

Math 269B, 2012 Winter, Homework 3

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1 Theory

1. (Strikwerda 5.1.2.) Show that the modified leapfrog scheme (5.1.6) is stable for ϵ satisfying

$$0 < \epsilon \leq 1 \quad \text{if} \quad 0 < a^2 \lambda^2 \leq \frac{1}{2}$$

and

$$0 < \epsilon \leq 4a^2 \lambda^2 (1 - a^2 \lambda^2) \quad \text{if} \quad \frac{1}{2} \leq a^2 \lambda^2 < 1.$$

Note that these limits are not sharp. It is possible to choose ϵ larger than these limits and still have the scheme be stable.

2. Derive the stability condition for the backward-time forward-space scheme

$$\frac{1}{k} (v_m^{n+1} - v_m^n) + \frac{a}{h} (v_{m+1}^{n+1} - v_{m-1}^{n+1}) = 0$$

used to approximate solutions to $u_t + au_x = 0$ with, say, $x \in [0, 1]$ and periodic boundary conditions. Give an example of an initial condition v_m^0 and an explicit expression for v_m^n that demonstrate unstable behavior for a particular λ (your choice) which fails to satisfy the stability condition. Does the growth in your example agree with your theoretical amplification factor?

3. Prove that numerical solutions to the Lax-Friedrichs scheme

$$\frac{1}{k} \left(v_m^{n+1} - \frac{1}{2} (v_{m+1}^n + v_{m-1}^n) \right) + \frac{a}{2h} (v_{m+1}^n - v_{m-1}^n) = 0$$

converge to solutions to the corresponding modified equation

$$u_t + au_x = \frac{h^2}{2k} \left(1 - \left(\frac{ak}{h} \right)^2 \right) u_{xx}$$

to second order accuracy in ℓ^∞ . I.e., show that $|v_m^n - u_{k,h}(t_n, x_m)| \rightarrow 0$ as $h, k \rightarrow 0$ (according to the stability criterion), where the subscripts on $u_{k,h}$ only indicate that the solution to the modified equation is parameterized by k, h .

4. (Strikwerda 4.1.2.) Show that the (2, 2) leapfrog scheme for $u_t + au_{xxx} = f$ (see (2.2.15)) given by

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\delta^2 \delta_0 v_m^n = f_m^n,$$

with $\nu = k/h^3$ constant, is stable if and only if

$$|a\nu| < \frac{2}{3^{3/2}}.$$

5. (Strikwerda 3.2.1.) Show that the (forward-backward) MacCormack scheme

$$\begin{aligned}\tilde{v}_m^{n+1} &= v_m^n - a\lambda (v_{m+1}^n - v_m^n) + kf_m^n, \\ v_m^{n+1} &= \frac{1}{2} (v_m^n + \tilde{v}_m^{n+1} - a\lambda (\tilde{v}_m^{n+1} - \tilde{v}_{m-1}^{n+1}) + kf_m^{n+1})\end{aligned}$$

is a second-order accurate scheme for the one-way wave equation (1.1.1). Show that for $f = 0$ it is identical to the Lax-Wendroff scheme (3.1.1).

2 Programming

1. Solve $u_t + au_x = 0$ numerically using the Lax-Friedrichs scheme. Take $a = 1$, $T = 1$, $x \in [0, 1]$ with periodic boundary conditions, and $u_0(x) = \sin 2\pi x$. Demonstrate convergence for various decreasing values of $\lambda := k/h$ (satisfying the stability criterion) by plotting the logarithm of the L^2 -norm of the error (between the analytic solution and the numerical solution) versus the logarithm of h . Verify that the slope suggested by your plot agrees with theory, and estimate the error constant C_λ in the relation $\text{error} = C_\lambda h^p$. Use enough values of λ to estimate the relation between C_λ and λ . What appears to happen to C_λ as $\lambda \rightarrow 0+$, i.e., as you shrink k relative to h ? What happens if, instead of taking $k = \lambda h$, you take $k = h^2$? Explain your numerical results in the context of the theoretical convergence analysis of the Lax-Friedrichs scheme.
2. For the one-way wave equation $u_t + au_x = 0$, investigate how close the numerical solution to a finite difference scheme is to the solution to the corresponding modified equation. To be concrete, suppose a pulse initial condition $u_0(x) = \frac{1}{2} (1 + |x|/x)$, $x \in [-1, 1]$, and periodic boundary conditions. Take $a = 1$, $k/h = 0.5$, and final time $T = 0.5$. Compare the following finite difference schemes: upwinding, Lax-Friedrichs, and Lax-Wendroff. Also, include a derivation of the respective corresponding modified equations. You may find solutions to the modified equations using any appropriate method (i.e., analytically or to a sufficiently high accuracy numerically).