1. *Consider the partial differential equation*

$$u_{tt} - u_{xx} - 2u_x = 0, \qquad 0 < x < 1, \quad t > 0,$$

with the boundary conditions $u_x(t,0) = u_x(t,1) = 0$*,* $t > 0$*, and initial conditions*

$$u(0,x) = \mathrm{e}^{-x}(\pi \cos \pi x + \sin \pi x), \qquad u_t(0,x) = 0, \qquad 0 < x < 1.$$

Show that a separation of variables leads to an eigenvalue problem in the variable x*. Determine the eigenvalues and the eigenfunctions for the eigenvalue problem. Find a solution that satisfies the boundary and initial conditions.*

Let $u(t,x) = T(t)X(x)$, then $\frac{T''}{T} = \frac{X''}{X} + 2\frac{X'}{X} = \lambda$. First we solve for $X(x)$:

$$\begin{aligned} X'' + 2X' - \lambda X &= 0 \\ w^2 + 2w - \lambda &= 0 \quad \Longrightarrow \quad w = -1 \pm \sqrt{1+\lambda}. \end{aligned}$$

Since u is periodic (by the boundary conditions), $X(x) = \sum_{n=1}^{\infty} a_n(x)\cos(n\pi x) + b_n(x)\sin(n\pi x)$. This requires that the imaginary part of w is $n\pi$, so

$$-(1+\lambda_n) = (n\pi)^2 \quad \Longrightarrow \quad \lambda_n = -1 - (n\pi)^2.$$

The eigenvalues are $\mathrm{e}^{-x}\cos(n\pi x)$ and $\mathrm{e}^{-x}\sin(n\pi x)$. The equation $T'' - \lambda_n T = 0$ implies

$$T(t) = c_n \cos\sqrt{1+(n\pi)^2}t + d_n \sin\sqrt{1+(n\pi)^2}t,$$

so the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} \mathrm{e}^{-x}\left[a_n \cos(n\pi x) + b_n \sin(n\pi x)\right]\left[c_n \cos\sqrt{1+(n\pi)^2}t + d_n \sin\sqrt{1+(n\pi)^2}t\right].$$

The boundary condition $u_t(0,x) = 0$ implies that $d_n = 0$. The initial condition gives a_n and b_n:

$$u(0,x) = \mathrm{e}^{-x}(\pi\cos\pi x + \sin\pi x) \qquad \Longrightarrow \qquad \begin{cases} a_1 = \pi \\ a_n = 0 & \text{for } n \neq 1 \\ b_1 = 1 \\ b_n = 0 & \text{for } n \neq 1 \end{cases}$$

The solution to the problem is $u(x,t) = \mathrm{e}^{-x}(\pi\cos\pi x + \sin\pi x)\cos\sqrt{1+\pi^2}t$.

2. *Let* $\varphi \in C^1(\mathbb{R}^2)$. *Solve the following Cauchy problem in* $\mathbb{R}^3$,

$$\begin{cases} x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u, \\ u(x_1, x_2, 0) = \varphi(x_1, x_2). \end{cases}$$

Use the method of characteristics with the initial parameterization $\Gamma = (s_1, s_2, 0, \varphi(s_1, s_2))$.

$$\begin{array}{rclcrclcrclcrcl}
x_1' & = & x_1 & & x_2' & = & 2x_2 & & x_3' & = & 1 & & u' & = & 3u \\
x_1(0) & = & s_1 & & x_2(0) & = & s_2 & & x_3(0) & = & 0 & & u(0) & = & \varphi(s_1, s_2) \\
x_1 & = & s_1 \mathrm{e}^t & & x_2 & = & s_2 \mathrm{e}^{2t} & & x_3 & = & t & & u & = & \varphi(s_1, s_2)\mathrm{e}^{3t} \\
s_1 & = & x_1 \mathrm{e}^{-x_3} & & s_2 & = & x_2 \mathrm{e}^{-2x_3} & & & & & & & &
\end{array}$$

So $u(x_1, x_2, x_3) = \varphi(x_1 \mathrm{e}^{-x_3}, x_2 \mathrm{e}^{-2x_3})\mathrm{e}^{3x_3}$.

3. *Let $u(x)$ be harmonic in the unit disc $|x| < 1$ in $\mathbb{R}^2$, and assume that $u \geq 0$. Prove the following Harnack's inequality:*

$$\frac{1-|x|}{1+|x|}u(0) \leq u(x) \leq \frac{1+|x|}{1-|x|}u(0), \qquad |x| < 1.$$

The Poisson integral formula is

$$u(re^{i\theta}) = \int_0^{2\pi} u(e^{i\varphi})\, P_r(\theta - \varphi)\, \frac{d\varphi}{2\pi}$$

where $P_r(\theta) = \dfrac{1-r^2}{1+r^2-2r\cos\theta}$, $r \in (0,1)$, $\theta \in [0, 2\pi]$. Note that

$$\frac{1-r}{1+r} \;=\; \frac{1-r^2}{1+2r+r^2} \;\leq\; P_r(\theta) \;\leq\; \frac{1-r^2}{1-2r+r^2} \;=\; \frac{1+r}{1-r}.$$

So, by the mean-value property,

$$u(x) \;\leq\; \frac{1+r}{1-r}\int_0^{2\pi} \frac{u(e^{i\varphi})}{2\pi}\, d\varphi \;=\; \frac{1+r}{1-r}u(0).$$

Similarly, $u(x) \;\geq\; \dfrac{1-r}{1+r}\displaystyle\int_0^{2\pi} \frac{u(e^{i\varphi})}{2\pi}\, d\varphi \;=\; \dfrac{1-r}{1+r}u(0)$. Therefore,

$$\frac{1-|x|}{1+|x|}u(0) \leq u(x) \leq \frac{1+|x|}{1-|x|}u(0), \qquad |x| < 1.$$

4. *Let $u(x,t) \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$ solve the Cauchy problem for the wave equation*

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \mathbb{R}^3, \quad t > 0, \\ u(x,0) = \varphi(x), & u_t(x,0) = \psi(x), \end{cases}$$

with $\varphi(x)$ and $\psi(x)$ being smooth compactly supported functions on $\mathbb{R}^3$. Use an explicit formula for the solution to show that there exists a constant $C > 0$ such that we have, uniformly in $x \in \mathbb{R}^3$,

$$|u(x,t)| \le \frac{C}{t}, \qquad t > 0.$$

The solution is given by Kirchoff's formula,

$$u(x,t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t \int_{|\xi|=1} \varphi(x + ct\xi)\,\mathrm{d}S_\xi \right) + \frac{t}{4\pi}\int_{|\xi|=1} \psi(x+ct\xi)\,\mathrm{d}S_\xi,$$

where the integrals are over the surface of the unit sphere and, for this problem, $c = 1$. The change of variables $z = x + t\xi$, $\mathrm{d}z = z^3\mathrm{d}\xi$, $\mathrm{d}S_z = t^2\mathrm{d}S_\xi$ yields

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t \int_{|z-x|=t} \varphi(z)\,\frac{\mathrm{d}S_z}{t^2}\right) + \frac{t}{4\pi}\int_{|z-x|=t}\psi(z)\,\frac{\mathrm{d}S_z}{t^2} \\ &= \frac{1}{4\pi}\frac{\partial}{\partial t}\left(\frac{1}{t}\int_{|z-x|=t}\varphi(z)\,\mathrm{d}S_z\right) + \frac{1}{4\pi}\frac{1}{t}\int_{|z-x|=t}\psi(z)\,\mathrm{d}S_z. \end{aligned}$$

Let $M = ||\varphi||_{L^\infty}$ and $N = ||\psi||_{L^\infty}$. The integrals are bounded,

$$\begin{aligned} \int_{|z-x|=t} |\varphi(z)|\,\mathrm{d}S_z &\le \int_{\operatorname{supp}\varphi} |\varphi(z)|\,\mathrm{d}z \;=\; ||\varphi||_{L^1} \;<\; \infty, \\ \int_{|z-x|=t} |\psi(z)|\,\mathrm{d}S_z &\le \int_{\operatorname{supp}\psi} |\psi(z)|\,\mathrm{d}z \;=\; ||\psi||_{L^1} \;<\; \infty. \end{aligned}$$

For $t \ge 1$,

$$\begin{aligned} |u(x,t)| \;\le\; \frac{1}{4\pi}\left|\frac{\partial}{\partial t}\frac{||\varphi||_{L^1}}{t}\right| + \frac{1}{4\pi}\frac{1}{t}||\psi||_{L^1} \quad &= \quad \frac{1}{4\pi}\frac{||\varphi||_{L^1}}{t^2} + \frac{1}{4\pi}\frac{1}{t}||\psi||_{L^1} \\ &\overset{(\text{since } t \ge 1)}{\le} \quad \frac{1}{4\pi}(||\varphi||_{L^1} + ||\psi||_{L^1})\frac{1}{t}. \end{aligned}$$

And for $0 \le t \le 1$,

$$\int_{|z-x|=t} |\varphi(z)|\,\mathrm{d}S_z \;\le\; 4\pi t^2 ||\varphi||_{L^\infty}, \qquad \int_{|z-x|=t} |\psi(z)|\,\mathrm{d}S_z \;\le\; 4\pi t^2 ||\psi||_{L^\infty},$$

which implies

$$\begin{aligned} |u(x,t)| \;\le\; \left|\frac{\partial}{\partial t}\frac{t^2||\varphi||_{L^\infty}}{t}\right| + \frac{t^2||\psi||_{L^\infty}}{t} \quad &= \quad ||\varphi||_{L^\infty} + t||\psi||_{L^\infty} \\ &\le \quad ||\varphi||_{L^\infty} + ||\psi||_{L^\infty} \;\le\; (||\varphi||_{L^\infty} + ||\psi||_{L^\infty})/t. \end{aligned}$$

Choose $C = \max\left\{\frac{1}{4\pi}(||\varphi||_{L^1} + ||\psi||_{L^1}),\ (||\varphi||_{L^\infty} + ||\psi||_{L^\infty})\right\}$.

5. *Solve the inhomogeneous problem for the Laplace operator in the unit disc* $\mathbf{D} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$,

$$\begin{cases} \Delta u = x^2 - y^2 & \textit{on } \mathbf{D} \\ u = 0 & \textit{on } \partial\mathbf{D}. \end{cases}$$

In polar coordinates, the problem is

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = r^2(\cos^2\theta - \sin^2\theta) & \text{on } 0 < r < 1,\ 0 \le \theta \le 2\pi, \\ u = 0 & \text{on } r = 1. \end{cases}$$

With the change of variables $r = \mathrm{e}^{-t}$, with $\frac{\mathrm{d}r}{\mathrm{d}t} = -\mathrm{e}^{-t} = -r$, $\frac{\mathrm{d}t}{\mathrm{d}r} = -\frac{1}{r}$, $t = -\log r$, yields

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}r}\right) \\ &= \frac{1}{r}\frac{\partial}{\partial r}(-u_t) = \frac{1}{r^2}u_{tt}. \end{aligned}$$

So $u_{tt} + u_{\theta\theta} = r^4(\cos^2\theta - \sin^2\theta)$, or equivalently, $u_{tt} + u_{\theta\theta} = \mathrm{e}^{-4t}\cos 2\theta$.

First we solve the homogeneous problem $u_{tt} + u_{\theta\theta} = 0$. With separation of variables, $u(t,\theta) = T(t)\Theta(\theta)$. Since Θ is periodic, the eigenfunctions of $\frac{\Theta''}{\Theta} = \lambda$ are $\cos n\theta$ and $\sin n\theta$. The eigenfunctions of $\frac{T''}{T} = -\lambda$ are e^{nt} and e^{-nt}, or in terms of r, r^{-n} and r^n. Thus the solution has the form

$$u_h(r,\theta) = \sum_{n=1}^{\infty} r^n(a_n\cos n\theta + b_n\sin n\theta).$$

Now we find a particular solution to $u_{tt} + u_{\theta\theta} = \mathrm{e}^{-4t}\cos 2\theta$. Substituting $u_p = C\mathrm{e}^{-4t}\cos 2\theta$ yields

$$C(-4)^2\mathrm{e}^{-4t}\cos 2\theta - C2^2\mathrm{e}^{-4t}\cos 2\theta = 12C\mathrm{e}^{-4t}\cos 2\theta \quad \Rightarrow \quad C = \tfrac{1}{12}.$$

So $u = u_h + u_p = \sum r^n(a_n\cos n\theta + \sin n\theta) + \frac{1}{12}r^4\cos 2\theta$. The boundary condition $u_{|r=1} = 0$ requires that $a_n, b_n = 0$ for all n except $a_2 = -\frac{1}{12}$. Therefore,

$$\begin{aligned} u &= -\tfrac{1}{12}r^2\cos 2\theta + \tfrac{1}{12}r^4\cos 4\theta \\ &= \tfrac{1}{12}\left[-r^2(\cos^2\theta - \sin^2\theta) + r^2\cdot r^2(\cos^2\theta - \sin^2\theta)\right] \\ &= \tfrac{1}{12}\left[-(x^2-y^2) + (x^2+y^2)(x^2-y^2)\right] = \tfrac{1}{12}(x^2+y^2-1)(x^2-y^2). \end{aligned}$$

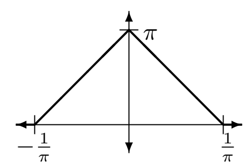
6. *Find the Fourier transform of the integrable function* $x \mapsto (\sin x)^2/x^2$. *Hint: Determine first the Fourier transform of* $x \mapsto x^{-1}\sin x$.

Let $f(x) = \begin{cases} b & |x|<a \\ 0 & |x|>a \end{cases}$, then

$$\hat{f}(\xi) = \int_{-a}^{a} \mathrm{e}^{-2\pi i \xi x} b \,\mathrm{d}x = \frac{-bi2\sin(2\pi\xi a)}{-2\pi i\xi} = \frac{b\sin(2\pi a\xi)}{\pi x i}.$$

Let $g(\xi) = \begin{cases} \pi & |\xi|<\frac{1}{2\pi} \\ 0 & |\xi|>\frac{1}{2\pi} \end{cases}$, then $\check{g}(x) = \frac{\sin x}{x}$. Therefore,

$$\begin{aligned} \widehat{\left(\frac{\sin^2 x}{x^2}\right)}(\xi) \;=\; \widehat{(\check{g}(x)\check{g}(x))}(\xi) &= (g * g)(\xi) \\ &= \begin{cases} \pi(1-|\pi x|) & |\xi| < \frac{1}{\pi}, \\ 0 & |\xi| > \frac{1}{\pi}. \end{cases} \end{aligned}$$



7. *Consider an autonomous system in* $\mathbb{R}^n$, $x'(t) = f(x(t))$, *where* $f = (f_1, f_2, \ldots, f_n)$ *is a smooth vector field, such that*

$$\sum_{k=1}^{n} x_k f_k(x) < 0 \quad \text{for } x \neq 0.$$

Show that $\lim_{t\to\infty} x(t) = 0$, *for each solution of the system, independent of the initial condition* $x(0)$.

Let $V(x) = \frac{1}{2}(x_1^2 + \cdots + x_n^2)$. Then V is positive definite and $V(x) \to \infty$ as $||x|| \to \infty$.

$$V^*(x) = \nabla V(x) \cdot f(x) = \sum_{k=1}^{n} x_k f_k(x) \leq 0 \quad \text{in } \mathbb{R}^n$$

The origin is the only invariant subset of the set $\{x : V^*(x) = 0\} = \{0\}$. Thus by Lyapunov's second method, the zero solution is globally asymptotically stable. So $\lim_{t\to\infty} x(t) = 0$ independent of the initial condition $x(0)$.