Math 269B, 2012 Winter, Final

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1 Theory

1. Suppose $u\colon [0,\infty) \times \mathbb{R} \to \mathbb{R}$ satisfies the inviscid Burger's equation,
$$0 = u_t + \frac{1}{2}\left(u^2\right)_x = u_t + uu_x, \quad u(0,x) = u_0(x). \tag{1}$$
Use the method of characteristics to show that u must then satisfy the implicit relation
$$u(t,x) = u_0\left(x - tu(t,x)\right). \tag{2}$$
[Hint: Begin by defining $\tilde{u}(t,X) := u\left(t,\varphi(t,X)\right)$ for some to-be-determined change of variables $\varphi\colon [0,\infty) \times \{X\} \to \{x\}$, and choose φ such that $\tilde{u}_t \equiv 0$.]

2. Suppose u_0' in (1) is bounded below, i.e., $u_0' \geq c$ for some constant c. Determine the maximal T such that a solution to (2) is guaranteed to exist for $t \in [0,T)$ (possibly with $T = \infty$). [Hint: Determine when one can guarantee that the function $u \mapsto u - u_0(x - tu)$ has a root.)

3. Solve (1) for $u_0(x) = ax + b$, where a, b are constants. [Hint: Use (2).]

4. Denote the solution to (1) by $u = F\left[u_0\right]$. Express $F\left[x \mapsto au_0(x) + b\right]$ in terms of $F\left[u_0\right]$.

   In other words, given u satisfying (1) for some u_0, determine the solution v (in terms of the aforementioned u) to
$$v_t + vv_x = 0, \quad v(0,x) = v_0(x) := au_0(x) + b.$$

5. Suppose u_0 is given as
$$u_0(x) := \begin{cases} u_0^L(x) := a_L x + b_L, & x < 0 \\ u_0^R(x) := a_R x + b_R, & x > 0 \end{cases}.$$
Determine the path $t \mapsto (t, x_S(t))$ of the (physically correct) shock in the solution u to (1) eminating from $(t,x) = (0,0)$. You may use the fact that
$$\frac{1}{2}\int \frac{b_L + b_R + \left(a_L b_R + a_R b_L\right)t}{\left(\left(1 + a_L t\right)\left(1 + a_R t\right)\right)^{3/2}} dt = \frac{\left(a_L b_R - a_R b_L\right)t + \left(b_R - b_L\right)}{\left(a_R - a_L\right)\sqrt{\left(1 + a_L t\right)\left(1 + a_R t\right)}} \quad \left[a_L \neq a_R\right].$$
[Hint: Recall that the shock speed $x_S'(t) = \frac{1}{2}\left(u^L + u^R\right)(t, x_S(t))$, thus allowing you to set up an ordinary differential equation for x_S.] Consider and explain the physical significance of the special cases $a_L = a_R$ and $b_L = b_R$.

6. Solve the weak form of (1) (i.e., give the entropy solution with rarefaction, and with any shocks propagating at the physically correct speed) on the *periodic* domain $[0,4]$ with the "pulse" initial condition
$$u_0(x) := \begin{cases} 0, & 0 \leq x < 1 \\ 2, & 1 < x < 2 \\ 0, & 2 < x \leq 4 \end{cases}. \tag{3}$$

Identify key points in time t when the character of the solution changes. (It will be natural to express the solution $u(t,x)$ piecewise with respect to x *and* t.) Confirm that $\int u(t,x)dx$ is conserved (i.e., $\int u(t,x)dx = \text{constant for all } t$), and determine $\lim_{t\to\infty} u(t,x)$.

7. (Strikwerda 6.3.9.) Consider a scheme for (6.1.1), $u_t = bu_{xx}$, of the form

$$v_m^{n+1} = (1 - 2\alpha_1 - 2\alpha_2)\, v_m^n + \alpha_1 \left(v_{m+1}^n + v_{m-1}^n\right) + \alpha_2 \left(v_{m+2}^n + v_{m-2}^n\right).$$

Show that when μ is constant, as k and h tend to zero, the scheme is inconsistent unless

$$\alpha_1 + 4\alpha_2 = b\mu.$$

Show that the scheme is fourth-order accurate in x is $\alpha_2 = -\alpha_1/16$.

2 Programming

1. Implement the following numerical schemes to solve (1) on the *periodic* domain $[0,4]$:

   - Godunov's method. At time level n, solve the Riemann problem assuming a piecewise constant initial condition v^n, then resample to determine v^{n+1}.
   - (Backward) Semi-Lagrangian. At time level $n+1$ and grid vertex m, trace the characteristic $t \mapsto x_m + v_m^n\,(t - t_{n+1})$ *backward* to time level n and linearly interpolate v^n to determine v_m^{n+1}.
   - (Forward) Semi-Lagrangian. Trace the characteristics $t \mapsto x_m + v_m^n\,(t - t_n)$ *forward* to time level $n+1$ and linearly interpolate the nearest characteristics at a given grid vertex m to determine v_m^{n+1}.
   - (Conservative) Lax-Friedrichs. Discretize the conservative form of (inviscid) Burger's equation:

$$\frac{1}{k}\left(v_m^{n+1} - \frac{1}{2}\left(v_{m+1}^n - v_{m-1}^n\right)\right) + \frac{1}{2}\cdot\frac{1}{2h}\left(\left(v_{m+1}^n\right)^2 - \left(v_{m-1}^n\right)^2\right) = 0$$

   - (Advective) Lax-Friedrichs. Discretize the advective form of (inviscid) Burger's equation:

$$\frac{1}{k}\left(v_m^{n+1} - \frac{1}{2}\left(v_{m+1}^n - v_{m-1}^n\right)\right) + v_m^n \frac{1}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) = 0$$

   Use the initial condition (3). For those schemes that appear to converge to the exact solution (derived previously), compute a numerical convergence rate. For those schemes that don't appear to converge to the exact solution, explain the discrepancy (e.g., incorrect rarefaction, non-physical shock speed, unstable). Which scheme do you think performs best for the given initial condition?

2. Use your implementation of the Thomas algorithm from Homework 4 to solve *periodic* tridiagonal systems:

$$a_i w_{i-1} + b_i w_i + c_i w_{i+1}, \quad i = 1, \ldots, m,$$

   with $w_0 = w_m$ and $w_{m+1} = w_1$. The following algorithm is described in Strikwerda. First, solve the following (non-periodic) tridiagonal systems:

$$\begin{aligned}
a_i x_{i-1} + b_i x_i + c_i x_{i+1} &= d_i, \quad x_0 = 0 \text{ and } x_{m+1} = 0;\\
a_i y_{i-1} + b_i y_i + c_i y_{i+1} &= 0, \quad y_0 = 1 \text{ and } y_{m+1} = 0;\\
a_i z_{i-1} + b_i z_i + c_i z_{i+1} &= 0, \quad z_0 = 0 \text{ and } z_{m+1} = 1;
\end{aligned}$$

   for $i = 1, \ldots, m$. Then w_i is given by

$$w_i = x_i + r y_i + s z_i$$

where

$$
\begin{aligned}
r &:= \frac{1}{D}\left(x_m\left(1-z_1\right)+x_1 z_m\right), \\
s &:= \frac{1}{D}\left(x_m y_1+x_1\left(1-y_m\right)\right), \\
D &:= \left(1-y_m\right)\left(1-z_1\right)-y_1 z_m.
\end{aligned}
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