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Basic Examination, S04

1. Let $\mathcal{S}$ denote the set of sequences $a = (a_1, a_2, \ldots)$, with $a_k = 0$ or 1. Show that the mapping $\theta : \mathcal{S} \to \mathbb{R}$ defined by
$$\theta((a_1, a_2, \ldots)) = \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots$$
is an injection. Include an explanation of why the infinite series converges.
Hint: if $a \neq b$, you may assume that
$$a = (a_1, \ldots, a_{n-1}, 0, a_{n+1}, \ldots),$$
$$b = (b_1, \ldots, b_{n-1}, 1, b_{n+1}, \ldots).$$

Solution

As in the hint, suppose $a \neq b$, $a, b \in \mathcal{S}$. Without loss of generality,
$$a = (a_1, \ldots, a_{n-1}, 0, a_{n+1}, \ldots),$$
$$b = (a_1, \ldots, a_{n-1}, 1, b_{n+1}, \ldots)$$
for some n, hence
$$\theta(b) - \theta(a) = \frac{1}{10^n} + \sum_{i=n+1}^{\infty} \frac{b_i - a_i}{10^i},$$
and
$$\left| \sum_{i=n+1}^{\infty} \frac{b_i - a_i}{10^i} \right| \leq \sum_{i=n+1}^{\infty} \frac{1}{10^i} = \frac{1}{9 \cdot 10^n},$$
so
$$\theta(b) - \theta(a) \geq \frac{1}{10^n} - \frac{1}{9 \cdot 10^n} = \frac{8}{9 \cdot 10^n} > 0$$
and $\theta(b) \neq \theta(a)$.
The infinite series converges since the partial sums are monotonically increasing and bounded above by
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{10^i} = \lim_{n\to\infty} \frac{\frac{1}{10} - \frac{1}{10^{n+1}}}{1 - \frac{1}{10}} = \frac{1}{9}.$$

2. Is $f(x) = \sqrt{x}$ uniformly continuous on $[0, \infty)$? Prove your assertion.

Solution

If $x, z \in [0, \infty)$, then
$$\sqrt{x+z} \leq \sqrt{x + z + 2\sqrt{xz}} = \sqrt{x} + \sqrt{z}.$$
Now given any $\epsilon > 0$, set $\delta = \epsilon^2$. Then for any $x, y \in [0, \infty)$ with $x \leq y < x + \delta$ we have, letting $z = y - x$ in the above inequality,
$$\sqrt{y} = \sqrt{x + (y - x)} \leq \sqrt{x} + \sqrt{y - x},$$
hence
$$0 \leq \sqrt{y} - \sqrt{x} \leq \sqrt{y - x} < \sqrt{\delta} = \epsilon,$$
proving that f is uniformly continuous on $[0, \infty)$.

3. (a) Carefully define when a function f on $[0, 1]$ is Riemann integrable.

(b) Show that if f_n are Riemann integrable functions on $[0,1]$ and f_n converges to f uniformly, then f is Riemann integrable.

Solution

(a) (F03.4)

(b) Given $\epsilon > 0$, choose N such that $\|f_n - f\|_\infty < \epsilon$ for $n > N$. Then $f_n(x) - \epsilon < f(x) < f_n(x) + \epsilon$ for all $x \in [0,1]$ and $n > N$, hence

$$\int_0^1 f_n dx - \epsilon < \underline{\int_0^1} f dx \leq \overline{\int_0^1} f dx < \int_0^1 f_n dx + \epsilon,$$

so

$$\overline{\int_0^1} f dx - \underline{\int_0^1} f dx < 2\epsilon.$$

Since ϵ was arbitrary, this shows that f is Riemann integrable.

Incidentally, a second application of the above inequalities shows that

$$\left| \int_0^1 f dx - \int_0^1 f_n dx \right| < \epsilon$$

for $n > N$, hence

$$\lim_{n\to\infty} \int_0^1 f_n dx = \int_0^1 f dx.$$

4. Are there infinite compact subsets of $\mathbb{Q}$? Prove your assertion.

Solution

Let

$$A = \{0\} \cup \bigcup_{n=1}^{\infty} \left\{ \frac{1}{n} \right\}.$$

We show that A is sequentially compact. Indeed, any sequence $\{x_n\}_{n=1}^\infty$ in A must itself converge to 0, for there are only finitely many elements of A outside of $B(0;\epsilon)$ for any $\epsilon > 0$, hence infinitely many of the x_n's within $B(0;\epsilon)$. It follows that A is compact.

5. Suppose that G is an open set in $\mathbb{R}^n$, $f : G \to \mathbb{R}^m$ is a function, and that $x_0 \in G$.

(a) Carefully define what is meant by $f'(x_0) : \mathbb{R}^n \to \mathbb{R}^m$.

(b) Suppose that I is a line segment in G such that $f'(x)$ is defined for all $x \in I$. Show that if f is differentiable at all the points of I, then for some point c in I

$$\|f(q) - f(p)\|_2 \leq \|f'(c)\| \|q - p\|_2.$$

Hint: let w be a unit vector with $\|f(q) - f(p)\|_2 = (f(q) - f(p)) \cdot w$.

Solution

(a) f is differentiable at x_0 if there exists a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{x\to x_0} \frac{f(x) - f(x_0) - T(x - x_0)}{\|x - x_0\|} = 0.$$

In this case, $f'(x_0) = T$.

(b) Let

$$w = f(q) - f(p)$$

and define $g : [0,1] \to \mathbb{R}$ by

$$g(t) = f((1-t)p + tq) \cdot w.$$

Then g is differentiable on $[0,1]$, so an application of the Mean Value Theorem yields $t \in [0,1]$ such that

$$g(1) - g(0) = g'(t) = f'((1-t)p + tq)(q-p) \cdot w = f'(c)(q-p) \cdot w,$$

and $c \in I$. But

$$g(1) - g(0) = f(q) \cdot w - f(p) \cdot w = (f(q) - f(p)) \cdot w = \|w\|^2,$$

hence, applying the Cauchy Schwarz inequality,

$$\|w\|^2 = f'(c)(q-p) \cdot w \le \|f'(c)\| \|q-p\| \|w\|$$

and it follows that

$$\|f(q) - f(p)\| = \|w\| \le \|f'(c)\| \|q-p\|.$$

6. Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$.

(a) Prove that there exists a constant d with $\|x\| \le d\|x\|_2$ for all $x \in \mathbb{R}^n$, and use this to show that $N(x) = \|x\|$ is continuous in the usual topology on $\mathbb{R}^n$.

(b) Prove that there exists a constant c with $\|x\| \ge c\|x\|_2$ (Hint: use the fact that N is continuous on the sphere $\{x : \|x\|_2 = 1\}$).

(c) Show that if L is an n-dimensional subspace of an arbitrary normed vector space V, then L is closed.

Solution

(a) Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$ with respect to $\|\cdot\|_2$, and let

$$x = \sum_{i=1}^{n} x_i e_i.$$

Then

$$\|x\| \le \sum_{i=1}^{n} \|x_i e_i\| = \sum_{i=1}^{n} |x_i| \|e_i\| \le M \sum_{i=1}^{n} |x_i| \le M \sum_{i=1}^{n} \|x\|_2 = nM\|x\|_2$$

where $M = \max_i \|e_i\|$. It follows that $N(x) = \|x\|$ is uniformly continuous with respect to $\|\cdot\|_2$. Indeed, given $\epsilon > 0$, for any $x, y \in \mathbb{R}^n$ with $\|x - y\|_2 < \epsilon/d$,

$$|N(x) - N(y)| = |\|x\| - \|y\|| \le \|x - y\| \le d\|x - y\|_2 < \epsilon.$$

(b) Since N is continuous on $S^n = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$, which is compact, N achieves its minimum $c = N(u)$ for some $u \in S^n$. Hence for any $x \in \mathbb{R}^n$,

$$\|x\| = \|x\|_2 \left\| \frac{x}{\|x\|_2} \right\| = \|x\|_2 N\left(\frac{x}{\|x\|_2}\right) \ge \|x\|_2 N(u) = c\|x\|_2.$$

(c) L must be isomorphic to $(\mathbb{R}^n, \|\cdot\|)$ for some norm $\|\cdot\|$ on $\mathbb{R}^n$. Thus we show L is closed if we can show $\mathbb{R}^n$ is closed with respect to $\|\cdot\|$. To this end, let $\{x_i\}_{i=1}^n$ be a Cauchy sequence in $\mathbb{R}^n$ with respect to $\|\cdot\|$. Then by (a) above, $\{x_i\}$ is a Cauchy sequence with respect to $\|\cdot\|_2$ as well, hence converges to some $x^* \in \mathbb{R}^n$ with respect to $\|\cdot\|_2$. By (b) above, then, $x_i \to x^*$ with respect to $\|\cdot\|$. Therefore, $\mathbb{R}^n$ is complete, so closed, with respect to $\|\cdot\|$.

7. Let V be a finite dimensional real vector space. Let $W_1, W_2 \subset V$ be subspaces. Show both of the following:

   (a) $W_1^\circ \cap W_2^\circ = (W_1 + W_2)^\circ$

   (b) $(W_1 \cap W_2)^\circ = W_1^\circ + W_2^\circ$

   (Note: W_i° is the annihilator of W_i.)

   Solution

   (a) (S03.7)

   (b) The claim follows from the fact that $W^{\circ\circ} = W$ for any subspace W of V.

8. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation about the axis $(1, 0, -1)$ by an angle of $30°$ (you can use either orientation).

   (a) Find the matrix representation $A \in M_3(\mathbb{R})$ of T in the standard basis. (You do not have to multiply out matrices but must evaluate inverses.)

   (b) Find all the eigenvalues of $A \in M_3(\mathbb{R})$.

   (c) Find all the eigenvalues of $A \in M_3(\mathbb{C})$.

   Solution

   (a) Let
   $$B_T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$
   Then regarding the columns of B_T as an orthonormal basis, the matrix representation of T in this basis is
   $$[T]_{B_T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 30° & \sin 30° \\ 0 & -\sin 30° & \cos 30° \end{pmatrix},$$
   so the matrix representation of T in the standard basis is
   $$T = B_T [T]_{B_T} B_T^{-1} = B_T [T]_{B_T} B_T^t.$$

   (b) The only real eigenvalue of T is 1.

   (c) Considering eigenvalues in $\mathbb{C}$, we have additionally that $e^{i30°}$ and $e^{-i30°}$ are eigenvalues of T.

9. Let V be a finite dimensional real inner product space under $(\cdot, \cdot)$ and $T : V \to V$ a linear operator. Show the following are equivalent:

   (a) $(Tx, Ty) = (x, y)$ for all $x, y \in V$.

   (b) $\|Tx\| = \|x\|$ for all $x \in V$.

   (c) $T^*T = I$, where T^* is the adjoint of T.

   (d) $TT^* = I$.

Solution

Clearly, (a) implies (b) by setting $y = x$. To show that (b) implies (a),

$$2(Tx, Ty) = (T(x+y), T(x+y)) - (Tx, Tx) - (Ty, Ty) = (x+y, x+y) - (x, x) - (y, y) = 2(x, y).$$

Suppose (a). Then for $x, y \in V$,

$$(x, T^*Ty - y) = (x, T^*Ty) - (x, y) = (Tx, Ty) - (x, y) = 0,$$

so letting $x = T^*Ty - y$ allows us to conclude that $T^*Ty - y = 0$ for all y, hence $T^*T = I$, and (c) is satisfied. Conversely, given (c),

$$(x, T^*Ty - y) = 0$$

for all $x, y \in V$, and the expansion above gives (a).

(c) and (d) imply each other, since a left (right) inverse is also a right (left) inverse.

10. Let T be a real symmetric matrix. Show that T is similar to a diagonal matrix.

(You cannot use the Spectral Theorem.)

Solution

(F01.9)