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Applied Differential Equations Qualifying Exam, Winter 2005

1. Consider the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} = 0,\ 0 < x < 1,\ t > 0, \qquad (1)$$

with the boundary conditions

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0,\ t > 0,$$

and initial conditions

$$u(0,x) = e^{-x}(\pi\cos\pi x + \sin\pi x),\ \frac{\partial u}{\partial t}(0,x) = 0,\ 0 < x < 1.$$

- Show that a separation of variables in (1) leads to an eigenvalue problem in the variable x.
- Determine the eigenvalues and the eigenfunctions for the eigenvalue problem in question.
- Determine a solution to (1) which satisfies the boundary and the initial conditions.

Solution

- We assume $u(x,t) = X(x)T(t)$, yielding

$$XT'' - X''T - 2X'T = 0 \ \Rightarrow\ \frac{T''}{T} = \frac{X'' + 2X'}{X} = \lambda$$

for some constant λ. We thus obtain an eigenvalue problem for X:

$$LX = X'' + 2X' = \lambda X$$

subject to $X'(0) = X'(1) = 0$.

- Multiplying the equation by e^x gives

$$\left(e^x X\right)'' = (\lambda + 1)\left(e^x X\right),$$

and hence the general solution is

$$X = e^{-x}\left(C_1 \sinh\left(\sqrt{\lambda+1}x\right) + C_2\cosh\left(\sqrt{\lambda+1}x\right)\right)$$

if $\lambda + 1 > 0$, while

$$X = e^{-x}\left(C_1 \sin\left(\sqrt{-(\lambda+1)}x\right) + C_2\cos\left(\sqrt{-(\lambda+1)}x\right)\right)$$

if $\lambda + 1 < 0$. In the former case ($\lambda + 1 > 0$), we compute that

$$\begin{aligned} X'(0) &= \sqrt{\lambda+1}C_1 - C_2; \\ X'(1) &= \frac{1}{e}\left(\left(\sqrt{\lambda+1}C_2 - C_1\right)\sinh\sqrt{\lambda+1} + \left(\sqrt{\lambda+1}C_1 - C_2\right)\cosh\sqrt{\lambda+1}\right). \end{aligned}$$

For these to simultaneously vanish (and avoid $C_1 = C_2 = 0$), we'd require $\lambda = 0$ or $\lambda = -1$, giving the two eigenfunctions

$$X_0(x) = 1,\ X_{-1}(x) = e^{-x}.$$

In the latter case above ($\lambda+1<0$), we compute

$$\begin{aligned} X'(0) &= \sqrt{-(\lambda+1)}C_1 - C_2; \\ X'(1) &= \frac{1}{e}\left(-\left(\sqrt{-(\lambda+1)}C_2 + C_1\right)\sin\sqrt{-(\lambda+1)} + \left(\sqrt{-(\lambda+1)}C_1 - C_2\right)\cos\sqrt{-(\lambda+1)}\right). \end{aligned}$$

For these to simultaneously vanish, we'd require $\sqrt{-(\lambda+1)} = k\pi$ for $k \geq 0$ integral, i.e., $\lambda_k = -(1+\pi^2 k^2)$, giving the remaining eigenfunctions

$$X_k(x) = e^{-x}\left(\sin k\pi x + k\pi \cos k\pi x\right).$$

- We note that $u(x,t) = X_1(x)$, hence we need only solve $T'' = \lambda_1 T = -(1+\pi^2)T$, yielding

$$T(t) = C_1 \sin\left((1+\pi^2)t\right) + C_2 \cos\left((1+\pi^2)t\right).$$

The condition $T'(0) = 0$ gives $T(t) = \cos\left((1+\pi^2)t\right)$. It follows that

$$u(x,t) = \cos\left((1+\pi^2)t\right) e^{-x}\left(\sin \pi x + \pi \cos \pi x\right).$$

2. Let $\phi \in C^1(\mathbb{R}^2)$. Solve the following Cauchy problem in $\mathbb{R}^3$:

$$\begin{cases} x_1 \partial_{x_1} u + 2x_2 \partial_{x_2} u + \partial_{x_3} u = 3u, \\ u(x_1, x_2, 0) = \phi(x_1, x_2) \end{cases}.$$

Solution

We use the method of characteristics, parametrizing the initial condition curve as $(s_1, s_2) \mapsto (s_1, s_2, 0, \phi(s_1, s_2))$. The system of ODEs results in

$$\begin{aligned} x_1' &= x_1; \\ x_2' &= 2x_2; \\ x_3' &= 1; \\ z' &= 3z. \end{aligned}$$

All equations may be solved immediately, giving

$$\begin{aligned} x_1(t) &= s_1 e^t; \\ x_2(t) &= s_2 e^{2t}; \\ x_3(t) &= t; \\ z(t) &= \phi(s_1, s_2) e^{3t}. \end{aligned}$$

We can solve for s_1, s_2, t in terms of x_1, x_2, x_3:

$$s_1 = x_1 e^{-x_3};\ s_2 = x_2 e^{-2x_3};\ t = x_3.$$

The solution is thus

$$u(x_1, x_2, x_3) = z = \phi\left(x_1 e^{-x_3}, x_2 e^{-2x_3}\right) e^{3x_3}.$$

3. Let $u(x)$ be harmonic in the unit disc $|x| < 1$ in $\mathbb{R}^2$, and assume that $u \geq 0$. Prove the following *Harnack's inequality*:

$$\frac{1-|x|}{1+|x|} u(0) \leq u(x) \leq \frac{1+|x|}{1-|x|} u(0),\ |x| < 1.$$

Solution

u is given by

$$u(x,y) = (1-x^2-y^2)\frac{1}{2\pi}\int\limits_{\xi^2+\eta^2=1} \frac{g(\xi,\eta)}{(x-\xi)^2+(y-\eta)^2} dS_{\xi,\eta}.$$

From the inequalities

$$1-\sqrt{x^2+y^2} \leq \sqrt{(x-\xi)^2+(y-\eta)^2} \leq 1+\sqrt{x^2+y^2},$$

and from the mean value property of u, we obtain

$$u(x,y) \leq \frac{1-x^2-y^2}{\left(1-\sqrt{x^2+y^2}\right)^2}\frac{1}{2\pi}\int g(\xi,\eta)dS_{\xi,\eta} = \frac{1+\sqrt{x^2+y^2}}{1-\sqrt{x^2+y^2}}u(0,0)$$

and

$$u(x,y) \geq \frac{1-x^2-y^2}{\left(1+\sqrt{x^2+y^2}\right)^2}\frac{1}{2\pi}\int g(\xi,\eta)dS_{\xi,\eta} = \frac{1-\sqrt{x^2+y^2}}{1+\sqrt{x^2+y^2}}u(0,0).$$

4. Let $u(x,t) \in C^\infty(\mathbb{R}^3 \times \mathbb{R})$ solve the Cauchy problem for the wave equation

$$\begin{cases} \left(\partial_t^2 - \Delta_x\right)u = 0, \ x \in \mathbb{R}^3, \ t > 0, \\ u|_{t=0} = \phi(x), \ \partial_t\phi|_{t=0} = \psi(x), \end{cases} \qquad (2)$$

with $\phi(x)$ and $\psi(x)$ being smooth compactly supported functions on $\mathbb{R}^3$. Use an explicit formula for the solution of (2) (the Kirchhoff's formula) to show that there exists a constant $C > 0$ such that we have, uniformly in $x \in \mathbb{R}^3$,

$$|u(x,t)| \leq \frac{C}{t}, \ t > 0.$$

Solution

u is given by

$$u(x,t) = \frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{|\xi|=1}\phi(x+t\xi)dS_\xi\right) + t\frac{1}{4\pi}\int_{|\xi|=1}\psi(x+t\xi)dS_\xi.$$

Let $R > 0$ be large enough such that $\phi(x) = \psi(x) = 0$ for $|x| \geq R$ (possible by compact support), and let $M > 0$ be such that $|\phi|, |\psi|, |\phi'| \leq M$ (possible by smoothness), where we denote $\phi'(x) = \phi_{x_1}(x) + \phi_{x_2}(x) + \phi_{x_3}(x)$. We first note that

$$\left|\frac{1}{4\pi}\int_{|\xi|=1}\psi(x+t\xi)dS_\xi\right| = \left|\frac{1}{4\pi t^2}\int_{|\eta+x|=t}\psi(\eta)dS_\eta\right| \leq \frac{R^2}{t^2}M$$

since the maximal surface area of $\{\eta \mid |\eta+x| = t \text{ and } |\eta+x| \leq R\}$ is $4\pi R^2$. Similarly,

$$\left|\frac{1}{4\pi}\int_{|\xi|=1}\phi(x+t\xi)dS_\xi\right|, \left|\frac{1}{4\pi}\int_{|\xi|=1}\phi'(x+t\xi)dS_\xi\right| \leq \frac{R^2}{t^2}M,$$

and so

$$\begin{aligned} |u(x,t)| &= \left|\frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{|\xi|=1}\phi(x+t\xi)dS_\xi\right) + t\frac{1}{4\pi}\int_{|\xi|=1}\psi(x+t\xi)dS_\xi\right| \\ &= \left|\frac{1}{4\pi}\int_{|\xi|=1}\phi(x+t\xi)dS_\xi + \frac{1}{4\pi}t\int_{|\xi|=1}\phi'(x+t\xi)dS_\xi + t\frac{1}{4\pi}\int_{|\xi|=1}\psi(x+t\xi)dS_\xi\right| \\ &\leq \left(\frac{1}{t^2}+\frac{1}{t}+\frac{1}{t}\right)\frac{M}{R^2} \\ &\leq \frac{C}{t} \end{aligned}$$

for some constant C with t bounded away from 0. Of course, we also have the bound

$$|u(x,t)| \leq M(2t+1),$$

which takes care of t near 0.

5. Solve the inhomogeneous problem for the Laplace operator in the unit disc $\mathbb{D} = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$,

$$\begin{cases} \Delta u = x^2 - y^2 \text{ in } \mathbb{D}, \\ u = 0 \text{ along } \partial\mathbb{D}. \end{cases}$$

Solution

We note that,

$$v(x,y) = \frac{1}{12}(x^4 - y^4)$$

satisfies $\Delta v = x^2 - y^2$, but fails to vanish on $\partial\mathbb{D}$. Hence we seek a harmonic function w agreeing with v on $\partial\mathbb{D}$. In polar coordinates,

$$v(r,\theta) = \frac{1}{12}r^4 \cos 2\theta,$$

so that

$$v(1,\theta) = \frac{1}{12}\cos 2\theta.$$

By inspection, we see that

$$w(x,y) = \frac{1}{12}(x^2 - y^2) = \frac{1}{12}r^2 \cos 2\theta$$

is harmonic and agrees with v on $\partial\mathbb{D}$. Hence

$$u(x,y) = v(x,y) - w(x,y) = \frac{1}{12}(x^4 - x^2 - y^4 + y^2).$$

Alternatively, to find w, we can change to polar coordinates, seeking w such that

$$0 = \Delta w = w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta}$$

subject to the boundary conditions $w(1,\theta) = \frac{1}{12}\cos 2\theta$. Setting $w(r,\theta) = R(r)\Theta(\theta)$ and separating variables yields

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \;\Rightarrow\; \frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

for some constant λ. From the periodic boundary conditions for Θ, we thus require $\lambda = k^2$ for $k \geq 0$ integral, giving

$$\Theta(\theta) = c_k \cos k\theta + s_k \sin k\theta.$$

We now examine the differential equation that R satisfies:

$$r^2R'' + rR' - k^2R = 0,$$

which has linearly independent solutions $R = r^k$ and $R = r^{-k}$, and, in the case of $k = 0$, $R = \log r$. Since we desire bounded solutions as $r \to 0$, we limit consideration to $R = r^k$. By linearity, then,

$$w(r,\theta) = \sum_{k\geq 0} r^k \left(c_k \cos k\theta + s_k \sin k\theta\right).$$

The boundary conditions give $c_2 = 1/12$ and all other coefficients 0, hence

$$w(r,\theta) = \frac{1}{12}r^2 \cos 2\theta = \frac{1}{12}(x^2 - y^2),$$

as before.

6. Find the Fourier transform of the integrable function $x \mapsto (\sin x)^2/x^2$.

   Hint. Determine first the Fourier transform of $x \mapsto x^{-1} \sin x$.

   Solution

   Let $\mathcal{F}$ denote the Fourier transformation, i.e., formally at least,

$$\mathcal{F}_x(f(x))(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x) dx.$$

   The following properties of the Fourier transform are easy to derive, at least formally:

$$\begin{aligned} \mathcal{F}_x(1)(\xi) &= 2\pi\delta(\xi); \\ \mathcal{F}_x\left(e^{iax}\right)(\xi) &= 2\pi\delta(\xi - a); \\ \mathcal{F}_x\left(xf(x)\right)(\xi) &= i\frac{d}{d\xi}\mathcal{F}_x(f(x))(\xi); \\ \mathcal{F}_x\left(f(x)g(x)\right)(\xi) &= \left(\mathcal{F}_x(f(x)) * \mathcal{F}_x(g(x))\right)(\xi). \end{aligned}$$

   We thus compute

$$\mathcal{F}_x(\sin x)(\xi) = \mathcal{F}_x\left(\frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\right)(\xi) = \frac{\pi}{i}\left(\delta(\xi - 1) - \delta(\xi + 1)\right).$$

   It follows that

$$\frac{\pi}{i}\left(\delta(\xi - 1) - \delta(\xi + 1)\right) = \mathcal{F}_x(\sin x)(\xi) = \mathcal{F}_x\left(x\frac{1}{x}\sin x\right)(\xi) = i\frac{d}{d\xi}\mathcal{F}_x\left(\frac{1}{x}\sin x\right)(\xi),$$

   hence

$$\mathcal{F}_x\left(\frac{1}{x}\sin x\right)(\xi) = -\pi\int_{-\infty}^{\xi}\left(\delta(\eta - 1) - \delta(\eta + 1)\right)d\eta = \begin{cases} 0, & \xi < -1 \\ \pi, & -1 < \xi < 1 \\ 0, & \xi > 1 \end{cases} = \pi\chi_{[-1,1]}(\xi).$$

   Note that the constant of integration is correct since $\mathcal{F}$ maps L^2 to L^2, so, as $(\sin x)/x$ is in L^2, its Fourier transform must also be in L^2, so must decay at $\pm\infty$. We therefore finally obtain

$$\begin{aligned} \mathcal{F}_x\left(\frac{\sin^2 x}{x^2}\right)(\xi) &= \pi^2\left(\chi_{[-1,1]} * \chi_{[-1,1]}\right)(\xi) \\ &= \pi^2\int_{-\infty}^{\infty}\chi_{[-1,1]}(\eta)\chi_{[-1,1]}(\xi - \eta)d\eta \\ &= \pi^2\int_{-1}^{1}\chi_{[-1,1]}(\xi - \eta)d\eta \\ &= \pi^2\int_{\xi-1}^{\xi+1}\chi_{[-1,1]}(\eta)d\eta \\ &= \begin{cases} 0, & \xi < -2 \\ \pi^2(2 + \xi), & -2 < \xi < 0 \\ \pi^2(2 - \xi), & 0 < \xi < 2 \\ 0, & \xi > 2 \end{cases}. \end{aligned}$$

7. Consider an autonomous system in $\mathbb{R}^n$, $x'(t) = f(x(t))$, where $f = (f_1, f_2, \ldots, f_n)$ is a smooth vector field, such that

$$\sum_{k=1}^{n} x_k f_k(x) < 0 \text{ for } x \neq 0.$$

show that $x(t) \to 0$ as $t \to \infty$, for each solution of the system, independently of the initial condition $x(0)$.

Solution

Notice that

$$\frac{d}{dt}\|x(t)\|_2^2 = 2x(t) \cdot x'(t) = 2x(t) \cdot f(x(t)) < 0$$

for $x(t) \neq 0$. It follows that any given trajectory $x(t)$ eventually leaves any compact set $K \subset \mathbb{R}^n$ not containing 0 (since the quantity $x \cdot f(x) < -\epsilon$ for some $\epsilon > 0$ on K), from which we conclude that $x(t) \to 0$ as $t \to \infty$.