# Math 269B, 2012 Winter, Homework 1 (Solutions)

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## 1 Theory

1. (Strikwerda 1.1.3.) Solve the initial value problem for

$$u_t + \frac{1}{1 + \frac{1}{2}\cos x} u_x = 0$$

Show that the solution is given by $u(t, x) = u_0(\xi)$, where $\xi$ is the unique solution of

$$\xi + \frac{1}{2}\sin\xi = x + \frac{1}{2}\sin x - t.$$

**Solution**

We use the method of characteristics, in which we use the change of variables $\tau = \tau(t, x)$, $\xi = \xi(t, x)$, and $\tilde{u}(\tau, \xi) = u\left(t(\tau, \xi), x(\tau, \xi)\right)$. Then

$$\tilde{u}_\tau = u_t t_\tau + u_x x_\tau = u_t + a(t, x) u_x \equiv 0$$

if

$$t_\tau \equiv 1, \quad x_\tau = a(t, x),$$

where

$$a(t, x) := \frac{1}{1 + \frac{1}{2}\cos x}.$$

We may solve the equation $t_\tau \equiv 1$ as $t = \tau$, while the equation for $x_\tau$ is an ordinary differential equation in $x = x(\tau)$:

$$\frac{dx}{d\tau} = \frac{1}{1 + \frac{1}{2}\cos x}, \quad x(\tau = 0) = \xi.$$

Via separation of variables, we obtain an implicit relation among $x$, $\tau = t$, and $\xi$:

$$x + \frac{1}{2}\sin x - \xi - \frac{1}{2}\sin\xi = \tau = t.$$

For each fixed $\xi$, we have $\tilde{u}_\tau = 0$, hence $\tilde{u}(\tau, \xi) = \tilde{u}(0, \xi) = u(0, \xi) = u_0(\xi)$. It follows that $u(t, x) = u_0(\xi)$, where $\xi = \xi(t, x)$ satisfies

$$\xi + \frac{1}{2}\sin\xi = x + \frac{1}{2}\sin x - t.$$

Note that the mapping $\xi \mapsto \xi + \frac{1}{2}\sin\xi$ is an automorphism on $\mathbb{R}$ (its derivative is uniformly bounded away from 0), hence such a $\xi$ always exists and is unique.

2. Solve the initial value problem

$$u_t + (\sin t)\, u_x = \frac{1}{1+t^2}, \quad u(0,x) = u_0(x), \quad x \in \mathbb{R}, \quad t > 0.$$

We use the method of characteristics, in which we use the change of variables $\tau = \tau(t,x)$, $\xi = \xi(t,x)$, and $\tilde{u}(\tau,\xi) = u\left(t(\tau,\xi), x(\tau,\xi)\right)$. Then

$$\tilde{u}_\tau = u_t t_\tau + u_x x_\tau = u_t + a(t,x)u_x = f(t,x)$$

if

$$t_\tau \equiv 1, \quad x_\tau = a(t,x),$$

where

$$a(t,x) := \sin t, \quad f(t,x) := \frac{1}{1+t^2}.$$

We may solve the equation $t_\tau \equiv 1$ as $t = \tau$, while the equation for $x_\tau$ is an ordinary differential equation in $x = x(\tau)$:

$$\frac{dx}{d\tau} = \sin t = \sin\tau, \quad x(\tau = 0) = \xi.$$

This easily solves to

$$x(\tau,\xi) = \xi + \int_0^\tau \sin\tau' d\tau' = 1 - \cos\tau + \xi.$$

For each fixed $\xi$, we have

$$\tilde{u}_\tau = \frac{1}{1+t^2} = \frac{1}{1+\tau^2}, \quad \tilde{u}(0,\xi) = u(0,\xi) = u_0(\xi),$$

which easily solves to

$$\tilde{u}(\tau,\xi) = \tilde{u}(0,\xi) + \int_0^\tau \frac{1}{1+(\tau')^2} d\tau' = \arctan\tau + u_0(\xi).$$

It follows that

$$u(t,x) = \arctan t + u_0\left(\cos t + x + 1\right).$$

3. Consider the first order system of PDEs of the form

$$\vec{u}_t + A\vec{u}_x = 0, \quad \vec{u}(0,x) = \vec{u}_0(x), \quad x \in [0,1], \quad t > 0.$$

(a) Give the solution to the initial value problem when

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(b) Describe appropriate boundary conditions at $x = 0$ and/or $x = 1$, if possible, which make the initial boundary value problem in (a) well-posed. Try to be as general as possible. How should such boundary conditions be presented to put the solution in a simple form?

(c) Give the solution to the initial value problem when

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

(d) Describe appropriate boundary conditions at $x = 0$ and/or $x = 1$, if possible, which make the initial boundary value problem in (c) well-posed. Try to be as general as possible. How should such boundary conditions be presented to put the solution in a simple form?

**Solution**

(a) The eigenvalues and corresponding eigenvectors of $A$ are

$$\lambda_\pm = 2 \pm 1, \quad \vec{u}_\pm = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix},$$

leading to the general solution

$$\vec{u}(t,x) = \begin{pmatrix} \vec{u}_+ & \vec{u}_- \end{pmatrix} \begin{pmatrix} c_+ (x - \lambda_+ t) \\ c_- (x - \lambda_- t) \end{pmatrix},$$

where the scalar functions $c_\pm$ are determined by the initial conditions and boundary conditions.

(b) Since both $\lambda_\pm > 0$, boundary conditions must only be specified at the $x = 0$ boundary. A simple way to present these boundary conditions is to simply specify $c_\pm$ when $x = 0$, i.e., $c_\pm(-\lambda_\pm t) = g^0_\pm(t)$ for $t > 0$.

(c) $A$ here has the same eigenvectors as in (a) but now with eigenvalues $\lambda_\pm = 2 \pm 3$. Otherwise, the solution takes the same general form.

(d) This time, $\lambda_+ > 0$ but $\lambda_- < 0$, so boundary conditions must be specified at both the $x = 0$ boundary and the $x = 1$ boundary. A simple way to present these boundary conditions is to specify $c_+$ when $x = 0$ in terms of $t$ and $c_-$, and vice versa, to specify $c_-$ when $x = 1$ in terms of $t$ and $c_+$:

$$\begin{aligned} c_+ (-\lambda_+ t) &= g^0_+ (t, c_- (-\lambda_- t)), \\ c_- (1 - \lambda_- t) &= g^1_- (t, c_+ (1 - \lambda_+ t)), \end{aligned}$$

for $t > 0$.

4. Derive the leading term of the local truncation error for the following finite difference schemes used to approximate solutions to the equation $u_t + a u_x = 0$.

(a)

$$\frac{1}{k}\left(v^{n+1}_m - v^n_m\right) + a\frac{1}{2h}\left(v^n_{m+1} - v^n_{m-1}\right) = 0.$$

(b)

$$\frac{1}{k}\left(v^{n+1}_m - \frac{1}{2}\left(v^n_{m+1} + v^n_{m-1}\right)\right) + a\frac{1}{2h}\left(v^n_{m+1} - v^n_{m-1}\right) = 0.$$

**Solution**

Unless otherwise noted, all functions are evaluated at a common point $(t, x)$.

(a) We use the following Taylor expansions about $(t, x)$:

$$\begin{aligned} \phi(t+k, x) &= \phi + k\phi_t + \frac{1}{2}k^2\phi_{tt} + O\left(k^3\right); \\ \phi(t, x \pm h) &= \phi \pm h\phi_x + \frac{1}{2}h^2\phi_{xx} \pm \frac{1}{6}h^3\phi_{xxx} + \frac{1}{24}h^4\phi_{xxxx} + O\left(h^5\right). \end{aligned}$$

Thus,

$$\begin{aligned} P_{k,h}\phi &= \frac{1}{k}\left(\phi(t+k,x) - \phi\right) + a\frac{1}{2h}\left(\phi(t,x+h) - \phi(t,x-h)\right) \\ &= \phi_t + a\phi_x + \frac{1}{2}k\phi_{tt} + \frac{1}{6}h^2\phi_{xxx} + O\left(k^2 + h^4\right), \end{aligned}$$

which agrees with $P\phi = (\partial_t + a\partial_x)\phi$ up to a truncation error of

$$P_{k,h}\phi - P\phi = \frac{1}{2}k\phi_{tt} + \frac{1}{6}h^2\phi_{xxx} + O\left(k^2 + h^4\right).$$

(b) Using the same Taylor expansions as above,

$$\begin{aligned}P_{k,h}\phi &= \frac{1}{k}\left(\phi(t+k,x) - \frac{1}{2}\left(\phi(t,x+h) + \phi(t,x-h)\right)\right) + a\frac{1}{2h}\left(\phi(t,x+h) - \phi(t,x-h)\right)\\ &= \phi_t + a\phi_x + \frac{1}{2}k\phi_{tt} - \frac{1}{2}\frac{h^2}{k}\phi_{xx} + O\left(k^2 + h^2\right),\end{aligned}$$

which agrees with $P\phi$ up to a truncation error of

$$P_{k,h}\phi - P\phi = \frac{1}{2}k\phi_{tt} - \frac{1}{2}\frac{h^2}{k}\phi_{xx} + O\left(k^2 + h^2\right).$$

5. Determine the stability region $\Lambda$ for each of the finite difference schemes in Problem 4.

**Solution**

(a) The amplification factor $g$ satisfies the equation

$$\frac{1}{k}(g-1) + a\frac{1}{2h}\left(e^{i\theta} - e^{-i\theta}\right) = 0,$$

so

$$g = 1 - i\frac{ak}{h}\sin\theta,$$

so $|g|^2 = 1 + \frac{a^2k^2}{h^2}\sin^2\theta \le 1 + O(k)$ if and only if $k = O\left(h^2\right)$.

(b) The amplification factor $g$ satisfies the equation

$$\frac{1}{k}\left(g - \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right) + a\frac{1}{2h}\left(e^{i\theta} - e^{-i\theta}\right) = 0,$$

so

$$g = \cos\theta - i\frac{ak}{h}\sin\theta,$$

so $|g| \le 1$ if and only if $|ak/h| \le 1$.

# 2 Programming

1. Implement the finite difference schemes in Problem 4. in the Theory section for $x \in [0,1]$, $t \in [0,T]$ for some final time $T$, $u(x,0) = u_0(x)$, and *periodic* boundary conditions.

2. Investigate the convergence of each scheme for $a = 1$ and $T = 1$. Set $k/h =: \lambda$ to be constant, and demonstrate which values of $\lambda$ cause the scheme to converge and which to diverge. If no such $\lambda$ gives convergence, find an alternate relation between $k$ and $h$ which does ensure convergence (if possible). Try using both a smooth initial condition (e.g., $u_0(x) = \sin(2\pi x)$); a non-smooth initial condition (e.g., $u_0(x) = 1 - 2|x - 1/2|$); and a discontinuous initial condition (e.g., $u_0(x) = 0$ if $|x - 1/2| > 1/4$ and $u_0(x) = 1$ if $|x - 1/2| < 1/4$). Use the discrete $L^2$ norm to measure the error between your numerical solution and the true solution:

$$\|w\|_h = \left(h\sum_m |w_m|^2\right)^{1/2}.$$

[Note: Due to periodicity, be sure to avoid double-counting the contributions at $x = 0$ and $x = 1$!] Plot the numerical solutions from each scheme at $t = T$ when $h = 1/100$. Summarize your results. Which scheme do you think is better and why?

**Solution**

Using the included code, the results of the following statements investigate the convergence of the Lax-Friedrichs scheme with the various initial conditions and $\lambda = 0.99$.

```
test_convergence( ...
    1, 1, @(x) sin(2*pi*x), ...
    @lax_friedrichs, "Lax-Friedrichs", ...
    2.^(-(9:0.5:12)), @(h) 0.99*h);
test_convergence( ...
    1, 1, @(x) 1 - 2*abs(x - 1/2), ...
    @lax_friedrichs, "Lax-Friedrichs", ...
    2.^(-(9:0.5:12)), @(h) 0.99*h);
test_convergence( ...
    1, 1, @(x) (1 + sign(1/4 - abs(x - 1/2)))/2, ...
    @lax_friedrichs, "Lax-Friedrichs", ...
    2.^(-(9:0.5:12)), @(h) 0.99*h);
```

The numerical convergence rates, based on fitting a linear regression through a log-log plot of $h$ vs. $L^2$-error, are found to be, respectively, 1.06, 0.80, and 0.19. For the smooth initial condition $u_0(x) = \sin(2\pi x)$, the numerical convergence rate, 1.06, agrees well with the theoretical convergence rate of 1. The fact that the numerical convergence rates for the non-smooth initial conditions are significantly less than 1 is not surprising given that our theoretical convergence analysis depended on the existence of some number of derivatives.

Additionally, as predicted by theory, numerical evidence indicates that the Lax-Friedrichs scheme diverges if $\lambda > 1$ (and, still, $a = 1$), while the forward-time central-space scheme diverges for any $\lambda$, which is consistent with our theoretical analysis.

We do get the forward-time central-space scheme to converge if, e.g., we let $k = h^2$, as demonstrated by the results of the following statement.

```
test_convergence( ...
    1, 1, @(x) sin(2*pi*x), ...
    @ftcs, "forward-time central-space", ...
    2.^(-(4:0.5:7)), @(h) h*h);
```

The numerical convergence rate is found to be 2.01, which agrees with the theoretical rate of convergence of 2.