Jeffrey Hellrung
Numerical Analysis Qualifying Exam, Fall 2002

1. Consider using the composite trapezoidal method to numerically evalute the following integral:

$$(I)\ \int_0^1 \frac{\sin t}{\sqrt{t}} dt.$$

Two different methods are employed:

(a) The composite trapezoidal method is directly applied to the integral (I) and the avlue of the integrand at $t = 0$ is taken to be 0.

(b) The composite trapezoidal method is applied to

$$(I')\ \int_0^1 2 \sin s^2 ds.$$

(This latter integral is obtained from (I) by using the change of variables $s = \sqrt{t}$.)

The errors in the numerical approximation for these computations are given in the following table.

Δx	error with computation (a)	error with computation (b)
	$-5.840e-04$	$7.204e-05$
	$-2.068e-04$	$1.800e-05$
	$-7.325e-05$	$4.500e-06$

(a) What is the expected rate of convergence for the composite trapezoidal method?

(b) Give an estimate, based on the results in the above table, of the rate of convergence for each of the computational procedures.

(c) If your estimate of convergence does not agree with the expected rate of convergence for either of these procedures, explain this discrepancy.

Solution

(a) The expected rate of convergence for the composite trapezoidal method is second-order (i.e., the error is $O(h^2)$).

(b) The errors in computation (a) seem to be $O(h^{1+\epsilon})$ for some relatively small $\epsilon > 0$; the errors for computation (b) seem to be $O(h^2)$.

(c) For computation (a), the integrand fails to be differentiable at $t = 0$, which would explain the less-than-expected rate of convergence.

2. Consider the two-point boundary-value problem over the interval $[0, 1]$:

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}(x)\right) = f(x),\ u(0) = u(1) = 0;$$

with $p(x) > 0$.

(a) Assuming you are using an equispaced set of grid points in $[0, 1]$, give a finite difference discretization of this equation that results in a *symmetric* linear system of equations.

(b) Derive the leading term of the truncation error for the discretization in (a).

Solution

(a) We use

$$p_{m-1/2}u_{m-1} - \left(p_{m-1/2} + p_{m+1/2}\right) u_m + p_{m+1/2}u_{m+1} = h^2 f_m,\ m = 1, \ldots, N-1;$$

where $p_{m\pm 1/2} = p((m \pm 1/2)h)$, $f_m = f(mh)$, u_m is the approximation to $u(mh)$, and $h = 1/N$. This may be rewritten in matrix form as

$$\begin{pmatrix} -(p_{\frac{1}{2}} + p_{\frac{3}{2}}) & p_{\frac{3}{2}} & & & \\ p_{\frac{3}{2}} & -(p_{\frac{3}{2}} + p_{\frac{5}{2}}) & p_{\frac{5}{2}} & & \\ & p_{\frac{5}{2}} & -(p_{\frac{5}{2}} + p_{\frac{7}{2}}) & p_{\frac{7}{2}} & \\ & & \ddots & \ddots & \ddots \\ & & & p_{N-\frac{3}{2}} & -(p_{N-\frac{3}{2}} + p_{N-\frac{1}{2}}) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-1} \end{pmatrix} = h^2 \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-1} \end{pmatrix}.$$

(b) We rewrite the system, where $x = mh$, as

$$\frac{1}{h}\left(p\left(x + \frac{h}{2}\right)\frac{u(x+h) - u(x)}{h} - p\left(x - \frac{h}{2}\right)\frac{u(x) - u(x-h)}{h}\right) = f(x).$$

By Taylor's Theorem,

$$\begin{aligned} \frac{u(x+h) - u(x)}{h} &= u'\left(x + \frac{h}{2}\right) + \frac{1}{24}u^{(3)}\left(x + \frac{h}{2}\right)h^2 + O(h^4), \\ \frac{u(x) - u(x-h)}{h} &= u'\left(x - \frac{h}{2}\right) + \frac{1}{24}u^{(3)}\left(x - \frac{h}{2}\right)h^2 + O(h^4), \end{aligned}$$

$$\begin{aligned} \frac{1}{h}\left(p\left(x + \frac{h}{2}\right)u'\left(x + \frac{h}{2}\right) - p\left(x - \frac{h}{2}\right)u'\left(x - \frac{h}{2}\right)\right) &= (pu')'(x) + O(h^2), \\ \frac{1}{h}\left(p\left(x + \frac{h}{2}\right)u^{(3)}\left(x + \frac{h}{2}\right) - p\left(x - \frac{h}{2}\right)u^{(3)}\left(x - \frac{h}{2}\right)\right) &= \left(pu^{(3)}\right)'(x) + O(h^2), \end{aligned}$$

so we finally get

$$\frac{1}{h}\left(p\left(x + \frac{h}{2}\right)\frac{u(x+h) - u(x)}{h} - p\left(x - \frac{h}{2}\right)\frac{u(x) - u(x-h)}{h}\right) = (pu')'(x) + O(h^2),$$

giving a truncation error $\in O(h^2)$.