BASIC EXAM, FALL 2003

Instructions: Do all of the following:

1:

Prove that $\mathbb{R}$ is uncountable. If you like to use the Baire category theorem, you have to prove it.

2:

Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely often differentiable function. Assume that for each element $x \in [0,1]$ there is a positive integer m, such that the m-th derivative of f at x is not zero.

Prove that there exists an integer M such that the following stronger statement holds: For each element $x \in [0,1]$ there is a positive integer m with $m \leq M$ such that the m-th derivative of f at x is not zero.

3:

Prove that the sequence $a_1, a_2,$ with

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

converges as $n \to \infty$.

4:

Let f $\mathbb{R} \to \mathbb{R}$ be a continuous function. State the definition of the Riemann integral

$$\int_0^1 f(x)\,dx$$

and prove that it exists.

5:

Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that all partial derivatives of order 3 exist and are continuous. Write down (explicitly in terms of partial derivatives of f) a quadratic polynomial $P(x,y)$ in x and y such that

$$|f(x,y) - P(x,y)| \leq C(x^2 + y^2)^{3/2}$$

for all (x,y) in some small neighborhood of $(0,0)$, where C is a number that may depend on f but not on x and y. Then prove the above estimate.

6:

Let $U = \{(x,y) : x^2 + y^2 < 1\}$ be the standard unit ball in $\mathbb{R}^2$ and let ∂U denote its boundary.

Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable and that the Jacobian determinant of F is everywhere non-zero. Suppose also that $F(x,y) \in U$ for some $(x,y) \in U$ and $F(x,y) \notin U \cup \partial U$ for all $(x,y) \in \partial U$. Prove that $U \subset F(U)$.

7:

Prove that the space of continuous functions on the closed interval $[0,1]$ with the metric

$$\mathrm{dist(f,g)} := \sup_{x \in [0,1]} |f(x) - g(x)| = \|f - g\|_\infty$$

is complete. You do not need to show that this is a metric space.

8:

Prove the following three statements. You certainly may choose an order of these statements and then use the earlier statements to prove the later statements.

a) If $T : V \to W$ is a linear tranformation between two finite dimensional real vector spaces V, W, then

$$\dim(im(T)) = \dim(V) - \dim(ker(T))$$

b) If $T : V \to V$ is a linear transformation on a finite dimensional real inner product space and T^* denotes its adjoint, then $im(T^*)$ is the orthogonal complement of $ker(T)$ in V.

c) Let A be a n by n real matrix, then the maximal number of linearly independent rows (row rank) in the matrix equals the maximal number of linearly independent columns (column rank).

9:

Consider a 3 by 3 real symmetric matrix with determinant 6. Assume $(1,2,3)$ and $(0,3,-2)$ are eigenvectors with eigenvalues 1 and 2. Give answers to a) and b) below and justify the answers.

a)

Give an eigenvector of the form $(1,x,y)$ for some real x, y which is linearly independent of the two vectors above.

b)

What is the eigenvalue of this eigenvector.

10:

a) Let $t \in \mathbb{R}$ such that t is not an integer multiple of π. For the matrix

$$A = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

prove that there does not exist a real valued matrix B such that BAB^{-1} is a diagonal matrix.

b) Do the same for the matrix

$$A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$.