DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

ALL PROBLEMS HAVE EQUAL VALUE. There are 7 problems.

Do 5 problems and only 2 of them from 1, 2, 3

[1] Let f(0), f(h) and f(2h) be the values of a real valued function at x=0, x=h and x=2h

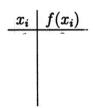
(a) Derive the coefficients c_0 , c_1 and c_2 so that

$$Df_h(x) = c_0 f(0) + c_1 f(h) + c_2 f(2h)$$

is as accurate an approximation to f'(0) as possible.

(b) Derive the leading term of a truncation error estimate for the formula you derived in (a).

[2](a) Find and solve the normal equations used to determine the coefficients for a straight line that fits the following data in the least squares sense.



- (b) Let A be an $m \times n$ matrix, with m > n and the columns of A being independent. Given the QR factorization of A, i.e. A=QR, where Q's columns are orthonormal and R is upper triangular, what equations must you solve to find the least squares solution of the over-determined system of equations $A\vec{x} = \vec{b}$?
- (c) Show that the Gram-Schmidt orthogonalization process applied to the columns of A leads to a QR factorization of the matrix A. (Specifically, give the elements of Q and R when the Gram-Schmidt process is written in matrix form.)

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[3] Consider the scalar function $f: \mathbf{R} \to \mathbf{R}$. Let x^* be a root of f and x^n be an approximation to that root.

- (a) Derive the formula for getting a "better" approximation to the root by setting x^{n+1} to be the root of the linear approximation to f obtained from the first two terms of the Taylor series approximation to f at x^n .
- (b) What is the common name for the method you have derived?
- (c) Consider $F: \mathbb{R}^n \to \mathbb{R}^n$. Using the approach in part (a), derive a vector iteration for solving $F(\vec{x}) = 0$.

[4] Consider the theta method

$$y_{i+1}$$
 $y_i + h \Big[\theta f(t_i, y_i) + (1 \quad \theta) f(t_{i+1}, y_{i+1}) \Big]$

to approximate the solution of the ordinary differential equation y' = f(t, y).

- (a) Find the order of the method as a function of the values of the parameter θ .
- (b) Determine all values of θ such that the theta method is A-stable.
- (c) What particular method is obtained for $\theta = 1$? Prove convergence of the method in this case $\theta = 1$ and state the necessary assumptions

$$u_t + au_x \quad 0 \text{ for } t > 0, \ 0 \le x \le 1$$

u(x,0) $\varphi(x)$ smooth, u periodic in x, u(x+1,t) u(x,t) we use:

$$\frac{1}{2\Delta t}[(v_j^{n+1}+v_{j+1}^{n+1}) \quad (v_j^n+v_{j+1}^n)] + \frac{a}{2\Delta x}[v_{j+1}^{n+1}-v_j^{n+1}+v_{j+1}^n \quad v_j^n] = 0$$

For what values of $\frac{\Delta t}{\Delta x}$, if any, does this converge? At what rate? Explain your answers.

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[6] Consider the differential equation

$$u_t = u_{xx} + u_{yy} + bu_{xy}$$
 for $t > 0$, $0 < x < 1$, $0 < y < 1$

with u = 0 on the boundary, and $u(x, y, 0) = \varphi(x, y)$, a smooth function.

- (a) For what values of b can you obtain a convergent, unconditionally stable finite difference scheme?
- (b) Construct such a scheme. Explain your answers.
- [7] (a) Develop and describe the piecewise linear Galerkin finite element approximation of

$$-\Delta u + b(x)u = f(x), \quad x = (x_1, x_2) \in \Omega$$
$$u = 2, \quad x \in \partial \Omega_1$$
$$\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + u = 2, \quad x \in \partial \Omega_2,$$

$$\begin{array}{lcl} \Omega &=& \{x|x_1>0,\ x_2>0,\ x_1+x_2<1\}\\ \partial \Omega_1 &=& \{x|x_1=0,\ 0\leq x_2\leq 1\} \cup \{x|x_2=0,\ 0\leq x_1\leq 1\}\\ \partial \Omega_2 &=& \{x|x_1>0,\ x_2>0,\ x_1+x_2=1\} \end{array}$$

$$0 < b \le b(x) \le B.$$

(b) Justify your approximation by analyzing the appropriate bilinear and linear forms. Give a weak formulation of the problem. Give a convergence estimate and quote the appropriate theorems for convergence