Basic Exam F01

Instructions: Do all of the following:

1. Let K be a compact set of real numbers and let $f(x)$ be a continuous function on K. Prove there exists $x_0 \in K$ such that $f(x) \leq f(x_0)$ for all $x \in K$.

2. Let N denote the positive integers, let $a_n = (-1)^n \frac{1}{n}$, and let α be any real number. Prove there is a one-to-one and onto mapping $\sigma : N \to N$ such that
$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \alpha.$$

3. Let E be a set of real numbers and let $\{f_n\}$ be a sequence of continuous real-valued functions on E. Prove that if $f_n(x)$ converges to $f(x)$ *uniformly* on E, then $f(x)$ is continuous on E. (Recall that $f_n(x)$ converges to $f(x)$ uniformly on E means that for every $\epsilon > 0$ there is N such that whenever $n > N$ and $x \in E$, $|f_n(x) - f(x)| < \epsilon$.)

4. Let S be the set of all sequences $(x_1, x_2, \ldots)$ such that for all n,
$$x_n \in \{0, 1\}.$$
Prove there does not exist a one-to-one mapping from the set $N = \{1, 2, \ldots\}$ *onto* the set S.

5. Suppose that $f : \mathbf{R}^2 \to \mathbf{R}$ is a continuous function such that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ of f exist everywhere and are continuous everywhere, and $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})$ and $\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$ also exist and are continuous everywhere. Prove that
$$\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$$
at every point of $\mathbf{R}^2$.

6. Suppose that $F : \mathbf{R}^2 \to \mathbf{R}^2$ is a continuously differentiable function with $F((0,0)) = (0,0)$ and with the Jacobian of F at $(0,0)$ equal to the identity matrix (i.e., if $F = (f_1, f_2)$ then $\frac{\partial f_i}{\partial x_j}\Big|_{(0,0)} = 1$ if $i = j$ and $= 0$ if $i \neq j$). Outline a proof that there exists $\delta > 0$ such that if $a^2 + b^2 < \delta$ then there is a point (x, y) in $\mathbf{R}^2$ with $F(x, y) = (a, b)$. (Prove this directly: do not just restate the Inverse Function Theorem. Your argument will be part of the proof of the Inverse Function Theorem. You may use any basic

estimation you need about the change in F being approximated by the differential of F without proof.)

7. If V is a real vector space and X is a subspace, let $V^* = \{f : V \to \mathbf{R} \,|\, f \text{ is linear }\}$ be the dual space of V and $X^o = \{f \in V^* \,|\, f(x) = 0 \text{ for all } x \in X\}$ be the annihilator of X. Let $T : V \to W$ be a linear transformation of finite dimensional real vector spaces. Recall that the *transpose* of T is the linear map $T^t : W^* \to V^*$ defined by $T^t(f) = f \circ T$. Prove the following:
   a. $\operatorname{im}(T)^o = \ker(T^t)$. [Here $\operatorname{im}(T)$ is the image or range of T and $\ker(T)$ is the kernel or null space of T.]
   b. $\dim \operatorname{im}(T) = \dim \operatorname{im}(T^t)$.

8. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the rotation by 60^o counterclockwise about the plane perpendicular to the vector $(1,1,1)$ and $S : \mathbf{R}^3 \to \mathbf{R}^3$ be the reflection about the plane perpendicular to the vector $(1,0,1)$. Determine the matrix representation of $S \circ T$ in the standard basis $\{(1,0,0), (0,1,0), (0,0,1)\}$. You do not have to multiply the resulting matrices but you must determine any inverses that arise.

9. Let A be a real symmetric matrix. Prove that there exists an invertible matrix P such that $P^{-1}AP$ is diagonal.
   [You cannot just quote a theorem, but must prove it from scratch.]

10. Let V be a complex vector space and $T : V \to V$ a linear transformation. Let $v_1, \ldots, v_n$ be non-zero vectors in V, each an eigenvector of a different eigenvalue. Prove that $\{v_1, \ldots, v_n\}$ is linearly independent.