

Simluation of Elasticity, Biomechanics, and Virtual Surgery

Problem Session II

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1 Elasticity

The goal in this problem session is to use a (variable-coefficient) Poisson solver (either one you coded from Problem Session I or the one we provide) to implement the Neo-Hookean model of elasticity in dimension $d = 1$. Recall that the equations of elasticity are given by the boundary value problem

$$\rho_0(\mathbf{X}) \frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla^{\mathbf{X}} \cdot \mathbf{P} + \mathbf{f}^{\text{ext}} \quad \in \Omega(t) \quad (1a)$$

$$\mathbf{u}(\cdot, t) = \mathbf{g}(\cdot, t) \quad \in \partial\Omega_d(t) \quad (1b)$$

$$(\mathbf{P} \cdot \hat{\mathbf{n}})(\cdot, t) = \mathbf{h}(\cdot, t) \quad \in \partial\Omega_n(t) \quad (1c)$$

where ρ_0 is the mass density (as a function of \mathbf{X} , the undeformed coordinates); $\mathbf{u} = \phi - \mathbf{X}$ is the unknown displacement; \mathbf{P} is the first Piola-Kirchoff stress (which takes a specific form for Neo-Hookean, to be given later); \mathbf{f}^{ext} is the given (external) force; and \mathbf{g} and \mathbf{h} specify the Dirichlet and Neumann boundary conditions, respectively. For simplicity, we'll assume a uniform mass density, i.e., $\rho_0 \equiv 1$.

2 Neo-Hookean

The Neo-Hookean model relates the stress \mathbf{P} to the deformation gradient \mathbf{F} via

$$\Psi(\mathbf{F}) = \frac{\mu}{2} (F_{ij}F_{ji} - 2) - \mu \log J + \frac{\lambda}{2} \log^2 J, \quad (2a)$$

$$\mathbf{P}(\mathbf{F}) = \frac{\partial \Psi}{\partial \mathbf{F}} = \mu \mathbf{F} + (\lambda \log J - \mu) \mathbf{F}^{-T}. \quad (2b)$$

(Recall that $J = \det \mathbf{F}$.) This manifests itself in dimension $d = 1$ in terms of the displacement u as

$$P(u) = \mu \left(\frac{du}{dX} + 1 \right) + \left(\lambda \log \left(\frac{du}{dX} + 1 \right) - \mu \right) \frac{1}{\frac{du}{dX} + 1}. \quad (3)$$

3 Inversion-Robust Neo-Hookean

Neo-Hookean as formulated above will not be robust to element inversions, due to the $\log J$ term. To remedy this, we replace the logarithm in (2a) with a cubic Taylor approximation around 1:

$$\begin{aligned}
r(x) &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \\
&= -\frac{11}{6} + 3x - \frac{3}{2}x^2 + \frac{1}{3}x^3,
\end{aligned} \tag{4a}$$

$$\begin{aligned}
r'(x) &= 1 - (x-1) + (x-1)^2 \\
&= 3 - 3x + x^2.
\end{aligned} \tag{4b}$$

$$\begin{aligned}
r''(x) &= -1 + 2(x-1) \\
&= -3 + 2x.
\end{aligned} \tag{4c}$$

This gives

$$\Psi(\mathbf{F}) = \frac{\mu}{2} (F_{ij}F_{ji} - 2) - \mu r(J) + \frac{\lambda}{2} r(J)^2, \tag{5a}$$

$$\mathbf{P}(\mathbf{F}) = \frac{\partial \Psi}{\partial \mathbf{F}} = \mu \mathbf{F} + (\lambda r(J) - \mu) r'(J) \frac{\partial J}{\partial \mathbf{F}}. \tag{5b}$$

Again, specializing this for $d = 1$ dimension, we obtain

$$P(u) = \mu \left(\frac{du}{dX} + 1 \right) + \left(\lambda r \left(\frac{du}{dX} + 1 \right) - \mu \right) r' \left(\frac{du}{dX} + 1 \right). \tag{6}$$

4 Quasistatics

We begin by studying eqs (1) at equilibrium, which is the basis for quasistatic evolution. This reduces the equations to

$$-\nabla^{\mathbf{X}} \cdot \mathbf{P} = \mathbf{f}^{\text{ext}}. \tag{7}$$

The equivalent weak formulation is

$$\int_{\Omega_0} w_{i,j} P_{ij} d\mathbf{X} = \int_{\partial\Omega_n} w_i h_i dS(\mathbf{X}) + \int_{\Omega_0} w_i f_i^{\text{ext}} d\mathbf{X} \tag{8}$$

for all test functions \mathbf{w} . Recall that summation over repeated indices is implied, and comma'ed indices indicate differentiation.

As for Poisson, we let the coordinates of \mathbf{w} vary over the nodal basis functions N_i . In dimension $d = 1$, this reduces to the system of equations

$$q_i(u) = \int_a^b \frac{\partial N_i}{\partial X} P(F(u(X))) dX - b_i = 0; \tag{9a}$$

$$b_i = \int_a^b N_i f_i^{\text{ext}} dX + [\text{Neumann boundary terms}] \tag{9b}$$

for each grid vertex i . For Neo-Hookean, P depends *non-linearly* on the displacement u , hence one must use a non-linear solver, such as Newton iteration, to solve (9) for u . The Newton step looks like

$$\frac{\partial q_i}{\partial u}(u^k) \Delta u + q_i(u^k) = 0; \tag{10a}$$

$$u^{k+1} = u^k + \Delta u \tag{10b}$$

where

$$\frac{\partial q_i}{\partial u_j}(u) = \int_a^b \frac{\partial N_i}{\partial X} \frac{\partial P}{\partial F}(F(u)) \frac{\partial N_j}{\partial X} dX \quad (11)$$

Thus, the computation of Δu in each Newton iteration amounts to solving a variable coefficient Poisson problem. This coefficient is $\partial P / \partial F = \partial P / \partial (du/dX)$, which we can express via (6) as

$$\frac{\partial P}{\partial F} = \mu + \lambda r' \left(\frac{du}{dX} + 1 \right)^2 + \left(\lambda r \left(\frac{du}{dX} + 1 \right) - \mu \right) r'' \left(\frac{du}{dX} + 1 \right). \quad (12)$$

This gives all the necessary pieces to implement a quasistatics evolution of the elasticity equations (1).

5 Backward Euler

In the event that inertial terms are non-negligible, we must use some time-stepping scheme to solve (1). Here, we outline the implementation of Backward Euler. Since (1) is second-order in time, we must introduce an additional variable, $\mathbf{v} = \partial \mathbf{u} / \partial t$, to convert (1) to an equivalent first-order system:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} &= \mathbf{v}, \\ \rho_0 \frac{\partial \mathbf{v}}{\partial t} &= \nabla^{\mathbf{X}} \cdot \mathbf{P} + \mathbf{f}^{\text{ext}}. \end{aligned} \quad (13)$$

The time discretization for Backward Euler thus gives:

$$\begin{aligned} \frac{\mathbf{u}^{m+1} - \mathbf{u}^m}{\Delta t} &= \mathbf{v}^{m+1}, \\ \rho_0 \frac{\mathbf{v}^{m+1} - \mathbf{v}^m}{\Delta t} &= (\nabla^{\mathbf{X}} \cdot \mathbf{P})_{\mathbf{u}^{m+1}} + \mathbf{f}^{\text{ext}}. \end{aligned} \quad (14)$$

To solve these equations, we use the first to substitute for \mathbf{v}^{m+1} in the second and rearrange terms to obtain (again, assuming $\rho \equiv 1$ for simplicity)

$$\mathbf{u}^{m+1} - \Delta t^2 (\nabla^{\mathbf{X}} \cdot \mathbf{P})_{\mathbf{u}^{m+1}} = \Delta t^2 \mathbf{f}^{\text{ext}} + \Delta t \mathbf{v}^m + \mathbf{u}^m \quad (15)$$

which turns out to be very similar to (7). Indeed, the difference between the two systems essentially amounts to the addition of an identity matrix.