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Basic Examination, F05 (practice)

1. Let (X, d) and (Y, ρ) be connected metric spaces, and give the product $X \times Y$ the metric

$$D((x, y), (x', y')) = d(x, x') + \rho(y, y').$$

Prove the metric space $X \times Y$ is connected.

Solution

Suppose $X \times Y$ is not connected. Then there exists nonempty sets A, B such that $X \times Y = A \cup B$ with $A \cap \overline{B} = \overline{A} \cap B = \emptyset$. For $x \in X$, define

$$P_x = \{(x, y) \in X \times Y \mid y \in Y\}.$$

Then P_x is isomorphic to Y; it follows that $A \cap P_x$ and $B \cap P_x$ form separated sets of P_x, thus, as Y is connected, either $A \cap P_x$ or $B \cap P_x$ is empty. Thus P_x is completely contained within either A or B for all $x \in X$. Likewise, if we define

$$Q_y = \{(x, y) \in X \times Y \mid x \in X\}$$

for $x \in X$, Q_y is completely contained in either A or B for every $y \in Y$.

Now A and B are both nonempty, hence there exists some $x_A, x_B \in X$ such that $P_{x_A} \subset A$ and $P_{x_B} \subset B$. Likewise, there exists some $y_A, y_B \in Y$ such that $Q_{y_A} \subset A$ and $Q_{y_B} \subset B$. Thus $(x_A, y_B) \in A \cap B = \emptyset$, a contradiction. It follows that $X \times Y$ is connected.

2. Let X and Y be Banach spaces and let $T : X \to Y$ be a linear map such that there are constants $0 < c < C < \infty$ such that for all $x \in X$,

$$c\|x\|_X \leq \|T(x)\|_Y \leq C\|x\|_X.$$

Prove that the range $T(X) = \{y \in Y : y = T(x), \text{ some } x \in X\}$ is a closed subset of Y. Note: You cannot assume T maps X *onto* Y.

Solution

Let $\{y_n\}_{n=1}^{\infty}$ be a Cauchy sequence in Y. Then for each y_n, there exists some x_n such that $T(x_n) = y_n$. Thus

$$\|x_n - x_m\|_X \leq \frac{1}{c}\|T(x_n) - T(x_m)\|_Y = \frac{1}{c}\|y_n - y_m\|_Y,$$

from which it follows that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence (in X) as well. Since X is a Banach space, there exists some x^* such that $x_n \to x^*$ as $n \to \infty$. Set $y^* = T(x^*)$. Then

$$\|y^* - y_n\|_Y \leq \|T(x^*) - T(x_n)\|_Y \leq C\|x^* - x_n\|_X,$$

from which it follows that $y_n \to y^*$ as $n \to \infty$. Therefore Y is closed.

3. Give an example of a function $F : \mathbb{R}^2 \to \mathbb{R}$ such that the partial derivatives D_1F and D_2F exist and are continuous, and such that the mixed partials derivatives $D_{1,2}F(0,0)$ and $D_{2,1}F(0,0)$ exist, but

$$D_{1,2}(0,0) \neq D_{2,1}F(0,0).$$

Solution

Define $F(0, 0) = 0$ and

$$F(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}.$$

4. Let $U \subset \mathbb{R}^2$ be open and let $f : U \to \mathbb{R}$ a function such that each partial derivative

$$\frac{\partial f}{\partial x_j} = \lim_{h \to 0} \frac{f(x + he_j) - f(x)}{h}$$

exists *and is continuous in* U. Prove $f \in C^1(U)$. Note: (Above e_j denotes the j^{th} unit vector in $\mathbb{R}^n$.)

Solution

Without loss of generality, let $0 \in U$; we show that f is differentiable at 0, from which the argument can be extended to show that f is differentiable on all of U.

Let $x = \sum_{i=1}^{n} x_i e_i$ in some open ball of 0 contained in U. Set $y_0 = 0$ and

$$y_j = \sum_{i=1}^{j} x_i e_i$$

for $j = 1, \ldots, n$. Multiple applications of the Mean Value Theorem yields points y_j' between y_{j-1} and y_j such that

$$\begin{aligned} f(y_1) - f(y_0) &= \frac{\partial f}{\partial x_1}(y_1')(x_1 - 0), \\ f(y_2) - f(y_1) &= \frac{\partial f}{\partial x_2}(y_2')(x_2 - 0), \\ &\vdots \\ f(y_n) - f(y_{n-1}) &= \frac{\partial f}{\partial x_n}(y_n')(x_n - 0). \end{aligned}$$

Since $y_0 = 0$ and $y_n = x$, we see that

$$f(x) - f(0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(y_i') x_i.$$

Since each $\frac{\partial f}{\partial x_i}$ is continuous in U, we can further restrict x to a neighborhood of 0 such that, given any $\epsilon > 0$,

$$\left| \frac{\partial f}{\partial x_i}(z) - \frac{\partial f}{\partial x_i}(0) \right| < \epsilon$$

for all z in this neighborhood. Thus

$$\lim_{x \to 0} \frac{\left| f(x) - f(0) - \sum_i \frac{\partial f}{\partial x_i}(0) x_i \right|}{\|x\|} \le \lim_{x \to 0} \sum_i \left| \frac{\partial f}{\partial x_i}(y_i') - \frac{\partial f}{\partial x_i}(0) \right| \frac{|x_i|}{\|x\|} < \epsilon \lim_{x \to 0} \sum_i \frac{|x_i|}{\|x\|} < n\epsilon,$$

and since ϵ was arbitrary, we conclude that

$$f'(0)x = \sum_i \frac{\partial f}{\partial x_i}(0) x_i$$

and, in general, for $t \in U$,

$$f'(t)x = \sum_i \frac{\partial f}{\partial x_i}(t) x_i.$$

The continuity of f' follows from the above equality and the continuity of the partials.

5. Let $f(x)$ be a bounded real function on the interval $[0,1]$ such that f has a finite set of discontinuities. Prove that f is integrable on $[0,1]$.

Solution

Enumerate the discontinuities of f on $[0,1]$ by x_i, $i=1,\dots,n$. Since f is bounded, there exists an M such that $|f|<M$ on $[0,1]$.

Let $\epsilon>0$ be given. Choose δ such that $\delta<\epsilon/8Mn$ and the closures of the intervals $(x_i-\delta,x_i+\delta)$ are disjoint (possible since there are finitely many x_i's). Set

$$E=[0,1]\backslash\bigcup_i(x_i-\delta,x_i+\delta),$$

which is evidently a finite union of closed intervals, on each of which f is continuous. Then there exists a partition P of E such that

$$U(P,f|_E)-L(P,f|_E)<\frac{\epsilon}{2}.$$

Let $P'=P\cup\{0,1\}$ be viewed now as a partition of $[0,1]$, and set m_i and M_i to be the infimum and supremum of f over $[x_i-\delta,x_i+\delta]\cap[0,1]$. Then $M_i-m_i<2M$, hence

$$U(P',f)-L(P',f)<\frac{\epsilon}{2}+\sum_i(M_i-m_i)2\delta<\frac{\epsilon}{2}+4Mn\delta<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon,$$

and it follows that f is Riemann integrable.

6. Assume $a_n\geq a_{n+1}\geq 0$ and $\lim a_n=0$. Prove the series

$$\sum_{n=1}^{\infty}(-1)^n a_n$$

converges to a real number S, and prove that the partial sums

$$S_N=\sum_{n=1}^{N}(-1)^n a_n$$

satisfy

$$|S_N-S|\leq|a_{N+1}|.$$

Solution

Consider

$$S_{n+k}-S_n=\sum_{i=n+1}^{n+k}(-1)^i a_i=(-1)^{n+1}\left(a_{n+1}-a_{n+2}+\cdots+(-1)^{k-1}a_{n+k}\right).$$

Suppose first that k is odd. Then

$$\begin{aligned} d &= a_{n+1}-a_{n+2}+\cdots+(-1)^{k-1}a_{n+k} \\ &= a_{n+1}-(a_{n+2}-a_{n+3})-(a_{n+4}-a_{n+5})-\cdots-(a_{n+k-1}-a_{n+k}) \quad\leq\quad a_{n+1} \end{aligned}$$

since $a_i\geq a_{i+1}$. But also

$$d=(a_{n+1}-a_{n+2})+(a_{n+3}-a_{n+4})+\cdots+(a_{n+k-2}-a_{n+k-1})+a_{n+k}\geq a_{n+k}\geq 0.$$

Similarly, for k even,

$$d=a_{n+1}-(a_{n+2}-a_{n+3})-(a_{n+4}-a_{n+5})-\cdots-(a_{n+k-2}-a_{n+k-1})-a_{n+k}\leq a_{n+1},$$

and

$$d = (a_{n+1} - a_{n+2}) + (a_{n+3} - a_{n+4}) + \cdots + (a_{n+k-1} - a_{n+k}) \geq 0.$$

Hence

$$|S_{n+k} - S_n| \leq a_{n+1}.$$

Since $a_n \to 0$, it follows that $\{S_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, hence has some limit S. By letting $k \to \infty$ in the above inequality, we arrive at

$$|S - S_n| \leq a_{n+1}.$$

7. Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ be two points in $\mathbb{R}^n$. Prove the Cauchy-Schwarz inequality

$$\left|\sum_{j=1}^{n} x_j y_j\right| \leq \left(\sum x_j^2\right)^{\frac{1}{2}} \left(\sum y_j^2\right)^{\frac{1}{2}},$$

and show that equality holds in this inequality if and only if x and y are linearly dependent.

Solution

For all $\lambda \in \mathbb{R}$,

$$0 \leq \|x - \lambda y\|^2 = \|x\|^2 + \lambda^2 \|y\|^2 - 2\lambda(x \cdot y).$$

Now if $y = 0$, the claim is trivial, so suppose $y \neq 0$. Set $\lambda = (x \cdot y)/\|y\|^2$. Then

$$0 \leq \|x\|^2 + \frac{(x \cdot y)^2}{\|y\|^2} - 2\frac{(x \cdot y)}{\|y\|^2} = \|x\|^2 - \frac{(x \cdot y)^2}{\|y\|^2},$$

from which the claim follows immediately. Equality is evidently achieved if and only if $x = \lambda y$.