Math 269B, 2012 Winter, Homework 4 (Solutions)

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1 Theory

1. Solve the heat equation $u_t = bu_{xx}$ on a interval $I \subset \mathbb{R}$ with *periodic* boundary conditions. How does $\int_I u(t,x)dx$ vary with time t?

Solution

Let us first suppose that $I = [-\pi, +\pi]$. We proceed as in the text for the heat equation over $\mathbb{R}$, but this time use the Fourier transform defined over periodic functions on $[-\pi, +\pi]$:

$$\hat{u}_m = (\mathcal{F}u)_m := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} e^{-imx} u(x)dx.$$

Applying this Fourier transform in space to $u_t = bu_{xx}$ yields $(\hat{u}_m)_t = -bm^2\hat{u}_m$, an ordinary differential equation in $t \mapsto \hat{u}_m(t)$ which easily solves to $\hat{u}_m(t) = e^{-bm^2 t}\hat{u}_m(0)$. An inverse Fourier transform thus yields

$$\begin{aligned} u(t,x) &= \left(\mathcal{F}^{-1}\hat{u}(t)\right)(x) \\ &:= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{+\infty} e^{imx}\hat{u}_m(t) \\ &= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{+\infty} e^{imx} e^{-bm^2 t}\hat{u}_m(0). \end{aligned}$$

A similar formula holds for general $I =: [\alpha, \beta]$. First, let $\tilde{u} : [-\pi, \pi] \to \mathbb{R}$ be defined as

$$\tilde{u}(t,\xi) := u\left(t, \gamma(\xi)\right), \quad \gamma : [-\pi, \pi] \to I, \xi \mapsto \frac{\alpha(\pi - \xi) + \beta(\xi + \pi)}{2\pi}.$$

Then $\tilde{u}$ and satisfies

$$b\tilde{u}_{\xi\xi} = b\left(\gamma'\right)^2 u_{xx} = \left(\gamma'\right)^2 u_t = \left(\gamma'\right)^2 \tilde{u}_t \quad \Rightarrow \quad \tilde{u}_t = \tilde{b}\tilde{u}_{\xi\xi}, \quad \tilde{b} = \left(\gamma'\right)^{-2} b.$$

(Note that $\gamma' \equiv (\beta - \alpha)/2\pi$.) Thus,

$$\tilde{u}(t,\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{+\infty} e^{im\xi} e^{-\tilde{b}m^2 t} \left(\mathcal{F}\tilde{u}(t=0,\cdot)\right)_m$$

where

$$\begin{aligned} \left(\mathcal{F}\tilde{u}(t=0,\cdot)\right)_m &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{+\pi} e^{-im\xi}\tilde{u}(0,\xi)d\xi \\ &= \frac{\sqrt{2\pi}}{\beta - \alpha} \int_I e^{-im\gamma^{-1}(x)} u_0(x)dx \\ &=: \left(\mathcal{F}u_0\right)_m. \end{aligned}$$

This finally gives us

$$u(t,x) = \tilde{u}\left(t, \gamma^{-1}(x)\right)$$

with $\tilde{u}$ as above.

Clearly, with the representation above, since $\int_{-\pi}^{+\pi} e^{im\xi} d\xi = 0$ except when $m = 0$,

$$\int_I u(t,x)dx = \frac{\beta-\alpha}{2\pi} \int_{-\pi}^{+\pi} \tilde{u}(t,\xi)d\xi = \frac{\beta-\alpha}{\sqrt{2\pi}} \left(\mathcal{F}u_0\right)_0 = \int_I u_0(x)dx.$$

2. (Strikwerda 6.1.4.) Use the representation (6.1.3) to verify the following estimates on the norms of $u(t,x)$:

$$\|u(t,\cdot)\|_1 \leq \|u_0\|_1\,,$$
$$\|u(t,\cdot)\|_\infty \leq \|u_0\|_\infty\,.$$

Show that if u_0 is nonnegative, then

$$\|u(t,\cdot)\|_1 = \|u_0\|_1\,.$$

Solution

For the first inequality, we make use of exchanging the order of integration:

$$\begin{aligned}
\|u(t,\cdot)\|_1 &= \int_{-\infty}^{+\infty} |u(t,x)|\,dx \\
&= \int_{-\infty}^{+\infty} \left| \frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4bt} u_0(y)dy \right| dx \\
&\leq \int_{-\infty}^{+\infty} |u_0(y)| \left(\frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4bt} dx \right) dy \\
&= \int_{-\infty}^{+\infty} |u_0(y)|\,dy \\
&= \|u_0\|_1\,.
\end{aligned}$$

Note that if $u_0 \geq 0$, then the inequality above is actually an equality, we have that $\|u(t,\cdot)\|_1 = \|u_0\|_1$.

The second inequality is derived similarly:

$$\begin{aligned}
\|u(t,\cdot)\|_\infty &= \sup_{x\in\mathbb{R}} |u(t,x)| \\
&= \sup_{x\in\mathbb{R}} \left| \frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4bt} u_0(y)dy \right| \\
&\leq \sup_{x\in\mathbb{R}} \frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4bt} \|u_0\|_\infty\,dy \\
&= \|u_0\|_\infty \sup_{x\in\mathbb{R}} \frac{1}{\sqrt{4\pi bt}} \int_{-\infty}^{+\infty} e^{-(x-y)^2/4bt} dy \\
&= \|u_0\|_\infty\,.
\end{aligned}$$

3. Determine the stability and accuracy of the following combination of the Lax-Wendroff and backward-time central-space schemes to solve $u_t + au_x = bu_{xx}$ (with $b > 0$):

$$\begin{aligned}
0 &= P_{k,h} v_m^n \\
&= \frac{1}{k}\left(v_m^{n+1} - v_m^n\right) + \frac{a}{2h}\left(v_{m+1}^n - v_{m-1}^n\right) - \frac{a^2k}{2h^2}\left(v_{m+1}^n - 2v_m^n + v_{m-1}^n\right) \\
&\quad - \frac{b}{h^2}\left(v_{m+1}^{n+1} - 2v_m^{n+1} + v_{m-1}^{n+1}\right).
\end{aligned}$$

Solution

The symbol corresponding to the differential operator $P_{k,h}$ is

$$\begin{aligned}
p_{k,h}(s,\xi) &:= P_{k,h}\left(e^{skn+imh\xi}\right)/e^{skn+imh\xi} \\
&= \frac{1}{k}\left(e^{sk}-1\right) + i\frac{a}{h}\sin h\xi + \frac{a^2k}{h^2}\left(1-\cos h\xi\right) + \frac{2b}{h^2}e^{sk}\left(1-\cos h\xi\right) \\
&= s + ia\xi + b\xi^2 + O\left(k+h^2\right) \\
&= p(s,\xi) + O\left(k+h^2\right),
\end{aligned}$$

where p is the symbol of the differential operator $P := \partial_t + a\partial_x - b\partial_x^2$. Thus, this scheme is accurate to order $(1,2)$.

Regarding stability, we need to find the roots of $p_{k,h}$ with respect to $g := e^{sk}$, giving

$$g = \frac{1 - ia\lambda\sin\theta - a^2\lambda^2\left(1-\cos\theta\right)}{1 + 2b\mu\left(1-\cos\theta\right)},$$

where $\lambda := k/h$ and $\mu := k/h^2$. Note that the numerator is precisely the amplification factor of the Lax-Wendroff scheme applied to $u_t + au_x = 0$, which we know is stable for $|a\lambda| \le 1$, while the denominator is always at least 1. Hence, $|a\lambda| \le 1$ is certainly sufficient for stability. However, notice that the numerator is $1 + O\left(\lambda + \lambda^2\right)$ while the denominator is $1 + O\left(\mu\right)$, and since $\lambda + \lambda^2 = (h+k)\mu$, we see that the denominator will bound the numerator (uniformly in θ) as $h, k \to 0$. Thus, this scheme is actually unconditionally stable.

2 Programming

1. Solve $u_t + au_x = 0$ numerically using the Lax-Friedrichs scheme. Take $a = 1$, $T = 1$, $x \in [0,1]$ with periodic boundary conditions, and $u_0(x) = \sin 2\pi x$. For each fixed λ within a decreasing sequence of λs (each satisfying the stability criterion), demonstrate convergence with $k/h =: \lambda$ by plotting the logarithm of the L^2-norm of the error (between the analytic solution and the numerical solution) versus the logarithm of h. Verify that the slope suggested by your plot agrees with theory, and estimate the error constant C_λ in the relation error $= C_\lambda h^p$. Use enough values of λ to estimate the relation between C_λ and λ. What appears to happen to C_λ as $\lambda \to 0+$, i.e., as you shrink k relative to h? What happens if, instead of taking $k = \lambda h$, you take $k = h^2$? Explain your numerical results in the context of the theoretical convergence analysis of the Lax-Friedrichs scheme.

   Solution

   The results of the following statements

   ```
   lambdas = 2.^(-(3:0.5:7));
   C = lax_friedrichs_convergence_constant(1, 1, @(x) sin(2*pi*x), lambdas);
   r = corr(log(lambdas), log(C));
   p = polyfit(log(lambdas), log(C), 1);
   plot(log(lambdas), log(C), "o", log(lambdas), polyval(p,log(lambdas)));
   ```

   yields a correlation coefficient of $r = -0.99999$ between $\log\lambda$ and $\log C_\lambda$, with the linear regression giving the relation $C_\lambda = 2.8266\lambda^{-0.61853}$. Clearly, $C_\lambda \to +\infty$ as $\lambda \to 0+$.

   Using code from Homework 1 allows one to easily see that Lax-Friedrichs does not converge if $k = h^2$. This is consistent with the truncation error being $O\left(h^2/k\right)$.

2. Implement the scheme from problem 3 in the Theory section and confirm numerically the theoretical rate of convergence. Use convenient (but non-trivial) initial and boundary conditions such that the solution takes a simple form.

Solution

Using the included code, the results of the following statements

```
test_convergence_lax_wendroff_btcs(1, 1/128, 1, 2.^(-(10:0.5:13)), @(h) 0.5*h);
test_convergence_lax_wendroff_btcs(1, 1/128, 1, 2.^(-(10:0.5:13)), @(h) 2*h);
```

give numerical convergence rates of 1.06 and 1.05, respectively, consistent with the theoretical convergence rate of 1. Note that we set the diffusion coefficient b to be 1/128 to avoid a nearly zero solution once $T = 1$, and that we get convergence even when $|a\lambda| > 1$.

3. Write a function implementing the Thomas algorithm presented in Strikwerda 3.5. Specifically, we solve the system of equations

$$a_i w_{i-1} + b_i w_i + c_i w_{i+1} = d_i, \quad i = 1, \ldots, m-1,$$

with $w_0 = \beta_0$ and $w_m = \beta_m$. The solution is given by

$$w_i = p_{i+1} w_{i+1} + q_{i+1}$$

where p_{i+1} and q_{i+1} are defined recursively by

$$\begin{aligned} p_{i+1} &= -\left(a_i p_i + b_i\right)^{-1} c_i, \\ q_{i+1} &= \left(a_i p_i + b_i\right)^{-1} \left(d_i - a_i q_i\right), \end{aligned}$$

and with p_1 and q_1 determined by the boundary conditions. For the next homework, be prepared to utilize your function implementing the Thomas algorithm to write a function which solves *periodic* tridiagonal systems.

Solution

One can test the included code as follows.

```
N = 10;
a = rand([N 1]); b = rand([N 1]); c = rand([N 1]);
A = spdiags([[a(2:N);0] b [0;c(1:N-1)]], [-1 0 +1], N, N);
x = rand([N 1]);
d = A*x;
y = solve_tridiag(a,b,c,d);
norm(x-y, "inf")
```