1. *Solve the following initial-boundary value problem for the wave equation with a potential term,*

$$\begin{cases} u_{tt} - u_{xx} + u = 0, \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad u_t(x,0) = 0, \quad 0 < x < \pi, \end{cases}$$

where

$$f(x) = \begin{cases} x & \text{if } x \in (0, \pi/2), \\ \pi - x & \text{if } x \in (\pi/2, \pi). \end{cases}$$

The answer should be given in terms of an infinite series of explicitly given functions.

Applying separation of variables, the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin nx.$$

Substituting into the PDE yields

$$\begin{aligned} \sum a_n''(t) \sin nx + \sum n^2 a_n(t) \sin nx + a_n(t) \sin nx &= 0 \\ a_n''(t) + (n^2+1) a_n(t) &= 0, \end{aligned}$$

which implies that $a_n(t) = c_n \cos \sqrt{n^2+1}t + d_n \sin \sqrt{n^2+1}t$. The initial condition $u_t(x,0) = 0$ determines that $d_n = 0$ for all n. From the other initial condition $u(x,0) = f(x)$, the c_n are the Fourier series coefficients for f, $f(x) = \sum_n c_n \sin nx$.

$$\begin{aligned} c_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, \mathrm{d}x = \frac{2}{\pi} \Big[\int_0^{\frac{\pi}{2}} x \sin nx \, \mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} (\pi - x) \sin nx \, \mathrm{d}x \Big] \\ &= \frac{2}{\pi} \left[\left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_0^{\frac{\pi}{2}} + \pi \left[-\frac{\cos nx}{n} \right]_{\frac{\pi}{2}}^{\pi} - \left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right]_{\frac{\pi}{2}}^{\pi} \right] \\ &= \frac{4}{\pi} \frac{\sin n\frac{\pi}{2}}{n^2} = \begin{cases} \frac{4}{\pi n^2} (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Thus the solution is

$$u(x,t) = \sum_{k=0}^{\infty} \frac{4(-1)^k}{\pi(2k+1)^2} \cos[t\sqrt{(2k+1)^2+1}\,] \sin[x(2k+1)].$$

2. *Let $u(x,t)$ be a bounded solution to the Cauchy problem for the heat equation*

$$\begin{cases} u_t = a^2 u_{xx}, \quad t>0, \quad x \in \mathbb{R}, \quad a>0, \\ u(x,0) = \varphi(x). \end{cases}$$

Here $\varphi(x) \in C(\mathbb{R})$ satisfies

$$\lim_{x\to+\infty} \varphi(x) = b, \qquad \lim_{x\to-\infty} \varphi(x) = c.$$

Compute the limit of $u(x,t)$ as $t \to +\infty$, $x \in \mathbb{R}$. Justify your argument carefully.

The solution is

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{4a^2\pi t}} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{(x-y)^2}{4a^2 t}} \varphi(y)\,\mathrm{d}y \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-z^2} \varphi(x + z\sqrt{4a^2 t})\,\mathrm{d}z \qquad \text{substitute } z = \tfrac{y-x}{\sqrt{4a^2t}},\ \mathrm{d}z = \tfrac{1}{\sqrt{4a^2t}}\mathrm{d}y \\ &= \frac{1}{\sqrt{\pi}} \left(\int_{-\infty}^{-\epsilon} + \int_{-\epsilon}^{\epsilon} + \int_{\epsilon}^{\infty} \right) \mathrm{e}^{-z^2} \varphi(x + z\sqrt{4a^2 t})\,\mathrm{d}z. \end{aligned}$$

The boundary conditions and continuity imply that φ is bounded, let $|\varphi(x)| \le M$. The integrand is dominated by $M\mathrm{e}^{-z^2}$, so by dominated convergence,

$$\begin{aligned} \int_{-\infty}^{-\epsilon} \mathrm{e}^{-z^2} \varphi(x + z\sqrt{4a^2 t})\,\mathrm{d}z + \int_{\epsilon}^{\infty} \mathrm{e}^{-z^2} \varphi(x + z\sqrt{4a^2 t})\,\mathrm{d}z \quad &\xrightarrow[t\to\infty]{} \quad \int_{-\infty}^{-\epsilon} \mathrm{e}^{-z^2} c\,\mathrm{d}z + \int_{\epsilon}^{\infty} \mathrm{e}^{-z^2} b\,\mathrm{d}z \\ &\xrightarrow[\epsilon\downarrow 0]{} \quad \frac{\sqrt{\pi}}{2}c + \frac{\sqrt{\pi}}{2}b. \end{aligned}$$

For any t we have

$$\int_{-\epsilon}^{\epsilon} \mathrm{e}^{-z^2} \varphi(x + z\sqrt{4a^2 t})\,\mathrm{d}z \ < \ \int_{-\epsilon}^{\epsilon} 1 \cdot M\,\mathrm{d}z \ = \ 2\epsilon M \ \longrightarrow \ 0 \qquad \text{as } \epsilon \downarrow 0.$$

Therefore,

$$\lim_{t\to\infty} u(x,t) \ = \ \frac{1}{\sqrt{\pi}} \left(\frac{\sqrt{\pi}}{2}c + 0 + \frac{\sqrt{\pi}}{2}b \right) \ = \ \frac{b+c}{2}.$$

3. *Consider the following damped wave equation,*

$$\begin{cases} u_{tt} - \Delta u + a(x) u_t = 0, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}, \\ u_{|t=0} = u_0, \quad u_{t|t=0} = u_1. \end{cases}$$

Here the damping coefficient $a \in C_0^\infty(\mathbb{R}^3)$ *is a nonnegative function and* $u_0, u_1 \in C_0^\infty(\mathbb{R}^3)$. *Show that the energy of the solution* $u(x,t)$ *at time* t,

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla_x u|^2 + |u_t|^2)$$

is a decreasing function of t *for* $t > 0$.

Claim: Fix $x_0 \in \mathbb{R}^3$ and $t_0 > 0$. Let $C = \{(x,t) \,:\, 0 \le t \le t_0, |x - x_0| \le |t_0 - t|\}$ be the cone of dependence for point (x_0, t_0) and define $B_\tau = C \cap \{t = \tau\}$. Then $u = u_t = 0$ in B_0 implies $u \equiv 0$ in C.

Proof: . Let $e(\tau) = \int_{B_\tau} \frac{1}{2}(|u_t|^2 + |\nabla u|^2)\,\mathrm{d}x$, then

$$\begin{aligned} e'(\tau) &= \int_{B_\tau} u_t u_{tt} + \nabla u \cdot \nabla u_t \\ &= \int_{B_\tau} u_t u_{tt} - \int_{B_\tau} u_t \Delta u + \int_{\partial B_\tau} u_t \tfrac{\partial u}{\partial n} \qquad \text{the third term is zero since } \partial B_\tau = \emptyset \\ &= \int_{B_\tau} u_t(\Delta u - a u_t) - \int_{B_\tau} u_t \Delta u \\ &= -\int_{B_\tau} a u_t^2 \le 0. \end{aligned}$$

Also observing that $e(0) = 0$ and $e(\tau) \ge 0$, we have $e(\tau) = 0$ for $\tau \ge 0$. So $u_t = \nabla u = u_{tt} \equiv 0$ in C and hence $u \equiv 0$ in C.

The claim implies that the solution has finite propagation speed: $u(x,t) \equiv 0$ for $d(x,K) > t$, where $K = \operatorname{supp} u_0 \cup \operatorname{supp} u_1$. For any fixed t, there exists an open bounded set K_t such that $u(x,t) \equiv 0$ on K_t^c. Therefore, $E(t)$ is a decreasing function of t,

$$\begin{aligned} E'(t) &= \int_{K_t} u_t u_{tt} + \nabla u \cdot \nabla u_t \\ &= \int_{K_t} u_t(\Delta u - a u_t) - \int_{K_t} u_t \Delta u + \int_{\partial K_t} \underbrace{u_t \tfrac{\partial u}{\partial n}}_{=0} \\ &= -\int_{K_t} a u_t^2 \le 0. \end{aligned}$$

4. *Prove that each solution (except $x_1 = x_2 = 0$) of the autonomous system*

$$\begin{cases} x_1' &= x_2 + x_1(x_1^2 + x_2^2) \\ x_2' &= -x_1 + x_2(x_1^2 + x_2^2) \end{cases}$$

blows up in finite time. What is the blow-up time for the solution which starts at the point $(1, 0)$ when $t = 0$?

Multiplying the first equation by x_1 and the second by x_2 and then adding them,

$$\begin{cases} x_1'x_1 &= x_1x_2 + x_1^2(x_1^2 + x_2^2) \\ x_2'x_2 &= -x_1x_2 + x_2^2(x_1^2 + x_2^2) \end{cases} \qquad \Longrightarrow \qquad \begin{aligned} x_1'x_1 + x_2'x_2 &= x_1^2(x_1^2 + x_2^2) + x_2^2(x_1^2 + x_2^2) \\ \tfrac{1}{2}(x_1^2 + x_2^2)' &= (x_1^2 + x_2^2)^2, \end{aligned}$$

yields the one-dimensional equation

$$\tfrac{1}{2}r' = r^2.$$

Then

$$\begin{aligned} \int \frac{\mathrm{d}r}{2r^2} &= \int \mathrm{d}t \\ \frac{-1}{2r} &= t + C \qquad\qquad \frac{-1}{2r(0)} = C \\ r(t) &= \frac{-1}{2(t - \frac{1}{2r(0)})}. \end{aligned}$$

With the exception of $x_1(0) = x_2(0) = 0$, the solution blows up at $t = \frac{1}{2(x_1(0)^2 + x_2(0)^2)} < \infty$. The blow-up time from the starting point $(1, 0)$ is $t = \frac{1}{2}$.

5. *Let us consider a generalized Volterra-Lotka system in the plane, given by*

$$x'(t) = f(x(t)), \qquad x(t) \in \mathbb{R}^2,$$

where $f(x) = (f_1(x), f_2(x)) = (ax_1 - bx_1x_2 - ex_1^2, -cx_2 + dx_1x_2 - fx_2^2)$*, and* a, b, c, d, e, f *are positive constants. Show that*

$$\nabla \cdot (\varphi f) \neq 0, \qquad x_1 > 0,\ x_2 > 0,$$

where $\varphi(x_1, x_2) = 1/(x_1x_2)$*. Using this observation, prove that the autonomous system has no closed orbits in the first quadrant.*

Indeed, $\nabla \cdot (\varphi f) \neq 0$:

$$\begin{aligned} \nabla \cdot (\varphi f) &= \nabla\varphi \cdot f + \varphi \nabla \cdot f \\ &= -\frac{f_1}{x_1^2 x_2} - \frac{f_2}{x_1 x_2^2} + \frac{1}{x_1x_2}(ax_1 - bx_1x_2 - ex_1^2 - cx_2 + dx_1x_2 - fx_2^2) \\ &= \frac{1}{x_1x_2}(-ex_1 - fx_2) < 0. \end{aligned}$$

Suppose there is a closed orbit in the first quadrant. Let Ω be a domain with the closed orbit as its boundary. Then

$$\begin{aligned} \int_\Omega \nabla \cdot (\varphi f) \;=\; \int_{\partial\Omega} (\varphi f) \cdot n \;&=\; \int_{\partial\Omega} \varphi(f_1, f_2) \cdot (-x_2', x_1') \\ &=\; \int_{\partial\Omega} \varphi(x_1', x_2') \cdot (-x_2', x_1') \;=\; 0. \end{aligned}$$

Since φ and f are continuous, this implies that $\nabla \cdot (\varphi f) = 0$ for some $(x_1, x_2) \in \Omega$; contradiction. Therefore, there is no closed orbit in the first quadrant.

6. *Let $q \in C_0^1(\mathbb{R}^3)$. Prove that the vector field*

$$u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{q(y)(x-y)}{|x-y|^3} \, \mathrm{d}y$$

enjoys the following properties:

1. $u(x)$ *is conservative*
2. $\nabla \cdot u(x) = q(x)$ *for all* $x \in \mathbb{R}^3$
3. $|u(x)| = \mathcal{O}(|x|^{-2})$ *for large* x.

Furthermore, prove that the properties (1), (2), and (3) above determine the vector field $u(x)$ *uniquely.*

A fundamental solution of $\Delta v = f$ in three dimensions is $K(x) = -\frac{1}{4\pi}\frac{1}{\sqrt{x_1^2+x_2^2+x_3^2}}$. Then $\nabla K(x) = \frac{1}{4\pi}\frac{(x_1,x_2,x_3)}{(\sqrt{x_1^2+x_2^2+x_3^2})^3} = \frac{1}{4\pi}\frac{x}{|x|^3}$, so $u = q * \nabla K = \nabla(q * K)$, which implies property (1).

For property (2), compute $\nabla \cdot u = \Delta(q * K) = q * \Delta K = q * \delta = q$.

For property (3), let R be sufficiently large such that $B_R(0) \supset \operatorname{supp} q$, then for $|x| \geq 2R$,

$$\begin{aligned}
|u(x)| \leq \frac{1}{4\pi}\int_{B_R(0)} \frac{|q(y)|}{|x-y|^2}\,\mathrm{d}y &\leq \frac{1}{4\pi}||q||_{L^\infty}\int_{B_R(0)} \frac{1}{||x|-R|^2}\,\mathrm{d}y \\
&\leq \frac{1}{4\pi}||q||_{L^\infty}|B_R(0)|\left(\frac{2}{|x|}\right)^2 \qquad \text{since } |x|-R \geq \tfrac{|x|}{2} \\
&\leq \frac{1}{\pi}||q||_{L^\infty}|B_R(0)|\,|x|^{-2} = \mathcal{O}(|x|^{-2}).
\end{aligned}$$

Consider the problems

$$\begin{cases} u = \nabla f_1 \\ \nabla \cdot u = q \\ |u| = \mathcal{O}(|x|^{-2}) \end{cases} \qquad \begin{cases} v = \nabla f_2 & f_1, f_2 \in C(\mathbb{R}^3) \\ \nabla \cdot v = q \\ |v| = \mathcal{O}(|x|^{-2}) & \text{for large } |x| \end{cases}$$

Let $w = u - v$ and $f = f_1 - f_2$, then

$$\begin{cases} w = \nabla f \\ \nabla \cdot w = 0 \\ |w| = \mathcal{O}(|x|^{-2}) \end{cases} \implies \begin{cases} \Delta f = 0 & (\text{since } w = \nabla f \text{ and } \nabla \cdot w = 0) \\ |f| = \mathcal{O}(|x|^{-1}) & \text{for large } |x| \end{cases}$$

Since f is continuous and $|f| = \mathcal{O}(|x|^{-1})$ for large $|x|$, f is bounded. By Liouville's theorem, bounded harmonic functions must be constant; $f \equiv C$. Then $w = \nabla C = 0$, so u is uniquely determined.

7. *Consider the partial differential equation*

$$uu_z + u_t + u = 0, \quad (z,t) \in \mathbb{R}^2.$$

- *Find the particular solution that satisfies the condition* $u(0,t) = \mathrm{e}^{-2t}$.
- *Show that at the point* $(z,t) = (1/9, \log 2)$, $u = 1/3$.

Applying the method of characteristics,

$$\begin{array}{rclcrclcrcl}
z' &=& u & & t' &=& 1 & & u' &=& -u \\
z(0) &=& 0 & & t(0) &=& s & & u(0) &=& \mathrm{e}^{-2s} \\
z &=& \mathrm{e}^{-2s} - \mathrm{e}^{-\tau-2s} & & t &=& s+\tau & & u &=& \mathrm{e}^{-\tau-2s} \\
\mathrm{e}^{-s} &=& \frac{1}{2}(\mathrm{e}^{-t} + \sqrt{\mathrm{e}^{-2t}+4z}) & & \tau &=& t-s & & u &=& \mathrm{e}^{-t}\mathrm{e}^{-s}
\end{array}$$

we have $u(z,t) = \frac{1}{2}\mathrm{e}^{-t}(\mathrm{e}^{-t} + \sqrt{\mathrm{e}^{-2t}+4z})$.

At $(z,t) = (1/9, \log 2)$,

$$\begin{aligned}
u &= \tfrac{1}{2}\mathrm{e}^{-\log 2}\left(\mathrm{e}^{-\log 2} + \sqrt{\mathrm{e}^{-2\log 2} + \tfrac{4}{9}}\right) \\
&= \tfrac{1}{4}\left(\tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \tfrac{4}{9}}\right) = \tfrac{1}{4}(\tfrac{1}{2} + \tfrac{5}{6}) = \tfrac{1}{3}.
\end{aligned}$$

8. *The function $y(x,t)$ satisfies the partial differential equation*

$$xy_x + y_{xt} + 2y = 0,$$

and the boundary conditions

$$y(x,0) = 1, \quad y(0,t) = \mathrm{e}^{-at},$$

where $a \geq 0$. Find the Laplace transform, $\overline{y}(x,s)$, of the solution, and hence derive an expression for $y(x,t)$ in the domain $x \geq 0$, $t \geq 0$.

In terms of $\overline{y}(x,s)$, the problem is

$$\begin{cases} x\overline{y}_x + s\overline{y}_x - y_x(0) + 2\overline{y} = 0 \\ \overline{y}(0,s) = \frac{1}{s+a} \end{cases}$$

where $y_x(0) = 0$ by the condition $y(x,0) = 1$. Therefore,

$$\begin{aligned} (x+s)\overline{y}_x + 2\overline{y} &= 0 \\ \int \frac{\mathrm{d}\overline{y}}{\overline{y}} &= -2\int \frac{\mathrm{d}x}{x+s} \\ \log \overline{y} &= -2\log(x+s) + C \\ \overline{y}(x,s) &= \frac{C'}{(x+s)^2} \qquad\qquad \frac{1}{a+s} = \frac{C'}{s^2} \\ \overline{y}(x,s) &= \frac{s^2}{(a+s)(x+s)^2}. \end{aligned}$$

We can now find the solution $y(x,t)$ either using transform pairs ($\mathrm{e}^{-\alpha t}u(t) \leftrightarrow \frac{1}{s+\alpha}$, $t\mathrm{e}^{-\alpha t}u(t) \leftrightarrow \frac{1}{(s+\alpha)^2}$) or using complex contour integration:

$$\begin{aligned} y(x,t) &= \frac{1}{2\pi i}\int_{-i\infty+1}^{+i\infty+1} \frac{s^2\mathrm{e}^{st}}{(a+s)(x+s)^2}\,\mathrm{d}s \\ &= \mathrm{Res}\left(\tfrac{s^2\mathrm{e}^{st}}{(a+s)(x+s)^2}; s=-a\right) + \mathrm{Res}\left(\tfrac{s^2\mathrm{e}^{st}}{(a+s)(x+s)^2}; s=-x\right) \\ &= \frac{a^2\mathrm{e}^{-at}}{(a-x)^2} + \left(\tfrac{\mathrm{d}}{\mathrm{d}s}\tfrac{s^2\mathrm{e}^{st}}{a+s}\right)\Big|_{s=-x} \\ &= \frac{a^2\mathrm{e}^{-at}}{(a-x)^2} + \left(\tfrac{2s\mathrm{e}^{st}+s^2t\mathrm{e}^{st}}{a+s} - \tfrac{s^2\mathrm{e}^{st}}{(a+s)^2}\right)\Big|_{s=-x} \\ &= \frac{a^2\mathrm{e}^{-at}}{(a-x)^2} + \frac{-2x\mathrm{e}^{-xt} + x^2t\mathrm{e}^{-xt}}{a-x} - \frac{x^2\mathrm{e}^{-xt}}{(a-x)^2}. \end{aligned}$$