Basic Examination **May 2003**

1. (a) Suppose $f : (0,1) \to \mathbb{R}$ is a continuous function. Define what it means for f to be **uniformly continuous.**

   (b) Show that if $f : (0,1) \to \mathbb{R}$ is uniformly continuous, then there is a continuous function $F : [0,1] \to \mathbb{R}$ with $F(x) = f(x)$ for all $x \in (0,1)$.

2. Prove: If $a_1, a_2, a_3, \ldots$ is a sequence of real numbers with

$$\sum_{j=1}^{+\infty} |a_j| < +\infty,$$

then $\lim\limits_{N \to +\infty} \sum\limits_{j=1}^{N} a_j$ exists.

3. Find a subset S of the real numbers $\mathbb{R}$ such that both (i) and (ii) hold for S:

   (i) S is not the countable union of closed sets

   (ii) S is not the countable intersection of open sets.

4. Consider the following equation for a function $F(x,y)$ on $\mathbb{R}^2$

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2} \qquad (*)$$

   (a) Show that if a function F has the form $F(x,y) = f(x+y) + g(x-y)$ where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are twice differentiable, then F satisfies the equation (*).

   (b) Show that if $F(x,y) = ax^2 + bxy + cy^2$, $a, b, c \in \mathbb{R}$, satisfies (*) then $F(x,y) =$
$f(x+y)$ $f(x,y) + g(x-y)$ for some polynomials f and g in one variable.

5. Consider the function $F(x,y) = ax^2 + 2bxy + cy^2$ on the set $A = \{(x,y) : x^2 + y^2 = 1\}$.

   (a) Show that F has a maximum and minimum on A.

   (b) Use Lagrange multipliers to show that if the maximum of F on A occurs at a point (x_0, y_0), then the vector (x_0, y_0) is an eigenvector of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$

6. Formulate some reasonably general conditions on a function $f : \mathbb{R}^2 \to \mathbb{R}$ which guarantee that

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

and prove that your conditions do in fact guarantee that this equality holds.

7. Let V be a finite dimensional real vector space. If $W \subset V$ be a subspace let $W^o := \{f : V \to F \text{ linear}, | \, f = 0 \text{ on } W\}$. Let $W_i \subset V$ be subspaces for $i = 1, 2$. Prove that

$$W_1{}^o \cap W_2{}^o = (W_1 + W_2)^o.$$

8. Let V be an n-dimensional complex vector space and $T : V \to V$ a linear operator. Suppose that the characteristic polynomial of T has n distinct roots. Show that there is a basis B for V such that the matrix representation of T is the basis B is diagonal. (Make sure that you prove that your choice of B is in fact a basis.)

9. Let $A \in \mathbf{M}_3(\mathbf{R})$ satisfy $\det(A) = 1$ and $A^tA = I = AA^t$ where I is the 3×3 identity matrix. Prove that the characteristic polynomial of A has 1 as a root.

10. Let V be a finite dimensional real inner product space and $T : V \to V$ a hermitian linear operator. Suppose the matrix representation of T^2 in the standard basis has trace zero. Prove that T is the zero operator.