Qualifying Exam, Fall 2003

NUMERICAL ANALYSIS

DO NOT FORGET TO WRITE YOUR SID NUMBER ON YOUR EXAM.

Do all 7 problems.

Problems 1-3 are worth 5 points; problems 4-7 are worth 10 points.

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[1] **(5 Pts.)** For a single panel, Simpson's rule

$$\int_a^{a+2h} f(x)dx \approx \frac{h}{3}[f(a) + 4f(a+h) + f(a+2h)]$$

is fifth order accurate.

(a) What is the order of accuracy of the composite Simpson's rule?

(b) I have written a program that implements composite Simpson's rule for integrating functions over the interval $[0, 1]$. In checking the program for correctness I test the routine on the integral $\int_0^1 x^5 dx$ and I obtain the following results

M	approximation	error
8	0.24169	0.04169
16	0.22083	0.02083
32	0.21041	0.01041
64	0.20520	0.00520

What is the factor by which the errors should decrease as the number of panels, M, is doubled?

(c) On the basis of the above computational results can I conclude that my program is incorrect? Explain your answer

[2] **(5 Pts.)** Consider the following iterative method

$$\mathrm{A}\,\vec{x}^{k+1} \quad \mathrm{B}\,\vec{x}^{k} + \vec{c}$$

where $\vec{c}$ is the vector $(1,1)^t$ and A and B are the matrices

$$\mathrm{A} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \mathrm{B} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(a) Assume the iteration converges; to what vector $\vec{x}$ does the iteration converge?

(b) Does this iteration converge for arbitrary initial vectors, $\vec{x}^0$? Justify your answer.

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[3] **(5 Pts.)** **(a)** Give the cubic polynomial that interpolates the function $f(x) = 2^x$ at the points $x = 0$, $x = 1$, $x = 2$ and $x = 3$.

(b) Give the value of your interpolant at $x = \frac{1}{2}$, and hence derive an approximation to $\sqrt{2} = 2^{\frac{1}{2}}$

[4] **(10 Pts.)** **(a)** Construct a two-stage second order Runge-Kutta method for the ODE

$$y' \quad f(y),\ y(0) \quad y_0,$$

and find its region of absolute stability.

(b) Give an equivalent first-order system for the second-order differential equation:

$$y'' - 21y' + 20 \quad 0.$$

(c) Give the stability time-step restriction if 2nd order Runge-Kutta is used to compute solutions to the first order system.

[5] **(10 Pts.)** Consider the differential equation

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)^2 \varphi \quad c^2\frac{\partial^2 \varphi}{\partial x^2} \quad 0$$

to be solved for $0 \leq x \leq 1$, $t > 0$, with periodic boundary conditions in x and initial data

$$\begin{aligned} \varphi(x,0) &= \varphi_0(x) \\ \varphi_t(x,0) &= \varphi_1(x) \end{aligned}$$

Here $u, c > 0$ are positive constants. Give a convergent, second order accurate, finite difference approximation to this equation. Be sure to justify that your approximation is second order accurate and convergent.

[6] **(10 Pts.)** Consider the one-dimensional diffusion equations

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2}, \quad \alpha > 0$$

to be solved for $0 \leq x \leq 1,\ t > 0$, with periodic boundary conditions in x and initial data

$$v(x,0) \quad v_0(x)$$

Assume one uses the Dufort Frankel method:

$$\frac{v_m^{n+1} - v_m^{n-1}}{2\Delta t} - \alpha \left(\frac{v_{m+1}^n - (v_m^{n+1} + v_m^{n-1}) + v_{m-1}^n}{\Delta x^2} \right)$$

as a means of computing approximate solutions to this equation.

(a) Determine the truncation error associated with this approximation. Under what conditions does the scheme provide a consistent approximation to the diffusion equation? Would the condition required for consistency be difficult to satisfy in a set of computational experiments where Δx is repeatedly halved?

(b) Surprisingly, this scheme is explicit and unconditionally stable. Show this, and explain why this does not violate the CFL condition.

[7] **(10 Pts.)** Develop and describe the piecewise-linear Galerkin finite element approximation of

$$\begin{array}{rl} \Delta u + u & f(x,y), \quad (x,y) \in T, \\ u & g_1(x), \quad (x,y) \in T_1, \\ u & g_2(y), \quad (x,y) \in T_2, \\ \dfrac{\partial u}{\partial n} & h(x,y), \quad (x,y) \in T_3 \end{array}$$

where

$$\begin{array}{rcl} T & = & \{(x,y)|\ x > 0,\ y > 0,\ x + y < 1\} \\ T_1 & = & \{(x,y)|\ y = 0,\ 0 < x < 1\} \\ T_2 & & \{(x,y)|\ x = 0,\ 0 < y < 1\} \\ T_3 & & \{(x,y)|\ x > 0,\ y > 0,\ x + y = 1\} \end{array}$$

Justify your approximation by analyzing the appropriate bilinear and linear forms. Give a weak formulation of the problem. Give a convergence estimate and quote the appropriate theorems for convergence.