Math 269B, 2012 Winter, Homework 3 (Solutions)

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1 Theory

1. (Strikwerda 5.1.2.) Show that the modified leapfrog scheme (5.1.6) is stable for ϵ satisfying

$$0 < \epsilon \leq 1 \quad \text{if} \quad 0 < a^2\lambda^2 \leq \frac{1}{2}$$

and

$$0 < \epsilon \leq 4a^2\lambda^2\left(1 - a^2\lambda^2\right) \quad \text{if} \quad \frac{1}{2} \leq a^2\lambda^2 < 1.$$

Note that these limits are not sharp. It is possible to choose ϵ larger than these limits and still have the scheme be stable.

Solution

Continuing from the text, we find the amplification factors to be

$$g_{\pm}(\theta) = -ia\lambda\sin\theta \pm \sqrt{1 - a^2\lambda^2\sin^2\theta - \epsilon\sin^4\frac{1}{2}\theta}.$$

If the expression under the $\sqrt{\cdot}$ is nonnegative, then

$$\left|g_{\pm}(\theta)\right|^2 = 1 - \epsilon\sin^4\frac{1}{2}\theta \leq 1,$$

hence the scheme is stable. We thus wish to satisfy

$$0 \leq 1 - a^2\lambda^2\sin^2\theta - \epsilon\sin^4\frac{1}{2}\theta =: \alpha(\theta).$$

We compute that

$$\alpha'(\theta) = -\frac{1}{2}\sin\theta\left(\left(4a^2\lambda^2 - \epsilon\right)\cos\theta + \epsilon\right),$$

and hence the extrema of α occur when $\sin\theta = 0$ or $\cos\theta = \epsilon/\left(\epsilon - 4a^2\lambda^2\right)$. Values of θ satisfying $\sin\theta = 0$ give $\alpha = 1$ or $\alpha = 1 - \epsilon$, requiring that $\epsilon \leq 1$. Values of θ satisfying $\cos\theta = \epsilon/\left(\epsilon - 4a^2\lambda^2\right)$ exist if and only if $\left|\epsilon/\left(\epsilon - 4a^2\lambda^2\right)\right| \leq 1$, which is equivalent to $\epsilon \leq 2a^2\lambda^2$. For such θ, we get $\alpha = 1 - 4a^4\lambda^4/\left(4a^2\lambda^2 - \epsilon\right)$, and for this to be nonnegative, we must have $\epsilon \leq 4a^2\lambda^2\left(1 - a^2\lambda^2\right)$. In particular, we must have $|a\lambda| < 1$.

So far, we have deduced that, at a minimum, $0 < \epsilon \leq 1$. Furthermore, if $\epsilon \leq 2a^2\lambda^2$, then we must additionally satisfy $\epsilon \leq 4a^2\lambda^2\left(1 - a^2\lambda^2\right)$. Now, in the instance that $2a^2\lambda^2 \leq 4a^2\lambda^2\left(1 - a^2\lambda^2\right)$, we would automatically satisfy the second condition, and this latter inequality is equivalent to $a^2\lambda^2 \leq \frac{1}{2}$. It follows that

- If $0 < a^2\lambda^2 \leq \frac{1}{2}$, it is sufficient to take $0 < \epsilon \leq 1$.

– If $\frac{1}{2} \le a^2\lambda^2 < 1$, it is sufficient to take $0 < \epsilon \le 4a^2\lambda^2\left(1 - a^2\lambda^2\right)$.

2. Derive the stability condition for the backward-time forward-space scheme

$$\frac{1}{k}\left(v_m^{n+1} - v_m^n\right) + \frac{a}{h}\left(v_{m+1}^{n+1} - v_m^{n+1}\right) = 0$$

used to approximate solutions to $u_t + au_x = 0$ with, say, $x \in [0,1]$ and periodic boundary conditions. Give an example of an initial condition v_m^0 and an explicit expression for v_m^n that demonstrate unstable behavior for a particular λ (your choice) which fails to satisfy the stability condition. Does the growth in your example agree with your theoretical amplification factor?

Solution

Our difference operator is

$$P_{k,h}v_m^n = \frac{1}{k}\left(v_m^{n+1} - v_m^n\right) + \frac{a}{h}\left(v_{m+1}^{n+1} - v_m^{n+1}\right)$$

which has symbol

$$\begin{aligned} p_{k,h}(s,\xi) &= P_{k,h}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi} \\ &= \frac{1}{k}\left(e^{sk} - 1\right) + \frac{a}{h}e^{sk}\left(e^{ih\xi} - 1\right). \end{aligned}$$

We determine stability by finding the roots of the symbol as a function of $g := e^{sk}$, yielding

$$g = \frac{1}{1 + a\lambda\left(e^{i\theta} - 1\right)}$$

where $\lambda := k/h$ and $\theta := h\xi$. We find that

$$|g|^{-2} = 1 + 2a\lambda\left(a\lambda - 1\right)\left(1 - \cos\theta\right),$$

hence the scheme is stable ($|g| \le 1$) if and only if $a \le 0$ or $a\lambda \ge 1$.

If, for example, $a\lambda = \frac{1}{4}$, then

$$|g|^{-2} = 1 - \frac{3}{8}\left(1 - \cos\theta\right) = \frac{5}{8} + \frac{3}{8}\cos\theta.$$

Choosing, for example, $\theta = \pi$ ought to give an amplication factor of exactly $g = 2$ of the pure mode $v_m = e^{i\theta m} = (-1)^m$. Indeed, one can quickly verify that $v_m^n = 2^n(-1)^m$ satisfies the difference equation:

$$\begin{aligned} kP_{k,h}v_m^n &= v_m^{n+1} - v_m^n + a\lambda\left(v_{m+1}^{n+1} - v_m^{n+1}\right) \\ &= 2^{n+1}(-1)^m - 2^n(-1)^m + \frac{1}{4}\left(2^{n+1}(-1)^{m+1} - 2^{n+1}(-1)^m\right) \\ &= 2^n(-1)^m\left(2 - 1 + \frac{1}{4}(-2-2)\right) \\ &= 0. \end{aligned}$$

One final remark: Notice that if $a\lambda = \frac{1}{2}$, $|g|$ is *unbounded* near $\theta = \pi$. This corresponds to a null space in the resulting system of equations for v^{n+1} induced by the difference operator, and this null space is spanned precisely by the mode corresponding to $\theta = \pi$, $v_m = (-1)^m$.

3. Prove that numerical solutions to the Lax-Friedrichs scheme

$$\frac{1}{k}\left(v_m^{n+1}-\frac{1}{2}\left(v_{m+1}^n+v_{m-1}^n\right)\right)+\frac{a}{2h}\left(v_{m+1}^n-v_{m-1}^n\right)=0$$

converge to solutions to the corresponding modified equation

$$u_t+au_x=\frac{h^2}{2k}\left(1-\left(\frac{ak}{h}\right)^2\right)u_{xx}$$

to second order accuracy in L^∞. I.e., show that $\left|v_m^n-u_{k,h}\left(t_n,x_m\right)\right|\to 0$ as $h,k\to 0$ (according to the stability criterion), where the subscripts on $u_{k,h}$ only indicate that the solution to the modified equation is parameterized by k,h.

Solution

The difference operator for the Lax-Friedrichs scheme is

$$P_{k,h}v_m^n=\frac{1}{k}\left(v_m^{n+1}-\frac{1}{2}\left(v_{m+1}^n+v_{m-1}^n\right)\right)+\frac{a}{2h}\left(v_{m+1}^n-v_{m-1}^n\right)$$

which has symbol

$$\begin{aligned}p_{k,h}(s,\xi)&=P_{k,h}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi}\\&=\frac{1}{k}\left(e^{sk}-\cos h\xi\right)+i\frac{a}{h}\sin h\xi\\&=\frac{1}{k}\left(1+sk+\frac{1}{2}s^2k^2-1+\frac{1}{2}h^2\xi^2\right)+i\frac{a}{h}\left(h\xi\right)+O\left(k^2+h^2+h^4k^{-1}\right)\\&=s+ia\xi+\frac{k}{2}s^2+\frac{h^2}{2k}\xi^2+O\left(k^2+h^2+h^4k^{-1}\right).\end{aligned}$$

4. (Strikwerda 4.1.2.) Show that the $(2,2)$ leapfrog scheme for $u_t+au_{xxx}=f$ (see (2.2.15)) given by

$$\frac{v_m^{n+1}-v_m^{n-1}}{2k}+a\delta^2\delta_0v_m^n=f_m^n,$$

with $\nu=k/h^3$ constant, is stable if and only if

$$|a\nu|<\frac{2}{3^{3/2}}.$$

Solution

5. (Strikwerda 3.2.1.) Show that the (forward-backward) MacCormack scheme

$$\begin{aligned}\tilde{v}_m^{n+1}&=v_m^n-a\lambda\left(v_{m+1}^n-v_m^n\right)+kf_m^n,\\v_m^{n+1}&=\frac{1}{2}\left(v_m^n+\tilde{v}_m^{n+1}-a\lambda\left(\tilde{v}_m^{n+1}-\tilde{v}_{m-1}^{n+1}\right)+kf_m^{n+1}\right)\end{aligned}$$

is a second-order accurate scheme for the one-way wave equation (1.1.1). Show that for $f=0$ it is identical to the Lax-Wendroff scheme (3.1.1).

Solution

2 Programming

1. For the one-way wave equation $u_t + au_x = 0$, investigate how close the numerical solution to a finite difference scheme is to the solution to the corresponding modified equation. To be concrete, suppose a convenient initial condition for which you can solve the modified equation explicitly with periodic boundary conditions. Take $a = 1$, $k/h = 0.5$, and final time $T = 0.5$. Compare the following finite difference schemes: upwinding, Lax-Friedrichs, and Lax-Wendroff. Also, include a derivation of the respective corresponding modified equations.