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Analysis Bootcamp, Homework 1

1. *Let $\mathbb{F}$ be an ordered field such that $\mathbb{Q} \subset \mathbb{F}$ (with the same ordering and same field operations) and such that $\mathbb{Q}$ is dense in $\mathbb{F}$. Prove the following properties of $\mathbb{F}$ are equivalent:*

   (a) *Every nonempty subset of $\mathbb{F}$ that is bounded above has a least upper bound in $\mathbb{F}$.*

   (b) *If $A \subset \mathbb{F}$ is such that $A \neq \emptyset$, $\mathbb{F} \backslash A \neq \emptyset$, $x < y \in A \Rightarrow x \in A$, and $x \in A \Rightarrow \exists y \in A, y > x$, then there is a unique $\alpha \in \mathbb{F}$ such that $A = \{x \in \mathbb{F} : x < \alpha\}$.*

   (c) *Every bounded increasing sequence in $\mathbb{F}$ converges to a point in $\mathbb{F}$, where $x_n$ converges to $X$ means that for every $\mathbb{F} \ni \epsilon > 0$ there is $N$ so that $x - \epsilon < x_n < x + \epsilon$ whenever $n > N$.*

   (d) *Every Cauchy sequence in $\mathbb{F}$ converges to a point of $\mathbb{F}$, where the sequence $x_n$ is Cauchy in $\mathbb{F}$ if for every $\mathbb{F} \ni \epsilon > 0$ there is $N$ so that $x_m - \epsilon < x_n < x_m + \epsilon$ whenever $n > N$ and $m > N$.*

   **Solution**

   - *(a) $\Rightarrow$ (b)*
     Let $A$ be as in the hypotheses of *(b)*. $A$ is nonempty (by hypothesis). Further, $\mathbb{F} \backslash A$ is nonempty as well (by hypothesis), hence there exists some $z \in \mathbb{F}$ such that $z \notin A$. $z$ is an upper bound of $A$, for if $x > z$ were such that $x \in A$, then, since $A$ satisfies the condition "$x < y \in A \Rightarrow x \in A$" (by hypothesis), we'd conclude that, in fact, $z \in A$, but we chose $z$ specifically such that $z \notin A$. Thus $A$ satisfies the hypotheses of *(a)*, so has a least upper bound $\alpha \in \mathbb{F}$. Further, $\alpha \notin A$, for, since $A$ satisfies the condition "$x \in A \Rightarrow \exists y \in A, y > x$" (by hypothesis), the inclusion of $\alpha \in A$ would imply the existence of some $x \in A$ but with $x > \alpha$, violating the fact that $\alpha$ is a least upper bound of $A$. This establishes that $A \subset \{x \in \mathbb{F} : x < \alpha\}$. To show the opposite inclusion, suppose $x < \alpha$. There must exist some $y \in A$ such that $x < y < \alpha$, for if such was not the case, $x$ would be an upper bound of $A$, contradicting the fact that $\alpha$ is the least upper bound. But then the existence of such a $y$ and the condition on $A$ that "$x < y \in A \Rightarrow x \in A$" implies that $x \in A$, which establishes the desired opposite inclusion. Therefore, $A = \{x \in \mathbb{F} : x < \alpha\}$.
   - *(b) $\Rightarrow$ (c)*
     Let $\{x_n\}_{n=1}^{\infty}$ be a sequence as in the hypotheses of *(c)*. Let $A_n = \{x \in \mathbb{F} : x < x_n\}$ and set $A = \bigcup_n A_n$. Clearly, $A \neq \emptyset$ as each $A_n \neq \emptyset$. Further, $\{x_n\}$ is bounded (by hypothesis), say, by $z \in \mathbb{F}$, thus each $A_n \subset \{x \in \mathbb{F} : x < z\}$, so $A \subset \{x \in \mathbb{F} : x < z\}$, from which it follows that $\mathbb{F} \backslash A \supset \{x \in \mathbb{F} : x \geq z\} \neq \emptyset$. Further, suppose $y \in A$. Then $y \in A_n$ for some $n$, and since $A_n \supset \{x \in \mathbb{F} : x < y\}$, any $x \in \mathbb{F}$ with $x < y$ will be in $A_n$, hence $A$. This establishes the condition "$x < y \in A \Rightarrow x \in A$". Lastly, suppose $x \in A$. Then, again, $x \in A_n$ for some $n$, so $x < x_n$ and there exists a $y \in \mathbb{F}$ such that $x < y < x_n$. $y \in A_n$, so $y \in A$, establishing the condition "$x \in A \Rightarrow \exists y \in A, y > x$". Thus $A$ satisfies the hypotheses of *(b)*, so there exists a unique $\alpha \in \mathbb{F}$ such that $A = \{x \in \mathbb{F} : x < \alpha\}$.

     Now let $\epsilon \in \mathbb{F}$, $\epsilon > 0$ be given, and set $\alpha_- = \alpha - \epsilon \in A$. Then $\alpha_- \in A_N$ for some $N$. Indeed, since $\{x_n\}$ is an increasing sequence (by hypothesis), $A_m \subset A_{m+1}$, hence $\alpha_- \in A_n$ for all $n \geq N$. Thus $\alpha_- < x_n$ for all $n \geq N$, or, equivalently, $\alpha - \epsilon < x_n$ for all $n \geq N$. Clearly, $x_n < \alpha < \alpha + \epsilon$ for all $n$, showing that $x_n \to \alpha \in \mathbb{F}$.
   - *(c) $\Rightarrow$ (d)*
     Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence as in the hypotheses of *(d)*. Let $\epsilon_i = 2^{-i}$ for integral $i \geq 1$. Now since $\{x_n\}$ is Cauchy, for each $i$, there exists an $N_i$ such that for $n, m \geq N_i$, $|x_n - x_m| < \epsilon_i$. Further, we can choose $\{N_i\}_{i=1}^{\infty}$ to be an increasing sequence of integers. It follows that, for each $i$, $x_{N_i} - \epsilon_i < x_n < x_{N_i} + \epsilon_i$ whenever $n \geq N_i$. Set $y_i = x_{N_i} - \sum_{j=i}^{\infty} \epsilon_j$. Note that since $\{N_i\}$ is an increasing sequence, $N_{i+1} > N_i$, so

$$x_{N_i} - \epsilon_i < x_{N_{i+1}},$$

hence

$$y_i = x_{N_i} - \epsilon_i - \sum_{j=i+1}^{\infty} \epsilon_j < x_{N_{i+1}} - \sum_{j=i+1}^{\infty} \epsilon_j = y_{i+1},$$

showing that $\{y_i\}_{i=1}^{\infty}$ is an increasing sequence. Further,

$$y_i = x_{N_i} - \sum_{j=i}^{\infty} \epsilon_j < x_{N_i} < x_{N_1} + \epsilon_1,$$

showing that $\{y_i\}$ is bounded. Thus $\{y_i\}$ satisfies the hypotheses of *(c)*, so $y_i \to y$ for some $y \in \mathbb{F}$. Now let $\epsilon \in \mathbb{F}$, $\epsilon > 0$ be given. Choose $I_1$ such that $\epsilon_i < \epsilon$ for $i \geq I_1$ (possible since $\epsilon_i = 2^{-i} \to 0$). Further, by the conclusions of *(c)* above, there exists an $I_2$ such that $|y_i - y| < \epsilon$ for $i \geq I_2$. Set $I = \max\{I_1, I_2\}$. Now

$$\sum_{j=i}^{\infty} \epsilon_j = \sum_{j=i}^{\infty} 2^{-j} = 2^{1-i} = 2\epsilon_i < 2\epsilon$$

for $i \geq I$, so

$$\epsilon > |y_i - y| = \left| x_{N_i} - \sum_{j=i}^{\infty} \epsilon_j - y \right| > |x_{N_i} - y| - \left| \sum_{j=i}^{\infty} \epsilon_j \right| > |x_{N_i} - y| - 2\epsilon,$$

for $i \geq I$, hence

$$|x_{N_i} - y| < 3\epsilon,$$

for $i \geq I$. In particular, choose $i = I$. Then for $n > N_I$, we know that

$$|x_n - y| < |x_n - x_{N_I}| + |x_{N_I} - y| < \epsilon_I + 3\epsilon < 4\epsilon,$$

proving that $x_n \to y \in \mathbb{F}$.

- *(d)* $\Rightarrow$ *(a)*
  Let $A \subset \mathbb{F}$ satisfy the hypotheses of *(a)*. Construct two sequences, $\{x_n\}$ and $\{z_m\}$ (finite or infinite) as follows. $A$ is nonempty (by hypothesis), so there exists, say, $x_1 \in A$. Further, $A$ is bounded (by hypothesis), so there exists, say, $z_1 \in \mathbb{F}$ such that $x < z_1$ for all $x \in A$. Set $n_1 = 1$, $m_1 = 1$, and $y_1 = \frac{1}{2}(x_1 + z_1)$. Now if $y_1$ is an upper bound of $A$, set $n_2 = n_1$, $m_2 = m_1 + 1$, $z_2 = y_1$, and $y_2 = \frac{1}{2}(x_1 + z_2)$. On the other hand, if $y_1$ is not an upper bound of $A$, set $n_2 = n_1 + 1$, $m_2 = m_1$, $x_2 = y_1$, and $y_2 = \frac{1}{2}(x_2 + z_1)$. Continue the previous step on $y_2$, obtaining $y_3$, and so forth. Explicitly, given $y_i$,
  - if $y_i$ is an upper bound of $A$, set $n_{i+1} = n_i$, $m_{i+1} = m_i + 1$, $z_{m_{i+1}} = y_i$;
  - if $y_i$ is not an upper bound of $A$, set $n_{i+1} = n_i + 1$, $m_{i+1} = m_i$, $x_{n_{i+1}} = y_i$;

  and finally set $y_{i+1} = \frac{1}{2}\left(x_{n_{i+1}} + z_{m_{i+1}}\right)$.
  Suppose first that both of $\{x_n\}$ and $\{z_m\}$ are infinite; that is, $n_i \to \infty$ and $m_i \to \infty$. We make the following observations. For each $i$, $x_{n_i} < y_i < z_{m_i}$, which can easily be proved inductively. Indeed, each $z_m$ is an upper bound of $A$ while each $x_n$ is not, so, in fact, $x_n < z_m$ for all pairs $n, m$. Further, $\{z_{m_i}\}$ is a decreasing sequence while $\{x_{n_i}\}$ is an increasing sequence. Also, one can show that

  $$z_{m_i} - x_{n_i} = \frac{1}{2}\left(z_{m_{i-1}} - x_{n_{i-1}}\right),$$

  hence

  $$z_{m_i} - x_{n_i} = 2^{-i+1}\left(z_{m_1} - x_{n_1}\right) = 2^{-i}d$$

for $d = 2(z_{m_1} - x_{n_1})$. Now given $\epsilon \in \mathbb{F}$, $\epsilon > 0$, choose $I$ large enough such that $2^{-i}d < \epsilon$ for $i \geq I$. Then for any $j, k \geq I$,

$$\left|z_{m_j} - z_{m_k}\right| < \left(z_{m_I} - z_{m_j}\right) + \left(z_{m_I} - z_{m_k}\right) < \left(z_{m_I} - x_{n_I}\right) + \left(z_{m_I} - x_{n_I}\right) = 2\left(2^{-i}d\right) < 2\epsilon.$$

Now since both $\{n_i\}$ and $\{m_i\}$ are nondecreasing sequences, and, as sets, $\{n_i\} = \{m_i\} = \mathbb{N}$, we have that

$$|z_r - z_s| < 2\epsilon$$

for $r, s \geq m_I$ (since $r = m_j$ for some $j \geq I$, and similarly for $s$), establishing that $\{z_m\}$ is a Cauchy sequence in $\mathbb{F}$. By *(d)*, then, $\{z_m\}$ converges to a point $\alpha \in \mathbb{F}$.

A similar argument shows that $\{x_n\}$ is a Cauchy sequence, too, hence converges as well. More importantly, though, $x_n \to \alpha$ as well, for take $\epsilon \in \mathbb{F}$, $\epsilon > 0$. We can choose $I$ large enough such that $z_{m_i} - x_{n_i} = 2^{-i}d < \epsilon$ for $i \geq I$. Since $\alpha < z_m$, we see that $|\alpha - x_n| = \alpha - x_n < \epsilon$ for $n \geq n_I$.

We show first that $\alpha$ is an upper bound of $A$. For suppose some $y \in A$ was such that $y > \alpha$; we can can choose $m$ large enough such that $\alpha < z_m < y$, contradicting the fact that $z_m$ is an upper bound for $A$. Thus $\alpha$ is an upper bound of $A$. To show that $\alpha$ is the least upper bound, suppose $y < \alpha$ was also an upper bound for $A$. We can choose $n$ large enough such that $y < x_n < \alpha$, contradicting the fact that $x_n$ is not an upper bound of $A$. This establishes that $\alpha$ is the least upper bound of $A$.

We now address the case when one of $\{x_n\}$ or $\{z_m\}$ are finite. Suppose that $\{x_n\}$ was finite, and set $N = \max\{n_i\}$. The observations above still apply. In particular, $\{z_m\}$ is still a Cauchy sequence which converges to some $\alpha \in \mathbb{F}$, and $\alpha$ is still an upper bound for $A$. Additionally, $\alpha$ must be the least upper bound, for suppose $y < \alpha$ was also an upper bound for $A$. We can choose $m$ large enough such that $z_m - x_N < \alpha - y$, implying that $x_N > y$ (since $z_m > \alpha$), contradicting the fact that $x_N$ is not an upper bound of $A$.

The case for $\{z_m\}$ finite is similar. Set $M = \max\{m_i\}$. $\{x_n\}$ is still a Cauchy sequence which converges to $\alpha \in \mathbb{F}$. Again, $\alpha$ is an upper bound for $A$, for suppose some $y \in A$ was such that $y > \alpha$. We can choose $n$ large enough such that $z_M - x_n < y - \alpha$, implying that $z_M < y$ (since $x_n < \alpha$), contradicting the fact that $z_M$ is an upper bound of $A$.

Therefore, in call cases, $\alpha$ is the least upper bound of $A$

2. *Gamelin and Greene, page 8. 4, 10, 11, 12.*

   4. *Show that the semiopen interval* $(0, 1]$ *is neither open nor closed in* $\mathbb{R}$.

      **Solution**

      $1 \in (0, 1]$ is not an interior point of $(0, 1]$ (since $B(1; r) \cap (0, 1] = (1, 1 + r) \neq \emptyset$ for any $r > 0$), while $0 \notin (0, 1]$ is adherent to $(0, 1]$ (since $B(0; r) \cap (0, 1] = (0, r) \neq \emptyset$ for any $r > 0$), so $(0, 1]$ is neither open nor closed in $\mathbb{R}$.

   10. *A point* $x \in X$ *is a* limit point *of a subset* $S$ *of* $X$ *if every ball* $B(x; r)$ *contains infinitely many points of* $S$. *Show that* $x$ *is a limit point of* $S$ *if and only if there is a sequence* $\{x_j\}_{j=1}^{\infty}$ *in* $S$ *such that* $x_j \to x$ *and* $x_j \neq x$ *for all* $j$. *Show that the set of limit points of* $S$ *is closed.*

       **Solution**

       Let $r_j = \frac{1}{j}$ for $j \geq 1$. Set $x_j$ to be some point of $B(x; r_j) \cap S \backslash \{x\}$ (possible since this set is given to contain infinitely many points). Then given any $\epsilon > 0$, there exists a $J$ such that $r_j < \epsilon$ for $j > J$, hence $d(x_j, x) < \epsilon$ for $j > J$ (since $x_j \in B(x; r_j)$), showing that $x_j \to x$.

       Denote the set of limit points of $S$ by $S'$. Let $x$ be adherent to $S'$, and choose $r > 0$. $x$ adherent to $S'$ implies that $B(x; r) \cap S' \neq \emptyset$, hence there exists some $y \in B(x; r) \cap S'$. Let $h = r - d(x, y)$. $y$ is a limit point of $S$, hence $B(y; h)$ contains infinitely many points of $S$. Since $B(y; h) \subset B(x; r)$, $B(x; r)$ also contains infinitely many points of $S$. As $r$ was arbitrary, we conclude that $x$ is a limit point of $S$ as as well, i.e., $x \in S'$. Therefore $S'$ is closed.

11. *A point $x \in S$ is an* isolated point *of $S$ if there exists $r > 0$ such that $B(x;r) \cap S = \{x\}$. Show that the closure of a subset $S$ of $X$ is the disjoint union of the limit points of $S$ and the isolated points of $S$.*

    **Solution**

    First note that the properties of being a limit point and of being an isolated point are mutually exclusive (that is, a point cannot be both a limit point and an isolated point). Thus, we aim to show that every adherent point of $S \subset X$ is either a limit point of $S$ or an isolated point of $S$. Indeed, suppose $x \in X$ was adherent to $S$, i.e., $B(x;r) \cap S \neq \emptyset$ for all $r > 0$. Either $x$ is a limit point of $S$ (in which case we're done), or there exists some $r$ such that $B(x;r)$ contains finitely many points of $S$. Set $h = \min_{y \in B(x;r) \cap S \setminus \{x\}} d(x,y)$, or $h = r$ in the case that $B(x;r) \cap S \setminus \{x\} = \emptyset$. Note that $h > 0$ and $B(x;h) \cap S$ has no points in common with $B(x;r) \cap S \setminus \{x\}$. Yet $B(x;h) \subset B(x;r)$, and $B(x;h) \cap S \neq \emptyset$ (since $x$ is adherent to $S$), so we must have that $B(x;h) = \{x\}$, showing that $x$ is, in fact, an isolated point of $S$.

12. *Two metrics on $X$ are* equivalent *if they determine the same open subsets. Show that two metrics $d, \rho$ on $X$ are* equivalent *if and only if the convergent sequences in $(X,d)$ are the same as the convergent sequences in $(X,\rho)$.*

    **Solution**

    Suppose $d, \rho$ are equivalent metrics on $X$. Let $\{x_n\}_{n=1}^{\infty}$ be a convergent sequence in $(X,d)$, and suppose it converges to $x \in X$. Let $\epsilon > 0$ be given. $B_\rho(x,\epsilon) = \{y \in X : \rho(y,x) < \epsilon\}$ is an open subset of $(X,\rho)$, hence is also an open subset of $(X,d)$ (since $d, \rho$ are equivalent on $X$), and in particular, contains $x$. Thus, there exists some $\delta > 0$ such that $B_d(x,\delta) \subset B_\rho(x,\epsilon)$, and correspondingly some $N$ such that $x_n \in B_d(x,\delta)$ for $n > N$. It follows that $x_n \in B_\rho(x,\epsilon)$ for $n > N$, i.e., $\rho(x_n,x) < \epsilon$ for $n > N$, hence $\{x_n\}$ converges to $x$ in $(X,\rho)$. Similarly, any convergent sequence in $(X,\rho)$ can be shown to be a convergent sequence in $(X,d)$.

    Now suppose the convergent sequences in $(X,d)$ are the same as in $(X,\rho)$, and let $Y$ be an open subset of $(X,d)$. Suppose, for the sake of contradiction, that $Y$ is not an open subset of $(X,\rho)$, that is, there exists some $x \in Y$ such that $B_\rho(x,r) \not\subset Y$ for all $r > 0$. Let $r_n = \frac{1}{n}$ for $n \geq 1$, and choose $x_n \in B_\rho(x,r_n) \setminus Y \neq \emptyset$. Then $\{x_n\}_{n=1}^{\infty}$ converges to $x$ in $(X,\rho)$, hence also converges to $x$ in $(X,d)$ (by hypothesis). But this means that for each $r > 0$, there exists some $N$ such that $x_n \in B_d(x,r)$ for $n > N$, and each $x_n \notin Y$, contradicting the fact that $x$ is an interior point of $Y$ with respect to $d$ (since $x \in Y$ and $Y$ is open in $(X,d)$). It follows that $Y$ must also be an open subset of $(X,\rho)$.

3. *Gamelin and Greene, page 12. 3, 5, 7, 8.*

    3. *Prove that the set of isolated points of a countable complete metric space $X$ forms a dense subset of $X$.*

        **Solution**

        Let $T$ be the set of isolated points of $X$. Now suppose, for the sake of contradiction, that $T$ is not dense in $X$. Then there exists some $x \in X$ and $r > 0$ such that $B(x;2r)$ contains no points of $T$. Then $Y = \overline{B(x;r)} \subset B(x;2r)$ contains no points of $T$ as well. $Y$ is a closed subspace of $X$, which is complete, hence $Y$ is complete as well, by Theorem 2.3, and since $Y \subset X$, $Y$ is at most countable.

        Further, $Y$ contains no isolated points with respect to $Y$. For suppose there existed some $y \in Y$ and $h > 0$ such that $B(y;h) \cap Y = \{y\}$. If $d(x,y) < r$, then $y$ is an interior point of $Y$, i.e., $B(y;r - d(x,y)) \subset Y$. Set $k = \min\{h, r - d(x,y)\}$. Then $B(y;k) \subset Y$, so $\{y\} \subset B(y;k) = B(y;k) \cap Y \subset B(y;h) \cap Y = \{y\}$, hence $B(y;k) = \{y\}$ and $y$ would also be an isolated point with respect to $X$. But this contradicts the construction of $Y$ containing no isolated points of $X$. On the other hand, suppose $d(x,y) = r$. Then $y \notin B(x;r)$, and since $B(y;h) \cap B(x;r) \subset B(y;h) \cap Y = \{y\}$, we must conclude that $B(y;h) \cap B(x;r) = \emptyset$. But this implies that $y$ is not adherent to $B(x;r)$, contradicting the fact that $y \in Y = \overline{B(x;r)}$. Clearly, $d(x,y) \le r$ if $y \in Y$, so $Y$ cannot contain any isolated points with respect to $Y$.

For $y \in Y$, set $U_y = Y \backslash \{y\}$. As no $y \in Y$ is an isolated point of $Y$, each $U_y$ is dense in $Y$. Further, each $z \in U_y$ is an interior point of $U_y$ with respect to $Y$, since $B(z; d(z,y)) \cap Y \subset U_y$, hence each $U_y$ is open with respect to $Y$. As established earlier, $Y$ is countable, hence $\{U_y\}$ can be arranged into a sequence, and $Y$ is complete, so it follows from Theorem 2.6 (Baire Category Theorem) that $\bigcap_{y \in Y} U_y$ is dense in $Y$. But $\bigcap_{y \in Y} U_y = \emptyset$, so this implies that $Y = \overline{\emptyset} = \emptyset$, which contradicts the fact that $x \in Y$, by construction. This proves that, indeed, $T$ must be dense in $X$.

(Note: Easier to take each $U_y = Y \backslash \{y\}$ for $y \in Y \backslash T$.)

5. *Prove that any countable union of sets of the first category in $X$ is again of the first category in $X$.*

**Solution**

A subset of the first category of a metric space $X$ is a countable union of nowhere dense subsets, hence a countable union of subsets of the first category is a countable union of countable unions of nowhere dense subsets, hence is a countable union of nowhere dense subsets (a countable union of countable unions is again a countable union), hence is again of the first category.

7. *Let $(X,d)$ be a metric space and let $S$ be the set of Cauchy sequences in $S$. Define a relation "$\sim$" in $X$ by declaring "$\{s_k\} \sim \{t_k\}$" to mean that $d(s_k, t_k) \to 0$ as $k \to \infty$.*

   (a) *Show that the relation "$\sim$" is an equivalence relation.*

   (b) *Let $\tilde{X}$ denote the set of equivalence classes of $S$ and let $\tilde{s}$ denote the equivalence class of $s = \{s_k\}_{k=1}^{\infty}$. Show that the function*

$$\rho(\tilde{s}, \tilde{t}) = \lim_{k \to \infty} d(s_k, t_k), \quad \tilde{s}, \tilde{t} \in \tilde{X}$$

   *defines a metric on $\tilde{X}$.*

   (c) *Show that $(\tilde{X}, \rho)$ is complete.*

   (d) *For $x \in X$, define $\tilde{x}$ to be the equivalence class of the constant sequence $\{x, x, \ldots\}$. Show that the function $x \to \tilde{x}$ is an isometry of $X$ onto a dense subset of $\tilde{X}$. (By an isometry, we mean that $d(x,y) = \rho(\tilde{x}, \tilde{y})$, $x, y \in X$.)*

   Note: *If a complete metric space $Y$ contains $X$ as a dense subspace, we say that $Y$ is a* completion *of $X$. The space $\tilde{X}$ of Exercise 7 can be regarded as a completion of $X$ by identifying each $x \in X$ with the constant sequence $\{x, x, \ldots\}$. The next part of the exercise shows that the completion of $X$ is unique, up to isometry.*

   (e) *Show that when $Y$ is a completion of $X$, then the inclusion map $X \to Y$ extends to an isometry of $\tilde{X}$ onto $Y$.*

**Solution**

(a) Let $\{r_k\}_{k=1}^{\infty}$, $\{s_k\}_{k=1}^{\infty}$, and $\{t_k\}_{k=1}^{\infty}$ be sequences in $X$.
$d(r_k, r_k) = 0$ for all $k$, hence $d(r_k, r_k) \to 0$ as $k \to \infty$, establishing that $\{r_k\} \sim \{r_k\}$ and "$\sim$" is reflexive.
$d(r_k, s_k) = d(s_k, r_k)$ for all $k$, hence $d(r_k, s_k) \to 0 \Leftrightarrow d(s_k, r_k) \to 0$ (as $k \to \infty$), establishing that $\{r_k\} \sim \{s_k\} \Leftrightarrow \{s_k\} \sim \{r_k\}$ and "$\sim$" is symmetric.
$d(r_k, t_k) \le d(r_k, s_k) + d(s_k, t_k)$ for all $k$, hence $d(r_k, s_k) \to 0$ and $d(s_k, t_k) \to 0$ implies that $d(r_k, t_k) \to 0$ (as $k \to \infty$), establishing that $\{r_k\} \sim \{s_k\}, \{s_k\} \sim \{t_k\} \Rightarrow \{r_k\} \sim \{t_k\}$ and "$\sim$" is transitive.

(b) Let $\tilde{r}, \tilde{s}, \tilde{t} \in \tilde{X}$, $r = \{r_k\}_{k=1}^{\infty}$, $s = \{s_k\}_{k=1}^{\infty} \in \tilde{s}$, and $t = \{t_k\}_{k=1}^{\infty} \in \tilde{t}$.
Let $s' = \{s'_k\}_{k=1}^{\infty} \in \tilde{s}$ and $t' = \{t'_k\}_{k=1}^{\infty} \in \tilde{t}$. Now

$$d(s_k, t_k) \le d(s_k, s'_k) + d(s'_k, t'_k) + d(t'_k, t_k),$$

and since $d(s_k, s'_k) \to 0$ and $d(t'_k, t_k) \to 0$ (as $k \to \infty$), we obtain

$$\lim_{k\to\infty} d(s_k, t_k) \leq \lim_{k\to\infty} d(s'_k, t'_k).$$

The above argument is completely symmetrical, which allows us to obtain the reverse inequality and conclude that

$$\lim_{k\to\infty} d(s_k, t_k) = \lim_{k\to\infty} d(s'_k, t'_k)$$

and $\rho$ is well-defined.
$d(s_k, t_k) \geq 0$, so it follows that $\rho(\tilde{s}, \tilde{t}) = \lim_{k\to\infty} d(s_k, t_k) \geq 0$ as well.
$\tilde{s} = \tilde{t}$ if and only if $\{s_k\} \sim \{t_k\}$ if and only if $\rho(\tilde{s}, \tilde{t}) = \lim_{k\to\infty} d(s_k, t_k) = 0$.
$d(s_k, t_k) = d(t_k, s_k)$, hence $\rho(\tilde{s}, \tilde{t}) = \lim_{k\to\infty} d(s_k, t_k) = \lim_{k\to\infty} d(t_k, s_k) = \rho(\tilde{t}, \tilde{s})$.

$$\begin{aligned}\rho(\tilde{r}, \tilde{t}) &= \lim_{k\to\infty} d(r_k, t_k) \leq \lim_{k\to\infty} (d(r_k, s_k) + d(s_k, t_k)) \\ &= \lim_{k\to\infty} d(r_k, s_k) + \lim_{k\to\infty} d(s_k, t_k) = \rho(\tilde{r}, \tilde{s}) + \rho(\tilde{s}, \tilde{t}).\end{aligned}$$

(c) Let $\{\tilde{s}_n\}_{n=1}^\infty$ be a Cauchy sequence in $\tilde{X}$. Let $\{s_n^k\}_{k=1}^\infty \in \tilde{s}_n$ for each $n$. Now since each $\{s_n^k\} \in S$, there exists a $K_n$ such that $d(s_n^i, s_n^j) < \frac{1}{n}$ for $i, j \geq K_n$. Set $s = \left\{s_k^{K_k}\right\}_{k=1}^\infty$.
Let $\epsilon > 0$ be given. Now since $\{\tilde{s}_n\}$ is a Cauchy sequence, there exists an $N$ such that $\rho(\tilde{s}_n, \tilde{s}_m) < \epsilon$ for $n, m > N$. Set $M = \max\left\{N, \left\lceil\frac{1}{\epsilon}\right\rceil\right\}$, so that $\frac{1}{M} < \epsilon$. Now, for the moment, fix $n, m > M$. Since $\epsilon > \rho(\tilde{s}_n, \tilde{s}_m) = \lim_{k\to\infty} d(s_n^k, s_m^k)$, there exists a $K$ such that $d(s_n^k, s_m^k) < \epsilon$ for $k \geq K$. Assume, without loss of generality, that $K_n \leq K_m$, and set $K'_m = \max\{K_m, K\}$. Then

$$d\left(s_n^{K_n}, s_m^{K_m}\right) \leq d\left(s_n^{K_n}, s_n^{K'_m}\right) + d\left(s_n^{K'_m}, s_m^{K'_m}\right) + d\left(s_m^{K'_m}, s_m^{K_m}\right) < \frac{1}{n} + \epsilon + \frac{1}{m} < 3\epsilon.$$

Since the only requirements of $n, m$ are that $n, m > M$, this establishes that $\tilde{s} \in \tilde{X}$ (that is, $s$ is a Cauchy sequence). Further, fix $n > M$. Then for $k > \max\{K_n, M\}$,

$$d\left(s_n^k, s_k^{K_k}\right) \leq d\left(s_n^k, s_n^{K_n}\right) + d\left(s_n^{K_n}, s_k^{K_k}\right) < \frac{1}{n} + 3\epsilon < 4\epsilon,$$

hence $\rho(\tilde{s}_n, \tilde{s}) = \lim_{k\to\infty} d\left(s_n^k, s_k^{K_k}\right) < 4\epsilon$. Since the only requirement on $n$ is that $n > M$, this establishes that $\tilde{s}_n \to \tilde{s} \in \tilde{X}$, i.e., that $\{\tilde{s}_n\}$ converges. Therefore, $(\tilde{X}, \rho)$ is complete.

(d) Let $x, y \in X$, $x_k = x$ and $y_k = y$ for $k \geq 1$, $\tilde{x} = \{x_k\}_{k=1}^\infty$, and $\tilde{y} = \{y_k\}_{k=1}^\infty$.
$d(x_k, y_k) = d(x, y)$, hence $\rho(\tilde{x}, \tilde{y}) = \lim_{k\to\infty} d(x_k, y_k) = d(x, y)$, showing that $x \to \tilde{x}$ is an isometry of $(X, d)$ onto $(\tilde{X}, \rho)$.
Let $\tilde{s} \in \tilde{X}$, $\{s_k\}_{k=1}^\infty \in \tilde{s}$, and $r > 0$. Since $\{s_k\}$ is a Cauchy sequence, there exists a $K$ such that $d(s_i, s_j) < r$ for $i, j \geq K$. Set $\tilde{x} = \{x, x, \ldots\}$ for $x = s_K$. Then $\rho(\tilde{s}, \tilde{x}) = \lim_{k\to\infty} d(s_k, x) < r$, hence $\tilde{x} \in B_\rho(\tilde{s}, r)$. Since $\tilde{x}$ is in the image of $X$ of the mapping $x \to \tilde{x}$, this shows that the said image is dense in $\tilde{X}$.

(e) Define $f : \tilde{X} \to Y$ by $f\left(\tilde{s} = \{s_k\}_{k=1}^\infty\right) = \lim_{k\to\infty} s_k$, where each $s_k$ is mapped by the inclusion $X \to Y$.

8. *The* diameter *of a nonempty subset $E$ of a metric space $(X, d)$ is defined to be*

$$\operatorname{diam}(E) = \sup\{d(x, y) : x, y \in E\}.$$

*Show that if $\{E_k\}_{k=1}^\infty$ is a decreasing sequence of closed nonempty subsets of a complete metric space whose diameters tend to zero, then $\bigcap_{k=1}^\infty E_k$ consists of precisely one point. How much* *of the conclusion remains true if $X$ is not complete? Can this property be used to characterize complete metric spaces? Justify your answer.*

**Solution**

Suppose $x \in \bigcap_{k=1}^{\infty} E_k$. Then for any $y \in X$, there exists some $K$ such that $\operatorname{diam}(E_k) < d(x, y)$ for $k > K$, hence $y \notin E_k$ for $k > K$ (since $x \in E_k$), hence $y \notin \bigcap E_k$. This shows that $\bigcap E_k$ consists of at most one point.

Choose $x_k \in E_k \cap (X \backslash E_{k+1})$ (possible since each $E_k \neq \emptyset$). $\{x_k\}$ is a Cauchy sequence, since, given some $\epsilon > 0$, there exists a $K > 0$ such that $\operatorname{diam}(E_k) < \epsilon$ for $k \geq K$, hence $d(x_i, x_j) < \epsilon$ for $i, j \geq K$ (since $x_i, x_j \in E_K$ for $i, j \geq K$). Thus, $x_k \to x$ for some $x \in X$, since $X$ is complete. Fix $k$ and $r > 0$, and notice that there exists some $I$ such that $d(x_i, x) < r$ for $i \geq I$. Since $x_i \in E_k$ for $i \geq \max\{I, k\}$, and $r$ was arbitrary, that $x$ is adherent to $E_k$, hence $x \in \overline{E_k} = E_k$. Since $k$ was arbitrary, we conclude that $x \in \bigcap E_k$, and combined with the previous argument, in fact $\{x\} = \bigcap E_k$.

If $X$ is not complete, the first argument remains valid since it did not utilize the fact that $X$ was complete, so all we can conclude is that $\bigcap E_k$ consists of at most one point. Indeed, if we take $E_k = \left(0, \frac{1}{k}\right]$ within the metric space $X = (0, 1]$, then $\bigcap E_k = \emptyset$.

Suppose a metric space $X$ was such that all such decreasing sequences of nonempty closed subsets whose diameters tended to 0 had precisely one point in their intersection. Let $\{x_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $X$, set $E = \overline{\bigcup_{k=1}^{\infty} \{x_k\}}$, and set $E_k = E \backslash \bigcup_{i=1}^{k} \{x_i\}$. Each $E_k$ is closed (the removal of a finite number of points from a closed set is still closed), $E_k \supset E_{k+1}$, and $\operatorname{diam}(E_k) \to 0$, hence, by hypothesis, $\bigcap E_k = \{x\}$ for some $x \in X$. But $x$ satisfies the conditions of being a convergent point of $\{x_k\}$, and we conclude that $X$ is complete.

4. *Gamelin and Greene, page 15. 4, 6, 9.*

    4. *Prove that every open subset of $\mathbb{R}$ is a union of disjoint open intervals (finite, semi-infinite, or infinite).*

       **Solution**

       Let $Y \subset \mathbb{R}$ be open, and for $x \in Y$, set $I_x$ be the union of all open intervals containing $x$ and contained by $Y$. $I_x$ is then open as well (by Theorem 1.3). Further, given $x, y \in Y$, suppose some $z \in I_x \cap I_y$. Then some open interval containing $x, z$ is contained in $Y$, and some open interval containing $y, z$ is contained in $Y$, so it follows that some open interval containing $x, y$ is contained in $Y$. Thus any open interval containing $x$ and contained by $Y$ can be extended to an open interval containing $x, y$ and contained by $Y$, and similarly for $y$, hence $I_x = I_y$. Thus each pair of $I_x$'s is either disjoint or identical, and their union is all of $Y$.

    6. *Prove that the set of rational numbers cannot be expressed as the intersection of a sequence of open subsets of $\mathbb{R}$.*

       **Solution**

       Suppose that there exists some sequence $\{E_n\}_{n=1}^{\infty}$ of open subsets of $\mathbb{R}$ such that $\bigcap_{n=1}^{\infty} E_n = \mathbb{Q}$. Then $(\bigcap_{n=1}^{\infty} E_n) \cap \left(\bigcap_{r \in \mathbb{Q}} \mathbb{R} \backslash r\right) = \emptyset$, which is not dense in $\mathbb{R}$. Each of the sets in the intersection is open, hence, by Theorem 2.6 (Baire Category Theorem), one of them is not dense in $\mathbb{R}$. Each $\mathbb{R} \backslash r$ is dense in $\mathbb{R}$ for $r \in \mathbb{Q}$, hence one of the $E_n$'s is not dense in $\mathbb{R}$, i.e., there exists an open interval $I \subset \mathbb{R}$ not in one of the $E_n$'s. But then $\bigcap_n E_n$ would not contain $I$ either, and this contradicts the fact that $\bigcap_n E_n = \mathbb{Q}$ is dense in $\mathbb{R}$. Therefore, the set of rational numbers cannot be expressed as the intersection of a sequence of open subsets of $\mathbb{R}$.

    9. *Determine the interior, the closure, the limit points, and the isolated points of each of the following subsets of $\mathbb{R}$:*

       *(a) the interval* $[0, 1)$,

       *(b) the set of rational numbers,*

*(c)* $\{m + n\pi : m \text{ and } n \text{ positive integers}\}$,
*(d)* $\left\{\frac{1}{m} + \frac{1}{n} : m \text{ and } n \text{ positive integers}\right\}$.

**Solution**

(a)
- $\operatorname{int}([0,1)) = (0,1)$
- $\overline{[0,1)} = [0,1]$
- limit points of $[0,1) = [0,1]$
- isolated points of $[0,1) = \emptyset$

(b)
- $\operatorname{int}(\mathbb{Q}) = \emptyset$
- $\overline{\mathbb{Q}} = \mathbb{R}$
- limit points of $\mathbb{Q} = \mathbb{R}$
- isolated points of $\mathbb{Q} = \emptyset$

(c)
- $\operatorname{int}(\{m + n\pi\}) = \emptyset$
- $\overline{\{m + n\pi\}} = \{m + n\pi\}$
- limit points of $\{m + n\pi\} = \emptyset$
- isolated points of $\{m + n\pi\} = \{m + n\pi\}$

(d)
- $\operatorname{int}\left(\left\{\frac{1}{m} + \frac{1}{n}\right\}\right) = \emptyset$
- $\overline{\left\{\frac{1}{m} + \frac{1}{n}\right\}} = \left\{\frac{1}{m} + \frac{1}{n}\right\} \cup \{0\}$
- limit points of $\left\{\frac{1}{m} + \frac{1}{n}\right\} = \left\{\frac{1}{n}\right\} \cup \{0\}$
- isolated points of $\left\{\frac{1}{m} + \frac{1}{n}\right\} = \left\{\frac{1}{m} + \frac{1}{n}\right\} \setminus \left\{\frac{1}{n}\right\}$