Math 269B, 2012 Winter, Final (Solutions)

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1 Theory

1. Suppose $u\colon [0,\infty)\times\mathbb{R}\to\mathbb{R}$ satisfies the inviscid Burger's equation,

$$0 = u_t + \frac{1}{2}\left(u^2\right)_x = u_t + uu_x, \quad u(0,x) = u_0(x). \tag{1}$$

Use the method of characteristics to show that u must then satisfy the implicit relation

$$u(t,x) = u_0\left(x - tu(t,x)\right). \tag{2}$$

[Hint: Begin by defining $\tilde{u}(t,X) := u\left(t,\varphi(t,X)\right)$ for some to-be-determined change of variables $\varphi\colon [0,\infty)\times\{X\}\to\{x\}$, and choose φ such that $\tilde{u}_t \equiv 0$.]

Solution

Let $\tilde{u}$ and φ be as in the hint. Then

$$\tilde{u}_t = u_t + \varphi_t u_x = 0$$

if

$$\varphi_t(t,X) = \tilde{u}(t,X) = u(t,\varphi(t,X)).$$

Of course, if $\tilde{u}_t = 0$, then

$$\tilde{u}(t,X) = \tilde{u}(0,X) = u(0,\varphi(0,X)) = u_0(X)$$

so long as $\varphi(0,X) = X$. Thus,

$$\varphi_t(t,X) = u_0(X) \quad \Rightarrow \quad \varphi(t,X) = X + tu_0(X) = X + t\tilde{u}(t,X) = X + tu\left(t,\varphi(t,X)\right)$$

and hence

$$u\left(t,\varphi(t,X)\right) = \tilde{u}(t,X) = u_0(X) = u_0\left(\varphi(t,X) - tu\left(t,\varphi(t,X)\right)\right).$$

If $x = \varphi(t,X)$, this reduces to

$$u(t,x) = u_0\left(x - tu(t,x)\right).$$

2. Suppose u_0' in (1) is bounded below, i.e., $u_0' \geq c$ for some constant c. Determine the maximal T such that a solution to (2) is guaranteed to exist for $t\in[0,T)$ (possibly with $T=\infty$). [Hint: Determine when one can guarantee that the function $u\mapsto u - u_0(x-tu)$ has a unique root.)

Solution

Note that (1) has a solution up to time T if (2) has a solution u for each $(t,x)\in[0,T)\times\mathbb{R}$, and the latter is equivalent to $\alpha_{t,x}(u) := u - u_0(x-tu)$ having a unique root. A sufficient condition for a unique root is $\alpha_{t,x}'$ continuous and bounded away from zero. As such, we compute $\alpha_{t,x}'(u) = 1 + tu_0'(x-tu)$. Now given that $u_0'\geq c$, if $c\geq 0$ then $\alpha_{t,x}'\geq 1$ for all $(t,x)\in[0,\infty)\times\mathbb{R}$, implying that a solution to (1) exists for all $t\in[0,\infty)$, i.e., $T=\infty$. Otherwise, if $c<0$, we conclude that a solution is guaranteed to exist only up to time $T = -1/c$.

3. Solve (1) for $u_0(x) = ax + b$, where a, b are constants. [Hint: Use (2).]

Solution

Using (2), we solve

$$u = u_0\left(x - tu\right) = a\left(x - tu\right) + b,$$

giving

$$u(t, x) = \frac{ax + b}{1 + at}.$$

Indeed, one can quickly verify that this does, in fact, solve (1).

4. Denote the solution to (1) by $u = F\left[u_0\right]$. Express $F\left[x \mapsto au_0(x) + b\right]$ in terms of $F\left[u_0\right]$.

In other words, given u satisfying (1) for some u_0, determine the solution v (in terms of the aforementioned u) to

$$v_t + vv_x = 0, \quad v(0, x) = v_0(x) := au_0(x) + b.$$

Solution

Let u denote the solution to (1) with initial condition u_0, and let v denote the solution to (1) with initial condition $v_0(x) := au_0(x) + b$. Using (2), we wish to solve

$$v = v_0\left(x - tv\right) = au_0\left(x - tv\right) \quad \Leftrightarrow \quad \frac{1}{a}\left(v - b\right) = u_0\left(\left(x - tb\right) - \left(at\right)\frac{1}{a}\left(v - b\right)\right),$$

which suggests that

$$\frac{1}{a}\left(v(t, x) - b\right) = u\left(at, x - tb\right) \quad \Leftrightarrow \quad v(t, x) = au\left(at, x - tb\right) + b.$$

Indeed, one can quickly verify that this does, in fact, solve (1) with initial condition v_0. In other words,

$$F\left[x \mapsto au_0(x) + b\right](t, x) = aF\left[u_0\right]\left(at, x - tb\right) + b.$$

5. Suppose u_0 is given as

$$u_0(x) := \begin{cases} u_0^L(x) := a_L x + b_L, & x < 0 \\ u_0^R(x) := a_R x + b_R, & x > 0 \end{cases}.$$

Determine the path $t \mapsto (t, x_S(t))$ of the (physically correct) shock in the solution u to (1) eminating from $(t, x) = (0, 0)$ (assume $b_L \geq b_R$). You may use the fact that

$$\frac{1}{2}\int \frac{b_L + b_R + \left(a_L b_R + a_R b_L\right)t}{\left(\left(1 + a_L t\right)\left(1 + a_R t\right)\right)^{3/2}} dt = \frac{\left(a_L b_R - a_R b_L\right)t + \left(b_R - b_L\right)}{\left(a_L - a_R\right)\sqrt{\left(1 + a_L t\right)\left(1 + a_R t\right)}} \qquad \left[a_L \neq a_R\right].$$

[Hint: Recall that the shock speed $x_S'(t) = \frac{1}{2}\left(u^L + u^R\right)(t, x_S(t))$, thus allowing you to set up an ordinary differential equation for x_S.] Consider and explain the physical significance of the special cases $a_L = a_R$ and $b_L = b_R$.

Solution

We know that

$$u^L(t, x) = \frac{a_L x + b_L}{1 + a_L t}, \quad u^R(t, x) = \frac{a_R x + b_R}{1 + a_R t}$$

describe the solution u to the left and right of the shock, respectively. From the hint, it follows that

$$\begin{aligned} x_S'(t) &= \frac{1}{2}\left(u^L + u^R\right)(t, x_S) \\ &= \frac{1}{2}\left(\frac{a_L x_S + b_L}{1 + a_L t} + \frac{a_R x_S + b_R}{1 + a_R t}\right). \end{aligned}$$

Rearranging gives

$$x_S' - \frac{1}{2}\left(\frac{a_L}{1+a_Lt} + \frac{a_R}{1+a_Rt}\right)x_S = \frac{1}{2}\frac{b_L+b_R+(a_Lb_R+a_Rb_L)\,t}{(1+a_Lt)\,(1+a_Rt)}.$$

One can solve this via multiplication of the integrating factor

$$\exp\left(\int -\frac{1}{2}\left(\frac{a_L}{1+a_Lt} + \frac{a_R}{1+a_Rt}\right)dt\right) = ((1+a_Lt)\,(1+a_Rt))^{-1/2}$$

giving

$$\left(\frac{x_S}{\sqrt{(1+a_Lt)\,(1+a_Rt)}}\right)' = \frac{1}{2}\frac{b_L+b_R+(a_Lb_R+a_Rb_L)\,t}{((1+a_Lt)\,(1+a_Rt))^{3/2}}$$

and hence

$$x_S = \frac{1}{2}\sqrt{(1+a_Lt)\,(1+a_Rt)}\int_0^t \frac{b_L+b_R+(a_Lb_R+a_Rb_L)\,\tau}{((1+a_L\tau)\,(1+a_R\tau))^{3/2}}d\tau.$$

Now, if $a_L \neq a_R$, the integral on the right evaluates to

$$\frac{1}{2}\int_0^t \frac{b_L+b_R+(a_Lb_R+a_Rb_L)\,\tau}{((1+a_L\tau)\,(1+a_R\tau))^{3/2}}d\tau = \frac{(a_Lb_R-a_Rb_L)\,t+(b_R-b_L)}{(a_L-a_R)\sqrt{(1+a_Lt)\,(1+a_Rt)}} - \frac{b_R-b_L}{a_L-a_R}$$

giving

$$x_S = \frac{1}{a_L-a_R}\left((a_Lb_R-a_Rb_L)\,t+(b_R-b_L)\left(1-\sqrt{(1+a_Lt)\,(1+a_Rt)}\right)\right).$$

On the other hand, if $a_L = a_R =: a$, then the integral is significantly simpler:

$$\frac{1}{2}\int_0^t \frac{b_L+b_R+(a_Lb_R+a_Rb_L)\,\tau}{((1+a_L\tau)\,(1+a_R\tau))^{3/2}}d\tau = \frac{1}{2}(b_L+b_R)\int_0^t \frac{1}{(1+a\tau)^2}d\tau = \frac{b_L+b_R}{2a}\left(1-\frac{1}{1+at}\right)$$

giving the particularly simple trajectory

$$x_S = \frac{1}{2}(b_L+b_R)\,t.$$

The shock trajectory for this latter case, when $a_L = a_R$, indicates that the shock travels exactly along the path where $(u_L+u_R)/2 = (b_L+b_R)/2$.

When $b_L = b_R =: b$, the solution is actually *continuous*, i.e., there is no shock; nonetheless, the computed shock trajectory $x_S(t) = bt$ correctly tracks the interface between u_L and u_R.

6. Solve the weak form of (1) (i.e., give the entropy solution with rarefaction, and with any shocks propagating at the physically correct speed) on the *periodic* domain $[0,4]$ with the "pulse" initial condition

$$u_0(x) := \begin{cases} 0, & 0 \le x < 1 \\ 2, & 1 < x < 2 \\ 0, & 2 < x \le 4 \end{cases}. \tag{3}$$

Identify key points in time t when the character of the solution changes. (It will be natural to express the solution $u(t,x)$ piecewise with respect to x *and* t.) Confirm that $\int u(t,x)dx$ is conserved (i.e., $\int u(t,x)dx =$ constant for all t), and determine $\lim_{t\to\infty} u(t,x)$.

Solution

With the given initial condition u_0, we immediately have both a rarefaction, between $x = 1$ and $x = 1+2t$; and a shock, at $x = 2+t$, where we compute the speed of the shock to be 1 based on the average of the left (2) and right (0) values of the solution, which both remain constant. The solution continues with the rarefaction and the constant-speed shock propagation until the rarefaction region reaches the shock at time $t = 1$ (when $1 + 2t = 2 + t$). Thus, for $t \in [0, 1]$, we have the solution

$$u(t,x) = \begin{cases} 0, & 0 \le x \le 1 \\ \frac{x-1}{t}, & 1 \le x \le 1+2t \\ 2, & 1+2t \le x < 2+t \\ 0, & 2+t < x \le 4 \end{cases} \quad [0 \le t \le 1].$$

We quickly verify that

$$\int u(t,x)dx = \int_1^{1+2t} \frac{x-1}{t} dx + \int_{1+2t}^{2+t} 2dx = 2t + 2(1-t) = 2 \quad [0 \le t \le 1].$$

Once the rarefaction region reaches the shock, we can use our previous derivation to compute the trajectory of the shock until the shock reaches $x = 5 \cong 1$, the left edge of the rarefaction region. This gives

$$u(t,x) = \begin{cases} \frac{x-1}{t}, & 1 \le x < 1+2\sqrt{t} \\ 0, & 1+2\sqrt{t} < x \le 5 \end{cases} \quad [1 \le t \le 4],$$

where we have taken the liberty to identify $[0, 4] \cong [1, 5]$ via periodicity. We again verify that

$$\int u(t,x)dx = \int_1^{1+2\sqrt{t}} \frac{x-1}{t} dx = 2.$$

Once the shock reaches the left edge of the rarefaction region at $x = 1$, it travels at the constant speed $1/2$, giving

$$u(t,x) = \left\{ \frac{x-1}{t}, \quad \tfrac{1}{2}t - 1 < x < \tfrac{1}{2}t + 3 \quad [4 \le t]. \right.$$

We again verify that

$$\int u(t,x)dx = \int_{\frac{1}{2}t-1}^{\frac{1}{2}t+3} \frac{x-1}{t} dx = 2.$$

For time $t \ge 4$, the solution $u(t, x)$ is affine save for a discontinuity at the shock, where the solution increases from $\frac{1}{2} - \frac{2}{t}$ to the right of the shock up to $\frac{1}{2} + \frac{2}{t}$ to the left of the shock. Hence, $\lim_{t\to\infty} u(t, x) = \frac{1}{2}$.

7. (Strikwerda 6.3.9.) Consider a scheme for (6.1.1), $u_t = bu_{xx}$, of the form

$$v_m^{n+1} = (1 - 2\alpha_1 - 2\alpha_2) v_m^n + \alpha_1 \left(v_{m+1}^n + v_{m-1}^n\right) + \alpha_2 \left(v_{m+2}^n + v_{m-2}^n\right).$$

Show that when μ is constant, as k and h tend to zero, the scheme is inconsistent unless

$$\alpha_1 + 4\alpha_2 = b\mu.$$

Show that the scheme is fourth-order accurate in x if $\alpha_2 = -\alpha_1/16$.

Solution

Denote by $P_{k,h}$ the (scaled) difference operator

$$P_{k,h}v_m^n := \frac{1}{k}\left(v_m^{n+1} - v_m^n\right) - \frac{\alpha_1}{k}\left(v_{m+1}^n - 2v_m^n + v_{m-1}^n\right) - \frac{\alpha_2}{k}\left(v_{m+2}^n - 2v_m^n + v_{m-2}^n\right).$$

The corresponding symbol is then

$$\begin{aligned}
p_{k,h}(s,\xi) &:= P_{k,h}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi} \\
&= \frac{1}{k}\left(e^{sk}-1\right) + \frac{2\alpha_1}{k}\left(1-\cos h\xi\right) + \frac{2\alpha_2}{k}\left(1-\cos 2h\xi\right) \\
&= s + \left(\alpha_1+4\alpha_2\right)\frac{h^2}{k}\xi^2 - \frac{1}{12}\left(\alpha_1+16\alpha_2\right)\frac{h^4}{k}\xi^4 + O\left(k^2 + \left(\alpha_1+\alpha_2\right)\frac{h^6}{k}\right)
\end{aligned}$$

which is consistent with the symbol $p = s + b\xi^2$ of the differential operator $P = \partial_t - b\partial_x^2$ as long as

$$\left(\alpha_1+4\alpha_2\right)\frac{h^2}{k} = b \quad \Leftrightarrow \quad \alpha_1 + 4\alpha_2 = b\mu,$$

with $\mu := k/h^2$. Indeed, we see that one gets fourth-order accuracy in space if, additionally, $\alpha_1+16\alpha_2 = 0$.

2 Programming

1. Implement the following numerical schemes to solve (1) on the *periodic* domain $[0, 4]$:

   – Godunov's method. At time level n, solve the Riemann problem assuming a piecewise constant initial condition v^n, then resample to determine v^{n+1}.

   – (Backward) Semi-Lagrangian. At time level $n+1$ and grid vertex m, trace the characteristic $t \mapsto x_m + v_m^n\left(t - t_{n+1}\right)$ *backward* to time level n and linearly interpolate v^n to determine v_m^{n+1}.

   – (Forward) Semi-Lagrangian. Trace the characteristics $t \mapsto x_m + v_m^n\left(t - t_n\right)$ *forward* to time level $n+1$ and linearly interpolate the nearest characteristics at a given grid vertex m to determine v_m^{n+1}.

   – (Conservative) Lax-Friedrichs. Discretize the conservative form of (inviscid) Burger's equation:

   $$\frac{1}{k}\left(v_m^{n+1} - \frac{1}{2}\left(v_{m+1}^n - v_{m-1}^n\right)\right) + \frac{1}{2}\cdot\frac{1}{2h}\left(\left(v_{m+1}^n\right)^2 - \left(v_{m-1}^n\right)^2\right) = 0$$

   – (Advective) Lax-Friedrichs. Discretize the advective form of (inviscid) Burger's equation:

   $$\frac{1}{k}\left(v_m^{n+1} - \frac{1}{2}\left(v_{m+1}^n - v_{m-1}^n\right)\right) + v_m^n\frac{1}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) = 0$$

   Use the initial condition (3). For those schemes that appear to converge to the exact solution (derived previously), compute a numerical convergence rate. For those schemes that don't appear to converge to the exact solution, explain the discrepancy (e.g., incorrect rarefaction, non-physical shock speed, unstable). Which scheme do you think performs best for the given initial condition?

   Solution

   – Godunov's method. Numerical evidence suggests that Godunov's method converges with a numerical convergence rate of 0.42:

   ```
   test_convergence( ...
       6, ...
       @godunov, "Godunov's Method", ...
       2.^(-(7:0.5:10)), @(h) h/2);
   ```

– (Backward) Semi-Lagrangian. For small enough k, the (backward) semi-Lagrangian scheme is equivalent to

$$v_m^{n+1} = v_m^n \left(1 - \frac{k}{h}\left(\begin{cases} v_m^n - v_{m-1}^n, & v_m^n > 0 \\ v_{m+1}^n - v_m^n, & v_m^n < 0 \end{cases}\right)\right),$$

which is just the upwinding scheme. Numerical evidence suggests that this scheme *diverges*. Shock speeds are not captured accurately. In particular, any zero values at the n^{th} time step are propagated *directly* to the $(n+1)^{\text{th}}$ time step, hence if $u^L > 0$ and $u^R = 0$ around a shock, the shock will remain stationary.

– (Forward) Semi-Lagrangian. For small enough k, the (forward) semi-Lagrangian scheme is equivalent to

$$v_m^{n+1} = v_m^n \left(1 + \frac{k}{h}\left(\begin{cases} v_m^n - v_{m-1}^n, & v_m^n > 0 \\ v_{m+1}^n - v_m^n, & v_m^n < 0 \end{cases}\right)\right)^{-1}.$$

Numerical evidence suggests that this scheme *diverges*. Again, shock speeds are not captured accurately, as in the case where $v_m = 0$.

– (Conservative) Lax-Friedrichs. Numerical evidence suggests that the (conservative) Lax-Friedrichs scheme converges with a numerical convergence rate of 0.53:

```
test_convergence( ...
    6, ...
    @lax_friedrichs_conservative, "(Conservative) Lax-Friedrichs", ...
    2.^(-(7:0.5:10)), @(h) h/2);
```

– (Advective) Lax-Friedrichs. Numerical evidence suggests that the (advective) Lax-Friedrichs scheme *diverges*. Although shock speeds appear to be captured accurately, the scheme introduces substantial oscillations in the numerical solution as $k, h \to 0$.

Clearly, Godunov's method and the (conservative) Lax-Friedrichs schemes are the only worthwhile schemes to use for solving (1). It appears that the (conservative) Lax-Friedrichs scheme gives a slightly better numerical convergence rate for the initial condition (3). On the other hand, Godunov's method retains the sharpness of the shocks better.

2. Use your implementation of the Thomas algorithm from Homework 4 to solve *periodic* tridiagonal systems:

$$a_i w_{i-1} + b_i w_i + c_i w_{i+1}, \quad i = 1, \ldots, m,$$

with $w_0 = w_m$ and $w_{m+1} = w_1$. The following algorithm is described in Strikwerda. First, solve the following (non-periodic) tridiagonal systems:

$$\begin{aligned} a_i x_{i-1} + b_i x_i + c_i x_{i+1} &= d_i, \quad x_0 = 0 \text{ and } x_{m+1} = 0; \\ a_i y_{i-1} + b_i y_i + c_i y_{i+1} &= 0, \quad y_0 = 1 \text{ and } y_{m+1} = 0; \\ a_i z_{i-1} + b_i z_i + c_i z_{i+1} &= 0, \quad z_0 = 0 \text{ and } z_{m+1} = 1; \end{aligned}$$

for $i = 1, \ldots, m$. Then w_i is given by

$$w_i = x_i + r y_i + s z_i$$

where

$$\begin{aligned} r &:= \frac{1}{D}\left(x_m\left(1 - z_1\right) + x_1 z_m\right), \\ s &:= \frac{1}{D}\left(x_m y_1 + x_1\left(1 - y_m\right)\right), \\ D &:= \left(1 - y_m\right)\left(1 - z_1\right) - y_1 z_m. \end{aligned}$$

Solution

We can test the included code as follows.

```
N = 10;
a = rand([N 1]); b = rand([N 1]); c = rand([N 1]);
A = spdiags([[a(2:N);0] b [0;c(1:N-1)]], [-1 0 +1], N, N);
A(1,N) = a(1);
A(N,1) = c(N);
x = rand([N 1]);
d = A*x;
y = solve_periodic_tridiag(a,b,c,d);
norm(x-y, "inf")
```