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Applied Differential Equations Qualifying Exam, Winter 2004

1. Consider the differential equation:

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) + \lambda u(x,y) = 0 \qquad (1)$$

in the strip $\{(x,y) \mid 0 < y < \pi,\ -\infty < x < \infty\}$ with boundary conditions

$$u(x,0) = 0,\ u(x,\pi) = 0. \qquad (2)$$

Find all bounded solutions of the boundary value problem (1),(2) when

(a) $\lambda = 0$,
(b) $\lambda > 0$,
(c) $\lambda < 0$.

Solution

We assume $u(x,y) = X(x)Y(y)$, and separate variables to get

$$0 = X''Y + XY'' + \lambda XY \ \Rightarrow\ \frac{Y''}{Y} = -\frac{X'' + \lambda X}{X} = \mu$$

for some constant μ. The boundary conditions on Y, namely, $Y(0) = Y(\pi) = 0$, imply that $Y = \sin(ky)$ with $\mu_k = -k^2$ for $k \geq 1$ integral. The equation that X then satisfies

$$X'' + (\lambda - k^2)X = 0,$$

subject to the condition that $X(x)$ be bounded on $-\infty < x < \infty$. Such nontrivial solutions only exist when $\lambda \geq k^2$. It follows that, if $\lambda \geq 1$,

$$u(x,y) = \sum_{1 \leq k \leq \sqrt{\lambda}} c_k X_k(x) \sin(ky)$$

for some constants c_k. Otherwise, the only bounded solution is the trivial solution $u \equiv 0$.

2. Let $C^2\left(\overline{\Omega}\right)$ be the space of twice continuously differentiable functions in the bounded smooth closed domain $\overline{\Omega} \subset \mathbb{R}^2$. Let $u_0(x,y)$ be the function that minimizes the functional

$$D(u) = \iint_\Omega \left(\left(\frac{\partial u}{\partial x}(x,y)\right)^2 + \left(\frac{\partial u}{\partial y}(x,y)\right)^2 + f(x,y)u(x,y) \right) dxdy + \int_{\partial\Omega} a(s)u(x(s),y(s))^2 ds,$$

where $f(x,y)$ and $a(s)$ are given continuous functions and ds is the arclength element on $\partial\Omega$.

Find the differential equation and the boundary condition that u_0 satisfies.

Solution

If $u = u_0$ minimizes D, then $g(\epsilon) = D(u+\epsilon v)$ must have vanishing derivative at $\epsilon = 0$ for all $v \in C^2\left(\overline{\Omega}\right)$. We have

$$g(\epsilon) = D(u+\epsilon v) = \int_\Omega \left(|\nabla(u+\epsilon v)|^2 + f(u+\epsilon v)\right) + \int_{\partial\Omega} a(u+\epsilon v)^2,$$

so we compute

$$\begin{aligned} g'(0) &= \int_\Omega (2\nabla u \cdot \nabla v + fv) + \int_{\partial\Omega} 2auv \\ &= \int_\Omega (-2\Delta u + f)\, v + \int_{\partial\Omega} (2au + \nabla u \cdot \nu)\, v. \end{aligned}$$

For $g'(0)$ to vanish for all v, we'd thus require $\Delta u = \frac{1}{2} f$ on Ω and $2au + \partial u / \partial \nu = 0$ on $\partial\Omega$.

3. Let $f(x_1, x_2)$ be a continuous function with compact support. Define

$$u(x_1, x_2) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} \frac{f(y_1, y_2)}{z - w} dy_1 dy_2$$

where $z = x_1 + ix_2$, $w = y_1 + iy_2$. Prove that

$$\frac{\partial u}{\partial x_1} + i \frac{\partial u}{\partial x_2} = f(x_1, x_2) \text{ in } \mathbb{R}^2.$$

Solution

By a change of variables,

$$u(x_1, x_2) = \frac{1}{2\pi} \iint \frac{f(x_1 - z_1, x_2 - z_2)}{z_1 + iz_2} dz_1 dz_2.$$

Due to the integrability of the integrand near 0, we can say that

$$u(x_1, x_2) = \lim_{\epsilon \searrow 0} \frac{1}{2\pi} \iint_{|z| \geq \epsilon} \frac{f(x_1 - z_1, x_2 - z_2)}{z_1 + iz_2} dz_1 dz_2.$$

We thus compute, for $\epsilon > 0$, using integration by parts (and keeping in mind that f vanishes for large $|z|$),

$$\begin{aligned}
&\frac{\partial}{\partial x_1} \left(\frac{1}{2\pi} \iint_{|z| \geq \epsilon} \frac{f(x_1 - z_1, x_2 - z_2)}{z_1 + iz_2} dz_1 dz_2 \right) \\
&= \frac{1}{2\pi} \iint_{|z| \geq \epsilon} \frac{\frac{\partial}{\partial x_1} f(x_1 - z_1, x_2 - z_2)}{z_1 + iz_2} dz_1 dz_2 \\
&= -\frac{1}{2\pi} \iint_{|z| \geq \epsilon} \frac{\frac{\partial}{\partial z_1} f(x_1 - z_1, x_2 - z_2)}{z_1 + iz_2} dz_1 dz_2 \\
&= -\frac{1}{2\pi} \int_{|z| = \epsilon} f(x_1 - z_1, x_2 - z_2) \frac{1}{z_1 + iz_2} \cdot \nu_1 ds_{z_1, z_2} \\
&\quad + \frac{1}{2\pi} \iint_{|z| \geq \epsilon} f(x_1 - z_1, x_2 - z_2) \frac{\partial}{\partial z_1} \frac{1}{z_1 + iz_2} dz_1 dz_2.
\end{aligned}$$

Now

$$\frac{\partial}{\partial z_1} \frac{1}{z_1 + iz_2} = -\frac{1}{(z_1 + iz_2)^2},$$

while, since $\nu_1 = -z_1/\epsilon$ (the inward normal) and $z_1^2 + z_2^2 = \epsilon^2$ on $|z| = \epsilon$,

$$\frac{1}{z_1 + iz_2} \cdot \nu_1 = \frac{z_1 - iz_2}{z_1^2 + z_2^2} \cdot \left(-\frac{z_1}{\epsilon} \right) = -\frac{z_1^2}{\epsilon^3}.$$

We thus obtain

$$\begin{aligned}
&\frac{\partial}{\partial x_1} \left(\frac{1}{2\pi} \iint_{|z| \geq \epsilon} \frac{f(x_1 - z_1, x_2 - z_2)}{z_1 + iz_2} dz_1 dz_2 \right) \\
&= \frac{1}{2\pi\epsilon^3} \int_{|z| = \epsilon} f(x_1 - z_1, x_2 - z_2) z_1^2 ds_{z_1, z_2} \\
&\quad - \frac{1}{2\pi} \iint_{|z| \geq \epsilon} f(x_1 - z_1, x_2 - z_2) \frac{1}{(z_1 + iz_2)^2} dz_1 dz_2.
\end{aligned}$$

A similar derivation gives

$$
\begin{aligned}
& i\frac{\partial}{\partial x_2}\left(\frac{1}{2\pi}\iint_{|z|\geq\epsilon}\frac{f(x_1-z_1,x_2-z_2)}{z_1+iz_2}dz_1dz_2\right) \\
= \ & \frac{1}{2\pi\epsilon^3}\int_{|z|=\epsilon} f(x_1-z_1,x_2-z_2)z_2^2 ds_{z_1,z_2} \\
& +\frac{1}{2\pi}\iint_{|z|\geq\epsilon} f(x_1-z_1,x_2-z_2)\frac{1}{(z_1+iz_2)^2}dz_1dz_2,
\end{aligned}
$$

and so

$$
\begin{aligned}
& \left(\frac{\partial}{\partial x_1}+i\frac{\partial}{\partial x_2}\right)\left(\frac{1}{2\pi}\iint_{|z|\geq\epsilon}\frac{f(x_1-z_1,x_2-z_2)}{z_1+iz_2}dz_1dz_2\right) \\
= \ & \frac{1}{2\pi\epsilon}\int_{|z|=\epsilon} f(x_1-z_1,x_2-z_2)ds_{z_1,z_2} \\
\to \ & f(x_1,x_2)
\end{aligned}
$$

as $\epsilon \searrow 0$. The claim then follows:

$$\frac{\partial u}{\partial x_1}+i\frac{\partial u}{\partial x_2}=f(x_1,x_2).$$

4. Consider the boundary value problem on $[0,\pi]$:

$$
\begin{aligned}
y''(x)+p(x)y(x)=f(x), \quad 0<x<\pi, \qquad (1) \\
y(0)=0, \quad y'(\pi)=0. \qquad (2)
\end{aligned}
$$

Find the smallest λ_0 such that the boundary value problem (1),(2) has a unique solution whenever $p(x)>\lambda_0$ for all x. Justify your answer.

Solution

Denote by L the linear differential operator defined by $Ly=y''+py$, with the given boundary conditions. Then it is easy to see that L is self-adjoint in the usual L^2-inner product (because of the boundary conditions on y):

$$(Ly,z)=\int(y''+py)z=\int y(z''+pz)=(y,Lz),$$

hence the eigenfunctions of L form an orthogonal basis, demonstrating existence of a solution to (1),(2). Uniqueness requires that the null space of L be trivial. Suppose $Ly=0$. Then

$$0=(Ly,y)=\int(y''+py)y=\int(py^2-(y')^2);$$

If $p\leq 0$, then we'd be able to conclude that $y=0$, and we'd get uniqueness.

5. Consider the Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\ y>0,\ -\infty<x<\infty \qquad (1)$$

with the boundary condition

$$\frac{\partial u}{\partial y}(x,0)-u(x,0)=f(x),$$

where $f(x) \in C_0^\infty(\mathbb{R}^1)$. Find a bounded solution $u(x,y)$ of (1),(2) and show that $u(x,y) \to 0$ when $|x| + y \to \infty$.

Solution

[F06.4]

6. Consider the first-order system $u_t - u_x = v_t + v_x = 0$ in the diamond-shaped region $-1 < x + t < 1$, $-1 < x - t < 1$. For each of the following boundary value problems, state whether this problem is well-posed. If it is well-posed, find the solution.

   (a) $u(x,t) = u_0(x+t)$ on $x - t = -1$, $v(x-t) = v_0(x-t)$ on $x + t = -1$.

   (b) $v(x,t) = v_0(x+t)$ on $x - t = -1$, $u(x-t) = u_0(x-t)$ on $x + t = -1$.

   Solution

   We note that the characteristics for u lie on $x + t = \text{const}$, while the characteristics for v lie on $x - t = \text{const}$.

   (a) The initial condition curves for u and v lie nontangentially (in fact, orthogonally) to their respective characteristic curves, hence this is well-posed, with solutions $u(x,t) = u_0(x+t)$ and $v(x,t) = v_0(x-t)$.

   (b) The initial condition curves for u and v lie along their respective characteristic curves, hence this is not well-posed.

7. For the two-point boundary value problem $Lf = f_{xx} - f$ on $-\infty < x < \infty$ with $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$, the Green's function $G(x,x')$ solves $LG = \delta(x - x')$, in which L acts on the variables x.

   (a) Show that $G(x,x') = G(x - x')$.

   (b) For each x', show that

   $$G(x,x') = \begin{cases} a_- e^x, & \text{for } x < x' \\ a_+ e^{-x}, & \text{for } x' < x \end{cases},$$

   in which $a_\pm$ are functions that depend only on x'.

   (c) Using (a), find the x' dependence of $a_\pm$.

   (d) Finish finding $G(x,x')$ by using the jump conditions to find the remaining unknowns in $a_\pm$.

   Solution

   (a) By the definition of G,

   $$f(x') = \int L(G(\cdot, x'))(x) f(x) dx = \int G(x,x')(Lf)(x) dx.$$

   Now consider $(x,x') \mapsto G(x - x', 0)$. By changing variables,

   $$\int G(x - x', 0)(Lf)(x) dx = \int G(y,0)(Lf)(y + x') dy = \int G(y,0) L(f \circ \tau_{x'})(y) dy,$$

   where $\tau_{x'}$ denotes the shift map $y \mapsto y + x'$ (and it commutes with L, justifying the last equality). But by the first set of equalities, it follows that

   $$\int G(y,0) L(f \circ \tau_{x'})(y) dy = (f \circ \tau_{x'})(0) = f(x'),$$

i.e.,

$$\int G(x,x')(Lf)(x)dx = \int G(x-x',0)(Lf)(x)dx,$$

and since this is true for all f, we find that $G(x,x') = G(x-x',0) = G(x-x')$. Alternatively, one can note that

$$\begin{aligned}
\int G_x(x,x')(Lf)(x)dx &= -\int G(x,x')(Lf')(x)dx \\
&= -f'(x') \\
&= -\frac{d}{dx'}\int G(x,x')(Lf)(x)dx \\
&= -\int G_{x'}(x,x')(Lf)(x)dx,
\end{aligned}$$

so that $G_x + G_{x'} = 0$, hence $G(x,x') = G(x-x')$.

(b) Since

$$L(G(\cdot,x'))(x) = \delta(x-x') = 0$$

for x away from x', $G(x,x')$ must satisfy $G_{xx}(x,x') - G(x,x') = 0$ for $x < x'$ and $x > x'$. The general solution is $C_1(x')e^x + C_2(x')e^{-x}$, and the decay requirements of $G(x,x')$ at $x = \pm\infty$ dictates that

$$G(x,x') = \begin{cases} a_-(x')e^x, & x < x' \\ a_+(x')e^{-x}, & x > x' \end{cases}.$$

(c) From (a),

$$G(x,x') = G(x-x',0) = \begin{cases} a_-(0)e^{x-x'}, & x-x' < 0 \\ a_+(0)e^{-(x-x')}, & x-x' > 0 \end{cases},$$

so we find that $a_\pm(x') = a_\pm(0)e^{\pm x'}$.

(d) We have that

$$f(x') = \int_{-\infty}^{\infty} G(x-x')(Lf)(x)dx = \lim_{\epsilon\searrow 0}\int_{|x-x'|>\epsilon} G(x-x')(f''(x) - f(x))dx.$$

We compute

$$\begin{aligned}
&\int_{|x-x'|>\epsilon} G(x-x')(f''(x)-f(x))dx \\
&= -G(x-x')f'(x)\Big|_{x'-\epsilon}^{x'+\epsilon} + \int_{|x-x'|>\epsilon} G'(x-x')(-f'(x)-f(x))dx \\
&= G(-\epsilon)f'(x'-\epsilon) - G(\epsilon)f'(x'+\epsilon) + \int_{|x-x'|>\epsilon} G'(x-x')(-f'(x)-f(x))dx.
\end{aligned}$$

We desire the above boundary term to vanish (in the limit as $\epsilon \searrow 0$), hence we require G to be continuous at 0, hence $a_-(0) = a_+(0)$. We continue applying integration by parts one more time:

$$\begin{aligned}
\int_{|x-x'|>\epsilon} G'(x-x')(-f'(x)-f(x))dx &= G'(x-x')f(x)\Big|_{x'-\epsilon}^{x'+\epsilon} \\
&= G'(\epsilon)f(x'+\epsilon) - G'(-\epsilon)f(x'-\epsilon) \\
&= -a_+(0)e^{-\epsilon}f(x'+\epsilon) - a_-(0)e^{\epsilon}f(x'-\epsilon).
\end{aligned}$$

We desire this boundary term to tend to $f(x')$ as $\epsilon \searrow 0$, giving $a_+(0) + a_-(0) = -1$. It follows that $a_\pm(0) = -1/2$, and

$$G(x - x') = \begin{cases} -\frac{1}{2}e^{x-x'}, & x < x' \\ -\frac{1}{2}e^{-(x-x')}, & x > x' \end{cases} = -\frac{1}{2}e^{-|x-x'|}.$$

8. For the ODE

$$\begin{aligned} u_t &= u - v^2; \\ v_t &= v - u^2; \end{aligned}$$

do all of the following:

(a) Find all stationary points.

(b) Analyze their type.

(c) Show that $u = v$ is an invariant set for this ODE, i.e., if $u(0) = v(0)$, then $u(t) = v(t)$ for all t.

(d) Draw the phase plane for this system.

Solution

(a) Let $F(u, v) = (u - v^2, v - u^2)$. Then a stationary point $(u, v)^*$ satisfies $F((u, v)^*) = 0$, hence $(u, v)^* \in \{(0, 0), (1, 1)\}$.

(b) We compute

$$DF(u, v) = \begin{pmatrix} 1 & -2v \\ -2u & 1 \end{pmatrix}.$$

- $(u, v)^* = (0, 0)$. The sole eigenvalue of

$$DF(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  is $\lambda = 1$. Thus, $(0, 0)$ is an unstable node.
- $(u, v)^* = (1, 1)$. The eigenvalues of

$$DF(1, 1) = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$$

  are $\lambda_\pm = 1 \pm 2$. The corresponding eigenvalues are

$$v_\pm = \begin{pmatrix} 1 \\ \mp 1 \end{pmatrix}.$$

  Thus, $(1, 1)$ is a saddle.

(c) Let $w = u - v$. Then it is easy to get that w satisfies

$$w_t = (1 + u + v)w.$$

If $w(0) = 0$, then by uniqueness, $w(t) = 0$ for all t. In other words, if $u(0) = v(0)$, then $u(t) = v(t)$ for all t.

(d)

9. Consider the initial value problem

$$u_{tt} = \Delta u$$

for $x \in \mathbb{R}^d$ and $t > 0$, and with $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$, in which $u_0(x) = u_1(x) = 0$ for $|x| > R_1$ and $|x| > R_2$. For $d = 2$ and $d = 3$, find the largest set $\Omega_0 \subset \{x \in \mathbb{R}^d,\ t > 0\}$ on which $u = 0$ for any choice of u_0.

Solution

Let $R = \max\{R_1, R_2\}$. First consider $d = 2$. The domain of a dependence for a point $(x_0, t_0) \in \mathbb{R}^2 \times \mathbb{R}$ is the interior of the disc $|x - x_0| \leq t_0$ in $\mathbb{R}^2$. Thus, the largest set Ω_0 on which we can be sure that $u = 0$ is $\Omega_0 = \{(x,t) \in \mathbb{R}^2 \times \mathbb{R} \mid |x| > R + t\}$. On the other hand, for the case $d = 3$, the domain of dependence for a point $(x_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}$ is the surface of the sphere $|x - x_0| = t_0$ in $\mathbb{R}^3$. Thus, the largest set Ω_0 on which we can be sure that $u = 0$ is $\Omega_0 = \{(x,t) \in \mathbb{R}^3 \times \mathbb{R} \mid |x| > R + t \text{ or } |x| < t - R\}$.