

# Math 269B, 2012 Winter, Homework 4

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## 1 Theory

1. Solve the heat equation  $u_t = bu_{xx}$  on a interval  $I \subset \mathbb{R}$  with *periodic* boundary conditions. How does  $\int_I u(t, x) dx$  vary with time  $t$ ?
2. (Strikwerda 6.1.4.) Use the representation (6.1.3) to verify the following estimates on the norms of  $u(t, x)$ :

$$\|u(t, \cdot)\|_1 \leq \|u_0\|_1,$$

$$\|u(t, \cdot)\|_\infty \leq \|u_0\|_\infty.$$

Show that if  $u_0$  is nonnegative, then

$$\|u(t, \cdot)\|_1 = \|u_0\|_1.$$

3. Determine the stability and accuracy of the following combination of the Lax-Wendroff and Backward-Time Central-Space schemes to solve  $u_t + au_x = bu_{xx}$  (with  $b > 0$ ):

$$\begin{aligned} 0 &= P_{k,h} v_m^n \\ &= \frac{1}{k} (v_m^{n+1} - v_m^n) + \frac{a}{2h} (v_{m+1}^n - v_{m-1}^n) - \frac{a^2 k}{2h^2} (v_{m+1}^n - 2v_m^n + v_{m-1}^n) \\ &\quad - \frac{b}{h^2} (v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}). \end{aligned}$$

## 2 Programming

1. Solve  $u_t + au_x = 0$  numerically using the Lax-Friedrichs scheme. Take  $a = 1$ ,  $T = 1$ ,  $x \in [0, 1]$  with periodic boundary conditions, and  $u_0(x) = \sin 2\pi x$ . For each fixed  $\lambda$  within a decreasing sequence of  $\lambda$ s (each satisfying the stability criterion), demonstrate convergence with  $k/h =: \lambda$  by plotting the logarithm of the  $L^2$ -norm of the error (between the analytic solution and the numerical solution) versus the logarithm of  $h$ . Verify that the slope suggested by your plot agrees with theory, and estimate the error constant  $C_\lambda$  in the relation  $\text{error} = C_\lambda h^p$ . Use enough values of  $\lambda$  to estimate the relation between  $C_\lambda$  and  $\lambda$ . What appears to happen to  $C_\lambda$  as  $\lambda \rightarrow 0+$ , i.e., as you shrink  $k$  relative to  $h$ ? What happens if, instead of taking  $k = \lambda h$ , you take  $k = h^2$ ? Explain your numerical results in the context of the theoretical convergence analysis of the Lax-Friedrichs scheme.
2. Implement the scheme from problem 3 in the Theory section and confirm numerically the theoretical rate of convergence.
3. Write a function implementing the Thomas algorithm presented in Strikwerda 3.5. Specifically, we solve the system of equations

$$a_i w_{i-1} + b_i w_i + c_i w_{i+1} = d_i, \quad i = 1, \dots, m-1,$$

with  $w_0 = \beta_0$  and  $w_m = \beta_m$ . The solution is given by

$$w_i = p_{i+1}w_{i+1} + q_{i+1}$$

where  $p_{i+1}$  and  $q_{i+1}$  are defined recursively by

$$\begin{aligned} p_{i+1} &= -(a_i p_i + b_i)^{-1} c_i, \\ q_{i+1} &= (a_i p_i + b_i)^{-1} (d_i - a_i q_i), \end{aligned}$$

and with  $p_1$  and  $q_1$  determined by the boundary conditions. For the next homework, be prepared to utilize your function implementing the Thomas algorithm to write a function which solves *periodic* tridiagonal systems.