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Basic Examination, F05

1. A real number α is said to be *algebraic* if for some finite set of integers $a_0, \ldots, a_n$, not all 0,
$$a_0 + a_1\alpha + \cdots + a_n\alpha^n = 0.$$
Prove that the set of algebraic real numbers is countable.
Solution
(S05.5)

2. State some reasonable conditions on a real-valued function $f(x, y)$ on $\mathbb{R}^2$ which guarantee that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ at every point of $\mathbb{R}^2$. Then prove that your conditions do in fact guarantee this equality.
Solution
(F01.5)

3. (a) Prove that if $f_j : [0, 1] \to \mathbb{R}$ is a sequence of continuous functions which converges uniformly on $[0, 1]$ to a (necessarily continuous) function $F : [0, 1] \to \mathbb{R}$, then
$$\int_0^1 F^2(x)dx = \lim_{j\to\infty} \int_0^1 f_j^2(x)dx.$$
(b) Give an example of a sequence $f_j : [0, 1] \to \mathbb{R}$ of continuous functions which converges to a continuous function $F : [0, 1] \to \mathbb{R}$ pointwise and for which
$$\lim_{j\to\infty} \int_0^1 f_j^2(x)dx \text{ exists, but}$$
$$\lim_{j\to\infty} \int_0^1 f_j^2(x)dx \neq \int_0^1 F^2(x)dx$$
(f_j converges to F "pointwise" means that for each $x \in [0, 1]$, $F(x) = \lim_{j\to\infty} f_j(x)$).

Solution

(a) Since F is continuous on $[0, 1]$, a compact set, F is bounded, say, $|F| < M$ on $[0, 1]$. Let $\epsilon > 0$ be given. Then there exists a J such that $\sup_{[0,1]} |F - f_j| < \epsilon$ for all $j > J$. Thus
$$\left| \int_0^1 F^2(x)dx - \int_0^1 f_j^2(x)dx \right| \leq \int_0^1 |F(x) + f_j(x)||F(x) - f_j(x)|dx \leq (2M + \epsilon)\epsilon$$
for all $j > J$, so since ϵ was arbitrary, we conclude that
$$\lim_{j\to\infty} \int_0^1 f_j^2(x)dx = \int_0^1 F^2(x)dx.$$
(b) Define
$$f_j(x) = \begin{cases} \sqrt{j^2 x}, & 0 \leq x \leq \frac{1}{j} \\ \sqrt{2j - j^2 x}, & \frac{1}{j} \leq x \leq \frac{2}{j} \\ 0, & \frac{2}{j} \leq x \leq 1 \end{cases}$$
for $j = 1, 2, \ldots$. Then $f_j(x) \to 0$ for all $x \in [0, 1]$, but
$$\int_0^1 f_j^2(x)dx = 1.$$

4. Suppose $F : [0,1] \to [0,1]$ is a C^2 function with $F(0) = 0$, $F(1) = 0$, and $F''(x) < 0$ for all $x \in [0,1]$. Prove that the arc length of the curve $\{(x, F(x)) : x \in [0,1]\}$ is less than 3. (Suggestion: Remember that $\sqrt{a^2 + b^2} < |a| + |b|$ when you are looking at the arc length formula - and at a picture of what $\{(x, F(x))\}$ could look like.)

   Solution

   By the Mean Value Theorem, since $F(0) = F(1) = 0$, there exists an $x_0 \in (0,1)$ such that $F'(x_0) = 0$. Since $F'' < 0$ on $[0,1]$, it follows that $F'(x) > 0$ for $x < x_0$ and $F'(x) < 0$ for $x > x_0$.

   Define $\gamma : [0,1] \to \mathbb{R}^2$ by $\gamma(t) = (t, F(t))$. Then the image of γ is the curve in question, and, since $\gamma \in C^1$, the length of this curve is given by

$$\begin{array}{rcl} \Lambda(\gamma) & = & \int_0^1 |\gamma'(t)|dt \\ & = & \int_0^1 \sqrt{1 + F'(t)^2}dt \\ & \leq & \int_0^1 (1 + |F'(t)|)\, dt \\ & = & \int_0^{x_0} (1 + F'(t))dt + \int_{x_0}^1 (1 - F'(t))dt \\ & = & (x_0 - 0 + F(x_0) - F(0)) + (1 - x_0 - F(1) + F(x_0)) \\ & = & 1 + 2F(x_0) \end{array},$$

   where we have used the fact that $\sqrt{a^2 + b^2} \leq \sqrt{a^2 + b^2 + 2|a||b|} = |a| + |b|$ for $a, b \in \mathbb{R}$. But $F(x_0) \in [0,1]$, hence

$$\Lambda(\gamma) \leq 3.$$

5. Prove carefully that $\mathbb{R}^2$ is not a (countable) union of sets S_i, $i = 1, 2, \ldots$, with each S_i being a subset of some straight line L_i in $\mathbb{R}^2$.

   Solution

   The closure of each S_i is contained in L_i, and each L_i has empty interior, hence each S_i is nowhere dense. By a corollary to the Baire Category Theorem, $\bigcup_i S_i$ also has empty interior, hence certainly cannot be all of $\mathbb{R}^2$.

6. (a) Prove that if P is a real-coefficient polynomial and if A is a real symmetric matrix, then the eigenvalues of $P(A)$ are exactly the numbers $P(\lambda)$, where λ is an eigenvalue of A.

   (b) Use part (a) to prove that if A is a real symmetric matrix, then A^2 is nonnegative definite.

   (c) Check part (b) by verifying directly that $\det A^2$ and $\operatorname{tr} A^2$ are nonnegative when A is real symmetric.

   Solution

   (a) Clearly, every eigenvalue λ of A gives a corresponding eigenvalue $P(\lambda)$ of $P(A)$ (with the same eigenvector). To get the converse, note that A and $P(A)$ commute, and if A is real symmetric, so is $P(A)$. Thus the Spectral Theorem allows us to construct an orthonormal basis of eigenvectors of both A and $P(A)$. Let μ be an eigenvalue of $P(A)$, and x a corresponding eigenvector from the aforementioned basis. Then x is also an eigenvector of A, and let its corresponding eigenvalue be λ. Then it follows that

$$P(A)x = P(\lambda)x,$$

   and we conclude $\mu = P(\lambda)$, establishing the converse.

   (b) As discussed in part (a), there exists an orthornormal basis $\{x_i\}_{i=1}^n$ of eigenvectors of both A and A^2. Each eigenvector x_i of A^2 has a corresponding eigenvalue $\mu_i = \lambda_i^2$, where λ_i is an eigenvalue of A. Thus, for any $x = \sum_i c_i x_i$,

$$(A^2 x, x) = \left(A^2 \sum_i c_i x_i, \sum_j c_j x_j \right) = \left(\sum_i c_i \lambda_i^2 x_i, \sum_j c_j x_j \right) = \sum_i c_i^2 \lambda_i^2 \geq 0,$$

where we have taken advantage of the orthonormality of the x_i's.

(c) We have that

$$\det A^2 = \prod_i \lambda_i^2 \geq 0,$$

$$\operatorname{tr} A^2 = \sum_i \lambda_i^2 \geq 0.$$

7. Let A be a real $n \times m$ matrix. Prove that the maximum number of linearly independent rows of $A =$ the maximum number of linearly independent columns. ("row rank = column rank")

Solution

(F01.7)

8. For a real $n \times n$ matrix A, let $T_A : \mathbb{R}^n \to \mathbb{R}^n$ be the associated linear mapping. Set $\|A\| = \sup_{x \in \mathbb{R}^n, \|x\|=1} \|Ax\|$ (here $\|x\|$ = the usual Euclidean norm, i.e.,

$$\|(x_1, \ldots, x_n)\| = \left(x_1^2 + \cdots + x_n^2\right)^{1/2}\Big).$$

(a) Prove that $\|A+B\| \leq \|A\| + \|B\|$.

(b) Use part (a) to check that the set M of all $n \times n$ matrices is a metric space if the distance function d is defined by

$$d(A,B) = \|B - A\|.$$

(c) Prove that M is a complete metric space with this "distance function".
(Suggestion: The ij^{th} element of $A = (T_A e_j, e_i)$, where $e_i = (0, \ldots, 1, \ldots 0)$, 1 in the i^{th} position.)

Solution

(a) Let $x \in \mathbb{R}^n$ with $\|x\| = 1$. Then

$$\|(A+B)x\| = \|Ax + Bx\| \leq \|Ax\| + \|Bx\| \leq \|A\| + \|B\|,$$

and taking the supremum over all $x \in \mathbb{R}^n$ with $\|x\| = 1$ establishes the claim.

(b) d is certainly symmetric and positive definite. The triangle inequality follows from

$$\|A - C\| = \|(A - B) + (B - C)\| \leq \|A - B\| + \|B - C\|.$$

(c) Let $\{A_k\}_{k=1}^\infty$ be a Cauchy sequence in (M, d). Let a_{ij}^k be the ij^{th} entry of A_k. Then

$$\begin{array}{rcl} \left|a_{ij}^k - a_{ij}^\ell\right| & = & (A_k e_j, e_i) - (A_\ell e_j, e_i) \\ & = & ((A_k - A_\ell)e_j, e_i) \\ & \leq & \|(A_k - A_\ell)e_j\| \|e_i\| \\ & \leq & \|A_k - A_\ell\| \end{array},$$

where we have applied the Cauchy Schwarz inequality and the fact that $\|e_i\| = 1$. It follows that $\{a_{ij}^k\}_{k=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$, so converges to some a_{ij}. Let $A \in M$ have ij^{th} entry equal to a_{ij}. For $\epsilon > 0$, let K be large enough such that $\left|a_{ij}^k - a_{ij}\right| < \epsilon$ for all i, j and $k > K$ (possible since there are finitely many such combinations of i, j). Then for any $x = \sum_i x_i e_i$ such that $\|x\| = 1$,

$$\begin{array}{rcl} \|(A - A_k)x\| & = & \left\|\sum_j x_j (A - A_k) e_j\right| \\ & \leq & \sum_j |x_j| \|(A - A_k)e_j\| \\ & \leq & \sum_j \|(A - A_k)e_j\| \\ & = & \sum_j \sqrt{\sum_i (a_{ij} - a_{ij}^k)^2} \\ & < & \sum_j \sqrt{n\epsilon^2} \\ & = & n^{3/2}\epsilon \end{array},$$

so it follows that $\|A - A_k\| < n^{3/2}\epsilon$ for all $k > K$, and since ϵ was arbitrary, we conclude that $A_k \to A$ with respect to d.

9. Suppose V_1 and V_2 are subspaces of a finite-dimensional vector space V.

   (a) Show that
   $$\dim(V_1 \cap V_2) = \dim(V_1) + \dim(V_2) - \dim(\mathrm{span}(V_1 \cup V_2)),$$
   where $\mathrm{span}(V_1 \cup V_2)$ is by definition the smallest subspace that contains both V_1 and V_2.

   (b) Let $n = \dim(V)$. Use part (a) to show that, if $k < n$, then an intersection of k subspaces of dimension $n-1$ always has dimension at least $n-k$.
   (Suggestion: Do induction on k.)

Solution

(a) Let $\{u_1, \ldots, u_j\}$ be a basis for $V_1 \cap V_2$, and extend this basis with $\{v_1^i, \ldots, v_{k_i}^i\}$ to form a basis for V_i, $i = 1, 2$. Then the claim is that $\{u_1, \ldots, u_j, v_1^1, \ldots, v_{k_1}^1, v_1^2, \ldots, v_{k_2}^2\}$ is a basis for $\mathrm{span}(V_1 \cup V_2)$. Indeed, any vector in $\mathrm{span}(V_1 \cup V_2)$ can be written as a linear combination of vectors in the claimed basis. Further, if we suppose some trivial linear combination,
$$\sum_i c_i u_i + \sum_i c_i^1 v_i^1 + \sum_i c_i^2 v_i^2 = 0,$$
Then
$$\sum_i c_i u_i + \sum_i c_i^1 v_i^1 = 0,$$
$$\sum_i c_i^2 v_i^2 = 0,$$
since the former resides in V_1 while the latter resides in the complement. Thus each $c_i^2 = 0$, and a similar argument leads to $c_i^1 = 0$, from which it follows that $c_i = 0$ as well. Thus the claimed basis is also linearly independent, so is a basis as claimed. Therefore,
$$\begin{aligned} \dim(V_1) + \dim(V_2) - \dim(\mathrm{span}(V_1 \cup V_2)) &= (j + k_1) + (j + k_2) - (j + k_1 + k_2) \\ &= j \\ &= \dim(V_1 \cap V_2) \end{aligned}.$$

(b) The claim is trivial for $k = 1$, so suppose the claim for $k-1$. Let $V_1, \ldots, V_k$ be subspaces of V of dimension $n-1$. By the induction hypothesis,
$$\dim(V_1 \cap \cdots \cap V_{k-1}) \geq n - (k-1),$$
hence, by part (a),
$$\begin{aligned} \dim(V_1 \cap \cdots \cap V_k) &= \dim(V_1 \cap \cdots \cap V_{k-1}) + \dim(V_k) - \dim(\mathrm{span}((V_1 \cap \cdots \cap V_{k-1}) \cup V_k)) \\ &\geq (n-k+1) + (n-1) - n \\ &= n-k \end{aligned},$$
which proves the claim for k, hence the claim is proved in general by induction.

10. (a) For each $n = 2, 3, 4, \ldots$, is there an $n \times n$ matrix A with $A^{n-1} \neq 0$ but $A^n = 0$? (Give example or proof of nonexistence.)

   (b) Is there an $n \times n$ upper triangular matrix A with $A^n \neq 0$ but $A^{n+1} = 0$? (Give an example or proof of nonexistence.)
   (Note: A square matrix is *upper triangular* if all the entries below the main diagonal are 0.)

Solution

(a) Let $A \in M_{n\times n}(\mathbb{R})$ be such that the ij^{th} entry of A, $(A)_{ij}$, is 0 for $j-i \leq 0$ and 1 for $j-i \geq 1$. We prove by induction that $(A^m)_{ij} > 0$ if and only if $j-i \geq m$. Indeed, the claim is true for $m=1$ by definition, and, assuming the claim for $m-1$, we know that

$$(A^m)_{ij} = \sum_k (A^{m-1})_{ik}(A)_{kj}.$$

Now $(A^{m-1})_{ik} > 0$ if and only if $k-i \geq m-1$, by the inductive hypothesis, and $(A)_{kj} > 0$ if and only if $j-k \geq 1$. It follows that there are nonzero terms in the above sum if and only if there exists a k, $1 \leq k \leq n$, such that

$$k - i \geq m - 1$$

and

$$j - k \geq 1$$

simulataneously. Note that this occurs only if $j-i \geq m$ (by adding the inequalities), hence $(A^m)_{ij}$ is nonzero only if $j-i \geq m$. To see the converse, notice that if $j-i \geq m$, the term in the sum corresponding to $k = j-1$ is nonzero, and all nonzero terms must be positive, hence in this case, $(A^m)_{ij} > 0$. This proves the claim by induction.

It follows immediately that $A^{n-1} \neq 0$ (since $(A^{n-1})_{1n} > 0$), while $A^n = 0$.

(b) Denote by

$$K_m = \ker A^m.$$

Note first that A maps K_{m+1} into K_m, since $x \in K_{m+1}$ implies that

$$0 = A^{m+1}x = A^m(Ax),$$

so $Ax \in K_m$. It follows that A maps the quotient space K_{m+2}/K_{m+1} into the quotient space K_{m+1}/K_m. Indeed, this mapping is injective. For suppose $x, y \in K_{m+2}/K_{m+1}$ were such that $Ax = Ay \pmod{K_m}$. But then $A(x-y) \in K_m$, i.e.,

$$0 = A^m(A(x-y)) = A^{m+1}(x-y),$$

so $x - y \in K_{m+1}$ and $x = y \pmod{K_{m+1}}$.

Now suppose K_{n+1}/K_n was nontrivial, i.e., there existed some $x \in K_{n+1}$ with $x \neq 0 \pmod{K_n}$. Then, by the previous argument, $Ax \in K_n/K_{n-1}$ with $Ax \neq 0 \pmod{K_{n-1}}$, and, in general, $A^m x \in K_{n-m+1}/K_{n-m}$ with $A^m x \neq 0 \pmod{K_{n-m}}$ for $m = 0, \ldots, n$. Note that then, by the containment $K_m \subset K_{m+1}$, we obtain a sequence of vectors $A^m x \in K_{n-m+1}$ such that $A^m x \notin K_k$ for $k = 0, \ldots, n-m$, hence it follows that the $A^m x$'s are linearly independent. But there are $n+1$ such values of m, implying the existence of a set of $n+1$ linearly independent vectors in $\mathbb{R}^n$, an absurdity. It follows that K_{n+1}/K_n is the trivial vector space, i.e., $K_{n+1} = K_n$.

Now if we have $A \in M_{n\times n}(\mathbb{R})$ such that $A^{n+1} = 0$, then $K_n = K_{n+1} = \mathbb{R}^n$, hence $A^n = 0$ as well.