

# Math 269B, 2012 Winter, Homework 1

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## 1 Theory

1. (Strikwerda 1.1.3) Solve the initial value problem for

$$u_t + \frac{1}{1 + \frac{1}{2} \cos x} u_x = 0$$

Show that the solution is given by  $u(t, x) = u_0(\xi)$ , where  $\xi$  is the unique solution of

$$\xi + \frac{1}{2} \sin \xi = x + \frac{1}{2} \sin x - t.$$

2. Solve the initial value problem

$$u_t + (\sin t) u_x = \frac{1}{1 + t^2}, \quad u(0, x) = u_0(x), \quad x \in \mathbb{R}, \quad t > 0.$$

3. Consider the first order system of PDEs of the form

$$\vec{u}_t + A \vec{u}_x = 0, \quad \vec{u}(0, x) = \vec{u}_0(x), \quad x \in [0, 1], \quad t > 0.$$

- (a) Give the solution to the initial value problem when

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

- (b) Describe appropriate boundary conditions at  $x = 0$  and/or  $x = 1$ , if possible, which make the initial boundary value problem in (a) well-posed. Try to be as general as possible. How should such boundary conditions be presented to put the solution in a simple form?

- (c) Give the solution to the initial value problem when

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}.$$

- (d) Describe appropriate boundary conditions at  $x = 0$  and/or  $x = 1$ , if possible, which make the initial boundary value problem in (c) well-posed. Try to be as general as possible. How should such boundary conditions be presented to put the solution in a simple form?

4. Derive the leading term of the local truncation error for the following finite difference schemes used to approximate solutions to the equation  $u_t + au_x = 0$ .

- (a)

$$\frac{1}{k} (v_m^{n+1} - v_m^n) + a \frac{1}{2h} (v_{m+1}^n - v_{m-1}^n) = 0.$$

- (b)

$$\frac{1}{k} \left( v_m^{n+1} - \frac{1}{2} (v_{m+1}^n + v_{m-1}^n) \right) + a \frac{1}{2h} (v_{m+1}^n - v_{m-1}^n) = 0.$$

5. Determine the stability region  $\Lambda$  for each of the finite difference schemes in Problem 4.

## 2 Programming

1. Implement the finite difference schemes in Problem 4. in the Theory section for  $x \in [0, 1]$ ,  $t \in [0, T]$  for some final time  $T$ ,  $u(x, 0) = u_0(x)$ , and *periodic* boundary conditions.
2. Investigate the convergence of each scheme for  $a = 1$  and  $T = 1$ . Use an appropriate relation between  $k$  and  $h$  to ensure convergence. Try using both a smooth initial condition (e.g.,  $u_0(x) = \sin(2\pi x)$ ) and a non-smooth initial condition (e.g.,  $u_0(x) = 1 - 2|x - 1/2|$ ). Use the discrete  $L^2$  norm to measure the error between your numerical solution and the true solution:

$$\|w\|_h = \left( h \sum_m |w_m|^2 \right)^{1/2}.$$

[Note: Due to periodicity, be sure to avoid double-counting the contributions at  $x = 0$  and  $x = 1$ !]  
Plot the numerical solutions from each scheme at  $t = T$  when  $h = 1/100$ . Which scheme do you think is better and why?