Jeffrey Hellrung
Applied Differential Equations Qualifying Exam, Fall 2005

1. Consider the initial value problem

$$u_t = v;\ v_t = |u|^\alpha;$$
$$u(t=0) = u_0;\ v(t=0) = 0.$$

For what constant values of $u_0 \geq 0$ and $\alpha \geq 0$ is this problem well-posed, (a) locally in time, or (b) globally in time? Prove your answer.

Solution

Let $F(u,v) = (v, |u|^\alpha)$.

(a) We're guaranteed well-posedness at least locally in time whenever F is locally Lipschitz around $(u_0, 0)$, which is the case whenever $u_0 > 0$ or $\alpha \geq 1$.

(b) Assume we're in the case of well-posedness at least locally in time, i.e., $u_0 > 0$ or $\alpha \geq 1$. Clearly if $u_0 = 0$ (and $\alpha \geq 1$), the solution is $u(t) = v(t) = 0$, hence this case is well-posed globally in time. We therefore restrict our attention to $u_0 > 0$ (and hence begin with no assumptions on α aside from $\alpha \geq 0$).

Note that $u_{tt} = |u|^\alpha \geq 0$, and $u_t(0) = v(0) = 0$, hence u_t remains nonnegative as t increases, from which we can conclude that $u \geq u_0 > 0$ for all t. We can thus drop absolute value signs in what follows. Further, since $v_t = u^\alpha \geq u_0^\alpha$, we get that $v \geq u_0^\alpha t$. We can use this to conclude that $u \geq u_0 + \frac{1}{2}u_0^\alpha t^2$, and so on. This implies that u, v dominate t^n as $t \to \infty$ for all finite n.

Since u remains away from 0 for all t, well-posedness globally in time can only fail if we have finite-time blow-up, since the problem is well-posed everywhere locally in time. We thus aim to estimate u. Starting with $u_{tt} = u^\alpha$, we can multiply by u_t, integrate, and apply initial conditions to obtain the relation

$$v^2 = \frac{2}{\alpha+1}\left(u^{\alpha+1} - u_0^{\alpha+1}\right).$$

It follows that

$$u_t = v = \left(\frac{2}{\alpha+1}\left(u^{\alpha+1} - u_0^{\alpha+1}\right)\right)^{1/2}.$$

By the previous remarks, $u \to \infty$ as $t \to \infty$, hence there exists a $T > 0$ such that $u^{\alpha+1} > 2u_0^{\alpha+1}$ for $t \geq T$. For $t \geq T$, then,

$$u_t \geq \left(\frac{1}{\alpha+1}u^{\alpha+1}\right)^{1/2}.$$

Suppose $\alpha > 1$. Then separating variables gives

$$u(t) \geq \left(u(T)^{-(\alpha-1)/2} - \frac{2}{(\alpha-1)(\alpha+1)^{1/2}}(t-T)\right)^{-2/(\alpha-1)}$$

for $t \geq T$. Clearly the large paranthesized expression,

$$u(T)^{-(\alpha-1)/2} - \frac{2}{(\alpha-1)(\alpha+1)^{1/2}}(t-T),$$

vanishes for some finite $t > T$, indicating a finite-time blow-up. Hence the case $\alpha > 1$ is not well-posed globally in time.

On the other hand, if $0 \leq \alpha < 1$, then

$$u_t = \left(\frac{2}{\alpha+1}\left(u^{\alpha+1} - u_0^{\alpha+1}\right)\right)^{1/2} \leq \left(\frac{2}{\alpha+1}u^{\alpha+1}\right)^{1/2}$$

gives, when separating variables,

$$u(t) \le \left(u_0^{(1-\alpha)/2} + \frac{2}{(1-\alpha)(1+\alpha)^{1/2}} t \right)^{2/(1-\alpha)},$$

which is finite for all t. Hence the case $0 \le \alpha < 1$ is well-posed globally in time.

For the case $\alpha = 1$, one can solve the system directly, giving $u = u_0 \cosh t$, which is also finite for all t. Hence the case $\alpha = 1$ is well-posed globally in time.

2. Consider the two-point boundary value operator L defined for $u = u(x)$ by

$$Lu = u'' + u' - a(1+x^2)u$$

defined on the interval $x \in [0,1]$ with boundary conditions

$$u(0) = u(1) = 0$$

with $a > 0$. Let λ_{a0} be the eigenvalue of smallest absolute value for L and let u_{a0} be the corresponding eigenfunction. Do the following:

(a) Find an inner product in terms of which L is self-adjoint.

(b) Show that $\lambda_{a0} < 0$.

(c) Show that $|\lambda_{a0}|$ is an increasing function of a, i.e., if $0 < a_1 < a_2$, then $|\lambda_{a10}| < |\lambda_{a20}|$.

Solution

(a) We try using a weighted L^2-inner product with weighted function ϕ, and compute, using integration by parts,

$$\begin{aligned}
(Lu, v)_\phi &= \int_0^1 (Lu)v\phi dx \\
&= \int_0^1 \left(u'' + u' - a(1+x^2)u\right) v\phi dx \\
&= \int_0^1 \left(u(v\phi)'' - u(v\phi)' - a(1+x^2)uv\phi\right) dx \\
&= \int_0^1 u\left((v\phi)'' - (v\phi)' - a(1+x^2)v\phi\right) dx \\
&= \int_0^1 u\left(v''\phi + 2v'\phi' + v\phi'' - v'\phi - v\phi' - a(1+x^2)v\phi\right) dx \\
&= \int_0^1 u\left(v'' + \left(2\frac{\phi'}{\phi} - 1\right) v' + \frac{\phi'' - \phi'}{\phi} v - a(1+x^2)v\right) \phi dx.
\end{aligned}$$

We'd like this to equal $(u, Lv)_\phi$, so for the paranthesized expression to be Lv, we'd require

$$\begin{aligned}
2\frac{\phi'}{\phi} - 1 = 1 \quad &\Rightarrow \quad \phi' = \phi; \\
\frac{\phi'' - \phi'}{\phi} = 0 \quad &\Rightarrow \quad \phi'' = \phi'.
\end{aligned}$$

We see that these are compatible conditions, both giving $\phi(x) = e^x$.

(b) Let (λ, u) be an eigenvalue/eigenfunction pair, and suppose u is normalized such that $\max u = 1$. Let $x_0 \in [0,1]$ be the point at which u attains its maximum. Then $u'(x_0) = 0$ and $u''(x_0) \leq 0$. Thus

$$0 = Lu - \lambda u = u'' + u' - \big(a(1+x^2) + \lambda\big)\, u,$$

and upon evaluation at x_0, we find that

$$a(1+x_0)^2 + \lambda \leq 0 \;\Rightarrow\; \lambda < 0.$$

Alternatively, we can compute

$$\begin{aligned} \lambda \|u\|_\phi^2 &= (\lambda u, u)_\phi \\ &= (Lu, u)_\phi \\ &= \int_0^1 \big(u''u + u'u - a(1+x^2)u^2\big)\, e^x dx \\ &= -\int_0^1 \big((u')^2 + a(1+x^2)u^2\big)\, e^x dx \\ &< 0, \end{aligned}$$

since, by integration by parts,

$$\int_0^1 u'' u e^x dx = -\int_0^1 \big((u')^2 + u'u\big)\, e^x dx.$$

We again conclude that $\lambda < 0$.

(c) λ_{a0} is given by the Rayleigh Quotient

$$\lambda_{a0} = \sup \frac{(Lu, u)_\phi}{(u, u)_\phi}.$$

Since $\lambda_{a0} < 0$, the claim is shown once we demonstrate that $(Lu, u)_\phi$ decreases as a increases for fixed u.. But this is clear from the expression in (b):

$$(Lu, u)_\phi = -\int_0^1 \big((u')^2 + a(1+x^2)u^2\big)\, e^x dx.$$

3. For the ODE $f'' - f(f^2 - 1) = 0$ do the following:

(a) Find the stationary points and classify their type.

(b) Find all periodic orbits and all orbits that connect stationary points.

(c) Draw a picture of the phase plane.

Solution

(a) Rewrite the system as

$$(f, f')' = (f', f(f^2 - 1)) = F(f, f').$$

Stationary points $(f, f')^*$ satisfy $F((f, f')^*) = 0$, giving $(f, f')^* \in \{(0,0), (\pm 1, 0)\}$. We compute

$$DF(f, f') = \begin{pmatrix} 0 & 1 \\ 3f^2 - 1 & 0 \end{pmatrix}.$$

- $(f, f')^* = (0, 0)$. The eigenvalues of

$$DF(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

are $\lambda_\pm = \pm i$. Thus, $(0,0)$ is a center.
- $(f, f')^* = (1, 0)$. The eigenvalues of

$$DF(1,0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

are $\lambda_\pm = \pm\sqrt{2}$, with corresponding eigenvectors

$$v_\pm = \begin{pmatrix} 1 \\ \pm\sqrt{2} \end{pmatrix}.$$

Thus, $(1,0)$ is a saddle.
- $(f, f')^* = (-1, 0)$. Same as for $(1, 0)$.

(b) Multiplying by f' and integrating gives

$$(f')^2 - \frac{1}{4}f^4 + \frac{1}{2}f^2 = C.$$

Attempting to solve for f' in terms of f yields

$$f' = \pm\sqrt{C + \frac{1}{4}f^4 - \frac{1}{2}f^2}.$$

Periodic orbits correspond to $0 < C < 1/2$, and the orbits that connect the saddle points $(\pm 1, 0)$ corresopnd to $C = 1/2$.

(c)

4. Consider the heat equation

$$u_t = u_{yy}$$

on the real line with initial data $u_0 = 1,\ y < 0,\ u_0 = 0,\ y > 0$. (a) Show that the solution $u(y,t)$ satisfies $\lim_{t\to\infty} u(y,t) = 1/2$. (b) Is the limit uniform in y? Prove your answer.

Solution

(a) The solution is given by

$$\begin{aligned} u(y,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(y-x)^2/4t} u_0(x)\,dx \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{0} e^{-(y-x)^2/4t}\,dx, \qquad \left[z = \frac{x-y}{2\sqrt{t}}\right] \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-y/2\sqrt{t}} e^{-z^2}\,dz, \end{aligned}$$

which, as $t \to \infty$ with y fixed, tends to

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-z^2}\,dz = \frac{1}{2}.$$

(b) No, since, for fixed $t > 0$, we still have $u(y,t) \to 1$ as $y \to -\infty$ and $u(y,t) \to 0$ as $y \to \infty$.

5. The Cahn-Hilliard equation for phase separation of a binary alloy is

$$u_t + \Delta\left(\epsilon\Delta u - \frac{1}{\epsilon}W'(u)\right) = 0,$$

where $W(u)$ is a smooth function of u. Show that

$$E(u) = \epsilon\frac{1}{2}\int |\nabla u|^2 dx + \frac{1}{\epsilon}\int W(u)dx$$

is a monotonically decreasing quantity for smooth solutions of the Cahn-Hilliard equation on the torus $\mathbb{T}^n$.

Solution

Using integration by parts (and noting that $\partial\mathbb{T}^n = \emptyset$), we compute

$$\begin{aligned}
\frac{d}{dt}E(u) &= \frac{d}{dt}\left(\epsilon\frac{1}{2}\int |\nabla u|^2 dx + \frac{1}{\epsilon}\int W(u)dx\right) \\
&= \epsilon\frac{1}{2}\int (2\nabla u \cdot \nabla(u_t))\, dx + \frac{1}{\epsilon}\int W'(u)u_t dx \\
&= -\epsilon\int \Delta u u_t dx + \frac{1}{\epsilon}\int W'(u)u_t dx \\
&= \int \left(\frac{1}{\epsilon}W'(u) - \Delta u\right) u_t dx \\
&= \int \left(\frac{1}{\epsilon}W'(u) - \Delta u\right)\Delta\left(\frac{1}{\epsilon}W'(u) - \Delta u\right) dx \\
&= -\int \left|\nabla\left(\frac{1}{\epsilon}W'(u) - \Delta u\right)\right|^2 dx \\
&\leq 0.
\end{aligned}$$

6. Let f be a smooth function defined on $\mathbb{R}^3$ and suppose that $\Delta\Delta f = 0$ for $|x| \leq a$. Show that

$$\left(4\pi a^2\right)^{-1}\int_{|x|=a} f(x)ds = f(0) + \frac{a^2}{6}\Delta f(0).$$

Hint: Do this first for spherically symmetric f, i.e., for $f(x) = f(r = |x|)$, for which $\Delta = r^{-2}\partial_r\left(r^2\partial_r\right)$.

Solution

Let

$$I(a) = \frac{1}{4\pi a^2}\int_{|x|\leq a} f(x)ds = \frac{1}{4\pi}\int_{|\xi|=1} f(a\xi)ds(\xi).$$

We compute $I'(a)$:

$$\begin{aligned}
I'(a) &= \frac{1}{4\pi}\int_{|\xi|=1} \nabla f(a\xi)\cdot\xi ds(\xi) \\
&= \frac{1}{4\pi}\int_{|\xi|=1} \nabla f(a\xi)\cdot\nu ds(\xi) \\
&= \frac{1}{4\pi a^2}\int_{|\xi|=1} \nabla f(x)\cdot\nu ds \\
&= \frac{1}{4\pi a^2}\int_{|\xi|\leq 1} \Delta f(x)dx,
\end{aligned}$$

where we have used the Divergence Theorem in the last equality. But Δf is harmonic, hence satisfies the mean value property, so we have

$$I'(a) = \frac{a}{3}\Delta f(0).$$

Upon integrating and noting that $I(0) = f(0)$, we obtain the claim:

$$I(a) = f(0) + \frac{a^2}{6}\Delta f(0).$$

7. Find the (entropy) solution for all time $t > 0$ of the inviscid Burgers equation $u_t + \frac{1}{2}(u^2)_x = 0$ with initial condition

$$u(x,0) = \begin{cases} 0, & x < -1 \\ x+1, & -1 < x < 0 \\ 1 - \frac{1}{2}x, & 0 < x < 2 \\ 0, & x > 2 \end{cases}.$$

Solution

Denote $g(x) = u(x,0)$, and, for the purposes of applying the method of characteristics, let $y = t$, so that the PDE is $uu_x + u_y = 0$. The initial condition curve may be parametrized by $s \mapsto (s, 0, g(s)) = (x_0, y_0, z_0)$. As mentioned, we use the method of characteristics, giving the system of ODEs

$$\begin{aligned} x'(t) &= z; \\ y'(t) &= 1; \\ z'(t) &= 0. \end{aligned}$$

We can solve for y and z immediately:

$$\begin{aligned} y(t) &= t + y_0 = t; \\ z(t) &= z_0 = g(s). \end{aligned}$$

It follows that

$$x(t) = g(s)t + x_0 = g(s)t + s.$$

To solve for s, t in terms of x, y, we have immediately that $t = y$, defining s implicitly by

$$0 = g(s)y + s - x.$$

This is invertible precisely when $-1 < y < 2$ (this ensures that the s-derivative of the right-hand side never vanishes). Since

$$0 = g(s)y + s - x = \begin{cases} s - x, & s < -1 \\ (1+y)s - x + y, & -1 < s < 0 \\ \left(1 - \frac{1}{2}y\right)s - x + y, & 0 < s < 2 \\ s - x, & s > 2 \end{cases},$$

we find that

$$s = s(x,y) = \begin{cases} x, & x < -1 \\ \frac{x-y}{1+y}, & -1 < x < y \\ \frac{x-y}{1-\frac{1}{2}y}, & y < x < 2 \\ x, & x > 2 \end{cases}$$

and the resulting solution is

$$u(x,y) = z = g(s(x,y)) = \begin{cases} 0, & x < -1 \\ \frac{1+x}{1+y}, & -1 < x < y \\ \frac{2-x}{2-y}, & y < x < 2 \\ 0, & x > 2 \end{cases}.$$

This solution evidently forms a shock at (particularly) $y = 2$:

$$u(x,2) = \begin{cases} 0, & x < -1 \\ \frac{1}{3}(1+x), & -1 < x < 2 \\ 0, & x > 2 \end{cases}.$$

To determine the propagation of the shock, we return to the original PDE and integrate with respect to x between $x = a$ and $x = b$ to obtain

$$\frac{1}{2}u(b,y)^2 - \frac{1}{2}u(a,y)^2 + \frac{d}{dy}\int_a^b u(x,y)dx = 0.$$

If the shock propagates along $x = \xi(y)$, then taking $a < \xi(y) < b$ gives

$$\begin{aligned} 0 &= \frac{1}{2}u(b,y)^2 - \frac{1}{2}u(a,y)^2 + \frac{d}{dy}\int_a^b u(x,y)dx \\ &= \frac{1}{2}u(b,y)^2 - \frac{1}{2}u(a,y)^2 + \frac{d}{dy}\left(\int_a^{\xi(y)} u(x,y)dx + \int_{\xi(y)}^b u(x,y)dx\right) \\ &= \frac{1}{2}u(b,y)^2 - \frac{1}{2}u(a,y)^2 \\ &\quad +\xi'(y)u_\ell(\xi(y),y) + \int_a^{\xi(y)} u_y(x,y)dx - \xi'(y)u_r(\xi(y),y) + \int_{\xi(y)}^b u_y(x,y)dx, \end{aligned}$$

and letting $a \nearrow \xi(y)$ and $b \searrow \xi(y)$ gives

$$\xi'(y) = \frac{\frac{1}{2}u_r(\xi(y),y)^2 - \frac{1}{2}u_\ell(\xi(y),y)^2}{u_r(\xi(y),y) - u_\ell(\xi(y),y)} = \frac{1}{2}\left(u_r(\xi(y),y) + u_\ell(\xi(y),y)\right),$$

where u_r and u_ℓ denote the values of u to the right and left of the shock, respectively. In this case,

$$\begin{aligned} u_r(\xi(y),y) &= 0; \\ u_\ell(\xi(y),y) &= \frac{1+\xi(y)}{1+y}; \end{aligned}$$

so

$$\xi'(y) = \frac{1+\xi(y)}{2(1+y)}$$

with $\xi(2) = 2$. Solving for ξ yields

$$\xi(y) = \sqrt{3(1+y)} - 1.$$

The solution is thus (for $y > 2$)

$$u(x,y) = \begin{cases} 0, & x < -1 \\ \frac{1+x}{1+y}, & -1 < x < \sqrt{3(1+y)} - 1 \\ 0, & x > \sqrt{3(1+y)} - 1 \end{cases}.$$

8. Consider the "eikonal" equation in $\mathbb{R}^2$:

$$\phi_x^2 + \phi_y^2 = 1$$

on the domain $0 < x < 2\pi$ and $0 \leq y < \infty$, with periodic boundary conditions in x and boundary data

$$\phi(x, 0) = \cos x.$$

Find a solution in an implicit form.

Solution

For the purposes of applying the method of characteristics, let $u = \phi$, so that the PDE is $u_x^2 + u_y^2 = 1$. We identify $F(x, y, p, q) = (p^2 + q^2 - 1)/2 = 0$, and parametrize the initial condition curve by $s \mapsto (s, 0, \cos(s)) = (x_0, y_0, z_0)$. To determine $p_0 = \phi$ and $q_0 = \psi$, we require

$$0 = F(x_0, y_0, z_0, \phi, \psi) = \frac{1}{2}\left(\phi^2 + \psi^2 - 1\right)$$

and

$$0 = z_0' - \phi x_0' - \psi y_0' = -\sin(s) - \phi,$$

yielding $\phi(s) = -\sin(s)$, $\psi(s) = \pm\cos(s)$. As mentioned, we use the method of characteristics, giving the system of ODEs

$$\begin{aligned}
x' &= F_p = p; \\
y' &= F_q = q; \\
z' &= px' + qy' = p^2 + q^2 = 1; \\
p' &= -F_x - F_z p = 0; \\
q' &= -F_y - F_z q = 0.
\end{aligned}$$

We can solve for z, p, and q immediately:

$$\begin{aligned}
z(t) &= t + z_0 = t + \cos(s); \\
p(t) &= p_0 = -\sin(s); \\
q(t) &= q_0 = \pm\cos(s).
\end{aligned}$$

It follows that

$$\begin{aligned}
x(t) &= -t\sin(s) + x_0 = -t\sin(s) + s; \\
y(t) &= \pm t\cos(s) + y_0 = \pm t\cos(s).
\end{aligned}$$

We can derive implicit equations that give s, t in terms of x, y. First, eliminating t in the equations for x and y, we find that

$$x = s \mp y\tan(s).$$