Math 269B, 2012 Winter, Homework 2 - Solutions

Professor Joseph Teran Jeffrey Lee Hellrung, Jr.

February 13, 2012

1 Theory

1. (Strikwerda 2.1.9.) Finite Fourier Transforms. For a function v_m defined on the integers, $m = 0, 1, \dots, M-1$, we can define the Fourier transform as

$$\hat{v}_\ell = \sum_{m=0}^{M-1} e^{-2i\pi\ell m/M} v_m \quad \text{for} \ell = 0, \dots, M-1.$$

For this transform prove the Fourier inversion formula

$$v_m = \frac{1}{M} \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \hat{v}_\ell,$$

and the Parseval's relation

$$\sum_{m=0}^{M-1} |v_m|^2 = \frac{1}{M} \sum_{\ell=0}^{M-1} |\hat{v}_\ell|^2.$$

Note that v_m and $\hat{v}_\ell$ can be defined for all integers by making them periodic with period M.

Solution

Substituting in for $\hat{v}_\ell$ and swapping the order of summation yields

$$\begin{aligned}
&\frac{1}{M} \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \hat{v}_\ell \\
=&\frac{1}{M} \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \sum_{m'=0}^{M-1} e^{-2i\pi\ell m'/M} v_{m'} \\
=&\frac{1}{M} \sum_{m'=0}^{M-1} v_{m'} \sum_{\ell=0}^{M-1} e^{2i\pi\ell\left(m-m'\right)/M}.
\end{aligned}$$

Now, if $m \neq m'$, then

$$\sum_{\ell=0}^{M-1} e^{2i\pi\ell\left(m-m'\right)/M} = \frac{1 - e^{2i\pi M\left(m-m'\right)/M}}{1 - e^{2i\pi(m-m')/M}} = 0;$$

on the other hand, if $m = m'$, then the summand is identically 1 and hence the sum is M. The result follows immediately:

$$\frac{1}{M} \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \hat{v}_\ell = v_m.$$

Regarding the Parseval's relation,

$$\begin{aligned}
\sum_{m=0}^{M-1} |v_m|^2 &= \frac{1}{M^2} \sum_{m=0}^{M-1} \left| \sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \hat{v}_\ell \right|^2 \\
&= \frac{1}{M^2} \sum_{m=0}^{M-1} \left(\sum_{\ell=0}^{M-1} e^{2i\pi\ell m/M} \hat{v}_\ell \right) \left(\overline{\sum_{\ell'=0}^{M-1} e^{2i\pi\ell' m/M} \hat{v}_{\ell'}} \right) \\
&= \frac{1}{M^2} \sum_{\ell=0}^{M-1} \sum_{\ell'=0}^{M-1} \sum_{m=0}^{M-1} e^{2i\pi\left(\ell-\ell'\right)m/M} \hat{v}_\ell \overline{\hat{v}_{\ell'}} \\
&= \frac{1}{M} \sum_{\ell=0}^{M-1} |\hat{v}_\ell|^2 ,
\end{aligned}$$

as desired (since, when $\ell \neq \ell'$, the inner sum over m sums out to 0).

2. Prove convergence for the Beam-Warming scheme

$$u_m^{n+1} = u_m^n - \frac{ak}{2h}\left(3u_m^n - 4u_{m-1}^n + u_{m-2}^n\right) + \frac{a^2k^2}{2h^2}\left(u_m^n - 2u_{m-1}^n + u_{m-2}^n\right)$$

used to approximate solutions to $u_t + au_x = 0$ for $a > 0$.

Solution

We rewrite the difference operator as

$$P_{k,h}u_m^n = \frac{1}{k}\left(u_m^{n+1} - u_m^n\right) + \frac{a}{2h}\left(3u_m^n - 4u_{m-1}^n + u_{m-2}^n\right) - \frac{a^2k}{2h^2}\left(u_m^n - 2u_{m-1}^n + u_{m-2}^n\right).$$

Its symbol is thus

$$\begin{aligned}
p_{k,h}(s,\xi) &= P_{k,h}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi} \\
&= \frac{1}{k}\left(e^{sk}-1\right) + \frac{a}{2h}\left(3 - 4e^{-ih\xi} + e^{-2ih\xi}\right) - \frac{a^2k}{2h^2}\left(1 - 2e^{-ih\xi} + e^{-2ih\xi}\right) \\
&= \frac{1}{k}\left(sk + \frac{1}{2}s^2k^2 + O\left(k^2\right)\right) + \frac{a}{2h}\left(2ih\xi + O\left(h^3\right)\right) - \frac{a^2k}{2h^2}\left(-h^2\xi^2 + O\left(h^3\right)\right) \\
&= \left(1 + \frac{k}{2}\left(s - ia\xi\right)\right)\left(s + ia\xi\right) + O\left(k^2 + kh + h^2\right);
\end{aligned}$$

since the symbol of the differential operator $P = \partial_t + a\partial_x$ is $p = s + ia\xi$, this demonstrates that $P_{k,h}$ is consistent with P (to second order if $k/h = O(1)$). Incidentally, it also suggests that the symbol $r_{k,h}$ of the difference operator $R_{k,h}$ should be (for example)

$$\begin{aligned}
r_{k,h}(s,\xi) &= 1 + \frac{k}{2}\left(s - ia\xi\right) + O\left(k^2 + kh + h^2\right) \\
&= \frac{1}{2}\left(e^{sk} + 1\right) - \frac{ak}{2h}\left(1 - e^{-ih\xi}\right),
\end{aligned}$$

giving

$$R_{k,h}f_m^n = \frac{1}{2}\left(f_m^{n+1} + f_m^n\right) - \frac{ak}{2h}\left(f_m^n - f_{m-1}^n\right).$$

Regarding stability, we substitute $g := e^{sk}$ and solve $p_{k,h} = 0$ for g, giving

$$\begin{aligned}
g &= 1 - \frac{ak}{2h}\left(3 - 4e^{-i\theta} + e^{-2i\theta}\right) + \frac{a^2k^2}{2h^2}\left(1 - 2e^{-i\theta} + e^{-2i\theta}\right) \\
&= 1 - e^{-i\theta}\frac{ak}{2h}\left(\left(3 - \frac{ak}{h}\right)e^{i\theta} - 2\left(2 - \frac{ak}{h}\right) + \left(1 - \frac{ak}{h}\right)e^{-i\theta}\right) \\
&= 1 + e^{-i\theta}\frac{ak}{h}\left(\left(2 - \frac{ak}{h}\right)(1 - \cos\theta) - i\sin\theta\right) \\
&= e^{-i\theta}\left(\cos\theta + \frac{ak}{h}\left(2 - \frac{ak}{h}\right)(1 - \cos\theta) + i\left(1 - \frac{ak}{h}\right)\sin\theta\right) \\
&= e^{-i\theta}\left(1 - \left(1 - \frac{ak}{h}\right)^2(1 - \cos\theta) + i\left(1 - \frac{ak}{h}\right)\sin\theta\right).
\end{aligned}$$

If we let $\alpha = 1 - ak/h$, then

$$\begin{aligned}
|g|^2 &= \left|1 - \alpha^2(1 - \cos\theta) + i\alpha\sin\theta\right|^2 \\
&= 1 + \alpha^4(1 - \cos\theta)^2 + \alpha^2\sin^2\theta - 2\alpha^2(1 - \cos\theta) \\
&= 1 - \alpha^2(1 - \cos\theta)^2\left(1 - \alpha^2\right),
\end{aligned}$$

hence $|g| \le 1$ if and only if $|\alpha| \le 1$, which is equivalent to $0 \le ak/h \le 2$.

3. (Strikwerda 2.2.4.) Show that the box scheme

$$\frac{1}{2k}\left(\left(v_m^{n+1} + v_{m+1}^{n+1}\right) - \left(v_m^n + v_{m+1}^n\right)\right) + \frac{a}{2h}\left(\left(v_{m+1}^{n+1} - v_m^{n+1}\right) + \left(v_{m+1}^n - v_m^n\right)\right) = f_m^n$$

is consistent with the one-way wave equation $u_t + au_x = f$ and is stable for all values of λ.

Solution

The symbols corresponding to the difference operators $P_{k,h}$ and $R_{k,h}$ are

$$\begin{aligned}
p_{k,h}(s,\xi) &= P_{k,h}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi} \\
&= \frac{1}{2k}\left(e^{sk} - 1\right)\left(e^{ih\xi} + 1\right) + \frac{a}{2h}\left(e^{sk} + 1\right)\left(e^{ih\xi} - 1\right) \\
&= \frac{1}{2k}\left(sk + \frac{1}{2}s^2k^2 + O\left(k^3\right)\right)\left(2 + ih\xi + O\left(h^2\right)\right) \\
&\quad + \frac{a}{2h}\left(2 + sk + O\left(k^2\right)\right)\left(ih\xi - \frac{1}{2}h^2\xi^2 + O\left(h^3\right)\right) \\
&= \left(1 + \frac{1}{2}(sk + ih\xi)\right)(s + ia\xi) + O\left(h^2 + hk + k^2\right); \\
r_{k,h}(s,\xi) &\equiv 1.
\end{aligned}$$

Since the symbol of the differential operator $P = \partial_t + a\partial_x$ is $p = s + ia\xi$, this shows that the box scheme as given by the difference operators $P_{k,h}$ and $R_{k,h}$ is consistent with P to first order in k and h. It also shows that we can actually achieve second order accuracy by modifying $r_{k,h}$ to

$$\begin{aligned}
r'_{k,h}(s,\xi) &= 1 + \frac{1}{2}(sk + ih\xi) + O\left(k^2 + kh + h^2\right) \\
&= \frac{1}{4}\left(1 + e^{sk}\right)\left(1 + e^{ih\xi}\right),
\end{aligned}$$

giving

$$R'_{k,h} f_m^n = \frac{1}{4}\left(f_m^n + f_m^{n+1} + f_{m+1}^n + f_{m+1}^{n+1}\right).$$

Regarding stability, we substitute $g := e^{sk}$ and solve $p_{k,h} = 0$ for g, giving

$$\begin{aligned} g &= \frac{e^{i\theta} + 1 - a\lambda\left(e^{i\theta} - 1\right)}{e^{i\theta} + 1 + a\lambda\left(e^{i\theta} - 1\right)} \\ &= \frac{e^{i\theta/2} + e^{-i\theta/2} - a\lambda\left(e^{i\theta/2} - e^{-i\theta/2}\right)}{e^{i\theta/2} + e^{-i\theta/2} + a\lambda\left(e^{i\theta/2} - e^{-i\theta/2}\right)}, \end{aligned}$$

which clearly shows that $|g| \equiv 1$ (the numerator and denominator are complex conjugates).

4. (Strikwerda 2.2.6.) Determine the stability of the following scheme, sometimes called the Euler backward scheme, for $u_t + au_x = f$:

$$\begin{aligned} v_m^{n+1/2} &= v_m^n - \frac{a\lambda}{2}\left(v_{m+1}^n - v_{m-1}^n\right) + kf_m^n, \\ v_m^{n+1} &= v_m^n - \frac{a\lambda}{2}\left(v_{m+1}^{n+1/2} - v_{m-1}^{n+1/2}\right) + kf_m^{n+1}. \end{aligned}$$

The variable $v^{n+1/2}$ is a temporary variable, as is $\tilde{v}$ in Example 2.2.5.

Solution

The symbol $p_{k,h}^{1/2}$ of the difference operator $P_{k,h}^{1/2}$ of the first half-step is

$$\begin{aligned} p_{k,h}^{1/2}(s,\xi) &= P_{k,h}^{1/2}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi} \\ &= \frac{1}{k}\left(e^{sk} - 1\right) + \frac{a}{2h}\left(e^{ih\xi} - e^{-ih\xi}\right); \end{aligned}$$

the symbol $p_{k,h}$ of the full difference operator $P_{k,h}$ is then

$$\begin{aligned} p_{k,h}(s,\xi) &= P_{k,h}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi} \\ &= \frac{1}{k}\left(e^{sk} - 1\right) + \frac{a}{2h}\left(e^{ih\xi} - e^{-ih\xi}\right) v^{n+1/2}\left(e^{imh\xi}\right) \\ &= \frac{1}{k}\left(e^{sk} - 1\right) + \frac{a}{2h}\left(e^{ih\xi} - e^{-ih\xi}\right)\left(1 - \frac{a\lambda}{2}\left(e^{ih\xi} - e^{-ih\xi}\right)\right). \end{aligned}$$

Substituting $g := e^{sk}$ and solving $p_{k,h} = 0$ for g yields

$$\begin{aligned} g &= 1 - \frac{a\lambda}{2}\left(e^{i\theta} - e^{-i\theta}\right)\left(1 - \frac{a\lambda}{2}\left(e^{i\theta} - e^{-i\theta}\right)\right) \\ &= 1 - a\lambda i \sin\theta - a^2\lambda^2 \sin^2\theta, \end{aligned}$$

hence

$$\begin{aligned} |g|^2 &= \left(1 - a^2\lambda^2\sin^2\theta\right)^2 + a^2\lambda^2\sin^2\theta \\ &= 1 - a^2\lambda^2\sin^2\theta\left(1 - a^2\lambda^2\sin^2\theta\right) \end{aligned}$$

from which we conclude that $|g| \le 1$ (independent of θ) if and only if $|a\lambda| \le 1$.

2 Programming

1. (Strikwerda 2.3.3.) Solve the initial value problem for equation

$$u_t + \left(1 + \frac{1}{4}(3-x)(1+x)\right) u_x = 0$$

on the interval $[-1, 3]$ with the Lax-Friedrichs scheme (2.3.1) with λ equal to 0.8. Demonstrate that the instability phenomena occur where $|a(t,x)\lambda|$ is greater than 1 and where there are discontinuities in the solution. Use the same initial data as in Exercise 2.3.1. Specify the solution to be 0 at both boundaries. Compute up to the time of 0.2 and use successively smaller values of h to show the location of the instability.

2. Investigate (via numerical evidence) the convergence (or lack thereof) of the forward-time central-space scheme

$$\frac{1}{k}\left(u_m^{n+1} - u_m^n\right) + \frac{a}{2h}\left(u_{m+1}^n - u_{m-1}^n\right) = 0$$

in the L^∞-norm. Use the same scenarios from Homework 1, e.g., compare your results using smooth, continuous-but-non-smooth, and discontinuous initial conditions. Be sure to restrict the relation between k and h appropriately. Compare convergence in the L^∞-norm with convergence in the L^2-norm.