Basic Exam S02

Prove that the closed interval $[0, 1]$ is connected.

2. Show that the set Q of rational numbers in $\mathbb{R}$ is not expressible as the intersection of a countable collection of open subsets of $\mathbb{R}$.

3. Suppose that X is a compact metric space (in the covering sense of the word compact) Prove that every sequence $\{x_n \quad x_n \in X, n \quad 1, 2, 3 \quad\}$ has a convergent subsequence. [Prove this directly. Do not just quote a theorem.

4. (a) Define *uniform continuity* of a function $F : X \to \mathbb{R}$, X a metric space.

   (b) Prove that a function $f \quad (0, 1) \to \mathbb{R}$ is the restriction to $(0, 1$ of a continuous function $F \quad [0, 1] \to \mathbb{R}$ if and only if f is uniformly continuous on $(0, 1$

5 State some reasonable conditions under which a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

everywhere on $\mathbb{R}^2$ and prove this equality under the conditions you give.

6. Suppose $f \quad \mathbb{R}^3 \to \mathbb{R}$ is a continuously differentiable function with grad $f \neq \vec{0}$ at $\vec{0}$ ($\vec{0} = (0, 0, 0)$ in $\mathbb{R}^3$). Show that there are two other continuously differentiable functions $g : \mathbb{R}^3 \to \mathbb{R}$, $h : \mathbb{R}^3 \to \mathbb{R}$ such that the function

$$(x, y, z) \to (f(x, y, z), g(x, y, z), h(x, y, z)$$

from $\mathbb{R}^3$ to $\mathbb{R}^3$ is one-to-one on some neighborhood of $\vec{0}$.

Suppose $F : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable and that the Jacobian matrix of F is everywhere nonsingular. Suppose also that $F(\vec{0}) \quad \vec{0}$ and that $\|F((x,y))\| \geq 1$ for all (x,y) with $\|(x,y)\| = 1$.

Prove that $F(\{(x,y) \quad \|(x,y)\| < 1\}) \supset \{(x,y) : \|(x,y)\| < 1$

(Hint. Show, with $U = \{(x,y) : \|(x,y)\| < 1\}$, that $F(U) \cap U$ is both open and closed in U).

8. Let V be a finite dimensional real vector space. Let $W \subset V$ be a subspace and $W^o := \{f : V \to F \text{ linear} \mid f = 0 \text{ on } W\}$. Prove that

$$\dim(V) = \dim(W) + \dim(W^o)$$

9. Find the matrix representation in the standard basis for either rotation by an angle θ in the plane perpendicular to the subspace spanned by the vectors (1,1,1,1) and (1,1,1,0) in $\mathbf{R}^4$.

   [You do not have to multiply the matrices out but must compute any inverses.]

10. Let V be a complex inner product space and W a finite dimensional subspace. Let $v \in V$. Prove that there exists a unique vector $v_W \in W$ such that

$$||v - v_W|| \leq ||v - w||$$

for all $w \in W$. Deduce that equality holds if and only if $w \quad v_W$

11 Let V be a finite dimensional real inner product space and $T, S \quad V \to V$ two commuting hermitian linear operators. Show that there exists an orthonormal basis for V consisting of vectors that are simultaneously eigenvectors of T and S.