x',y,y) \\
&\qquad + u_t(x',-y,y)^2 + u_{x_n}(x',-y,y)^2 - 2u_t(x',-y,y)u_{x_n}(x',-y,y)\big)\, dy\Bigg)\, dx' \\
&= \frac{1}{2}\int_{\mathbb{R}^{n-1}}\Bigg(\int_{-T}^{T}\left(u_t(x',y,T)^2 + u_{x_n}(x',y,T)^2\right) dy \\
&\quad - \int_0^T \left(\left(u_t(x',y,y) + u_x(x',y,y)\right)^2 + \left(u_t(x',-y,y) - u_x(x',-y,y)\right)^2\right) dy\Bigg)\, dx' \\
&= \frac{1}{2}\int_{\mathbb{R}^{n-1}}\Bigg(\int_{-T}^{T}\left(u_t(x',y,T)^2 + u_{x_n}(x',y,T)^2\right) dy \\
&\quad - \int_0^T \left(\left(\frac{\partial}{\partial y}u(x',y,y)\right)^2 + \left(\frac{\partial}{\partial y}u(x',-y,y)\right)^2\right) dy\Bigg)\, dx'.
\end{aligned}$$

Similar to the previous simplification of I_{x_j}, since $u(x',y,y) = 0$ for $y \in [0,T]$, we have that

$$\frac{\partial}{\partial y}u(x',y,y) = 0,$$

and the difference $I_t - I_n$ simplifies to

$$I_t - I_n = \frac{1}{2}\int_{\mathbb{R}^{n-1}}\int_{-T}^{T}\left(u_t(x',y,T)^2 + u_{x_n}(x',y,T)^2\right) dydx'.$$

We finally get, then, that

$$0 = I = I_t - \sum_{j=1}^{n} I_j = \frac{1}{2}\int_{\mathbb{R}^{n-1}}\int_{-T}^{T}\left(u_t(x',y,T)^2 + \sum_{j=1}^{n} u_{x_j}(x',y,T)^2\right) dydx'.$$

As the integrand is nonnegative, we therefore infer that (replacing $(x',y) = (x',x_n) = x$)

$$u_t(x,T) = u_{x_1}(x,T) = \cdots = u_{x_n}(x,T) = 0.$$

In particular, since $u(x',T,T) = u(x',-T,T) = 0$, we must conclude that $u(x,T) = 0$. But since T was arbitrary, we obtain $u \equiv 0$.