Simluation of Elasticity, Biomechanics, and Virtual Surgery
Problem Session II

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1 Elasticity

The goal in this problem session is to use a (variable-coefficient) Poisson solver (either one you coded from Problem Session I or the one we provide) to implement the Neo-Hookean model of elasticity in dimension $d = 1$. Recall that the equations of elasticity are given by the boundary value problem

$$\rho_0(\mathbf{X})\frac{\partial^2 \mathbf{u}}{\partial t^2} = \nabla^{\mathbf{X}} \cdot \mathbf{P} + \mathbf{f}^{\text{ext}} \quad \in \Omega(t) \tag{1a}$$

$$\mathbf{u}(\cdot, t) = \mathbf{g}(\cdot, t) \quad \in \partial\Omega_d(t) \tag{1b}$$

$$(\mathbf{P} \cdot \hat{\mathbf{n}})(\cdot, t) = \mathbf{h}(\cdot, t) \quad \in \partial\Omega_n(t) \tag{1c}$$

where ρ_0 is the mass density (as a function of $\mathbf{X}$, the undeformed coordinates); $\mathbf{u} = \phi - \mathbf{X}$ is the unknown displacement; $\mathbf{P}$ is the first Piola-Kirchoff stress (which takes a specific form for Neo-Hookean, to be given later); $\mathbf{f}^{\text{ext}}$ is the given (external) force; and $\mathbf{g}$ and $\mathbf{h}$ specify the Dirichlet and Neumann boundary conditions, respectively. For simplicity, we'll assume a uniform mass density, i.e., $\rho_0 \equiv 1$.

2 Neo-Hookean

The Neo-Hookean model relates the stress $\mathbf{P}$ to the deformation gradient $\mathbf{F}$ via

$$\Psi(\mathbf{F}) = \frac{\mu}{2}(F_{ij}F_{ji} - 2) - \mu \log J + \frac{\lambda}{2}\log^2 J, \tag{2a}$$

$$\mathbf{P}(\mathbf{F}) = \frac{\partial \Psi}{\partial \mathbf{F}} = \mu\mathbf{F} + (\lambda \log J - \mu)\mathbf{F}^{-T}. \tag{2b}$$

(Recall that $J = \det \mathbf{F}$.) This manifests itself in dimension $d = 1$ in terms of the displacement u as

$$P(u) = \mu\left(\frac{du}{dX} + 1\right) + \left(\lambda \log\left(\frac{du}{dX} + 1\right) - \mu\right)\frac{1}{\frac{du}{dX} + 1}. \tag{3}$$

3 Inversion-Robust Neo-Hookean

Neo-Hookean as formulated above will not be robust to element inversions, due to the $\log J$ term. To remedy this, we replace the logarithm in (2a) with a cubic Taylor approximation around 1:

$$
\begin{aligned}
r(x) &= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \\
&= -\frac{11}{6} + 3x - \frac{3}{2}x^2 + \frac{1}{3}x^3,
\end{aligned} \tag{4a}
$$

$$
\begin{aligned}
r'(x) &= 1 - (x-1) + (x-1)^2 \\
&= 3 - 3x + x^2.
\end{aligned} \tag{4b}
$$

$$
\begin{aligned}
r''(x) &= -1 + 2(x-1) \\
&= -3 + 2x.
\end{aligned} \tag{4c}
$$

This gives

$$
\Psi(\mathbf{F}) = \frac{\mu}{2}\left(F_{ij}F_{ji} - 2\right) - \mu r(J) + \frac{\lambda}{2} r(J)^2, \tag{5a}
$$

$$
\mathbf{P}(\mathbf{F}) = \frac{\partial \Psi}{\partial \mathbf{F}} = \mu \mathbf{F} + \left(\lambda r(J) - \mu\right) r'(J) \frac{\partial J}{\partial \mathbf{F}}. \tag{5b}
$$

Again, specializing this for $d = 1$ dimension, we obtain

$$
P(u) = \mu\left(\frac{du}{dX} + 1\right) + \left(\lambda r\left(\frac{du}{dX} + 1\right) - \mu\right) r'\left(\frac{du}{dX} + 1\right). \tag{6}
$$

4 Quasistatics

We begin by studying eqs (1) at equilibrium, which is the basis for quasistatic evolution. This reduces the equations to

$$
-\nabla^{\mathbf{X}} \cdot \mathbf{P} = \mathbf{f}^{\text{ext}}. \tag{7}
$$

The equivalent weak formulation is

$$
\int_{\Omega_0} w_{i,j} P_{ij} d\mathbf{X} = \int_{\partial\Omega_n} w_i h_i dS(\mathbf{X}) + \int_{\Omega_0} w_i f_i^{\text{ext}} d\mathbf{X} \tag{8}
$$

for all test functions $\mathbf{w}$. Recall that summation over repeated indices is implied, and comma'ed indices indicate differentiation.

As for Poisson, we let the coordinates of $\mathbf{w}$ vary over the nodal basis functions N_i. In dimension $d = 1$, this reduces to the system of equations

$$
q_i(u) = \int_a^b \frac{\partial N_i}{\partial X} P(F(u(X))) dX - b_i = 0; \tag{9a}
$$

$$
b_i = \int_a^b N_i f^{\text{ext}} dX + [\text{Neumann boundary terms}] \tag{9b}
$$

for each grid vertex i. For Neo-Hookean, P depends *non-linearly* on the displacement u, hence one must use a non-linear solver, such as Newton iteration, to solve (9) for u. The Newton step looks like

$$
\frac{\partial q_i}{\partial u}\left(u^k\right) \Delta u + q_i\left(u^k\right) = 0; \tag{10a}
$$

$$
u^{k+1} = u^k + \Delta u \tag{10b}
$$

where

$$\frac{\partial q_i}{\partial u_j}(u) = \int_a^b \frac{\partial N_i}{\partial X} \frac{\partial P}{\partial F} (F(u)) \frac{\partial N_j}{\partial X} dX \tag{11}$$

Thus, the computation of Δu in each Newton iteration amounts to solving a variable coefficient Poisson problem. This coefficient is $\partial P / \partial F = \partial P / \partial \left(du/dX \right)$, which we can express via (6) as

$$\frac{\partial P}{\partial F} = \mu + \lambda r' \left(\frac{du}{dX} + 1 \right)^2 + \left(\lambda r \left(\frac{du}{dX} + 1 \right) - \mu \right) r'' \left(\frac{du}{dX} + 1 \right). \tag{12}$$

This gives all the necessary pieces to implement a quasistatics evolution of the elasticity equations (1).

5 Backward Euler

In the event that inertial terms are non-negligible, we must use some time-stepping scheme to solve (1). Here, we outline the implementation of Backward Euler. Since (1) is second-order in time, we must introduce an additional variable, $\mathbf{v} = \partial \mathbf{u} / \partial t$, to convert (1) to an equivalent first-order system:

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} &= \mathbf{v}, \\ \rho_0 \frac{\partial \mathbf{v}}{\partial t} &= \nabla^{\mathbf{X}} \cdot \mathbf{P} + \mathbf{f}^{\text{ext}}. \end{aligned} \tag{13}$$

The time discretization for Backward Euler thus gives:

$$\begin{aligned} \frac{\mathbf{u}^{m+1} - \mathbf{u}^m}{\Delta t} &= \mathbf{v}^{m+1}, \\ \rho_0 \frac{\mathbf{v}^{m+1} - \mathbf{v}^m}{\Delta t} &= \left(\nabla^{\mathbf{X}} \cdot \mathbf{P} \right)_{\mathbf{u}^{m+1}} + \mathbf{f}^{\text{ext}}. \end{aligned} \tag{14}$$

To solve these equations, we use the first to substitute for $\mathbf{v}^{m+1}$ in the second and rearrange terms to obtain (again, assuming $\rho \equiv 1$ for simplicity)

$$\mathbf{u}^{m+1} - \Delta t^2 \left(\nabla^{\mathbf{X}} \cdot \mathbf{P} \right)_{\mathbf{u}^{m+1}} = \Delta t^2 \mathbf{f}^{\text{ext}} + \Delta t \mathbf{v}^m + \mathbf{u}^m \tag{15}$$

which turns out to be very similar to (7). Indeed, the difference between the two systems essentially amounts to the addition of an identity matrix.